Spectral analysis for linearizations of the Allen-Cahn equation around rescaled stationary solutions with triple-junction

Dissertation

zur Erlangung des Doktorgrades der Naturwissenschaften

(Dr. rer. nat.) der Fakultät für Mathematik

der Universität Regensburg

vorgelegt von

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Regensburg

2006

Promotionsgesuch eingereicht am: 10.01.2006

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Introduction

Phase-field models

Via formal calculations, Bronsard and Reitich, [BR], studied the asymptotic behavior of the vector-valued Ginzburg-Landau equation

$$\frac{\partial}{\partial t} u^\epsilon = 2\epsilon^2 \triangle u^\epsilon - (DW(u^\epsilon))^T \tag{1}$$

$$\frac{\partial}{\partial \nu} u^\epsilon|_{\partial\Omega} = 0 \text{ or } u^\epsilon(x,t)|_{\partial\Omega} = h(x) \tag{2}$$

$$u^\epsilon(x,0) = g(x) \tag{3}$$

as $\epsilon \to 0$. We consider this equation on an open domain $\Omega \subset \mathbb{R}^n$ and for $u^\epsilon : \Omega \times \mathbb{R}_+ \to \mathbb{R}^m$, where $m, n \geq 2$. The potential $W : \mathbb{R}^m \to \mathbb{R}$ is smooth and attains its minimum value zero at exactly three distinct points a, b, and c, so as to model a three-phase physical system. Instead of equation (1), we can also consider the vector-valued Allen-Cahn equation

$$\epsilon^2 \frac{\partial}{\partial t} v^\epsilon = \epsilon^2 \triangle v^\epsilon - \left(D\hat{W}(v^\epsilon) \right)^T. \tag{4}$$

Equation (1) equals (4) via

$$\hat{W}(x) := \frac{1}{2} W(x),$$

and

$$v^\epsilon(x,t) := u^\epsilon \left(x, \frac{1}{2\epsilon^2} t \right).$$

The question is how the solution u^ϵ of (1), (2), and (3) behaves as $\epsilon \to 0$. The phase-field parameter $\epsilon > 0$ represents the thickness of the transition layer between different phases. Therefore, we expect that u^ϵ approaches a sharp interface model as $\epsilon \to 0$. One such sharp interface model is the mean-curvature flow. Roughly speaking, this is a family $(\Gamma_t)_{t \in [0,t]}$ of smooth manifolds in $\mathbb{R}^n$ such that the signed distance function $d(.,t)$ of Γ_t fulfills

$$\triangle d(x,t) = \frac{\partial}{\partial t} d(x,t), \quad t \in [0,T], \quad x \in \Gamma_t.$$

A precise definition is given in [AS]. For $m = 1$, i.e. the scalar Allen-Cahn equation, de Mottoni and Schatzman, [deMS], proved that there exist initial data for the Allen-Cahn equation such that the corresponding solutions converge to the minima of W uniformly outside each tubular neighborhood of Γ_t as $\epsilon \to 0$. Essentially, the proof is a rigorous justification of formal asymptotic expansion, i.e. it is supposed that in a tubular neighborhood of $(\Gamma_t)_{t\in[0,T]}$ the solution u^ϵ is approximately given by the asymptotic expansion

$$u_A^\epsilon(x,t) = \sum_{i=0}^{N} \epsilon^i u_i \left(\frac{d_\epsilon(x,t)}{\epsilon}, x, t \right), \quad t \in [0,T], \quad x \in \Gamma_t(\delta).$$

Note that $\Gamma_t(\delta) := \{x \in \Omega : \operatorname{dist}(x, \Gamma_t) < \delta\}$. The function d_ϵ is the modified distance function, i.e.

$$d_\epsilon(x,t) = d(x,t) + \sum_{i=1}^{N} \epsilon^i d_i(x,t), \quad t \in [0,T], \quad x \in \Gamma_t(\delta).$$

If one puts u_A^ϵ into the Allen-Cahn equation, expands the term $DW\left(u_A^\epsilon\right)$ via Taylor expansion, and arranges the terms according to their ϵ-power, the results are equations for the u_i of the form

$$L_0 u_i = R_{i-1}\left(d_{i-1}\right), \tag{5}$$

where $R_{i-1}\left(d_{i-1}\right)$ depends only on known quantities and the function d_{i-1} which is not determined so far. The operator L_0 has domain $H^2(\mathbb{R}, \mathbb{C})$ and is given by

$$L_0 u = -u'' + D^2 W(\theta_0) u.$$

The function θ_0 is the unique increasing solution of

$$-\theta'' + DW(\theta) = 0, \quad \theta(0) = 0,$$

that connects the two distinct minima of W. Equation (5) has a solution if and only if

$$R_{i-1}\left(d_{i-1}\right) \in \ker(L_0)^\perp.$$

This determines d_{i-1}, as $\dim \ker(L_0) = 1$. As the solutions of (5) decay at an exponential rate, the approximate solution u_A^ϵ can be extended to Ω. The result is a family of approximate solutions $\left(u_A^\epsilon\right)_{\epsilon\in(0,1)}$ such that

$$u_A^\epsilon(x,t) = \theta_0\left(\frac{d(x,t)}{\epsilon}\right) + O(\epsilon^2), \quad x \in \Gamma_t(\delta),$$

and

$$\epsilon^2 \frac{\partial}{\partial t} u_A^\epsilon - \epsilon^2 \triangle u_A^\epsilon + DW\left(u_A^\epsilon\right) = O\left(\epsilon^k\right), \quad \epsilon \to 0.$$

The integer $k \in \mathbb{N}$ grows with the length of the asymptotic expansion. Important for the proof of the convergence $u_A^\epsilon \to u^\epsilon$ is to analyze the behavior of the smallest eigenvalue λ_1^ϵ of the operator

$$L_\epsilon = -\frac{d^2}{dz^2} + D^2W(\theta_0) \tag{6}$$

that is equipped with Neumann boundary conditions in $L^2\left(\left(-\frac{1}{\epsilon}, \frac{1}{\epsilon}\right), \mathbb{C}\right)$. This delivers the $[deMS]$-estimate for the Allen-Cahn operator, i.e. the smallest eigenvalue of

$$-\epsilon^2 \triangle + D^2W\left(u_A^\epsilon\right)$$

behaves like $O(\epsilon^2)$, $\epsilon \to 0$. The operator that is given by the differential expression

$$-\epsilon^2 \triangle + D^2W\left(u_A^\epsilon\right)$$

is called Allen-Cahn operator. It represents the linearization of the Allen-Cahn equation around the approximate solution u_A^ϵ.

Concerning the vector valued Allen-Cahn equation, Bronsard and Reitich proved short time existence for the problem of three curves Γ_i moving by mean curvature such that the three curves meet at a triple-junction $m(t)$, and the other end point of each curve lies on the boundary of Ω - cf. figure 1.

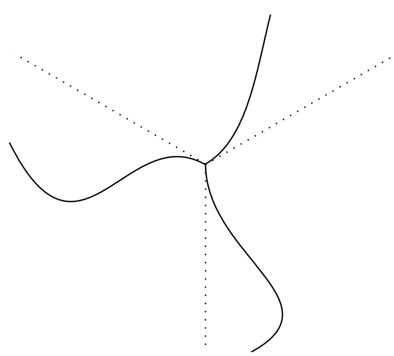

Figure 1: Three-phase boundary motion.

Via formal asymptotic expansion, Bronsard and Reitich obtained the evolution laws of three-phase boundary motion derived by material scientists. At the triple junction $m(t)$, they used the expansion

$$u^\epsilon(x,t) \approx \sum_{i=0}^{N} \epsilon^i u_i\left(\frac{x - m(t)}{\epsilon}, t\right).$$

For the function u_0, the expansion leads to the equation

$$-\triangle u_0 + \left(DW(u_0)\right)^T = 0.$$

Moreover, in directions tangentially to the interfaces, one expects that u_0 approaches the standing wave solution that connects two minima of W. The existence of such an u_0 was rigorously proved in the work of [BGS], details given in chapter 3. This is the first step in the proof of rigorous convergence to the limiting flow. If one pursues the formal calculation to determine the u_i's, he is led to equations of the form

$$\mathcal{L}_0 u_i = R_{i-1}. \tag{7}$$

The function R_{i-1} depends only on known quantities, and $\mathcal{L}_0$ is given by the differential expression

$$-\triangle + D^2 W(u_0).$$

The operator $\mathcal{L}_0$ was introduced in [BGS]. It's domain is given by the set of all elements in $\left(H^2(\mathbb{R}^2, \mathbb{C})\right)^2$ that are equivariant with respect to the symmetry group G of the equilateral triangle. A byproduct of the proofs in [BGS] is that $\mathcal{L}_0$ is self-adjoint and positive semidefinite.

Target of the endevours

Now, we consider the case $m = 2$. In [BGS], they proved the existence of a solution θ_0 of

$$-\theta_0'' + (DW(\theta_0))^t = 0$$

which connects two distinct global minima of W and fulfills

$$\sup_{x\in\mathbb{R}} |u_0(x,y) - \theta_0(x)| \to 0, \quad y \to \infty. \tag{8}$$

In this work, we show that the convergence in (8) produces a strong connection between the essential spectrum of $\mathcal{L}_0$ and the spectrum of the operators L_ϵ^{odd}, $\epsilon \geq 0$. The operator L_ϵ^{odd} is given by the restriction of the vector valued version of L_ϵ (cf. (6)) to a certain subspace.

Set $\lambda_1^{\epsilon,odd} = \min \sigma(L_\epsilon^{odd})$. The first main result of this work is the following Theorem (Theorem 3.1).

Theorem *Suppose* dim ker $(L_0) = 1$. *Then the following statements hold:*

1. *We have*

$$\min \sigma_e(\mathcal{L}_0) = \liminf_{\epsilon\to 0} \lambda_1^{\epsilon,odd} > 0,$$

and

$$\sigma(L_0^{odd}) \subset \sigma_e(\mathcal{L}_0).$$

2. *For each* $\lambda \in \sigma_p(\mathcal{L}_0) \cap (-\infty, \min \sigma_e(\mathcal{L}_0))$, *and* $\delta \in (\frac{1}{2}, 1)$, *there exists a constant* $C > 0$ *such that for each normalized* $\psi \in \ker(\mathcal{L}_0 - \lambda)$, *we have*

$$\forall x \in \mathbb{R}^2 : |\psi(x)| \leq Ce^{-(1-\delta)\sqrt{\min \sigma_e(\mathcal{L}_0)-\lambda}|x|}.$$

3. *Suppose* $E < \min \sigma_e(\mathcal{L}_0)$, $\psi \in D_{\mathcal{L}_0}$, *and* $R \in L^2_G(\mathbb{R}^2)$ *such that*

$$(\mathcal{L}_0 - E)\psi = R.$$

Assume there exist $c, a > 0$ *such that*

$$|R(x)| \leq ce^{-a|x|}$$

for a.e. $x \in \mathbb{R}^2$. *Then there exists a constant* $C > 0$ *and* $\delta \in (\frac{1}{2}, 1)$ *such that*

$$|\psi(x)| \leq Ce^{-(1-\delta)\sqrt{\min \sigma_e(\mathcal{L}_0)-E}|x|}$$

for each $x \in \mathbb{R}^2$.

Further, we introduce sesquilinear forms $\mathcal{T}_\epsilon$ in the Hilbert space $L^2_G(T_\epsilon)$ where T_ϵ is the equilateral triangle of edge length $\frac{2}{\epsilon}$. The space $L^2_G(T_\epsilon)$ contains the elements of $(L^2(T_\epsilon, \mathbb{C}))^2$ that are equivariant with respect to the symmetry group of the equilateral triangle.

Definition *Let* $\epsilon \in (0, 1)$. *Define*

$$D_{\mathcal{T}_\epsilon} := H^1_G(T_\epsilon),$$

and

$$\mathcal{T}_\epsilon[u, v] := \int_{T_\epsilon} \sum_{j=1,2} \langle \nabla u_j, \nabla v_j \rangle + \left\langle D^2W(u_0)u, v \right\rangle dx$$

for $u, v \in D_{\mathcal{T}_\epsilon}$. *Set*

$$\nu_1^\epsilon := \inf_{\substack{u \in D_{\mathcal{T}_\epsilon} \\ \|u\|_{L^2_G(T_\epsilon)}=1}} \mathcal{T}_\epsilon[u, u].$$

Set $\mu_1^0 = \min \sigma(\mathcal{L}_0)$, and denote the radius of the incircle of T_ϵ with $\varrho(\epsilon)$. The second main result is given by the following Theorem (Theorem 3.2), which concerns the behavior of ν_1^ϵ as $\epsilon \to 0$. A motivation for the study of this problem is given in chapter 4.

Theorem *Suppose* $\dim \ker(L_0) = 1$*. Then the following statements hold:*

1. *If* $[0, \min \sigma_e(\mathcal{L}_0)) \cap \sigma_d(\mathcal{L}_0) = \emptyset$*, then*
$$\liminf_{\epsilon \to 0} \nu_1^\epsilon \geq \min \sigma_e(\mathcal{L}_0).$$

2. *If* $\mu_1^0 \in [0, \min \sigma_e(\mathcal{L}_0)) \cap \sigma_d(\mathcal{L}_0) \neq \emptyset$*, then*
$$\left|\nu_1^\epsilon - \mu_1^0\right| = O\left(\varrho(\epsilon) e^{-\frac{\sqrt{\min \sigma_e(\mathcal{L}_0) - \mu_1^0}}{8}\varrho(\epsilon)}\right), \quad \epsilon \to 0.$$

Description of the work

Chapter 1
This chapter deals with vector-valued Sturm-Liouville operators. We study the tunneling effect, i.e. the exponential decay of eigenfunctions. We prove that the strength of the tunneling effect does not depend on the length of the underlying interval, provided the coefficients are uniformly bounded in some Banach spaces. This result is proved in section two. The proof is a generalization of results in chapter 3 of [HS] to the case of vector-valued Sturm-Liouville operators in weighted L^2-spaces on not necessarily unbounded domains. In order to obtain exponential decay for higher order derivatives, we analyze the range space of the operators. In section four of chapter one, we investigate how the eigenvalues of the operators behave as the length of the underlying interval tends to infinity.

Chapter 2
This chapter starts with an existence result for standing wave solutions that connect two distinct global minima of a potential W. Symmetry is considered. The crucial point of chapter 2 is Lemma 2.1, especially for the considerations in chapter 3. Essentially, it deals with the convergence of the eigenvalues $\lambda_1^\epsilon < \lambda_2^\epsilon < ...$ of L_ϵ to the eigenvalues $\lambda_1 < \lambda_2 < ... < \min \sigma_e(L_0)$ of the operator L_0. In contrast to [deMS, C, ABC], the operators L_ϵ, $\epsilon \geq 0$, are vector-valued. A few statements in Lemma 2.1 are given in [C] for the scalar case. Some ideas of the proofs enter into Lemma 2.1. But we can not take over the proofs, as the argumentation is based on Harnacks principle, comparison principle, etc. We also investigate the limit $\lambda_1^{\epsilon,odd} \to \lambda_1^{0,odd}$, which is important for the proofs in chapter three.

Chapter 3
The third chapter starts with a general consideration of Sobolev spaces that own a symmetry. In section 2, we repeat the results of [BGS] that we need for this work. The main results of chapter three are Theorem 3.1 and Theorem 3.2. First, we prove statement one of Theorem 3.1. The statement on exponential decay in Theorem 3.1 is the analogue of the results in section 1 of chapter 1. But

a generalization of the proofs does not lead to the result, as in general there is no ball such that the potential $D^2W(u_0)$ of $\mathcal{L}_0$ is positive definite outside this ball. In this case the tunneling effect is produced by an operator-valued barrier. We introduce operators $\mathcal{L}_\epsilon$ in the Hilbert space $L^2_G(\Omega_\epsilon)$ where Ω_ϵ is a suitable regularized version of the equilateral triangle T_ϵ. The space $L^2_G(\Omega_\epsilon)$ contains the elements of $\left(L^2(\Omega_\epsilon, \mathbb{C})\right)^2$ that are equivariant with respect to the symmetry group of the equilateral triangle.

Definition *Set*

$$D_{\mathcal{L}_\epsilon} := \left\{ u \in H^2(\Omega_\epsilon, \mathbb{C}) : \frac{\partial u}{\partial \nu} = 0 \ \textit{on} \ \partial\Omega_\epsilon \right\}^2 \cap L^2_G(\Omega_\epsilon),$$

and

$$\mathcal{L}_\epsilon := -\triangle + \left[D^2W(u_0)\right]$$

for $\epsilon > 0$.

We prove that the tunneling effect (i.e. exponential decay of the eigenfunctions) for $\mathcal{L}_\epsilon$, $\epsilon \geq 0$, is uniform in ϵ (Lemma 3.2).

Lemma *Let* $\beta > 0$. *There exists* $\epsilon_0 \in (0, 1)$ *such that for each* $\delta \in (\frac{1}{2}, 1)$, *there exists a constant* $C > 0$ *so that*

$$\forall \epsilon \in [0, \epsilon_0) : \forall E \in \sigma_p(\mathcal{L}_\epsilon) \cap (-\infty, \min \sigma_e(\mathcal{L}_0) - \beta] : \forall \psi \in \ker(\mathcal{L}_\epsilon - E), \textit{normalized} :$$

$$\forall x \in \Omega_\epsilon : |\psi(x)| \leq C e^{-(1-\delta)\sqrt{\min \sigma_e(\mathcal{L}_0) - E}|x|}.$$

This proves statement two of Theorem 3.1. Then we investigate the behavior of $\mu_1^\epsilon := \min \sigma(\mathcal{L}_\epsilon)$, $\epsilon > 0$, in the limit $\epsilon \to 0$ and obtain the following Lemma (Lemma 3.3).

Lemma

1. *If* $[0, \min \sigma_e(\mathcal{L}_0)) \cap \sigma_d(\mathcal{L}_0) = \emptyset$, *then*

$$\liminf_{\epsilon \to 0} \mu_1^\epsilon \geq \min \sigma_e(\mathcal{L}_0).$$

2. *If* $\mu_1^0 \in [0, \min \sigma_e(\mathcal{L}_0)) \cap \sigma_d(\mathcal{L}_0) \neq \emptyset$, *then*

$$\left|\mu_1^\epsilon - \mu_1^0\right| = O\left(\varrho(\epsilon) e^{-\frac{\sqrt{\min \sigma_e(\mathcal{L}_0) - \mu_1^0}}{8} \varrho(\epsilon)} \right), \quad \epsilon \to 0.$$

With the help of this lemma, we obtain Theorem 3.2. In section 4, we prove that all the stated assumptions are fulfilled for a typical potential in the theory of phase transitions.

Chapter 4
The work closes with chapter 4. On a formal level, we outline a possible application of the results which might help to prove the convergence of solutions of the Allen-Cahn equation.

Appendix
The appendix compiles parts of measure theory and spectral theory in Hilbert spaces of particular relevance for this work. Especially, the connection between sesquilinear forms and the discrete spectrum of the corresponding operators is outlined.

Acknowledgment

Finally, I want to thank Prof. Dr. Harald Garcke for supervising this work, especially for the suggestion of the example in section 3.4.

Notations

Numbers and vector spaces

For the different sets of numbers, we use the notations

$$
\begin{aligned}
&\mathbb{N} := \{1, 2, 3, ...\},\\
&\mathbb{N}_0 := \{0, 1, 2, 3, ...\},\\
&\mathbb{R}_\pm := \{x \in \mathbb{R} : \pm x > 0\},\\
&\mathbb{R}_+^\infty := \mathbb{R}_+ \cup \{+\infty\},\\
&\mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}.
\end{aligned}
$$

For a complex numer $z \in \mathbb{C}$, $\mathrm{Re}(z)$ denotes the real part of z and $\mathrm{Im}(z)$ is the imaginary part. If $z \in \mathbb{R}$, then $\mathrm{sign}(z)$ denotes the sign of z.

The standard scalar product in $\mathbb{C}^n$ is denoted by $\langle ., . \rangle$ and the corresponding norm with $|.|$. If $z \in \mathbb{R}$ is a real number, then $[z]$ is the largest integer equal or smaller than z.

Sets and mappings

For an arbitrary set X, we denote the power set of X by $\mathbb{P}(X)$. If $V, W \subset \mathbb{R}^n$ are open subsets, then we define

$$V \subset\subset W :\Leftrightarrow V \text{ bounded }, \overline{V} \subset W.$$

For a subset $U \subset \mathbb{R}^n$, define

$$\mathrm{diam}(U) := \sup\{|x - y| : x, y \in U\},$$

and denote the interior of U with $\mathring{U}$. Moreover, define $U^c := \mathbb{R}^n \setminus U$. Assume $f : X \to Y$ is a mapping between sets X and Y. Define

$$\mathrm{im}(f) := \{f(a) : a \in X\},$$

and set

$$\ker(f) := \{a \in X : f(a) = 0\},$$

provided X and Y are vector spaces. Assume X is a topological space and Y a vector space. Then define

$$\operatorname{supp}(f) := \overline{\{a \in X : f(a) \neq 0\}}.$$

Let $g : (0, a) \to \mathbb{R}$ for $a > 0$. Then $O(g(\epsilon))$ is the representative for a function $h : (0, a) \to \mathbb{R}$ such that

$$\exists C, \epsilon_0 > 0 : \forall \epsilon \in (0, \epsilon_0) \cap X : |h(\epsilon)| \leq C|g(\epsilon)|.$$

In this case, we write $h(\epsilon) = O(g(\epsilon)), \quad \epsilon \to 0$.

Banach and Hilbert spaces

Assume $(B_i, \|.\|_i)$, $i = 1, 2$, are Banach spaces over $\mathbb{K}$. If B_1 is finite dimensional, then dim B_1 denotes the dimension of B_1. For a subset $M \subset B_1$, we define

$$\text{lin } M := \left\{ \sum_{i}^{N} \lambda_i b_i : N \in \mathbb{N}, \lambda_i \in \mathbb{K}, b_i \in M \right\}.$$

For $x \in B_1$ and $r > 0$, we set

$$B_r(x) := \{y \in B_1 : \|x - y\|_1 < r\}.$$

If $(X, \|.\|_X)$ is a Banach space, then

$$\|z\|_{X^n} := \left(\sum_{i=1}^{n} \|z_i\|_X^2 \right)^{\frac{1}{2}}, \quad z \in X^n,$$

is a norm on X^n. For simplicity, we denote this norm also with $\|.\|_X$. If X is a Hilbert space with scalar product $\langle ., . \rangle_X$, then

$$\langle y, z \rangle_{X^n} := \sum_{i=1}^{n} \langle y_i, z_i \rangle_X, \quad y, z \in X^n,$$

is a scalar product on X^n which we denote by $\langle ., . \rangle_X$. If a sequence x_n in a Hilbert space X converges weakly to $x \in X$, then we write $x_n \xrightarrow{w} x$.

Operators

Assume $(B_i, \|.\|_i)$, $i = 1, 2$, are Banach spaces over $\mathbb{K}$. A operator T from B_1 to B_2 is a linear map $T : D_T \to B_2$ such that D_T is a linear subspace of B_1. The set D_T is called the domain of T. If $B_1 = B_2$, then T is called operator in B_1. Suppose T and S are operators from B_1 to B_2. Then T is called a restriction of S ($T \subset S$) if $D_T \subset D_S$ and $T = S|_{D_T}$. If T_i, $i = 1, ..., n$, are operators from B_1 to B_2, then $\otimes_{i=1}^n T_i$ is the operator from ${B_1}^n$ to ${B_2}^n$ with domain

$$D_{\otimes_{i=1}^n T_i} := \times_{i=1}^n D_{T_i},$$

and

$$\otimes_{i=1}^n T_i \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} T_1 x_1 \\ \vdots \\ T_n x_n \end{pmatrix}, \quad x_i \in D_{T_i}, \quad i = 1, ..., n.$$

Suppose B_i, $i = 1, 2$, are Hilbert spaces and T is an operator from B_1 to B_2 that is densely defined, i.e. D_T is dense in B_1. Define

$$D_{T^*} := \{ y \in B_2 : x \in D_T \longmapsto \langle Tx, y \rangle \text{ is continuous } \}$$

and

$$T^* y = z :\Leftrightarrow \forall x \in D_T : \langle Tx, y \rangle = \langle x, z \rangle$$

for $y \in D_{T^*}$ and $z \in B_1$. Then T^* is an operator from B_2 to B_1 and is called the adjoint operator of T. The set of all continuous linear maps $T : B_1 \to B_2$ is denoted by $\mathcal{L}(B_1, B_2)$. We use the convention $\mathcal{L}(B_1) := \mathcal{L}(B_1, B_1)$. For $T \in \mathcal{L}(B_1, B_2)$, define the norm

$$\|T\|_{\mathcal{L}(B_1, B_2)} := \sup_{x \neq 0} \frac{\|Tx\|_2}{\|x\|_1}.$$

Moreover, we define

$$M(m, \mathbb{K}) := \mathcal{L}\left(\mathbb{K}^m\right),$$

$$Gl(m, \mathbb{K}) := \{ T \in M(m, \mathbb{K}) : T \text{ is invertible } \},$$

$$O(m) := \{ T \in M(m, \mathbb{R}) : T^* T = I \}.$$

The set of all symmetric matrices of $M(m, \mathbb{R})$ is denoted by $S(\mathbb{R}^m)$. The vector space $M(m, \mathbb{K})$ becomes a Hilbert space together with the scalar product

$$\langle A, B \rangle_{tr} := tr(B^* A), \quad A, B \in M(m, \mathbb{K}).$$

The corresponding norm is denoted by $\|.\|_{tr}$. The identity operator in $M(m, \mathbb{K})$ is denoted by $I_{\mathbb{K}^m}$. For $A \in M(m, \mathbb{K})$, denote the determinant of A by $\det(A)$.

Hölder spaces

For an open set $\Omega \subset \mathbb{R}^n$ and a Banach space $(B, \|.\|)$, we define the following function spaces:

$C^k(\Omega, B)$	Set of functions $u : \Omega \to B$ that are k-th times continuously differentiable.
$C_b^k(\Omega, B)$	Elements u of $C^k(\Omega, B)$ such that the derivative $D^\alpha u$ is bounded on Ω for all $\lvert\alpha\rvert \leq k$.
$C^k(\overline{\Omega}, B)$	Elements u of $C^k(\Omega, B)$ such that the derivative $D^\alpha u$ is uniformly continuous for all $\lvert\alpha\rvert \leq k$.

If $u \in C_b^k(\Omega, B)$, we define the norm

$$\|u\|_{C^k(\Omega)} := \max_{0 \leq |\alpha| \leq k} \sup_{x \in \Omega} \|D^\alpha u(x)\| .$$

Further, we set

$$C^\infty(\Omega, B) := \bigcap_{k \in \mathbb{N}_0} C^k(\Omega, B),$$

and

$$C_0^\infty(\Omega, B) := \{u \in C^\infty(\Omega, B) : \operatorname{supp}(u) \subset\subset \Omega\} .$$

Sobolev spaces

For a measurable set $\Omega \subset \mathbb{R}^n$ and a measurable function $J : \Omega \to \mathbb{R}_+$, we denote the space of all measurable functions $u : \Omega \to \mathbb{K}$ such that

$$\int_\Omega |u(x)|^2 J(x) dx < \infty$$

with $L_J^2(\Omega, \mathbb{K})$. Equipped with the norm

$$\|u\|_{L_J^2(\Omega)} = \|u\|_{L_J^2} := \left(\int_\Omega |u(x)|^2 J(x) dx \right)^{\frac{1}{2}} ,$$

$L_J^2(\Omega, \mathbb{K})$ becomes a Hilbert space. Hence $L^2(\Omega, \mathbb{K}) = L_1^2(\Omega, \mathbb{K})$. The scalar product of $L_J^2(\Omega, \mathbb{K})$ and $L^2(\Omega, \mathbb{K})$ is denoted by $\langle ., . \rangle_{L_J^2}$ and $\langle ., . \rangle_{L^2}$, respectively. Moreover, we define

$$L^\infty(\Omega, \mathbb{K}) := \{u : \Omega \to \mathbb{K} : u \text{ measurable}, \exists C > 0 : |f| \leq C \text{ a.e. } \} .$$

The norm is given by

$$\|u\|_{L^\infty(\Omega)} := \operatorname{ess\,sup} |u|.$$

For $k \in \mathbb{N}_0$ and an open subset $\Omega \subset \mathbb{R}^n$, we define the following spaces:

$H^k(\Omega, \mathbb{K})$	Space of k-th times weakly differentiable functions $u : \Omega \to \mathbb{K}$ such that the derivatives are square integrable.
$H^{k,\infty}(\Omega, \mathbb{K})$	Vector space of k-th times weakly differentiable functions $u : \Omega \to \mathbb{K}$ with derivatives in $L^\infty(\Omega)$.
$H_{loc}^{k,\infty}(\Omega, \mathbb{K})$	Space of k-th times weakly differentiable functions $u : \Omega \to \mathbb{K}$ such that $u \in H^{k,\infty}(V, \mathbb{K})$ for each $V \subset\subset \Omega$.

The norm on $H^k(\Omega, \mathbb{K})$ is given by

$$\|u\|_{H^k(\Omega)} := \left(\sum_{0\leq|\alpha|\leq k} \|D^\alpha u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}.$$

Moreover, set

$$\|u\|_{H^{k,\infty}(\Omega)} := \max_{0\leq|\alpha|\leq k} \|D^\alpha u\|_{L^\infty(\Omega)}$$

for $u \in H^{k,\infty}(\Omega, \mathbb{K})$. Finally, the closure of $C_0^\infty(\Omega, \mathbb{K})$ with respect to $\|.\|_{H^k(\Omega)}$ is denoted by $\overset{\circ}{H^k}(\Omega, \mathbb{K})$.

Differential expressions

Assume $\Omega \subset \mathbb{R}^n$ is open. For the (weak) derivatives, we use the following notations:

$$
\begin{array}{rl}
D^\alpha u := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u & \text{for } u \in H^k_{loc}(\Omega, \mathbb{K}) \text{ and a mulitindex } \alpha,\ |\alpha| \leq k. \\
Du := \left(\frac{\partial}{\partial x_j} u_i \right)_{\substack{i=1,\dots,m \\ j=1,\dots,n}} & u \in (H^1_{loc}(\Omega, \mathbb{K}))^m. \\
D^2 u := \left(\frac{\partial^2}{\partial x_i \partial x_j} u \right)_{i,j=1,\dots,n} & \text{if } u \in H^2_{loc}(\Omega, \mathbb{K}). \\
\nabla u := \left(\frac{\partial}{\partial x_i} u \right)_{i=1,\dots,n} & \text{for } u \in H^1_{loc}(\Omega, \mathbb{K}). \\
\operatorname{div} u := \sum_{i=1}^n \frac{\partial}{\partial x_i} u_i & \text{provided } u \in (H^1_{loc}(\Omega, \mathbb{K}))^n. \\
\triangle u := \operatorname{div} \nabla u & \text{for } u \in H^2_{loc}(\Omega, \mathbb{K}).
\end{array}
$$

Suppose T is an operator in $H^k(\Omega, \mathbb{K})$. If there exist $a_{ij}, b_i, c \in L^\infty(\Omega, \mathbb{R})$ such that we have $C_0^\infty(\Omega, \mathbb{K}) \subset D_T$ and

$$(Tu)(x) = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} u(x) + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i} u(x) + c(x)u(x)$$

for each $u \in C_0^\infty(\Omega, \mathbb{K})$, then $[T]$ denotes the mapping

$$[T] : H^{k+2}_{loc}(\Omega, \mathbb{K}) \to H^k_{loc}(\Omega, \mathbb{K}),$$

given by

$$u \longmapsto \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} u + \sum_{i=1}^n b_i \frac{\partial}{\partial x_i} u + cu$$

for each $u \in H^{k+2}_{loc}(\Omega, \mathbb{K})$.

Boundaries and Green's formula

Assume $\Omega \subset \mathbb{R}^n$ is an open, bounded subset and $k \in \mathbb{N}_0$. We say $\partial\Omega \in C^k$ if for each x_0 there exists $f \in C^k(\mathbb{R}^{n-1}, \mathbb{R})$ and $r > 0$ such that - up to a coordinate transformation - we have

$$\Omega \cap B_r(x_0) = \{x \in B_r(x_0) : x_n > f(x_1, ..., x_{n-1})\}.$$

Assume $\partial\Omega \in C^1$. Then there exists a continuous outward pointing unit normal

$$\nu : \partial\Omega \to \mathbb{R}^n.$$

For $m \in \mathbb{N}$, the normal derivative with respect to ν is given by

$$\frac{\partial}{\partial \nu} : \left(C^1(\overline{\Omega}, \mathbb{K})\right)^m \to \left(C^0(\partial\Omega, \mathbb{K})\right)^m,$$

$$\left(\frac{\partial}{\partial \nu} u\right)(x) := \left(\langle \nabla u_j(x), \nu(x)\rangle\right)_{j=1,\dots,m}, \quad u \in \left(C^1(\overline{\Omega}, \mathbb{K})\right)^m, \quad x \in \partial\Omega.$$

The mapping extends to

$$\frac{\partial}{\partial \nu} : \left(H^2(\Omega, \mathbb{K})\right)^m \to \left(L^2(\partial\Omega, \mathbb{K})\right)^m.$$

Instead of $\frac{\partial}{\partial \nu} u$ we also write $\frac{\partial u}{\partial \nu}$. According to [E, Theorem 3, p. 628], we have

$$\sum_{j=1}^{m} \int_\Omega \langle \nabla v_j, \nabla u_j \rangle \, dx = -\sum_{j=1}^{m} \int_\Omega \triangle v_j \overline{u_j} dx + \int_{\partial\Omega} \left\langle \frac{\partial v}{\partial \nu}, u \right\rangle dS \tag{9}$$

for $u_j \in H^1(\Omega, \mathbb{K})$, $v_j \in H^2(\Omega, \mathbb{K})$, $j = 1, ..., m$. We refer to equation (9) by "Green's formula" or "integration by parts".

Chapter 1

Vector-valued Sturm-Liouville operators

In this section, we prove that the tunneling effect for vector-valued Sturm-Liouville operators with uniformly bounded coefficients does not depend on the length of the underlying interval. We use these results to investigate the convergence of eigenvalues when the endpoints of the interval tend to infinity. In this section, we always consider complex Hilbert spaces, i.e. the Sobolev spaces consist of complex-valued functions.

Definition 1.1. *For $a \in \mathbb{R}_+^\infty$, let*

$$V_a \in C_b^0\left((-a,a), S\left(\mathbb{R}^m\right)\right),$$

$$J_a \in C_b^1\left((-a,a), \mathbb{R}\right).$$

The smallest eigenvalue of $V_a(x)$, $|x| < a$, is denoted by $\lambda_a(x)$. Suppose the family $(J_a, V_a)_{a \in \mathbb{R}_+^\infty}$ has the following properties:

1. $\exists K \in \mathbb{R}_+ : \forall a \in \mathbb{R}_+^\infty : \|V_a\|_{C^0((-a,a))} \leq K$.
2. $\lambda_\pm := \liminf_{x \to \pm\infty} \lambda_\infty(x), > 0$.
3. $\forall \beta > 0 : \exists a_0 \in \mathbb{R}_+ : \forall a > a_0 : \forall a > |x| \geq a_0 : V_a(x) \geq \min\left(\lambda_+, \lambda_-\right) - \beta$.
4. $\exists m, M \in \mathbb{R}_+ : \forall a \in \mathbb{R}_+^\infty : m \leq J_a \leq M, \|J_a\|_{C^1((-a,a))} \leq M$.

Then we define the operator P_a in $L^2_{J_a}((-a,a), \mathbb{C})$ by

$$D_{P_a} := H^1((-a,a), \mathbb{C}),$$

and

$$P_a u := iu'$$

for $a \in \mathbb{R}_+^\infty$. Set

$$H_a := \otimes_{i=1}^m P_a^* P_a + [V_a] .$$

Further, set $\lambda := \min(\lambda_+, \lambda_-)$, and define

$$F_E^a := \left\{ |x| < a : \lambda_a(x) \geq \frac{E + \lambda}{2} \right\}$$

for $E \in \mathbb{R}$.

