The Membership Problem for quadratic modules with focus on the one dimensional case

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Introduction

The Membership Problem for quadratic modules

For a subset $Q \subseteq K[X] = K[X_1, ..., X_n]$ of the polynomial ring over a field K in the indeterminates $X_1, ..., X_n$ the Membership Problem asks the following:

Is there an algorithm to decide whether a given polynomial $f \in K[X]$ lies in Q?

This means that the Membership Problem asks for a computational procedure which on input the coefficient vector of f stops after finitely many steps with output YES if $f \in Q$ and output NO if $f \notin Q$.

If Q is an ideal then the Membership Problem was solved affirmatively by Grete Hermann [He] in 1926 and algorithms for this problem, which are mainly based on the theory of Gröbner bases, are widely studied (see e.g. [C-L-S]).

When working over real closed fields R, which are fields sharing the algebraic properties of the field of real numbers, it is not enough to study polynomial equalities. Instead, polynomial inequalities are of central importance in real algebraic geometry. Thus in real algebraic geometry varieties, the fundamental geometric objects of classical algebraic geometry, are replaced by semialgebraic sets which are the solution sets of polynomial inequalities. If $G = \{g_1, ..., g_s\} \subseteq R[X]$ is finite then the set

$$S(G) := \{x \in R^n \mid g_i(x) \geq 0 \ (1 \leq i \leq s)\}$$

is called the basic closed semialgebraic set generated by G. For the set

$$\mathcal{P}(S(G)) := \{f \in R[X] \mid f|_{S(G)} \geq 0\}$$

the Membership Problem is solvable in the affirmative due to a groundbreaking result of Tarski [T].

Tarski proved that the theory of real closed fields in the first order language of ordered rings $L = \{+, -, \cdot, 0, 1, <\}$ is decidable. This means for the real closed field R, provided R is given in some explicitly computable manner, there is an algorithm which on input a sentence Φ in the language of ordered rings decides the truth or falsity of Φ. A sentence Φ being an expression that is built up using the operations $+, -, \cdot$, the relations $=, <$ and the boolean connectives as well as quantifiers over variables which range over the elements of R.

Therefore one way of giving a positive answer to the Membership Problem for a set $Q \subseteq R[X]$ is to prove that Q is definable. This is a new notion introduced in this thesis to express that for any general polynomial $f(X, Y) \in \mathbb{Z}[X, Y]$ there is an L-formula $\varphi(Y)$ such that a coefficient vector $c \in R^Y$ fulfills $f(X, c) \in Q$ if and only if $\varphi(c)$ is true. Being definable is for $Q \subseteq R[X]$ equivalent to being

weakly semialgebraic, which was introduced by Knebusch in [K1] and means that the intersection of Q with every finite dimensional subspace of $R[X]$ is semialgebraic.

The set $\mathcal{P}(S(G))$ for some finite $G \subseteq R[X]$ is a particular example of a weakly semialgebraic subset of $R[X]$. Thus Tarski's result provides an explicit algorithm for deciding whether a polynomial f lies in $\mathcal{P}(S(G))$, i.e. whether f is nonnegative on the basic closed semialgebraic set $S(G)$. However Tarski's algorithm is intractable for problems with a large number of variables since the complexity for any general decision procedure for the theory of real closed fields is at least doubly exponential in the number of variables (see [D-H]).

One possibility to overcome this complexity drawback of Tarski's method, which is theoretically so powerful, is to approximate $\mathcal{P}(S(G))$ by a set which consists of polynomials that are nonnegative on $S(G)$ and whose nonnegativity is witnessed by the fact that they can be written in the following form:

$$QM(G) := \{\sigma_0 + \sigma_1 g_1 + ... + \sigma_s g_s \mid \sigma_i \in \sum R[X]^2 \ (0 \leq i \leq s)\}$$

where $\sum R[X]^2$ denotes the set of all finite sums of squares of polynomials.
If $f \in QM(G)$ then we say that f possesses a certificate for nonnegativity on $S(G)$. The set $Q = QM(G)$ is not just an arbitrary subset of $R[X]$, it is a subset containing 1, being closed under addition and under multiplication with squares of polynomials. A subset with these properties is called a quadratic module and plays a very important role in real algebraic geometry. In fact Q belongs to the class of finitely generated quadratic modules which are exactly those quadratic modules of the form $QM(G)$ with associated semialgebraic set $S(G)$ for some finite $G \subseteq R[X]$.
The set $\mathcal{P}(S(G))$ is also a particular quadratic module, namely a multiplicatively closed quadratic module. Quadratic modules with this property are called preorderings. Similar to the way that ideals correspond to varieties in algebraic geometry, preorderings correspond to semialgebraic sets in real algebraic geometry. However quadratic modules and preorderings are much harder to study than ideals because they tend not to be finitely generated.

The reason that makes the finitely generated quadratic module $Q = QM(G)$ in view of computational aspects more attractive than $\mathcal{P}(S(G))$ is that testing membership in Q can be done in polynomial time if Q is stable. Stability means that the degree of the sums of squares used in the representation of an element f of Q can be bounded by a number which depends only on the degree of f. In this case the Membership Problem for Q translates into a semidefinite programming problem which can be solved in polynomial time by using interior point methods (see [N-N]).

