Milnor K-theory of local rings

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Wann, wenn nicht jetzt, sollen wir den Stein schleudern
gegen Goliaths Stirn?
Primo Levi

Summary

This thesis examines Milnor K-theory of local rings. We will prove the Beilinson-Lichtenbaum conjecture relating Milnor K-groups of equicharacteristic regular local rings with infinite residue fields to motivic cohomology groups, the Gersten conjecture for Milnor K-theory and in the finite residue field case we will show that (n, n)-motivic cohomology of an equicharacteristic regular local ring is generated by elements of degree 1.

Milnor K-theory of fields originated in Milnor's seminal Inventiones article from 1970 [28]. There he defined Milnor K-groups and proposed his famous conjectures, now known as the Milnor conjectures, which on the one hand relate Milnor K-theory to quadratic forms and on the other hand to Galois cohomology. Following Milnor's ideas the theory of Milnor K-groups of fields developed swiftly. Starting with Bass and Tate [2] a norm homomorphism for K-groups of finite field extensions was defined and Milnor K-groups of local and global fields were calculated. In arithmetic it was observed by Parshin, Bloch, Kato and Saito in the late 1970s that Milnor K-groups could be used to define class groups of arithmetic schemes. Already then it became obvious that for a satisfying higher global class field theory it was necessary to consider Milnor K-groups of local rings and the Milnor K-sheaf for some Grothendieck topology instead of working only with K-groups of fields [17].

In another direction it was observed by Suslin in the early 1980s that up to torsion Milnor K-groups of fields are direct summands of Quillen K-groups. Later Suslin, revisiting his earlier work, observed in collaboration with Nesterenko [29] that his results could easily be generalized to Milnor K-groups of local rings. The latter type of result led Beilinson and Lichtenbaum to their conjecture on the existence of a motivic cohomology theory of smooth varieties [3] which they predicted should be related to Milnor K-groups of local rings. More precisely they conjectured that for an essentially smooth local rings A over a field there should be an isomorphism

$$K_n^M(A) \xrightarrow{\sim} H^n_{mot}(A, \mathbb{Z}(n)) \tag{$\aleph$}$$

between Milnor K-groups and motivic cohomology.

In these two directions, in which Milnor K-groups of local rings were first introduced, a naive generalization of Milnor's original definition for fields was used. Namely for a local ring A we let $T(A^\times)$ be the tensor algebra over the units of A and define the graded ring $K_*^M(A)$ to be the quotient of $T(A^\times)$ by the two-sided ideal generated by elements of the form $a \otimes (1-a)$ with $a, 1-a \in A^\times$. Nevertheless, it was observed by the experts that this is not a proper K-theory if the residue field of A is very small (contains less than 4 elements) [13, Appendix]; for example the map in ($\aleph$) is not an isomorphism then.

Our aim in this thesis is twofold. Firstly, we will prove in Chapter 3 that there is an isomorphism ($\aleph$) if the residue field of A is infinite, establishing a conjecture of Beilinson and Lichtenbaum. Secondly, we will show in Chapter 4 that if we factor out more relations in the definition of Milnor K-groups of local rings we get a sensible theory for arbitrary residue fields. The former result will be deduced from the exactness of the Gersten complex for Milnor

K-theory: Let A be an excellent local ring, $X = Spec(A)$ with generic point η and $X^{(i)}$ the set of points of X of codimension i. Then Kato [16] constructed a so called Gersten complex

$$0 \longrightarrow K_n^M(A) \longrightarrow K_n^M(k(\eta)) \longrightarrow \oplus_{x \in X^{(1)}} K_{n-1}^M(k(x)) \longrightarrow \cdots$$

The exactness of the Gersten complex for A regular, equicharacteristic and with infinite residue field, also known as the Gersten conjecture for Milnor K-theory, is of independent geometric interest and one of the further main results of this thesis. For a detailed overview of our results we refer to Sections 3.1 and 4.1.

The first two chapters are preliminary. Chapter 1 recalls some results on inverse limits of schemes and sketches a definition of motivic cohomology of regular schemes along the lines of Voevodsky's approach. This construction seems to be well known to the experts but is nowhere explicated in the literature. Chapter 2 contains a collection of motivational results on Milnor K-theory of fields some of which have been generalized at least conjecturally to local rings. We will prove a part of these conjectures in this theses.

The results which are proved in this thesis will be published in [19] and [20].

Acknowledgment

I developed the idea for the proof of the Gersten conjecture for Milnor K-theory during the work on my diploma thesis at the University of Mainz. I would like to thank Stefan Müller-Stach, the advisor for my diploma thesis, who contributed many ideas exploited here. Also I am deeply indebted to Burt Totaro for supporting me with a lot of mathematical improvements and the opportunity to work in Cambridge where part of this thesis was written. Stephen Lichtenbaum and Burt Totaro explained to me that even in the finite residue case (n, n)-motivic cohomology of a regular local ring should have a symbolic description and that this was expected as part of the fantastic Beilinson-Lichtenbaum program on motivic cohomology. This was in fact the initial motivation for Chapter 4. I am indebted to Wilberd van der Kallen for explaining me the different presentations available for K_2 of a local ring. Finally, but most cordially, I thank Uwe Jannsen for many helpful comments and his encouragement during the work on this thesis.

During the last years I profited from a scholarship of the Studienstiftung des deutschen Volkes.

Contents

Chapter 1

Preliminaries

One of the basic ingredients of our proof of the Gersten conjecture for Milnor K-theory as presented in Chapter 3 will be a variant of Noether normalization due to Ofer Gabber, although in the weak form that we state it is not clear what it has to do with Noether's theorem. A proof of a more general version can be found in [4].

Proposition 1.0.1 (Gabber)**.** *If X is an affine smooth connected variety of dimension d over an infinite field k and $Z \subset X$ is a finite set of closed points then there exists a k-morphism $f : X \to \mathbb{A}^d_k$ which is étale around the points in Z and induces an isomorphism of (reduced) schemes $Z \xrightarrow{\sim} f(Z)$.*

A further ingredient in Chapter 3 will be the reduction of the 'regular' problem to a smooth problem over a finite field. This is accomplished by a fascinating method due to Popescu. A proof of the next proposition can be found in [37]. Recall that a homomorphism of Noetherian rings $A \to B$ is called regular if the geometric fibers of $Spec(B) \to Spec(A)$ are regular.

Proposition 1.0.2 (Popescu)**.** *If the homomorphism $f : A \to B$ of Noetherian rings is regular there exists a filtering direct system $f_i : A_i \to B_i$ of smooth homomorphisms of Noetherian rings with $\lim_{\to} f_i = f$.*

The version of Popescu's theorem that we need is the following. A ring is called essentially smooth over a field k if it is the localization of a smooth affine k-algebra.

Corollary 1.0.3. *Let A be a regular semi-local ring containing a field k which is finite over its prime field. Then A is the filtering direct limit of essentially smooth semi-local rings A_i/k.*

Proof. By the proposition we can construct a filtering direct limit A'_i/k of smooth affine k-algebras with $\lim_{\to} A'_i = A$. Let A_i be the localization of A'_i at the inverse image of the maximal ideals of A. □

When we use Popescu's theorem in a reduction argument we have to assure that our cohomology theories commute with filtering direct limits of rings. This commutativity is validated by means of Grothendieck's fancy limit theorem [SGA IV/2, Exposé VI, Theorem 8.7.3].

Proposition 1.0.4 (Grothendieck's limit theorem)**.** *Let I be a small filtering category and $p : (F \to I, A)$, $q : (G \to I, B)$ ringed fibred topoi. Let $m : p \to q$ be a morphism of topoi* *such that for $f : i \to j$ in I the derived homomorphisms $R^n f_* : \mathrm{Mod}(F_i, A_i) \to \mathrm{Mod}(F_j, A_j)$ and $R^n m_{i*} : \mathrm{Mod}(F_i, A_i) \to \mathrm{Mod}(G_i, B_i)$ commute with small filtering direct limits. Then for every A-module $i \mapsto M_i$ in $Top(p)$ we have*

$$R^n m_* Q^*(j \mapsto M_j) \cong \varinjlim_{I^\circ} \mu_j^* R^n m_{j*}(M_j)$$

where $Top(p)$ is the total topos of the fibered topos. By definition Q and μ_i are the morphisms of topoi from the diagram

$$\begin{array}{ccc} \overset{p}{\leftarrow} & \xrightarrow{\mu_i} & F_i \\ {\scriptstyle Q}\downarrow & & \\ Top(p) & & \end{array}$$

From this proposition we can extract:

Corollary 1.0.5. *Let X_i be a filtering inverse limit of affine Noetherian schemes with $\lim_{\to} X_i = X$ Noetherian and let $(F_i)_i$ be a compatible system of Zariski sheaves on the schemes X_i with limit sheaf F on X. Then the natural map*

$$\varinjlim H^n(X_i, F_i) \longrightarrow H^n(X, F)$$

is an isomorphism.

Proof. Let in the proposition $p : F \to I$ be the fibered topos of sheaves on the schemes X_i ($i \in I$) and $q : G \to I$ the constant fibered topos of sets. The ring objects A and B are just set to be $\mathbb{Z}$. We know from [**?**] that for a Noetherian scheme Y and a filtering direct limit of sheaves $(G_i)_i$ on Y with limit G we have for $n \geq 0$

$$\varinjlim H^n(Y, G_i) \cong H^n(Y, G).$$

It follows immediately from this continuity of Zariski cohomology that $R^n f_*$ and $R^n m_{i*}$ are continuous. Furthermore we claim that the ringed topos $\overset{p}{\leftarrow}$ is isomorphic to the ringed topos of Zariski sheaves on X. In order to see this let $\pi_i : X \to X_i$ be the projection. The map which associates to an inverse system $(G_i)_i \in \overset{p}{\leftarrow}$ of sheaves on the schemes X_i the sheaf $\lim_{\longrightarrow} \pi_i^*(G_i)$ is an isomorphism of topoi because X is Noetherian – same argument as before. □

Let F be a covariant functor from rings to abelian groups.

Definition 1.0.6. *The functor F is called continuous if for every filtering direct limit of rings*

$$A = \varinjlim A_i$$

the natural homomorphism

$$\varinjlim F(A_i) \longrightarrow F(A)$$

is an isomorphism.

A (pre-)sheaf on a subcategory of the category of schemes is called continuous if its restriction to affine schemes is continuous in the above sense.

Our final and most important aim in this preliminary chapter is to define motivic cohomology of regular schemes. As our primary interest is in Milnor K-theory and not in motivic cohomology we do only sketch the necessary constructions. Unfortunately, a comprehensive account of the theory over general base schemes has not yet appeared. In case we are interested in smooth varieties over fields a good reference is [25]. In the following few paragraphs we will generalize the theory explained there to regular base schemes.

Let S be a regular scheme, recall that this means in particular that S is Noetherian, and let $Sm(S)$ be the category of schemes smooth, separated and of finite type over S. For $X \in Sm(X)$ we will denote by $c_0(X/S)$ the free abelian group generated by the closed irreducible subschemes of X which are finite over S and dominate an irreducible component of S. Consider a Cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ T & \xrightarrow[g]{} & S \end{array}$$

with $X \in Sm(S)$ and $Y \in Sm(T)$. Here $g : T \to S$ is an arbitrary morphism of regular schemes. Then using Serre's Tor-formula or any other device which produces multiplicities in this generality one can define in a canonical way a functorial pullback $f^* : c_0(X/S) \to c_0(Y/T)$, for details we refer to [5, Section 1] or [25, Appendix 1A]. There does also exist a functorial pushforward. If X is as above we let $\mathbb{Z}_{tr}(X)$ be the presheaf on $Sm(S)$ defined by

$$U \mapsto c_0(X \times_S U/U) \; .$$

This is in fact a Zariski sheaf. By $\mathbb{G}_m$ we mean the sheaf $\mathbb{Z}_{tr}(\mathbb{A}^1_X - \{0\})$ and by $\mathbb{G}_m^{\wedge n}$ we mean the quotient sheaf of $\mathbb{Z}_{tr}((\mathbb{A}^1_X - \{0\})^{\times n})$ by the subsheaf generated by the embeddings $(\mathbb{A}^1_X - \{0\})^{\times n-1} \to (\mathbb{A}^1_X - \{0\})^{\times n}$ where one has the constant map 1 at one factor of the image. For any presheaf F we let $C_*(F)$ be simplicial presheaf $C_i(F)(U) = F(\mathbb{A}^i_U)$. Then Voevodsky's motivic complex of Zariski sheaves $\mathbb{Z}(n)$ on $Sm(S)$ is defined to be the ascending cochain complex associated to the chain complex $C_*\mathbb{G}_m^{\wedge n}$ and we shift $\mathbb{G}_m^{\wedge n}$ to degree n.

Definition 1.0.7. *For a regular scheme S motivic cohomology $H^m_{mot}(S, \mathbb{Z}(n))$ is defined as the Zariski hypercohomology $\mathbb{H}^m(S, \mathbb{Z}(n))$.*

Sometimes $H^m_{mot}(S, \mathbb{Z}(n))$ is denoted by $H^{m,n}(S)$ and if $S = Spec(A)$ is affine we write $H^m_{mot}(A, \mathbb{Z}(n))$ for the motivic cohomology of S. One can define a bigraded ring structure on motivic cohomology. The following lemma is standard.

Lemma 1.0.8. *Motivic cohomology is continuous on regular rings.*

Proof. Let G be an arbitrary complex of Zariski sheaves on a scheme X and let $\tau_{\geq i}(G)$ be the brutal truncation, i.e. we have $\tau_{\geq i}(G)_j = 0$ for $j < i$ and $\tau_{\geq i}(G)_j = G_j$ for $j \geq i$. Then the complexes $(\tau_{\geq i}(G))_{i \in \mathbb{Z}}$ form a direct system and we know from [EGA III, Chapter 0, Lemma 11.5.1] that Zariski hypercohomology commutes with this limit, i.e. we have for $n \in \mathbb{Z}$

$$\varinjlim_i \mathbb{H}^n(X, \tau_{\geq i}(G)) \cong \mathbb{H}^n(X, G) \; .$$

Applying this to our situation we see that it is sufficient to prove that for fixed $i \in \mathbb{Z}$ and for a filtering direct limit of regular affine schemes S_j with regular limit S the map

$$\varinjlim_j \mathbb{H}^m(S_j, \tau_{\geq i}(\mathbb{Z}(n))) \longrightarrow \mathbb{H}^m(S, \tau_{\geq i}(\mathbb{Z}(n)))$$

is an isomorphism. By the convergent spectral sequence

$$E_1^{l,k} = H^l(S_j, (\tau_{\geq i}(\mathbb{Z}(n)))_k) \Longrightarrow \mathbb{H}^{l+k}(S_j, \tau_{\geq i}(\mathbb{Z}(n)))$$

we are reduced to show continuity of the following functors on regular schemes $X \mapsto H^m(X, \mathbb{Z}(n)_i)$ for all $m, n, i \in \mathbb{Z}$. But since the sheaves $\mathbb{Z}(n)_i$ commute with filtering direct limits of regular schemes (the sections are just given by certain cycles which are defined by a finite number of equations) the lemma follows from Corollary 1.0.5. □

Proposition 1.0.9. *For an essentially smooth semi-local ring A over a field, $X = Spec(A)$, and $m, n \geq 0$ the Gersten complex*

$$\begin{aligned} 0 \longrightarrow H^m_{mot}(Spec(A), \mathbb{Z}(n)) &\longrightarrow \oplus_{x \in X^{(0)}} H^m_{mot}(x, \mathbb{Z}(n)) \\ &\longrightarrow \oplus_{x \in X^{(1)}} H^{m-1}_{mot}(x, \mathbb{Z}(n-1)) \longrightarrow \cdots \end{aligned}$$

is universally exact.

For the construction of the Gersten complex as well as its exactness see the elaboration of arguments of Gabber in [4]. Here $X^{(i)}$ is the set of points of codimension i in X and $H^m_{mot}(x, \mathbb{Z}(n))$ is the motivic cohomology of the residue field $k(x)$. Observe that $\mathbb{Z}(n) = 0$ for $n < 0$.

For the convenience of the reader we recall the definition of universal exactness from [4].

Definition 1.0.10. *Let*

$$A' \longrightarrow A \longrightarrow A''$$

be a sequence of abelian groups. We say this sequence is universally exact if

$$F(A') \longrightarrow F(A) \longrightarrow F(A'')$$

*is exact for every additive functor F : **Ab** $\to$ **B** which commutes with filtering small colimits. Here we assume **B** is an abelian category satisfying AB5 (see [10]).*

For a regular ring A we have a natural map

$$A^\times \longrightarrow H^{1,1}(Spec(A))$$

defined by sending $a \in A^\times$ to the constant correspondence $a \in (\mathbb{A}^1_A - \{0\})(A)$.

Proposition 1.0.11. *If the regular ring A contains a field the map*

$$A^\times \to H^{1,1}(Spec(A))$$

is an isomorphism.

Proof. Let k be the prime field in A and let A_i/k be a filtering direct system of smooth affine algebras with direct limit A. Then $A^\times \to H^{1,1}(Spec(A))$ is the direct limit of the maps $A_i^\times \to H^{1,1}(Spec(A_i))$ which we know are isomorphisms by [25, Lecture 4] □

Chapter 2

Milnor K-theory of fields

In this chapter we recall some properties of Milnor K-theory of fields. Milnor K-theory of fields started with Milnor's influential article [28]. There he defined the K-groups, explained their connection to quadratic forms and Galois cohomology and stated his fundamental Milnor conjecture which was proved by Voevodsky [43]. This chapter is divided into an elementary part and a motivic part. Here elementary means that we collect together a few simple properties of Milnor K-groups which can be proved by straightforward symbolic arguments. On the other hand motivic properties of Milnor K-groups are those which are predicted by or connected to the Beilinson-Lichtenbaum program on motivic cohomology [3] or which have a geometric flavour.

We will follow [28] and [45] in our presentation of this well known material. Especially, we refer to these two treatises for proofs or further references.

2.1 Elementary theory

For a field F we let

$$T(F^\times) = \mathbb{Z} \oplus F^\times \oplus (F^\times \otimes F^\times) \oplus \cdots$$

be the tensor algebra over the $\mathbb{Z}$-module $F^\times$. Let I be the two-sided homogeneous ideal in $T(F^\times)$ generated by elements $a \otimes (1-a)$ with $a, 1-a \in F^\times$. Elements of I are usually called Steinberg relations.