The spaces $H^k((-a, a), \mathbb{C})$, $\overset{\circ}{H}{}^k ((-a, a), \mathbb{C})$ and $S(\mathbb{R}^m)$ are defined on p. 16 ff., and $[V_a]$ denotes the bounded operator in $\left(L^2_{J_a}((-a, a), \mathbb{C})\right)^m$ that is given by the multiplication with V_a - cf. definition B.6.

Proposition 1.1. *The operator H_a, $a \in \mathbb{R}_+^\infty$, is self-adjoint in $\left(L^2_{J_a}((-a, a), \mathbb{C})\right)^m$, and*

$$D_{H_a} = \left\{ u \in H^2((-a, a), \mathbb{C}) : u'(\pm a) = 0 \right\}^m .$$

Proof. According to corollary B.2, the operator $[V_a]$ is bounded. In view of corollary B.3, it is self-adjoint. We only have to prove that $P_a^* P_a$ is self-adjoint. This follows from [We1, Satz 4.11], as P_a is closed. Moreover,

$$D_{P_a^*} = \overset{\circ}{H}{}^1 ((-a, a), \mathbb{C}),$$

and

$$P_a^* u = i \frac{1}{J} (Ju)'.$$

Suppose $v \in \overset{\circ}{H}{}^1 ((-a, a), \mathbb{C})$. If $u \in D_{P_a}$, then we have

$$\langle P_a u, v \rangle_{L^2_{J_a}} = \langle iu', J_a v \rangle_{L^2((-a,a))} = \left\langle u, i \frac{1}{J_a} (J_a v)' \right\rangle_{L^2_{J_a}} .$$

This proves $\overset{\circ}{H}{}^1 ((-a, a), \mathbb{C}) \subset D_{P_a^*}$, and

$$P_a^* u = i \frac{1}{J_a} (J_a u)', \quad u \in \overset{\circ}{H}{}^1 ((-a, a), \mathbb{C}).$$

Conversely, assume $v \in D_{P_a^*}$. This implies that

$$u \in H^1((-a, a), \mathbb{C}) \longmapsto \langle P_a u, v \rangle_{L^2_{J_a}}$$

is continuous with respect to $\|.\|_{L^2_{J_a}}$. On account of [We1, Satz 2.15], there exists $w \in L^2_{J_a}((-a, a), \mathbb{C})$ such that

$$\forall \varphi \in H^1((-a, a), \mathbb{C}) : \langle \varphi', i J_a v \rangle_{L^2((-a,a))} = \langle \varphi, w \rangle_{L^2((-a,a))} . \tag{1.1}$$

This proves $J_a v \in H^1((-a,a),\mathbb{C})$. According to assumption 4 in definition 1.1, we have $v \in H^1((-a,a),\mathbb{C})$. It is left to show $v(\pm a) = 0$. Choose $\varphi_n^\pm \in C^\infty(\mathbb{R},\mathbb{R})$ such that

$$1_{B_{1/n}(\pm a)} \leq \varphi_n^\pm \leq 1_{B_{2/n}(\pm a)}.$$

Green's formula and (1.1) yield

$$\left\langle \varphi_n^\pm, w + i(J_a v)' \right\rangle_{L^2((-a,a))} = i\varphi_n^\pm J_a v|_{-a}^{+a} = iJ_a(\pm a)v(\pm a). \tag{1.2}$$

With theorem A.1, we conclude that the left side in (1.2) converges to zero as $n \to \infty$. Hence $v \in \overset{\circ}{H}{}^1((-a,a),\mathbb{C})$. $\square$

1.1 Exponential L^2-bounds

If $\Omega \subset \mathbb{R}^n$ is a nonempty open subset, $J : \Omega \to \mathbb{R}_+$ a weight of the Lebesgue measure, then we say that a measurable function $g : \Omega \to \mathbb{R}$ is exponential L^2-bounded by a measurable function $G : \Omega \to \mathbb{R}_+$ (with respect to Jdx) if and only if

$$\int_\Omega |g(x)|^2 e^{2G(x)} J(x)dx < +\infty.$$

In this context, we denote G the exponential bound of g. The aim of this subsection is to prove uniform exponential L^2-boundedness for eigenfunctions of H_a with eigenvalues λ_a such that $\sup_a \lambda_a < \lambda = \min(\lambda_+, \lambda_-)$. It turns out that there is some clearance for the choice of the uniform exponential bound G.

Definition 1.2. *Define*

$$f_\alpha(x) := \frac{(1-\delta)\sqrt{\lambda - E}|x|}{1 + \alpha(1-\delta)\sqrt{\lambda - E}|x|}$$

for $E \in (-\infty, \lambda]$, $\alpha \geq 0$, *and* $\delta \in \left(\frac{1}{2}, 1\right]$.

It is easy to see that $f_0 \in H^{1,\infty}_{loc}(\mathbb{R},\mathbb{R})$. Moreover, we have $f_\alpha \in H^{1,\infty}(\mathbb{R},\mathbb{R})$, for $\alpha > 0$. Precisely, $f_\alpha \leq \frac{1}{\alpha}$, $\alpha > 0$, and $|f_\alpha'|^2 \leq \lambda - E$, $\alpha \geq 0$. To motivate the following definition, let us neglect the potential and assume

$$-\triangle\psi = \lambda\psi.$$

Formally

$$-e^{f_\alpha}\triangle e^{-f_\alpha}\left(e^{f_\alpha}\psi\right) = \lambda\left(e^{f_\alpha}\psi\right).$$

Thus, we determine a L^2-bound for the eigenfunctions of $-U^{-1}\triangle U$, where $Ux := e^{-f_\alpha}x$.

Definition 1.3. *For $a \in \mathbb{R}_+^\infty$, $\alpha \geq 0$, $\delta \in \left(\frac{1}{2}, 1\right]$, and $E \in \mathbb{R}$, we define*

$$B_{a,\alpha}[u,v] := \sum_{j=1}^{m} \left\langle -iu_j' + if_\alpha' u_j, -iv_j' - if_\alpha' v_j \right\rangle_{L^2_{J_a}} + \langle [V_a - E]\, u, v \rangle_{L^2_{J_a}},$$

and

$$D_{B_{a,\alpha}} := \left(H^1((-a,a),\mathbb{C})\right)^m.$$

Proposition 1.2. *Let $a \in \mathbb{R}_+^\infty$, $\phi \in \left(H^1((-a,a),\mathbb{C})\right)^m$, and $E \in (-\infty, \lambda]$, such that*

$$\forall j \in \{1,...,m\} : \mathrm{supp}(\phi_j) \subset F_E^a.$$

Then we have

$$\forall \delta \in \left(\frac{1}{2}, 1\right] : \forall \alpha \geq 0 : \mathrm{Re}\left(B_{a,\alpha}[\phi]\right) \geq (2\delta - 1)\frac{\lambda - E}{2}\|\phi\|^2_{L^2_{J_a}}.$$

Proof. For $u \in H^1((-a,a),\mathbb{C})$, calculation yields the equation

$$\langle -iu' + if_\alpha' u, -iu' - if_\alpha' u \rangle_{L^2_{J_a}} =$$

$$\|u'\|^2_{L^2_{J_a}} + 2i\mathrm{Im}\left(\langle u', f_\alpha' u\rangle_{L^2_{J_a}}\right) - \|f_\alpha' u\|^2_{L^2_{J_a}}.$$

We obtain

$$B_{a,\alpha}[\phi] := \|\phi'\|^2_{L^2_{J_a}} + 2i\mathrm{Im}\left(\langle \phi', f_\alpha' \phi\rangle_{L^2_{J_a}}\right) - \|f_\alpha' \phi\|^2_{L^2_{J_a}} + \langle [V_a - E]\,\phi, \phi\rangle_{L^2_{J_a}}.$$

Moreover,

$$\|f_\alpha'\phi\|^2_{L^2_{J_a}} \leq \sum_{j=1}^{m} \left\langle |f_0'|^2 \phi_j, \phi_j \right\rangle_{L^2_{J_a}} \leq (1-\delta)\,\langle (\lambda - E)\phi, \phi\rangle_{L^2_{J_a}}.$$

Taking

$$V_a(x) \geq \lambda_a, \quad |x| < a,$$

into account, we obtain

$$\mathrm{Re}(B_{a,\alpha}[\phi]) \geq (\delta - 1)\,\langle (\lambda - E)\phi, \phi\rangle_{L^2_{J_a}} + \langle (\lambda_a - E)\phi, \phi\rangle_{L^2_{J_a}}.$$

In view of

$$\mathrm{supp}(\phi_j) \subset F_E^a,$$

we have

$$\langle (\lambda_a - E)\phi, \phi\rangle_{L^2_{J_a}} \geq \frac{\lambda - E}{2}\|\phi\|^2_{L^2_{J_a}}.$$

This completes the proof. ∎

A definition of the spectral parts of a self-adjoint operator is given in the appendix.

Proposition 1.3. *Let $a \in \mathbb{R}_+^\infty$. Suppose the function $\eta \in C^\infty((-a,a),\mathbb{R})$ fulfills $\eta' \in C_0^\infty((-a,a),\mathbb{R})$. Then for each $E \in \sigma_p(H_a)$, $E \le \lambda$, $\psi \in \ker(H_a - E)$, $\alpha > 0$, and $\delta \in \left(\frac{1}{2}, 1\right]$, we have*

$$B_{a,\alpha}[\eta e^{f_\alpha}\psi] = \left\langle \zeta e^{2f_\alpha}\psi, \psi \right\rangle_{L^2_{J_a}},$$

where

$$\zeta_\alpha := |\eta'|^2 + 2\eta\eta' f_\alpha', \quad \alpha > 0,$$

has compact support.

Proof. Note that

$$(J_a(\eta\psi_j)')' = (J_a\eta')'\psi_j + 2J_a\eta'\psi_j' + \eta(J_a\psi_j')', \tag{1.3}$$

for $j = 1, ..., m$. For each $u \in H^1((-a,a),\mathbb{C})$ and $\alpha > 0$, we have

$$e^{f_\alpha}\left(e^{-f_\alpha}u\right)' = i(-iu' + if_\alpha' u),$$

and

$$e^{-f_\alpha}\left(e^{f_\alpha}u\right)' = i(-iu' - if_\alpha' u).$$

The last two equations imply

$$B_{a,\alpha}[\eta e^{f_\alpha}\psi] = -\sum_{j=1}^m \left\langle (J_a(\eta\psi_j)')', \eta e^{2f_\alpha}\psi_j \right\rangle_{L^2((-a,a))} + \langle [V_a - E]\,\phi, \phi\rangle_{L^2_{J_a}} +$$

$$\sum_{j=1}^m \eta e^{2f_\alpha}\psi_j J_a(\eta\psi_j)'\Big|_{-a}^{a}.$$

The boundary terms vanish. In view of (1.3) and

$$H_a\psi = E\psi,$$

we obtain

$$B_{a,\alpha}[\eta e^{f_\alpha}\psi] = -\sum_{j=1}^m \left\langle (J_a\eta')'\,\psi_j, \eta e^{2f_\alpha}\psi_j \right\rangle_{L^2((-a,a))} - \tag{1.4}$$

$$2\sum_{j=1}^m \left\langle J_a\eta'\psi_j', \eta e^{2f_\alpha}\psi_j \right\rangle_{L^2((-a,a))}.$$

Integration by parts shows

$$-\left\langle (J_a\eta')'\,\psi_j, \eta e^{2f_\alpha}\psi_j \right\rangle_{L^2((-a,a))} = \left\langle J_a\eta'\psi_j, \eta' e^{2f_\alpha}\psi_j \right\rangle_{L^2((-a,a))} +$$

$$\left\langle J_a\eta', 2\eta f_\alpha' e^{2f_\alpha}\psi_j^2 \right\rangle_{L^2((-a,a))} + \left\langle J_a\eta', 2\eta e^{2f_\alpha}\psi_j\psi_j' \right\rangle_{L^2((-a,a))}. \tag{1.5}$$

The sum of the third addends in the right side of (1.5) cancels with the second sum in (1.4). Summation yields the assertion. $\square$

As already mentioned above, the eigenvalues have to stay away a fixed distance $\beta > 0$ from λ in order to obtain a uniform exponential bound for the corresponding eigenfunctions. Essentially, this bound is given by β.

Proposition 1.4. *Let $\beta > 0$. Then there exists $a_0 \in \mathbb{R}_+$ such that for each $\frac{1}{2} < \delta \leq 1$*

$$\exists C = C(\delta, \lambda, K, \beta, a_0) > 0 : \forall a \in \mathbb{R}_+^\infty, a > a_0 : \forall E \in \sigma_p(H_a) \cap (-\infty, \lambda - \beta] :$$

$$\forall \psi \in \ker(H_a - E), \|\psi\|_{L^2_{J_a}} = 1 : \left\|e^{f_0}\psi\right\|^2_{L^2_{J_a}} \leq C.$$

Proof. Throughout this proof, we choose $\alpha > 0$. On account of assumption 3 in definition 1.1, applied with $\frac{\beta}{2}$, there exists $\tilde{a}_0 \in \mathbb{R}_+$ such that

$$\{x \in \mathbb{R} : \tilde{a}_0 \leq |x| < a\} \subset F_E^a$$

for each $a > \tilde{a}_0$ and $E \leq \lambda - \beta$. Choose a cut-off function $\eta \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\eta \equiv 1 \quad \text{on} \quad \mathbb{R}^n \backslash B_{\tilde{a}_0+1}(0),$$
$$\eta \equiv 0 \quad \text{on} \quad \overline{B_{\tilde{a}_0}(0)},$$

and

$$0 \leq \eta \leq 1 \quad \text{on} \quad \mathbb{R}.$$

Set $a_0 := \tilde{a}_0 + 1$. Choose $\frac{1}{2} < \delta \leq 1$ arbitrarily, and define

$$C(\delta, \lambda, K, \beta, a_0) := e^{2a_0\sqrt{\lambda+K}} \left(1 + \frac{2}{(2\delta - 1)\beta} \sup_{|x| \leq a_0} |\eta'| \Big(|\eta'| + 2\sqrt{2\lambda + \beta}\Big)\right).$$

In what follows, we need

$$\sup_{|x| \leq a_0} e^{2f_0(x)} \leq e^{2a_0\sqrt{\lambda+K}},$$

and

$$\sup_{|x| \leq a_0} \left|\zeta e^{2f_0(x)}\right| \leq e^{2a_0\sqrt{\lambda+K}} \sup_{|x| \leq a_0} |\eta'| \Big(|\eta'| + 2\sqrt{2\lambda + \beta}\Big).$$

For $a > a_0$,

$$E \in \sigma_p(H_a), E \leq \lambda - \beta,$$

and

$$\psi \in \ker(H_a - E),$$

we have

$$F_E^a \supset \operatorname{supp}(\eta\psi) \cap K_a(0).$$

Set

$$\phi_\alpha := \eta e^{f_\alpha}\psi, \quad \alpha > 0.$$

Proposition 1.2 and proposition 1.3 impliy

$$(2\delta - 1)\frac{\beta}{2}\|\phi_\alpha\|^2_{L^2_{J_a}} \le \left|\left\langle \zeta_\alpha e^{2f_\alpha}\psi, \psi\right\rangle_{L^2_{J_a}}\right|,$$

where

$$\zeta_\alpha := |\eta'|^2 + 2\eta\eta' f'_\alpha, \quad \alpha > 0.$$

In view of

$$|\zeta_\alpha| \le |\eta'|^2 + 2|\eta'|\sqrt{\lambda + K},$$

and

$$f_\alpha \le f_0,$$

we use Bepo Levi and send $\alpha \to 0$. It follows that

$$\left\|\eta e^{f_0}\psi\right\|^2_{L^2_{J_a}} \le \frac{2}{\beta(2\delta - 1)} \sup_{|x|\le a_0} |\eta'|\Big(|\eta'| + 2\sqrt{2\lambda + \beta}\Big)e^{2f_0(x)}.$$

This implies

$$\int_{-a}^{a} e^{2f_0(x)}|\psi(x)|^2 J_a(x)dx \le$$

$$\int_{\{|x|<a:\eta(x)=1\}} \eta^2 e^{2f_0(x)}|\psi(x)|^2 J_a(x)dx + \int_{|x|\le a_0} e^{2f_0(x)}|\psi(x)|^2 J_a(x)dx \le$$

$$\frac{2}{(2\delta - 1)\beta}\sup_{|x|\le a_0}|\zeta|e^{2f_0(x)} + \sup_{|x|\le a_0} e^{2f_0(x)} \le C(\delta, \lambda, K, \beta, a_0).$$

□

1.2 Pointwise exponential bounds

In order to obtain pointwise exponential bounds out of the L^2-bound, we have to prove that the derivative of an eigenfunction of H_a is locally estimable by its L^2-norm. Later we need the commutator of two operators. Let X be a Hilbert space and A, B operators in X. The commutator of A and B is given by

$$[A, B] := AB - BA.$$

Proposition 1.5. *Let $a \in \mathbb{R}^\infty_+$, and $\chi \in C^\infty_0(\mathbb{R}, \mathbb{R})$ such that $\chi'(\pm a) = 0$. Then for each $E \in \mathbb{R}$, there exists a constant $C > 0$, estimable from above in terms of $m, M, K, E, \|\chi\|_{C^2(\mathbb{R})}$, such that for each $\psi \in D_{H_a}$, and $\theta \in \left(L^2_{J_a}((-a, a), \mathbb{C})\right)^m$, the equation*

$$(H_a - E)\psi = \theta$$

implies

$$\|(\chi\psi)'\|_{L^2((-a,a))} \le C\Big[\|\psi\|_{L^2(K)} + \|\theta\|_{L^2(K)}\Big],$$

where

$$K := \operatorname{supp}(\chi) \cap (-a, a).$$

Proof. As $(H_a - E)\psi = \theta$ if and only if $(\otimes_{i=1}^n P_a^* P_a)\,\psi = \theta + [E - V_a]\psi$, the problem reduces to a single component. Define

$$T_a := P_a^* P_a,$$

and

$$C := \|\chi\|_{C^2(\mathbb{R})} \frac{M}{m}\left(4 + \frac{M}{m}\right).$$

Suppose $\theta \in L^2((-a,a),\mathbb{C})$ and $\psi \in D_{T_a}$ fulfill

$$T_a\psi = \theta.$$

Set $\mu := -1$. Clearly $\mu \in \rho(T_a)$, and $\chi\psi \in D_{T_a}$. It follows that

$$P_a(\chi\psi) = P_a(T_a+1)^{-1}(T_a+1)(\chi\psi) = \tag{1.6}$$

$$P_a(T_a+1)^{-1}\Big[(\theta+\psi)\chi + [T_a,[\chi]]\,\psi\Big].$$

The commutator $[T_a,[\chi]]$ has domain D_{T_a}, and

$$[T_a,[\chi]]\,u = i2P_a^*[\chi']u + \left[\chi'' + \chi'\frac{J'}{J}\right]u, \quad u \in D_{T_a}. \tag{1.7}$$

In view of

$$\|u\|_{L^2_{J_a}}^2 + \|P_a u\|_{L^2_{J_a}}^2 = \langle (T_a+1)u, u\rangle_{L^2_{J_a}} = \left\|(T_a+1)^{\frac{1}{2}}u\right\|_{L^2_{J_a}}^2, \quad u \in D_{T_a},$$

we have

$$P_a(T_a+1)^{-\frac{1}{2}} \in \mathcal{L}\left(L^2_{J_a}((-a,a),\mathbb{C})\right),$$

and

$$\left\|(T_a+1)^{-\frac{1}{2}}\right\|_{\mathcal{L}\left(L^2_{J_a}((-a,a),\mathbb{C})\right)}, \left\|P_a(T_a+1)^{-\frac{1}{2}}\right\|_{\mathcal{L}\left(L^2_{J_a}((-a,a),\mathbb{C})\right)} \leq 1. \tag{1.8}$$

On account of [We1, Satz 2.43], we conclude

$$(T_a+1)^{-\frac{1}{2}}P_a^* \subset \left(P_a(T_a+1)^{-\frac{1}{2}}\right)^*.$$

As the operator on the left side is densely defined, we get

$$\left(P_a(T_a+1)^{-\frac{1}{2}}\right)^* = \overline{(T_a+1)^{-\frac{1}{2}}P_a^*}.$$

If follows

$$P_a(T_a+1)^{-1}P_a^* \subset P_a(T_a+1)^{-\frac{1}{2}}\left(P_a(T_a+1)^{-\frac{1}{2}}\right)^*,$$

and as the operator on the left side is densely defined, we obtain

$$\overline{P_a(T_a+1)^{-1}P_a^*} = P_a(T_a+1)^{-\frac{1}{2}}\left(P_a(T_a+1)^{-\frac{1}{2}}\right)^*.$$

On account of [We1, Satz 2.36] and (1.8), we conclude

$$\left\|\overline{P_a(T_a+1)^{-1}P_a^*}\right\|_{\mathcal{L}\left(L^2_{J_a}((-a,a),\mathbb{C})\right)} \leq 1.$$

Owing to (1.7), we have

$$P_a(T_a+1)^{-1}\left[T_a,[\chi]\right] = i2\overline{P_a(T_a+1)^{-1}P_a^*}[\chi'] + P_a(T_a+1)^{-1}\left[\chi'' + \chi'\frac{J'}{J}\right] \quad (1.9)$$

on D_{T_a}. Via a priori estimates, we obtain from (1.6), (1.7), (1.8), and (1.9) that

$$\|(\chi\psi)'\|_{L^2(K)} \leq \|\chi\|_{C^2(\mathbb{R})}\frac{M}{m}\left[\|\theta\|_{L^2(K)} + 4\|\psi\|_{L^2(K)} + \frac{M}{m}\|\psi\|_{L^2(K)}\right] \leq$$

$$C\Big[\|\psi\|_{L^2(K)} + \|\theta\|_{L^2(K)}\Big].$$

□

Lemma 1.1. *Suppose $\beta > 0$. There exists $a_0 > 0$ such that for each $\frac{1}{2} < \delta \leq 1$, there exists a constant $C > 0$, estimable from above in terms of m, M, K, λ, a_0, δ, and β, such that*

$$\forall a \in \mathbb{R}_+^\infty, > a_0 : \forall E \in \sigma_p(H_a) \cap (-\infty, \lambda - \beta] : \forall \psi \in \ker(H_a - E) :$$

$$\forall |x| < a : |\psi(x)| \leq C\|\psi\|_{L^2_{J_a}} e^{-(1-\delta)\sqrt{\lambda - E}|x|}.$$

Proof. We cover the interval $(-a, a)$ with the support of cut-off functions. For $a \in \mathbb{R}_+$,$a > 1$, we define

$$x_{a,i} := \frac{a-1}{[a]}i, \quad i \in \mathbb{Z}, \quad |i| \leq [a],$$

and

$$x_{a,\pm([a]+1)} := \pm a.$$

For $a = \infty$, set

$$x_{\infty,i} := i, \quad i \in \mathbb{Z}.$$

Choose $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$1_{B_{1/2}(0)} \leq \chi \leq 1_{B_1(0)}.$$

Define

$$\chi_{a,i}(x) := \chi(x - x_{a,i})$$

for $a \in \mathbb{R}_+^\infty$, $a > 1$, $i \in \mathbb{Z}$, $|i| \leq [a] + 1$, and $x \in \mathbb{R}$. We have

$$(-a, a) \subset \bigcup_{|i| \leq [a]+1} \{\chi_{a,i} = 1\},$$

and D_{H_a} is an invariant subspace of $[\chi_{a,i}]$. According to [B, Theorem 4], there exists a constant $C_1 > 0$ such that we have

$$\max_{x \in K_{1/2}(x_{a,i})} |\psi(x)| \leq C_1 \left\|\chi_{a,i}\psi\right\|_{H^1((-a,a))}$$

for all $a \in \mathbb{R}_+^\infty$, $a > 1$, $i \in \mathbb{Z}$, $|i| \leq [a] + 1$, and $\psi \in H^1((-a, a), \mathbb{C})$. In view of proposition 1.5, there exists a constant $C_2 > 0$, estimable from above in terms of $\|\chi\|_{C^2(\mathbb{R})}$, m, M, K, λ, such that we have

$$\|\chi_{a,i}\psi\|_{H^1((-a,a))} \leq C_2 \|\psi\|_{L^2(K_1(x_{a,i}) \cap (-a,a))}$$

for all $a \in \mathbb{R}_+^\infty$, $E \in \sigma_p(H_a), \leq \lambda - \beta$, $\psi \in \ker(H_a - E)$, and $i \in \mathbb{Z}$ such that $|i| \leq [a] + 1$. Choose $\frac{1}{2} < \delta \leq 1$. Owing to proposition 1.4, there exists $\tilde{a}_0 > 0$ and a constant $C_3 = C_3(\delta, \lambda, K, \beta, \tilde{a}_0) > 0$ such that

$$\forall a \in \mathbb{R}_+^\infty, > \tilde{a}_0 : \forall E \in \sigma_p(H_a), E \leq \lambda - \beta : \forall \psi \in \ker(H_a - E) : \left\|e^{f_0}\psi\right\|_{L^2_{J_a}} \leq C_3 \|\psi\|_{L^2_{J_a}}.$$

Set

$$C := \frac{C_1 C_2 C_3}{m} e^{2(1-\delta)\sqrt{\lambda + K}},$$

and

$$a_0 := \max(1, \tilde{a}_0).$$

Choose $a \in \mathbb{R}_+^\infty, > a_0$, $E \in \sigma_p(H_a), E \leq \lambda - \beta$, and $\psi \in \ker(H_a - E)$ arbitrarily. Let $x \in (-a, a) \cap \{\chi_{a,i} = 1\}$. Then

$$\left|\psi(x) e^{(1-\delta)\sqrt{\lambda - E}|x|}\right| \leq C_1 \sup_{y \in K_1(x_{a,i})} e^{f_0(y)} \|\chi_{a,i}\psi\|_{H^1((-a,a))} \leq$$

$$C_1 C_2 \sup_{y \in K_1(x_{a,i})} e^{f_0(y)} \sup_{z \in K_1(x_{a,i})} e^{-f_0(z)} \inf_{w \in K_1(x_{a,i})} e^{f_0(w)} \|\psi\|_{L^2(K_1(x_{a,i}) \cap (-a,a))} \leq$$

$$\frac{C_1 C_2}{m} \sup_{|y| \leq 2} e^{(1-\delta)\sqrt{\lambda + K}|y|} \left\|e^{f_0}\psi\right\|_{L^2_{J_a}} \leq C \|\psi\|_{L^2_{J_a}}.$$

This completes the proof. □

1.3 The range of Sturm-Liouville sytems

Throughout this section, we consider the Schrödinger operator H_∞. Let us set

$$U_{a,r,C} := \left\{u \in (L_2(\mathbb{R},\mathbb{C}))^m : \forall |z| \geq r : |u(z)| \leq Ce^{-a|z|}\right\}$$

for $a, r, C > 0$, and define the subspace

$$U := \bigcup_{a,r,C \in \mathbb{R}_+} U_{a,r,C}$$

of all exponentially fast decreasing functions. Our aim is to prove

$$(H_\infty - E)^{-1}(U) \subset U$$

for all $E < \lambda$. Analogous to the foregoing section, we prove L^2-boundedness to obtain pointwise bounds.

Corollary 1.1. *Let $E < \lambda$, $\psi \in D_{H_\infty}$, and $\theta \in (L^2(\mathbb{R},\mathbb{C}))^m$ such that $\theta \in U_{a,r,C}$, and*

$$H_\infty \psi = E\psi + \theta.$$

Suppose $\eta \in C^\infty(\mathbb{R},\mathbb{R})$ fulfills $\eta' \in C_0^\infty(\mathbb{R},\mathbb{R})$. Define

$$\zeta_\alpha := |\eta'|^2 + 2\eta\eta' f_\alpha',$$

and

$$\phi_\alpha := \eta e^{f_\alpha}\psi, \quad \alpha > 0.$$

Then

$$B_{\infty,\alpha}[\phi_\alpha] = \left\langle \zeta_\alpha e^{2f_\alpha}\psi, \psi \right\rangle_{L^2_{J_\infty}} + \left\langle \theta e^{2f_\alpha}, \eta^2\psi \right\rangle_{L^2_{J_\infty}}$$

for each $\alpha > 0$.

Proof. Again, we have

$$B_{\infty,\alpha}[\phi_\alpha] = -\sum_{j=1}^m \left\langle \left(J_\infty (\eta\psi_j)'\right)', \eta e^{2f_\alpha}\psi_j \right\rangle_{L^2(\mathbb{R})} + \langle [V_\infty - E]\,\phi, \phi\rangle_{L^2_{J_\infty}}$$

for $\alpha > 0$. In view of (1.3) and $H_\infty\psi = E\psi + \theta$, we obtain

$$B_\alpha[\phi_\alpha] = -\sum_{j=1}^m \left\langle (J_\infty\eta')'\,\psi_j, \eta e^{2f_\alpha}\psi_j \right\rangle_{L^2(\mathbb{R})} -$$

$$2\sum_{j=1}^m \left\langle J_\infty\eta'\psi_j', \eta e^{2f_\alpha}\psi_j \right\rangle_{L^2(\mathbb{R})} + \left\langle \eta e^{2f_\alpha}\theta, \eta\psi \right\rangle_{L^2_{J_\infty}}$$

for $\alpha > 0$. On account of (1.5), summation yields the assertion. $\square$

Proposition 1.6. *Let $E < \lambda$, $\theta \in U_{a,r,C}$, and $\psi \in D_{H_\infty}$ such that*

$$H_\infty \psi = E\psi + \theta.$$

Suppose either

$$a \geq \sqrt{\lambda - E}$$

and $\frac{1}{2} < \delta \leq 1$, or

$$a < \sqrt{\lambda - E}$$

and

$$1 \geq \delta > 1 - \frac{a}{2\sqrt{\lambda - E}}, > \frac{1}{2}.$$

Then we have $e^{f_0}\theta, e^{f_0}\psi \in \left(L^2_{J_\infty}(\mathbb{R}, \mathbb{C})\right)^m$, and there exists a constant $C > 0$, estimable from above in terms of m, M, K, $\lambda - E$, $\|e^{f_0}\psi\|_{L^2_{J_\infty}}$, and $\left\|e^{f_0}\theta\right\|_{L^2_{J_\infty}}$, such that

$$\forall z \in \mathbb{R} : |\psi(z)| \leq Ce^{-(1-\delta)\sqrt{\lambda - E}|z|}.$$

Proof. First we establish that $\theta e^{2f_0} \in \left(L^2(\mathbb{R}, \mathbb{C})\right)^m$. There exists a constant $C > 0$ such that

$$\left|\theta(z)e^{2f_0(z)}\right| \leq Ce^{(-a+2(1-\delta)\sqrt{\lambda - E})|z|}$$

for $|z| \geq r$. Clearly,

$$-a + 2(1-\delta)\sqrt{\lambda - E} < 0$$

if one of the cases above holds. Owing to assumption 3 in definition 1.1, there exists $R > 0$ such that

$$\overline{K_R(0)}^c \subset F_E^\infty.$$

Choose $\eta \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

1. $0 \leq \eta \leq 1$, on $\mathbb{R}$,
2. $\eta \equiv 0$ on $\overline{B_R(0)}$,
3. $\eta \equiv 1$ on $\mathbb{R} \backslash B_{R+1}(0)$.

For each $\alpha > 0$, we define

$$\phi_\alpha := \eta e^{f_\alpha}\psi.$$

We obtain

$$\mathrm{supp}(\phi_\alpha) \subset F_E^\infty.$$

Set $\beta := \lambda - E$. In view of proposition 1.2, we get

$$\mathrm{Re}(B_{\infty,\alpha}[\phi_\alpha]) \geq (2\delta - 1)\frac{\beta}{2}\|\phi\|^2_{L^2_{J_\infty}}.$$

Owing to proposition 1.1, we obtain

$$\mathrm{Re}(B_{\infty,\alpha}[\phi]) \leq \left| \left\langle \zeta e^{2f_\alpha}\psi, \psi \right\rangle_{L^2_{J_\infty}} \right| + \left| \left\langle \theta e^{2f_\alpha}, \psi\eta^2 \right\rangle_{L^2_{J_\infty}} \right|.$$

This delivers

$$(2\delta - 1)\frac{\beta}{2}\|\phi_\alpha\|^2_{L^2_{J_\infty}} \leq \sup_{|x|\leq R+1} |\zeta e^{2f_0}| \|\psi\|^2_{L^2_{J_\infty}} + \|\theta e^{2f_0}\|_{L^2}\|\psi\|_{L^2_{J_\infty}}.$$

On account of lemma A.1, we send α to zero. It follows

$$(2\delta - 1)\frac{\beta}{2}\left\|\eta e^{f_0}\psi\right\|_{L^2_{J_\infty}} \leq \sup_{|x|\leq R+1} |\zeta e^{2f_0}| \|\psi\|^2_{L^2_{J_\infty}} + \|\theta e^{2f_0}\|_{L^2}\|\psi\|_{L^2_{J_\infty}}.$$

Finally,

$$\int_{\mathbb{R}} e^{2f_0(x)}|\psi(x)|^2 J_\infty(x)dx \leq$$

$$\int_{\{x:\eta(x)=1\}} \eta^2(x)e^{2f_0(x)}|\psi(x)|^2 J_\infty(x)dx + \int_{-(R+1)}^{R+1} e^{2f_0(x)}|\psi(x)|^2 J_\infty(x)dx \leq$$

$$\left(\frac{2}{(2\delta-1)\beta}\sup_{|x|\leq R+1}\left|\zeta e^{2f_0}\right| + \sup_{|x|\leq R+1} e^{2f_0}\right)\|\psi\|^2_{L^2_{J_\infty}} + \frac{2}{(2\delta-1)\beta}\left\|\theta e^{2f_0}\right\|_{L^2}\|\psi\|_{L^2_{J_\infty}}.$$

We have proved

$$e^{f_0}\psi \in \left(L^2_{J_\infty}(\mathbb{R},\mathbb{C})\right)^m.$$

Choose χ, $x_{\infty,i}$, and $\chi_{\infty,i}$, $i \in \mathbb{Z}$, according to the proof of lemma 1.1. On account of [B, Theorem 4], there exists a constant $C_1 > 0$ such that we have

$$\max_{x\in K_{\frac{1}{2}}(x_{\infty,i})} |\psi(x)| \leq C_1\|\chi_{\infty,i}\psi\|_{H^1(K_1(x_{\infty,i}))}$$

for each $a \in \mathbb{R}^\infty_+$ and $i \in \mathbb{Z}$. On account of proposition 1.5, there exists a constant $C_2 > 0$, estimable from above in terms of m, M, K, E, and $\|\chi\|_{C^2(\mathbb{R})}$, such that

$$\|\chi_{\infty,i}\psi\|_{H^1((-a,a))} \leq C_2\left[\|\psi\|_{L^2(K_1(x_{\infty,i}))} + \|\theta\|_{L^2(K_1(x_{\infty,i}))}\right]$$

for each $a \in \mathbb{R}^\infty_+$ and $i \in \mathbb{Z}$. Define $C := e^{2(1-\delta)\sqrt{\lambda+K}}\frac{C_1C_2}{m}\left(\left\|e^{f_0}\psi\right\|_{L^2_{J_\infty}} + \left\|e^{f_0}\theta\right\|_{L^2_{J_\infty}}\right)$. Then a similar calculation as above shows

$$\max_{x\in K_{\frac{1}{2}}(x_{\infty,i})} |\psi(x)e^{f_0(x)}| \leq C, \quad a \in \mathbb{R}^\infty_+, \quad i \in \mathbb{Z}.$$

It follows

$$|\psi(x)| \leq Ce^{-(1-\delta)\sqrt{\lambda-E}|x|}, \quad x \in \mathbb{R}.$$

□

Corollary 1.2. *Let $k \in \mathbb{N}_0$. Suppose $V_\infty \in C_b^k(\mathbb{R}, S(\mathbb{R}^m))$, $J_\infty \in C_b^{k+1}(\mathbb{R}, \mathbb{R})$, $E < \lambda$, $\psi \in D_{H_\infty}$, $\theta \in (H^k(\mathbb{R}, \mathbb{C}))^m$, $\theta^{(k)} \in U_{a_0,r,C_0}$, $\psi \in U_{a_1,r,C_1}$, and*

$$H_\infty \psi = E\psi + \theta.$$

Then $\psi \in (H^{k+2}(\mathbb{R}, \mathbb{C}))^m$ and $\psi^{(k+2)} \in U_{\min(a_0,a_1),r,C}$, and $C > 0$ is estimable from above in terms of C_i, a_i, $i = 0, 1$, $\|V_\infty\|_{C^k(\mathbb{R})}$, and $\|J_\infty\|_{C^{k+1}(\mathbb{R})}$.