Up to now we have seen two classes of finitely generated quadratic modules of $R[X]$ for which the Membership Problem is solvable affirmatively. The first consists of the saturated preorderings, which are equal to $\mathcal{P}(S(G))$ for some finite $G \subseteq R[X]$,

and the second consists of the stable quadratic modules.

Examples of not finitely generated quadratic modules for which there is a positive answer to the Membership Problem are the orderings of $\mathbb{R}[X]$. This is due to the Marker-Steinhorn theorem ([M-S] Theorem 2.1) which says that all orderings of $\mathbb{R}[X]$ are weakly semialgebraic.

Motivation

The Membership Problem for quadratic modules is in itself from a theoretical viewpoint an interesting problem. Its solution however is also of interest for applied mathematics as many problems can be formulated using just polynomial inequalities. As indicated above one way to overcome the drawback of Tarski's algorithm as regards complexity is to approximate the solution to such problems by using certificates for nonnegativity expressed as the membership in certain quadratic modules. We illustrate this approach with the optimization algorithm of Lasserre [L]. The optimization problem in consideration is the minimization of a polynomial f over a nonempty compact basic closed semialgebraic set $S(G)$. Equivalently one can compute the largest real number a such that $f-a$ is nonnegative on $S(G)$. The key idea is now to replace the nonnegativity of $f-a$ on $S(G)$ by the algebraic nonnegativity certificate $f-a \in QM(G)$. By successively increasing the degree of the sums of squares used for representations of elements of $QM(G)$ Lasserre obtains a sequence of semidefinite programs of increasing size. The convergence of the solutions of these semidefinite programs to the solution of the original optimization problem is given by a theorem of Putinar about positivity of polynomials ([Pu] Lemma 4.1).

Aim of this work and main results

The aim of this work is to investigate the Membership Problem for quadratic modules of the ring of polynomials over a real closed field.

For the case of finitely generated quadratic modules of $\mathbb{R}[X_1]$ we succeed and solve the Membership Problem affirmatively (Theorem 2.20). The proof of this also shows that the definability of a finitely generated quadratic module is not equivalent to stability.

Furthermore we obtain a positive solution of the Membership Problem for finitely generated quadratic modules not just in $\mathbb{R}[X_1]$ but in $R[X_1]$ where R is an arbitrary real closed field if the associated semialgebraic set is finite (Theorem 2.37).

We mention that our proof of these two results essentially uses that the quadratic modules are finitely generated and can not be extended to the not finitely generated case.

Another important part of the thesis is the generalization of the model theoretic

concept of heirs which plays an important role in the solution of the Membership Problem for orderings. We define the heir of an arbitrary subset Q of $R[X]$ on a real closed field $R' \supseteq R$ as a certain subset of $R'[X]$ such that the definability of Q becomes equivalent to the existence of a unique heir on every real closed extension field of R. This is a main tool for a possible affirmative answer to the Membership Problem. For finitely generated quadratic modules $QM(G) \subseteq \mathbb{R}[X_1]$ with nonempty bounded $S(G)$ we explicitly compute the heirs on $R \supseteq \mathbb{R}$ (Theorem 3.11).

Outline of this thesis

In Chapter 1 we introduce the notion of being definable or equivalently of being weakly semialgebraic which is fundamental for our approach to solve the Membership Problem. Then we describe why finitely generated saturated preorderings and finitely generated stable quadratic modules are weakly semialgebraic. For a stable quadratic module Q we give a description of an algorithm based on semidefinite programming which decides membership in Q and goes back to a work of Powers and Wörmann [P-W].

The key result for the positive solution of the Membership Problem for a finitely generated quadratic module Q in $\mathbb{R}[X_1]$ in Section 2.1 is the explicit description of the membership in Q in the case that the associated semialgebraic set is bounded. We obtain this by first characterizing the finitely generated quadratic modules in the formal power series ring $R[[X_1 - a]]$ for $a \in R$ and then using a local-global principle due to Scheiderer. From our description of the finitely generated quadratic modules in the formal power series rings we furthermore derive, by again using the local-global principle, that every finitely generated quadratic module of $\mathbb{R}[X_1]$ whose associated semialgebraic set is bounded is, in fact, a preordering. At the end of Section 2.1 we characterize when a finitely generated quadratic module $Q \subseteq \mathbb{R}[X_1]$ whose associated semialgebraic set is bounded can be generated by just one polynomial and describe an algorithm that produces in general at most three generators of Q. Furthermore we show that a finitely generated quadratic module of $\mathbb{R}[X_1]$ with nonempty bounded semialgebraic set S is completely determined by two vectors: one which encodes the boundary points of S and one which encodes order conditions attached to these points.
In Section 2.2 we prove a local-global principle for quadratic modules of $R[X_1]$, where R is an arbitrary real closed field, under the assumption that the associated semialgebraic set is finite. This enables us to solve the Membership Problem affirmatively for those quadratic modules and to give a description of their support.
In Section 2.3 we are concerned with positivity divisors of $(R[X], Q)$, i.e. elements h of Q such that for every $f \in R[X]$ the fact that $hf \in Q$ implies that $f \in Q$, and convexity divisors, which are positivity divisors h having the additional property that the principal ideal $hR[X]$ is convex. We use the explicit description of the

membership in a finitely generated quadratic module Q of $\mathbb{R}[X_1]$ whose associated semialgebraic set is bounded from Section 2.1 to determine the positivity and convexity divisors of $(\mathbb{R}[X_1], Q)$. This enables us to give a proof of a second local-global principle of Scheiderer for this special situation.