Definition 2.1.1. *The Milnor K-groups of a field F are defined to be the graded ring*

$$K_*^M(F) = T(F^\times)/I\ .$$

The residue class of an element $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ in $K_n^M(A)$ is denoted $\{a_1, a_2, \ldots, a_n\}$. It is immediate that $K_0^M(F) = \mathbb{Z}$ and that $K_1^M(F) = F^\times$. For an inclusion of field $F \hookrightarrow E$ there is a natural homomorphism of graded rings $K_*^M(F) \to K_*^M(E)$.

Lemma 2.1.2. *The following relations hold.*

- *If $x \in K_n^M(F)$ and $y \in K_m^M(F)$*

$$x\,y = (-1)^{n\,m} y\,x\ .$$

- *For* $a \in F^\times$ *we have* $\{a, -a\} = \{a, -1\}$.
- *If* $a_1, \ldots, a_n \in F^\times$ *and* $a_1 + \cdots + a_n$ *is either* 0 *or* 1 *we have*

$$0 = \{a_1, \ldots, a_n\} \in K_n^M(F) .$$

Examples 2.1.3. *For special fields we know the following.*

- *For a finite field* F *we have* $K_n^M(F) = 0$ *for* $n > 1$.
- *For a number field* F *we have* $K_n^M(F) = (\mathbb{Z}/2)^{r_1}$ *for* $n > 2$ *where* r_1 *is the number of embeddings* $F \hookrightarrow \mathbb{R}$.
- *For a local field* F *with finite residue field* $K_n^M(F)$ *is uniquely divisible for* $n > 2$ *and*

$$K_2^M(F) = \mu_\infty(F) \oplus div$$

where div *is uniquely divisible and* $\mu_\infty(F)$ *are the roots of unity of* F.

For a discretely valued field (F, v) with ring of integers A and prime element π there exists a unique group homomorphisms

$$K_n^M(F) \xrightarrow{\partial} K_{n-1}^M(A/(\pi))$$

such that for $u_i \in A^\times$

$$\partial\{\pi, u_2, \ldots, u_n\} = \{\bar{u}_2, \ldots, \bar{u}_n\} \quad \text{and} \quad \partial\{u_1, \ldots, u_n\} = 0 .$$

The next proposition is one of the basic results in Milnor K-theory due to Milnor [28].

Proposition 2.1.4 (Milnor)**.** *For a field* F *the sequence*

$$0 \longrightarrow K_n^M(F) \longrightarrow K_n^M(F(t)) \xrightarrow{\partial} \bigoplus_{\pi} K_{n-1}^M(F[t]/(\pi)) \longrightarrow 0$$

is split exact. Here the sum is over all irreducible, monic $\pi \in F[t]$.

A fundamental step in Chapter 3 will be the generalization of this sequence from a field F to a local ring.

In a standard way this sequence allows us to define a norm $N_{E/F} : K_n^M(E) \to K_n^M(F)$, also called transfer, for a finite extension $F \subset E$ of fields, compare Section 3.5. In fact this norm depends a priori on generators of E over F but Kazuya Kato [15] showed that the norm is independent of the choice of these generators. Our constructions in Chapter 3 will allow us to generalize this norm to a norm of Milnor K-groups of finite etale extensions of local rings, but unfortunately we cannot show that this norm is independent from the choice of generators unless the local rings are equicharacteristic, i.e. contain a field. For further elaborations of elementary Milnor K-theory the reader is referred to the literature or to [18].

2.2 Motivic theory

In the very last paragraph of Chapter 1 we explained that for a regular ring A there exists a homomorphism $A^{\times} \to H^1_{mot}(A, \mathbb{Z}(1))$. It is shown in [25, Proposition 5.9] that for a field F the resulting map

$$T(F^{\times})_n \to H^n_{mot}(F, \mathbb{Z}(n))$$

factors through $K^M_n(F)$.

Proposition 2.2.1. *For a field F the canonical map*

$$K^M_n(F) \xrightarrow{\sim} H^n_{mot}(F, \mathbb{Z}(n))$$

is an isomorphism.

A direct proof of this proposition can be found in [25, Theorem 5.1]. Originally, it was shown by Nesterenko and Suslin [29] and Totaro [39] who used Bloch's higher Chow groups in order to define motivic cohomology. It was shown later that both versions of motivic cohomology are isomorphic. In chapter 3 we will generalize this proposition to regular local rings containing a field.

Earlier Suslin and Soulé [34] had already obtained the following result which, as will be explained below, in modern terms can be seen as a version with rational coefficients of the last proposition.

Proposition 2.2.2. *For a field F there are natural homomorphisms $K^M_n(F) \to K_n(F)$ and $K_n(F) \to K^M_n(F)$ from Milnor K-theory to Quillen K-theory and vice versa such that the composition*

$$K^M_n(F) \to K_n(F) \to K^M_n(F)$$

is multiplication by $(n-1)!$. The image of $K^M_n(F) \otimes \mathbb{Q} \to K_n(F) \otimes \mathbb{Q}$ is the the subgroup $F^n_{\gamma} K_n(F) \otimes \mathbb{Q}$ given by the γ-filtration on Quillen K-theory.

The connection between the two propositions is given by the algebraic Atiyah-Hirzebruch spectral sequence

$$E^{p,q}_2 = H^{p-q}_{mot}(Spec(F), \mathbb{Z}(-q)) \Longrightarrow K_{-p-q}(F)$$

which degenerates up to torsion showing that Proposition 2.2.1 implies Proposition 2.2.2. But it is rather straightforward to generalize Proposition 2.2.2 to regular local rings, see [29], and the same is true for the spectral sequence, so a rational version of Proposition 2.2.1 for local rings is well known to the experts. This is some motiviation why we are interested in generalizing Proposition 2.2.1 to regular local rings even for the torsion part.

In [16] Kato constructed in a straightforward manner a Gersten complex of Zariski sheaves for Milnor K-theory of an excellent scheme X

$$\oplus_{x \in X^{(0)}} i_{x*}(K^M_n(x)) \longrightarrow \oplus_{x \in X^{(1)}} i_{x*}(K^M_{n-1}(x)) \longrightarrow \oplus_{x \in X^{(2)}} i_{x*}(K^M_{n-2}(x)) \longrightarrow \cdots$$

Here i_x is the morphism of schemes from the spectrum of the residue field at x to X. It can be shown that this Gersten complex is compatible via the isomorphism of Proposition 2.2.1 to the Gersten complex constructed in Chapter 1. This shows the following:

Proposition 2.2.3. *The above Gersten complex for Milnor K-theory is exact except in codimension 0 if X is a regular variety.*

The main aim of this thesis will be the determination of the kernel of the left arrow in the complex.

Another important construction in Milnor K-theory is the so called Galois symbol. Let F be a field of characteristic prime to some natural number l. Kummer theory gives a map

$$F^\times/(F^\times)^l \xrightarrow{\sim} H^1(F,\mathbb{Z}/l(1))$$

from $K_1(F)/l$ to Galois cohomology where $\mathbb{Z}/l(n)$ is the Galois module $\mu_l^{\otimes n}$. Hilbert's theorem 90 implies that this map is an isomorphism. Using the cup-product in Galois cohomology we get a map

$$T(F^\times)_n/l \longrightarrow H^n(F,\mathbb{Z}/l(n)) \ .$$

Lemma 2.2.4 (Tate)**.** *The above map induces a homomorphism of graded rings*

$$\chi_n : K_n^M(F)/l \longrightarrow H^n(F,\mathbb{Z}/l(n))$$

Proof. We have to show that the Steinberg relation $a\otimes(1-a)$ for all $a\in F-\{0,1\}$ maps to zero or in other words that the cup product $a\cup(1-a)$ vanishes. So let $t^l-a=\prod_i f_i\in F[t]$ be a factorization into irreducible polynomials. Let x_i be a zero value of f_i in some algebraic closure of F. It is well known that $f_i(1)=N_{F(x_i)/F}(1-x_i)$, so that we get $1-a=\prod_i N_{F(x_i)/F}(1-x_i)$. This implies

$$\begin{aligned} a\cup(1-a) &= \sum_i a\cup N_{F(x_i)/F}(1-x_i)=\sum_i N_{F(x_i)/F}(a\cup(1-x_i)) \\ &= l\sum_i N_{F(x_i)/F}(x_i\cup(1-x_i))=0\ . \end{aligned}$$

□

Conjecture 2.2.5 (Bloch-Kato)**.** *The norm residue homomorphism*

$$\chi_n : K_n^M(F)/l \longrightarrow H^n(F,\mathbb{Z}/l(n))$$

is an isomorphism for all fields F whose characteristic does not divide l and $n\geq 0$.

A proof of the Bloch-Kato conjecture has been announced by Voevodsky and Rost [44]. The case $n=2$ is known due to Merkurijev and Suslin [27]. The case $l=2$ and n arbitrary is part of the Milnor conjectures and was proved by Voevodsky, see Theorem 2.2.6.

In [28] Milnor considered beside the Galois symbol a map from Milnor K-groups to the graded Witt ring. Let us denote the Witt ring of a field F of characteristic different from 2 by $W(F)$ and the fundamental ideal by $I_F\subset W(F)$. Then Milnor defines a homomorphism of graded rings $K_*^M/2\to I_F^*/I_F^{*+1}$.

Theorem 2.2.6 (Voevodsky et al.)**.** *For a field F of characteristic different from 2 the two maps*

$$K_n^M(F)/2 \xrightarrow{\sim} H^n(F,\mathbb{Z}/2(n)) \quad \text{and} \quad K_n^M(F)/2 \xrightarrow{\sim} I_F^n/I_F^{n+1}$$

are isomorphisms for all $n\geq 0$.

For a proof of the first isomorphism see [43], for a proof of the second isomorphism see [30].

Chapter 3

Gersten conjecture

3.1 Overview

The aim of this chapter is to prove of a conjecture due to Alexander Beilinson [3] relating Milnor K-theory and motivic cohomology of local rings and to prove the Gersten conjecture for Milnor K-theory.

Theorem A (Beilinson's conjecture)**.** *For Voevodsky's motivic complexes of Zariski sheaves $\mathbb{Z}(n)$ on the category of smooth schemes over an infinite field the natural map*

$$\mathcal{K}_n^M \xrightarrow{\sim} \mathcal{H}^n(\mathbb{Z}(n)) \tag{3.1}$$

is an isomorphism of cohomology sheaves for all $n \geq 0$.

Here $\mathcal{K}_*^M$ is the Zariski sheaf of Milnor K-groups (see Definition 2.1) and $\mathbb{Z}(n)$ is the motivic complex defined in Chapter 1.

The surjectivity of the map in the theorem has been proven by Gabber [7] and Elbaz-Vincent/Müller-Stach [6], but only very little was known about injectivity at least if we are interested in torsion elements. Suslin/Yarosh proved the injectivity for discrete valuation rings of geometric type over an infinite field and $n = 3$ [36].

We deduce Beilinson's conjecture from the Gersten conjecture for Milnor K-theory, i.e. the exactness of the Gersten complex

$$0 \longrightarrow \mathcal{K}_n^M|_X \longrightarrow \oplus_{x \in X^{(0)}} i_{x*}(K_n^M(x)) \longrightarrow \oplus_{x \in X^{(1)}} i_{x*}(K_{n-1}^M(x)) \longrightarrow \cdots$$

for a regular excellent scheme X over an infinite field. This can be done because the isomorphism (3.1) is known in the field case, Proposition 2.2.1, and there is an exact Gersten complex for motivic cohomology of smooth schemes, Proposition 1.0.9.

As a consequence of Gersten's conjecture one deduces a Bloch formula relating Milnor K-theory and Chow groups

$$H^n(X, \mathcal{K}_n^M) = CH^n(X)$$

which was previously known only up to torsion [34] and for $n = 1, 2, \dim(X)$ due to Kato and Quillen [16], [32].

Furthermore one can deduce Levine's generalized Bloch-Kato conjecture for semi-local equicharacteristic rings [23] from the Bloch-Kato conjecture for fields, as well as the Milnor conjecture on quadratic forms over local rings.

Theorem B (Levine's Bloch-Kato conjecture)**.** *Assume the Bloch-Kato conjecture, Conjecture 2.2.5. The norm residue homomorphism*

$$\chi_n : K_n^M(A)/l \longrightarrow H_{et}^n(A, \mu_l^{\otimes n})$$

is an isomorphism for $n > 0$ and all semi-local rings A containing a field k of characteristic not dividing l with $|k| = \infty$.

The proof of the Gersten conjecture is in a sense elementary and uses a mixture of methods due to Ofer Gabber, Andrei Suslin, and Manuel Ojanguren. There are two new ingredients:

In Section 3.3 we construct a co-Cartesian square motivated by motivic cohomology which was suggested to hold by Gabber [7].

Section 3.4 extends the Milnor sequence, see Section 2.1, to semi-local rings. This provides norm maps on Milnor K-groups for finite, étale extensions of semi-local rings which are constructed in Section 3.5. The existence of these generalizations was conjectured by Bruno Kahn [12] and Elbaz-Vincent/Müller-Stach.

In Section 3.6 our main theorem is proved namely:

Theorem C. *Let A be a regular connected semi-local ring containing a field with quotient field F. Assume that each residue field of A is infinite. Then the map*

$$i_n : K_n^M(A) \longrightarrow K_n^M(F)$$

is universally injective for all $n \geq 0$.

The applications described above are discussed in Section 3.7. We should remark that the proof of the universality of the injection, but not the simple injectivity itself, requires the use of motivic cohomology.

The strategy of our proof of the main theorem is as follows:
First we reduce the proof to the case in which A is defined over an infinite perfect field k and A is the semi-local ring associated to a collection of closed points of an affine, smooth variety X/k. This reduction is accomplished by a Néron-Popescu desingularization [37] and using the norms constructed in Section 3.5. Then we apply induction on $d = \dim(A)$ for all n at once.

By the co-Cartesian square and Gabber's geometric presentation theorem one can assume $X = \mathbb{A}_k^d$.

Using the generalized Milnor-Bass-Tate sequence and the induction assumption that injectivity is already proved for rings of lower dimension one gets injectivity in dimension d.

Gabber used a similar mechanism to prove the surjectivity of the map (3.1) in [7]. His proof as well as the proof of Elbaz-Vincent/Müller-Stach for this statement can be simplified using the methods developed in Section 3.4, compare [18], [21].

3.2 Milnor K-Theory of local rings

In this section we recall the definition of Milnor K-Theory of semi-local rings, generalizing Definition 2.1.1, and some properties needed later – following [29] and [36].

Let A be a unital commutative ring, $T(A^\times)$ the $\mathbb{Z}$-tensor algebra over the units of A. Let I be the homogeneous ideal in $T(A^\times)$ generated by elements $a \otimes (1-a)$ with $a, 1-a \in A^\times$. Elements of I are usually called Steinberg relations.

Definition 3.2.1. *With the above notation we define the Milnor K-ring of A to be* $K_*^M(A) = T(A^\times)/I$.

By $\mathcal{K}_*^M$ we denote the associated Zariski sheaf of the presheaf

$$U \mapsto K_*^M(\Gamma(U, \mathcal{O}_U))$$

on the category of schemes.

The residue class of an element $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ in $K_n^M(A)$ is denoted $\{a_1, a_2, \ldots, a_n\}$. It is obvious that $K_0^M(A) = \mathbb{Z}$ and $K_1^M(A) = A^\times$ for any ring A. In [40] it is shown that if A is local and has more than 5 elements in its residue field the natural map

$$K_2^M(A) \to K_2(A)$$

from Milnor K-theory to algebraic K-theory, as defined for example in [32], is an isomorphism.

In what follows we will be concerned with the Milnor ring of a localization of a semi-local ring with sufficiently many elements in the residue fields. Sufficiently many will always depend on the context. Although results are usually discussed only for infinite residue fields, an argument in Sections 3.6 and 3.7 uses Milnor K-groups of semi-local rings with finite residue fields.

The next lemma is a generalization of [29, Lemma 3.2].

Lemma 3.2.2. *Let A be a semi-local ring with infinite residue fields and B a localization of A. For* $a, a_1, a_2 \in B^\times$ *we have*

$$\{a, -a\} = 0$$

and

$$\{a_1, a_2\} = -\{a_2, a_1\}\ .$$

During the next proof we misuse notation and write elements of A and the associated induced elements in B by the same symbols.

Proof. For simplicity we discuss only the case A local. It is clear that the second relation follows from the first since

$$\{a_1, a_2\} + \{a_2, a_1\} = \{a_1 a_2, -a_1 a_2\} - \{a_2, -a_2\} - \{a_1, -a_1\}\ . \tag{3.2}$$

The proof of the relation $\{a, -a\} = 0 \in K_2^M(B)$ for $a \in A^\times$, understood to mean the element induced in $B^\times$, goes as follows. If $1 - a \in A^\times$ write

$$-a = \frac{1-a}{1-1/a} \tag{3.3}$$

so that

$$\{a, -a\} = \{a, 1-a\} - \{a, 1-a^{-1}\} = 0\ .$$

If $1-a \notin A^\times$ but $a \in A^\times$, notice that for $s \in A^\times$, $\bar{s} \neq 1$ we have $1 - as \in A^\times$ so that

$$\begin{aligned} 0 = \{as, -as\} &= \{a,-a\} + \{s,-s\} + \{a,s\} + \{s,a\} \\ &= \{a,-a\} + \{a,s\} + \{s,a\} \ . \end{aligned}$$

So if we choose $s_1, s_2 \in A^\times$ with $\bar{s}_1 \neq 1 \neq \bar{s}_2$ and $\bar{s}_1\bar{s}_2 \neq 1$ we get from the last equations

$$\begin{aligned} \{a,-a\} = -\{a, s_1s_2\} - \{s_1s_2, a\} &= -\{a,s_1\} - \{s_1,a\} - \{a,s_2\} - \{s_2,a\} \\ &= \{a,-a\} + \{a,-a\} \ . \end{aligned}$$

Suppose now $a \in A$, $a \in B^\times$ but $a \notin A^\times$. Then $1-a \in A^\times$ and $1 - a^{-1} \in B^\times$. So we can write $-a$ as in (3.3). which again gives

$$\{a,-a\} = \{a, 1-a\} - \{a, 1-a^{-1}\} = 0 \ .$$

In the general case let $a = b/c$ for $b, c \in A$ and $b, c \in B^\times$

$$\{a,-a\} = \{b/c, -b/c\} = \{b,-b\} + \{c,c\} - \{c,-b\} - \{b,c\} \ .$$

What we have already proved together with (3.2) gives $\{c,c\} = \{c,(-1)(-c)\} = \{c,-1\}$ and

$$\{a,-a\} = \{c,-1\} - \{c,-b\} + \{c,b\} = 0 \ .$$

□

Let as before A be a semi-local ring with infinite residue fields.