Proof. Let us start with $k = 0$. In view of

$$H_\infty \psi = E\psi + \theta, \tag{1.10}$$

we obtain

$$(J_\infty \psi')' = J_\infty (V_\infty \psi - E\psi) - J_\infty \theta,$$

and thus

$$\psi'(x) = \frac{1}{J_\infty(x)} \int_{-\infty}^{x} J_\infty (V_\infty \psi - E\psi) - J_\infty \theta dz$$

which implies $\psi' \in U_{\min(a_0,a_1),r,\tilde{C}}$. The constant $\tilde{C}$ is estimable from above in terms of C_i, a_i, $i = 0, 1$, $\|V_\infty\|_{C^0(\mathbb{R})}$, and $\|J_\infty\|_{C^1(\mathbb{R})}$. If we use (1.10) again, the assertion follows for $k = 0$. Suppose now the assertion is already proved for $k \in \mathbb{N}_0$. Let $V_\infty \in C_b^{k+1}(\mathbb{R}, S(\mathbb{R}^m))$, $J_\infty \in C_b^{k+2}(\mathbb{R})$, $\theta^{(k+1)} \in U_{a_0,r,C_0}$. If we differentiate equation (1.10) k-th times, we obtain

$$-\left(\psi^{(k)}\right)'' = R,$$

where $R \in (H^1(\mathbb{R}, \mathbb{C}))^m$ is the sum of products of $\psi^{(l)}$, $V_\infty^{(r)}$, $\theta^{(t)}$, and $J_\infty^{(s)}$, where $l, s \leq k+1$ and $r, t \leq k$. Owing to the induction hypothesis, we have $R' \in U_{\min(a_0,a_1),r,C}$, and C has the stated properties. It follows $\psi \in (H^{k+3}(\mathbb{R}, \mathbb{C}))^m$, and

$$-\left(\psi^{(k+1)}\right)'' = R'$$

which yields the assertion for $k + 1$. □

1.4 Convergence of the spectrum

For later applications, we have to prove that eigenvalues of $\sigma_d(H_a)$ can produce eigenvalues of $\sigma_p(H_\infty)$. This is specified by the following lemma.

Lemma 1.2. *Suppose*

1. *$J_a(x) \to J_\infty(x)$, and $J_a'(x) \to J_\infty'(x)$, $a \to \infty$, for each $x \in \mathbb{R}$,*
2. *$V_a(x) \to V_\infty(x)$, $a \to \infty$, for each $x \in \mathbb{R}$.*

Assume $(a_n)_{n\in\mathbb{N}}$ is a sequence such that $a_n \to \infty$, $n \to \infty$. Suppose further that $E_n \in \sigma_d(H_{a_n})$, $n \in \mathbb{N}$, fulfills

$$\sup_{n\in\mathbb{N}} E_n < \lambda,$$

and $\psi_n \in \ker(H_{a_n} - E_n)$ is normalized. Then there exists a subsequence (without loss of generality the sequence itself), such that

$$\exists E := \lim_{n\to\infty} E_n \in \sigma_p(H_\infty),$$

and

$$\forall K \subset\subset \mathbb{R} : \psi_n \text{ conv. in } \left(C^1(K, \mathbb{C})\right)^m.$$

Moreover, the pointwise limit

$$\psi := \lim_{n\to\infty} \psi_n$$

is normalized in $\left(L^2_{J_\infty}(\mathbb{R}, \mathbb{C})\right)^m$ and fulfills

$$\psi \in \ker(H_\infty - E).$$

Proof. Let us prove that

$$\sup_{n\in\mathbb{N}} \|\psi_n\|_{H^2((-a_n,a_n))} < \infty.$$

In view of

$$H_{a_n}\psi_n = E_n\psi_n, \tag{1.11}$$

we have

$$\left\langle P^*_{a_n} P_{a_n}\psi_n, \psi_n\right\rangle_{L^2_{J_{a_n}}} = \langle (E_n - V_{a_n})\psi_n, \psi_n\rangle_{L^2_{J_{a_n}}}.$$

It follows that

$$m\|\psi_n'\|_{L^2((-a_n,a_n))} \le \|P_{a_n}\psi_n\|_{L^2_{J_{a_n}}} \le (K + \lambda). \tag{1.12}$$

In view of (1.11), we obtain

$$\sup_{n\in\mathbb{N}} \|\psi_n\|_{H^2((-a_n,a_n))} < \infty.$$

For each $K \subset\subset \mathbb{R}$, the embedding $H^2(K, \mathbb{C}) \hookrightarrow C^1(\overline{K}, \mathbb{C})$ is compact. It follows that for each $k \in \mathbb{N}$, there exists a subsequence $\left(\psi_n^{(k)}\right)_{n\ge N(k)}$, $N(k) \in \mathbb{N}$, that converges in $(C^1((-k, k), \mathbb{C}))^m$. The diagonal selection procedure delivers a subsequence ψ_{n_k} such that for each $K \subset\subset \mathbb{R}$, there exists $N(K) \in \mathbb{N}$ so that

$(\psi_{n_k})_{k\geq N(K)}$ converges in $(C^1(\overline{K},\mathbb{C}))^m$. On account of (1.11), this convergence holds in $(C^2(\overline{K},\mathbb{C}))^m$. Hence the pointwise limit

$$\psi(x) := \lim_{\substack{k\to\infty \\ k\geq N((x-1,x+1))}} \psi_{n_k}(x), \quad x \in \mathbb{R},$$

fulfills

$$\psi \in (C^2(\mathbb{R},\mathbb{C}))^m.$$

Sending $n \to \infty$ in (1.11) delivers

$$-\frac{1}{J_\infty}(J_\infty \psi')' + (V_\infty - E)\psi = 0. \tag{1.13}$$

It remains to prove that ψ is normalized in $\left(L^2_{J_\infty}(\mathbb{R},\mathbb{C})\right)^m$. There exists $\beta > 0$ such that $E_n \leq \lambda - \beta$. Lemma 1.1 delivers an exponentially fast decreasing envelope for all the ψ_{n_k}. Theorem A.1 yields

$$\|\psi\|^2_{L^2_{J_\infty}} = \lim_{k\to\infty} \|\psi_{n_k}\|^2_{L^2_{J_{a_n}}} = 1.$$

If we send $n \to \infty$ in (1.12), we obtain with lemma A.1 $\psi \in \left(H^1(\mathbb{R},\mathbb{C})\right)^m$. Owing to (1.13), we get $\psi \in \left(H^2(\mathbb{R},\mathbb{C})\right)^m$. $\square$

Chapter 2

Spectral analysis for a two-phase transition

As already mentioned in the introduction, the first term in the asymptotic expansion is given by the solution of the equation

$$-\frac{d^2}{dz^2}u + (DW(u))^T = 0.$$

This is the Euler-Lagrange equation of the functional

$$E(u) = \int_{\mathbb{R}} \frac{1}{2}|u'|^2 + W(u)dz.$$

In this section, we analyze the spectrum of the operator L_0 which is generated by the second directional derivative of E along a test function. We introduce self-adjoint realizations L_ϵ of $[L_0]$ in $\left(L^2\left(\left(-\frac{1}{\epsilon}, \frac{1}{\epsilon}\right), \mathbb{C}\right)\right)^m$ with Neumann boundary conditions. Moreover, we use the results of chapter 1 to investigate the limit $\sigma(L_\epsilon) \to \sigma_d(L_0)$, $\epsilon \to 0$.

2.1 Standing wave solutions

This subsection is devoted to existence results for minimizers of E. Throughout section 2.1, the Sobolev spaces consist of real-valued functions.

Definition 2.1. *Let $k \in \mathbb{N}$, $k \geq 2$, $W \in C^k(\mathbb{R}^m, \mathbb{R})$, and $a \neq b \in \mathbb{R}^m$. Assume $W(a) = W(b) = 0$ and $W(x) \geq 0$, $x \in \mathbb{R}^m$. Suppose $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ is odd such that*

$$\varphi(z) = \operatorname{sign}(z), \quad |z| \geq 1.$$

Set

$$\phi := \frac{a+b}{2} + \varphi\frac{b-a}{2},$$

$$M := \left(H^1(\mathbb{R},\mathbb{R})\right)^m + \phi,$$

and

$$E(u) := \int_{\mathbb{R}} \frac{1}{2}|u'|^2 + W(u)dz$$

for $u \in M$.

For the case $k \geq 3$, $m = 2$, $W(x) > 0, x \notin \{a, b\}$, it is proved in [S, Lemma] that if

1. D^2W is positive definite at $x = a, b$,
2. there exist positive constants c_1, c_2 and m, and a number $p \geq 2$ such that

$$c_1|x|^p \leq W(x) \leq c_2|x|^p, \quad |x| \geq m,$$

3. $W(x + r(\cos\theta, \sin\theta)) = r^2 + O\left(r^3\right)$ for r sufficiently small, $\theta \in [0, 2\pi)$, and $x = a, b$,

then there exists a minimizer $\zeta \in M$ of E that attains the limits $\lim_{\tau\to-\infty}\zeta(\tau) = a$ and $\lim_{\tau\to+\infty}\zeta(\tau) = b$ at an exponential rate. The minimizer fulfills

$$\zeta'' = (DW(\zeta))^T.$$

In the following, it is important to consider the case when W fulfills the symmetry condition $W \circ \gamma = W$ such that $\gamma \in O(m)$ is the reflection on a hyperplane of $\mathbb{R}^m$.

Definition 2.2. *Let* $a \in \mathbb{R}_+^\infty$, $\gamma \in O(m)$, *and* $u : (-a, a) \to \mathbb{K}^m$.

$$u \quad \textit{is} \quad \left\{ \begin{array}{c} \gamma\textit{-odd} \\ \gamma\textit{-even} \end{array} \right\} :\Leftrightarrow \left\{ \begin{array}{c} u(-x) = \gamma u(x) \\ -u(-x) = \gamma u(x) \end{array} \right.$$

The content of the following corollary is a slight modification of the results given in section 2 of [BGS].

Corollary 2.1. *Let* $W \in C^2(\mathbb{R}^m, \mathbb{R})$. *Assume* W *has two distinct non degenerate global minima* a *and* b *of equal depth zero. Suppose*

$$\exists\lambda, \rho > 0 : \forall x \in B_\rho(a) \cup B_\rho(b) : D^2W(x) \geq \lambda.$$

Assume $\gamma \in O(m)$ *is the reflection on the hyperplane* $\Gamma \subset \mathbb{R}^m$ *such that*

$$\gamma(a) = b.$$

Suppose that $W \circ \gamma = W$ *and that there exists a Lipschitz continuous function* $f : \mathbb{R}^m \to \mathbb{R}^m$ *which leaves* a *and* b *invariant and has bounded image. Further, assume*

$$\|Df(x)\|_{tr} \leq 1 \quad \textit{for a.e. } x \in \mathbb{R}^2,$$

and

$$W(f(x)) \leq W(x), \quad x \in \mathbb{R}^2.$$

Then there exists a global minimizer u of E over M which is γ-odd and fulfills the constraint $u(0) \in \Gamma$. In addition,

$$u \in C^3(\mathbb{R}, \mathbb{R}^m) \cap \left(\left(H^2(\mathbb{R}, \mathbb{R}) \right)^m + \phi \right).$$

The minimizer u fulfills the equation

$$-u'' + (DW(u))^T = 0$$

and attains the values a and b at an exponential rate as $z \to \pm\infty$. If in addition $W \in C^\infty(\mathbb{R}^m, \mathbb{R})$, then $u \in C^\infty(\mathbb{R}, \mathbb{R}^m)$.

Proof. We minimize E over the set M, where E and M are defined as above. Suppose there is a minimizing sequence $(u_n)_{n\in\mathbb{N}} \subset M$, i.e.

$$E(u_n) \searrow m := \inf_{v \in M} E(v).$$

Without loss of generality, we assume $\mathrm{im}(u_n) \subset \mathrm{im}(f)$, otherwise we consider the sequence $(f(u_n))_{n\in\mathbb{N}}$ which fulfills $E(f(u_n)) \leq E(u_n)$. Define

$$\vec{n} := \frac{b-a}{|b-a|}.$$

For $n \in \mathbb{N}$, set

$$x_n^- := \sup\{x \in \mathbb{R} : \forall z < x : \langle u_n(z), \vec{n} \rangle \leq 0\},$$

$$x_n^+ := \inf\{x \in \mathbb{R} : \forall z > x : \langle u_n(z), \vec{n} \rangle \geq 0\},$$

and

$$U_n := (x_n^-, x_n^+).$$

As each u_n is continuous, we have $\langle u_n(x_n^\pm), \vec{n} \rangle = 0$. Define

$$\tilde{u}_n(z) := \begin{cases} u_n(z), & z \notin (x_n^-, x_n^+) \\ u_n(z), & z \in (x_n^-, x_n^+) \wedge \langle u_n(z), \vec{n} \rangle \geq 0 \\ \gamma u_n(z), & z \in (x_n^-, x_n^+) \wedge \langle u_n(z), \vec{n} \rangle < 0 \end{cases}$$

As $\langle u_n, \vec{n} \rangle \in H^1(U_n, \mathbb{R})$, we have

$$\langle u_n, \vec{n} \rangle_+, \langle u_n, \vec{n} \rangle_- \in H^1(U_n, \mathbb{R}).$$

Let $b_1, ..., b_{m-1} \in \Gamma$ be an orthonormal basis of Γ. It follows that

$$\tilde{u}_n = \sum_{j=1}^{m-1} \langle u_n, b_j \rangle b_j + \chi_{U_n} \left(\langle u_n, \vec{n} \rangle_+ + \langle u_n, \vec{n} \rangle_- \right) \vec{n} + \chi_{U_n^c} \langle u_n, \vec{n} \rangle \vec{n}.$$

As $\tilde{u_n}$ is continuous, this implies $\tilde{u}_n \in M$. In view of $W \circ \gamma = W$, we obtain $E(\tilde{u}_n) = E(u_n)$. Owing to the transformation lemma, we further assume

$$\forall z \in \mathbb{R}_- : \langle u_n(z), \vec{n} \rangle \leq 0,$$

and

$$\forall z \in \mathbb{R}_+ : \langle u_n(z), \vec{n} \rangle \geq 0.$$

Especially

$$u_n(0) \in \Gamma.$$

Finally, we consider the values

$$E_n^- := \int_{\mathbb{R}_-} \frac{1}{2} |u_n'|^2 + W(u_n) dx,$$

and

$$E_n^+ := \int_{\mathbb{R}_+} \frac{1}{2} |u_n'|^2 + W(u_n) dx$$

for each $n \in \mathbb{N}$. We distinguish between the cases

$$E_n^- \leq E_n^+,$$

and

$$E_n^+ < E_n^-.$$

Provided the first case occurs, we define

$$\tilde{u}_n(z) := \begin{cases} u_n(z) & , z \in \mathbb{R}_- \\ u_n(0), & z = 0 \\ \gamma u_n(-z) & , z \in \mathbb{R}_+ \end{cases}.$$

It is easy to see that $\tilde{u}_n$ is γ-even. Moreover, as $u_n(0) \in \Gamma$, and $\gamma(a) = b$, we conclude that $\tilde{u}_n - \phi$ is continuous and weakly differentiable in $\mathbb{R}_\pm$. Thus we obtain

$$\tilde{u}_n \in M.$$

Similar, one can treat the second case. In view of

$$\kappa := \inf \left\{ \frac{W(x)}{\min\left(|x-a|^2, |x-b|^2\right)} : x \in \operatorname{im}(f) \right\} > 0,$$

which follows from Taylor expansion around the points a, b, we obtain a bound for $(u_n - \phi)_{n\in\mathbb{N}}$ in $(H^1(\mathbb{R},\mathbb{R}))^m$. Note that $\int_{\mathbb{R}} |u_n'|^2 dz$ is bounded. Therefore,

$$E(u_n) = \int_{\mathbb{R}} \frac{1}{2}|u_n'|^2 + W(u_n)dx \geq$$

$$-C_1 + \int_{\mathbb{R}} \frac{1}{2}|(u_n - \phi)'|^2 + W(u_n)dx \geq$$

$$-C_1 + \int_{\mathbb{R}} \frac{1}{2}|(u_n - \phi)'|^2 dx + \kappa \left[\int_{-\infty}^{0} |u_n - a|^2 dx + \int_0^{+\infty} |u_n - b|^2 dx\right] \geq$$

$$-C_2 + \int_{\mathbb{R}} \frac{1}{2}|(u_n - \phi)'|^2 dx + \kappa \int_{\mathbb{R}} |u_n - \phi|^2 dx.$$

This implies

$$\exists C > 0 : \forall n \in \mathbb{N} : \|u_n - \phi\|_{H^1(\mathbb{R})} \leq C.$$

Hence,

$$\forall s \in \mathbb{N} : \forall n \in \mathbb{N} : \|u_n - \phi\|_{H^1((-s,s))} \leq C.$$

With the diagonal selection procedure, we obtain a subsequence u_{n_k} and a element $u \in (H^1(\mathbb{R},\mathbb{R}))^m$ such that

$$\forall s \in \mathbb{N} : u_{n_k} - \phi \xrightarrow{w} u \quad \text{in} \quad \left(H^1((-s,s),\mathbb{R})\right)^m.$$

The embedding

$$H^1((-s,s),\mathbb{R}) \hookrightarrow L^2((-s,s),\mathbb{R})$$

is compact, and for each $s \in \mathbb{N}$, there exists a subsequence that converges a.e. to u. Thus, the diagonal selection procedure delivers a subsequence $u_{n_{k_j}}$ such that

$$u_{n_{k_j}} - \phi \xrightarrow{w} u \quad \text{in} \quad \left(H^1((-s,s),\mathbb{R})\right)^m,$$

$$u_{n_{k_j}} - \phi \to u \quad \text{in} \quad \left(L^2((-s,s),\mathbb{R})\right)^m,$$

and

$$u_{n_{k_j}} - \phi \to u \quad \text{a.e. in } \mathbb{R}$$

for each $s \in \mathbb{N}$. Suppose this subsequence is the sequence itself. Set $v := u + \phi$. Taylor expansion yields that there exists a constant $C > 0$ such that we have

$$|W(u_n(x)) - W(v(x)) - DW(v(x))(u_n(x) - v(x))| \leq C\,|u_n(x) - v(x)|^2$$

for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Hence,

$$E(u_n) \geq \int_{-s}^{s} \frac{1}{2}|(u_n - v)'|^2 dx + \int_{-s}^{s} \langle (u_n - v)', v' \rangle\, dx +$$

$$\int_{-s}^{s} DW(v)(u_n - v)dx - C\int_{-s}^{s} |v-u_n|^2 dx + \int_{-s}^{s} \frac{1}{2}|v'|^2 + W(v)dx.$$

We conclude

$$\liminf_{n\to\infty} E(u_n) \geq \int_{-s}^{s} \frac{1}{2}|v'|^2 + W(v)dx.$$

Lemma A.1 implies

$$m = \liminf_{n\to\infty} E(u_n) \geq E(v).$$

It follows that v is a global minimizer. Consider the derivative of E along a test function $\phi \in (C_0^\infty(\mathbb{R},\mathbb{R}))^m$ to obtain

$$-v'' + (DW(v))^T = 0$$

in the sense of distributions. As $DW(v)$ is square integrable, we have

$$v \in \left(H^2(\mathbb{R},\mathbb{R})\right)^m + \phi,$$

and

$$-v'' + (DW(v))^T = 0$$

in $(L^2(\mathbb{R},\mathbb{R}))^m$. Therefore

$$v \in C^3(\mathbb{R},\mathbb{R}^m) \cap \left(\left(H^2(\mathbb{R},\mathbb{R})\right)^m + \phi\right).$$

Following the idea in [BGS], exponential decay is proved with a comparison principle for

$$Z(x) := |v(x) - a|^2.$$

There exists a mapping $\epsilon : \mathbb{R}^m \to \mathcal{L}(\mathbb{R}^m,\mathbb{R})$ such that

$$DW(x) = D^2W(a)(x-a) + |x-a|\epsilon(x), \quad x \in \mathbb{R}^m,$$

and

$$\lim_{x\to a} \epsilon(x) = 0.$$

Choose $x_0 > 0$ large enough such that

$$\sup_{y\in\{v(x):x\leq -x_0\}} |\epsilon(y)| \leq \frac{\lambda}{2}.$$

Taylor expansion yields

$$Z''(x) = 2\left\langle (v-a), (DW(v))^T \right\rangle + 2|v'(x)|^2 \geq \lambda Z(x).$$

We compare the function Z with the solution of

$$\hat{Z}'' - \lambda\hat{Z} = 0, \hat{Z}(-\infty) = 0, \hat{Z}(-x_0) = Z(-x_0).$$

It is given by

$$\hat{Z}(x) = Z(-x_0)e^{\sqrt{\lambda}(x+x_0)}.$$

Suppose

$$c := \inf_{x \leq -x_0} \left(\hat{Z}(x) - Z(x) \right) < 0.$$

By continuity of $\hat{Z}$ and Z, we obtain the existence of $y \in (-\infty, -x_0)$ such that

$$c = \hat{Z}(y) - Z(y)$$

is a global minimum of the difference in $(-\infty, x_0)$. Thus

$$0 \geq -\left(\hat{Z}(y) - Z(y) \right)'' \geq -\lambda c > 0,$$

which is a contradiction. We have proved

$$\hat{Z}(x) \geq Z(x), \quad \forall x \leq -x_0.$$

$\square$

2.2 Linearizations around standing waves

The investigations of this section involve the following self-adjoint operators.

Definition 2.3. *Let $W \in C^2(\mathbb{R}^m, \mathbb{R})$ fulfill the assumptions of definition 2.1. Suppose further that*

$$\exists \lambda, \rho > 0 : \forall x \in B_\rho(a) \cup B_\rho(b) : D^2W(x) \geq \lambda.$$

Let $\theta_0 \in C^3(\mathbb{R}, \mathbb{R}^m) \cap \left((H^2(\mathbb{R}, \mathbb{R}))^m + \phi \right)$ be a global minimizer of the energy E over M which attains the limits $\lim_{z \to \pm\infty} \theta_0(z)$ at an exponential rate.

1. *Define*

$$I_\epsilon := \left(-\frac{1}{\epsilon}, \frac{1}{\epsilon} \right), \quad \epsilon > 0,$$

and

$$I_0 := \mathbb{R}.$$

2. *Set*

$$D_{L_\epsilon} := \left\{ u \in H^2(I_\epsilon, \mathbb{C}) : u' \left(\pm\frac{1}{\epsilon} \right) = 0 \right\}^m$$

for $\epsilon > 0$, and

$$D_{L_0} := \left(H^2(\mathbb{R}, \mathbb{C}) \right)^m.$$

Define

$$L_\epsilon u := -u'' + \left[D^2W(\theta_0) \right] u, \quad u \in D_{L_\epsilon},$$

for $\epsilon \geq 0$.

3. The closed quadratic form that is generated by L_ϵ is denoted with S_ϵ.

For later applications, we have to clarify how $\sigma_d(L_\epsilon)$ converges to $\sigma_p(L_0)$ as $\epsilon \to 0$.

2.2.1 Spectral analysis

Before we start to investigate the convergence of L_ϵ's ground state, we need an exact information on the minimum of the essential spectrum of L_0. If in addition θ_0 is γ-odd for some $\gamma \in O(m)$, we call this situation the symmetric case.

Definition 2.4. *Let $\gamma \in O(m)$. Define the spaces*

$$L^2_{odd}(I_\epsilon) := \left\{ u \in \left(L^2(I_\epsilon, \mathbb{C})\right)^m : u \text{ is } \gamma\text{-odd} \right\},$$

and

$$H^k_{odd}(I_\epsilon) := \left(H^k(I_\epsilon, \mathbb{C})\right)^m \cap L^2_{odd}(I_\epsilon)$$

for $\epsilon \geq 0$. Let L^{odd}_ϵ be the restriction of L_ϵ to $L^2_{odd}(I_\epsilon)$, $\epsilon \geq 0$.

According to [Wel, Satz 1.41], each convergent sequence in L^2 contains a subsequence that converges almost everywhere. This implies that $L^2_{odd}(I_\epsilon)$ is a closed subspace of $(L^2(I_\epsilon, \mathbb{C}))^m$, hence a Hilbert space. Note that L^{odd}_ϵ is a operator in $L^2_{odd}(I_\epsilon)$ with domain

$$D_{L^{odd}_\epsilon} = D_{L_\epsilon} \cap L^2_{odd}(I_\epsilon).$$

Corollary 2.2. *1. For each $\epsilon \geq 0$ and $u \in L^2_{odd}(I_\epsilon)$, we have*

$$\|u\|^2_{L^2_{odd}(I_\epsilon)} = 2\|u\|^2_{L^2(I_\epsilon \cap \mathbb{R}_\pm)}.$$

2. The operator L^{odd}_ϵ, $\epsilon \geq 0$, is self-adjoint in $L^2_{odd}(I_\epsilon)$.

Proof. 1. The transformation lemma yields

$$\|u\|^2_{L^2_{odd}(I_\epsilon)} = \int_{I_\epsilon} |u(t)|^2 dt = \int_{I_\epsilon \cap \mathbb{R}_\pm} |u(t)|^2 dt + \int_{I_\epsilon \cap \mathbb{R}_\mp} |u(t)|^2 dt =$$

$$\int_{I_\epsilon \cap \mathbb{R}_\pm} |u(t)|^2 + |u(-t)|^2 dt = 2\|u\|^2_{L^2(I_\epsilon \cap \mathbb{R}_\pm)}.$$

2. The orthogonal projection onto the γ-odd functions in $(L^2(I_\epsilon, \mathbb{C}))^m$ is given by

$$(R_\epsilon u)(x) := \frac{1}{2}\left(u(x) + \gamma u(-x)\right), \quad u \in \left(L^2(I_\epsilon, \mathbb{C})\right)^m, \quad x \in I_\epsilon.$$

We know that R_ϵ and L_ϵ commute, because

$$\gamma D^2 W(\theta_0(-x))u(-x) = \gamma D^2 W(\gamma\theta_0(x))\gamma\gamma u(-x) = D^2 W(\theta_0(x))\gamma u(-x).$$

The assertion follows with corollary B.1.

□

For later considerations, it is essential that the essential spectrum of L_0 and L_0^{odd} coincide in case of symmetry.

Proposition 2.1. *Suppose W fulfills the assumptions of definition 2.1. We have*

$$L_0 \geq 0,$$

and

$$\sigma_e(L_0) = \left[\min\left(\sigma(D^2W(a)) \cup \sigma(D^2W(b))\right), +\infty\right).$$

If in addition $W \circ \gamma = W$, and θ_0 is γ-odd, then

$$\sigma_e(L_0^{odd}) = \sigma_e(L_0).$$

Proof. The first assertion follows from the fact that the quadratic form of L_0 is given by the second directional derivative of E. More precisely, for some $\phi \in (C_0^\infty(\mathbb{R}, \mathbb{R}))^m$, the mapping

$$i_\phi(\tau) := E(\theta_0 + \tau\phi), \quad \tau \in \mathbb{R},$$

is twice differentiable at $\tau = 0$ with

$$i'_\phi(\tau) = \int_{\mathbb{R}} \langle \theta'_0, \phi' \rangle + \tau|\phi'|^2 + DW(\theta_0 + \tau\phi)\phi dx,$$

and

$$S_0[\phi] = i''_\phi(0) = \int_{\mathbb{R}} |\phi'|^2 + \left\langle D^2W(\theta_0)\phi, \phi \right\rangle dx.$$

As $\tau = 0$ is a global minimum of i_ϕ we must have $i''_\phi(0) \geq 0$. Hence $S_0 \geq 0$ on the set $(C_0^\infty(\mathbb{R}, \mathbb{R}))^m$. For $\phi_i \in (C_0^\infty(\mathbb{R}, \mathbb{R}))^m$, $i = 1, 2$, we have

$$S_0[\phi_1 + i\phi_2] = S_0[\phi_1] + S_0[\phi_2].$$

Hence, $S_0 \geq 0$ on the set $(C_0^\infty(\mathbb{R}, \mathbb{C}))^m$ which is a core of S_0. Now we prove

$$\sigma_e(L_0) \supset \left[\min\left(\sigma(D^2W(a)) \cup \sigma(D^2W(b))\right), +\infty\right). \tag{2.1}$$

Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$, and assume

$$1_{[1,+\infty)} \leq \chi \leq 1_{\mathbb{R}_+}.$$

Define the operator K as the multiplication with the function f on $\mathbb{R}$ given by

$$f(x) := \left[D^2W(b) - D^2W(\theta_0(x))\right]\chi + \left[D^2W(a) - D^2W(\theta_0(x))\right]\chi(-x). \tag{2.2}$$

As $f \to 0$, $x \to \pm\infty$, proposition B.2 implies that K is a compact operator from $(H^2(\mathbb{R}, \mathbb{C}))^m$ in $(L^2(\mathbb{R}, \mathbb{C}))^m$. A relatively compact perturbation does not change the essential spectrum, therefore

$$\sigma_e(L_0) = \sigma_e(L_0 + K).$$

Suppose

$$\min \sigma(D^2W(b)) = \min \left(\sigma(D^2W(a)) \cup \sigma(D^2W(b))\right) =: \lambda_+.$$

Let $U \in O(m)$ be such that $U^*D^2W(b)U$ is a diagonal matrix, without loss of generality λ_+ its first entry. Let us construct a Weyl sequence for $L_0 + K$ and $\lambda_+ + s$, $s \geq 0$. Choose $\phi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$1_{\mathbb{R}_-} \leq \phi \leq 1_{(-\infty,1]}.$$

For $m \in \mathbb{N}$, define

$$\phi_m(x) := \phi(|x - m^2 + 1| - m),$$

which fulfills

$$\operatorname{supp}(\phi_m) \subset \left[(m-2)^2, (m+2)^2\right], \quad m \geq 2. \tag{2.3}$$

Set

$$f_m(x) = \frac{1}{\sqrt{2m}} \phi_m(x) e^{i\sqrt{s}x} U e_1, \quad s \geq 0, \tag{2.4}$$

where $e_1 = (1, 0, ..., 0)^T \in \mathbb{R}^m$. Calculations yields

$$1 \leq \|f_m\|^2_{L^2(\mathbb{R})} \leq 4.$$

For $m \in \mathbb{N}$ sufficiently large such that $\operatorname{supp}(\phi_m) \subset [1, +\infty)$, we have

$$(L_0 + K - \lambda_+ - s) f_m = -\frac{d^2}{dz^2} f_m - s f_m + UU^* D^2W(b) f_m - \lambda_+ f_m =$$

$$-\frac{d^2}{dz^2} f_m - s f_m = -\frac{2i\sqrt{s}}{\sqrt{2m}} e^{i\sqrt{s}x} \phi_m'(x) U e_1 - \frac{1}{\sqrt{2m}} \phi_m''(x) e^{i\sqrt{s}x} U e_1$$

which converges to zero in $(L^2(\mathbb{R}, \mathbb{C}))^m$. In view of (2.3) and the fact that f_m is uniformly bounded, f_m converges weakly to zero in $(L^2(\mathbb{R}, \mathbb{C}))^m$. It follows that the normalized sequence $\frac{f_m}{\|f_m\|}$ is a Weyl sequence for $L_0 + K$ and $\lambda_+ + s$, $s \geq 0$. Theorem B.1 implies (2.1). Define the operator-valued lower bound of L_0 by

$$D_L := \left(H^2(\mathbb{R}, \mathbb{C})\right)^m$$

and

$$Lu := -\frac{d^2}{dz^2} u + [\lambda \cdot I_{\mathbb{C}^m}] u,$$

where $\lambda(x)$ is the smallest eigenvalue of $D^2W(\theta_0(x))$. We have

$$s_L \leq s_{L_0},$$

and lemma C.2 implies

$$\min \sigma_e(L) \leq \min \sigma_e(L_0).$$

Moreover, it is easy to see that

$$L = \otimes_{i=1}^m \hat{L}, \quad \hat{L} = -\frac{d^2}{dz^2} + [\lambda], \quad D_{\hat{L}} = H^2(\mathbb{R}, \mathbb{C}),$$

hence

$$\sigma_e(L) = \sigma_e(\hat{L}).$$

Owing to corollary B.4, we get

$$\min \sigma_e(\hat{L}) = \sup_{\substack{K \subset \mathbb{R}^d \\ \text{compact}}} \inf \left\{ \frac{\left\langle \hat{L}\varphi, \varphi \right\rangle}{\|\varphi\|^2} \middle| \varphi \in C_0^\infty(\mathbb{R} \backslash K, \mathbb{C}), \varphi \neq 0 \right\} \geq$$

$$\sup_{n \in \mathbb{N}} \inf_{|x| \geq n} \lambda(x) = \min_{x_0 = \pm\infty} \liminf_{x \to x_0} \lambda(x) = \min \left(\sigma(D^2 W(a)) \cup \sigma(D^2 W(b)) \right) = \lambda_+.$$

This completes the proof. Suppose now $W \circ \gamma = W$ for some reflection $\gamma \in O(m)$. Assume that θ_0 is γ-odd. The function f defined in (2.2) fulfills

$$\gamma f(-x) \gamma = f(x), \quad x \in \mathbb{R}.$$

This implies $f \cdot u \in L^2_{odd}(\mathbb{R})$ for $u \in L^2_{odd}(\mathbb{R})$. Therefore

$$K \in \mathcal{L}\left(L^2_{odd}(\mathbb{R})\right)$$

is L_0^{odd}-compact, and we conclude with lemma B.1

$$\sigma_e(L_0^{odd}) = \sigma_e(L_0^{odd} + K).$$

Suppose f_m is given by (2.4) and R_0 as in the proof of corollary 2.2. Define

$$F_m := R_0 f_m.$$

In view of statement one of corollary 2.2, we obtain

$$\left\| L_0^{odd} + K - \lambda_+ - s \right\|^2_{L^2_{odd}(\mathbb{R})} = 2 \left\| L_0 + K - \lambda_+ - s \right\|^2_{L^2(\mathbb{R}_+)} \to 0.$$

Thus, up to normalizing, the F_m deliver a Weyl sequence for L_0^{odd} and $\lambda_+ + s$, $s \in \mathbb{R}_+$. It follows that

$$\sigma_e(L_0) \subset \sigma_e(L_0^{odd}).$$

Clearly, the converse is also true, and the proof is finished. $\square$

2.2.2 Convergence of the ground state

It is not difficult to see that L_ϵ and L_0 fulfill the assumptions in definition 1.1. For $\epsilon > 0$, we only have to set $a := \frac{1}{\epsilon}$,

$$V_a := D^2W(\theta_0)|_{(-a,a)}, \quad V_\infty := D^2W(\theta_0),$$

$$J_a := 1, \quad J_\infty := 1,$$

and

$$H_a := L_\epsilon, \quad H_\infty := L_0.$$

As the limits $\lim_{z\to\pm\infty}\theta_0(z)$ are attained at an exponential rate, we have

$$\lambda_- = \min\sigma\left(D^2W(a)\right),$$

and

$$\lambda_+ = \min\sigma\left(D^2W(b)\right).$$

Definition 2.5. *1. Let $\epsilon \geq 0$. Define*

$$\lambda_1^\epsilon := \min\sigma(L_\epsilon),$$

and

$$\lambda_1^{\epsilon,odd} := \min\sigma(L_\epsilon^{odd}).$$

We denote normalized eigenfunctions that correspond to L_ϵ and L_ϵ^{odd} by ψ_1^ϵ and $\psi_1^{\epsilon,odd}$, respectively.

2. Suppose that $\phi_1, ..., \phi_r$ and $\phi_1^{odd}, ..., \phi_s^{odd}$ is an orthonormal basis of $\ker(L_0)$ *and* $\ker(L_0^{odd} - \lambda_1^{0,odd})$*, respectively. Set*

$$\left(P_\epsilon x\right)(t) := \sum_{j=1}^{r} \langle x, \phi_j\rangle_{L^2(I_\epsilon)}\, \phi_j(t), \quad x \in \left(L^2(I_\epsilon,\mathbb{C})\right)^m, t \in I_\epsilon,$$

for $\epsilon \geq 0$. In the same way, we define P_ϵ^{odd} with respect to $\phi_1^{odd}, ..., \phi_s^{odd}$.