In Chapter 3 we deal with heirs of subsets of $R[X]$. We develop the notion of a (weak resp. dual weak) heir of $Q \subseteq R[X]$ on $R' \supseteq R$ such that Q is definable if and only if it has a unique heir on every real closed extension field $R' \supseteq R$. The property of being stable can also be recognized with the help of heirs. A result of Scheiderer translated into the language of heirs says that a quadratic module Q generated by $g_1, ..., g_s \in R[X]$ is stable if and only if for every real closed extension $R' \supseteq R$ the unique heir of Q on R' equals the quadratic module generated by $g_1, ..., g_s$ in $R'[X]$. This implies that the finitely generated quadratic modules $QM(G)$ of $R[X_1]$ with the property that $S(G)$ is finite are stable. For a finitely generated quadratic module $Q = QM(G) \subseteq \mathbb{R}[X_1]$ we give, under the assumption that $S(G)$ is not empty and bounded, an explicit description of the heir of Q on a real closed field $R \supseteq \mathbb{R}$. From this we deduce that if in addition $S(G)$ has a nonempty interior then Q is stable if and only if it is saturated. Hence there are a lot of examples of definable but not stable quadratic modules which shows that the notion of stability is strictly stronger than the notion of definability. In particular it follows that the preordering $QM((1-X_1^2)^3) \subseteq \mathbb{R}[X_1]$ is not stable which has already been proved by Stengle [St2] using approximation theory. Using the upper and lower bounds given by Stengle in his paper [St2] we show that the heir of this preordering is not finitely generated.
In Section 3.3 we consider tame extensions $R' \supseteq R$ which are those extensions where the embedding $R \hookrightarrow \mathcal{O} \to \mathcal{O}/\mathfrak{m}$ is onto with $\mathcal{O}$ being the convex hull of R in R' and $\mathfrak{m}$ the maximal ideal of $\mathcal{O}$. The place $\lambda : \mathcal{O} \to R$ is in this case called the standard part map. We prove that the image of the weak heir and of the dual weak heir of a quadratic module $Q \subseteq R[X]$ on R' under the standard part map is equal to $Q^{(\ddagger)}$. This quadratic module was introduced by Kuhlmann, Marshall and Schwartz in [K-M-S] and plays an important role in the solution of the moment problem.

In Chapter 1 and 2 we showed that in dimension 1 the Membership Problem for finitely generated quadratic modules is solvable in the affirmative over arbitrary real closed fields if the associated semialgebraic set S is either not bounded or has empty interior. Within the remaining case we restrict ourselves in Chapter 4 to the case that the quadratic module Q is generated by finitely many elements $g_1, ..., g_s \in \mathcal{O}[X_1]$ where $\mathcal{O}$ is the convex hull of $\mathbb{R}$ in R. For the quadratic module $Q_{\mathcal{O}} \subseteq Q \cap \mathcal{O}[X_1]$ which is generated by $g_1, ..., g_s$ in $\mathcal{O}[X_1]$ we reduce in the case that $\operatorname{Spec} \mathcal{O} = \{\{0\}, \mathfrak{m}\}$ the question when a polynomial $f \in \mathcal{O}[X_1]$ lies in $Q_{\mathcal{O}}$ to finitely many membership questions in formal power series rings over $\mathcal{O}$ given some extra assumptions. These additional assumptions make it on the one hand possible to use a local-global principle which we establish over $\mathcal{O}$. On the other hand they ensure that $Q_{\mathcal{O}}$ is archimedean which has been proved for the case of a preordering

by Prestel (see [P-D]). For our proof we use that the semi-real spectrum of $\mathcal{O}[X_1]$ equals the real spectrum of $\mathcal{O}[X_1]$ which follows from our description of $\mathrm{Sper}\mathcal{O}[X_1]$ in Section 4.1. We conclude the thesis with a list of open problems which are topics for future research.

We defer the proofs of some results about definability and heirs which use deeper model theory to the appendix to make the thesis readable also for those who are not that acquainted with model theory.

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0 Notation and Prerequisites

In this preliminary chapter we fix some notation and state well-established results about quadratic modules and preorderings which will be used later on.

We denote the set of natural numbers with $\mathbb{N}$, the natural numbers including 0 with $\mathbb{N}_0$, the ring of integers with $\mathbb{Z}$, the field of rational numbers with $\mathbb{Q}$ and the field of real numbers with $\mathbb{R}$.

For some ordered field K and $a, b \in K$ with $a < b$ we use $]a, b[$ to denote the open interval $\{x \in K \mid a < x < b\}$. Similarly the closed interval is denoted by $[a, b]$ and the half-open intervals by $[a, b[$ and $]a, b]$.

The set of isolated points of a subset S of some topological space is denoted by S_{isol}.

If A is a ring and $f \in A$ then fA denotes the ideal generated by f in A.

The quotient field of an integral domain A is denoted by $\mathrm{Quot}(A)$.

The letter R is always used for real closed fields.