Proposition 3.2.3. *Let $a_1, \ldots, a_n$ be in $A^\times$ such that $a_1 + \cdots + a_n = 1$, then*

$$\{a_1, \ldots, a_n\} = 0 \in K_n^M(A) \ .$$

Proof. If the reader is interested she can find a proof in [36, Corollary 1.7]. □

Later we will need another simple lemma. Let B be a localization of a semi-local ring.

Lemma 3.2.4. *For $a_1, a_2, a_1 + a_2 \in B^\times$ we have*

$$\{a_1, a_2\} = \{a_1 + a_2, -\frac{a_2}{a_1}\} \ .$$

Proof. We have

$$\begin{aligned} 0 &= \{\frac{a_1}{a_1+a_2}, \frac{a_2}{a_1+a_2}\} \\ &= \{a_1,a_2\} - \{a_1, a_1+a_2\} - \{a_1+a_2, a_2\} + \{a_1+a_2, a_1+a_2\} \\ &= \{a_1,a_2\} - \{a_1+a_2, -\frac{a_2}{a_1}\} \ . \end{aligned}$$

The first equation is the standard Steinberg relation, the third equation comes from the relations of Lemma 2.2. □

Remark 3.2.5. *We do not know whether Proposition 2.3 holds in case $a_1, \ldots, a_n$ are elements in $B^\times$ with $a_1 + \cdots + a_n = 1$.*

3.3 A co-Cartesian square

The theorem we prove in this section was suggested to hold by Gabber [7]. In order to motivate it consider the following geometric data:
Let $f : X' \to X$ be an étale morphism of smooth varieties and $Z \subset X$ a closed subvariety such that $f^{-1}(Z) \to Z$ is an isomorphism. Let $U' = X' - f^{-1}(Z)$ and $U = X - Z$. Then in the derived category of mixed motives DM_{gm}^{eff} over a perfect field [42] there is a distinguished triangle of the form

$$M_{gm}(U') \longrightarrow M_{gm}(X') \oplus M_{gm}(U) \longrightarrow M_{gm}(X) \longrightarrow M_{gm}(U')[1] \ .$$

This can be easily deduced from [41, Proposition 5.18].

Therefore in case X is semi-local the sequence

$$H^n_{mot}(X, \mathbb{Z}(n)) \longrightarrow H^n_{mot}(X', \mathbb{Z}(n)) \oplus H^n_{mot}(U, \mathbb{Z}(n)) \longrightarrow H^n_{mot}(U', \mathbb{Z}(n)) \longrightarrow 0 \qquad (3.4)$$

is exact, because $H^{n+1}_{mot}(X, \mathbb{Z}(n)) = 0$ as X is semi-local. In fact the Zariski sheaf $\mathbb{Z}(n)$ vanishes in degrees greater than n so that the vanishing of $H^{n+1}_{mot}(X, \mathbb{Z}(n))$ follows from the spectral sequence

$$\mathbb{H}^l_{Zar}(X, \mathcal{H}^{k,n}_{mot}) \Longrightarrow H^{l+k}_{mot}(X, \mathbb{Z}(n))$$

and [41, Lemma 4.28].

Let $A \subset A'$ be a local extension of factorial semi-local rings with infinite residue fields, i.e. the morphism $Spec(A') \to Spec(A)$ is dominant, maps closed points to closed points and is surjective on the latter. Let $f, f_1 \neq 0$ be in A such that $f_1 | f$ and $A/(f) \cong A'/(f)$. Denote the localization of A with respect to $\{1, f, f^2, \ldots\}$ resp. $\{1, f_1, f_1^2, \ldots\}$ by A_f resp. A_{f_1}.

As according to the Beilinson conjectures the n-th Milnor K-group of a reasonably good ring – for example a localizations of a smooth local rings – coincides with its (n, n)-motivic cohomology the exact sequence (3.4) motivates:

Theorem 3.3.1. *The diagram*

$$\begin{array}{ccc} K^M_n(A_{f_1}) & \longrightarrow & K^M_n(A_f) \\ \downarrow & & \downarrow \\ K^M_n(A'_{f_1}) & \longrightarrow & K^M_n(A'_f) \end{array}$$

is co-Cartesian.

Proof. For simplicity we restrict to the case A, A' local. Let $\pi \in A$ be an irreducible factor of f/f_1, $f = \pi f'$, and B resp. B' the localization $A_{f'}$ resp. $A'_{f'}$. By induction it is clearly sufficient to show

$$\begin{array}{ccc} K^M_n(B) & \longrightarrow & K^M_n(B_\pi) \\ \downarrow & & \downarrow \\ K^M_n(B') & \longrightarrow & K^M_n(B'_\pi) \end{array} \qquad (3.5)$$

is co-Cartesian. In order to see this one has to construct a multilinear map

$$\lambda : ((B'_\pi)^\times)^{\times n} \longrightarrow K^M_n(B') \oplus K^M_n(B_\pi)/K^M_n(B)$$

which induces an isomorphism compatible with (3.5)

$$K_n^M(B'_\pi) \cong K_n^M(B') \oplus K_n^M(B_\pi)/K_n^M(B) \ .$$

Because $B'^\times = B^\times(1+\pi A')$ one can write each element of an n-tuple

$$(a_1, \dots, a_n) \in (B'_\pi)^{\times n}$$

as

$$a_i = \pi^{j_i} y_i (1+\pi x_i) \tag{3.6}$$

$i = 1, \dots, n$, with $j_i \in \mathbb{Z}$, $y_i \in B^\times$ and $x_i \in A'^\times$. The element x_i can be assumed to be invertible in A' since if it was not invertible one could write

$$y_i(1+\pi x_i) = y_i/(1+\pi)\,[1+\pi(1+x_i+\pi x_i)] \ .$$

Now we translate some results from [36] into our setting. Define the multiplicative group $A'^\times_{(1)}$ as $1+\pi A'$, the set $A'^\times_{(inv)}$ as $1+\pi A'^\times$ and the map

$$\rho : A'^\times_{(inv)} \times ((B'_\pi)^\times)^{\times n-1} \longrightarrow K_n^M(B')$$

by

$$\rho((1+\pi x), \pi^{j_2} w_2, \dots, \pi^{j_n} w_n) = \{(1+\pi x), \frac{w_2}{(-x)^{j_2}}, \frac{w_n}{(-x)^{j_n}}\}$$

for $(w_i, \pi) = 1$, $i = 2, \dots, n$.

Now let U be the union of $(A'^\times_{(1)}) \times ((B'_\pi)^\times)^{\times n-1}$, $(B'_\pi)^\times \times (A'^\times_{(1)}) \times ((B'_\pi)^\times)^{\times n-2}$ etc.

Lemma 3.3.2. *The map ρ extends uniquely to a well defined skew-symmetric multilinear map*

$$U \longrightarrow K_n^M(B') \ .$$

Proof. From Sublemma 3.3 we deduce that we can extend ρ to a canonical multilinear map from its original domain of definition $(A'^\times_{(inv)}) \times ((B'_\pi)^\times)^{\times n-1}$ to the domain $(A'^\times_{(1)}) \times ((B'_\pi)^\times)^{\times n-1}$.

Sublemma 3.3.3. *For $1+\pi x = (1+\pi x_1)(1+\pi x_2)$ and $x, x_1 \in A'^\times$, $x_2 \in A'$ with $x_2 \in (B')^\times$*

$$\{1+\pi x, 1/(-x)\} = \{1+\pi x_1, 1/(-x_1)\} + \{1+\pi x_2, 1/(-x_2)\} \ .$$

Proof. For sake of completeness we recall the proof from [36, Lemma 3.5]. Let

$$\eta = \{1+\pi x, -x\} - \{1+\pi x_1, -x_1\} - \{1+\pi x_2, -x_2\} \ .$$

We have

$$\begin{aligned} \eta &= \{1+\pi x_1, \frac{x}{x_1}\} + \{1+\pi x_2, \frac{x}{x_2}\} \\ &= \{-\frac{x_1}{x_2}, \frac{x}{x_1}\} + \{-\frac{x_2}{x_1}, \frac{x}{x_2}\} \\ &= \{-\frac{x_1}{x_2}, x\} + \{x_2, x_1\} + \{-\frac{x_2}{x_1}, x\} + \{x_1, x_2\} \\ &= 0 \end{aligned}$$

where the second equation follows from

$$\begin{aligned} \frac{x}{x_1} &= 1+\frac{x_2}{x_1}(1+\pi x_1) \\ \frac{x}{x_2} &= 1+\frac{x_1}{x_2}(1+\pi x_2)\ . \end{aligned}$$

□

Next we have to check what happens if there are two entries of $A'^{\times}_{(1)}$ in an n-tupel. The next sublemma shows that the definition of ρ does not depend on how we eliminate the factors of π from our n-tupel by using either of the two distinguished $A'^{\times}_{(1)}$ entries.

Sublemma 3.3.4. *For $x_1, x_2 \in A'^{\times}$ one has*

$$\{1-\pi x_1, 1-\pi x_2, \frac{1}{x_1}\} = \{1-\pi x_1, 1-\pi x_2, \frac{1}{x_2}\}$$

Proof. Because of Proposition 2.3 we have

$$\begin{aligned} \{1-\pi x_1, 1-\pi x_2, \frac{x_2}{x_1}\} &= \{-\frac{x_2}{x_1}(1-\pi x_1), 1-\pi x_2, \frac{x_2}{x_1}\} \\ &= 0 \end{aligned}$$

□

□

As we saw above $((B'_\pi)^\times)^{\otimes n}$ is generated by U and $V = ((B_\pi)^\times)^{\otimes n}$. So one defines λ on U by ρ and on V by the natural surjection $V \to K_n^M(B_\pi)$.

It is immediately clear that λ does not depend on the factorization (3.6) or what is the same on the special decomposition of an element of $((B'_\pi)^\times)^{\otimes n}$ into elements of U and V.

It is more difficult to show that λ maps the Steinberg relations to zero. Denote by Λ the subgroup of $((B'_\pi)^\times)^{\otimes n}$ generated by elements of the form $a_1 \otimes \cdots \otimes a_n$ with $a_i + a_j = 1$ for some $i \neq j$. We have to show $\lambda(\Lambda) = 0$.

Lemma 3.3.5. *The group Λ is generated by elements of the form*

(i) $a_1 \otimes \cdots \otimes a_n$ *with* $a_1, \ldots, a_n \in (B'_\pi)^\times$ *and* $a_i + a_j = 0$ *for some* $i \neq j$.

(ii) $a \otimes (1-a) \otimes a_3 \otimes \cdots \otimes a_n$ *with* $a, 1-a, a_3, \ldots, a_n \in (B_\pi)^\times$

(iii) $a\pi \otimes (1-a\pi) \otimes a_3 \otimes \cdots \otimes a_n$ *with* $a \in A'$, $a \in B'^\times$ *and* $a_i \in (B'_\pi)^\times$ *for* $i = 3, \ldots, n$.

(iv) $a\pi^i \otimes (1-a\pi^i) \otimes (1-f^\infty x) \otimes a_4 \otimes \cdots \otimes a_n$ *with* $i \geq 0$, $a, 1-a\pi^i \in B'^\times$, $a_4, \ldots, a_n \in (B'_\pi)^\times$ *and* $x \in A'^\times$.

Remember that f^∞ means an arbitrarily fixed power of f.

Proof. We have to recall the five-term relation whose proof is left to the reader.

Sublemma 3.3.6 (Five-term relation)**.** *With*

$$[a] = a \otimes (1-a) \in (B'_\pi)^\times \otimes (B'_\pi)^\times$$

we have

$$[x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)] = \\ x \otimes (1-x)/(1-y) + (1-x)/(1-y) \otimes x \tag{3.7}$$

if $x, y, 1-x, 1-y, x-y \in (B'_\pi)^\times$.

We use induction on n.

$\mathbf{n=2}$: Let $a, 1-a \in (B'_\pi)^\times$. We have to express $[a]$ in terms of relations (i)-(iii). Choose $x' \in A'^\times$ such that $y = (1+f^\infty x')a \in B_\pi$ and let x be $1+f^\infty x'$.

The five-term relation gives (modulo the relations (i)) $[y/x] = [a]$ in terms of $[y]$ which is covered by (ii) and $[x], [(1-x)/(1-y)], [(1-x^{-1})/(1-y^{-1})]$ which are covered by (iii) as we will show now.

The latter elements are of the form $[1+\pi^i a]$ with $a \in A'$ and $a \in B'^\times$, $i > 0$. We will see by induction on i that we can suppose $i = 1$. Set $x = 1+\pi$ and $y = (1+\pi)(1+\pi^i a)$. If we substitute this into the five-term relation (3.7) we get the result.

$\mathbf{n=3}$:

Modulo relation (i) we have to show that an element $a \otimes (1-a) \otimes b$ with $a, 1-a, b \in (B'_\pi)^\times$ can be expressed in terms of relations (ii)-(iv). According to what we proved for the case $n=2$ we can assume either $a, 1-a \in (B_\pi)^\times$ or $a/\pi, 1-a \in A'$ and $\in B'^\times$ without denominators. The latter case is comprehended by (iii), the former by (ii) and (iv) if we factor b in the form $(B'_\pi)^\times = (1-A'^\times f^\infty) \cdot (B_\pi)^\times$.

$\mathbf{n>3}$:

This is simple if we proceed in analogy to case $n=3$. □

Compatibility of λ with (i): Assume without loss of generality $n=2$. Given an element $\pi^i a(1-\pi x) \otimes -\pi^i a(1-\pi x) \in (B'_\pi)^\times \otimes (B'_\pi)^\times$ with $x \in A'^\times$ and $a \in B^\times$ we get

$$\begin{aligned} \lambda(\pi^i a(1-\pi x) \otimes -\pi^i a(1-\pi x)) &= [\{1-\pi x, -\frac{a}{x^i}\} + \{1-\pi x, 1-\pi x\} \\ &\quad +\{\frac{a}{x^i}, 1-\pi x\}] \oplus \{\pi^i a, -\pi^i a\} \\ &= 0 \oplus 0\,. \end{aligned}$$

Compatibility of λ with (ii): Clear.

Compatibility of λ with (iii): This follows from Sublemma 3.3 because we have

$$\lambda(a\pi \otimes (1-a\pi) \otimes a_3 \otimes \cdots \otimes a_n) = \{a/a, 1-a\pi, \ldots\} \oplus 0 = 0 \oplus 0\,.$$

Compatibility of λ with (iv): If $i=0$ this is trivial, therefore assume $i>0$. Write $a = a_1/a_2$ with $a_1, a_2 \in A'$, $a_1, a_2 \in B'^\times$ and $1-a_2 \in A'^\times$. Write further

$$1 - f^\infty x = \frac{(1-\pi^i a_1)(1-f^\infty x)}{1-\pi^i a_1} = \frac{1-\pi^i[a_1 + x f^\infty \pi^{-i}(1-\pi^i a_1)]}{1-\pi^i a_1}$$

So it is sufficient to show

$$\zeta = \lambda(a\pi^i \otimes (1 - a\pi^i) \otimes (1 - \pi^i a_1) \otimes a_4 \otimes \cdots \otimes a_n) = 0 \tag{3.8}$$

$$\lambda(a\pi^i \otimes (1 - a\pi^i) \otimes (1 - \pi^i [a_1 + x f^\infty \pi^{-i}(1 - \pi^i a_1)]) \otimes a_4 \otimes \cdots \otimes a_n) = 0 \tag{3.9}$$

The demonstration of (3.9) is almost identical to that of (3.8), so we restrict to (3.8).

We know from the compatibility of λ with (iii) and the proof of Lemma 3.5 that

$$\lambda(a_1\pi^i \otimes (1 - a_1\pi^i)) = 0 \ .$$

This gives the first equality in

$$\begin{aligned} \zeta &= \{\frac{1}{a_2}, 1 - \frac{a_1}{a_2}\pi^i, 1 - a_1\pi^i\} \oplus 0 = \{\frac{1}{a_2}, -a_2(1 - \frac{a_1}{a_2}\pi^i), 1 - a_1\pi^i\} \oplus 0 \\ &= \{\frac{1}{a_2}, a_1\pi^i - a_2, 1 - a_1\pi^i\} \oplus 0 \\ &= \{\frac{1}{a_2}, 1 - a_2, -\frac{1 - a_1\pi^i}{a_1\pi^i - a_2}\} \oplus 0 = 0 \ . \end{aligned}$$

The fourth equality follows from Lemma 2.4.

This finishes the proof of Theorem 3.1 as the reader checks without difficulties that λ is an inverse to

$$K_n^M(B') \oplus K_n^M(B_\pi)/K_n^M(B) \longrightarrow K_n^M(B'_\pi) \ .$$

□

3.4 Generalized Milnor sequence

The most fundamental result in Milnor K-theory of fields is the short exact sequence due to Milnor, see Proposition 2.1.4,

$$0 \longrightarrow K_n^M(F) \longrightarrow K_n^M(F(t)) \longrightarrow \oplus_\pi K_{n-1}^M(F[t]/(\pi)) \longrightarrow 0 \tag{3.10}$$

where F is a field and the direct sum is over all irreducible, monic $\pi \in F[t]$. It calculates Milnor K-groups of the function field of a projective line. In order to prove Beilinson's conjecture we generalize this sequence to the realm of local rings. Let A be a semi-local domain with infinite residue fields, F its quotient field. Furthermore we assume A to be factorial in order to simplify our notation. For a description of the general case, which is not needed in the proof of our main theorem, compare Section 3.5.

For a local ring version of (3.10) one has to replace the group $K_n^M(F(t))$ by a group of symbols in general position denoted $K_n^t(A)$.

Definition 3.4.1. *An n-tuple of rational functions*

$$(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n}) \in F(t)^n$$

with $p_i, q_i \in A[t]$ and p_i/q_i a reduced fraction for $i = 1, \ldots n$ is called feasible if the highest nonvanishing coefficients of p_i, q_i are invertible in A and for irreducible factors u of p_i or q_i and v of p_j or q_j ($i \neq j$), $u = av$ with $a \in A^\times$ or $(u, v) = 1$.

Before coming to the definition of $K_n^t(A)$ we have to replace ordinary tensor product.

Definition 3.4.2. *Define*

$$\mathcal{T}_n^t(A) = \mathbb{Z} < \{(p_1, \ldots, p_n) | (p_1, \ldots, p_n) \text{ feasible, } p_i \in A[t] \text{ irreducible or unit}\} > /Linear$$

Here $Linear$ denotes the subgroup generated by elements

$$(p_1, \ldots, ap_i, \ldots, p_n) - (p_1, \ldots, a, \ldots, p_n) - (p_1, \ldots, p_i, \ldots, p_n)$$

with $a \in A^\times$.