Let us agree that we always choose real valued eigenfunctions of L_ϵ^{odd} and L_ϵ.

Lemma 2.1. *The following statements hold:*

1. $\lambda_1^\epsilon = O\left(e^{-\frac{\sqrt{\min\sigma_e(L_0)}}{4\epsilon}}\right), \epsilon \to 0.$

2. *Suppose $W\circ\gamma = W$, $\gamma \in O(m)$, and θ_0 is γ-odd.*

   (a) If $\lambda_1^{0,odd} < \min\sigma_e(L_0^{odd})$, then

$$\left|\lambda_1^{0,odd} - \lambda_1^{\epsilon,odd}\right| = O\left(e^{-\frac{\sqrt{\min\sigma_e(L_0)-\lambda_1^{0,odd}}}{4\epsilon}}\right), \quad \epsilon \to 0,$$

   and if $\dim\ker(L_0) = 1$*, we have $\lambda_1^{0,odd} > 0$.*

(b) If $\lambda_1^{0,odd} = \min \sigma_e(L_0^{odd})$, then

$$\liminf_{\epsilon \to 0} \lambda_1^{\epsilon,odd} \geq \min \sigma_e(L_0).$$

Remark 2.1. *The proof of lemma 2.1 is based on lemma 1.2. If there is a sequence that fulfills the assumptions of lemma 1.2, then there exists a subsequence that delivers an eigenvalue and a corresponding eigenfunction. Throughout the proof, we never mention again that we always suppose the subsequence is the sequence itself.*

Proof. 1. Set

$$\beta_\epsilon := \frac{1}{\|\theta_0'\|_{L^2(I_\epsilon)}}, \quad \epsilon \geq 0.$$

In view of corollary C.1, we have

$$\lambda_1^\epsilon \leq \beta_\epsilon^2 S_\epsilon[\theta_0'] = \beta_\epsilon^2 \theta_0'' \theta_0'|_{\partial I_\epsilon}.$$

Differentiation of the Euler Lagrange equation of E shows that $\theta_0' \in \ker(L_0)$. Lemma 1.1 ($\delta = 3/4$) yields

$$|\theta_0'(z)| \leq C e^{-\frac{\sqrt{\min \sigma_e(L_0)}}{4}|z|}$$

for some constant $C > 0$. Hence

$$\lambda_1^\epsilon \leq C e^{-\frac{\sqrt{\min \sigma_e(L_0)}}{4\epsilon}}, \tag{2.5}$$

as $\beta_\epsilon \to 1$. Now we prove

$$\exists \epsilon_0 > 0 : \exists C > 0 : \forall \epsilon \in (0, \epsilon_0) :$$

$$\forall \psi_1^\epsilon \in \ker\left(L_\epsilon - \lambda_1^\epsilon\right), \text{ normalzied} : \langle \psi_1^\epsilon, P_\epsilon \psi_1^\epsilon \rangle_{L^2(I_\epsilon)} \geq C.$$

If the contrary holds, we obtain a sequence

$$\epsilon_n \longrightarrow 0,$$

and normalized $\psi_1^{\epsilon_n} \in \ker\left(L_{\epsilon_n} - \lambda_1^{\epsilon_n}\right)$ such that

$$\sum_{j=1}^m \left| \langle \psi_1^{\epsilon_n}, \phi_j \rangle_{L^2(I_{\epsilon_n})} \right|^2 < \frac{1}{n}.$$

Hence, by (2.5) and lemma 1.2, there exist

$$\psi := \lim_{n \to \infty} \psi_1^{\epsilon_n},$$

and

$$\lambda := \lim_{n\to\infty} \lambda_1^{\epsilon_n}$$

which fulfill

$$\psi \in \ker(L_0 - \lambda),$$

and

$$\|\psi\|_{L^2(\mathbb{R})} = 1.$$

Due to (2.5), we must have $\lambda \leq 0$. But according to proposition 2.1, we have $L_0 \geq 0$. It follows $\lambda = 0$ and

$$\psi \in \ker(L_0).$$

Owing to (2.5) and lemma 1.1, we have the uniform envelope $e^{-\frac{\sqrt{\min \sigma_e(L_0)}}{8}|z|}$ for each eigenfunction $\psi_1^{\epsilon_n}$, $\epsilon_n > 0$ sufficiently small. Theorem A.1 implies

$$\langle \psi, \phi_j \rangle_{L^2(\mathbb{R})} = \lim_{n\to\infty} \langle \psi_1^{\epsilon_n}, \phi_j \rangle_{L^2(I_{\epsilon_n})} = 0$$

for each $j \in \{1, ..., m\}$. It follows

$$\psi \in \ker(L_0)^{\perp},$$

which implies

$$\psi = 0.$$

This is a contradiction. Now we draw the conclusion:

$$|\lambda_1^\epsilon|\, C \leq \left| \langle L_\epsilon \psi_1^\epsilon, P_0^\epsilon \psi_1^\epsilon \rangle_{L^2(I_\epsilon)} \right| =$$

$$\left| \sum_{j=1}^{m} \overline{\langle \psi_1^\epsilon, \phi_j \rangle_{L^2(I_\epsilon)}} \, \langle L_\epsilon \psi_1^\epsilon, \phi_j \rangle_{L^2(I_\epsilon)} \right| =$$

$$\left| \sum_{j=1}^{m} \overline{\langle \psi_1^\epsilon, \phi_j \rangle_{L^2(I_\epsilon)}} \left[\langle \psi_1^\epsilon, [L_\epsilon]\, \phi_j \rangle_{L^2(I_\epsilon)} + \psi_1^{\epsilon\prime} \phi_j|_{\partial I_\epsilon} - \psi_1^\epsilon \phi_j{}'|_{\partial I_\epsilon} \right] \right| =$$

$$O\left(e^{\frac{-\sqrt{\min \sigma_e(L_0)}}{4\epsilon}} \right), \epsilon \to 0.$$

The last step follows from corollary 1.2, where we have proved that ϕ_j and ϕ_j' decay with the same rate.

2-(a) The quadratic form that is associated to L_ϵ^{odd} is given by the restriction of S_ϵ to $H^1_{odd}(I_\epsilon)$. Suppose $\lambda_1^{0,odd} \in \sigma_d(L_0^{odd})$. Then $\lambda_1^{0,odd}$ is in the discrete spectrum of L_0. Due to proposition 2.1 and lemma 1.1, each corresponding eigenfunction $\psi_1^{0,odd}$ has the envelope $e^{-\frac{\sqrt{\min\sigma_e(L_0)-\lambda_1^{0,odd}}}{4}|x|}$. Set

$$\gamma_\epsilon := \frac{1}{\|\psi_1^{0,odd}\|_{L^2(I_\epsilon)}}, \quad \epsilon \geq 0.$$

Then we have

$$\lambda_1^{\epsilon,odd} \leq \gamma_\epsilon^2 S_\epsilon[\psi_1^{odd}] = \lambda_1^{0,odd} + O\left(e^{-\frac{\sqrt{\min\sigma_e(L_0)-\lambda_1^{0,odd}}}{4\epsilon}}\right), \quad \epsilon \to 0.$$

First, we prove

$$\exists\epsilon_0 > 0 : \exists C > 0 : \forall\epsilon \in (0,\epsilon_0) : \forall\psi_1^{\epsilon,odd} \in \ker(L_\epsilon^{odd} - \lambda_1^{\epsilon,odd}), \text{ normalized } :$$

$$\left\langle \psi_1^{\epsilon,odd}, P_\epsilon^{odd}\psi_1^{\epsilon,odd}\right\rangle_{L^2(I_\epsilon)} \geq C.$$

If the contrary holds, there exists a sequence $\epsilon_n \to 0$ and normalized

$$\psi_1^{\epsilon_n,odd} \in \ker(L_{\epsilon_n}^{odd} - \lambda_1^{\epsilon_n,odd})$$

such that

$$\sum_{j=1}^{m}\left|\left\langle \psi_1^{\epsilon_n,odd}, \phi_j^{odd}\right\rangle_{L^2(I_{\epsilon_n})}\right|^2 < \frac{1}{n}. \tag{2.6}$$

With lemma 2.1, we obtain an eigenvalue

$$\lambda \in \sigma_d(L_0), \quad \lambda \leq \lambda_1^{0,odd} < \min\sigma_e(L_0),$$

and a normalized eigenfunction

$$\psi \in \ker(L_0 - \lambda)$$

which is γ-odd. Hence

$$\psi \in \ker\left(L_0^{odd} - \lambda\right),$$

and

$$\lambda = \lambda_1^{0,odd}.$$

In view of (2.6), lemma 1.1, and theorem A.1, we conclude

$$\left\langle \psi, \phi_j^{odd}\right\rangle_{L^2(\mathbb{R})} = \lim_{n\to\infty}\left\langle \psi_1^{\epsilon_n,odd}, \phi_j^{odd}\right\rangle_{L^2(I_{\epsilon_n})} = 0,$$

i.e.

$$\psi \in \ker\left(L_0^{odd} - \lambda_1^{odd}\right)^{\perp}.$$

This is a contradiction. With partial integration, we obtain

$$\lambda_1^{\epsilon,odd}\left\langle \psi_1^{\epsilon,odd}, P_\epsilon^{odd}\psi_1^{\epsilon,odd}\right\rangle_{L^2(I_\epsilon)} = \lambda_1^{0,odd}\left\langle \psi_1^{\epsilon,odd}, P_\epsilon^{odd}\psi_1^{\epsilon,odd}\right\rangle_{L^2(I_\epsilon)} -$$

$$\sum_{j=1}^{s} \overline{\left\langle \psi_1^{\epsilon,odd}, \phi_j^{odd}\right\rangle_{L^2(I_\epsilon)}} \psi_1^{\epsilon,odd}\phi_j^{odd\prime}\Big|_{\partial I_\epsilon}.$$

This implies

$$\left|\lambda_1^{0,odd} - \lambda_1^{\epsilon,odd}\right| C \le O\left(e^{-\frac{\sqrt{\min \sigma_e(L_0) - \lambda_1^{0,odd}}}{4\epsilon}}\right), \quad \epsilon \to 0.$$

In view of $\dim \ker(L_0) = 1$, we have $\psi_1^{0,odd} \in \ker(L_0)^\perp$. Corollary C.1 implies

$$0 < \lambda_1^{0,odd}.$$

2-(b) Suppose the contrary holds. Then there exists $\beta > 0$ and a sequence $\epsilon_n \to 0$ such that $\lambda_1^{\epsilon_n,odd} \le \min \sigma_e(L_0) - \beta$. If $\psi_1^{\epsilon_n,odd}$ is a corresponding sequence of normalized eigenfunctions, then lemma 1.2 implies that

$$\lambda := \lim_{n\to\infty} \lambda_1^{\epsilon_n,odd} \in \sigma_p(L_0), \quad \lambda < \min \sigma_e(L_0),$$

and

$$\psi := \lim_{n\to\infty} \psi_1^{\epsilon_n,odd}$$

fulfill

$$\psi \in \ker(L_0^{odd} - \lambda).$$

This implies $\lambda_1^{0,odd} < \min \sigma_e(L_0^{odd})$, which is a contradiction.

□

Chapter 3

Spectral analysis at the triple-junction

In this section, we analyze the spectrum of the linearization of the Allen-Cahn equation around a rescaled stationary solution. We always consider Sobolev spaces with complex valued functions.

3.1 Sobolev spaces with symmetry

Let us first define the different notions of symmetry used in this chapter.

Definition 3.1. *Let $m \in \mathbb{N}$. Suppose $G \subset Gl(m, \mathbb{R})$ is a subgroup.*

1. *A subset $\Omega \subset \mathbb{R}^m$ is called G-invariant if*

$$\forall g \in G : g \cdot \Omega = \Omega.$$

2. *If $u : \Omega \to \mathbb{C}$ and $v : \Omega \to \mathbb{C}^m$ are defined on a G-invariant subset Ω, then u is called G-invariant if*

$$\forall g \in G : u \circ g = u,$$

and v is called G-equivariant if

$$\forall g \in G : g \circ v = v \circ g.$$

3. *Let $V : \Omega \to M(m, \mathbb{C})$ where $\Omega \subset \mathbb{R}^m$ is G-invariant. The mapping V is called G-normal if*

$$\forall g \in G : \forall x \in \Omega : V(x) = gV\left(g^{-1}x\right)g^{-1}.$$

4. *If $G = \langle g \rangle$ is a cyclic subgroup of $Gl(m)$, we write g-normal instead of $\langle g \rangle$-normal, etc.*

Our aim is to construct the Sobolev spaces that own these symmetries. We define them as the range space of orthogonal projections.

Definition 3.2. *Let $m \in \mathbb{N}$. Assume $G \subset O(m)$ is a finite subgroup, and $\Omega \subset \mathbb{R}^m$ is G-invariant. Then for each $u \in \left(L^2(\Omega, \mathbb{C})\right)^m$, we define*

$$\left(P_\Omega^G u\right)(x) := \frac{1}{|G|} \sum_{g \in G} g u\left(g^{-1} x\right), \quad x \in \Omega,$$

and for $u \in L^2(\Omega, \mathbb{C})$, we set

$$\left(Q_\Omega^G u\right)(x) := \frac{1}{|G|} \sum_{g \in G} u\left(g^{-1} x\right), \quad x \in \Omega.$$

Corollary 3.1. *Under the assumptions of definition 3.2, the mapping P_Ω^G is the orthogonal projection onto the G-equivariant functions in $\left(L^2(\Omega, \mathbb{C})\right)^m$, and Q_Ω^G is the orthogonal projection onto the subspace of G-invariant elements in $L^2(\Omega, \mathbb{C})$.*

Proof. For $u \in \left(L^2(\Omega, \mathbb{C})\right)^m$, we have

$$\left(\left(P_\Omega^G\right)^2 u\right)(x) = \frac{1}{|G|} \sum_{g \in G} g\left(P_\Omega^G u\right)\left(g^{-1} x\right) =$$

$$\frac{1}{|G|^2} \sum_{g, h \in G} (gh) u\left((gh)^{-1} x\right) = \left(P_\Omega^G u\right)(x).$$

Hence P_Ω^G is a projection. Let us show that P_Ω^G is self-adjoint. This follows from the transformation lemma. For $u, v \in \left(L^2(\Omega, \mathbb{C})\right)^m$, we have

$$\left\langle P_\Omega^G u, v\right\rangle_{L^2(\Omega)} = \frac{1}{|G|} \sum_{g \in G} \int_\Omega \left\langle g u(g^{-1} x), v(x)\right\rangle dx =$$

$$\frac{1}{|G|} \sum_{g \in G} \int_\Omega \langle u(x), g^* v(gx)\rangle \, dx = \left\langle u, P_\Omega^G v\right\rangle_{L^2(\Omega)}.$$

The proof for Q_Ω^G is similar. □

Now we are able to define the Sobolev spaces that own this symmetry. Later, we will see that this is the natural setting at the triple-junction.

Definition 3.3. *Let $k \in \mathbb{N}$ and $m \in \mathbb{N}$. Suppose $G \subset O(m)$ is a finite subgroup and $\Omega \subset \mathbb{R}^m$ is G-invariant.*

1. *Define*
$$C^\infty_{0,G}(\Omega) := P^G_\Omega\left((C^\infty_0(\Omega, \mathbb{C}))^m\right),$$
and
$$L^2_G(\Omega) := P^G_\Omega\left(\left(L^2(\Omega, \mathbb{C})\right)^m\right).$$

2. *Set*
$$H^k_G(\Omega) := P^G_\Omega\left(\left(H^k(\Omega, \mathbb{C})\right)^m\right),$$
and
$$\overset{\circ}{H}{}^k_G(\Omega) := P^G_\Omega\left(\left(\overset{\circ}{H}{}^k(\Omega, \mathbb{C})\right)^m\right).$$

3. *Use the notation*
$$\overset{\circ}{H}{}^0_G(\Omega) = H^0_G(\Omega) := L^2_G(\Omega).$$

Later, we often make use of the fact that it suffices to compute the L^2-norm of a G-equivariant function on a part of its domain.

Corollary 3.2. *Suppose $\Omega \subset \mathbb{R}^m$ is open and G-invariant. Assume $U \subset \Omega$ is open and $\gamma = (\gamma_{ij})_{i,j=1,\dots,m} \in G$. If A is an operator in $L^2_G(\Omega)$, $u \in D_A$, and $v \in L^2_G(\Omega)$, then we have*
$$\langle Au, v\rangle_{L^2(U)} = \langle Au, v\rangle_{L^2(\gamma^* U)}.$$
Moreover,
$$\|u\|_{H^1(U)} = \|u\|_{H^1(\gamma^* U)}, \quad u \in H^1_G(\Omega),$$
and
$$\|u\|_{H^2(U)} = \|u\|_{H^2(\gamma^* U)}, \quad u \in H^2_G(\Omega).$$

Proof. As Au and v are G-equivariant, the transformation lemma implies
$$\langle Au, v\rangle_{L^2(U)} =$$
$$\int_U \langle (Au)(x), v(x)\rangle \, dx = \int_{\gamma^* U} \langle \gamma(Au)(x), \gamma v(x)\rangle \, dx =$$
$$\langle Au, v\rangle_{L^2(\gamma^* U)}.$$
Choose $u \in H^1_G(\Omega)$. Due to $\gamma \circ \phi = \phi \circ \gamma$, we have
$$\gamma D\phi(x) = D\phi(\gamma x)\gamma \tag{3.1}$$

for a.e. $x \in \Omega$. Together with the first assertion, it follows that

$$\|u\|_{H^1(U)}^2 - \|u\|_{L^2(U)}^2 = \int_U \|Du(x)\|_{tr}^2 dx = \int_{\gamma^* U} \|Du(\gamma x)\|_{tr}^2 dx =$$

$$\int_{\gamma^* U} \|\gamma Du(x)\gamma^*\|_{tr}^2 dx = \int_{\gamma^* U} \|Du(x)\|_{tr}^2 dx = \|u\|_{H^1(\gamma^* U)}^2 - \|u\|_{L^2(\gamma^* U)}.$$

In order to prove the last statement, note that we have

$$\gamma^T D^2 u_i(\gamma x)\gamma = \sum_{j=1}^{m} \gamma_{ij} D^2 u_j(x), \tag{3.2}$$

for each $u \in H_G^2(\Omega)$, $i, j = 1, ..., m$, and a.e. $x \in \Omega$. Furthermore,

$$\|u\|_{H_G^2(U)}^2 = \sum_{i=1}^{m} \int_U \left\|D^2 u_i(x)\right\|_{tr}^2 dx + \|u\|_{H^1(U)}^2.$$

As the trace is invariant under orthogonal transformations, we obtain with (3.2)

$$\sum_{i=1}^{m} \int_U \left\|D^2 u_i(x)\right\|_{tr}^2 dx = \sum_{\substack{r,s=1 \\ r \neq s}}^{m} \left(\sum_{i=1}^{m} \gamma_{ir}\gamma_{is}\right) \int_{\gamma^* U} \left\langle D^2 u_r(x), D^2 u_s(x)\right\rangle_{tr} dx +$$

$$\sum_{i=1}^{m} \int_{\gamma^* U} \left\|D^2 u_i(x)\right\|_{tr}^2 dx.$$

As $\gamma \in O(m)$, the assertion follows. □

Proposition 3.1. *Let $k \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Assume $G \subset O(m)$ is a finite subgroup, and $\Omega \subset \mathbb{R}^m$ is open and G-invariant. Suppose that*

$$V \in C_b^k\left(\Omega, M(m, \mathbb{C})\right)$$

is G-normal. Then

$$[V] \in \mathcal{L}\left(H_G^k(\Omega)\right).$$

If $u \in G$,

$$W_0 \in C_b^k\left(\Omega, M(m, \mathbb{C})\right)$$

is u-normal, $E \subset G$ is a generator of G such that $E^{-1} \subset E$, and for each $e \in E$, the mapping $\tau : x \in E \longmapsto exu$ is a bijection of E, then

$$W(x) := \sum_{e \in E} eW_0\left(e^{-1}x\right)e^{-1}, \quad x \in \Omega,$$

is G-normal.

Proof. In order to prove the first statement, we have to show that $V \cdot u \in H_G^k(\Omega)$ for $u \in H_G^k(\Omega)$. Clearly, the components of $V \cdot u$ are contained in $H^k(\Omega, \mathbb{C})$. For $x \in \Omega$ and $g \in G$, we have

$$(V \cdot u)(gx) = V(gx)u(gx) = gV(x)g^{-1}gu(x) = gV(x)u(x) = g(V \cdot u)(x),$$

i.e. $V \cdot u$ is G-equivariant. In order to prove the second assertion, choose $g \in E$ arbitrarily. Then

$$W(g^{-1}x) = g^{-1} \sum_{e \in E} (ge) W_0 \left((ge)^{-1}x\right)(ge)^{-1}g =$$

$$g^{-1} \sum_{e \in E} (geu) W_0 \left((geu)^{-1}x\right)(geu)^{-1}g.$$

As $\tau : E \to E$ is bijective, we have

$$W(g^{-1}x) = g^{-1}W(x)g, \quad g \in E.$$

For each $g \in G$, there exist $e_1, ..., e_n \in E$ such that $g = e_1 \cdot ... \cdot e_n$. Define

$$\Pi_j := e_1 \cdot ... \cdot e_j, \quad j \in \{1, ..., n\}.$$

Recursively, we obtain

$$gW(g^{-1}x)g^{-1} = \Pi_{n-1}W(\Pi_{n-1}^{-1}x)\Pi_{n-1}^{-1} = ... = W(x).$$

□

Let us consider the Laplace operator in L_G^2.

Definition 3.4. *Suppose $G \subset O(m)$ is a finite subgroup, $m \in \mathbb{N}$, and let $\Omega \subset \mathbb{R}^m$ be a open, bounded, and G-invariant subset such that $\partial\Omega \in C^2$. Define*

$$D_{\triangle} := \left\{ u \in H^2(\Omega, \mathbb{C}) : \frac{\partial u}{\partial \nu} = 0 \ \text{on} \ \partial\Omega \right\},$$

and

$$\triangle u := \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} u.$$

Corollary 3.3. *The operator $\otimes_{i=1}^m \triangle$ commutes with P_Ω^G, i.e.*

$$P_\Omega^G \circ (\otimes_{i=1}^m \triangle) \subset (\otimes_{i=1}^m \triangle) \circ P_\Omega^G.$$

Especially, the restriction of $\otimes_{i=1}^m \triangle$ on $L_G^2(\Omega)$ is self-adjoint.

Proof. A straightforward calculation yields

$$\otimes_{i=1}^{m} \triangle(Au \circ B) = A\left(\otimes_{i=1}^{m} \triangle u\right) \circ B$$

for $A, B \in O(m)$ and $u \in (D_{\triangle})^m$. This implies

$$P_{\Omega}^{G}\left(\otimes_{i=1}^{m} \triangle\right) u = \frac{1}{|G|} \sum_{g \in G} g\left(\otimes_{i=1}^{m} \triangle u\right) \circ g^{-1} =$$

$$\frac{1}{|G|} \sum_{g \in G} \left(\otimes_{i=1}^{m} \triangle\right) gu \circ g^{-1} = \left(\otimes_{i=1}^{m} \triangle\right) P_{\Omega}^{G} u$$

for $u \in (D_{\triangle})^m$. In view of corollary B.1, the restriction of $\otimes_{i=1}^{m} \triangle$ to $L_G^2(\Omega)$ has domain

$$\left\{ u \in H_G^2(\Omega) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}$$

and is self-adjoint. □

In order to simplify the notation, let us write $\triangle$ instead of $\otimes_{i=1}^{m} \triangle$ as long as no confusion occurs.

3.2 Rescaled stationary solutions

From now on, we consider the case $m = 2$ and the symmetry group of the equilateral triangle. This is refined in the following definition.

Definition 3.5. *Set*

$$\varphi := \frac{\pi}{3},$$

and

$$x_i := (-1)^{(2-\delta_3(i))} \begin{pmatrix} \cos\left(\frac{i}{2}\varphi\right) \\ \sin\left(\frac{i}{2}\varphi\right) \end{pmatrix}, \quad i \in \{1, 3, 5\}.$$

1. *For $i = 1, 3, 5$, let*

$$R_i := \begin{pmatrix} \cos(i\varphi) & \sin(i\varphi) \\ \sin(i\varphi) & -\cos(i\varphi) \end{pmatrix}$$

be the reflection on the subspace $\operatorname{lin}\{x_i\}$, *and*

$$D_j := \begin{pmatrix} \cos(j\varphi) & -\sin(j\varphi) \\ \sin(j\varphi) & \cos(j\varphi) \end{pmatrix}, \quad j \in \{0, 2, 4\},$$

the rotation by 0, 120, *and* 240 *degrees. Let G be the group generated by* $\{R_1, R_3, R_5\}$.

2. *Define*

$$R_{15} := \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R}, \sqrt{3}y \geq |x|\},$$

and

$$R_{35} := D_2 R_{15}, \quad R_{13} := D_4 R_{15}.$$

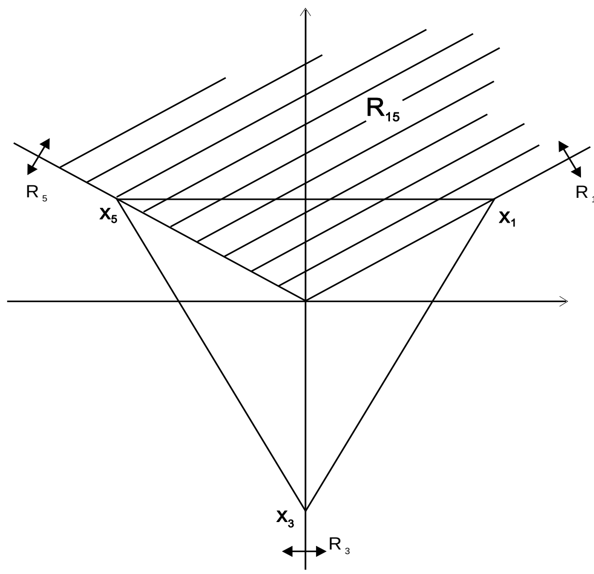

Figure 3.1: The equilateral triangle

The points x_1, x_3, x_5 are the corners of the equilateral triangle of edge length $\sqrt{3}$, centered at $x = 0$. Calculation yields that

$$G = \{D_0, D_2, D_4, R_1, R_3, R_5\}.$$

Throughout this section, we always assume that

(H1) $W \in C^2(\mathbb{R}^2, \mathbb{R})$ is G-invariant.

(H2) W has exactly the three global minima x_i, $i = 1, 3, 5$, such that

$$W(x_i) = 0, \quad i \in \{1, 3, 5\}.$$

Moreover, there exists $\lambda, \rho > 0$ such that

$$D^2 W(x) \geq \lambda$$

for each $x \in B_\rho(x_1) \cup B_\rho(x_3) \cup B_\rho(x_5)$.

(H3) There exists a Lipschitz continuous function $f : \mathbb{R}^2 \longmapsto \mathbb{R}^2$ such that f is G-equivariant,

$$\|Df(x)\|_{tr} \leq 1, \quad \text{for a.e. } x \in \mathbb{R}^2,$$
$$f(x_i) = x_i, \quad i \in \{1, 3, 5\},$$
$$W(f(x)) \leq W(x), \quad x \in \mathbb{R}^2,$$

and $\mathrm{im}(f)$ is bounded.

Remark 3.1. *If we set $b := x_1$ and $a := x_5$, then W fulfills the assumptions of definition 2.1. Thus, we consider the one-dimensional minimization problem associated to W, given by the energy E and the set M in definition 2.1.*

Definition 3.6. *The set of all R_3-odd global minimizers of E over M is denoted by $\mathcal{M}$.*

If the assumptions (H1)-(H3) are fulfilled, it follows from [BGS, Theorem 2.2], [BGS, Theorem 2.3], [BGS, Theorem 2.8], and [BGS, Theorem 4.7] that

1. $\mathcal{M} \neq \emptyset$.

2. If $u \in \mathcal{M}$, then $\mathrm{im}(u) \subset \overset{\circ}{R}_{15} \cap \mathrm{im}(f)$,

$$u \in C^3(\mathbb{R}, \mathbb{R}^2) \cap \left(\phi + \left(H^2(\mathbb{R}, \mathbb{R})\right)^2\right),$$

   and u attains the limits $\lim_{z \to \pm\infty} u(z)$ at an exponential rate.

3. There exists a non-trivial G-equivariant solution $u_0 \in C_b^2(\mathbb{R}^2, \mathbb{R}^2)$ of

$$-\triangle u_0 + (DW(u_0))^T = 0. \tag{3.3}$$

   The function $u_0(r\cos\theta, r\sin\theta)$ converges to $a = x_5$ as $r \to 0$, uniformly in

$$\left\{\theta : \frac{\pi}{2} + \delta \leq \theta \leq \frac{7\pi}{6} - \delta\right\}$$

   for $0 < \delta < \frac{\pi}{3}$. If $\mathcal{M}$ has finitely many elements, there exists $\theta_0 \in \mathcal{M}$ such that

$$\sup_{x \in \mathbb{R}} |u_0(x, y) - \theta_0(x)| \to 0, \quad y \to \infty.$$

In this section, we always assume that

(H4) $\mathcal{M}$ is finite, and if $u_0 \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ is the solution of (3.3), then

$$\sup_{x \in \mathbb{R}} |u_0(x, y) - \theta_0(x)| \to 0, \quad y \to \infty,$$

for fixed $\theta_0 \in \mathcal{M}$.

Note that $u_0\left(\frac{x}{\epsilon}\right)$ is a stationary solution of the Allen-Cahn equation.

3.3 Linearizations around rescaled stationary solutions

In the proof of the main theorem, we apply proposition 3.1 with the generator $E = \{R_1, R_3, R_5\}$. If $u = R_j$, $j \in \{1,3,5\}$, then the mapping τ is a bijection of E. This can be seen in the table bellow.

$\nearrow$	1	R_1	R_3	R_5	D_2	D_4
1	1	R_1	R_3	R_5	D_2	D_4
R_1	R_1	1	D_4	D_2	R_5	R_3
R_3	R_3	D_2	1	D_4	R_1	R_5
R_5	R_5	D_4	D_2	1	R_3	R_1
D_2	D_2	R_3	R_5	R_1	D_4	1
D_4	D_4	R_5	R_1	R_3	1	D_2

Table 3.1: The left row multiplied with the upper line.

3.3.1 Spectral analysis

In this section, we define the operator $\mathcal{L}_0$ that represents the linearization of the Allen-Cahn equation around u_0. From now on, the spaces $H^k_{odd}(I_\epsilon)$, $\epsilon \geq 0$, $k \in \mathbb{N}_0$, are defined with respect to R_3.

Definition 3.7. *Let*

$$D_{\mathcal{L}_0} := H^2_G\left(\mathbb{R}^2\right),$$

and

$$\mathcal{L}_0 := -\triangle + \left[D^2W\left(u_0\right)\right].$$

Further, define $D_{L_0} := \left(H^2(\mathbb{R},\mathbb{C})\right)^2$ *and*

$$L_0 u := -u'' + \left[D^2W(\theta_0)\right] u$$

for $u \in D_{L_0}$. *Denote the closed form that is associated to* $\mathcal{L}_0$ *and* L_0 *by* $\mathcal{S}_0$ *and* S_0, *respectively. The operator* L_0^{odd} *is the restriction of* L_0 *to* $L^2_{odd}(\mathbb{R})$. *Define*

$$\lambda_1^{0,odd} := \min \sigma(L_0^{odd}).$$

If $\lambda_1^{0,odd} \in \sigma_p(L_0^{odd})$, *denote corresponding normalized eigenfunctions with* $\psi_1^{0,odd}$.

Remark 3.2. *As* $W \circ R_3 = W$, *we conclude from proposition 2.1 that*

$$\sigma_e(L_0^{odd}) = \sigma_e(L_0).$$

The matrix-valued function $D^2W(u_0)$ *is* G-*normal, as* W *is* G-*invariant and* u_0 G-*equivariant.*

We denote the interior of the equilateral triangle that has edge length 1 and center $x = 0$ with T. For $\epsilon \in \mathbb{R}_+$, we consider the expanded version

$$T_\epsilon := \frac{2}{\epsilon} T \text{ and } T_0 := \mathbb{R}^2.$$

If $\epsilon > 0$, then T_ϵ has edge length $\frac{2}{\epsilon}$ and its incircle the radius

$$\varrho(\epsilon) := \frac{1}{\sqrt{3}\epsilon}.$$

The circumcircle of T_ϵ has the radius

$$r(\epsilon) := 2\varrho(\epsilon).$$

The triangle T_ϵ is the analogue of I_ϵ in chapter 1. In order to apply well-known regularity theory, we regularize the corners of T_ϵ. This modification is unessential for later considerations, as the corners of T_ϵ lie in the domain where $D^2W(u_0)$ is positive definite for ϵ small. Newton interpolation suggests the following definition.

Definition 3.8. *Define*

$$F(x) := \frac{1}{2} + \left(-120 + \frac{1155}{16}\sqrt{3}\right) x^4 + \left(1280 - 693\sqrt{3}\right) x^6 +$$

$$\left(-5760 + 2970\sqrt{3}\right) x^8 + \left(12288 - 6160\sqrt{3}\right) x^{10} + \left(-10240 + 5040\sqrt{3}\right) x^{12}$$

for $x \in \mathbb{R}$. Set

$$N := \left\{(x, y) \in \mathbb{R}^2 : |x| \leq \frac{1}{2}, \sqrt{3}x < y \leq \frac{1}{2}\sqrt{3}\right\},$$

and

$$M := \left\{(x, y) \in \mathbb{R}^2 : |x| \leq \frac{1}{2}, F(x) < y \leq \frac{1}{2}\sqrt{3}\right\}.$$

Define

$$N_\epsilon := \bigcup_{i=1,3,5} R_i \left(N - r(\epsilon)e_2\right),$$

$$M_\epsilon := \bigcup_{i=1,3,5} R_i \left(M - r(\epsilon)e_2\right),$$

and

$$\Omega_\epsilon := T_\epsilon \backslash N_\epsilon \cup M_\epsilon$$

for $\epsilon \in (0, 1]$. Set

$$\Omega_0 := \mathbb{R}^2.$$

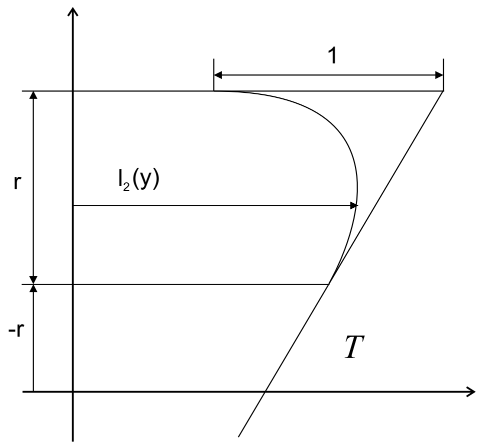

Figure 3.2: A regularized corner of T_ϵ

Calculation yields

$$M \subset N,$$

and

$$\partial\Omega_\epsilon \in C^4, \quad \epsilon \in (0,1].$$

Definition 3.9. *Assume $\epsilon > 0$.*

1. *Set*

$$D_{\mathcal{L}_\epsilon} := \left\{ u \in H^2_G(\Omega_\epsilon) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_\epsilon \right\},$$

   and

$$\mathcal{L}_\epsilon := -\triangle + \left[D^2 W(u_0)\right].$$

   The associated closed form is denoted by $\mathcal{S}_\epsilon$.

2. *Define $D_{L_\epsilon} := \left\{ u \in H^2(I_\epsilon, \mathbb{C}) : u'\left(\pm\frac{1}{\epsilon}\right) = 0 \right\}^2$ and*

$$L_\epsilon u := -u'' + \left[D^2 W(\theta_0)\right] u$$

   for $u \in D_{L_\epsilon}$. Let S_ϵ be the associated closed form. We define L_ϵ^{odd} as the restriction of L_ϵ to $L^2_{odd}(I_\epsilon)$. Set

$$\lambda_1^{\epsilon,odd} := \min \sigma(L_\epsilon^{odd}),$$

   and denote the corresponding normalized eigenfunctions with $\psi_1^{\epsilon,odd}$.

3. *Let*

$$P_\epsilon := P^G_{\Omega_\epsilon}$$

   for $\epsilon \in [0,1)$.