If v is a valuation of a field K then we write $\mathcal{O}$ for the valuation ring corresponding to v and $\mathfrak{m}$ for the maximal ideal of $\mathcal{O}$. The residue field $\mathcal{O}/\mathfrak{m}$ is denoted by $\overline{\mathcal{O}}$ and the residue map $\mathcal{O} \to \overline{\mathcal{O}}$ as well as its extension to the polynomial rings $\mathcal{O}[X] \to \overline{\mathcal{O}}[X]$ by λ.
If v is not stated explicitly and $R' \supseteq R$ is an extension of real closed fields then $\mathcal{O}$ is defined to be the convex hull of R in R' and v is the valuation corresponding to the valuation ring $\mathcal{O}$. In this case $\mathfrak{m}$ consists of the elements of R' which are infinitesimal with respect to R and the residue field $\overline{\mathcal{O}}$ is a real closed subfield of $\mathbb{R}$ which is isomorphic to $\mathbb{R}$ if $\mathbb{R} \subseteq R$.

For the remaining of this chapter A is a commutative ring with 1.

The basic algebraic objects of our study are quadratic modules and preorderings of A which are defined as follows.

Definition 0.1
A subset $Q \subseteq A$ is called a quadratic module of A if

$$Q + Q \subseteq Q, 1 \in Q \text{ and } A^2 Q \subseteq Q.$$

A subset $P \subseteq A$ is called a preordering of A if

$$P + P \subseteq P, P \cdot P \subseteq P \text{ and } A^2 \subseteq P.$$

Preorderings are special quadratic modules, namely those quadratic modules which are closed under multiplication.

If $Q \subseteq A$ is a quadratic module then the intersection $Q \cap -Q =: \mathrm{supp}(Q)$ is called the support of Q which is an ideal if $\frac{1}{2} \in A$.

A quadratic module Q is finitely generated if there are finitely many elements $g_1, ..., g_s \in A$ such that Q is the smallest quadratic module containing these elements. The quadratic module generated by $g_1, ..., g_s$ is denoted by $QM(g_1, ..., g_s)$ and is given by

$$QM(g_1, ..., g_s) = \{\sigma_0 + \sigma_1 g_1 + ... + \sigma_s g_s \mid \sigma_i \in \sum A^2 \ (0 \leq i \leq s)\}$$

where $\sum A^2$ denotes the set of all finite sums of squares of elements from A.
The preordering generated by $g_1, ..., g_s$ is similarly denoted by $PO(g_1, ..., g_s)$ and is given by

$$PO(g_1, ..., g_s) = \{ \sum_{\epsilon \in \{0,1\}^s} \sigma_\epsilon g_1^{\epsilon_1} \cdots g_s^{\epsilon_s} \mid \sigma_\epsilon \in \sum A^2 \ \forall \epsilon \in \{0,1\}^s\}.$$

If it is not clear from the context we write $PO_A(g_1, ..., g_s)$, and $QM_A(g_1, ..., g_s)$ to indicate that the sums of squares are formed with elements from A.

Quadratic modules and preorderings of A which have the two additional properties of the following definition are of special importance.

Definition 0.2
A preordering $\alpha \subseteq A$ is an ordering of A if

$$\alpha \cup -\alpha = A \text{ and } \mathrm{supp}(\alpha) \text{ is a prime ideal.}$$

A quadratic module $\beta \subseteq A$ is a semiordering of A if

$$\beta \cup -\beta = A \text{ and } \mathrm{supp}(\beta) \text{ is a prime ideal.}$$

The set

$$\mathrm{Sper}\, A := \{\alpha \subseteq A \mid \alpha \text{ is an ordering of } A\}$$

is called the real spectrum of A and similarly

$$\mathrm{SemiSper}\, A := \{\beta \subseteq A \mid \beta \text{ is a semiordering of } A\}$$

is called the semi-real spectrum of A.

For some set $T \subseteq A$ we define

$$\begin{array}{lcl} H(T) & := & \{\alpha \in \operatorname{Sper} A \mid T \subseteq \alpha \setminus (-\alpha)\}, \\ \overline{H}(T) & := & \{\alpha \in \operatorname{Sper} A \mid T \subseteq \alpha\}, \\ H_{Semi}(T) & := & \{\beta \in \operatorname{SemiSper} A \mid T \subseteq \beta \setminus (-\beta)\}, \\ \overline{H}_{Semi}(T) & := & \{\beta \in \operatorname{SemiSper} A \mid T \subseteq \beta\}. \end{array}$$

If Sper$A$ is provided with the Harrison topology which has $\{H(a) \mid a \in A\}$ as a subbasis of open sets then Sper$A$ is quasi-compact. Similarly SemiSper A provided with the topology generated by the subbasis $\{H_{Semi}(a) \mid a \in A\}$ forms a quasi-compact space. For more details about the real spectrum we refer for example to [K-S] and for the semi-real spectrum see for example [J1].

For every ring A the real spectrum is the subset of the semi-real spectrum consisting of those semiorderings which are closed under multiplication. If A is the ring of polynomials over a real closed field in one indeterminate then both sets coincide.

Proposition 0.3
If R is a real closed field and X denotes one indeterminate then

$$\operatorname{SemiSper} R[X] = \operatorname{Sper} R[X].$$

Proof:
We show that every semiordering of $R[X]$ is already an ordering.
For that purpose we use the bijection between the set of semiorderings of an arbitrary commutative ring A with 1 and the set of tuples $(\mathfrak{p}, \gamma)$ where $\mathfrak{p} \in \operatorname{Spec} A$ is a prime ideal of A and γ is a semiordering of the quotient field $\operatorname{Quot}(A/\mathfrak{p})$ of $A/\mathfrak{p}$.