By bilinear factorization the element

$$(p_1, \ldots, p_n) \in \mathcal{T}_n^t(A)$$

is defined for every feasible n-tuple with $p_i \in F(t)$.

Now define the subgroup $St \subset \mathcal{T}_n^t(A)$ to be generated by feasible n-tuples

$$(p_1, \ldots, p, 1-p, \ldots, p_n) \tag{3.11}$$

and

$$(p_1, \ldots, p, -p, \ldots, p_n) \tag{3.12}$$

with $p_i, p \in F(t)$.

Definition 3.4.3. *Define*

$$K_n^t(A) = \mathcal{T}_n^t(A)/St$$

We denote the image of $(p_1, \ldots, p_n)$ in $K_n^t(A)$ by $\{p_1, \ldots, p_n\}$.

Now the main theorem of this section reads:

Theorem 3.4.4. *There exists a split exact sequence*

$$0 \longrightarrow K_n^M(A) \longrightarrow K_n^t(A) \longrightarrow \oplus_\pi K_{n-1}^M(A[t]/(\pi)) \longrightarrow 0 \tag{3.13}$$

where the direct sum is over all monic, irreducible $\pi \in A[t]$.

The first map in sequence (3.13) is induced by the inclusion $A \to F(t)$. The second is a generalization of the tame symbol whose construction will be given below.

In the proof of the Gersten conjecture we need a slightly refined version of this theorem. Let $0 \neq p \in A[t]$ be an arbitrary monic polynomial. Define the group $K_n^t(A, p)$ in analogy to $K_n^t(A)$ but this time a tuple

$$(p_1/q_1, p_2/q_2, \ldots, p_n/q_n)$$

is feasible if additionally all p_i, q_i are coprime to p. The proof of the following theorem is almost identical to the proof of Theorem 4.4.

Theorem 3.4.5. *The sequence*

$$0 \longrightarrow K_n^M(A) \longrightarrow K_n^t(A, p) \longrightarrow \oplus_\pi K_{n-1}^M(A[t]/(\pi)) \longrightarrow 0$$

is split exact where the direct sum is over all $\pi \in A[t]$ monic and irreducible with $(\pi, p) = 1$.

Lemma 3.4.6. *For every feasible n-tuple* $(p_1, \dots, p_n)$ *and* $1 \le i < n$ *we have*

$$\{p_1, \dots, p_i, p_{i+1}, \dots, p_n\} = -\{p_1, \dots, p_{i+1}, p_i, \dots, p_n\} \in K_n^t(A) .$$

Proof. We can suppose $n = 2$ and $p_1, p_2 \in A[t]$ irreducible or units, then

$$\{p_1, p_2\} + \{p_2, p_1\} = \{p_1p_2, -p_1p_2\} - \{p_2, -p_2\} - \{p_1, -p_1\} = 0 .$$

□

Proof of Theorem 4.4. Step 1: The homomorphism $i_n : K_n^M(A) \to K_n^t(A)$ *is injective.*

We construct a left inverse ψ_n of i_n by associating to a polynomial its highest coefficient (specialization at infinity).
This gives a well defined map $\psi_n : \mathcal{T}_n^t(A) \to K_n^M(A)$.
We have to show ψ_n maps the Steinberg relations to zero. As concerns relation (3.12) one gets

$$\psi_n((p_1, \dots, p, -p, \dots, p_n)) = \{\psi_1(p_1), \dots, \psi_1(p), -\psi_1(p), \dots, \psi_1(p_n)\} = 0 .$$

For relation (3.11) one has to distinguish several cases. Given $p, q \in A[t]$, $\deg(p) > \deg(q)$ we have

$$\begin{aligned} \psi_n((p_1, \dots, p/q, 1 - p/q, \dots, p_n)) &= \psi_n((p_1, \dots, p/q, (q-p)/q, \dots, p_n)) \\ &= \{\psi_1(p_1), \dots, \psi_1(p)/\psi_1(q), -\psi_1(p)/\psi_1(q), \dots, \psi_1(p_n)\} \\ &= 0 \end{aligned}$$

for $\deg(p) < \deg(q)$

$$\begin{aligned} \psi_n((p_1, \dots, p/q, 1 - p/q, \dots, p_n)) &= \psi_n((p_1, \dots, p/q, (q-p)/q, \dots, p_n)) \\ &= \{\psi_1(p_1), \dots, \psi_1(p)/\psi_1(q), 1, \dots, \psi_1(p_n)\} \\ &= 0 \end{aligned}$$

for $\deg(p) = \deg(q) = \deg(q - p)$

$$\begin{aligned} \psi_n((p_1, \dots, p/q, 1 - p/q, \dots, p_n)) &= \psi_n((p_1, \dots, p/q, (q-p)/q, \dots, p_n)) \\ &= \{\psi_1(p_1), \dots, \psi_1(p)/\psi_1(q), 1 - \psi_1(p)/\psi_1(q), \dots, \psi_1(p_n)\} \\ &= 0 \end{aligned}$$

for $\deg(q) = \deg(p) > \deg(p - q)$

$$\begin{aligned} \psi_n((p_1, \dots, p/q, 1 - p/q, \dots, p_n)) &= \psi_n((p_1, \dots, p/q, (q-p)/q, \dots, p_n)) \\ &= \{\psi_1(p_1), \dots, 1, \psi_1(q-p)/\psi_1(q), \dots, \psi_1(p_n)\} \\ &= 0 . \end{aligned}$$

Therefore $\psi_n : K_n^t(A) \to K_n^M(A)$ is well defined and $\psi_n \circ i_n = id$.

Step 2: Constructing the homomorphisms $K_n^t(A) \to K_{n-1}^M(A[t]/(\pi))$.

Let $\pi \in A[t]$ be a monic irreducible. For every such π one constructs group homomorphisms

$$\partial_\pi : K_n^t(A) \longrightarrow K_{n-1}^M(A[t]/(\pi))$$

such that

$$\partial_\pi(\{\pi, p_2, \dots, p_n\}) = \{\bar{p}_2, \dots, \bar{p}_n\}$$

for $p_i \in A[t]$ and $(p_i, \pi) = 1$, $i = 2, \dots, n$. Clearly the last equation characterizes ∂_π uniquely. So one has to show existence. We give only a very sloppy construction; the details are left to the reader.

Introduce a formal element ξ with $\xi^2 = \xi\{-1\}$ and $\deg(\xi) = 1$. Define a formal map which is clearly not well defined

$$\theta_\pi : \mathcal{T}_*^t(A) \longrightarrow K_*^M(A[t]/(\pi))[\xi]$$

by

$$\theta_\pi(u_1\pi^{i_1}, \dots, u_n\pi^{i_n}) = (i_1\xi + \{\bar{u}_1\}) \cdots (i_n\xi + \{\bar{u}_n\}) \ .$$

We define ∂_π by taking the (right-)coefficient of ξ. This is a well defined homomorphism. So what remains to be shown is that ∂_π factors over the Steinberg relations.

Let $x = (\pi^i u, -\pi^i u)$ be feasible, then

$$\begin{aligned} \theta_\pi(x) &= (i\xi + \{\bar{u}\})(i\xi + \{-\bar{u}\}) \\ &= i\xi\{-1\} - i\xi\{\bar{u}\} + i\xi\{-\bar{u}\} + \{\bar{u}, -\bar{u}\} = 0 \ . \end{aligned}$$

For $i > 0$ and $x = (\pi^i u, 1 - \pi^i u)$ feasible one has

$$\theta_\pi(x) = (i\xi + \{\bar{u}\})\{1\} = 0 \ .$$

For $i < 0$ and $x = (\pi^i u, 1 - \pi^i u)$ feasible one has

$$\begin{aligned} \theta_\pi(x) &= (i\xi + \{\bar{u}\})(i\xi + \{-\bar{u}\}) \\ &= i\xi\{-1\} + i\xi\{-\bar{u}\} - i\xi\{\bar{u}\} + \{\bar{u}, -\bar{u}\} = 0 \ . \end{aligned}$$

Step 3: The filtration $L_d \subset K_n^t(A)$.

Let L_d be the subgroup of $K_n^t(A)$ generated by feasible n-tuples of polynomials of degree at most d. According to step 1 $L_0 = K_n^M(A)$. Moreover from the construction of step 2 we see that if π is of degree d one has $\partial_\pi(L_{d-1}) = 0$.

In order to finish the proof one shows that for $d > 0$

$$L_d/L_{d-1} \longrightarrow \oplus_{\deg(\pi)=d} K_{n-1}^M(A[t]/(\pi)) \tag{3.14}$$

is an isomorphism.

Step 4: The homomorphism $h_\pi : K_{n-1}^M(A[t]/(\pi)) \to L_d/L_{d-1}$.

For $\deg(\pi) = d$ and $\bar{g} \in A[t]/(\pi)$ let $g \in A[t]$ be the unique representative with $\deg(g) < d$.

Then there exists a unique homomorphism $h_\pi : K^M_{n-1}(A[t]/(\pi)) \to L_d/L_{d-1}$ such that

$$h_\pi(\{\bar g_2, \dots, \bar g_n\}) = \{\pi, g_2, \dots, g_n\}$$

for $(\pi, g_2, \dots, g_n)$ feasible.

According to the appendix the uniqueness is clear. We now show existence. Assume $\deg(\pi) > 2$. The case $\deg(\pi) = 2$ is similar but one has to factor everyting into three polynomials. Given $\{\bar g_2, \dots, \bar g_n\} \in K^M_{n-1}(A[t]/(\pi))$ choose for every $i = 2, \dots, n$ a generic factorization

$$g_i = f\,\pi + g_i' g_i''$$

as in the appendix. Because the factorization is generic all the elements $(g_2^*, \dots, g_n^*)$ are feasible ($*$ means primed respectively double primed). Set

$$h_\pi((\bar g_2, \dots, \bar g_n)) = \sum_{*^{n-1}} \{\pi, g_2^*, \dots, g_n^*\}$$

where the sum is over the 2^{n-1} maps from the set $\{2, \dots, n\}$ to $\{',''\}$.

One has to show that this gives a well defined homomorphism

$$(A[t]/(\pi))^{*\otimes n-1} \longrightarrow L_d/L_{d-1}\ .$$

In order to simplify the notation we do the case $n = 2$.

Let $g = f\pi + g'g''$ be a generic factorization as in the appendix with g feasible one has

$$\{\pi, g\} = \{\pi, g'\} + \{\pi, g''\} \in L_d/L_{d-1}$$

because of the Steinberg relation associated to

$$\frac{g}{g'g''} + \frac{-f\pi}{g'g''} = 1\ .$$

Finally we show the compatibility with the Steinberg relations. Observe that with a generic factorization $g = f\,\pi + g'g''$

$$h_\pi(\bar g \otimes -\bar g) = \{\pi\}(\{g'\} + \{g''\})(\{-g'\} + \{g''\}) = 0 \tag{3.15}$$

In order to show $h_\pi(\bar g \otimes (1 - \bar g)) = 0$ one can assume g generic of degree $d - 1$. This follows from the five term relation (7,Sublemma 3.6) and (3.15).

Step 5: $h : \oplus_{\deg(\pi)=d} K^M_{n-1}(A[t]/(\pi)) \to L_d/L_{d-1}$ is surjective.

Consider the symbol $\{p_1, \dots, p_n\} \in K^t_n(A)$ with $p_i \in A[t]$ prime and $\deg(p_i) \le d$. Use induction on the number of p_i which are of degree d. We can restrict to $n = 2$. We show that $\{\pi, f\} \in L_d$ lies in the image of this homomorphism for irreducible coprime $\pi, f \in A[t]$ of degree $d > 2$. As shown in the appendix (modulo the complication that $\deg(f) = d$) choose a generic factorization

$$f = f'\,\pi + f_1 f_2$$

with $(f'\pi, f, f_1, f_2)$ feasible and $\deg(f_1) = \deg(f_2) = \deg(f') + 1 = d - 1$. The Steinberg relation associated to

$$\frac{f}{f_1 f_2} + \frac{-f'\,\pi}{f_1 f_2} = 1$$

gives $\{\pi, f\} \in im(h) \mod L_{d-1}$.

Conclusion:

It is obvious that $\partial_\pi \circ h_\pi = 1$. Step 5 shows $\sum_\pi (h_\pi \circ \partial_\pi)$ is the identity on L_d/L_{d-1}. Because for every $d > 0$

$$L_d/L_{d-1} \longrightarrow \oplus_{\deg(\pi)=d} K_{n-1}^M(A[t]/(\pi))$$

is an isomorphism

$$K_n^t(A)/L_0 \longrightarrow \oplus_\pi K_{n-1}^M(A[t]/(\pi))$$

is an isomorphism too. This finishes the proof of Theorem 4.4. □

The relation between the exact sequence (3.13) and the classical Milnor sequence for $F = Q(A)$ is explained in the following proposition.

Proposition 3.4.7. *The following diagram commutes*

$$\begin{array}{ccccc} K_n^M(A) & \longrightarrow & K_n^t(A,p) & \longrightarrow & \oplus_\pi K_{n-1}^M(A[t]/(\pi)) \\ \downarrow & & \downarrow & & \downarrow \\ K_n^M(F) & \longrightarrow & K_n^M(F(t)) & \longrightarrow & \oplus_\pi K_{n-1}^M(F[t]/(\pi)) \end{array}$$

In the upper row the sum is over all $\pi \in A[t]$ irreducible, monic, and prime to p, in the lower row over all $\pi \in F[t]$ irreducible and monic.

Proof. The commutativity of the left square is clear. For the right square project the lower direct sum onto $K_{n-1}^M(F[t]/(\pi))$. An element

$$\{\pi, g_2, \dots, g_n\} \in K_n^t(A)$$

with $(g_i, \pi) = 1$ maps to $\{\bar{g}_2, \dots, \bar{g}_n\} \in K_{n-1}^M(F[t]/(\pi))$ in any case. □

3.5 Transfer

In this section we explain how to construct a transfer – also called norm – for finite, étale extensions of semi-local rings with infinite residue fields. Such extensions are exactly those which are of the form $B = A[t]/(\pi)$ with π monic and $\mathrm{Disc}(\pi) \in A^\times$.

Definition 3.5.1. *A polynomial $p \in A[t]$ is called feasible if the highest non-vanishing coefficient of p is invertible in A. It is called irreducible if it cannot be factored nontrivially into polynomials with highest coefficients invertible.*

Because $A[t]$ is not necessarily factorial we generalize Definition 4.1 by attaching as additional data to the p_i, q_i $i = 1, \dots, n$ a factorization up to units into irreducible polynomials with highest coefficients invertible. Furthermore we demand that $\mathrm{Disc}(p_i) \in A^\times$ and $\mathrm{Disc}(q_i) \in A^\times$. Later we will need the latter conditions to ensure nice functoriality properties for our generalized Milnor K-theory.

Definition 3.5.2. *An n-tuple of rational functions*

$$(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots, \frac{p_n}{q_n}) \in F(t)^n$$

with $p_i, q_i \in A[t]$ feasible together with factorizations of p_i, q_i

$$\begin{aligned} p_i &= a_i p_i^{(1)} \cdots p_i^{(j_i)} \\ q_i &= b_i q_i^{(1)} \cdots q_i^{(j_i)} \end{aligned}$$

with $a_i, b_i \in A^\times$ and $p_i^{(j)}, q_i^{(j)}$ monic irreducible is called feasible if:

- *In the obvious sense the p_i/q_i are reduced fractions.*
- *For $i \neq i'$ we have either $p_i^{(j)}, q_i^{(j)}$ equal or coprime to $p_{i'}^{(j')}, q_{i'}^{(j')}$ for $j \in \{1, \dots, j_i\}$ and $j' \in \{1, \dots, j_{i'}\}$.*
- $\mathrm{Disc}(p_i), \mathrm{Disc}(q_i) \in A^\times$.

Now that we have the notion of a feasible n-tupel we can immediately generalize Definition 4.2 to get a group $\mathcal{T}_n^{et}(A)$. Furthermore we define the subgroup $St^{et} \subset \mathcal{T}_n^{et}(A)$, in analogy to St, to be generated by feasible n-tupels

$$(p_1, \dots, \frac{p}{q}, \frac{q-p}{q}, \dots, p_n)$$

and

$$(p_1, \dots, \frac{p}{q}, -\frac{p}{q}, \dots, p_n)$$

with $p_i, p, q \in A[t]$ and $(p, q) = 1$, $(q - p, q) = 1$. Here we may attach arbitrary factorizations to $p_i, p, q, q - p$ such that the n-tupel is feasible.

Definition 3.5.3. *Define*

$$K_n^{et}(A) = \mathcal{T}_n^{et}(A)/St$$

The proof of the next theorem is analogous to the proof of Theorem 4.4.

Theorem 3.5.4. *There exists a split exact sequence*

$$0 \longrightarrow K_n^M(A) \longrightarrow K_n^{et}(A) \longrightarrow \oplus_\pi K_{n-1}^M(A[t]/(\pi)) \longrightarrow 0$$

where the direct sum is over all monic irreducible $\pi \in A[t]$ with $\mathrm{Disc}(\pi) \in A^\times$.

Now one defines a transfer as in the field case

$$N_{B/A} : K_n^M(B) \longrightarrow K_n^M(A)$$

by setting:

Definition 3.5.5. *For $x \in K_n^M(B)$ choose $x' \in K_{n+1}^t(A)$ with $\partial_\pi(x') = x$ and $\partial_{\pi'}(x') = 0$ for all monic irreducible $\pi' \neq \pi \in A[t]$. Define*

$$N_{B/A}(x) = -\partial_\infty(x')$$

where ∂_∞ is the infinite tame symbol uniquely defined by the following property: Consider a symbol $x := \{p_1, \dots, p_{n+1}\} \in K_{n+1}^M(A)$. Set $\partial_\infty(x) = 0$ if $\deg(p_1) = \cdots = \deg(p_{n+1}) = 0$ and set $\partial_\infty(x) = \psi_n(\{p_2, \dots, p_{n+1}\})$ if $p_1 = 1/t$ and $\deg(p_2) = \cdots = \deg(p_{n+1}) = 0$. Here the residue symbol ψ_n is defined analogously to step 1 in the proof of Theorem 4.4.

Assume given an arbitrary homomorphism – not necessarily local – of semi-local rings $i : A \to A'$. Fix as additional data a factorization into monic irreducible polynomials for every polynomial $i(p) \in A'[t]$ where $p \in A[t]$ is monic irreducible. Let $\pi \in A[t]$ be a monic irreducible polynomial with $\mathrm{Disc}(\pi) \in A^\times$ and let $i(\pi) = \prod_j \pi_j$ be the associated complete factorization. Denote $B = A[t]/(\pi)$ and $B'_j = A'[t]/(\pi_j)$.