Remark 3.3. *As the assumptions of definition 2.3 are fulfilled, all the results of lemma 2.1 are applicable to* L_ϵ*. That* $\mathcal{L}_0 \geq 0$ *was proved in [BGS]. In view of corollary 3.3, the operators* $\mathcal{L}_\epsilon$*,* $\epsilon \in [0,1]$*, are self-adjoint.*

Theorem 3.1. *Suppose* (H1)-(H4) *hold and* $\dim \ker (L_0) = 1$*. Then the following statements hold:*

1. *We have*

$$\min \sigma_e(\mathcal{L}_0) = \liminf_{\epsilon \to 0} \lambda_1^{\epsilon, odd} > 0,$$

and

$$\sigma(L_0^{odd}) \subset \sigma_e(\mathcal{L}_0).$$

2. *For each* $\lambda \in \sigma_p(\mathcal{L}_0) \cap (-\infty, \min \sigma_e(\mathcal{L}_0))$*, and* $\delta \in (\frac{1}{2}, 1)$*, there exists a constant* $C > 0$ *such that for each normalized* $\psi \in \ker(\mathcal{L}_0 - \lambda)$*, we have*

$$\forall x \in \mathbb{R}^2 : |\psi(x)| \leq C e^{-(1-\delta)\sqrt{\min \sigma_e(\mathcal{L}_0) - \lambda}|x|}.$$

3. *Suppose* $E < \min \sigma_e(\mathcal{L}_0)$*,* $\psi \in D_{\mathcal{L}_0}$*, and* $R \in L^2_G(\mathbb{R}^2)$ *such that*

$$(\mathcal{L}_0 - E)\psi = R.$$

Assume there exist $c, a > 0$ *such that*

$$|R(x)| \leq c e^{-a|x|}$$

for a.e. $x \in \mathbb{R}^2$*. Then there exists a constant* $C > 0$ *and* $\delta \in (\frac{1}{2}, 1)$ *such that*

$$|\psi(x)| \leq C e^{-(1-\delta)\sqrt{\min \sigma_e(\mathcal{L}_0) - E}|x|}$$

for each $x \in \mathbb{R}^2$*.*

Remark 3.4. *At this point, we prove statement 1 of theorem 3.1. Statement 2 is a consequence of more general considerations in subsection 3.3.2 - cf. lemma 3.2. Part 3 of this theorem is proved in subsection 3.3.3.*

Proof. Let us first prove that

$$\min \sigma_e(\mathcal{L}_0) \geq \liminf_{\epsilon \to 0} \lambda_1^{\epsilon, odd}. \tag{3.4}$$

Choose $\epsilon_0 > 0$ small enough so that

$$v(\epsilon_0) := \inf_{\epsilon \leq \epsilon_0} \lambda_1^{\epsilon, odd} > 0. \tag{3.5}$$

This is possible according to lemma 2.1. Set $\lambda := \|D^2 W\,(u_0)\|_{C^0(\mathbb{R}^2)}$, and define

$$K(\epsilon_0) := \left[(v(\epsilon_0) + \lambda) 1_{T_{\epsilon_0}}\right].$$

Let us show that

$$K(\epsilon_0) \in \mathcal{L}\left(L^2_G(\mathbb{R}^2)\right)$$

is $\mathcal{L}_0$-compact. Suppose $u_n \in D_{\mathcal{L}_0}$ is a sequence that is bounded in the graph norm of $\mathcal{L}_0$, i.e. it is bounded in $H^2_G(\mathbb{R}^2)$. Then the sequence $u_n|_{T_{\epsilon_0}}$ is bounded in $H^2_G(T_{\epsilon_0})$. In view of [Alt, Satz 8.9], there exists a subsequence u_{n_k} such that $K(\epsilon_0)u_{n_k}$ converges in $L^2_G(\mathbb{R}^2)$. According to lemma B.1, we have

$$\sigma_e(\mathcal{L}_0) = \sigma_e(\mathcal{L}_0 + K(\epsilon_0)). \tag{3.6}$$

Next we give a lower bound for $\sigma(\mathcal{L}_0 + K(\epsilon_0))$. The closed form that is associated to $\mathcal{L}_0 + K(\epsilon_0)$ is given by $D_{t_{\epsilon_0}} := H^1_G(\mathbb{R}^2)$ and

$$t_{\epsilon_0}[u] := \int_{\mathbb{R}^2} \sum_{j=1,2} |\nabla u_j|^2 + \left\langle D^2W\left(u_0\right) u, u\right\rangle + 1_{T_{\epsilon_0}}(v(\epsilon_0) + \lambda)|u|^2 dx.$$

In view of proposition 3.1 and corollary 3.2, we have

$$\frac{1}{3} t_{\epsilon_0}[u] = \int_{R_{15}} \sum_{j=1,2} |\nabla u_j|^2 + \left\langle D^2W\left(u_0\right) u, u\right\rangle + 1_{T_{\epsilon_0}}(v(\epsilon_0) + \lambda)|u|^2 dx \tag{3.7}$$

for $u \in D_{t_{\epsilon_0}}$. Choose $u \in C^\infty_{0,G}(\mathbb{R}^2)$ arbitrarily, and integrate in (3.7) over $R_{15} \cap T_{\epsilon_0}$ and $R_{15} \cap (T_{\epsilon_0})^c$. Due to the choice of λ, we obtain

$$\frac{1}{3} t_{\epsilon_0}[u] \geq \int_{\varrho(\epsilon_0)}^{+\infty} \int_{|x| \leq \sqrt{3}y} \sum_{j=1,2} |\nabla u_j|^2 + \left\langle D^2W\left(u_0\right) u, u\right\rangle dxdy + v(\epsilon_0)\|u\|^2_{L^2(R_{15} \cap T_{\epsilon_0})} =$$

$$\int_{\varrho(\epsilon_0)}^{+\infty} \int_{|x| \leq \sqrt{3}y} \left|\frac{\partial}{\partial x} u\right|^2 + \left\langle D^2W\left(\theta_0(x)\right) u, u\right\rangle dxdy + v(\epsilon_0)\|u\|^2_{L^2(R_{15} \cap T_{\epsilon_0})} +$$

$$\int_{\varrho(\epsilon_0)}^{+\infty} \int_{|x| \leq \sqrt{3}y} \left|\frac{\partial}{\partial y} u\right|^2 + \left\langle \left[D^2W\left(u_0(x,y)\right) - D^2W\left(\theta_0(x)\right)\right] u, u\right\rangle dxdy.$$

This implies

$$\frac{1}{3} t_{\epsilon_0}[u] \geq \int_{\varrho(\epsilon_0)}^{+\infty} S_{\frac{1}{\sqrt{3}y}}[u(.,y)]dy - \sup_{\substack{y \geq \varrho(\epsilon_0) \\ |x| \leq \sqrt{3}y}} \left\|D^2W\left(u_0(x,y)\right) - D^2W\left(\theta_0(x)\right)\right\|_{tr} \|u\|^2_{L^2(R_{15})} +$$

$$v(\epsilon_0) \int_0^{\varrho(\epsilon_0)} \int_{|x| \leq \sqrt{3}y} |u|^2 dxdy.$$

Set $\epsilon := \frac{1}{\sqrt{3}y}$ for $y \geq \varrho(\epsilon_0)$. Then $\epsilon \leq \epsilon_0$. As $u(-x,y) = R_3 u(x,y)$ for $(x,y) \in R_{15}$, we have

$$S_{\frac{1}{\sqrt{3}y}}[u(.,y)] \geq \lambda_1^{\frac{1}{\sqrt{3}y},odd} \int_{|x| \leq \sqrt{3}y} |u(x,y)|^2 dx \geq v(\epsilon_0) \int_{|x| \leq \sqrt{3}y} |u(x,y)|^2 dx$$

in view of (3.5). Thus,

$$t_{\epsilon_0}[u] \geq \left[v(\epsilon_0) - \sup_{\substack{y \geq \varrho(\epsilon_0) \\ |x| \leq \sqrt{3}y}} \left\| D^2W\left(u_0(x,y)\right) - D^2W\left(\theta_0(x)\right) \right\|_{tr} \right] \|u\|^2_{L^2_G(\mathbb{R}^2)} \quad (3.8)$$

for each $u \in C^\infty_{0,G}(\mathbb{R}^2)$. As $C^\infty_{0,G}(\mathbb{R}^2)$ is a core of t_{ϵ_0}, inequality (3.8) holds on $H^1_G(\mathbb{R}^2)$. According to equation (3.6), we have

$$\min \sigma_e(\mathcal{L}_0) \geq \min \sigma(\mathcal{L}_0 + K(\epsilon_0)) =$$

$$\min_{\substack{u \in H^1_G(\mathbb{R}^2) \\ \|u\|=1}} t_{\epsilon_0}[u] \geq v(\epsilon_0) - \sup_{\substack{y \geq \varrho(\epsilon_0) \\ |x| \leq \sqrt{3}y}} \left\| D^2W\left(u_0(x,y)\right) - D^2W\left(\theta_0(x)\right) \right\|_{tr}.$$

The range of u_0 and θ_0 is contained in a compact set K, and D^2W is uniformly continuous on K. If we send $\epsilon_0 \to 0$, then we obtain with (H4) inequality (3.4). Our task is now to prove

$$\left[\min \sigma(D^2W(a)), +\infty\right) \subset \sigma_e(\mathcal{L}_0).$$

In order to do this, we diagonalize the operator $\mathcal{L}_0$ in cones along the directions x_i, $i = 1, 3, 5$. Define

$$C(x, \phi) := \left\{ y \in \mathbb{R}^2 : \arccos\left(\frac{\langle x, y\rangle}{|x||y|}\right) \in [0, \phi] \right\}$$

for $x \in \mathbb{R}^2$ and $\phi \in [0, \pi)$. Set

$$C(i) := C(x_i, \pi/12), \quad i = 1, 3, 5.$$

Suppose $\chi \in C^\infty(\mathbb{R}^2, \mathbb{R})$ is R_5-equivariant such that

$$\chi = 1, \text{ on } C(5) \cap K_2(0)^c,$$

and

$$\chi = 0, \text{ outside } C(x_5, \pi/6) \cap K_1(0)^c.$$

Define

$$V_0(x) := \left[D^2W(x_5) - D^2W\left(u_0(x)\right)\right] \chi(x), \quad x \in \mathbb{R}^2,$$

and

$$V(x) := \sum_{i=1,3,5} R_i V_0\left(R_i x\right) R_i, \quad x \in \mathbb{R}^2.$$

As W is G-invariant, u_0 G-equivariant, and χ R_5-invariant, it follows that V_0 is R_5-normal. On account of proposition 3.1, V is G-normal. Define

$$K := [V].$$

Due to (H4), we have

$$\forall \delta > 0 : \exists n \in \mathbb{N} : \sup_{|x| \geq n} |V(x)| = \sup_{\substack{|x| \geq n \\ C(x_5, \pi/6)}} |V(x)| < \delta.$$

Proposition B.2 implies that $K \in \mathcal{L}\left(H^2_G(\mathbb{R}^2), L^2_G(\mathbb{R}^2)\right)$ is compact. Therefore, we conclude from lemma B.1

$$\sigma_e(\mathcal{L}_0) = \sigma_e(\mathcal{L}_0 + K).$$

We have

$$D^2W(x_i) = D^2W(u_0) + V \tag{3.9}$$

in the set $C(i) \cap K_2(0)^c$ for each $i \in \{1, 3, 5\}$. Assume $T \in O(2)$ is the matrix that diagonalizes $D^2W(x_5)$, i.e.

$$T^* D^2W(x_5) T = \mathrm{diag}(\alpha, \beta), \quad \alpha \leq \beta. \tag{3.10}$$

It follows that $D^2W(x_i)$, $i = 1, 3$, is diagonalized to $\mathrm{diag}(\alpha, \beta)$ by $R_j T$, $j \in \{1, 3, 5, \} \backslash \{5, i\}$. The next goal is to construct a Weyl sequence for $\mathcal{L}_0$ and $\alpha + s$, $s \geq 0$, that is supported in $\cup_{i=1,3,5} C(i)$. Suppose $\phi \in C^\infty(\mathbb{R}, \mathbb{R})$ fulfills

$$1_{\mathbb{R}_-} \leq \phi \leq 1_{(-\infty, 1]}.$$

For $m \in \mathbb{N}$, set $r_m := \frac{1}{4} m \sin\left(\frac{1}{12}\pi\right)$, and $l_m := m^2$. Define

$$\phi_m(x) := \phi\left(\left|x - \frac{x_5}{|x_5|} l_m\right| - r_m\right), \quad x \in \mathbb{R}^2, \quad m >> 1,$$

and

$$f_m(x, y) := \frac{1}{r_m} \phi_m(x, y) e^{i\sqrt{\frac{s}{2}}(x+y)}, \quad (x, y) \in \mathbb{R}^2.$$

Calculation yields

$$\nabla \phi_m(x) = \phi'\left(\left|x - \frac{x_5}{|x_5|} l_m\right| - r_m\right) \frac{x - \frac{x_5}{|x_5|} l_m}{\left|x - \frac{x_5}{|x_5|} l_m\right|}, \tag{3.11}$$

and

$$\triangle \phi_m(x) = \phi''\left(\left|x - \frac{x_5}{|x_5|} l_m\right| - r_m\right) + \frac{\phi'\left(\left|x - \frac{x_5}{|x_5|} l_m\right| - r_m\right)}{\left|x - \frac{x_5}{|x_5|} l_m\right|}. \tag{3.12}$$

The functions $\nabla \phi_m$ and $\triangle \phi_m$ are supported in

$$\left\{x \in \mathbb{R}^2 : r_m \leq \left|x - \frac{x_5}{|x_5|} l_m\right| \leq r_m + 1\right\}.$$

We have

$$\operatorname{supp}(f_m) \subset C(5) \cap K_m(0)^c, \quad m >> 1. \tag{3.13}$$

For $(x,y) \in \mathbb{R}^2$, we obtain

$$\triangle f_m(x,y) =$$

$$-sf_m(x,y) + 2i\frac{\sqrt{\frac{s}{2}}}{r_m}\langle \nabla\phi(x,y), e_1 + e_2\rangle e^{i\sqrt{\frac{s}{2}}(x+y)} + \frac{1}{r_m}\triangle\phi_m(x,y)e^{i\sqrt{\frac{s}{2}}(x+y)}.$$

This implies

$$\|-\triangle f_m - sf_m\|^2_{L^2(\mathbb{R}^2)} \leq \frac{4s}{r_m^2}\|\langle\nabla\phi_m, e_1 + e_2\rangle\|^2_{L^2(\mathbb{R}^2)} + \frac{2}{r_m^2}\|\triangle\phi_m\|^2_{L^2(\mathbb{R}^2)}.$$

In view of (3.11) and (3.12), there exists a constant $C > 0$ such that

$$\|-\triangle f_m - sf_m\|^2_{L^2(\mathbb{R}^2)} \leq C\frac{1}{r_m} \to 0, \quad m \to \infty.$$

Set

$$F_m := 3P_0(Tf_m e_1)$$

for $m \in \mathbb{N}$. On account of corollary 3.2, (3.9), (3.10), and (3.13), we have

$$\|(\mathcal{L}_0 + K - \alpha - s)F_m\|^2_{L^2_G(\mathbb{R}^2)} = 3\|(\mathcal{L}_0 + K - \alpha - s)Tf_m e_1\|^2_{L^2(C(5))} =$$

$$3\left\|-\triangle Tf_m e_1 + T\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}f_m e_1 - (\alpha+s)Tf_m e_1\right\|^2_{L^2(C(5))} =$$

$$\|-\triangle f_m - sf_m\|^2_{L^2(\mathbb{R}^2))} \to 0, \quad m \to \infty,$$

for $s \geq 0$. As F_m is uniformly bounded and has support outside $K_m(0)$, theorem A.1 implies that

$$F_m \xrightarrow{w} 0.$$

Moreover, the sequence F_m does not converge to zero in L^2, because

$$3\pi \leq \|F_m\|^2_{L^2(\mathbb{R}^2)} = 3\|f_m\|^2_{L^2(C(5))} \leq 6\pi.$$

Without loss of generality, we can assume that the sequence is normalized. Hence, F_m is a Weyl sequence for $\alpha + s$, $s \geq 0$. It follows that

$$\mathbb{R}_+ + \alpha = \left[\min\sigma(D^2W(a)), +\infty\right) \subset \sigma_e(L).$$

In view of proposition 2.1, we have

$$\sigma_e(L_0^{odd}) \subset \sigma_e(\mathcal{L}_0).$$

Let us now prove

$$\sigma_d(L_0^{odd}) \subset \sigma_e(\mathcal{L}_0). \tag{3.14}$$

Suppose $\lambda \in \sigma_d(L_0^{odd})$ and $\psi \in \ker(L_0^{odd} - \lambda)$ is normalized. Choose cut-off functions $\chi, \chi_m \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$1_{K_{1/2}(0)} \leq \chi \leq 1_{K_1(0)},$$

and

$$1_{K_m(0)} \geq \chi_m \geq 1_{K_{m-1}(0)},$$

where χ_m is even and

$$\chi_m(t + (m-1)) = \chi(t), \quad t \in [0,1].$$

Define

$$g_m(x,y) := \frac{1}{\sqrt{m}} \chi_m \left(y - m^2\right) \psi(x) \chi_m(x)$$

for $y \in \mathbb{R}_+$, $|x| \leq \sqrt{3}y$, and $m \in \mathbb{N}$. Set $g_m(x,y) := 0$ otherwise. Calculation yields

$$2\frac{m-1}{m} \int_{-(m-1)}^{m-1} |\psi(x)|^2 dx \leq \int_{R_{15}} |g_m(x,y)|^2 dx dy \leq 2 \int_m^m |\psi(x)|^2 dx$$

for $m >> 1$. Theorem A.1 implies

$$\int_{-(m-1)}^{m-1} |\psi(x)|^2 dx \to 1, \quad m \to \infty.$$

This implies that we can normalize the functions

$$G_m := P_0 g_m.$$

As g_m is R_3-equivariant, we have

$$G_m(x) = \frac{1}{3} g_m(x)$$

for $x \in R_{15}$. Choose $y \in \mathbb{R}_+$ and $|x| \leq \sqrt{3}y$. We have

$$\Big| \left([\mathcal{L}_0 - \lambda] g_m\right)(x,y) - \left([L_0 - \lambda] g_m(.,y)\right)(x) \Big| \leq$$

$$\left| \frac{\partial^2}{\partial y^2} g_m(x,y) \right| + \sup_{\substack{y \geq m \\ |x| \leq \sqrt{3}y}} \left\| D^2 W\left(u_0(x,y)\right) - D^2 W\left(\theta_0(x)\right) \right\|_{tr} |g_m(x,y)|.$$

Calculation yields

$$\frac{\partial^2}{\partial y^2} g_m(x,y) = \frac{1}{\sqrt{m}} \chi_m''(y-m^2)\psi(x)\chi_m(x),$$

therefore

$$\left\| \frac{\partial^2}{\partial y^2} g_m \right\|^2_{L^2(\mathbb{R}^2)} \leq \frac{C}{m} \int_{-m}^{m} |\psi(x)|^2 dx \to 0, \quad m \to \infty.$$

Moreover,

$$\left([L_0 - \lambda] g_m(.,y)\right)(x) = -\frac{1}{\sqrt{m}} \chi_m(y-m^2) \left[2\psi'(x)\chi_m'(x) + \psi(x)\chi_m''(x)\right].$$

In view of corollary 1.2, there exists $C > 0$ such that we have

$$|\psi(x)| + |\psi'(x)| \leq C e^{-\frac{\sqrt{\min \sigma_e(L_0) - \lambda}}{4}|x|}$$

for all $x \in \mathbb{R}$. It follows that

$$\|(L_0 - \lambda) g_m(.,y)\|^2_{L^2(\mathbb{R}^2)} \leq C \int_{m \geq |x| \geq m-1} |\psi(x)|^2 + |\psi'(x)|^2 dx =$$

$$O\left(e^{-\frac{\sqrt{\min \sigma_e(L_0) - \lambda}}{4} m}\right), \quad m \to \infty.$$

Owing to (H4), we obtain

$$\sup_{\substack{y \geq m \\ |x| \leq \sqrt{3} y}} \left\| D^2 W\left(u_0(x,y)\right) - D^2 W\left(\theta_0(x)\right) \right\|_{tr} \|g_m\|_{L^2_G(\mathbb{R}^2)} \to 0, \quad m \to \infty.$$

It follows that

$$\|(\mathcal{L}_0 - \lambda) G_m\|^2_{L^2(\mathbb{R}^2)} = \frac{1}{3} \|(\mathcal{L}_0 - \lambda) g_m\|^2_{L^2(\mathbb{R}^2)} \to 0,$$

which implies that G_m is a Weyl sequence for L and λ, i.e.

$$\lambda \in \sigma_e(\mathcal{L}_0).$$

We have proved

$$\sigma(L_0^{odd}) \subset \sigma_e(\mathcal{L}_0).$$

It remains to show that

$$\min \sigma_e(\mathcal{L}_0) = \liminf_{\epsilon \to 0} \lambda_1^{\epsilon, odd}.$$

If there is no eigenvalue of L_0^{odd} bellow $\min \sigma_e(L_0)$, then lemma 2.1 implies

$$\liminf_{\epsilon \to 0} \lambda_1^{\epsilon, odd} \geq \min \sigma(D^2 W(a)) \geq \min \sigma_e(\mathcal{L}_0).$$

If there is an eigenvalue $\lambda_1^{0,odd}$ of L_0^{odd} bellow the essential spectrum $\sigma_e(L_0)$, then (3.14) implies

$$\lambda_1^{0,odd} \geq \min \sigma_e(\mathcal{L}_0).$$

On the other hand, we have

$$\liminf_{\epsilon \to 0} \lambda_1^{\epsilon,odd} = \lambda_1^{0,odd}$$

in view of lemma 2.1. Together with (3.4), this completes the proof of statement 1 of theorem 3.1. □

In order to prove the statement on the exponential decay for the eigenfunctions of $\mathcal{L}_0$, we have to strike out our considerations, which is done in the following.

3.3.2 Exponential decay of eigenfunctions

One could presume that an extension of the proofs in section one leads to the result. Unfortunately, this does not work, because the potential $D^2W(u_0)$ is not uniformly definite outside a ball $B_r(0)$. What happens is a kind of tunneling effect trough a positive definite operator-valued barrier, given by L_0.

Definition 3.10. *Let $\delta \in (0,1)$, $\alpha \geq 0$, and $E < \min \sigma_e(\mathcal{L}_0)$. Define*

$$f_\alpha(x) := \frac{(1-\delta)\sqrt{\min \sigma_e(\mathcal{L}_0) - E}|x|}{1 + \alpha(1-\delta)\sqrt{\min \sigma_e(\mathcal{L}_0) - E}|x|}.$$

For $\epsilon \in [0,1)$, we set

$$D_{Q_{\epsilon,\alpha}} := H_G^1(\Omega_\epsilon),$$

and

$$Q_{\epsilon,\alpha}[u,v] := \sum_{j=1,2} \int_{\Omega_\epsilon} \langle -i\nabla u_j(x) + i\nabla f_\alpha(x) u_j(x), -i\nabla u_j(x) - i\nabla f_\alpha(x) u_j(x) \rangle \, dx +$$

$$\left\langle \left(D^2W\left(u_0\right) - E\right) u, u \right\rangle_{L^2(\Omega_\epsilon)}.$$

Moreover, define

$$D_{q_{\epsilon,\alpha}} := H^1\left(\Omega_\epsilon \cap \overset{\circ}{R}_{15}, \mathbb{C}\right),$$

and

$$q_{\epsilon,\alpha}[u,v] := \sum_{j=1,2} \int_{\Omega_\epsilon \cap \overset{\circ}{R}_{15}} \langle -i\nabla u_j(x) + i\nabla f_\alpha(x) u_j(x), -i\nabla u_j(x) - i\nabla f_\alpha(x) u_j(x) \rangle \, dx +$$

$$\left\langle \left(D^2W\left(u_0\right) - E\right) u, u \right\rangle_{L^2(\Omega_\epsilon \cap R_{15})}.$$

Remark 3.5. *As f_α, $\alpha \geq 0$, is Lipschitz-continuous with Lipschitz constant $(1-\delta)\sqrt{\min \sigma_e(\mathcal{L}_0) - E}$, we have*

$$f_\alpha \in H^{1,\infty}_{loc}(\mathbb{R}^2, \mathbb{R}).$$

If in addition $\alpha > 0$, then

$$|f_\alpha(x)| \leq \frac{1}{\alpha}, \quad x \in \mathbb{R}^2,$$

therefore $f_\alpha \in H^{1,\infty}(\mathbb{R}^2, \mathbb{R})$, $\alpha > 0$. Note that

$$H^{k,\infty}(\Omega, \mathbb{C}) \cdot H^k(\Omega, \mathbb{C}) \subset H^k(\Omega, \mathbb{C})$$

for $\emptyset \neq \Omega \subset \mathbb{R}^n$ open and $k \in \mathbb{N}_0$.

Energy estimates for $Q_{\epsilon,\alpha}$ can be reduced to those for $q_{\epsilon,\alpha}$. This is specified in the following proposition.

Proposition 3.2. *For each $\delta \in (0,1)$, $\alpha \geq 0$, and $\epsilon \in [0,1)$, we have*

$$\forall u \in D_{Q_{\epsilon,\alpha}} : Q_{\epsilon,\alpha}[u] = 3q_{\epsilon,\alpha}[u].$$

Proof. Choose $\phi \in D_{Q_{\epsilon,\alpha}}$ arbitrarily. Then we have

$$Q_{\epsilon,\alpha}[\phi] = \sum_{j=1,2} \|\nabla \phi_j\|^2_{L^2(\Omega_\epsilon)} - \sum_{j=1,2} \|\nabla f_\alpha \phi_j\|^2_{L^2(\Omega_\epsilon)} +$$

$$\sum_{j=1,2} 2i\mathrm{Im}\left(\langle \nabla \phi_j, \nabla f_\alpha \phi_j\rangle_{L^2(\Omega_\epsilon)}\right) + \left\langle \left(D^2 W\left(u_0\right) - E\right)\phi, \phi\right\rangle_{L^2(\Omega_\epsilon)}.$$

Let $U = \overset{\circ}{R}_{13} \cap \Omega_\epsilon$ or $U = \overset{\circ}{R}_{35} \cap \Omega_\epsilon$. Then there exists $\gamma \in G$ such that $U = \gamma(\overset{\circ}{R}_{15} \cap \Omega_\epsilon)$. As f_α is G-invariant and ϕ G-equivariant, we have

$$\nabla f_\alpha(\gamma x)\phi_1(\gamma x) = \gamma \nabla f_\alpha(x)\left(\gamma_{11}\phi_1(x) + \gamma_{12}\phi_2(x)\right), \tag{3.15}$$

and

$$\nabla f_\alpha(\gamma x)\phi_2(\gamma x) = \gamma \nabla f_\alpha(x)\left(\gamma_{21}\phi_1(x) + \gamma_{22}\phi_2(x)\right). \tag{3.16}$$

This implies

$$\|\nabla f_\alpha \phi\|^2_{L^2(U)} = \int_{\gamma^T U} \left(\gamma_{11}^2 + \gamma_{12}^2\right)|\nabla f_\alpha \phi_1|^2 + 2\left(\gamma_{11}\gamma_{12} + \gamma_{21}\gamma_{22}\right)\langle \nabla f_\alpha \phi_1, \nabla f_\alpha \phi_2\rangle +$$

$$\left(\gamma_{21}^2 + \gamma_{22}^2\right)|\nabla f_\alpha \phi_2|^2 dx = \|\nabla f_\alpha \phi\|^2_{L^2(\gamma^T U)},$$

and

$$\sum_{j=1,2} \langle \nabla\phi_j, \nabla f_\alpha \phi_j \rangle_{L^2(U)} = \int_{\gamma^T U} (\gamma_{11}^2 + \gamma_{12}^2) \langle \nabla\phi_1(x), \nabla f_\alpha(x)\phi_1(x)\rangle \, dx +$$

$$\int_{\gamma^T U} (\gamma_{21}^2 + \gamma_{22}^2) \langle \nabla\phi_2(x), \nabla f_\alpha(x)\phi_2(x)\rangle \, dx +$$

$$\int_{\gamma^T U} (\gamma_{11}\gamma_{12} + \gamma_{21}\gamma_{22}) \langle \nabla\phi_1(x), \nabla f_\alpha(x)\phi_2(x)\rangle \, dx +$$

$$\int_{\gamma^T U} (\gamma_{11}\gamma_{12} + \gamma_{21}\gamma_{22}) \langle \nabla\phi_2(x), \nabla f_\alpha(x)\phi_1(x)\rangle \, dx =$$

$$\sum_{j=1,2} \langle \nabla\phi_j, \nabla f_\alpha \phi_j \rangle_{L^2(\gamma^T U)} .$$

According to proposition 3.1 and corollary 3.2, we have

$$\left\langle \left(D^2 W(u_0) - E\right)\phi, \phi \right\rangle_{L^2(U)} = \left\langle \left(D^2 W(u_0) - E\right)\phi, \phi \right\rangle_{L^2(\gamma^T U)} .$$

In view of

$$q_{\epsilon,\alpha}[\phi] = \sum_{j=1,2} \|\nabla\phi_j\|^2_{L^2(\overset{\circ}{R}_{15}\cap\Omega_\epsilon)} - \sum_{j=1,2} \|\nabla f_\alpha \phi_j\|^2_{L^2(\overset{\circ}{R}_{15}\cap\Omega_\epsilon)} +$$

$$\sum_{j=1,2} 2i\mathrm{Im}\left(\langle \nabla\phi_j, \nabla f_\alpha \phi_j \rangle_{L^2(\overset{\circ}{R}_{15}\cap\Omega_\epsilon)} \right) + \left\langle \left(D^2 W(u_0) - E\right)\phi, \phi \right\rangle_{L^2(\overset{\circ}{R}_{15}\cap\Omega_\epsilon)} ,$$

the proof is complete. □

We intend to obtain an energy estimate for $Q_{\epsilon,\alpha}$ similar to proposition 1.2. It is at this point where the operator valued barrier L_0^{odd} plays the decisive role.

Lemma 3.1. *Suppose* (H1)-(H4) *hold and* $\dim\ker(L_0) = 1$. *Then there exist* $\epsilon_0 \in (0,1)$ *such that for all* $\delta \in [\frac{1}{2}, 1)$ *and* $E < \min \sigma_e(\mathcal{L}_0)$, *we have*

$$\forall \alpha \geq 0 : \forall \epsilon \in [0, \epsilon_0) : \forall \phi \in D_{Q_{\epsilon,\alpha}} :$$

$$\mathrm{supp}(\phi) \subset (T_{\epsilon_0})^c \Rightarrow \mathrm{Re}(Q_{\epsilon,\alpha}[\phi]) \geq (2\delta - 1)\frac{\min \sigma_e(\mathcal{L}_0) - E}{2} \|\phi\|^2_{L^2(\Omega_\epsilon)}.$$

Proof. Due to the C^4-regularization of T_ϵ (cf. figure 3.2), there exists $r > 0$ and a function $f \in C^4([0,r],\mathbb{R})$, $f(0) = 0$, such that the sets

$$\Omega_1^\epsilon := \left\{ (x,y) \in \mathbb{R}^2 : 0 \leq y < \varrho(\epsilon) - r, |x| \leq \sqrt{3}y \right\},$$

and

$$\Omega_2^\epsilon := \left\{ (x,y) \in \mathbb{R}^2 : y \in [\varrho(\epsilon) - r, \varrho(\epsilon)], |x| \leq f(y - (\varrho(\epsilon) - r)) + \sqrt{3}(\varrho(\epsilon) - r) \right\}$$

fulfill

$$\overline{\Omega_\epsilon} \cap R_{15} = \Omega_1^\epsilon \quad \dot{\cup} \quad \Omega_2^\epsilon.$$

Define

$$l_1(y) := \sqrt{3}y, \quad 0 \leq y < \varrho(\epsilon) - r,$$

and

$$l_2(y) := f(y - (\varrho(\epsilon) - r)) + \sqrt{3}(\varrho(\epsilon) - r), \quad y \in [\varrho(\epsilon) - r, \varrho(\epsilon)].$$

Owing to theorem 3.1, there exists $\epsilon_1 > 0$ such that

$$\inf_{\mu \leq \epsilon_1} \lambda_1^{\mu, odd} - E \geq \frac{3}{4} \left(\min \sigma_e(\mathcal{L}_0) - E\right) \tag{3.17}$$

for each $\epsilon \leq \epsilon_2$. In view of (H4), there exists $\epsilon_2 > 0$ such that

$$\sup_{\substack{y \geq \varrho(\epsilon) \\ |x| \leq \sqrt{3}y}} \left\| D^2 W\left(u_0(x,y)\right) - D^2 W\left(\theta_0(x)\right) \right\|_{tr} \leq \frac{1}{4} \left(\min \sigma_e(\mathcal{L}_0) - E\right). \tag{3.18}$$

Set

$$\delta_0 := \frac{3}{4} \text{ and } \epsilon_0 := \frac{\min_{i=1,2} \epsilon_i}{1 + \min_{i=1,2} \epsilon_i}. \tag{3.19}$$

Remark that the regularization of T_ϵ reduces the maximum length $\frac{1}{\epsilon}$ to $\frac{1}{\epsilon}$-1 (cf. figure 3.2). Choose $\delta \in [\delta_0, 1)$ and $\epsilon \in [0, \epsilon_0)$. Set $\Omega := \overset{\circ}{R}_{15} \cap \Omega_\epsilon$. Suppose $\phi \in C_{0,G}^\infty(\mathbb{R}^2)$ has support outside the triangle T_{ϵ_0}. Note that

$$|\nabla f_\alpha|^2 \leq |\nabla f_0|^2 \leq (1 - \delta)(\min \sigma_e(\mathcal{L}_0) - E), \quad a.e.$$

Calculation yields

$$\mathrm{Re}(q_{\epsilon,\alpha}[\phi]) = \sum_{j=1,2} \|\nabla \phi_j\|_{L^2(\Omega)}^2 - \sum_{j=1,2} \|\nabla f_\alpha \phi_j\|_{L^2(\Omega)}^2 + \left\langle \left(D^2 W\left(u_0\right) - E\right) \phi, \phi \right\rangle_{L^2(\Omega)} \geq$$

$$\int_{\substack{(x,y) \in \Omega \\ y \geq \varrho(\epsilon_0)}} \left| \frac{\partial}{\partial x} \phi(x,y) \right|^2 + \left\langle (D^2 W(\theta_0(x)) - E) \phi(x,y), \phi(x,y) \right\rangle dxdy - \tag{3.20}$$

$$\left[(1 - \delta)(\min \sigma_e(\mathcal{L}_0) - E) + \sup_{\substack{y \geq \varrho(\epsilon_0) \\ |x| \leq \sqrt{3}y}} \left\| D^2 W\left(u_0(x,y)\right) - D^2 W\left(\theta_0(x)\right) \right\|_{tr} \right] \|\phi\|_{L^2(\Omega)}^2.$$

The term in (3.20) equals

$$\int_{\varrho(\epsilon_0)}^{\varrho(\epsilon) - r} S_{\frac{1}{l_1(y)}} \left[\phi(.,y)\right] dy + \int_{\varrho(\epsilon) - r}^{\varrho(\epsilon)} S_{\frac{1}{l_2(y)}} \left[\phi(.,y)\right] dy - E \|\phi\|_{L^2(\Omega)}^2.$$

We have $\phi(.,y) \in D_{s_l}$ for each $l \geq 0$. As $\phi(.,y)$ is R_3-odd, we have

$$\int_{\varrho(\epsilon_0)}^{\varrho(\epsilon)-r} S_{\frac{1}{l_1(y)}}\left[\phi(.,y)\right] dy \geq \int_{\varrho(\epsilon_0)}^{\varrho(\epsilon)-r} \lambda_1^{\frac{1}{l_1(y)},odd} \|\phi(.,y)\|^2_{L^2(I_{l_1(y)-1})} dy \geq$$

$$\inf_{\mu \leq \epsilon_1} \lambda_1^{\mu,odd} \|\phi\|^2_{L^2(\Omega_1)}.$$

Similarly, we conclude

$$\int_{\varrho(\epsilon)-r}^{\varrho(\epsilon)} S_{\frac{1}{l_2(y)}}\left[\phi(.,y)\right] dy \geq \inf_{\mu \leq \epsilon_1} \lambda_1^{\mu,odd} \|\phi\|^2_{L^2(\Omega_2)}.$$

It follows that

$$\mathrm{Re}(q_{\epsilon,\alpha}[\phi]) \geq (2\delta - 1)\frac{\min \sigma_e(\mathcal{L}_0) - E}{2} \|\phi\|^2_{L^2(\Omega)} \tag{3.21}$$

for each $\phi \in C^\infty_{0,G}(\mathbb{R}^2)$ such that $\mathrm{supp}(\phi) \subset (T_{\epsilon_0})^c$. In view of proposition 3.2, we obtain

$$\mathrm{Re}(Q_{\epsilon,\alpha}[\phi]) \geq (2\delta - 1)\frac{\min \sigma_e(\mathcal{L}_0) - E}{2} \|\phi\|^2_{L^2(\Omega_\epsilon)} \tag{3.22}$$

for each $\phi \in C^\infty_{0,G}(\mathbb{R}^2)$ such that $\mathrm{supp}(\phi) \subset (T_{\epsilon_0})^c$. In view of [Alt, A 6.7 Lemma], we know that $\{u|_\Omega : u \in C^\infty_0(\mathbb{R}^2, \mathbb{C})\}$ is dense in $H^1(\Omega, \mathbb{C})$. It follows that $\left\{u|_\Omega : u \in C^\infty_{0,G}(\mathbb{R}^2)\right\}$ is a core of $Q_{\epsilon,\alpha}$. This implies that equation (3.22) holds for each $\phi \in D_{Q_{\epsilon,\alpha}}$ that has support in $(T_{\epsilon_0})^c$. $\square$

Proposition 3.3. *Let $\epsilon \in [0,1)$, $\alpha > 0$, and $\delta \in (0,1)$. Suppose $\eta \in C^\infty(\Omega_\epsilon, \mathbb{R})$ is G-invariant such that $\nabla\eta \in (C^\infty_0(\Omega_\epsilon, \mathbb{R}))^2$. Then*

$$\forall E \in \sigma_p(\mathcal{L}_\epsilon) : \forall \psi \in \ker(\mathcal{L}_\epsilon - E) : Q_{\epsilon,\alpha}\left[\eta e^{f_\alpha}\psi\right] = \left\langle \zeta_\alpha e^{f_\alpha}\psi, \psi \right\rangle_{L^2(\Omega_\epsilon)},$$

where

$$\zeta_\alpha := |\nabla\eta|^2 + 2\eta \left\langle \nabla\eta, \nabla f_\alpha \right\rangle.$$

Proof. Choose $E \in \sigma_p(\mathcal{L}_\epsilon)$ and $\psi \in \ker(\mathcal{L}_\epsilon - E)$. For each $u \in H^1(\Omega_\epsilon, \mathbb{C})$ and $\alpha > 0$, we have

$$e^{f_\alpha}\nabla\left(e^{-f_\alpha}u\right) = i\left(-i\nabla u + i\nabla f_\alpha u\right),$$

and

$$e^{-f_\alpha}\nabla\left(e^{f_\alpha}u\right) = i\left(-i\nabla u - i\nabla f_\alpha u\right).$$

It follows that

$$Q_{\epsilon,\alpha}\left[\eta e^{f_\alpha}\psi\right] = \sum_{j=1,2} \left\langle \nabla\left(\eta\psi_j\right), \nabla\left(\eta e^{2f_\alpha}\psi_j\right)\right\rangle_{L^2(\Omega_\epsilon)} + \left\langle \left(D^2W\left(u_0\right) - E\right)\eta\psi, \eta e^{2f_\alpha}\psi\right\rangle_{L^2(\Omega_\epsilon)}.$$

As the gradient of η has compact support, $\eta\psi_j$ fulfills Neumann boundary conditions. As $\psi \in \ker(\mathcal{L}_\epsilon - E)$, integration by parts yields

$$Q_{\epsilon,\alpha}\left[\eta e^{f_\alpha}\psi\right] = -2\sum_{j=1,2}\left\langle \nabla\eta\nabla\psi_j, \eta e^{2f_\alpha}\psi_j\right\rangle_{L^2(\Omega_\epsilon)} + \sum_{j=1,2}\left\langle \nabla\eta, \nabla\left(\eta e^{2f_\alpha}\psi_j^2\right)\right\rangle_{L^2(\Omega_\epsilon)}.$$

Differentation of the term $\eta e^{2f_\alpha}\psi_j^2$ yields the assertion. □

We want to prove that eigenfunctions ψ of $\mathcal{L}_\epsilon$ are uniformly bounded in ϵ. In order to obtain this result, we show that the L^2-norm of $\nabla(\chi\psi)$ is estimable from above by $\|\chi\|_{C^2}$. Let us first construct cut-off functions (χ_ϵ^i) such that their C^2-norm is uniformly bounded in ϵ and their support covers Ω_ϵ. In order to simplify the proofs, it is useful to consider the following operators.