There are three different kinds of prime ideals of $R[X]$.
If $\mathfrak{p} = ((X-a)^2 + b^2)R[X]$ for some $a, b \in R$ with $b \neq 0$ then -1 is a sum of squares in the quotient field of $R[X]/\mathfrak{p}$ which implies that there is no semiordering on it.
If $\mathfrak{p} = (X-a)R[X]$ for some $a \in R$ then the quotient field of $R[X]/\mathfrak{p}$ is isomorphic to R which has one unique semiordering, the unique ordering of R.
If finally $\mathfrak{p} = \{0\}$ then the quotient field of $R[X]/\mathfrak{p}$ is $R(X)$. In this case Prestel showed that every semiordering of $R(X)$ is already an ordering ([Pr] Theorem 3.6).

Thus in all three cases we just get the elements of the real spectrum which means that $\operatorname{SemiSper} R[X] = \operatorname{Sper} R[X]$.

Prop. 0.3 □

The elements of the (semi-)real spectrum of the ring of polynomials in one indeterminate over a real closed field are well-known (e.g. [K-S] III.3 Example 2).

SemiSper $R[X] = \text{Sper } R[X]$ consists of the orderings $a^{\pm}$ ($a \in R$), $\pm\infty$ and orderings ξ corresponding to free Dedekind cuts which all have support $\{0\}$ and the orderings α_a corresponding to evaluation in $a \in R$ with support $(X-a)R[X]$.

If Y is a subset of the real or semi-real spectrum of A then the elements of A can be considered as generalized functions on Y and the sign of some $f \in A$ on Y is given in the following way:

$$\begin{array}{lcl} f > 0 \text{ on } Y & \Leftrightarrow & f \in \alpha \setminus (-\alpha) \text{ for every } \alpha \in Y \\ f \geq 0 \text{ on } Y & \Leftrightarrow & f \in \alpha \text{ for every } \alpha \in Y \\ f = 0 \text{ on } Y & \Leftrightarrow & f \in \text{supp}(\alpha) \text{ for every } \alpha \in Y \end{array}$$

The abstract Stellensatz for quadratic modules Q (resp. preorderings P) characterizes those elements of A which are positive, nonnegative or zero on $\overline{H}_{Semi}(Q)$ (resp. $\overline{H}(P)$) in an algebraic way using sums of squares. For the abstract Stellensatz for quadratic modules we refer to Jacobi ([J1],[J2]). The Stellensatz for preorderings was discovered by Krivine [Kr] in 1964 and rediscovered by Stengle [St1] in 1974.

Theorem 0.4 (Abstract Stellensatz for quadratic modules)
Let $Q \subseteq A$ be a quadratic module and $f \in A$.
Then the following is true:

i) $f > 0$ *on* $\overline{H}_{Semi}(Q) \Leftrightarrow pf = 1 + q$ *for some* $p \in \sum A^2, q \in Q$

ii) $f \geq 0$ *on* $\overline{H}_{Semi}(Q) \Leftrightarrow pf = f^{2m} + q$ *for some* $m \in \mathbb{N}_0, p \in \sum A^2, q \in Q$

iii) $f = 0$ *on* $\overline{H}_{Semi}(Q) \Leftrightarrow -f^{2m} \in Q$ *for some* $m \in \mathbb{N}_0$

iv) $\overline{H}_{Semi}(Q) = \emptyset \Leftrightarrow -1 \in Q$

Theorem 0.5 (Abstract Stellensatz for preorderings)
Let $P \subseteq A$ be a preordering and $f \in A$.
Then the following is true:

i) $f > 0$ *on* $\overline{H}(P) \Leftrightarrow pf = 1 + q$ *for some* $p, q \in P$

ii) $f \geq 0$ *on* $\overline{H}(P) \Leftrightarrow pf = f^{2m} + q$ *for some* $m \in \mathbb{N}_0, p, q \in P$

iii) $f = 0$ *on* $\overline{H}(P) \Leftrightarrow -f^{2m} \in P$ *for some* $m \in \mathbb{N}_0$

iv) $\overline{H}(P) = \emptyset \Leftrightarrow -1 \in P$

If $A = R[X] = R[X_1, ..., X_n]$ and $P = PO(g_1, ..., g_s) \subseteq R[X]$ is a finitely generated preordering then $\overline{H}(P)$ is just $\widetilde{S}$, the constructible subset of $\operatorname{Sper} R[X]$ associated to the basic closed semialgebraic set

$$S := S(g_1, ..., g_s) := \{x \in R^n \mid g_i(x) \geq 0 \ (1 \leq i \leq s)\}.$$

In this situation the Stellensatz can be formulated in the following way.

Theorem 0.6 (Stellensatz for preorderings)
Let R be a real closed field, $g_1, ..., g_s \in R[X] = R[X_1, ..., X_n]$, $P = PO(g_1, ...g_s)$, $S = S(g_1, ..., g_s)$ and $f \in R[X]$.
Then the following is true:

i) $f > 0$ *on* $S \Leftrightarrow pf = 1 + q$ *for some* $p, q \in P$

ii) $f \geq 0$ *on* $S \Leftrightarrow pf = f^{2m} + q$ *for some* $m \in \mathbb{N}_0$ *and* $p, q \in P$

iii) $f = 0$ *on* $S \Leftrightarrow -f^{2m} \in P$ *for some* $m \in \mathbb{N}_0$

iv) $S = \emptyset \Leftrightarrow -1 \in P$.