Proposition 4.7 generalizes to:

Proposition 3.5.6. *The following diagram of exact sequences from Theorem 5.4 commutes:*

$$\begin{array}{ccccc} K_n^M(A) & \longrightarrow & K_n^{et}(A) & \longrightarrow & \oplus_\pi K_{n-1}^M(A[t]/(\pi)) \\ \downarrow & & \downarrow & & \downarrow \\ K_n^M(A') & \longrightarrow & K_n^{et}(A') & \longrightarrow & \oplus_{\pi'} K_{n-1}^M(A'[t]/(\pi')) \end{array}$$

The right vertical map is defined by the natural homomorphism

$$K_{n-1}^M(A[t]/(\pi)) \longrightarrow \oplus_{\pi_j} K_{n-1}^M(A'[t]/(\pi_j)) \ .$$

One should remark that the existence of the middle vertical arrow is guaranteed by the condition that all our polynomials have non-vanishing discriminant.

Now our main compatibility result states:

Proposition 3.5.7. *The diagram*

$$\begin{array}{ccc} K_n^M(B) & \longrightarrow & \oplus_j K_n^M(B'_j) \\ {\scriptstyle N_{B/A}}\downarrow & & \downarrow{\scriptstyle \oplus_j N_{A'_j/A'}} \\ K_n^M(A) & \longrightarrow & K_n^M(A') \end{array}$$

is commutative.

Remark 3.5.8. *By Remark 8.4 and the above construction for every $n \geq 0$ there clearly exists an $M_n \in \mathbb{N}$ such that for $B = A[t]/(\pi)$, $\deg(\pi) = 2, 3$, π irreducible and monic, $\mathrm{Disc}(\pi) \in A^\times$ and A a semi-local ring with more than M_n elements in each residue field there exist norms*

$$N : K_n^M(B) \to K_n^M(A)$$

that satisfy $N \circ i_* = \deg(\pi)$ where $i : A \to B$ is the embedding.

In principle one could go through the construction of the transer in order to determine a possible choice for M_n. In this paper we will not be concerned with this problem.

3.6 Main theorem

The main result is:

Theorem 3.6.1. *Let A be a regular connected semi-local ring containing a field with quotient field F. Assume that each residue field of A has more than M_n elements (see Remark 5.8). Then the map*

$$i_n : K_n^M(A) \longrightarrow K_n^M(F)$$

is universally injective.

The definition of universal injectivity is recalled in Definition 1.0.10.

Proof. First we prove ordinary injectivity under the assumption that k is an infinite perfect field and A is the semi-local ring associated to a finite set of closed points of a smooth, affine variety X/k of dimension d. In order to prove this special case use induction on d. The case $d = 0$ is trivial.

Suppose given $x \in K_n^M(A)$ such that $i_n(x) = 0$. Then there is $0 \neq f \in A$ such that $i'_n(x) = 0$ with $i'_n : K_n^M(A) \longrightarrow K_n^M(A_f)$ the canonical map.

Use Gabber's geometric presentation theorem [4] to construct a k-morphism $\phi : X \to \mathbb{A}_k^d$. Let A' be the semi-local ring at the images under ϕ of the points correspondig to A. Denote these points by $y_1, \ldots, y_l \in \mathbb{A}_k^d$. After shrinking X we can assume ϕ satisfies the following properties:

(i) The map $V(f) \to \mathbb{A}_k^d$ is an embedding.

(ii) ϕ is étale.

(iii) If $f' \in A'$ is chosen according to (i) such that $A/(f) \cong A'/(f')$, then $A/(f) = A \otimes_{A'} A'/(f')$.

Consider the commutative diagram

$$\begin{array}{ccc} K_n^M(A') & \longrightarrow & K_n^M(A'_{f'}) \\ \downarrow & & \downarrow \\ K_n^M(A) & \longrightarrow & K_n^M(A_f) \end{array}$$

Theorem 3.1 shows that it is co-Cartesian. According to a well known property of co-Cartesian squares the lower horizontal arrow is injective if the upper horizontal arrow is injective.

So we have to prove

$$i_n : K_n^M(A') \longrightarrow K_n^M(k(t_1, \ldots, t_d))$$

is injective.

Let again x be in the kernel of this homomorphism and denote by $p_1, \ldots, p_m \in k[t_1, \ldots, t_d]$ the irreducible different polynomials appearing in the symbols of x, $p_i \in A'^{\times}$.

Denote for $i = 1, \ldots, m$ by $W_i \subset V(p_i)$ the join of the singular locus of $V(p_i)$ with $\bigcup_{j \neq i} V(p_i) \cap V(p_j)$. Because we assumed k to be perfect $\dim(W_i) < d - 1$.

Use a slight generalization of Noether normalization to choose a linear projection

$$p : \mathbb{A}_k^d \longrightarrow \mathbb{A}_k^{d-1}$$

such that $p|_{V(p_j)}$ is finite and $p(y_i) \notin p(W_j)$ for $i = 1, \ldots, l$ and $j = 1, \ldots, m$.

Let A'' be the semi-local ring associated to the points $p(y_1), \ldots, p(y_l) \in \mathbb{A}_k^{d-1}$. Then $A'' \subset A'$ is a local ring extension and because $V(p_i)$ is finite integral over A'' one sees that $p_i \in A''[t]$ can be chosen to be monic and irreducible. Choose a monic $q \in A''[t]$ such that $V(q) \cap p^{-1}(p(y_i))$ consists exactly of the points from $\{y_1, \ldots, y_l\}$ which are in the fibre over $p(y_i)$ for all $i = 1, \ldots, l$. It follows that $(q, p_i) = 1$ for $i = 1, \ldots, m$.

There exists a natural map $K_n^t(A'', q) \to K_n^M(A')$. Now x is induced by an element $x' \in K_n^t(A'', q)$. Consider the commutative diagram with exact rows from Proposition 4.7

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_n^M(A'') & \longrightarrow & K_n^t(A'', q) & \xrightarrow{\alpha} & \oplus_\pi K_{n-1}^M(A''[t]/(\pi)) & \longrightarrow & 0 \\ & & \downarrow \gamma & & \downarrow \beta & & \downarrow \delta & & \\ 0 & \longrightarrow & K_n^M(F) & \longrightarrow & K_n^M(F(t)) & \longrightarrow & \oplus_\pi K_{n-1}^M(F[t]/(\pi)) & \longrightarrow & 0 \end{array}$$

where the notation $F = Q(A'')$ is used. By assumption $\beta(x') = 0$. By induction the relevant summands of δ are injective so that $\alpha(x') = 0$. But then x' comes from an element $x'' \in K_n^M(A'')$ with $\gamma(x'') = 0$. It follows again by induction that $x'' = 0$ and $x' = 0$.

This finishes the proof of Theorem 5.1, without the universality property, in case A is the semi-local ring at closed points of an smooth, affine variety X/k and k is infinite and perfect.

Next we show injectivity for a semi-local ring A corresponding to an arbitrary system of points $y_1, \ldots, y_l$ of a smooth affine variety X over an infinite perfect field k. Let F be the quotient field of A. Given

$$x \in \ker(K_n^M(A) \longrightarrow K_n^M(F))$$

choose for every y_i, $i = 1, \ldots, l$ a closed point $y_i' \in \overline{\{y_i\}}$ such that if A' denotes the semi-local ring corresponding to these points x is induced by an $x' \in K_n^M(A')$ under the natural map $K_n^M(A') \to K_n^M(A)$.

Because

$$x' \in \ker(K_n^M(A') \longrightarrow K_n^M(F))$$

and this map is injective we deduce $x' = 0$ and $x = 0$.

At this point we can write down the isomorphism

$$K_n^M(A) \xrightarrow{\sim} H^n(Spec(A), \mathbb{Z}(n))$$

for a ring A as in the previous paragraph. In fact what we have proved so far together with a theorem of Nesterenko/Suslin and Totaro, Proposition 2.2.1, implies this isomorphism. This will be explained in the proof of Theorem 7.5. But as

$$H^n(Spec(A), \mathbb{Z}(n)) \longrightarrow H^n(Spec(F), \mathbb{Z}(n))$$

is universally injective according to Proposition 1.0.9 the corresponding injection of Milnor K-groups is universally injective too.

For the general case of our theorem we use Néron-Popescu desingularization, Lemma 1.0.3. In fact one has to show

$$K_n^M(A) \longrightarrow K_n^M(A_f)$$

is universally injective for $0 \neq f \in A$. As A is the filtered inductive limit of smooth semi-local rings of geometric type over a prime field, it is sufficient to restrict to the case in which A is the semi-local ring at some points of a smooth, affine variety X over a prime field k_0 and the residue fields have more than M_n elements. For the argument below take M_n from Remark 5.8.

If $\mathrm{char}(k_0) > 0$ one has to use a norm trick to reduce to the case of a ground field which is an infinite algebraic extension of k_0. Let $k_1 \subset A$ be the algebraic closure of k_0 in A. Now argue as follows:

Fix $p = 2$ or $p = 3$. Choose a tower of finite extensions $k_1 \subset k_2 \subset k_3 \subset \cdots \subset k_\infty$ with $k_\infty = \cup_i k_i$ and $\dim_{k_i}(k_{i+1}) = p$, $i = 1, 2, \ldots$

From Remark 5.8 one deduces the existence of norms

$$N : K_n^M(A \otimes_{k_1} k_{i+1}) \longrightarrow K_n^M(A \otimes_{k_1} k_i)$$

which satisfy $N \circ i_* = p$ for the natural map $i : A \otimes k_i \to A \otimes k_{i+1}$.

Consider the commutative diagram

$$\begin{array}{ccc} K_n^M(A \otimes_{k_1} k_\infty) & \longrightarrow & K_n^M(F \otimes_{k_1} k_\infty) \\ \uparrow{\scriptstyle \alpha} & & \uparrow \\ K_n^M(A) & \xrightarrow[\beta]{} & K_n^M(F) \end{array}$$

with $F = Q(A)$. The upper arrow is universally injective according to what we proved above. Because of the existence of a norm $\alpha \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ is universally injective so that $\beta \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ is universally injective.

This implies β is universally injective, since $p = 2$ or 3 and for a functor F as in Definition 6.2 and an abelian group G we have

$$F(G \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]) = F(G) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \; .$$

$\square$

3.7 Applications

We will give some consequences of Theorem 6.1.

Assumption: All schemes and rings in this section up to Theorem 7.6 are excellent.

Recall that Kato constructed a Gersten complex of Zariski sheaves for Milnor K-theory of a scheme X

$$0 \to \mathcal{K}_n^M|_X \to \oplus_{x \in X^{(0)}} i_{x*}(K_n^M(x)) \to \oplus_{x \in X^{(1)}} i_{x*}(K_{n-1}^M(x)) \to \cdots \tag{3.16}$$

where $K_*^M(x) := K_*^M(k(x))$ and i_x is the embedding of the point x [16].

By Proposition 2.2.1 is is known that Milnor K_n^M of a field coincides with (n, n)-motivic cohomology – for the latter the exactness of the Gersten complex is well known, Proposition 1.0.9, if X is smooth. Moreover the differentials of (3.16) are equal to the ones constructed from the coniveau spectral sequence in motivic cohomology. This implies that (3.16) is exact except at the first two places if X is regular and of algebraic type over an arbitrary field. An elementary proof of this fact can be found in [33].

The question whether (3.16) is exact at the second place was settled independently by Gabber [7] and Elbaz-Vincent/Müller-Stach [6], for a short proof see [21].

From Theorem 6.1 and Panin's method [31] we conclude the Gersten conjecture is true in an equicharacteristic context:

Theorem 3.7.1 (Gersten conjecture)**.** *The Gersten complex (3.16) for Milnor K-theory is exact if X is regular, contains a field, and all residue fields of X contain more than M_n elements (see Remark 5.8).*

For the definition of universal exactness see Definition 1.0.10. Our proof is a translation of [31] into Milnor K-theory.

Proof. We have to prove the exactness in codimension > 0 of the complex of Zariski sheaves

$$g_n(X) = (\oplus_{x\in X^{(0)}} i_{x*}(K_n^M(x)) \longrightarrow \oplus_{x\in X^{(1)}} i_{x*}(K_{n-1}^M(x)) \longrightarrow \cdots)$$

and we have to show that the kernel of the left arrow is the Zariski sheaf $\mathcal{K}_n^M|_X$. if $X = Spec(A)$ with A regular and equicharacteristic. Here $\oplus_{x\in X^{(0)}} i_{x*}(K_n^M(x))$ is understood to be placed in degree zero. We use induction on $d = \dim(X)$.

Let $f \in A$ be a local parameter and $Z = Spec(A/(f))$. Then we have a short exact sequence

$$0 \longrightarrow g_{n-1}(Z)[-1] \longrightarrow g_n(X) \longrightarrow g_n(X_f) \longrightarrow 0 \tag{3.17}$$

as in [31]. Our induction assumption implies that $H^i(g_{n-1}[-1](Z)) = 0$ for $i \geq 2$ and $H^1(g_{n-1}[-1](Z)) = K_{n-1}^M(Z)$. Furthermore, because $\dim(X_f) < d$, $g_n(X_f)$ is the global section complex associated to a flabby resolution of $\mathcal{K}_n^M$. In other words:

$$H^i(g_n(X_f)) = H^i(X_f, \mathcal{K}_n^M)\ .$$

The latter cohomology groups can be calculated by going down to a smooth world:

Lemma 3.7.2. *We have $H^i(X_f, \mathcal{K}_n^M) = 0$ for $i > 0$ and $H^0(X_f, \mathcal{K}_n^M) = K_n^M(A_f)$.*

Proof. Using a Néron-Popescu desingularization, Corollary 1.0.3, and Grothendieck's limit theorem, Corollary 1.0.5, we can assume X to be essentially smooth over a prime field with a residue field with more than M_n elements. But then, reading our argument backwards and using the fact that we know from Proposition 1.0.9, [7], [6] and Theorem 6.1 that the Gersten conjecture is true for smooth varieties, we have $H^i(g_n(X_f)) = 0$ for $i > 0$ because of the long exact cohomology sequence associated to (3.17). Furthermore (3.17) induces a short exact sequence

$$0 \longrightarrow K_n^M(A) \longrightarrow \mathcal{K}_n^M(X_f) \longrightarrow K_{n-1}^M(A/(f)) \longrightarrow 0\ .$$

As a consequence of Theorem 6.1 we get an analogous sequence with $\mathcal{K}_n^M(A_f)$ replaced by $K_n^M(A_f)$:

Sublemma 3.7.3. *The canonical sequence*

$$0 \longrightarrow K_n^M(A) \longrightarrow K_n^M(A_f) \longrightarrow K_{n-1}^M(A/(f)) \longrightarrow 0$$

is exact for an arbitrary equicharacteristic regular local ring A and irreducible element f.

Proof. The injectivity of $K_n^M(A) \longrightarrow K_n^M(A_f)$ follows from Theorem 6.1. The rest is elementary and left to the reader. □

Putting the last two short exact sequences together we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_n^M(A) & \longrightarrow & K_n^M(A_f) & \longrightarrow & K_{n-1}^M(A/(f)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & K_n^M(A) & \longrightarrow & \mathcal{K}_n^M(X_f) & \longrightarrow & K_{n-1}^M(A/(f)) & \longrightarrow & 0 . \end{array}$$

Finally, the five-lemma shows

$$H^0(X_f, \mathcal{K}_n^M) = \mathcal{K}_n^M(X_f) = K_n^M(A_f) .$$

□

The long exact cohomology sequence associated with (3.17) gives, inserting the calculations of Lemma 7.2, $H^i(g_n(X)) = 0$ for $i > 1$ and the exact sequence

$$0 \longrightarrow H^0(g_n(X)) \longrightarrow K_n^M(A_f) \xrightarrow{\partial} K_{n-1}^M(A/(f)) \longrightarrow H^1(g_n(X)) \longrightarrow 0 .$$

As, according to Sublemma 7.4, ∂ is surjective and has kernel $K_n^M(A)$ this finishes the proof of Theorem 7.1. □

Kato's original motivation for studying the Gersten complex was to obtain an elementary generalization of the formula

$$H^1(X, \mathcal{O}_X^\times) = CH^1(X)$$

by means of Milnor K-theory. He proved the following fact in case $n = \dim(X)$ and X is smooth of finite type over a Dedekind ring [16].

Theorem 3.7.4 (Bloch formula)**.** *There is a canonical isomorphism*

$$H^n(X, \mathcal{K}_n^M) \cong CH^n(X) .$$

for every $n \geq 0$ if X is as in Theorem 7.1.

In fact using the arguments explained in the next chapter it is not difficult to see that the assumption that X has more than M_n elements in each residue field is superfluous, it is enough so assume that X is regular and contains an arbitrary field.

Furthermore from the exactness of the Gersten complex one deduces one of the remaining Beilinson conjectures on motivic cohomology [24], [3]. Let $\mathcal{H}^{m,n}$ be the Zariski sheaf associated to the presheaf $U \mapsto H^m_{mot}(U, \mathbb{Z}(n))$ defined in Chapter 1. We claim that for a semi-local ring A containing a field there exists a canonical homomorphism

$$K_n^M(A) \to H^n_{mot}(A, \mathbb{Z}(n)) .$$

In fact in Section 2.2 we constructed a map $T(A^\times) \to H^{n,n}(A)$ and observed that if A is a field this map factors through $K_n^M(A)$. If A is essentially smooth over a field, $F = Q(A)$, the lower horizontal arrow in the diagram

$$\begin{array}{ccc} T(A^\times) & \longrightarrow & K_n^M(F) \\ \downarrow & & \downarrow \\ H^{n,n}(A) & \longrightarrow & H^{n,n}(F) \end{array}$$

is injective, because of Proposition 1.0.9. So for A essentially smooth over a field the map $T(A^\times) \to H^{n,n}(A)$ does also factor through $K_n^M(A)$. Finally, the general case follows from Popescu's theorem, see Corollary 1.0.3.

Theorem 3.7.5 (Beilinson's conjecture)**.** *For a regular local ring A containing a field with more than M_n elements the canonical map*

$$K_n^M(A) \longrightarrow H^{n,n}(A)$$

is an isomorphism.