Definition 3.11. *Let $\epsilon \in [0,1)$. Define the operator*

$$\mathcal{K}_\epsilon := -\triangle + \left[D^2W(u_0)\right]$$

in $\left(L^2(\Omega_\epsilon,\mathbb{C})\right)^2$ with Neumann boundary conditions.

As $\mathcal{L}_\epsilon$ is the restriction of $\mathcal{K}_\epsilon$ to $L^2_G(\Omega_\epsilon)$, we have

$$\sigma_p(\mathcal{L}_\epsilon) \subset \sigma_p(\mathcal{K}_\epsilon),$$

and

$$\ker(\mathcal{L}_\epsilon - \lambda) \subset \ker(\mathcal{K}_\epsilon - \lambda)$$

for each $\lambda \in \sigma_p(\mathcal{L}_\epsilon)$.

Proposition 3.4. *There exists $\epsilon_0 > 0$, a family of subsets $(J_\epsilon)_{\epsilon\in[0,\epsilon_0)} \subset \mathbb{P}(\mathbb{N}_0)$, subsets $(K_i^\epsilon)_{i\in J_\epsilon} \subset \mathbb{P}(\mathbb{R}^2)$, and functions $(\varphi_{j,i}^\epsilon)_{\substack{i\in J_\epsilon\\ j=1,2}} \subset C_b^2(\mathbb{R}^2,\mathbb{R})$ such that the following properties are fulfilled:*

1. *We have $\partial K_i^\epsilon \in C^2$ for $\epsilon \in [0,\epsilon_0)$ and $i \in J_\epsilon$. There exist open, bounded sets $M_1, ..., M_n \subset \mathbb{R}^2$ such that $\partial M_j \in C^2$, $j = 1, ..., n$. For each $\epsilon \in [0,\epsilon_0)$ and $i \in J_\epsilon$, there exist $E \in O(2)$, $x \in \mathbb{R}^2$, and $l \in \mathbb{N}, l \le n$, such that*

$$K_i^\epsilon = EM_l + x \subset \Omega_\epsilon.$$

2. *We have*

$$\sup_{\epsilon\in[0,\epsilon_0)}\sup_{i\in J_\epsilon}\left\|\varphi_{j,i}^\epsilon\right\|_{C^2(\mathbb{R}^2)} < +\infty, \quad j = 1,2.$$

3. *For each $\epsilon \in (0,\epsilon_0)$ and $i \in J_\epsilon$, we have*

$$\frac{\partial}{\partial\nu}\varphi_{j,i}^\epsilon = 0, \quad \text{on } \partial\Omega_\epsilon \cup \partial K_i^\epsilon.$$

Moreover,

$$\partial K_i^\epsilon \cap \Omega_\epsilon \subset \left\{\varphi_{j,i}^\epsilon = 0\right\},$$

$$\Omega_\epsilon \cap \mathit{supp}\left(\varphi_{j,i}^\epsilon\right) \subset K_i^\epsilon,$$

$$\mathit{supp}\left(\varphi_{2,i}^\epsilon\right) \text{ is compact },$$

and

$$\mathit{supp}(\varphi_{1,i}^\epsilon) \subset \{\varphi_{2,i}^\epsilon = 1\}$$

for each $\epsilon \in [0, \epsilon_0)$ and $i \in J_\epsilon$.

4. *For each $\epsilon \in [0, \epsilon_0)$,*

$$\overline{\Omega_\epsilon} \subset \bigcup_{i \in J_\epsilon} \left\{\varphi_{j,i}^\epsilon = 1\right\}, \quad j = 1, 2.$$

Remark 3.6. *Before we start with the proof, note that the domain of $\mathcal{K}_\epsilon$ is a invariant subspace of the multiplication operator $\left[\varphi_{j,i}^\epsilon\right]$ for each $\epsilon \in [0, \epsilon_0)$, $i \in J_\epsilon$ and $j = 1, 2$. Clearly, we have $\varphi_{j,i}^\epsilon \in C_b^2\left(\overline{\Omega_\epsilon}, \mathbb{R}\right)$. It follows that*

$$\varphi_{j,i}^\epsilon \cdot u \in \left(H^2\left(\Omega_\epsilon, \mathbb{C}\right)\right)^2, \quad u \in D_{\mathcal{K}_\epsilon}.$$

It remains to show that the boundary condition is preserved. In view of statement three of proposition 3.4, we obtain

$$\frac{\partial}{\partial \nu}\left(\varphi_{j,i}^\epsilon \cdot u\right) = \frac{\partial}{\partial \nu}\varphi_{j,i}^\epsilon \cdot u + \varphi_{j,i}^\epsilon \cdot \frac{\partial}{\partial \nu} u = 0$$

on $\partial\Omega_\epsilon$. This implies

$$\frac{\partial}{\partial \nu}\left(\varphi_{j,i}^\epsilon \cdot u\right) = 0$$

on ∂K_i^ϵ. Moreover,

$$\sup_{\epsilon \in (0,\epsilon_0)} \sup_{i \in J_\epsilon} \operatorname{diam}(K_i^\epsilon) < \infty.$$

Proof. First, we treat the case $\epsilon > 0$. The set $\overline{\Omega_\epsilon}$ is compact for each $\epsilon \in (0, 1)$. Moreover, we have $\partial\Omega_\epsilon \in C^4$. In view of [F, §15, Satz 2] there exists a continuous normal $\nu_\epsilon : \partial\Omega_\epsilon \to \mathbb{R}^2$. Consider the mapping

$$\Phi_\epsilon : (-\kappa(\epsilon), \kappa(\epsilon)) \times \partial\Omega_\epsilon \to \partial\Omega_\epsilon(\kappa(\epsilon)) := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \partial\Omega_\epsilon) < \kappa(\epsilon)\}.$$

It is well known -cf. [GT]- that there exists a $\kappa(\epsilon) > 0$ such that Φ_ϵ is a diffeomorphism. As the parametrization of $\partial\Omega_\epsilon$ is uniform up to translation, there exists $\kappa \in (0, \frac{1}{2})$ such that we have $\kappa(\epsilon) = \kappa$ for each $\epsilon \in (0, 1)$. Throughout this prove, denote the inverse of Φ_ϵ by (d_ϵ, S_ϵ). For each $\epsilon \in (0,1)$ and $x \in \mathbb{R} \times \mathbb{R}_-$, define the arc-length

$$l(x) := \operatorname{sign}(x_1) \int_0^{\min(|x_1|,\frac{1}{2})} \sqrt{1+(F'(t))^2} dt + 2 \operatorname{sign}(x_1) \left(\max\left(|x_1|, \frac{1}{2} \right) - \frac{1}{2} \right),$$

$$l^\epsilon_{max} := l\left(\left(\frac{1}{2\epsilon}, \frac{\sqrt{3}}{2\epsilon} \right) \right),$$

and

$$N(\epsilon) := [l^\epsilon_{max}] + 1.$$

The mapping $l : \mathbb{R} \times \mathbb{R}_- \to \mathbb{R}$ is fourth times differentiable. Let us define the sets M_j, $j = 1, 2$. Similar to the regularization of T_ϵ, we construct a convex, open, and bounded set $M_{0,1}$ such that $\partial M_{0,1} \in C^4$, and

$$[-2, 2] \times [-\kappa, 0] \subset M_{0,1} \subset [-3, 3] \times [-\kappa, 0].$$

Define

$$M_1 := R_1 M_{0,1},$$

and

$$M_2 := \Omega_{\frac{1}{10}}.$$

Our task is now to define the cut-off functions. Choose $\chi_{j,0}, \chi_{j,1} \in C_0^\infty(\mathbb{R}, \mathbb{R})$, $j = 1, 2$, such that

$$1_{B_{1+\frac{j-1}{2}}(0)} \leq \chi_{j,0} \leq 1_{B_{\frac{3}{2}+\frac{j-1}{2}}(0)},$$

and

$$1_{B_{\frac{j\kappa}{4}}(0)} \leq \chi_{j,1} \leq 1_{B_{\frac{j\kappa}{2}}(0)}.$$

Choose $\epsilon_0 > 0$ such that it is smaller than $\frac{1}{100}$. Let $j = 1, 2$.

1. If $i = 0$, then define

$$\chi^\epsilon_{j,i}(x) := \chi_{j,0}\left(\frac{l(S_\epsilon(x))}{5} \right) \chi_{j,1}(d_\epsilon(x))$$

   for each $x \in \partial\Omega_\epsilon(\kappa) \cap (\mathbb{R} \times \mathbb{R}_-)$ and zero otherwise. Set

$$x^\epsilon_i := -\left(r(\epsilon) - r(\epsilon_0) \right) e_2.$$

   Moreover, we define

$$K^\epsilon_i := M_2 + x^\epsilon_i.$$

2. If $i = 1, 2, ..., N(\epsilon) - 1$, set

$$l_i^\epsilon := \frac{l_{max}^\epsilon - 10}{[l_{max}^\epsilon] - 1}(i-1) + 6.$$

Define

$$\chi_{j,i}^\epsilon(x) := \chi_{j,0}(l(S_\epsilon(x)) - l_i^\epsilon)\chi_{j,1}(d_\epsilon(x)),$$

for each $x \in \partial\Omega_\epsilon(\kappa) \cap (\mathbb{R} \times \mathbb{R}_-)$ and zero otherwise. Moreover, set

$$x_i^\epsilon := \varrho(\epsilon)R_1 e_1 - (l_{max}^\epsilon - l_\epsilon^i)R_1 e_2,$$

and

$$K_i^\epsilon := M_1 + x_i^\epsilon.$$

3. If $i = N(\epsilon)$, then define

$$\chi_{j,i}^\epsilon(x) := \chi_{j,0}\left(\frac{l(S_\epsilon(x)) - l_{max}^\epsilon}{5}\right)\chi_{j,1}(d_\epsilon(x)),$$

for each $x \in \partial\Omega_\epsilon(\kappa) \cap (\mathbb{R} \times \mathbb{R}_-)$ and zero otherwise. Set

$$x_i^\epsilon := -(\varrho(\epsilon) - \varrho(\epsilon_0))\frac{x_5}{|x_5|},$$

and

$$M_i^\epsilon := M_2 + x_i^\epsilon.$$

4. Suppose $G = \{T_0, T_1, ..., T_5\}$. Define

$$\varphi_{j,i}^\epsilon(x) := \chi_{j,i-lN(\epsilon)-l}^\epsilon\left(T_l^{-1}x\right),$$

and

$$K_i^\epsilon := T_l K_{i-lN(\epsilon)-l}^\epsilon$$

for $j = 1, 2$, $l = 0, ..., 5$, $i = lN(\epsilon) + l, ..., (l+1)N(\epsilon) + l$, and $x \in \mathbb{R}^2$.

Figure 3.3 outlines the construction of the $\chi_{j,i}^\epsilon$'s. The set $\Omega_\epsilon \backslash \Omega_\epsilon(\kappa/4)$ is compact. It follows that there exists a index $M(\epsilon) \in \mathbb{N}$ and finitely many points x_i^ϵ, $i = N(\epsilon)+1, ..., M(\epsilon)$, such that

$$\overline{\Omega_\epsilon} \backslash \Omega_\epsilon(\kappa/4) \subset \bigcup_{i=N(\epsilon)+1}^{M(\epsilon)} B_{\kappa/12}(x_i^\epsilon).$$

Choose $\chi_j \in C^\infty(\mathbb{R}^2, \mathbb{R})$, $j = 1, 2$, such that

$$1_{B_{j\kappa/12}(0)} \leq \chi_j \leq 1_{B_{j\kappa/6}(0)}.$$

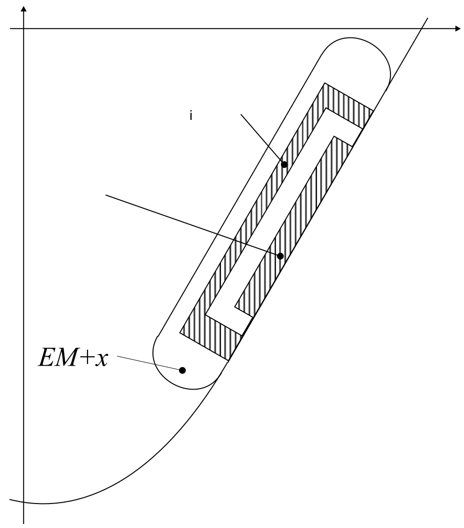

Figure 3.3: Construction of the cut-off functions

Set

$$M_3 := B_{\kappa/3}(0).$$

For $i = N(\epsilon)+1, ..., M(\epsilon)$, define

$$\varphi_{j,i}^{\epsilon}(x) := \chi_j(x - x_i^{\epsilon}), \quad x \in \mathbb{R}^2,$$

and

$$K_i^{\epsilon} := M_3 + x_i^{\epsilon}.$$

Define

$$J_{\epsilon} := \{n \in \mathbb{N}_0 : n \leq M(\epsilon)\},$$

for $\epsilon \in (0, \epsilon_0)$. It remains to show that the stated assertions are fulfilled. Statements 1 and 4 are direct consequences of the construction. We have to show that statement 2 holds. Let $i \in J_{\epsilon}$, $i \leq N(\epsilon)$. As the parametrization of $\partial\Omega_{\epsilon}$ is uniform up to translation, we obtain

$$\|S_{\epsilon}\|_{C^2(\partial\Omega_{\epsilon}(\kappa))} = \|S_{\epsilon_0}\|_{C^2(\partial\Omega_{\epsilon_0}(\kappa))},$$

and

$$\|d_{\epsilon}\|_{C^2(\partial\Omega_{\epsilon}(\kappa))} = \|d_{\epsilon_0}\|_{C^2(\partial\Omega_{\epsilon_0}(\kappa))}$$

for each $\epsilon \in (0, \epsilon_0)$. Moreover,

$$l \in C_b^2(\mathbb{R} \times \mathbb{R}_-).$$

In view of the construction, we have

$$\|\varphi_{j,i}^{\epsilon}\|_{C^2(\mathbb{R}^2)} = \|\chi_{j,i}^{\epsilon}\|_{C^2(\mathbb{R}^2)}$$

for each $\epsilon \in (0, \epsilon_0)$. Moreover, there exist $a, b \in \mathbb{R}$, $|a| \leq 1$, such that

$$\chi^{\epsilon}_{j,i}(x) = \chi_{j,0}(a(l^{\epsilon}(S_{\epsilon}(x)) - b))\chi_{j,1}(d_{\epsilon}).$$

It follows that $\|\varphi^{\epsilon}_{j,i}\|_{C^2(\mathbb{R}^2)}$ is estimable from above in terms of $\|S_{\epsilon}\|_{C^2(\partial\Omega_{\epsilon}(\kappa))}$, $\|d_{\epsilon}\|_{C^2(\partial\Omega_{\epsilon}(\kappa))}$, $\|l\|_{C^2(\mathbb{R}\times\mathbb{R}_-)}$, $\|\chi_0\|_{C^2(\mathbb{R}^2)}$, and $\|\chi_1\|_{C^2(\mathbb{R}^2)}$. This proves assertion 2. Finally, we have to show that statement 3 holds. Without loss of generality, we suppose $x \in \partial\Omega_{\epsilon} \cup \partial K^{\epsilon}_i$, $(-1)^{i-1}\langle x, e_i\rangle \geq 0$, and $l(x) \leq l^{\epsilon}_{max}$. This implies

$$\frac{\partial}{\partial\nu}\varphi^{\epsilon}_{j,i}(x) = \frac{\partial}{\partial\nu}\chi^{\epsilon}_{j,i}(x) =$$

$$a\chi'_{j,0}(a(l(S_{\epsilon}(x)) - b))\chi_{j,1}(d_{\epsilon}(x))\Big[Dl(S_{\epsilon}(x)) \circ DS_{\epsilon}(x)\Big](\nu_{\epsilon}(x)) +$$

$$\chi_{j,0}(a(l(S_{\epsilon}(x)) - b))\chi'_{j,1}(0).$$

On account of

$$DS_{\epsilon}(x)(\nu_{\epsilon}(x)) = 0,$$

and

$$\chi'_{j,1}(0) = 0,$$

we obtain

$$\frac{\partial}{\partial\nu}\varphi^{\epsilon}_{j,i}(x) = 0, \quad x \in \partial K^{\epsilon}_i \cup \partial\Omega_{\epsilon}.$$

Now, suppose $\epsilon = 0$. Let $\alpha : \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$ be bijective. Choose functions $\chi_{1,3}, \chi_{2,3} \in C^{\infty}(\mathbb{R}^2, \mathbb{R})$ such that

$$1_{B_j(0)} \leq \chi_{j,3} \leq 1_{B_{2j}(0)}, \quad j = 1, 2.$$

Define

$$M_3 := B_4(0), \quad J_{\epsilon} := \mathbb{N}_0,$$

and

$$\varphi^0_{j,i}(x) := \chi_{j,3}(x - \alpha(i)), \quad K^0_i := M_3 + \alpha(i)$$

for integers $i \in \mathbb{N}_0$ and $j = 1, 2$. □

Proposition 3.5. *Let $\epsilon_0 > 0$. Assume there exist $(J_{\epsilon})_{\epsilon\in[0,\epsilon_0)} \subset \mathbb{P}(\mathbb{N}_0)$, subsets $(K^{\epsilon}_i)_{i\in J_{\epsilon}} \subset \mathbb{P}(\mathbb{R}^2)$, and cut-off functions $(\varphi^{\epsilon}_{j,i})_{\substack{i\in J_{\epsilon}\\ j=1,2}} \subset C^2(\mathbb{R}^2, \mathbb{R})$ such that the assertions 1-3 in proposition 3.4 are fulfilled. There exists a constant $C > 0$ such that for each $\epsilon \in [0, \epsilon_0)$, we have*

$$\forall R \in \big(L^2(\Omega_{\epsilon}, \mathbb{C})\big)^2 : \forall E \in \big[-\|D^2W(u_0)\|_{C^0(\mathbb{R}^2)}, \min\sigma_e(\mathcal{L}_{\epsilon})\big] :$$

$$\forall\psi \in (\mathcal{K}_{\epsilon} - E)^{-1}(\{R\}) : \forall i \in J_{\epsilon} : \big\|\varphi^{\epsilon}_{1,i}\psi\big\|_{H^2(K^{\epsilon}_i)} \leq C\Bigg[\|\psi\|_{L^2(K^{\epsilon}_i)} + \|R\|_{L^2(K^{\epsilon}_i)}\Bigg].$$

Proof. Our first goal is to determine the constant $C > 0$. On account of statement three of [L, Theorem 3.1.1], there exists a constant $C_1 > 0$ such that we have

$$\|u\|_{H^2(M_j)} \leq C_1 \Big[\|u\|_{H^1(M_j)} + \|\triangle u\|_{L^2(M_j)} \Big]$$

for each $j = 1, ..., m$ and $u \in H^2(M_j, \mathbb{C})$ such that $\frac{\partial}{\partial \nu} u = 0$ on ∂M_j. As K_i^ϵ is the image of a M_j, $j = 1, ..., m$, under an euclidean transformations, we obtain

$$\|u\|_{H^2(K_i^\epsilon)} \leq C_1 \Big[\|u\|_{H^1(K_i^\epsilon)} + \|\triangle u\|_{L^2(K_i^\epsilon)} \Big] \tag{3.23}$$

for each $u \in \left(H^2(K_i^\epsilon, \mathbb{C})\right)^2$ such that $\frac{\partial}{\partial \nu} u = 0$ on ∂K_i^ϵ. Define

$$\mu := -\|D^2 W(u_0)\|_{C^0(\mathbb{R}^2)} - 1,$$

and set

$$C := 8 \cdot C_1 \cdot [\min \sigma_e(\mathcal{L}_0) - \mu] \cdot \sup_{\epsilon \in [0, \epsilon_0)} \sup_{i \in J_\epsilon} \|\varphi_{1,i}^\epsilon\|_{C^2(\mathbb{R}^2)} \left(1 + \|\varphi_{2,i}^\epsilon\|_{C^2(\mathbb{R}^2)}\right).$$

Omit the indices ϵ and i, i.e. $\varphi_{j,i}^\epsilon$ becomes φ_j etc. Choose $R \in \left(L^2(\Omega, \mathbb{C})\right)^2$, $-\|D^2 W(u_0)\|_{C^0(\mathbb{R}^2)} \leq E \leq \min \sigma_e(\mathcal{L}_0)$, and $\psi \in (\mathcal{K} - E)^{-1}(\{R\})$. It follows that

$$[\triangle, \otimes_{l=1,2}[\varphi_j]]\, \psi = -[\triangle \varphi_j \cdot I_{\mathbb{C}^2}]\psi + 2 \mathrm{div}\left(\otimes_{l=1,2} [\nabla \varphi_2]\right) \psi. \tag{3.24}$$

This implies

$$\nabla(\varphi_2 \psi) = \nabla(\mathcal{K} - \mu)^{-1} \Big[\varphi_2 (E - \mu)\psi + \varphi_2 R - [\triangle, \otimes_{l=1,2}[\varphi_j]]\, \psi \Big]. \tag{3.25}$$

In view of (3.24) and (3.25), we get

$$\nabla(\varphi_2 \psi) = (E - \mu) \nabla(\mathcal{K} - \mu)^{-1}(\varphi_2 \psi) + \nabla(\mathcal{K} - \mu)^{-1}(\varphi_2 R) +$$

$$\nabla(\mathcal{K} - \mu)^{-1} [\triangle \varphi_2 \cdot I_{\mathbb{C}^2}]\, \psi - 2\nabla(\mathcal{K} - \mu)^{-1} \mathrm{div}\left(\otimes_{l=1,2} [\nabla \varphi_2]\right) \psi.$$

As

$$\|u\|_{H^1(\Omega)} \leq \langle (\mathcal{K} - \mu) u, u \rangle_{L^2(\Omega)}, \quad u \in D_{\mathcal{K}},$$

we have

$$\nabla(\mathcal{K} - \mu)^{-\frac{1}{2}} \in \mathcal{L}(L^2(\Omega)),$$

and

$$\left\|\nabla(\mathcal{K} - \mu)^{-\frac{1}{2}}\right\|_{\mathcal{L}(L^2(\Omega))}, \left\|(\mathcal{K} - \mu)^{-\frac{1}{2}}\right\|_{\mathcal{L}(L^2(\Omega))} \leq 1.$$

In view of [We1, Satz 2.43] and corollary B.5, we have

$$i(\mathcal{K} - \mu)^{-\frac{1}{2}} \mathrm{div} \subset \left(i\nabla(\mathcal{K} - \mu)^{-\frac{1}{2}}\right)^*.$$

As the left side is bounded, we get

$$\left(i\nabla(\mathcal{K}-\mu)^{-\frac{1}{2}}\right)^* = i\overline{(\mathcal{K}-\mu)^{-\frac{1}{2}}\mathrm{div}}.$$

If follows that

$$-\overline{\nabla(\mathcal{K}-\mu)^{-1}\mathrm{div}} = i\nabla(\mathcal{K}-\mu)^{-\frac{1}{2}}\left(i\nabla(\mathcal{K}-\mu)^{-\frac{1}{2}}\right)^*.$$

Via a priori estimates, we obtain

$$\|\nabla(\varphi_2\psi)\|_{L^2(K)} \leq 4\|\varphi_2\|_{C^2(\mathbb{R}^2)}(\min\sigma_e(\mathcal{L}_0)-\mu)\Big[\|\psi\|_{L^2(K)}+\|R\|_{L^2(K)}\Big]. \quad (3.26)$$

As

$$\triangle(\varphi_1\psi) = \left[\varphi_1(D^2W(u_0)-E)\right]\psi + \left[\triangle\varphi_1\cdot I_{\mathbb{C}^2}\right]\psi + 2\sum_{j=1,2}\langle\nabla\varphi_1,\nabla(\varphi_2\psi_j)\rangle - \varphi_1 R,$$

we obtain

$$\begin{aligned}\|\triangle(\varphi_1\psi)\|_{L^2(K)} \leq{}& 2\|\varphi_1\|_{C^2(\mathbb{R}^2)}\left(\min\sigma_e(\mathcal{L}_0)-\mu\right)\|\psi\|_{L^2(K)}+ \\ & \|\varphi_1\|_{C^2(\mathbb{R}^2)}\|R\|_{L^2(K)} + 2\|\varphi_1\|_{C^2(\mathbb{R}^2)}\left\|\nabla(\varphi_2\psi)\right\|_{L^2(K)}.\end{aligned} \quad (3.27)$$

On account of (3.23), (3.26), and (3.27), we obtain the assertion. □

Now, we glue together proposition 3.3, and proposition 3.5 to obtain the desired pointwise exponential bound for eigenfunctions of $\mathcal{L}_\epsilon$.

Lemma 3.2. *Suppose* (H1)-(H4) *hold and* $\dim\ker(L_0)=1$. *Let* $\beta>0$. *Then there exists* $\epsilon_0\in(0,1)$ *such that for each* $\delta\in(\frac{1}{2},1)$, *there exists a constant* $C>0$ *so that*

$$\forall\epsilon\in[0,\epsilon_0):\forall E\in\sigma_p(\mathcal{L}_\epsilon)\cap(-\infty,\min\sigma_e(\mathcal{L}_0)-\beta]:$$

$$\forall\psi\in\ker(\mathcal{L}_\epsilon-E), normalized:\forall x\in\Omega_\epsilon:|\psi(x)|\leq Ce^{-(1-\delta)\sqrt{\min\sigma_e(\mathcal{L}_0)-E}|x|}.$$

Proof. Choose $\epsilon_0>0$ according to lemma 3.1 and proposition 3.5. Suppose $\varphi_{1,i}^\epsilon$ and K_i^ϵ fulfill the properties in proposition 3.5. Let $\eta\in C^\infty(\mathbb{R}^2,\mathbb{R})$ be radial symmetric, zero in T_{ϵ_0}, and $\eta=1$ outside $T_{\frac{\epsilon_0}{2}}$. Define

$$\zeta_\alpha := |\nabla\eta|^2 + 2\eta\langle\nabla\eta,\nabla f_\alpha\rangle, \quad \alpha\geq 0,$$

$$\mu := -\left\|D^2W(u_0)\right\|_{C^0(\mathbb{R}^2)} - 1,$$

and

$$C_1 := \frac{2}{\beta(2\delta-1)}\sup_{x\in T_{\frac{\epsilon_0}{2}}}|\nabla\eta(x)|\left(|\nabla\eta(x)|+2\sqrt{\min\sigma_e(\mathcal{L}_0)-\mu}\right)e^{2f_0(x)}+\sup_{x\in T_{\frac{\epsilon_0}{2}}}e^{2f_0(x)}.$$

Let us first prove that for each $\epsilon \in [0, \epsilon_0/2)$, $\forall E \in \sigma_p(\mathcal{L}_\epsilon)$, $E \leq \min \sigma_e(\mathcal{L}_0) - \beta$, and each normalized $\psi \in \ker(\mathcal{L}_\epsilon - E)$, we have

$$\left\| e^{f_0} \psi \right\|^2_{L^2_G(\Omega_\epsilon)} \leq C_1.$$

If ϵ, E, and ψ are chosen as above, we obtain with lemma 3.1 and proposition 3.3

$$\|\eta e^{f_\alpha} \psi\|^2_{L^2(\Omega_\epsilon)} \leq \frac{2}{\beta(2\delta - 1)} \left| \langle \zeta_\alpha e^{2f_\alpha} \psi, \psi \rangle \right| \leq \sup_{x \in T_{\frac{\epsilon_0}{2}}} \left| \zeta_\alpha(x) e^{2f_0(x)} \right| \frac{2}{\beta(2\delta - 1)}.$$

In view of lemma A.1, we send $\alpha \to 0$ and obtain

$$\left\| \eta e^{f_0} \psi \right\|^2_{L^2_G(\Omega_\epsilon)} \leq \sup_{x \in T_{\frac{\epsilon_0}{2}}} |\nabla \eta(x)| \left(|\nabla \eta(x)| + 2\sqrt{\min \sigma_e(\mathcal{L}_0) - \mu} \right) e^{2f_0(x)} \frac{4}{\beta}.$$

It follows that

$$\left\| e^{f_0} \psi \right\|^2_{L^2_G(\Omega_\epsilon)} \leq \int_{\Omega_\epsilon} \eta^2(x) e^{2f_0(x)} |\psi(x)|^2 dx + \int_{T_{\frac{\epsilon_0}{2}}} e^{2f_0(x)} |\psi(x)|^2 dx \leq C_1.$$

It remains to prove that the pointwise exponential bound exists. Choose $x \in \overline{\Omega_\epsilon}$ arbitrarily. Then there exists $i \in J_\epsilon$ such that $\varphi^\epsilon_{1,i}(x) = 1$. According to Sobolevs inequality, there exists a constant $C_3 > 0$, such that

$$\|.\|_{C^0(M_j)} \leq C_3 \|.\|_{H^2(M_j)}, \quad j = 1, ..., m.$$

The inequality above holds for K^ϵ_i instead of M_j. Thus

$$|\psi(x)| \leq C_3 \|\varphi^\epsilon_{1,i} \psi\|_{H^2(K^\epsilon_i)} \leq C_2 C_3 \|\psi\|_{L^2(K^\epsilon_i)},$$

where $C_2 > 0$ is the constant in proposition 3.5. This implies

$$\left| \psi(x) e^{f_0(x)} \right| \leq \sup_{y \in K^\epsilon_i} e^{f_0(y)} \sup_{z \in K^\epsilon_i} e^{-f_0(z)} C_2 C_3 \left\| e^{f_0} \psi \right\|^2_{L^2_G(\Omega)}$$

$$\leq \sqrt{C_1} C_2 C_3 e^{\sqrt{2 \min \sigma_e(L)} \mathrm{diam}(K^\epsilon_i)}.$$

Define

$$C := \sqrt{C_1} C_2 C_3 \sup_{\epsilon \in (0, \epsilon_0)} \sup_{i \in J_\epsilon} e^{\sqrt{2 \min \sigma_e(\mathcal{L}_0)} \mathrm{diam}(K^\epsilon_i)}.$$

As $x \in \overline{\Omega_\epsilon}$ was chosen arbitrarily, we obtain

$$|\psi(x)| \leq C e^{-f_0(x)}, x \in \overline{\Omega_\epsilon}.$$

□

Now we proof part 2 of theorem 3.1.

Proof.(theorem 3.1, 2) For $\lambda \in \sigma_p(\mathcal{L}_0)$, $\lambda < \min \sigma_e(\mathcal{L}_0)$, define

$$\beta := \min \sigma_e(\mathcal{L}_0) - \lambda.$$

Apply lemma 3.2 with $\delta \in (\frac{1}{2}, 1)$, $\epsilon = 0$, and $E = \lambda$.

□

3.3.3 The range space

The aim of this section is to investigate the behavior of solutions of

$$\mathcal{L}_0 u = R$$

in reliance on R. The proofs are slight modifications of those in the preceding section.