We note that the proof of this Stellensatz essentially uses Tarski's Transfer Principle which was formulated by Tarski [T] in 1931.

Theorem 0.7 (Tarski's Transfer Principle)
Suppose that R_1 and R_2 are two real closed fields inducing the same ordering on a common subfield K. If $p_1, ..., p_r \in K[X] = K[X_1, ..., X_n]$ and $\epsilon_1, ..., \epsilon_r \in \{-1, 0, 1\}$ then we have some $x \in R_1^n$ with $\operatorname{sign}(p_j(x)) = \epsilon_j$ $(j = 1, ..., r)$ if and only if there is some $x \in R_2^n$ satisfying the same system of polynomial equations and inequalities with coefficients from K.

The following Representation theorem is known in the literature as the Kadison-Dubois Theorem. The version for quadratic modules is due to Jacobi ([J2] Theorem 4). It gives denominator-free representations of polynomials which are strictly positive on $\overline{H}(Q)$ under the hypothesis that the quadratic module Q is archimedean.

Definition 0.8
A quadratic module Q of A is called archimedean if for every $f \in A$ there is some $N \in \mathbb{N}$ such that $N \pm f \in Q$.

Theorem 0.9 (Kadison-Dubois Theorem)
Suppose that $\frac{1}{2} \in A$, $f \in A$ *and* $Q \subseteq A$ *is an archimedean quadratic module. Then we have*

$$f > 0 \text{ on } \overline{H}(Q) \Rightarrow f \in Q.$$

Another result which we will need later on is a version of the Basic Lemma of Kuhlmann/Marshall/Schwartz ([K-M-S] Lemma 2.1) which can be found in [S5] ([S5] Proposition 2.4).

Theorem 0.10 (Basic Lemma)
Let $T \subseteq A$ *and* $\overline{H}(T)$ *a bounded subset of* $\operatorname{Sper} A$, *i.e. for every* $h \in A$ *there is some* $N \in \mathbb{N}$ *such that* $N \pm h \geq 0$ *on* $\overline{H}(T)$.
If $f, g, s, t \in A$ *such that* $f, g \geq 0$ *on* $\overline{H}(T)$ *and* $sf + tg = 1$ *then there are* $\sigma, \tau \in A$ *with* $\sigma f + \tau g = 1$ *and* $\sigma, \tau > 0$ *on* $\overline{H}(T)$.

1 The Membership Problem

1.1 Definability

The focus of this work is to answer the question regarding the definability of quadratic modules and preorderings in the polynomial ring over a real closed field.

The question of definability can be posed in a more general setting for subsets of $M[X]$ where M is an L-structure for some first order language L and X denotes a finite tuple of variables $(X_1, ..., X_n)$. We restrict the problem to subsets $Q \subseteq R[X]$ where R is a real closed field. Thus the first order language L in consideration is always the language of ordered rings $L_{or} = \{+, -, \cdot, 0, 1, <\}$.
$L(R)$ denotes the language obtained from L by adding a constant symbol to name each element of R, $\mathrm{Fml}L(R)$ is the set of $L(R)$-formulas and Y is a finite tuple of variables (of variable length).

Definition 1.1
$Q \subseteq R[X]$ is definable if and only if for every $f(X, Y) \in \mathbb{Z}[X, Y]$ there is a formula $\vartheta_f(Y) \in FmlL(R)$ such that for all $c \in R^Y$

$$f(X, c) \in Q \Leftrightarrow R \models \vartheta_f(c),$$

i.e. if and only if the set

$$D(f, Q) := \{c \in R^Y \mid f(X, c) \in Q\}$$

is definable (by an $L(R)$-formula $\vartheta_f(Y)$).

Although we concentrate on the case where Q is a preordering or a quadratic module this more general setup is very useful for example for the study of the set of polynomials of some quadratic module up to a certain degree d.

When working in the language of ordered rings with respect to the theory of real closed fields the definability of $D(f, Q)$ means nothing else than that this set is a semialgebraic subset of R^Y. In this setting the notion of definability given above is equivalent to the notion of being weakly semialgebraic which was introduced by Knebusch in [K1]. A proof of the equivalence is contained in the Appendix (Proposition A.2).

Definition 1.2
A subset Q of a finitely generated R-algebra A is called weakly semialgebraic if for every finite dimensional R-subspace U of A the set $Q \cap U$ is a semialgebraic subset of U.

Because of the mentioned equivalence we prefer to speak of weakly semialgebraic subsets Q if we have witnessed membership in the sets $D(f, Q)$ by $L_{or}(R)$-formulas as defined in Definition 1.1 since this notion is more precise what the underlying language and theory concerns.

The reason why we want to know whether Q is weakly semialgebraic or not is its impact on the solution of the Membership Problem for Q.

Definition 1.3
We say that the Membership Problem is solvable affirmatively for Q if and only if for every $f(X, Y) \in \mathbb{Z}[X, Y]$ there is an algorithm which decides upon input of $c \in R^Y$ whether $f(X, c) \in Q$ or not.

Since the theory of real closed fields in the language of ordered rings is by Tarski [T] decidable we know that the Membership Problem is solvable affirmatively for Q if Q is weakly semialgebraic and the input data is computable.

Though being weakly semialgebraic is sufficient, it is not necessary for the result that the Membership Problem is solvable affirmatively for a quadratic module.
We illustrate this with the following example.