Proof. If $X = Spec(A)$ is essentially smooth over a field with more than M_n elements the a diagram chase in the morphism of exact Gersten complexes of Zariski sheaves from Milnor K-theory to motivic cohomology

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{K}_n^M|_X & \longrightarrow & \oplus_{x\in X^{(0)}} i_{x*}(K_n^M(x)) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{H}^{n,n} & \longrightarrow & \oplus_{x\in X^{(0)}} i_{x*}(H^n(x,\mathbb{Z}(n))) & \longrightarrow & \cdots \end{array}$$

proves the theorem if we use the fact that for a field F the map $K_n^M(F) \to H^n_{mot}(Spec(F),\mathbb{Z}(n))$ is an isomorphism by Proposition 2.2.1. The general case follows from Popescu's desingularization, Corollary 1.0.3, using the fact that motivic cohomology commutes with filtering direct limits, Lemma 1.0.8. ☐

Using either the same Popescu trick as above or a generalization of Lemma 2.2.4 one constructs a Galois symbol

$$\chi_n : K_n^M(A)/l \longrightarrow H^n_{et}(A, \mu_l^{\otimes n})$$

for a regular local equicharacteristic ring A of characteristic prime to l generalizing the Galois symbol of Chapter 2 for fields. Marc Levine [23] and Bruno Kahn [13] conjectured the following generalized version of the Bloch-Kato conjecture, Conjecture 2.2.5. Levine showed even before the advent of modern motivic cohomology that it implies a form of the Quillen-Lichtenbaum conjecture.

Theorem 3.7.6 (Levine's Bloch-Kato conjecture)**.** *Assume the Bloch-Kato conjecture. The norm residue homomorphism*

$$\chi_n : K_n^M(A)/l \longrightarrow H^n_{et}(A, \mu_l^{\otimes n})$$

is an isomorphism for $n > 0$ and all semi-local rings A containing a field k of characteristic not dividing l with $|k| > M_n$.

Proof. Assume first that A is a smooth semi-local ring of geometric type over k. In this case the theorem follows from the morphism (up to a sign) of universally exact Gersten complexes, $X = Spec(A)$,

$$\begin{array}{ccccccc} 0 \longrightarrow & K_n^M(A)/l & \longrightarrow & K_n^M(Q(A))/l & \longrightarrow & \oplus_{x\in X^{(1)}} K_{n-1}^M(x)/l \\ & \downarrow & & \downarrow \wr & & \downarrow \wr \\ 0 \longrightarrow & H_{et}^n(A, \mu_l^{\otimes n}) & \longrightarrow & H_{et}^n(Q(A), \mu_l^{\otimes n}) & \longrightarrow & \oplus_{x\in X^{(1)}} H_{et}^n(k(x), \mu_l^{\otimes n-1}) \end{array}$$

The general cases uses a trick coined by Hoobler [11]. First of all because both Milnor K-theory and étale cohomology are continuous, see Definition 1.0.6, we can assume A to be of geometric type over k. Let $B \to A$ be surjective local morphism of semi-local rings with kernel I such that (B, I) is a henselian pair and B is ind-smooth over k. The homomorphism

$$K_n^M(B)/l \longrightarrow K_n^M(A)/l$$

is surjective. In [8] Gabber proves:

Lemma 3.7.7 (Gabber)**.**

$$H_{et}^n(B, \mu_l^{\otimes n}) \longrightarrow H_{et}^n(A, \mu_l^{\otimes n})$$

is an isomorphism.

Now the problem is reduced to the smooth case by the following commutative diagram

$$\begin{array}{ccc} K_n^M(B)/l & \longrightarrow & K_n^M(A)/l \\ \downarrow & & \downarrow \\ H_{et}^n(B, \mu_l^{\otimes n}) & \longrightarrow & H_{et}^n(A, \mu_l^{\otimes n}) \end{array}$$

□

For a local ring A let $W(A)$ be the Witt ring and I_A the fundamental ideal.

Theorem 3.7.8 (Generalized Milnor conjecture)**.** *Assume A is a local ring and contains a field k of characteristic different from two with $|k| > M_n$. Then the natural map*

$$K_n^M(A)/2 \longrightarrow I_A^n/I_A^{n+1}$$

is an isomorphism for $n \geq 0$.

Proof. Assume first that A is a smooth semi-local ring of geometric type over k, $X = Spec(A)$. We have a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & K_n^M(A)/2 & \longrightarrow & K_n^M(Q(A))/2 & \longrightarrow & \oplus_{x\in X^{(1)}} K_{n-1}^M(x)/2 \\ & \downarrow & & \downarrow \wr & & \downarrow \wr \\ 0 \longrightarrow & I_A^n/I_A^{n+1} & \longrightarrow & I_{Q(A)}^n/I_{Q(A)}^{n+1} & \longrightarrow & \oplus_{x\in X^{(1)}} I_x^{n-1}/I_x^n \end{array}$$

where the exactness of the lower sequence follows from the exactness of the upper sequence. This is because the left vertical map is surjective by standard facts, see [28], and the other vertical maps are isomorphisms by Voevodsky's theorem, Theorem 2.2.6. The exactness of the upper sequence is nothing but Theorem 7.1.

A diagram chase proves the theorem if A is essentially smooth over k. Choosing $B \to A$ as in the proof of the last theorem we have:

Lemma 3.7.9. *The natural homomorphism*

$$I_B^n/I_B^{n+1} \to I_A^n/I_A^{n+1}$$

is an isomorphism.

Proof. One can show that $W(B) \to W(A)$ is an isomorphism [1]. The lemma follows immediately. □

The following commutative diagram finishes the proof by reducing to the smooth case

$$\begin{array}{ccc} K_n^M(B)/2 & \longrightarrow & K_n^M(A)/2 \\ \downarrow & & \downarrow \\ I_B^n/I_B^{n+1} & \longrightarrow & I_A^n/I_A^{n+1} \end{array}$$

□

Finally, it follows that the transfer for Milnor K-groups of étale finite extensions B/A of semi-local rings constructed in Section 3.5 does not depend on any choice made if the rings are equicharacteristic.

Theorem 3.7.10. *If A contains an infinite field the transfer*

$$N_{B/A} : K_n^M(B) \longrightarrow K_n^M(A)$$

does not depend on the chosen generator of B over A and is functorial.

Proof. Let $B = A[t]/(\pi)$. Choose a regular, semi-local ring A' containing an infinite field and a map $i : A' \to A$ such that there exists a polynomial $\pi' \in A'[t]$ with $i(\pi') = \pi$. According to Proposition 5.3 and by choosing A' large we have to show that

$$N_{B/A} : K_n^M(B') \longrightarrow K_n^M(A')$$

does not depend on the generator of $B' = A'[t]/(\pi')$ over A'. The theorem follows from the diagram

$$\begin{array}{ccc} K_n^M(B') & \longrightarrow & K_n^M(Q(B')) \\ {\scriptstyle N}\downarrow & & \downarrow{\scriptstyle N} \\ K_n^M(A') & \longrightarrow & K_n^M(Q(A')) \end{array}$$

which according to Proposition 5.3 is commutative and by using Theorem 6.1 and Kato's results about the transfer in the field case [15]. □

3.8 Appendix

In this appendix we generalize a factorization result used by Gabber [7] in his proof of the surjectivity of the homomorphism of sheaves

$$\mathcal{K}_n^M \longrightarrow \mathcal{H}^{n,n}$$

on the big Zariski site of smooth varieties over an infinite field.

We use the expression feasible polynomial to mean a polynomial with highest non-vanishing coefficient invertible.

Theorem 3.8.1. *Let A be a semi-local ring with infinite residue fields, $\pi \in A[t]$ of degree $d > 2$ a monic polynomial. Then every $[p] \in (A[t]/(\pi))^\times$, $p \in A[t]$ and $\deg(p) < d$, can be written as*

$$p = f\,\pi + p_1\,p_2$$

with $f, p_1, p_2 \in A[t]$ feasible, $\deg(f) = d-2$, $\deg(p_i) = d-1$.
Furthermore we can achieve that in (p, f, π, p_1, p_2) every two elements are coprime and

$$\begin{aligned} \mathrm{Disc}_{2d-2}(p_1\,p_2) &\in A^\times \\ \mathrm{Disc}_{d-2}(f) &\in A^\times\,. \end{aligned}$$

Proof. Let m be the Jacobson radical of A and denote the image of a polynomial $g \in A[t]$ in $(A/m)[t]$ by $\bar{g}$. We first reduce to the case that A is a field. Suppose the result is known in this case and

$$\bar{p} = \bar{f}\,\bar{\pi} + \bar{p_1}\,\bar{p_2}$$

is such a factorization in A/m. We can choose $p_1 \in A[t]$ such that $\deg(p_1) = d-1$. Because by assumption $\mathrm{Res}_{d,d-1}(\pi, p_1) \in A^\times$ we see that we can choose $p_2, f \in A[t]$, $\deg(p_2) = d-1$, $\deg(f) = d-2$, such that

$$p = f\,\pi + p_1\,p_2\,.$$

As $f\,\pi, p_1, p_2$ are feasible the pairwise coprimeness of $(p, f\,\pi, p_1, p_2)$ is automatically satisfied, the same is true for the condition $\mathrm{Disc}_{2d-2}(p_1\,p_2) \in A^\times$.

Now we suppose A is an infinite field. We identify the space of polynomials of degree at most $d-1$ with $\mathbb{A}_A^{d-1}$. Then the set of such polynomials prime to an arbitrary non-vanishing polynomial in $A[t]$ is dense and open in $\mathbb{A}_A^{d-1}$. Furthermore it is clear that every dense open set in $\mathbb{A}_A^{d-1}$ contains an A-rational point, moreover the intersection of finitely many dense open sets is dense and open.

Therefore it is immediately clear that we always have a factorization

$$p = f\,\pi + p_1\,p_2\,.$$

where p_1, p_2 are of degree $d-1$ and $\mathrm{Disc}_{d-1}(p_1), \mathrm{Disc}_{d-1}(p_2) \neq 0$ (f is then automatically of degree $d-2$). Next we show that we can choose such a factorization generically so as to satisfy $\mathrm{Disc}_{2d-2}(p_1\,p_2) \neq 0$.

Case $\mathrm{char}(A) = 0$*:* The idea is to give a factorization with $p_1(0), p_2(0) \neq 0$ such that

$$\mathrm{Res}_{d-1,d-1}(t^{d-1}p_1(1/t), t^{d-1}p_2(1/t)) \neq 0 \tag{3.18}$$

Choose $x_0 \in A$ such that $p(x_0), \pi(x_0) \neq 0$ and let $f = p(x_0)/\pi(x_0)$. We can further assume $p(0) - f\,\pi(0) \neq 0$ and $p'(x_0) - f\,\pi'(x_0) \neq 0$ (the latter because $\mathrm{char}(A) = 0$).

Now let $p_1 = t - x_0$. It is obvious that (3.18) is satisfied for this choice and therefore it is generically satisfied. But generically p_1, p_2 are of degree $d-1$. In this case (3.18) is equivalent to

$$\mathrm{Res}_{d-1,d-1}(p_1, p_2) \neq 0\ .$$

This shows we generically have $\mathrm{Disc}_{2d-2}(p_1\,p_2) \neq 0$ and proves the theorem in case $\mathrm{char}(A) = 0$.

Case $\mathrm{char}(A) \neq 0$*:* The above proof works except in case $p' = \pi' = 0$. Now one can use a similar argument in order to show (3.18) is satisfied generically. We take f to be of degree 1 and two different points $x_0, x_1 \in A$ such that $p(x_0) - f(x_0)\,\pi(x_0) = 0$, $p(x_1) - f(x_1)\,\pi(x_1) = 0$, $\pi(x_0) \neq 0$, $\pi(x_1) \neq 0$, $p(0) - f(0)\,\pi(0) \neq 0$. Let $p_1 = (t - x_0)(t - x_1)$.

The fact that $f \in A[t]$ can be assumed to satisfy $\mathrm{Disc}_{d-2}(f) \in A^\times$ follows because over the algebraic closure of A there clearly exists at least one such factorization with no other conditions imposed so that $\mathrm{Disc}_{d-2}(f) \in A^\times$ is generically satisfied.

□

There is an equivalent theorem for $\deg(\pi) = 2$ which we state below.

Theorem 3.8.2. *Let A be a semi-local ring with infinite residue fields, $\pi \in A[t]$ a monic polynomial of degree two, $p \in A[t]$ an element coprime to π with $\deg(p) < 2$. Then there exists a factorization*

$$p = f\,\pi + p_1\,p_2\,p_3$$

with f, p_1, p_2, p_3 feasible,

$$\deg(f) = 1, \deg(p_1) = 1, \deg(p_2) = 1, \deg(p_3) = 1$$

$\mathrm{Disc}_3(p_1\,p_2\,p_3) \in A^\times$ *and such that in $(p, f, \pi, p_1, p_2, p_3)$ each two elements are coprime.*

Remark 3.8.3. *One can in fact show that given a monic polynomial $g \in A[t]$ the factorization in Theorem 8.1 and 8.2 can be chosen such that $f, p_1, p_2(, p_3)$ are coprime to g.*

Remark 3.8.4. *For given $d_1 > 2, d_2 > 0$ there exists an integer M such that a factorization (coprime in the above sense to a monic $g \in A[t]$ of degree d_2) as in Theorem 8.1 for any monic π of degree d_1 and any $p \in A[t]$ of degree smaller d_1 exists if the number of elements in each residue field of A is greater than M.*
Similarly for Theorem 8.2.

Chapter 4

Finite residue fields

4.1 Overview

It is well known that Milnor K-theory of fields is a very nice cohomology theory in the sense that it encodes important arithmetic information about the field in question. Or in fancy words it is part of a motivic cohomology theory of smooth varieties [25].

In view of this fact the following question, which is one of the motivations for this chapter, seems to be reasonable:

Question: How can we generalize Milnor K-theory from fields to commutative rings?

If we want to generalize Milnor K-theory to more general rings we could simply copy the symbolic definition proposed by Milnor [28] for fields to an arbitrary commutative ring A as was done in Definition 3.2.1. This is what we would like to call naive Milnor K-theory in this chapter. But there are at least two problems:

1. Thomason [38] showed that a good definition of Milnor K-theory of smooth varieties, which generalizes the one for fields, does not exist. Here good means that the theory should satisfy standard properties of a cohomology theory like for example $\mathbb{A}^1$-homotopy invariance and there should exist a functorial homomorphism to Quillen K-theory.
   This means that we can expect a good Milnor K-theory only for local rings.

2. Even if we restrict to local rings the functor K_*^M defined above is not what we would like to call Milnor K-theory. For example it does not satisfy the Gersten conjecture, compare Remark 4.2.9.

In spite of (2) the naive Milnor K-ring of a local ring with infinite residue field yields a good cohomology theory as the results proved in [29] and Chapter 3 suggest. So these considerations reduce our question posed above to the question how to define Milnor K-theory of local rings with finite residue fields. As the author hopes the reader will find a satisfying answer in Section 4.2.

There we show the following: Let F be an abelian sheaf on the big Zariski site of all schemes. Assume F is continuous, i.e. that it commutes with filtering inverse limits of affine schemes, see Definition 1.0.6, and has some kind a transfer for finite étale extensions of local rings with infinite residue fields – for an explanation see Section 4.2. In Theorem 4.2.5 we prove:

Theorem A. *There exists a universal transformation of continuous sheaves $F \to \hat{F}$ such that $\hat{F}$ has a transfer for finite étale extensions of local rings. Moreover for a local ring A with infinite residue field we have $F(A) = \hat{F}(A)$.*

In fact by what is proved in Chapter 3 we can take F to be the sheafification of K_*^M, which we denote by $\mathcal{K}_*^M$, and get some improved Milnor K-sheaf $\hat{\mathcal{K}}_*^M$. In order to convince the reader that this improved Milnor K-sheaf is in fact the correct one, we have collected some basic results on the latter in Proposition 4.2.8. We should remark that the sheaf $\hat{\mathcal{K}}_*^M$ does already appear in unpublished notes of Gabber [7], but without the tranfer map of Chapter 3 it is quite hard to study it.

The second aim of this article is to show that the improved Milnor K-ring is generated by symbols. In fact this is not at all clear in view of the definition of $\hat{\mathcal{K}}_*^M$ via Theorem A. Our main result, Theorem 4.2.7, whose proof unfortunately requires a very messy calculation in polynomial rings, says:

Theorem B. *The natural homomorphism of Zariski sheaves*

$$\mathcal{K}_*^M \longrightarrow \hat{\mathcal{K}}_*^M$$

is surjective.

Via the Milnor Conjecture proved by Voevodsky et al. [30] this result has some interesting application to quadratic forms over local rings. Furthermore let $H^m_{mot}(Spec(A), \mathbb{Z}(n))$ be the hypercohomology of the complex of Zariski sheaves $\mathbb{Z}(n)$ constructed in Chapter 1. We can deduce in Corollary 4.3.4:

Corollary. *The motivic cohomology ring*

$$(H^n_{mot}(Spec(A), \mathbb{Z}(n)))_{n\geq 0}$$

for a regular local equicharacteristic ring A is generated by elements of degree one.

This corollary was – at least implicitly – predicted by the Beilinson–Lichtenbaum conjectures on motivic cohomology.

4.2 Improved Milnor K-theory of local rings

For a ring A we will denote $\mathcal{K}_*^M(Spec\, A)$ also by $\mathcal{K}_*^M(A)$. Below we will improve the definition of Milnor K-theory in order to get a sensible theory for local rings with finite residue fields. Therefore K_*^M will usually be called naive K-theory.

The following facts are standard for Milnor K-groups of local rings with infinite residue fields, see [29], [36] and [40].

Proposition 4.2.1. *Let A be a local ring with infinite residue field. Then we have*

1. *The natural map $K_2^M(A) \to K_2(A)$ from Milnor K-theory to algebraic K-theory is an isomorphism.*

2. *The relation $\{a, -a\} = 0$ holds in $K_2^M(A)$ for $a \in A^\times$.*

3. *The ring $K_*^M(A)$ is skew-symmetric.*

4. *For $a_1, \dots, a_n \in A^\times$ with $a_1 + \cdots + a_n = 1$ the relation*

$$0 = \{a_1, \dots, a_n\} \in K_n^M(A)$$

   holds.

Moreover there exists a transfer for Milnor K-groups which is constructed in Chapter 3 and whose main properties we recall in the next proposition. The transfer will be essential for the constructions of this paper.

Let $i : A \to B$ be a finite étale extension of local rings with infinite residue fields. Fix an explicit presentation $B \cong A[t]/(f)$ which exists by EGA IV 18.4.5.

Proposition 4.2.2. *For a fixed presentation of B over A there exists a canonical transfer homomorphisms*

$$N_{B/A} : K_n^M(B) \longrightarrow K_n^M(A)$$

satisfying:

1. *$N_{B/A} : K_1^M(B) \to K_1^M(A)$ is the usual norm map on unit groups.*
   $N_{B/A} : K_0^M(B) \to K_0^M(A)$ is multiplication by $\deg(B/A)$.