Corollary 3.4. *Let $\alpha > 0$, $E < \min \sigma_e(\mathcal{L}_0)$, and $\delta \in (0,1)$. Suppose $\eta \in C^\infty(\mathbb{R}^2, \mathbb{R})$ is G-invariant such that $\nabla \eta \in (C_0^\infty(\mathbb{R}^2, \mathbb{R}))^2$. Assume $\psi \in D_{\mathcal{L}_0}$ and $R \in L_G^2(\mathbb{R}^2)$ such that*

$$(\mathcal{L}_0 - E)\,\psi = R.$$

Then

$$Q_{0,\alpha}\left[\eta e^{f_\alpha}\psi\right] = \left\langle \zeta_\alpha e^{f_\alpha}\psi, \psi\right\rangle_{L^2(\mathbb{R}^2)} + \left\langle e^{2f_\alpha}R, \eta^2\psi\right\rangle_{L^2(\mathbb{R}^2)},$$

where

$$\zeta_\alpha := |\nabla \eta|^2 + 2\eta \left\langle \nabla\eta, \nabla f_\alpha\right\rangle, \quad \alpha > 0.$$

Proof. Proceeding as in the proof of proposition 3.3, we obtain

$$Q_{0,\alpha}\left[\eta e^{f_\alpha}\psi\right] = \sum_{j=1,2} \left\langle \nabla\left(\eta\psi_j\right), \nabla\left(\eta e^{2f_\alpha}\psi_j\right)\right\rangle_{L^2(\mathbb{R}^2)} + \left\langle \left(D^2 W\left(u_0\right) - E\right)u, u\right\rangle_{L^2(\mathbb{R}^2)}.$$

In view of the assumption, we get

$$Q_{0,\alpha}\left[\eta e^{f_\alpha}\psi\right] = -2\sum_{j=1,2}\left\langle \nabla\eta\nabla\psi_j, \eta e^{2f_\alpha}\psi_j\right\rangle_{L^2(\mathbb{R}^2)} + \sum_{j=1,2}\left\langle \nabla\eta, \nabla\left(\eta e^{2f_\alpha}\psi_j^2\right)\right\rangle_{L^2(\mathbb{R}^2)} +$$

$$\left\langle e^{2f_\alpha}R, \eta^2\psi\right\rangle_{L^2(\mathbb{R}^2)}.$$

Differentation of the term $\eta e^{2f_\alpha}\psi_j^2$ and summation yield the assertion. $\square$

We are now in a position to proof part 3 of theorem 3.1.

Proof.(theorem 3.1, 3) Suppose either

$$a \geq \sqrt{\min \sigma_e(\mathcal{L}_0) - E}$$

and $\frac{1}{2} < \delta < 1$, or

$$a < \sqrt{\min \sigma_e(\mathcal{L}_0) - E}$$

and

$$1 > \delta > 1 - \frac{a}{2\sqrt{\min \sigma_e(\mathcal{L}_0) - E}}, > \frac{1}{2}.$$

First, we prove that $\theta e^{2f_0} \in L_G^2(\mathbb{R}^2)$. We have

$$\left|R(x)e^{2f_0(x)}\right| \leq Ce^{(-a+2(1-\delta)\sqrt{\min \sigma_e(\mathcal{L}_0) - E})|z|}.$$

If one of the cases above holds, then

$$-a+2(1-\delta)\sqrt{\min\sigma_e(\mathcal{L}_0)-E}<0.$$

Our task is now to show that $e^{f_0}\psi\in L^2_G(\mathbb{R}^2)$. In view of lemma 3.1, there exists $\epsilon_0>0$ such that we have

$$\mathrm{supp}(\phi)\subset (T_{\epsilon_0})^c\Rightarrow \mathrm{Re}\left(Q_{0,\alpha}[\phi]\right)\geq (2\delta-1)\frac{\min\sigma_e(\mathcal{L}_0)-E}{2}\|\phi\|^2_{L^2(\mathbb{R}^2)}$$

for each $\phi\in D_{Q_{0,\alpha}}$. Choose a G-invariant cut-off function $\eta\in C^\infty(\mathbb{R}^2,\mathbb{R})$ such that

$$\eta(x)=0,\quad x\in T_{\epsilon_0},$$

and

$$\eta(x)=1,\quad x\in\left(T_{\frac{\epsilon_0}{2}}\right)^c.$$

Define

$$\zeta_\alpha:=|\nabla\eta|^2+2\eta\left\langle\nabla\eta,\nabla f_\alpha\right\rangle,\quad \alpha>0.$$

Let $\alpha>0$. Owing to corollary 3.4, we obtain

$$\|\eta e^{f_\alpha}\psi\|^2_{L^2(\mathbb{R}^2)}\leq\frac{(\delta-\frac{1}{2})^{-1}}{\min\sigma_e(\mathcal{L}_0)-E}\left[\left|\left\langle\zeta_\alpha e^{f_\alpha}\psi,\psi\right\rangle_{L^2(\mathbb{R}^2)}\right|+\left|\left\langle e^{2f_\alpha}R,\eta^2\psi\right\rangle_{L^2(\mathbb{R}^2)}\right|\right].$$

Moreover,

$$\left|\left\langle\zeta_\alpha e^{f_\alpha}\psi,\psi\right\rangle_{L^2(\mathbb{R}^2)}\right|\leq\sup_{x\in T_{\frac{\epsilon_0}{2}}}|\nabla\eta(x)|\left(|\nabla\eta(x)|+2\sqrt{\min\sigma_e(\mathcal{L}_0)-E}\right)e^{f_0(x)}\|\psi\|^2_{L^2(\mathbb{R}^2)},$$

and

$$\left|\left\langle e^{2f_\alpha}R,\eta^2\psi\right\rangle_{L^2(\mathbb{R}^2)}\right|\leq\left\|e^{2f_0}R\right\|_{L^2(\mathbb{R}^2)}\|\psi\|_{L^2(\mathbb{R}^2)}.$$

Now we send $\alpha\to 0$. In view of lemma A.1, we obtain

$$\|\eta e^{f_0}\psi\|^2_{L^2(\mathbb{R}^2)}<\infty.$$

It follows that

$$\|e^{f_0}\psi\|^2_{L^2(\mathbb{R}^2)}\leq\|\eta e^{f_0}\psi\|^2_{L^2(\mathbb{R}^2)}+\int_{T_{\frac{\epsilon_0}{2}}}e^{2f_0(x)}|\psi(x)|^2dx\leq$$

$$\|\eta e^{f_0}\psi\|^2_{L^2(\mathbb{R}^2)}+\sup_{x\in T_{\frac{\epsilon_0}{2}}}e^{2f_0(x)}\|\psi\|^2_{L^2(\mathbb{R}^2)}<\infty.$$

It remains to show that the pointwise exponential bound exists. Let $\varphi^0_{1,i}$, K^0_i, $i\in\mathbb{N}_0$ fulfill statements 1-4 in proposition 3.4. According to Sobolevs inequality, there exists a constant $C_3>0$ such that

$$\|.\|_{C(M_j)}\leq C_3\|.\|_{H^2(M_j)},\quad j=1,...,m.$$

As we obtain each K_i^0 by translation and rotation out of M_j, the inequality above holds for K_i^0 instead of M_j. Thus

$$|\psi(x)| \leq C_3 \|\varphi_{1,i}^0 \psi\|_{H^2(K_i^0)} \leq C_2 C_3 \left[\|\psi\|_{L^2(K_i^0)} + \|R\|_{L^2(K_i^0)} \right],$$

where $C_2 > 0$ is the constant in proposition 3.5. This implies

$$\left|\psi(x) e^{f_0(x)}\right| \leq \sup_{y \in K_i^0} e^{f_0(y)} \sup_{z \in K_i^0} e^{-f_0(z)} C_2 C_3 \left[\left\|e^{f_0}\psi\right\|_{L_G^2(\Omega)} + \left\|e^{f_0} R\right\|_{L_G^2(\Omega)} \right] \leq$$

$$\left[\left\|e^{f_0}\psi\right\|_{L_G^2(\Omega)} + \left\|e^{f_0} R\right\|_{L_G^2(\Omega)} \right] C_2 C_3 e^{\sqrt{2 \min \sigma_e(\mathcal{L}_0)} \operatorname{diam}(K_i^0)}.$$

Define

$$C := \left[\left\|e^{f_0}\psi\right\|_{L_G^2(\Omega)} + \left\|e^{f_0} R\right\|_{L_G^2(\Omega)} \right] C_2 C_3 \sup_{i \in \mathbb{N}_0} e^{\sqrt{2 \min \sigma_e(\mathcal{L}_0)} \operatorname{diam}(K_i^0)}.$$

As $x \in \overline{\Omega_\epsilon}$ was chosen arbitrarily, we obtain

$$|\psi(x)| \leq C e^{-f_0(x)}, \quad x \in \mathbb{R}^2.$$

□

3.3.4 Convergence of the ground state

Proposition 3.6. *Suppose* (H1)-(H4) *hold,* $\dim \ker (L_0) = 1$*, and* $W \in C^4(\mathbb{R}^2, \mathbb{R})$*. Let* $\epsilon_n \to 0$*,* $n \to \infty$*. Assume* $E_n \in \sigma_p(\mathcal{L}_{\epsilon_n})$*,* $E_n \leq \min \sigma_e(\mathcal{L}_0) - \beta$*, for some* $\beta > 0$*. Let* $\psi_n \in \ker(\mathcal{L}_{\epsilon_n} - E_n)$ *be normalized. Then there exists a subsequence (without loss of generality the sequence itself) such that*

$$\exists \lim_{n \to \infty} E_n =: E, \in \sigma_p(\mathcal{L}_0),$$

and

$$\forall K \subset\subset \mathbb{R}^2 : \psi_n \text{ converges in } C^2(K, \mathbb{C}),$$

with the property that the pointwise limit

$$\psi := \lim_{n \to \infty} \psi_n \in \ker(\mathcal{L}_0 - E)$$

is normalized in $L_G^2(\mathbb{R}^2)$*.*

Proof. Let $K \subset\subset \mathbb{R}^2$ and $N \in \mathbb{N}$, such that

$$\forall n \geq N : \{x \in \mathbb{R}^2 : \operatorname{dist}(x, K) \leq 2\} \subset \Omega_{\epsilon_n}.$$

There exist finitely many $y_i \in K$, $i = 1, ..., l$, such that

$$K \subset \bigcup_{i=1}^{l} B_{\frac{1}{2}}(y_i).$$

Choose $\chi \in C^\infty(\mathbb{R}^2, \mathbb{R})$ G-invariant such that

$$1_{B_{\frac{1}{2}}(0)} \leq \chi \leq 1_{B_1(0)}.$$

Define

$$\chi_i(x) := \frac{\chi(x - y_i)}{\sum_{j=1}^{l} \chi(x - y_j)}, \quad x \in \mathbb{R}^2, \quad i = 1, ..., l.$$

On account of proposition 3.5, we obtain a constant $C > 0$ such that

$$\|\chi_i \psi_n\|_{H^2(B_1(y_i))} \leq C \|\chi_i \psi_n\|_{L^2(B_1(y_i))}$$

for $n \in \mathbb{N}$ sufficiently large and $i = 1, ..., l$. According to [E, Theorem 5, p.323], applied on the differential equation

$$-\triangle(\chi_i \psi_n) = -\chi_i (D^2 W(u_0) - \lambda_n) \psi_n - [\triangle, [\chi_i]] \, \psi_n,$$

there exist constants $\tilde{C}_k$, $k = 0, 1, 2$, such that

$$\|\chi_i \psi_n\|_{H^{2+k}(B_1(y_i))} = \|\chi_i(. + y_i) \psi_n(. + y_i)\|_{H^{2+k}(B_1(0))} \leq$$

$$\tilde{C}_k \|\chi_i(. + y_i) \psi_n(. + y_i)\|_{H^{2+k-1}(B_1(0))} = \|\chi_i \psi_n\|_{H^{2+k-1}(B_1(y_i))}.$$

This implies

$$\|\chi_i \psi_n\|_{H^4(B_1(y_i))} \leq C \tilde{C}_1 \tilde{C}_2, \quad i = 1, ..., l.$$

Since the embedding $H^4(K, \mathbb{C}) \hookrightarrow C^2(\overline{K}, \mathbb{C})$ is compact, there exists a subsequence ψ_{n_k} such that $\chi_i \psi_{n_k}$ converges in $C^2(\overline{B_1(y_i)}, \mathbb{C})$ for each $i = 1, ..., l$. It follows that

$$\psi_{n_k}|_K = \sum_{i=1}^{l} (\chi_i \psi_{n_k})\,|_K$$

converges in $C^2(\overline{K}, \mathbb{C})$. Hence, for each $n \in N$ there exists a index $N(n) \in \mathbb{N}$ and a subsequence ψ_{n_m}, $m \geq N(n)$, that converges in $C^2(\overline{B_n(0)}, \mathbb{C})$. The diagonal selection procedure delivers a sequence ψ_{n_k} that converges in $C^2(\overline{K}, \mathbb{C})$ for each $K \subset\subset \mathbb{R}^2$. It follows that the pointwise limit

$$\psi := \lim_{k \to \infty} \psi_{n_k}$$

exists in $C_b^2(\mathbb{R}^2, \mathbb{C})$. As each ψ_n has the envelope $e^{-\frac{\sqrt{\beta}}{2}|x|}$, the function ψ decays with the same rate. Lebesgues theorem implies

$$\|\psi\|_{L^2(\mathbb{R}^2)}^2 = \lim_{k \to \infty} \|\psi_{n_k}\|_{L^2(\Omega_{\epsilon_{n_k}})}^2 = 1.$$

Finally, we take the limit of the equations

$$\mathcal{L}_{\epsilon_n}\psi_n = E_n\psi_n \tag{3.28}$$

and obtain

$$-\triangle\psi = D^2W(u_0)\psi - E\psi.$$

Furthermore, (3.28) implies

$$\|\nabla\psi_n\|^2_{L^2_G(\Omega_{\epsilon_n})} \leq \left\|D^2W(u_0)\right\|_{C^0(\mathbb{R}^2)} + \min\sigma_e(L),$$

and lemma A.1 delivers

$$\|\nabla\psi\|^2_{L^2_G(\mathbb{R}^2)} \leq \left\|D^2W(u_0)\right\|_{C^0(\mathbb{R}^2)} + \min\sigma_e(L).$$

In view of (3.28), we obtain

$$\overline{s_{\mathcal{L}_0}}[\psi,\varphi] = E\,\langle\psi,\varphi\rangle_{L^2(\mathbb{R}^2)}\,,\quad \varphi\in C^\infty_{0,G}(\mathbb{R}^2).$$

On account of remark C.1, we conclude

$$\psi\in\ker(\mathcal{L}_0 - E).$$

□

Let us prove that if $0\in\sigma_p(\mathcal{L}_0)$, then the smallest eigenvalue of $\mathcal{L}_\epsilon$ converges to zero at an exponential rate.

Definition 3.12. *1. Define*

$$\mu_1^0 := \min\sigma(\mathcal{L}_0).$$

If $\mu_1^0\in\sigma_p(\mathcal{L}_0)$, denote the corresponding normalized eigenfunctions with ψ_1^0.

2. Let $\epsilon > 0$. Assume $\phi_1,...,\phi_r$ is an orthonormal basis of $\ker(\mathcal{L}_0-\mu_1^0)$. Define

$$P_\epsilon u := \sum_{i=1}^r \langle u,\phi_i\rangle_{L^2(\Omega_\epsilon)}\phi_i,$$

for $u\in L^2_G(\Omega_\epsilon)$.

3. Suppose $\epsilon > 0$. Define

$$\mu_1^\epsilon := \min\sigma(\mathcal{L}_\epsilon),$$

and denote the corresponding normalized eigenfunctions with ψ_1^ϵ.

Lemma 3.3. *Assume* (H1)-(H4) *hold,* dim ker $(L_0) = 1$, *and* $W \in C^4(\mathbb{R}^2, \mathbb{R})$. *Then the following statements hold:*

1. *If* $[0, \min \sigma_e(\mathcal{L}_0)) \cap \sigma_d(\mathcal{L}_0) = \emptyset$, *then*

$$\liminf_{\epsilon \to 0} \mu_1^\epsilon \geq \min \sigma_e(\mathcal{L}_0).$$

2. *If* $[0, \min \sigma_e(\mathcal{L}_0)) \cap \sigma_d(\mathcal{L}_0) \neq \emptyset$, *then*

$$\left|\mu_1^\epsilon - \mu_1^0\right| = O\left(\varrho(\epsilon) e^{-\frac{\sqrt{\min \sigma_e(\mathcal{L}_0) - \mu_1^0}}{8} \varrho(\epsilon)}\right), \quad \epsilon \to 0.$$

Proof. 1. Suppose the contrary holds, i.e.

$$\liminf_{\epsilon \to 0} \mu_1^\epsilon < \min \sigma_e(\mathcal{L}_0).$$

Then there exists $\beta > 0$ and a sequence $\epsilon_n \to 0$ such that

$$\mu_1^{\epsilon_n} \leq \min \sigma_e(\mathcal{L}_0) - \beta.$$

In view of proposition 3.6, we obtain

$$\mu := \lim_{n \to \infty} \mu_1^{\epsilon_n}, \quad \mu < \min \sigma_e(\mathcal{L}_0),$$

and

$$\psi := \lim_{n \to \infty} \psi_1^{\epsilon_n}$$

such that

$$\mu \in \sigma_p(\mathcal{L}_0), \quad \psi \in \ker(\mathcal{L}_0 - \mu), \text{ normalized.}$$

It follows that $[0, \min \sigma_e(\mathcal{L}_0)) \cap \sigma_d(\mathcal{L}_0) \neq \emptyset$ which is a contradiction.

2. Suppose there exists an eigenvalue below $\min \sigma_e(\mathcal{L}_0)$. Let $\psi_1^0 \in \ker(\mathcal{L}_0 - \mu_1^0)$ be normalized, and define

$$\beta_\epsilon := \frac{1}{\|\psi_1^0\|_{L^2(\Omega_\epsilon)}}, \quad \epsilon \geq 0.$$

With corollary C.1 and Green's formula, we obtain

$$\mu_1^\epsilon \leq \beta_\epsilon^2 \mathcal{S}_\epsilon \left[\psi_1^0\right] = \beta_\epsilon \left(\int_{\Omega_\epsilon} \left|\nabla \psi_1^0\right|^2 + \left\langle D^2 W(u_0) \psi_1^0, \psi_1^0 \right\rangle dx\right) =$$

$$\beta_\epsilon^2 \left(\int_{\Omega_\epsilon} - \left\langle \triangle \psi_1^0, \psi_1^0 \right\rangle + \left\langle D^2 W(u_0) \psi_1^0, \psi_1^0 \right\rangle dx + \right.$$

$$\int_{\partial\Omega_\epsilon}\left\langle\frac{\partial}{\partial\nu}\psi_1^0,\psi_1^0\right\rangle dS\Bigg) =$$

$$\mu_1^0+\beta_\epsilon^2\int_{\partial\Omega_\epsilon}\left\langle\frac{\partial}{\partial\nu}\psi_1^0,\psi_1^0\right\rangle dS.$$

As $W\in C^4(\mathbb{R}^2,\mathbb{R})$, we have $\psi_1\in C_b^2(\mathbb{R}^2,\mathbb{R}^2)$. Thus, the trace of ψ_1 and its first order derivatives exist in the classical sense. It follows that there exists a constant $C>0$ such that

$$\left|\int_{\partial\Omega_\epsilon}\left\langle\frac{\partial}{\partial\nu}\psi_1^0,\psi_1^0\right\rangle dS\right|\leq C\int_{\partial\Omega_\epsilon}\left\|D\psi_1^0\right\|_{tr}\left|\psi_1^0\right| dS\leq C\varrho(\epsilon)e^{-\frac{\sqrt{\min\sigma_e(\mathcal{L}_0)-\mu_1^0}}{4}\varrho(\epsilon)}$$

for $\epsilon>0$ sufficiently small. The last inequality follows from theorem 3.1 applied with $\delta=3/4$. We have proved

$$\mu_1^\epsilon\leq\beta_\epsilon^2\mathcal{S}_\epsilon\left[\psi_1^0\right]=\mu_1^0+O\left(\varrho(\epsilon)e^{-\frac{\sqrt{\min\sigma_e(\mathcal{L}_0)-\mu_1^0}}{4}\varrho(\epsilon)}\right),\quad\epsilon\to 0.\qquad(3.29)$$

Our task is now to show that there exist $\epsilon_0, C>0$ such that

$$\forall\epsilon\in(0,\epsilon_0):\langle\psi_1^\epsilon,P_\epsilon\psi_1^\epsilon\rangle_{L^2(\Omega_\epsilon)}\geq C.$$

If the contrary holds, there exists a sequence $\epsilon_n\to 0$ such that by (3.29) $\mu_1^{\epsilon_n}\leq\frac{\min\sigma_e(\mathcal{L}_0)+\mu_1^0}{2}$, and

$$\sum_{i=1}^r\left|\langle\psi_1^{\epsilon_n},\phi_i\rangle_{L^2(\Omega_{\epsilon_n})}\right|^2=\langle\psi_1^{\epsilon_n},P_{\epsilon_n}\psi_1^{\epsilon_n}\rangle_{L^2(\Omega_{\epsilon_n})}<\frac{1}{n}.\qquad(3.30)$$

On account of proposition 3.6, this sequence delivers a normalized eigenfunction ψ of $\mathcal{L}_0$ with corresponding eigenvalue λ. More precisely, we have

$$\psi=\lim_{n\to\infty}\psi_1^{\epsilon_n},$$

and

$$\lambda=\lim_{n\to\infty}\mu_1^{\epsilon_n}.$$

In view of (3.29), we have $\lambda\leq\mu_1^0$, hence $\lambda=\mu_1^0$. As each $\psi_1^{\epsilon_n}$ has the envelope $e^{-\frac{\sqrt{\min\sigma_e(\mathcal{L}_0)-\mu_1^0}}{8}|x|}$ for sufficiently large n, the same holds true for ψ. Lebesgues theorem and (3.30) imply

$$\langle\psi,\phi_i\rangle_{L^2_G(\mathbb{R}^2)}=\lim_{n\to\infty}\langle\psi_1^{\epsilon_n},\phi_i\rangle_{L^2(\Omega_{\epsilon_n})}=0$$

for $i=1,...,r$. It follows that

$$\psi\in\ker(\mathcal{L}_0-\mu_1^0)^\perp$$

which is a contradiction. Integration by parts shows

$$\mu_1^\epsilon \langle \psi_1^\epsilon, P_\epsilon \psi_1^\epsilon \rangle_{L^2(\Omega_\epsilon)} = \langle \mathcal{L}_\epsilon \psi_1^\epsilon, P_\epsilon \psi_1^\epsilon \rangle_{L^2(\Omega_\epsilon)} =$$

$$\sum_{i=1}^{r} \overline{\langle \psi_1^\epsilon, \phi_i \rangle_{L^2(\Omega_\epsilon)}} \langle \mathcal{L}_\epsilon \psi_1^\epsilon, \phi_i \rangle_{L^2(\Omega_\epsilon)} =$$

$$\sum_{i=1}^{r} \overline{\langle \psi_1^\epsilon, \phi_i \rangle_{L^2(\Omega_\epsilon)}} \left[\mu_1^0 \langle \psi_1^\epsilon, \phi_i \rangle_{L^2(\Omega_\epsilon)} - \int_{\partial\Omega_\epsilon} \left\langle \psi_1^\epsilon, \frac{\partial}{\partial\nu} \phi_i \right\rangle dS \right] =$$

$$\mu_1^0 \langle \psi_1^\epsilon, P_\epsilon \psi_1^\epsilon \rangle_{L^2(\Omega_\epsilon)} - \sum_{i=1}^{r} \overline{\langle \psi_1^\epsilon, \phi_i \rangle_{L^2(\Omega_\epsilon)}} \int_{\partial\Omega_\epsilon} \left\langle \psi_1^\epsilon, \frac{\partial}{\partial\nu} \phi_i \right\rangle dS.$$

In view of lemma 3.2, we obtain

$$|\mu_1^\epsilon - \mu_1^0| C \leq O\left(\varrho(\epsilon) e^{-\frac{\sqrt{\min \sigma_e(\mathcal{L}_0) - \mu_1^0}}{8} \varrho(\epsilon)} \right), \quad \epsilon \to 0.$$

□

The regularization of T_ϵ via F is unessential for the desired spectral analysis. In the following, we refine this.

Definition 3.13. *Let $\epsilon \in (0,1)$. Define*

$$D_{\mathcal{T}_\epsilon} := H_G^1(T_\epsilon),$$

and

$$\mathcal{T}_\epsilon[u,v] := \int_{T_\epsilon} \sum_{j=1,2} \langle \nabla u_j, \nabla v_j \rangle + \left\langle D^2 W(u_0) u, v \right\rangle dx$$

for $u, v \in D_{\mathcal{T}_\epsilon}$. Set

$$\nu_1^\epsilon := \inf_{\substack{u \in D_{\mathcal{T}_\epsilon} \\ \|u\|_{L_G^2(\mathcal{T}_\epsilon)} = 1}} \mathcal{T}_\epsilon[u,u].$$

Remark 3.7. *As $u_n \xrightarrow{\mathcal{T}_\epsilon} u$ is equivalent to $u_n \to u$ in $H_G^1(\mathcal{T}_\epsilon)$, the sesquilinear form $\mathcal{T}_\epsilon$ is closed. Moreover, ν_1^ϵ is the smallest eigenvalue of the self-adjoint operator that is associated to $\mathcal{T}_\epsilon$. For details, the reader is referred to the appendix.*

Theorem 3.2. *Suppose* (H1)-(H4) *hold,* dim ker $(L_0) = 1$, *and $W \in C^4(\mathbb{R}^2, \mathbb{R})$. Then the following statements hold:*

1. *If $[0, \min \sigma_e(\mathcal{L}_0)) \cap \sigma_d(\mathcal{L}_0) = \emptyset$, then*

$$\liminf_{\epsilon \to 0} \nu_1^\epsilon \geq \min \sigma_e(\mathcal{L}_0).$$

2. If $[0, \min \sigma_e(\mathcal{L}_0)) \cap \sigma_d(\mathcal{L}_0) \neq \emptyset$*, then*

$$\left|\nu_1^\epsilon - \mu_1^0\right| = O\left(\varrho(\epsilon) e^{-\frac{\sqrt{\min \sigma_e(\mathcal{L}_0) - \mu_1^0}}{8} \varrho(\epsilon)}\right), \quad \epsilon \to 0.$$

Proof. Set

$$\epsilon' := \epsilon \left(1 + \frac{\epsilon\sqrt{3}}{4}\right)^{-1},$$

for each $\epsilon \in (0,1)$. It follows that

$$\Omega_\epsilon \subset T_\epsilon \subset \Omega_{\epsilon'}.$$

First, we establish that

$$\mathcal{T}_\epsilon[u] \geq \mathcal{S}_\epsilon[u], \quad u \in H_G^1(T_\epsilon), \tag{3.31}$$

for ϵ sufficiently small. Suppose $u \in H_G^1(T_\epsilon)$. On account of corollary 3.2, we obtain

$$\mathcal{T}_\epsilon[u] - \mathcal{S}_\epsilon[u] \geq \sum_{i=1,3,5} \int_{R_i(N\setminus M - r(\epsilon)e_2)} \left\langle D^2W(u_0)u, u\right\rangle dx =$$

$$3\int_{N\setminus M - r(\epsilon)e_2} \left\langle D^2W(u_0)u, u\right\rangle dx = 3\int_{N\setminus M} \left\langle D^2W(u_0(. + r(\epsilon)e_2)\tilde{u}, \tilde{u}\right\rangle dx,$$

where $\tilde{u}(x) := u(x + r(\epsilon)e_2)$. We have $D^2W(u_0(. + r(\epsilon)e_2)) \to D^2W(x_3)$ as $\epsilon \to 0$, in view of [BGS, Theorem 4.7]. This proves (3.31). Our task is now to prove

$$\mathcal{S}_{\epsilon'}[u] - \mathcal{T}_\epsilon[u] \geq 0, \quad u \in H_G^1(\Omega_{\epsilon'}), \tag{3.32}$$

for ϵ sufficiently small. Due to the C^4-regularization of $T_{\epsilon'}$, there exists $r > 0$ and a function $f \in C^4([0,r], \mathbb{R})$, $f(0) = 0$, such that the sets

$$\Omega_1^{\epsilon'} := \left\{(x,y) \in \mathbb{R}^2 : 0 \leq y < \varrho(\epsilon') - r, |x| \leq \sqrt{3}y\right\},$$

and

$$\Omega_2^{\epsilon'} := \left\{(x,y) \in \mathbb{R}^2 : y \in [\varrho(\epsilon') - r, \varrho(\epsilon')], |x| \leq f(y - (\varrho(\epsilon') - r)) + \sqrt{3}(\varrho(\epsilon') - r)\right\}$$

fulfill

$$\overline{\Omega_{\epsilon'}} \cap R_{15} = \Omega_1^{\epsilon'} \quad \dot{\cup} \quad \Omega_2^{\epsilon'}.$$

Define

$$l_1(y) := \sqrt{3}y, \quad 0 \leq y < \varrho(\epsilon') - r,$$

and

$$l_2(y) := f(y - (\varrho(\epsilon') - r)) + \sqrt{3}(\varrho(\epsilon') - r), \quad y \in [\varrho(\epsilon') - r, \varrho(\epsilon')].$$

Let $u \in C^\infty_{0,G}(\mathbb{R}^2)$. On account of corollary 3.2, calculation yields

$$\frac{1}{3}\left(\mathcal{S}_{\epsilon'}[u] - \mathcal{T}_\epsilon[u]\right) \geq$$

$$\int_{\varrho(\epsilon')-r}^{\varrho(\epsilon')} S_{l_2(y)^{-1}}[u(.,y)]dy - \sup_{\substack{y\geq\varrho(\epsilon')-r\\ |x|\leq l_2(y)}} \left\| D^2W(u_0(x,y)) - D^2W(\theta_0(x)) \right\|_{tr} \|u\|^2_{L^2(\Omega_2^{\epsilon'})} \geq$$

$$\left[\inf_{\varrho(\epsilon')-r\leq y\leq\varrho(\epsilon')} \lambda_1^{\frac{1}{l_2(y)},odd} - \sup_{\substack{y\geq\varrho(\epsilon')-r\\ |x|\leq l_2(y)}} \left\| D^2W(u_0(x,y)) - D^2W(\theta_0(x)) \right\|_{tr}\right] \|u\|^2_{L^2(\Omega_2^{\epsilon'})}.$$

In view of theorem 3.1 and (H4), the last expression in brackets becomes positive for ϵ sufficiently small. As $\{u|_{\Omega_\mu} : u \in C^\infty_{0,G}(\mathbb{R}^2)\}$ is dense in $H^1_G(\Omega_\mu)$, $\mu \in \{\epsilon, \epsilon'\}$, we obtain (3.32). We are now in a position to prove the assertion of the theorem. Owing to lemma C.1, we obtain from (3.31) and (3.32) the estimate

$$\mu_1^{\epsilon'} \geq \nu_1^\epsilon \geq \mu_1^\epsilon \tag{3.33}$$

for ϵ sufficiently small. Suppose $[0, \min\sigma_e(\mathcal{L}_0)) \cap \sigma_d(\mathcal{L}_0) = \emptyset$. On account of (3.33) and lemma 3.3, we have

$$\liminf_{\epsilon\to0} \nu_1^\epsilon \geq \liminf_{\epsilon\to0} \mu_1^\epsilon \geq \min\sigma_e(\mathcal{L}_0).$$

If $[0, \min\sigma_e(\mathcal{L}_0)) \cap \sigma_d(\mathcal{L}_0) \neq \emptyset$, then (3.33) implies

$$\left|\nu_1^\epsilon - \mu_1^0\right| \leq \max\left(\left|\mu_1^\epsilon - \mu_1^0\right|, \left|\mu_1^{\epsilon'} - \mu_1^0\right|\right)$$

for ϵ sufficiently small. The assertion follows with statement 2 of lemma 3.3. $\square$

3.4 Example

In this section, we prove that there exists a smooth potential that fulfills the assumptions (H1)-(H4) and dim ker $(L_0) = 1$. Consider the potential

$$V(x,y,z) := x^2y^2 + x^2z^2 + y^2z^2 + xyz^2 + xy^2z + x^2yz,$$

cf. [GHS]. We are interested in the restriction of V to

$$\Sigma := \{(x,y,z) \in \mathbb{R}^3 : x+y+z = 1\}.$$

A standard calculation yields that

$$f_1 := \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad f_2 := \frac{1}{2}\frac{\sqrt{2}}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

are orthonormal. Moreover,

$$\Sigma = \frac{1}{3}(e_1 + e_2 + e_3) + \text{lin}\{f_1, f_2\}.$$

We define

$$W(x,y) := V\left(\frac{1}{3}(e_1+e_2+e_3) + \frac{\sqrt{2}}{\sqrt{3}}xf_1 + \frac{\sqrt{2}}{\sqrt{3}}yf_2\right).$$

The graph of W is given in figure 3.4.