Example 1.4
Let $P := \{f \in \mathbb{R}[X] \mid f|_{\mathbb{Z}} \geq 0\}$ where X denotes one indeterminate.
Then the preordering P is not weakly semialgebraic because for the polynomial $f(X, Y) := 2(X - Y)^2 - 1 \in \mathbb{Z}[X, Y]$ the set $D(f, P)$ is equal to $\mathbb{Z} + \frac{1}{2}$ which is not semialgebraic.
However the Membership Problem is solvable affirmatively for P as the following pseudo code shows.

INPUT: coefficient vector $(c_0, ..., c_d)$ of a polynomial $f = f(X, c)$ of degree d
COMPUTE the Cauchy bound B for $(c_0, ..., c_d)$
COMPUTE $f(x)$ for all $x \in \mathbb{Z}, |x| < B$
IF $f(x) \geq 0$ for all $x \in \mathbb{Z}, |x| < B$
    OUTPUT: $f \in P$
ELSE
    OUTPUT: $f \notin P$

1.2 Saturated preorderings and stable quadratic modules

In this section we present two classes of finitely generated quadratic modules for which we easily can see that they are weakly semialgebraic.

Saturated preorderings

The first class of weakly semialgebraic quadratic modules consists of the finitely generated saturated preorderings.

A preordering P of some commutative ring A with 1 is saturated if and only if P is equal to the intersection of all orderings containing P, i.e. if $P = \bigcap_{\alpha \in \overline{H}(P)} \alpha$.

If $A = R[X] = R[X_1, ..., X_n]$ and P is finitely generated then being saturated translates by Tarski's Transfer Principle (Theorem 0.7) into the following:
$P = PO(g_1, ..., g_s) \subseteq R[X]$ is saturated if and only if every $f \in R[X]$ which is nonnegative on the basic closed semialgebraic set $S = S(g_1, ..., g_s) \subseteq R^n$ lies in P. With

$$\mathcal{P}(S) := \{f \in R[X] \mid f|_S \geq 0\}$$

this means that

$$P \text{ is saturated } \Leftrightarrow P = \mathcal{P}(S).$$

We give some important steps of the proof of this equivalence because they involve two important results which are part of the foundation of real algebraic geometry: Tarski's Transfer Principle (Theorem 0.7) and the famous Stellensatz for preorderings (Theorem 0.6).

If the preordering P is not proper, i.e. if $-1 \in P$, then $P = R[X]$ because we can write $f = (\frac{f+1}{2})^2 - (\frac{f-1}{2})^2 \in P$ for every $f \in R[X]$. On the other hand the Stellensatz for preorderings (Theorem 0.6 iv)) says that $S = \emptyset$ which implies that the set of all nonnegative polynomials on S is also equal to $R[X]$.
Thus the equivalence "P is saturated $\Leftrightarrow P = \mathcal{P}(S)$" is fulfilled if P is not proper.

Now we suppose that $-1 \notin P$.
The implication "$P = \mathcal{P}(S) \Rightarrow P$ saturated" is not hard to prove. We give the proof of the other implication.
Since clearly $P \subseteq \mathcal{P}(S)$ we take some $f \in R[X]$ with $f|_S \geq 0$ and show that $f \in P$. We suppose to the contrary that there is some ordering $\alpha \supseteq P$ with $f \notin \alpha$. This will be used to get elements $x_1, ..., x_n$ in some real closed extension field of R with $g_1(x) \geq 0, ..., g_s(x) \geq 0$ and $f(x) < 0$ for $x = (x_1, ..., x_n)$. The real closed field in consideration is the real closure $k(\alpha)$ of the quotient field of $R[X]/\text{supp}(\alpha)$.

This is a real closed extension field of R and the ordering of $k(\alpha)$ restricts to the unique ordering on R. In $k(\alpha)$ we have with $x_i := X_i + \mathrm{supp}(\alpha)$ $(1 \leq i \leq n)$ and $x = (x_1, ..., x_n) \in k(\alpha)^n$ that $g_j(x) \geq 0$ $(1 \leq j \leq s)$ as $P \subseteq \alpha$ and $f(x) < 0$ because by assumption $f \notin \alpha$. Thus we get by Tarski's Transfer Principle (Theorem 0.7) some $x \in R^n$ with $x \in S$ and $f(x) < 0$. This contradicts the fact that $f|_S \geq 0$ and proves the claim.

Thus for some saturated preordering $P = PO(g_1, ..., g_s) \subseteq R[X]$ and some polynomial $f(X, Y) \in \mathbb{Z}[X, Y]$ the formula $\vartheta_f^{sat}(Y)$ which defines membership in P is given by

$$\vartheta_f^{sat}(Y) := \forall X \left(\bigwedge_{i=1}^{s} g_i(X) \geq 0 \rightarrow f(X, Y) \geq 0 \right).$$

The parameters appearing in the formula $\vartheta_f^{sat}(Y) \in \mathrm{Fml}L(R)$ are just the coefficients of the polynomials $g_1, ..., g_s \in R[X]$.
If $f(X, Y) \in \mathbb{Z}[X, Y], g_1(X, Z), ..., g_s(X, Z) \in \mathbb{Z}[X, Z]$ then the L-formula

$$\vartheta^{sat}(Y, Z) := \forall X \left(\bigwedge_{i=1}^{s} g_i(X, Z) \geq 0 \rightarrow f(X, Y) \geq 0 \right)$$

defines uniformly the membership in the saturation which means that for every $c \in R^Y, b \in R^Z$

$$f(X, c) \in \mathcal{P}(S(g_1(X, b), ..., g_s(X, b)) \Leftrightarrow R \models \vartheta^{sat}(c, b).$$

If $P = PO(g_1(X, b), ..., g_s(X, b))$ is saturated than this formula defines the membership in P.