2. *The projection formula holds, i.e. for $x \in K_n^M(A)$, $y \in K_m^M(B)$ we have*

$$x\, N_{B/A}(y) = N_{B/A}(i_*(x)\, y) \in K_{n+m}^M(A)\,.$$

3. *If A contains a field the transfer does not depend on the presentation of B over A chosen.*

4. *Let $j : A \to A'$ be a homomorphism of local rings and let $i' : A' \to B' = B \otimes_A A'$ be the induced inclusion, for which we fix the induced presentation. Assume B' is local. Then the diagram*

$$\begin{array}{ccc} K_n^M(B) & \longrightarrow & K_n^M(B') \\ \downarrow{\scriptstyle N_{B/A}} & & \downarrow{\scriptstyle N_{B'/A'}} \\ K_n^M(A) & \xrightarrow[j_*]{} & K_n^M(A') \end{array}$$

   commutes.

Proof. This is proved in Section 3.5. □

Next we consider general abelian sheaves with a weak form of a transfer. In fact we will construct the improved Milnor K-groups axiomatically such that they have a transfer map.

Let $\mathfrak{S}$ be the category of abelian sheaves on the big Zariski site of all schemes. Let $\mathfrak{ST}$ be the full subcategory of $\mathfrak{S}$ such that a sheaf F is in $\mathfrak{ST}$ if for every finite étale extension of local rings $i : A \subset B$ there exists a compatible system of norms

$$\left[N_{B'/A'} : F(B') \to F(A')\right]_{A'}$$

where A' runs over all local A-algebras for which $B' = B \otimes_A A'$ is also local. Compatibility means that if $A' \to A''$ are both local A-algebras such that $B' = B \otimes_A A'$ and $B'' = B \otimes_A A''$ are local the diagram

$$\begin{array}{ccc} F(B') & \longrightarrow & F(B'') \\ {\scriptstyle N_{B'/A'}}\downarrow & & \downarrow{\scriptstyle N_{B''/A''}} \\ F(A') & \longrightarrow & F(A'') \end{array}$$

commutes. Furthermore we assume that if $i' : A' \to B'$ is the induced inclusion our norm $N_{B'/A'}$ satisfies

$$N_{B'/A'} \circ i'_* = \deg(B/A)\, id_{F(A')} \, .$$

Let $\mathfrak{ST}^\infty$ be the full subcategory of sheaves in $\mathfrak{S}$ which have such norms if we restrict to the system of local A-algebras A' with infinite residue fields.

Proposition 4.2.3. *The functor $\mathcal{K}_n^M$ is an object of $\mathfrak{ST}^\infty$ for all $n \geq 0$.*

Proof. Immediate from Proposition 4.2.2. □

Actually the Milnor K-functor should have some more global canonical transfer but at the moment we can define it only in the case of equicharacteristic schemes. We shall not be concerned with this problem here.

Proposition 4.2.4. *The Milnor K-sheaf $\mathcal{K}_*^M$ is continuous.*

Proof. It is clear from the definition that K_*^M is continuous. Furthermore a simple calculation shows that if a presheaf is continuous the associated sheaf in the Zariski topology is so too. □

Our existence and uniqueness result, which is motivated by a construction of improved Milnor K-theory due to Gabber [7], reads now:

Theorem 4.2.5. *For every continuous $F \in \mathfrak{ST}^\infty$ there exists a universal continuous $\hat{F} \in \mathfrak{ST}$ and a natural transformation $F \to \hat{F}$. That is for arbitrary continuous $G \in \mathfrak{ST}$ and natural transformation $F \to G$ there exists a unique natural transformation $\hat{F} \to G$ such that the diagram commutes.*

$$\begin{array}{ccc} F & \longrightarrow & \hat{F} \\ & \searrow \quad \swarrow & \\ & G & \end{array}$$

Moreover for a local ring A with infinite residue field we have $F(A) = \hat{F}(A)$.

Before we can give the proof we have to recall the construction of the rational function ring $A(t)$ over a ring A and some of its basic properties. For a commutative ring A we let $A(t_1, \dots, t_n)$ be the ring $A[t_1, \dots, t_n]_S$ where S is the multiplicative set consisting of all polynomials $\sum_I a_{(I)} t^I$ such that the ideal in A generated by all the coefficients $a_{(I)} \in A$ is A itself. If A is local with maximal ideal m the ring $A(t_1, \dots, t_n)$ is local too, in fact it is easy to see that for A local $S = A[t_1, \dots, t_n] - m_t$ where m_t is the prime ideal $m\,A[t_1, \dots, t_n]$. Denote by $i : A \to A(t)$ the natural ring homomorphism. Denote by i_1 resp. i_2 the natural ring homomorphism $A(t) \to A(t_1, t_2)$ which sends t to t_1 resp. t_2.

Lemma 4.2.6. *If $A \subset B$ is a finite étale extension of local rings there is a canonical isomorphism $B \otimes_A A(t_1, \dots, t_n) \tilde{\to} B(t_1, \dots, t_n)$*

Proof. For simplicity we restrict to $n = 1$. Let m be the maximal ideal of A. Consider the finite extension of rings $A[t] \to B[t]$. Let m_t^A be the prime ideal $m\,A[t]$ and m_t^B be the prime ideal $m\,B[t]$. The latter ideal is indeed a prime ideal, because $m\,B$ is the maximal ideal of B by assumption. This also implies that

$$B(t) = B[t]_{m_t^B} .$$

Moreover the finiteness of $A \subset B$ implies that m_t^B is the only prime ideal over m_t^A. Recall that according to standard facts $B[t] \otimes_{A[t]} A(t)$ is the semi-local ring whose maximal ideals correspond to the finite set of prime ideals in $B[t]$ which lie over m_t^A. But as we saw there is only one prime ideal over m_t^A, namely m_t^B, so

$$B \otimes_A A(t) = B[t] \otimes_{A[t]} A(t) \qquad \text{and} \qquad B(t) = B[t]_{m_t^B}$$

must be isomorphic. ☐

Proof of Theorem 4.2.5. For an arbitrary Zariski sheaf G we let $\hat{G}$ be the Zariski sheafification of the following presheaf defined on affine schemes:

$$Spec(A) \longmapsto ker[G(A(t)) \xrightarrow{i_{1*} - i_{2*}} G(A(t_1, t_2))]$$

We claim that if G is an object in $\mathfrak{S}\mathfrak{T}^\infty$ then $\hat{G}$ is an object in $\mathfrak{S}\mathfrak{T}$. The continuity of $\hat{G}$ follows because the presheaf on affine schemes just defined is clearly continuous and the Zariski sheafification of a continuous presheaf is continuous. For a finite étale extension of local rings $A \subset B$ Lemma 4.2.6 and the existence of a compatible system of norms allow us to write down the commutative diagram

$$\begin{array}{ccccc} G(B) & \longrightarrow & G(B(t)) & \xrightarrow{i_{1*} - i_{2*}} & G(B(t_1, t_2)) \\ & & \downarrow N & & \downarrow N \\ G(A) & \longrightarrow & G(A(t)) & \xrightarrow[i_{1*} - i_{2*}]{} & G(A(t_1, t_2)) \end{array}$$

So there exists a norm map $\hat{G}(B) \to \hat{G}(A)$ for which one easily verifies the compatibility with base change. This shows that $\hat{G}$ is an object in $\mathfrak{S}\mathfrak{T}$.

The next proposition will be essential for the proof of the universal property of $\hat{F}$.

Proposition 4.2.7. *Let $G \in \mathfrak{ST}$ (resp. $G \in \mathfrak{S}$) be continuous. Then for a local ring A (resp. a local ring with infinite residue field) we have $G(A) = \hat{G}(A)$.*

Proof. First we prove the statement in parenthesis. So let A have infinite residue field and let $G \in \mathfrak{S}$ be continuous. We will prove the injectivity of $G(A) \to \hat{G}(A)$ first. In the following $S' \subset S$ will always be some finitely generated submonoid where $S \subset A[t]$ is defined as above. So by continuity we clearly have

$$\hat{G}(A) \subset G(A[t]_S) = \varinjlim_{S'} G(A[t]_{S'})$$

So it is enough to show that $G(A) \to G(A[t]_{S'})$ is injective for every S'. For fixed S' we will explain how to choose an element $\alpha \in A$ such that for all $p \in S'$ we have $p(\alpha) \in A^\times$. Let $p_1, \dots, p_r \in S'$ be generators of the finitely generated monoid S'. Because the residue field of A is infinite, it is possible to find $\alpha \in A$ with $p_1(\alpha) \cdots p_r(\alpha) \in A^\times$. This is the element α we were looking for. Let $\pi : A[t]_{S'} \to A$ be the ring homomorphism such that t maps to α. As

$$G(A) \longrightarrow G(A[t]_{S'}) \xrightarrow{\pi_*} G(A)$$

is the identity the injectivity of the first arrow follows.

For the surjectivity of $G(A) \to \hat{G}(A)$ we argue similarly. Let $S' \subset A(t)^\times \cap A[t]$ and $S'' \subset A(t_1, t_2)^\times \cap A[t_1, t_2]$ be some finitely generated submonoids and $x \in G(A[t]_{S'})$ such that the arrow

$$G(A[t]_{S'}) \xrightarrow{i_{1*} - i_{2*}} G(A[t_1, t_2]_{S''})$$

is well defined and kills x. Since S' and S'' are finitely generated and the residue field of A is infinite, we can construct an element $\alpha \in A$ such that for all $p \in S'$ we have $p(\alpha) \in A^\times$ and for all $p \in S''$ we have $p(t, \alpha) \in A(t)^\times$. Denote by $\pi : A[t]_{S'} \to A$ resp. $\pi' : A[t_1, t_2]_{S''} \to A(t)$ the ring homomorphisms sending t to α resp. t_1 to t and t_2 to α. Now the sequence of equalities

$$i_* \circ \pi(x) = \pi' \circ i_{2*}(x) = \pi' \circ i_{1*}(x) = im(x) \in G(A(t))$$

proves the surjectivity of $G(A) \to \hat{G}(A)$.

Next we prove that for $G \in \mathfrak{ST}$ continuous and A local $G(A) \to \hat{G}(A)$ is an isomorphism. We prove injectivity first. Fix an arbitrary prime p. Consider a tower of finite étale extensions of A

$$A \subset A_1 \subset A_2 \subset \cdots \subset A_\infty$$

such that A_m is local, $[A_m : A_{m-1}] = p$ and $\cup_m A_m = A_\infty$. Now $G(A_\infty) = \hat{G}(A_\infty)$ according to the first part of the proof. Consider $x \in ker[G(A) \to \hat{G}(A)]$. So by continuity $x \in ker[G(A) \to G(A_m)]$ for some $m > 0$. The existence of a transfer homomorphism $N : G(A_m) \to G(A)$ with

$$G(A) \longrightarrow G(A_m) \xrightarrow{N} G(A)$$

equal to multiplication by p^m shows that $p^m x = 0$. As this holds for all primes p we have proved injectivity.

In order to prove surjectivity of $G(A) \to \hat{G}(A)$ consider $x \in \hat{G}(A)$ and fix a prime p and a tower of finite étale extensions as in the injectivity proof. Again observe that $G(A_\infty) = \hat{G}(A_\infty)$.

Because of the continuity of G there exists $x_m \in G(A_m)$ which has the same image in $\hat{G}(A_m)$ as x. The commutative diagram

$$\begin{array}{ccc} G(A_m) & \longrightarrow & \hat{G}(A_m) \\ {\scriptstyle N_{A_m/A}}\downarrow & & \downarrow{\scriptstyle N_{A_m/A}} \\ G(A) & \xrightarrow[i_*]{} & \hat{G}(A) \end{array}$$

implies that $i_* \circ N_{A_m/A}(x_m) = p^m x$. So if we choose two different primes p_1, p_2 we see that there exists an integer n with $p_1^n x \in im(i_*)$ and $p_2^n x \in im(i_*)$. Choose $\alpha, \beta \in \mathbb{Z}$ such that $\alpha\, p_1^n + \beta\, p_2^n = 1$. Then $x = \alpha\, p_1^n x + \alpha\, p_2^n x \in im(i_*)$, what we had to show. □

Now we can finish the proof. Let F and G be as in the statement of the theorem. Define the homomorphism $\hat{F} \to G$ such that the following diagram is commutative

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & \nearrow & \downarrow{\scriptstyle \alpha} \\ \hat{F} & \longrightarrow & \hat{G} \end{array}$$

where α is in isomorphism by Proposition 4.2.7. The uniqueness of the homomorphism $\hat{F} \to G$ follows form the commutative diagram

$$\begin{array}{ccccc} F(A(t)) & \xrightarrow{\beta} & \hat{F}(A(t)) & \longleftarrow & \hat{F}(A) \\ \downarrow & & \downarrow & & \downarrow \\ G(A(t)) & = & G(A(t)) & \xleftarrow{\gamma} & G(A) \end{array}$$

where A is a local ring, since by Proposition 4.2.7 β is an isomorphism and γ is injective.

□

The next proposition comprises basic information about our improved Milnor K-theory $\hat{\mathcal{K}}_*^M$. We will only sketch the proofs.

Proposition 4.2.8. *Let (A, m) be a local ring. Then:*

1. $\hat{\mathcal{K}}_1^M(A) = A^\times$.
2. *$\hat{\mathcal{K}}_*^M(A)$ has a natural graded commutative ring structure.*
3. *Proposition 4.2.1 and 4.2.2 remain true for any any local ring A if we replace K_*^M by $\hat{\mathcal{K}}_*^M$.*
4. *If F is a field we have $K_n^M(F) = \hat{\mathcal{K}}_n^M(F)$.*
5. *For every $n \geq 0$ there exists a universal natural number M_n such that if $|A/m| > M_n$ the natural homomorphism*
$$K_n^M(A) \longrightarrow \hat{\mathcal{K}}_n^M(A)$$
is an isomorphism.

6. *There exists a homomorphism*

$$K_n(A) \longrightarrow \hat{\mathcal{K}}_n^M(A)$$

such that the composition

$$\hat{\mathcal{K}}_n^M(A) \longrightarrow K_n(A) \longrightarrow \hat{\mathcal{K}}_n^M(A)$$

is multiplication by $(n-1)!$ *and the composition*

$$K_n(A) \longrightarrow \hat{\mathcal{K}}_n^M(A) \longrightarrow K_n(A)$$

is the Chern class $c_{n,n}$.

7. *If* (A, I) *is a Henselian pair in the sense of [9] and* $s \in \mathbb{N}$ *is invertible in* A/I *the map*

$$\hat{\mathcal{K}}_n^M(A)/s \longrightarrow \hat{\mathcal{K}}_n^M(A/I)/s$$

is an isomorphism.

8. *Let* A *be regular and equicharacteristic,* $F = Q(A)$ *and* $X = Spec\, A$. *The Gersten conjecture holds for Milnor* K*-theory, i.e. the Gersten complex*

$$0 \longrightarrow \hat{\mathcal{K}}_n^M(A) \longrightarrow K_n^M(F) \longrightarrow \oplus_{x \in X^{(1)}} K_{n-1}^M(k(x)) \longrightarrow \cdots$$

is exact.

9. *Let* X *be a regular scheme containing a field. There is a natural isomorphism*

$$H^n_{Zar}(X, \hat{\mathcal{K}}_n^M) \cong CH^n(X)\,.$$

10. *If* A *is equicharacteristic of characteristic prime to* 2 *the map*

$$i_A : \hat{\mathcal{K}}_3^M(A) \longrightarrow K_3(A)$$

is injective.

11. *Let* A *be regular and equicharacteristic. There is a natural isomorphism*

$$\hat{\mathcal{K}}_n^M(A) \xrightarrow{\sim} H^n_{mot}(Spec(A), \mathbb{Z}(n))$$

onto motivic cohomology.

Proof. (1): Since K_1^M is an objects in $\mathfrak{ST}$, the isomorphism $\hat{\mathcal{K}}_1^M(A) = K_1^M(A) = A^\times$ follows from Proposition 4.2.7.
(2): This follows immediately from the hat construction in the proof of Theorem 4.2.5.
(3): It is well known that the sheaf associated to $X \mapsto K_2(X)$ is an object in $\mathfrak{ST}$, so that $\hat{\mathcal{K}}_2(A) = K_2(A)$ by Proposition 4.2.7. But if A is a local ring with infinite residue field we have $K_2^M(A) = K_2(A)$ according to Proposition 4.2.1 (1) and the isomorphism of sheaves $\hat{\mathcal{K}}_2^M = K_2$

follows from the definition of the 'hat' in the proof of Theorem 4.2.5. The rest follows from the injectivity of

$$\hat{\mathcal{K}}_n^M(A) \longrightarrow \hat{\mathcal{K}}_n^M(A(t)) = K_n^M(A(t)) \, .$$

(4): If F is infinite this follows from Proposition 4.2.7, if F is finite then $K_n^M(F) = 0$ and so it suffices to show that $K_n^M(F) \to \hat{\mathcal{K}}_n^M(F)$ is surjective. Let $s_i : K_n^M(F(t_1, \dots, t_i)) \to K_n^M(F(t_1, \dots, t_{i-1}))$ be the specialization homomorphism which maps $\{t_i^j + a_{j-1} t_i^{j-1} + \cdots + a_0\}$ to 0 for $a_k \in F(t_1, \dots, t_{i-1})$ $(0 \leq k < j)$ so that

$$K_n^M(F(t_1, \dots, t_{i-1})) \stackrel{inc_{i-1}}{\longrightarrow} K_n^M(F(t_1, \dots, t_i)) \stackrel{s_i}{\longrightarrow} K_n^M(F(t_1, \dots, t_{i-1}))$$

is the identity. Then with the notation as in the proof of the theorem we have for $x \in \hat{\mathcal{K}}_n^M(F) \subset K_n^M(F(t))$

$$\hat{\mathcal{K}}_n^M(F) \ni inc_0 \circ s_1(x) = s_2 \circ i_{2*}(x) = s_2 \circ i_{1*}(x) = x \, .$$