Proposition 3.7. *The potential W fulfills* (H1)-(H4), *and* $\dim\ker(L_0) = 1$. *Moreover, $\mathcal{M} = \{\theta_0\}$ where*

$$\theta_0(z) = \left(\ \tfrac{\sqrt{3}}{2}\tanh\left(\tfrac{z}{\sqrt{6}}\right), \tfrac{1}{2}\ \right), \quad z \in \mathbb{R}.$$

Proof. Clearly $W \in C^\infty(\mathbb{R}^2, \mathbb{R})$. Calculation yields

$$W(x,y) = \frac{1}{9}\left[x^4 + (2y^2 - 2y - 1)x^2 + \frac{2}{3} - y^2 + y^4 + \frac{2}{3}y^3\right].$$

It follows that $W \circ R_3 = W$. In polar coordinates, we have

$$W(r,t) = \frac{1}{27}\left(3r^4 - 2\sin(3t)r^3 - 3r^2 + 2\right), \quad (r,t) \in \mathbb{R}_+ \times [0, 2\pi].$$

Hence

$$W(r,t) = W\left(r, t + \frac{2}{3}\pi\right)$$

which implies

$$W \circ D_2 = W.$$

But $\{R_3, D_2\}$ is a generator of G, hence W is G-invariant. It follows that (H1) is fulfilled. Obviously the x_i, $i = 1, 3, 5$, are zeros of W and up to a normal transformation

$$D^2 W(x_i) = \frac{2}{3}I.$$

In order to prove that there are no other zeros of W beside the x_i's, note that

$$W(r,t) \geq \frac{1}{27}\left(3r^4 - 2r^3 - 3r^2 + 2\right) = \frac{1}{9}(r-1)^2\left|r + \frac{2}{3} + \frac{\sqrt{2}}{3}i\right|^2 \geq \frac{4}{81}(r-1)^2.$$

Thus $W(r,t) > 0$ for $t \in [0, 2\pi]$, $r \neq 1$. If $r = 1$, then

$$W(r,t) = 0 \Leftrightarrow \sin(3t) = 1 \Leftrightarrow t = \frac{\pi}{6} + k\frac{2\pi}{3}, k = 0, 1, 2.$$

Hence (H2) holds. According to [BGS, p. 680], the condition (H3) is fulfilled if

$$\langle DW(x), x\rangle \geq 0, \quad |x| >> 1.$$

The latter holds true, as we have

$$\langle DW(x), x\rangle\,|_{x=(r,t)} = \frac{4}{9}r^4 - \frac{2\sin(3t)}{9}r^3 - \frac{2}{9}r^2$$

in polar coordinates. It remains to show that (H4) is fulfilled, especially that $\mathcal{M}$ contains exactly one element. Define the mapping

$$T : \mathbb{R}^2 \to \mathbb{R}^2,$$

by

$$(x, y) \longmapsto \left(x, \frac{1}{2}\right).$$

Calculation yields

$$\|DT(x, y)\|_{tr} = 1, \quad (x, y) \in \mathbb{R}^2.$$

For each $x \in \mathbb{R}$, the point $y_0 := \frac{1}{2}$ is the global minimum of $y \in \mathbb{R}_+ \longmapsto W(x, y)$. This holds true, because calculation yields

$$\frac{\partial}{\partial y}W(x, y) = \frac{2}{9}(2y - 1)(y^2 + y + x^2), \quad (x, y) \in \mathbb{R}^2.$$

It follows

$$\frac{\partial}{\partial y}W(x, y) = 0 \Leftrightarrow y = \frac{1}{2}$$

for $(x, y) \in \mathbb{R} \times \mathbb{R}_+$, and

$$\frac{\partial^2}{\partial y^2}W(x, y_0) = \frac{4}{9}x^2 + \frac{1}{3} > 0.$$

We conclude

$$W(T(x, y)) \leq W(x, y)$$

for $(x, y) \in \mathbb{R} \times \mathbb{R}_+$. Due to [BGS, Theorem 2.3], we have $\mathrm{im}(\theta) \subset \overset{\circ}{R}_{15}$ for each $\theta \in \mathcal{M}$. This implies that $T\theta \in M$ for each $\theta \in \mathcal{M}$. Let us prove that each minimizer in $\mathcal{M}$ has image in $\{(x, 1/2) : x \in \mathbb{R}\}$. Assume the contrary holds, i.e. there exists $\theta \in \mathcal{M}$ and there exists $z_0 \in \mathbb{R}$ such that $\theta(z_0)_2 \neq \frac{1}{2}$. Then there exists $\delta > 0$ such that

$$\forall z \in K_\delta(z_0) : \theta(z)_2 \neq \frac{1}{2}.$$

This implies

$$\frac{1}{2}\left|(T\theta)'\right|^2 + W(T\theta) < \frac{1}{2}\left|\theta'\right|^2 + W(\theta)$$

on $K_\delta(z_0)$. It follows $E(T\theta) < E(\theta)$ which is a contradiction. As the elements of $\mathcal{M}$ are R_3-odd, we must have

$$\theta(0) = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$$

for each $\theta \in \mathcal{M}$. By [BGS, Lemma 2.4] two elements of $\mathcal{M}$ are equal up to translation, and due to [BGS, Corollary 2.5] each minimizer is injective. It follows that $\mathcal{M}$ contains exactly one element. As a consequence, the minimizer $\theta_0 \in \mathcal{M}$ has the shape

$$\theta_0 = \left(\ \theta, \tfrac{1}{2} \ \right),$$

where θ is a global minimizer of

$$E_0(u) := \int_{\mathbb{R}} \frac{1}{2}|u'|^2 + F(u)dt$$

over the set

$$M_0 := H^1(\mathbb{R}, \mathbb{R}) + \frac{\sqrt{3}}{2} \tanh\left(\frac{\cdot}{\sqrt{6}}\right).$$

The potential is given by $F(x) := W(x, \frac{1}{2})$. According to corollary 2.1, the minimizer θ is the unique increasing solution of

$$-\theta'' + F'(\theta) = 0.$$

Calculation yields

$$\theta(z) = \frac{\sqrt{3}}{2} \tanh\left(\frac{z}{\sqrt{6}}\right), \quad z \in \mathbb{R}.$$

Finally, let $\theta_0 \in \mathcal{M}$. Then

$$D^2W(\theta_0) = D^2W\left(\theta, \frac{1}{2}\right) = \begin{pmatrix} \frac{4}{3}\theta^2 - \frac{1}{3} & 0 \\ 0 & \frac{4}{9}\theta^2 + \frac{1}{3} \end{pmatrix}.$$

This implies that L_0 is the product of two scalar Sturm-Liouville operators with bounded potentials. Define

$$L_1 := -\frac{d^2}{dz^2} + \frac{4}{3}\theta_{0,1}^2 - \frac{1}{3},$$

$$L_2 := -\frac{d^2}{dz^2} + \frac{4}{9}\theta_{0,1}^2 + \frac{1}{3},$$

and

$$D_{L_1} = D_{L_2} := H^2(\mathbb{R}, \mathbb{C}).$$

It follows that

$$L_0 = L_1 \otimes L_2.$$

Obviously, $L_2 \geq \frac{1}{3}$ is invertible, i.e. $\ker(L_2) = \{0\}$. Hence,

$$\dim \ker(L_0) = \dim \ker(L_1) = 1.$$

Note that the ground state of a scalar Sturm-Liouville operator has multiplicity one - cf. [We2, Satz 17.14]. $\square$

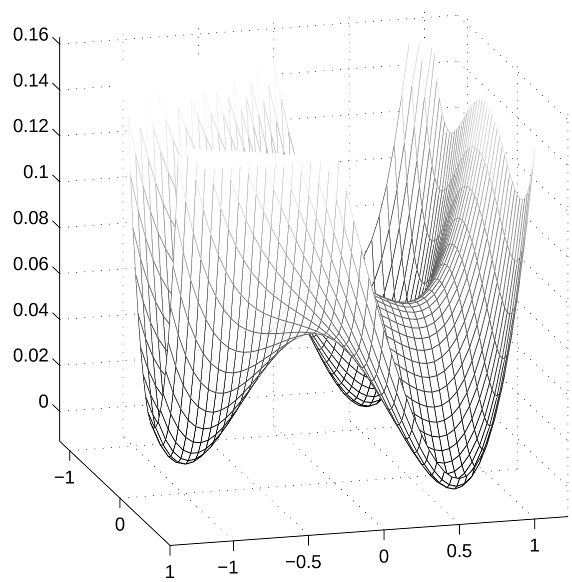

Figure 3.4: Graph of W

Remark 3.8. *Due to* (H4), *we have*

$$u_0(0, y) \to \left(0, \frac{1}{2}\right), \quad y \to \infty.$$

We obtain

$$D^2W(u_0(0, y)) \to \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}, \quad y \to \infty.$$

It follows that there exists no pair of numbers $R, \gamma > 0$ *such that*

$$D^2W(u_0(x)) \geq \gamma, \quad |x| \geq R.$$

Thus, a generalization of the technique in chapter 1 does not lead to the result on exponential decay.

Chapter 4

Discussion

In this section, we outline a possible application of the foregoing results to the rigorous convergence on a formal level . I want to emphasize that the considerations in this section should only give some ideas for further work.

As already mentioned in the introduction, the asymptotic expansion at the triple-junction leads to equations of the form

$$\mathcal{L}_0 u_i = R_{i-1}.$$

Due to theorem 3.1, we know that $\min \sigma_e(\mathcal{L}_0) > 0$. Thus, in order to obtain $R \in \ker(\mathcal{L}_0)^{\perp}$, at most finitely many conditions have to be fulfilled. Moreover, statement 3 of theorem 3.1 might deliver the proper matching conditions.

Let us consider the situation at a fixed time $t \in [0, T]$. Suppose there exist closed smooth manifolds M_i, $i = 1, 2, 3$, such that for two distinct triple-junctions m_i, $i = 1, 2$, the endpoint of each manifold is either m_1 or m_2 - cf. figure 4.1. Let $R \in O(2)$ be the reflection on the x-axis. Suppose that n_i is a continuous normal of M_i, $i = 1, 2, 3$. Suppose $\delta > 0$ is small so that the mapping

$$M_i \times (-\delta, \delta) \to \mathbb{R}^2,$$

$$(\sigma, \lambda) \to \sigma + \lambda n_i(\sigma),$$

is a diffeomorphism onto its image which we denote with $M_i(\delta)$. Let T be the equilateral triangle of edge length $\sqrt{3}$, centered at the origin. Following the formal calculations in [BR], we anticipate to obtain approximate solutions u_A^ϵ, $\epsilon \in (0, 1)$, such that the following local properties are fulfilled:

i) For each $x \in T_1 := R \cdot T + m_1$, we have

$$u_A^\epsilon(x, t) = u_0 \left(R \frac{x - m_1}{\epsilon} \right) + O\left(\epsilon^2\right),$$

and

$$u_A^\epsilon(x,t) = u_0\left(\frac{x-m_2}{\epsilon}\right) + O\left(\epsilon^2\right), \quad x \in T_2 := T + m_2.$$

Moreover,

$$u_A^\epsilon(R^{i-2}. + m_i, t)$$

is G-equivariant in T for $i = 1, 2$.

ii) Suppose θ_{ij} is the standing wave that connects the points x_i and x_j such that $\theta_{ij}(-\infty) = x_i$ and $\theta_{ij}(+\infty) = x_j$. We require

$$u_A^\epsilon(x,t) = \theta_{51}\left(\frac{d_1(x)}{\epsilon}\right) + O\left(\epsilon^2\right), \quad x \in M_1(\delta),$$

$$u_A^\epsilon(x,t) = \theta_{13}\left(\frac{d_2(x)}{\epsilon}\right) + O\left(\epsilon^2\right), \quad x \in M_2(\delta),$$

and

$$u_A^\epsilon(x,t) = \theta_{53}\left(\frac{d_3(x)}{\epsilon}\right) + O\left(\epsilon^2\right), \quad x \in M_3(\delta).$$

iii) For simplicity, suppose there exist compact submanifolds $S_i \subset M_i$, $i = 1, 2, 3$, such that, with $S_i(\delta) := \{x \in \mathbb{R}^2 : \text{dist}(x, S_i) < \delta\}$, we have

$$\forall x \in \Omega \setminus \left(\bigcup_{i=1,2,3} S_i(\delta) \,\dot{\cup}\, \bigcup_{i=1,2} T_i \right) : D^2 W\left(u_A^\epsilon(x,t)\right) \geq 0.$$

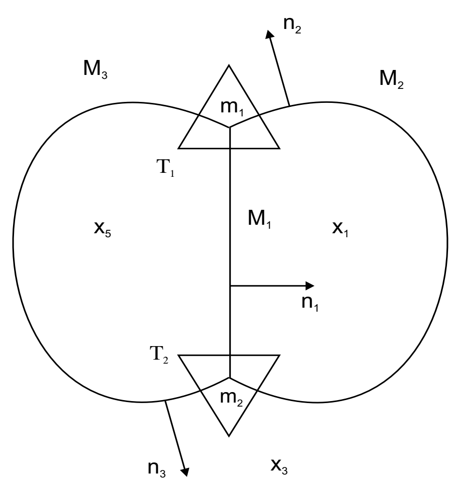

Figure 4.1: Triple-junction motion

Definition 4.1. *For each $\epsilon \in (0,1)$ define the quadratic form AC_ϵ by*

$$D_{AC_\epsilon} := \left\{ u \in \left(H^1(\Omega, \mathbb{R})\right)^2 : u(R^{i-2}. + m_i) \text{ is } G\text{-equivariant in } T \text{ for } i = 1,2 \right\},$$

and

$$AC_\epsilon[u] := \int_\Omega \epsilon |\nabla u|^2 + \frac{1}{\epsilon} \left\langle D^2 W\left(u_A^\epsilon\right) u, u \right\rangle dx.$$

In the case of the scalar Allen-Cahn equation, the following estimate is known as the [deMS]-estimate.

Lemma 4.1. *There exists $C > 0$ such that*

$$AC_\epsilon \geq -C\epsilon$$

for ϵ sufficiently small.

Proof. Let $\psi \in D_{AC_\epsilon}$. Then

$$AC_\epsilon[\psi] \geq \sum_{i=1,2,3} \int_{S_i(\delta)} \epsilon|\nabla\psi|^2 + \frac{1}{\epsilon}\left\langle D^2W\left(\theta_{kl}\left(\frac{d_i(x)}{\epsilon}\right)\right)\psi, \psi\right\rangle dx + O(\epsilon)\|\psi\|^2_{S_i(\delta)}$$

$$\sum_{i=1,2} \int_{T_i} \epsilon|\nabla\psi|^2 + \frac{1}{\epsilon}\left\langle D^2W\left(u_0\left(R^{i-2}\frac{x-m_i}{\epsilon}\right)\right)\psi, \psi\right\rangle dx + O(\epsilon)\left\|\psi\right\|^2_{L^2(T_i)},$$

where θ_{kl} is the standing wave that corresponds to S_i. Set $\tilde{\psi}(x) := \psi(\epsilon R^{i-2}x + m_i)$. The transformation lemma and lemma 3.2 deliver

$$\int_{T_i} \epsilon|\nabla\psi|^2 + \frac{1}{\epsilon}\left\langle D^2W\left(u_0\left(R^{i-2}\frac{x-m_i}{\epsilon}\right)\right)\psi, \psi\right\rangle dx =$$

$$\epsilon\mathcal{T}_\epsilon\left[\tilde{\psi}\right] \geq \epsilon\nu_1^\epsilon\left\|\tilde{\psi}\right\|^2_{L^2(T_\epsilon)} = \frac{1}{\epsilon}\nu_1^\epsilon\left\|\tilde{\psi}\right\|^2_{L^2(T_\epsilon)} \geq$$

$$-C\varrho(\epsilon)^2 e^{-\frac{\sqrt{\min\sigma_e(\mathcal{L}_0)-\mu_1^0}}{8}\varrho(\epsilon)}\left\|\psi\right\|^2_{L^2(T_i)} \geq$$

$$-C\epsilon\left\|\psi\right\|^2_{L^2(\Omega)}, \quad \epsilon \to 0.$$

Suppose that

$$\tau : M_2 \times (-\delta, \delta) \to M_2(\delta),$$
$$(\sigma, \lambda) \longmapsto \sigma + \lambda n_2(\sigma),$$

is a diffeomorphism. Set

$$\tilde{\psi}(\sigma, z) := \psi(\tau(\sigma, \epsilon z)), \quad z \in I_\epsilon,$$

and

$$J(\sigma, \lambda) := \left|\det(D\tau(\sigma, \lambda))\right|.$$

Define
$$J_\epsilon(\sigma,z) := J(\sigma,\epsilon z).$$
Assume $B_{\epsilon,\sigma}$ is the quadratic form associated to the self-adjoint operator $L_{\epsilon,\sigma}$ in $\left(L^2_{J_\epsilon(\sigma,.)}(I_\epsilon)\right)^m$ given by
$$-\frac{1}{J_\epsilon(\sigma,.)}\left(J_\epsilon(\sigma,.)u'\right)' + \left[D^2W(\theta_{13})\right]u, \quad u\in\left(H^2(I_\epsilon,\mathbb{C})\right)^m,$$
equipped with Neumann-boundary conditions. This family of operators fulfills the assumptions of definition 1.1 uniformly in $\sigma\in M_2$, with
$$\lambda=\lambda_+=\lambda_-=\min\sigma(D^2W(a)).$$
All the results of chapter 1 are applicable. Define
$$\lambda_1^{\epsilon,\sigma} := \inf_{\|u\|=1} B_{\epsilon,\sigma}[u],$$
and let us proof that
$$\lambda_1^{\epsilon,\sigma} = O\left(\epsilon^2\right), \quad \epsilon\to 0.$$
Following the calculations in [AF, C, deMS], we obtain
$$\int_{I_\epsilon} J_\epsilon\left|\left(\frac{1}{\sqrt{J_\epsilon}}u\right)'\right|^2 dz = \|u'\|^2_{L^2(I_\epsilon)} + O\left(\epsilon^2\right)\|u\|^2_{L^2(I_\epsilon)} - \left.\frac{J'_\epsilon}{2J_\epsilon}u^2\right|_{\partial I_\epsilon}, \quad \epsilon\to 0,$$
for each $u\in H^1(I_\epsilon,\mathbb{C})$. It follows that
$$B_{\epsilon,\sigma}\left[\frac{1}{\sqrt{J_\epsilon}(\sigma,.)}u\right] = s_\epsilon[u] + O\left(\epsilon^2\right)\|u\|^2_{L^2(I_\epsilon)} - \left.\frac{J'_\epsilon}{2J_\epsilon}u^2\right|_{\partial I_\epsilon}, \quad \epsilon\to 0,$$
for each $u\in H^1(I_\epsilon)$. Hence,
$$\lambda_1^{\epsilon,\sigma} \le B_{\epsilon,\sigma}\left[\frac{1}{\sqrt{J_\epsilon}(\sigma,.)}\frac{\theta'_{13}}{\|\theta'_{13}\|_{L^2(I_\epsilon)}}\right] = s_\epsilon\left[\frac{\theta'_{13}}{\|\theta'_{13}\|_{L^2(I_\epsilon)}}\right] + O\left(\epsilon^2\right) = O\left(\epsilon^2\right), \quad \epsilon\to 0.$$
On the other hand, we have
$$\lambda_1^{\epsilon,\sigma} = B_{\epsilon,\sigma}[u] = s_\epsilon\left[u\sqrt{J_\epsilon}(\sigma,.)\right] + O\left(\epsilon^2\right) \ge \lambda_1^\epsilon + O\left(\epsilon^2\right) = O\left(\epsilon^2\right), \quad \epsilon\to 0,$$
for a normalized eigenfunction $u\in\ker\left(L_{\epsilon,\sigma}-\lambda_1^{\epsilon,\sigma}\right)$. Keeping this in mind, we obtain with the transformation lemma
$$\int_{S_2(\delta)} \epsilon|\nabla\psi|^2 + \frac{1}{\epsilon}\left\langle D^2W\left(\theta_{13}\left(\frac{d_2(x,t)}{\epsilon}\right)\right)\psi,\psi\right\rangle dx$$
$$\ge \int_{S_2} B_{\epsilon,\sigma}\left[\tilde{\psi}\right] d\sigma \ge -C\epsilon\|\psi\|^2_{L^2(\Omega)}.$$
$\square$

Remark 4.1. *The calculations for $B_{\epsilon,\sigma}$ can also be used for the case without triple-junction in order to obtain an estimate for the Allen-Cahn operator.*

Appendix A

Measure theory

In this section, let (X, μ) be a measure space. The set of measurable, integrable functions $f : X \to \mathbb{R}$ is denoted by $L^1(X, \mu)$.

Definition A.1. *Suppose $T \subset \mathbb{R}$ contains infinitely many elements, $f : T \to \mathbb{R}$ is bounded and let $a \in \overline{T}$ be a cluster point of T. If $a \in \mathbb{R}$, we define*

$$\liminf_{x \to a} f(x) := \lim_{n \to \infty} \left(\inf_{x \in K_{\frac{1}{n}}(a) \cap T} f(x) \right),$$

and if $a = \pm\infty$, we set

$$\liminf_{x \to a} f(x) := \lim_{n \to \infty} \left(\inf_{\substack{x \in T \\ \pm x \geq n}} f(x) \right).$$

Lemma A.1. *Suppose $(f_n)_{n \in \mathbb{N}} \subset L^1(X, \mu)$ is such that $0 \leq f_n$ a.e.,*

$$\int_X f_n d\mu \leq C < \infty, \quad n \in \mathbb{N},$$

and

$$f_n \to f, \quad \mu - f.\ddot{u}.$$

Then we have

$$f \in L^1(X, \mu), \text{ and } \int_X f d\mu \leq \liminf_{n \to \infty} \int_X f_n d\mu.$$

This is precisely the content of [We1, Satz A.13].

Theorem A.1. *Let $f_n, g \in L^1(X, \mu)$, $n \in \mathbb{N}$, such that*

$$|f_n(x)| \leq g(x), \quad a.e.,$$

and

$$f_n \to f.$$

Then we have

$$f \in L^1(X, \mu),$$

and

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu.$$

This is Lebesgue's convergence theorem, cf. [We1, Satz A.12].

Appendix B

Operator theory

Throughout this section, $(H, \langle ., . \rangle)$ denotes a complex Hilbert space and $T : D_T \subset H \to H$ a densely defined operator in H. The operator T is called symmetric, if $T \subset T^*$, and self-adjoint if $T = T^*$. We have the following spectral parts for a self-adjoint operator.

Definition B.1. *1. The resolvent set*

$$\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is bijective, } (T-\lambda)^{-1} \in \mathcal{L}(H)\}.$$

2. We call

$$\sigma(T) := \mathbb{C} \backslash \rho(T)$$

the spectrum of T.

3. The essential spectrum $\sigma_e(T)$ is the set of all eigenvalues with infinite multiplicity and cluster points of $\sigma(T)$.

4. The discrete spectrum of T is given by

$$\sigma_d(T) := \sigma(T) \backslash \sigma_e(T).$$

5. The point spectrum of T is given by

$$\sigma_p(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not injective } \}.$$

Proposition B.1. *Let T be a self-adjoint operator in H that is bounded from bellow by $\gamma \in \mathbb{R}$, i.e.*

$$\forall x \in D_T : \langle Tx, x \rangle \geq \gamma |x|^2.$$

Suppose T has eigenvalues $\lambda_0 < \lambda_1 < ... < \min \sigma_e(T)$. Then, we have

$$\lambda_j = \inf_{\substack{x \in D_T \cap K_{j-1}^{\perp} \\ x \neq 0}} \frac{\langle Tx, x \rangle}{|x|^2},$$

for $j \in \mathbb{N}_0$, *where*

$$K_i := \bigcup_{0 \le j \le i} \ker(T - \lambda_j), \quad i \in \mathbb{N}_0,$$

and

$$K_{-1} := \{0\}.$$

This is precisely the content of [HS, Proposition 12.1].

Definition B.2. *If* T *is a self-adjoint operator in* H*, then a sequence* $(u_n)_{n\in\mathbb{N}}$ *in* D_T *is called a Weyl-sequence for* T *and* $\lambda \in \mathbb{C}$ *if*

1. $\|u_n\| = 1$.
2. $u_n \xrightarrow{w} 0$.
3. $(T - \lambda)u_n \to 0$.

Theorem B.1. *Let* T *be a self-adjoint operator in* H*. Then* $\lambda \in \mathbb{C}$ *is in* $\sigma_e(T)$ *if and only if there exists a Weyl-sequence for* T *and* λ*.*

This is Weyl's criterion, cf. [HS, Theorem 7.2].

Definition B.3. *Let* T *and* S *be operators in* H *such that* $D_T \subset D_S$*.* S *is called* T*-compact (relatively compact with respect to* T*) if each sequence* $u_n \in D_T$ *that is bounded in the graph norm of* T *contains a subsequence* u_{n_k} *such that* $(Su_{n_k})_{k\in\mathbb{N}}$ *is convergent.*

Lemma B.1. *Let* T *be self-adjoint and* S *symmetric such that* S *is* T*-compact. Then* $T + S$ *is self-adjoint and*

$$\sigma_e(T) = \sigma_e(T + S).$$

For a proof see [We1, Satz 9.14].

Definition B.4.

1. *A bounded operator* $P \in \mathcal{L}(H)$ *is called orthogonal projection if*
   - *(a)* P *is idempotent, i.e* $P^2 = P$*.*
   - *(b)* P *is self-adjoint.*
2. *If* T *is self-adjoint we say that* T *and* P *commute if* $PT \subset TP$*.*

Corollary B.1. *If* T *is self-adjoint and* P *an orthogonal projection such that* $PT \subset TP$*, then* $T|_{\mathrm{im}(P)}$ *is self-adjoint in* $(\mathrm{im}(P), \langle ., . \rangle)$*, where* $D_{T|_{\mathrm{im}(P)}} = D_T \cap \mathrm{im}(P)$*.*

The reader is referred to [We1, Satz 2.60] for a proof.

Definition B.5. *Let $\Omega \subset \mathbb{R}^d$ be a nonempty measurable subset. Assume $V : \Omega \to \mathbb{C}$ is measurable. Define the multiplication operator $[V]$ by*

$$D_{[V]} := \{u \in L^2(\Omega, \mathbb{C}) : V \cdot u \in L^2(\Omega, \mathbb{C})\},$$

and

$$[V]u := V \cdot u, \quad u \in D_{[V]}.$$

Corollary B.2. *The operator $[V]$ is densely defined and fulfills $[V]^* = [\overline{V}]$. If $V \in L^\infty(\Omega, \mathbb{C})$, then $V \in \mathcal{L}\left(L^2(\Omega, \mathbb{C})\right)$.*

The reader is referred to [We1, Satz 6.1] for the proof.

Definition B.6. *Let $m, d \in \mathbb{N}$, and suppose $\Omega \subset \mathbb{R}^d$ is an open set. A mapping*

$$V : \Omega \to M(m, \mathbb{C})$$

is called measurable if there exist measurable functions $V_{ij} : \Omega \to \mathbb{C}$, $1 \leq i, j \leq m$, such that

$$\forall x \in \Omega : V(x) = \left(V_{ij}(x)\right)_{1 \leq i,j \leq m}.$$

If $V : \Omega \to M(m, \mathbb{C})$ is measurable, we define the multiplication operator $[V]$ in $\left(L^2(\Omega, \mathbb{C})\right)^m$ by

$$D_{[V]} := \left\{u \in \left(L^2(\Omega, \mathbb{C})\right)^m : V \cdot u \in \left(L^2(\Omega, \mathbb{C})\right)^m\right\},$$

and

$$[V]u := \left(\sum_{j=1}^{d} [V_{ij}]u_j\right)_{i=1,\ldots,m}, \quad u \in D_{[V]}.$$

Corollary B.3. *If $V : \Omega \to S\left(\mathbb{R}^m\right)$ is measurable, then $[V]$ is symmetric.*

Proof. Let $u, v \in D_{[V]}$. Then

$$\langle [V]u, v \rangle_{L^2(\Omega)} = \int_\Omega \langle (V \cdot u)(x), v(x) \rangle \, dx =$$

$$\int_\Omega \langle u(x), (V \cdot v)(x) \rangle \, dx = \langle u, [V]v \rangle_{L^2(\Omega)}.$$

□

Proposition B.2. *Suppose $V \in C\left(\mathbb{R}^d, M(m, \mathbb{K})\right)$ is bounded, i.e.*

$$\sup_{x \in \mathbb{R}^d} \|V(x)\|_{tr} < +\infty,$$

and for each $\epsilon > 0$, there exists a compact set $K \subset \mathbb{R}^d$ such that

$$\sup_{x \in \mathbb{R}^d \setminus K} \|V(x)\|_{tr} < \epsilon.$$

Then,

$$[V] \in \mathcal{L}\left(\left(L^2(\mathbb{R}^d,\mathbb{C})\right)^m\right),$$

and for each $k \in \mathbb{N}$, the restriction of $[V]|_{(H^k(\mathbb{R}^d,\mathbb{C}))^m}$ is a compact operator from $\left(H^k(\mathbb{R}^d,\mathbb{C})\right)^m$ in $\left(L^2(\mathbb{R}^d,\mathbb{C})\right)^m$.

Proof. Without loss of generality, we restrict our considerations to the case $m = 1$. Consider the inclusion

$$i : H^k(\mathbb{R}^d,\mathbb{C}) \to L^2(\mathbb{R}^d,\mathbb{C}).$$

Choose a sequence $\chi_n \in C_0^\infty(\mathbb{R}^d,\mathbb{R})$, $n \in \mathbb{N}$, such that

$$1_{\overline{K_n(0)}} \leq \chi_n \leq 1_{K_{n+1}(0)}.$$

Set $V_n := \chi_n \cdot V$ and let us prove that

$$[V_n] \to [V], \quad n \to \infty, \tag{B.1}$$

in $\mathcal{L}\left(L^2(\mathbb{R}^d,\mathbb{C})\right)$. For arbitrary $u \in L^2(\mathbb{R}^d,\mathbb{C})$, we have

$$\|V(1-\chi_n)u\|_{L^2(\mathbb{R}^d)} \leq \sup_{|x|\geq n} \|V(x)\|_{tr}\|u\|_{L^2(\mathbb{R}^d)}.$$

Choose $\epsilon > 0$ arbitrarily. Then there exists $n \in \mathbb{N}$ such that

$$\sup_{|x|\geq n} |V(x)| < \epsilon,$$

hence

$$\|[V_n] - [V]\| < \epsilon.$$

This proves (B.1). Our task is now to prove that $[V_n]$ is a compact operator. Let $(\eta_\epsilon)_{\epsilon\in(0,1)}$ be any sequence of mollifiers. Then

$$V_n * \eta_\epsilon \to V_n$$

uniformly. Therefore

$$[V_n * \eta_\epsilon] \to [V_n]$$

in $\mathcal{L}\left(L^2(\mathbb{R}^d,\mathbb{C})\right)$. But $V_n * \eta_\epsilon \in C_0^\infty(\mathbb{R}^d,\mathbb{R})$. Hence $[V_n * \eta_\epsilon] \circ i$ is compact. We habe

$$[V_n * \eta_\epsilon] \circ i \to [V] \circ i.$$

As the set of compact operators is closed, we conclude that $[V]$ is a compact operator from $H^k(\mathbb{R}^d,\mathbb{C})$ in $L^2(\mathbb{R}^d,\mathbb{C})$. $\square$

Definition B.7. *Suppose $\Omega \subset \mathbb{R}^n$ is an open bounded subset such that $\partial\Omega \in C^2$. Define*

$$D_\triangle := \left\{ u \in H^2(\Omega, \mathbb{C}) : \frac{\partial}{\partial \nu} u = 0 \ \text{on} \ \partial\Omega \right\}.$$

If $\Omega = \mathbb{R}^n$, set

$$D_\triangle := H^2(\mathbb{R}^n, \mathbb{C}).$$

Define for each $u \in D_\triangle$

$$\triangle u := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u.$$

Corollary B.4. 1. *The operator $\triangle$ is self-adjoint.*

2. *If $\Omega = \mathbb{R}^n$, and $V \in L^\infty(\mathbb{R}^n, \mathbb{R})$, then the operator $T := -\triangle + [V]$ is self-adjoint and fulfills*

$$\inf \sigma_e(T) = \sup_{\substack{K \subset \mathbb{R}^d \\ \text{compact}}} \inf \left\{ \frac{\langle T\phi, \phi \rangle}{|\phi|^2} : \phi \in C_0^\infty(\mathbb{R}^d \backslash K), \neq 0 \right\}.$$

A proof for the statement on the essential spectrum can be found in [A, Theorem 3.2]. The self-adjointness follows from [L, Theorem 3.1.3].

Definition B.8. *Suppose $\Omega \subset \mathbb{R}^n$ is open.*

1. *Define*

$$D_\nabla := H^1(\Omega, \mathbb{C}),$$

and

$$\nabla u := \left(\frac{\partial}{\partial x_i} u \right)_{i=1,\dots,n}.$$

2. *Assume either that Ω is bounded and $\partial\Omega \in C^1$ or $\Omega = \mathbb{R}^n$. Accordingly, set*

$$D_{\mathrm{div}} := \left\{ u \in (H^1(\Omega, \mathbb{C}))^n : \langle u, \nu \rangle = 0 \ \text{on} \ \partial\Omega \right\},$$

or

$$D_{\mathrm{div}} := (H^1(\Omega, \mathbb{C}))^n,$$

and

$$\mathrm{div}\ u := \sum_{i=1}^{n} \frac{\partial}{\partial x_i} u_i.$$

Remark B.1. *Note that ∇ is an operator from $L^2(\Omega, \mathbb{C})$ to $\left(L^2(\Omega, \mathbb{C})\right)^n$, and* div *is an operator from $\left(L^2(\Omega, \mathbb{C})\right)^n$ to $L^2(\Omega, \mathbb{C})$.*

Corollary B.5. *Suppose either $\Omega \subset \mathbb{R}^n$ is open, bounded and $\partial\Omega \in C^1$ or $\Omega = \mathbb{R}^n$. Then*

1. $i\mathrm{div} \subset (i\nabla)^*$,
2. $-\triangle = (i\nabla)^*(i\nabla)$.

Proof. First, we prove part one. Choose $v \in D_{\mathrm{div}}$ and $u \in D_\nabla$. In view of Green's formula, we obtain

$$\langle i\nabla u, v\rangle_{L^2(\Omega)} = \int_\Omega u(x)\overline{i\mathrm{div}v}(x)dx + \int_{\partial\Omega} u\,\langle v,\nu\rangle\,dS = \langle u, i\mathrm{div}\ v\rangle_{L^2(\Omega)}\,.$$

This proves assertion one. Our task is now to prove statement two. The graph norm of ∇ is equivalent to $\|.\|_{H^1(\Omega)}$. It follows that ∇ is closed. On account of [We1, Satz 4.11], we conclude that $(i\nabla)^*(i\nabla)$ is self-adjoint. Owing to statement one of this corollary, we obtain

$$(i\nabla)^*(i\nabla) \supset -\triangle.$$

As both operators are self-adjoint, the assertion follows. □

Remark B.2. *As long as no confusion occurs, we denote $\otimes_{i=1}^m \nabla$ and $\otimes_{i=1}^m \mathrm{div}$ with ∇ and div, respectively.*

Appendix C

Sesquilinear forms

Definition C.1. *1. A sesquilinear form in H is a sesquilinear map*

$$s : D_s \times D_s \to \mathbb{C}$$

such that $D_s \subset H$ a dense linear subspace. We call D_s the domain of s.

2. *A sesquilinear form in H is called symmetric if*

$$s[x,y] = \overline{s[y,x]}, \quad x, y \in D_s.$$

We say that a symmetric sesquilinear form s is bounded from bellow with $\gamma \in \mathbb{R}$ ($s \geq \gamma$) if

$$s[x,x] \geq \gamma |x|^2, \quad x \in D_s.$$

3. *Suppose s is a sesquilinear form in H. Then s is called closed, if for each sequence $(u_n)_{n\in\mathbb{N}} \subset D_s$ and $u \in H$ such that*

$$u_n \to u,$$

and

$$s[u_n - u_m] \to 0,$$

i.e. $u_n \xrightarrow{s} u$, we have $u \in D_s$ and $s[u_n - u] \to 0$.

4. *A densely defined sesquilinear form s in H is called closable, if it has a closed extension, i.e. there exists a closed form t such that $s \subset t$. In this case the closure of s is the smallest closed extension of s and is denoted by $\overline{s}$.*

5. *Assume s is a closed form in H. Then a linear subspace $D \subset H$ is called a core of s if the closure of $s|_D$ is s.*

The following definition gives an explicit representation of the closure.

Definition C.2. *Assume s is a closable sesquilinear form in H. Then we define*

$$D_{\overline{s}} := \{u \in H : \exists (u_n)_{n\in\mathbb{N}} \subset D_s : u_n \xrightarrow{s} u\}.$$

If $u, v \in D_{\overline{s}}$ such that $u_n, v_n \in D_s$, $u_n \xrightarrow{s} u$, and $v_n \xrightarrow{s} v$, we define

$$\overline{s}[u,v] := \lim_{n\to\infty} s[u_n, v_n].$$

Definition C.3. *Suppose s is a closed symmetric sesquilinear form in H. Define*

$$D_{T_s} := \{x \in D_s : y \in D_s \longmapsto s[x,y] \text{ is continuous }\}.$$

If $x \in D_{T_s}$, then there exists a unique element $T_s x \in H$ such that

$$s[x,y] = \langle T_s x, y\rangle, \quad y \in D_s.$$

Lemma C.1.

1. *Let s and T_s be as above. Then T_s is a self-adjoint operator in H that is bounded from bellow. We call T_s the operator that is associated to s.*

2. *Assume T is a symmetric operator in H, bounded from bellow. Then the sesquilinear form*
$$D_{s_T} := D_T,$$
$$s_T[x,y] := \langle Tx, y\rangle$$
is closable. The operator $T_{\overline{s_T}}$ is called Friedrichs extension of T and fulfills $T_{\overline{s_T}} \subset T^*$.

For a proof, the reader is referred to [K, VI, Theorem 2.1] or [We1, Satz 4.14].

Remark C.1. *If T is self-adjoint and bounded from bellow, then*

$$T_{\overline{s_T}} = T.$$

Corollary C.1. *Let T be a self-adjoint operator in H, bounded from bellow by zero. Suppose T has eigenvalues $\lambda_0 < \lambda_1 < ... < \min \sigma_e(T)$. Then, we have*

$$\lambda_j = \inf_{\substack{x\in D_t\cap K_{j-1}^{\perp} \\ |x|=1}} \overline{s_T}[x], \quad j \in \mathbb{N}_0,$$

where K_j, $j \in \mathbb{Z}, \geq -1$, is defined as in proposition B.1.

Proof. Let P_j be the orthogonal projection on $K_{j-1}^{\perp}$. Chose $x \in D_t \cap K_{j-1}^{\perp}$ arbitrarily. As D_T is a core of t, there exists a sequence $x_n \in D_T$ such that $x_n \to x$ and

$$t[x_n] = \langle Tx_n, x_n\rangle \to t[x].$$

Without loss of generality we suppose $x_n \in K_{j-1}^{\perp}$, otherwise we consider $P_j x_n$. Note that P_j and T commute. In view of [K, Theorem 3.35], it follows that P_j and $T^{\frac{1}{2}}$ commute. Moreover, we can restrict our considerations to the case where x_n is normalized. Due to proposition B.1 we have

$$\langle T x_n, x_n \rangle \geq \lambda_j,$$

hence

$$t[x] \geq \lambda_j.$$

Thus

$$\lambda_j \leq \inf_{\substack{x \in D_t \cap K_{j-1}^{\perp} \\ |x|=1}} t[x], \quad j \in \mathbb{N}_0.$$

According to proposition B.1 the converse is also true. $\square$

Definition C.4. *Let S and T be self-adjoint operators in H, both being bounded from bellow. Denote the associated closed forms by s and t, respectively. We say*

$$S \leq T :\Leftrightarrow D_t \subset D_s \text{ and } s[x,x] \leq t[x,x], \quad x \in D_t.$$

Lemma C.2. *Suppose S and T are self-adjoint operators in H, both bounded from bellow. Then $S \leq T$ implies*

$$\min \sigma_e(S) \leq \min \sigma_e(T).$$

This lemma is part b of [We1, Satz 8.34].

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