(5): It was shown in Section 3.5 that there exists an $M_n \in \mathbb{N}$ such that the statement of Proposition 4.2.2 remains true if the local ring A has more than M_n elements in its residue field and if $\deg(B/A) \leq 3$. Now an analog of the second part of proof of Proposition 4.2.7 with $p_1 = 2$ and $p_2 = 3$ gives (5).
(6): This follows immediately from [29].
(7): An elementary calculation shows that

$$K_n^M(A)/s \longrightarrow K_n^M(A/I)/s$$

is an isomorphism for every local ring A with s invertible in A/I. Now a norm argument shows the analogous result for the improved Milnor K-groups. This is accomplished by choosing an étale local extension $A \subset A'$ of some prime power degree q, coprime to s, such that the residue field of A' has more than M_n elements. Here M_n is the natural number from part (5). Observe that $(A', A' I)$ is again a Henselian pair. Consider the commutative diagram

$$\begin{array}{ccc} \hat{\mathcal{K}}_n^M(A')/s & \longrightarrow & \hat{\mathcal{K}}_n^M(A'/I)/s \\ \downarrow N & & \downarrow N \\ \hat{\mathcal{K}}_n^M(A)/s & \longrightarrow & \hat{\mathcal{K}}_n^M(A/I)/s \end{array}$$

where the upper horizontal arrow is an isomorphism by what has been said so far. A simple diagram chase shows that the kernel and cokernel of $\hat{\mathcal{K}}_n^M(A)/s \longrightarrow \hat{\mathcal{K}}_n^M(A/I)/s$ have exponent q and must therefore vanish.
(8): This complex was constructed in [16]. Again if A has more than M_n elements in its residue field the result was proven in Section 3.7. If not one uses a norm trick as in (7).
(9): Immediate from (8).
(10): If A is a field this was shown by Kahn using Voevodsky's proof of the Milnor conjecture [14]. If A is regular it follows from the field case and (8). If A is not regular we first use the norm trick and can and will assume that A has more than M_3 elements in its residue field. Next we use Hoobler's trick [11] and choose some regular equicharacteristic local ring A' such that there exists an exact sequence

$$0 \longrightarrow I \longrightarrow A' \longrightarrow A \longrightarrow 0$$

such that (A', I) is a Henselian pair. Let x be in $ker(i_A)$. Then $2x = 0$ according to (6). Next choose $x' \in K_3^M(A') = \hat{\mathcal{K}}_3^M(A')$ which maps to x. An elementary argument left to the reader shows that we can choose x' such that $2x' = 0$. Now remember that the two torsion in $K_3(A')$ is isomorphic to the two torsion in $K_3(A)$ by the rigidity of algebraic K-theory, see [9], so that $i_{A'}(x') = 0$ and finally $x' = 0$ by the regular case proved above.
(11): Using Corollary 1.0.3 we are reduced to the case A essentially smooth over a field. Then improved Milnor K-theory fulfills the Gersten conjecture by (8) and motivic cohomology does so by Proposition 1.0.9, so the result follows from an easy diagram chase of Gersten complexes using the fact that (11) is known if A is a field, Proposition 2.2.1. The details can be found in Section 3.7. □

Remark 4.2.9. *In general the map*

$$K_n^M(A) \longrightarrow \hat{\mathcal{K}}_n^M(A)$$

is not an isomorphism. For example for $n = 2$ we have $\hat{\mathcal{K}}_2^M(A) = K_2(A)$ according to Proposition 4.2.8(3) and it was shown in the appendix to [13] that for $A = \mathbb{F}_2[t]_{(t)}$ the groups $K_2^M(A)$ and $K_2(A)$ are not isomorphic.

4.3 Generation by symbols

In this section we propose a conjectural determination of $\hat{\mathcal{K}}_*^M(A)$. Set

$$I = \ker[K_2^M(A) \to K_2(A)]\ .$$

One can show that $I = 0$ if $|A/m| > 3$, where m is the maximal ideal of A. In fact there are explicit descriptions of I by van der Kallen, Maazen and Stienstra [40], [26] and Kolster [22].

Conjecture 4.3.1. *For any local ring A the natural homomorphism of graded rings*

$$K_*^M(A)/(I) \longrightarrow \hat{\mathcal{K}}_*^M(A)$$

is an isomorphism.

We will prove the surjectivity as our main theorem, which is Theorem B of the introduction.

Theorem 4.3.2. *Let A be a local ring. Then the map*

$$\eta_A : K_*^M(A) \longrightarrow \hat{\mathcal{K}}_*^M(A)$$

is surjective.

In the proof we use the statement of the theorem for $n = 2$ which is classical modulo Proposition 4.2.8(3), see for example [22]:

Proposition 4.3.3 (Dennis–Stein)**.** *The map*

$$K_2^M(A) \longrightarrow K_2(A) = \hat{\mathcal{K}}_2^M(A)$$

is surjective.

An interesting application of our theorem concerns motivic cohomology of local rings.

Corollary 4.3.4. *Assume A to be regular local and equicharacteristic. The motivic cohomology ring*

$$[H^n_{mot}(A, \mathbb{Z}(n))]_{n \geq 0}$$

is generated by elements of degree one.

Proof. Immediate from Proposition 4.2.8(11). □

Finally, all the results in [21, Section 0] remain true for local rings with finite residue fields of characteristic different from 2. For the convenience of the reader we state them without proofs in the next proposition. Let $W(A)$ be the Witt ring of the local ring A and $I_A \subset W(A)$ the fundamental ideal. Assume that the residue field of A has characteristic different from 2.

Proposition 4.3.5. *If A is equicharacteristic the following holds:*

1. *If A is regular and $i : A \to F$ is the inclusion into the fraction field, $F = Q(A)$, the following conditions are equivalent:*
   (i) $q \in I^n_A \subset W(A)$.
   (ii) $i_*(q) \in I^n_F \subset W(F)$.

2. *If $A \subset A'$ is a finite étale extension with A' local the transfer*

$$N_{A'/A} : W(A') \to W(A)$$

   maps $I^n_{A'}$ to I^n_A.

4.4 Proof of Theorem 4.3.2

The heart of the proof will be to show that for a finite étale extension $A \subset B$ of local rings of degree $p = 2$ or 3 the transfer $N_{B/A} : \hat{\mathcal{K}}^M_n(B) \to \hat{\mathcal{K}}^M_n(A)$ maps the image of η_B to the image of η_A. In the proof of this fact an elementary but technical reasoning reduces us to $n = 1$ if $p = 2$ and $n = 2$ if $p = 3$ – this is the reason why we have to restrict to these two special primes. But in both cases we know that η_A is surjective, in fact for $n = 2$ this is the proposition due to Dennis and Stein mentioned above. Finally, using this key result the standard norm trick allows us to reduce the proof of the surjectivity of η_A to the case of infinite residue fields.

We start the proof by fixing $p = 2$ or 3. Consider a tower of finite étale extensions of A

$$A \subset A_1 \subset A_2 \subset \cdots \subset A_\infty$$

such that A_m is local, $[A_m : A_{m-1}] = p$ and $\cup_m A_m = A_\infty$.

Form Proposition 4.2.8(3) we know

1. $K^M_n(A_\infty) = \hat{\mathcal{K}}^M_n(A_\infty)$.

2. There exist transfers

$$N_{A_m/A_{m-1}} : \hat{\mathcal{K}}^M_n(A_m) \longrightarrow \hat{\mathcal{K}}^M_n(A_{m-1})$$

such that the composite

$$\hat{\mathcal{K}}_n^M(A_{m-1}) \longrightarrow \hat{\mathcal{K}}_n^M(A_m) \xrightarrow{N} \hat{\mathcal{K}}_n^M(A_{m-1})$$

is multiplication by p and such that the projection formula holds, compare Proposition 4.2.2 and the hat construction in the proof of theorem 4.2.5.

Now let x be in $\hat{\mathcal{K}}_n^M(A)$ and let x_m be the induced element in $\hat{\mathcal{K}}_n^M(A_m)$. By (1) there exists $x'_\infty \in K_n^M(A_\infty)$ with $\eta_{A_\infty}(x'_\infty) = x_\infty$. Because of the continuity of K_n^M and $\hat{\mathcal{K}}_n^M$ (Proposition 4.2.4) there exists $m \in \mathbb{N}$ and $x'_m \in K_n^M(A_m)$ with $\eta_{A_m}(x'_m) = x_m$. Now the next lemma produces $x' \in K_n^M(A)$ with $\eta_A(x') = p^m x$.

So making this construction for $p = 2$ and $p = 3$ we find $m_2, m_3 \geq 0$ and $x_2^*, x_3^* \in K_n^M(A)$ such that

$$\begin{aligned} \eta_A(x_2^*) &= 2^{m_2} x \\ \eta_A(x_3^*) &= 3^{m_3} x \end{aligned}$$

Choose $\alpha, \beta \in \mathbb{Z}$ satisfying

$$\alpha\, 2^{m_2} + \beta\, 3^{m_3} = 1\,.$$

Then $\eta_A(\alpha\, x_2^* + \beta\, x_3^*) = x$. So we deduce that $K_n^M(A) \to \hat{\mathcal{K}}_n^M(A)$ is surjective. In order to complete the proof we have to show:

Lemma 4.4.1. *With the notation of the theorem for $p = 2$ or 3 and $A \subset B$ a finite étale extension of local rings of degree p we have*

$$N_{B/A}(im\, \eta_B) \subset im\, \eta_A\,.$$

For $p = 2$ resp. $p = 3$ the proof of this lemma is reduced to the case $n = 1$ resp. $n = 2$ by the projection formula (Proposition 4.2.2(2)) and the next two sublemmas. But for $n \leq 2$ the lemma is clear as $K_1^M(A) = \hat{\mathcal{K}}_1^M(A)$ and as $K_2^M(A) \to \hat{\mathcal{K}}_2^M(A)$ is surjective by Proposition 4.3.3.

Sublemma 4.4.2. *For $p = 2$ the subgroup $im\, \eta_B \subset \hat{\mathcal{K}}_n^M(B)$ is generated by symbols*

$$\{a_1, a_2, \ldots, a_n\} \in \hat{\mathcal{K}}_n^M(B)$$

with $a_1 \in B^\times$ and $a_i \in A^\times$ for $i > 1$.

Sublemma 4.4.3. *For $p = 3$ the subgroup $im\, \eta_B \subset \hat{\mathcal{K}}_n^M(B)$ is generated by symbols*

$$\{a_1, a_2, \ldots, a_n\} \in \hat{\mathcal{K}}_n^M(B)$$

with $a_1, a_2 \in B^\times$ and $a_i \in A^\times$ for $i > 2$.

By EGA IV 18.4.5 we can write $B = A[t]/(f)$ where $f = t^p + \alpha_{p-1} t^{p-1} + \cdots + \alpha_0$ is irreducible modulo the maximal ideal $m \subset A$.

In the proof of the sublemmas we can by induction restrict to $n = 2$ for $p = 2$ and $n = 3$ for $p = 3$. Now the rest of the proof is by brute force.

Proof of Sublemma 4.4.2. We start with a symbol $\{a_1t+a_0, b_1t+b_0\} \in \hat{\mathcal{K}}_2^M(B)$, $a_1, a_0, b_1, b_0 \in A$ and have to show that it lies in the image of $K_1(A) \otimes_{\mathbb{Z}} K_1(B)$ in $\hat{\mathcal{K}}_2^M(B)$.
<u>1st step:</u> Reduce to $a_1 = b_1 = 1$.
If $a_1 \in A^\times$ then we write

$$\{a_1t + a_0\} = \{t + \frac{a_0}{a_1}\} + \{a_1\} \, .$$

and multiply from the right with $\{b_1t + b_0\}$. If $a_1 \notin A^\times$ write

$$\{a_1t + a_0\} = -\{t\} + \{t - \frac{a_1\alpha_0}{a_0 - a_1\alpha_1}\} + \{a_0 - a_1\alpha_1\} \in K_1^M(B) \, .$$

and multiply from the right with $\{b_1t + b_0\}$. Similarly we reduce to $b_1 = 1$.
<u>2nd step:</u> Reduce to $\bar{a}_0 \neq \bar{b}_0 \in A/m$ and $a_1 = b_1 = 1$.
By the first step we can assume $a_1 = b_1 = 1$ and $\bar{a}_0 = \bar{b}_0 \in A/m$.
<u>Case A:</u> $(t + \bar{a}_0)^2 \notin A/m$.

In this case write

$$\begin{aligned}\{t + a_0, t + b_0\} &= -\{t + a_0, t + a_0\} + \{t + a_0, t + \frac{a_0b_0 - \alpha_0}{a_0 + b_0 - \alpha_1}\} + \\ &\quad \{t + a_0, a_0 + b_0 - \alpha_1\}\end{aligned}$$

Remark that $\{t + a_0, t + a_0\} = \{t + a_0, -1\}$, $\bar{a}_0 + \bar{b}_0 - \bar{\alpha}_1 \neq 0$ by assumption and

$$\bar{a}_0 \neq \frac{\bar{a}_0\bar{b}_0 - \bar{\alpha}_0}{\bar{a}_0 + \bar{b}_0 - \bar{\alpha}_1} \, .$$

The latter because otherwise $\bar{a}_0$ would be a zero of

$$(t + \bar{a}_0)(t + \bar{b}_0) - \bar{f}$$

but $\bar{f}$ has no zeros in A/m.

<u>Case B:</u> $(t + \bar{a}_0)^2 \in A/m$.

Choose $c \in A$ with $\bar{c} \neq \bar{a}_0$. Then

$$\{t + a_0, t + b_0\} = -\{t + c, t + b_0\} + \{(a_0 + c - \alpha_1)t + a_0c_0 - \alpha_0, t + b_0\} \, .$$

Again as in the previous case $\bar{a}_0 + \bar{c} - \bar{\alpha}_1 \neq 0$ by assumption and

$$\bar{b}_0 \neq \frac{\bar{a}_0\bar{c} - \bar{\alpha}_0}{\bar{a}_0 + \bar{c}_0 - \bar{\alpha}_1} \, .$$

<u>3rd step:</u>
We have to show $\{t + a_0, t + b_0\} \in \hat{\mathcal{K}}_2^M(B)$, where $\bar{a}_0 \neq \bar{b}_0$, is induced by an element from $K_1^M(A) \otimes K_1^M(B)$. Write

$$\frac{t + a_0}{a_0 - b_0} + \frac{t + b_0}{b_0 - a_0} = 1$$

and correspondingly

$$0 = \{\frac{t + a_0}{a_0 - b_0}, \frac{t + b_0}{b_0 - a_0}\} \in \{t + a_0, t + b_0\} + K_1^M(A) \cdot K_1^M(B)$$

□

Proof of Sublemma 4.4.3. By a simple linear change of variables $t \mapsto t + c$, $c \in A$, we can and will assume $\alpha_1 \in A^\times$. We start with a symbol

$$\{a_2t^2 + a_1t + a_0, b_2t^2 + b_1t + b_0, c_2t^2 + c_1t + c_0\} \in \hat{\mathcal{K}}_3^M(B) .$$

1st step: Reduce to $a_2, b_2, c_2 \in A^\times \cup \{0\}$.
Let $a_2 \in m$. Then either $a_1 \in A^\times$ or $a_0 \in A^\times$. Assume for example $a_1 \in A^\times$. Then write

$$\{a_2t^2 + a_1t + a_0\} = -\{t\} + \{(a_1 - a_2\alpha_2)t^2 + (a_0 - a_2\alpha_1)t - a_2\alpha_0\} \in K_1^M(B)$$

Similarly for b and c.
2nd step: Reduce to $a_2 = b_2 = c_2 = 0$.
If $a_2 \in A^\times$ write

$$\{a_2t^2 + a_1t + a_0\} = -\{t + \frac{a_2\alpha_2 - a_1}{a_2}\} + \{\cdots\} \in K_1^M(B)$$

where $\cdots$ stands for some polynomial in $A[t]$ of degree one.
3rd step: Reduce to $a_1, b_1, c_1 \in A^\times$ and $a_2 = b_2 = c_2 = 0$.
If $a_1 \in m$ let $\beta = a_0 - a_1\alpha_2$ and $c = \frac{\beta\alpha_2 + a_1\alpha_1}{\beta}$ and write

$$\{a_1t + a_0\} = -2\{t\} - \{t + c\} + \{\cdots\} \in K_1^M(B)$$

where $\cdots$ stands for some polynomial of degree one with an invertible degree one coefficient. Here we use the fact $\alpha_1 \in A^\times$.
In the following we consider without restriction a symbol $\{t + a_0, t + b_0, t + c_0\} \in \hat{\mathcal{K}}_3^M(B)$.
4th step: Reduce to $\bar{a}_0 \neq \bar{b}_0$.
We can assume $\bar{a}_0 = \bar{b}_0 = \bar{c}_0$ because otherwise a permutation finishes the step. Now argue as follows: Choose $\bar{c} \in A/m$, $\bar{c} \neq \bar{a}_0$. Then we can find $\bar{d} \in A/m$ such that

$$(t + \bar{a}_0)(t + \bar{c})(t + \bar{d}) \equiv \bar{g} \quad mod \quad \bar{f}$$

with $\deg \bar{g} < 2$. If $\bar{d} = \bar{a}_0$ set $d = b_0$ and lift $\bar{c}$ to $c \in A$ such that

$$(t + a_0)(t + c)(t + d) \equiv g \quad mod \quad f$$

with $\deg g < 2$. If $\bar{d} \neq \bar{a}_0$ lift $\bar{c}$ and $\bar{d}$ arbitrarily to elements $c, d \in A$ such that with the notation as above $\deg g < 2$.

Case A: $\deg \bar{g} = 1$.

Observe that g is clearly coprime to $t + b_0$. So it is enough to write

$$\{t + a_0, t + b_0, t + c_0\} = (-\{t + d\} - \{t + c\} + \{g\})\{t + b_0, t + c_0\} .$$

Case B: $\deg \bar{g} = 0$.

Similar to Case A it is clearly enough to show

$$\{g, t + b_0, t + c_0\} \in K_1^M(A) \cdot K_2^M(B) \subset \hat{\mathcal{K}}_3^M(B) . \tag{4.1}$$

We have $g = q_1 t + q_0$, $q_1 \in m$. Let $\beta = q_0 + (1 - q_1)b_0 - c_0$ and write

$$g/\beta + (1 - q_1)(t + b_0)/\beta - (t + c_0)/\beta = 1$$

But Proposition 4.2.8(3) resp. Proposition 4.2.1(4) applied to the last equation gives (4.1).

5th step:
We have to show that $\{t + a_0, t + b_0, t + c_0\} \in \hat{\mathcal{K}}_3^M(B)$ with $\bar{a}_0 \neq \bar{b}_0$ is induced by an element from $K_1^M(A) \cdot K_2^M(B)$. Write

$$\frac{t + a_0}{a_0 - b_0} + \frac{t + b_0}{b_0 - a_0} = 1$$

and correspondingly

$$0 = \{\frac{t + a_0}{a_0 - b_0}, \frac{t + b_0}{b_0 - a_0}, t + c_0\} \in \{t + a_0, t + b_0, t + c_0\} + K_1^M(A) \cdot K_2^M(B) \, .$$

□

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