When is a preordering saturated?

In dimension 1 the basic closed set $S = S(g_1, ..., g_s) \subseteq R$ for some $g_1, ..., g_s \in R[X]$ is a finite union of intervals and points and being saturated is closely related to the set of natural generators of $\mathcal{P}(S)$.

If $S \neq \emptyset$ is a basic closed semialgebraic subset of R then the set of natural generators of $\mathcal{P}(S)$ is defined as

$$\begin{array}{rcl} \mathrm{Nat}(S) & := & \{X - a \mid a \text{ is a least element of } S\} \\ & \cup & \{a - X \mid a \text{ is a largest element of } S\} \\ & \cup & \{(X - a)(X - b) \mid a, b \in S, a < b \text{ and }]a, b[\cap S = \emptyset\} \end{array}$$

If $S = \emptyset$ we define $\mathrm{Nat}(\emptyset) := \{-1\}$.

The result of the next theorem is due Kuhlmann, Marshall and Schwartz ([K-M-S] Theorem 3.1). Since it is stated there for the field $\mathbb{R}$ and we need it later on for arbitrary real closed fields we state and prove it here. For the proof we need a lemma which can be found in [B-M].

Lemma 1.5 (Berg-Marserick, [B-M] Lemma 4)
If $a, c_1, c_2, b \in R$ with $a \leq c_1 \leq c_2 \leq b$ then

$$(X - c_1)(X - c_2) \in QM((X - a)(X - b)).$$

Proof:
If $c_1 = c_2$ then the claim is trivial.
From now on we suppose that $a \leq c_1 < c_2 \leq b$.
Since we are working in dimension 1 where every polynomial which is nonnegative on all of R is a sum of (two) squares (see for example [M1] Proposition 1.2.1) it is enough to show that there is some nonnegative element $\gamma \in R$ such that

$$f_\gamma(x) := (x - c_1)(x - c_2) - \gamma(x - a)(x - b) \geq 0$$

for every $x \in R$.
Because $f_\gamma(X)$ is a polynomial of degree two which has in the case $\gamma < 1$ a minimum for

$$x_0 = \frac{(c_1 + c_2) - \gamma(a + b)}{2(1 - \gamma)}$$

with value

$$f_\gamma(x_0) = -\left(\frac{(c_1 + c_2) - \gamma(a + b)}{2}\right)^2 \frac{1}{1 - \gamma} + c_1c_2 - \gamma ab$$

we are looking for some $\gamma_0 \in]0, 1[$ such that $f_{\gamma_0}(x_0) \geq 0$.

Case 1: $c_1 + c_2 > a + b$
Then $\gamma_0 := \frac{2}{b-a}(b - \frac{c_1+c_2}{2}) \in]0, 1[$ and $f_{\gamma_0}(x_0) = (b - c_1)(b - c_2) \geq 0$.

Case 2: $c_1 + c_2 < a + b$
Then $\gamma_0 := \frac{2}{b-a}(\frac{c_1+c_2}{2} - a) \in]0, 1[$ and $f_{\gamma_0}(x_0) = (c_1 - a)(c_2 - a) \geq 0$.

Case 3: $c_1 + c_2 = a + b$
Then $\gamma_0 := \frac{(c_2-c_1)^2}{(b-a)^2} \in]0, 1[$ and we have
$f_{\gamma_0}(x_0) = \frac{1}{4}(b - a)^2 \frac{(c_2-c_1)^2}{(b-a)^2} - \frac{1}{4}(c_1 + c_2)^2 + c_1c_2 = \frac{1}{4}(c_1 - c_2)^2 - \frac{1}{4}(c_1 - c_2)^2 = 0.$
Thus in any of these three cases we have found an appropriate γ_0 which proves the claim.

Lemma 1.5 □

Theorem 1.6
For a basic closed semialgebraic set $S \subseteq R$ we have

$$\mathcal{P}(S) = PO(Nat(S)).$$

Proof:
If $S = \emptyset$ then the set of nonnegative polynomials on S is $R[X]$ which is also equal to $PO(\mathrm{Nat}(\emptyset)) = PO(-1)$ because every polynomial in $R[X]$ can be written as a difference of two squares.

We suppose now that $S \neq \emptyset$.

The inclusion $\supseteq$ is clear since every element of $\mathrm{Nat}(S)$ is in a natural way nonnegative on S.
For the other inclusion we consider some $f \in R[X]$ with $f|_S \geq 0$ and prove by induction on the degree d of f that $f \in PO(\mathrm{Nat}(S))$.
For $d = 0$ the result is clear. Thus suppose that $d > 0$.
If f is nonnegative on all of R then f is a sum of two squares and hence in $PO(\mathrm{Nat}(S))$.
We suppose now that there is some $c \in R$ with $f(c) < 0$ and distinguish the following three cases.

Case 1: S has a least element a and $c < a$
Let c_0 be the least root of f in $]c, a]$ then $f = (X - c_0)g$ with
$X - c_0 = X - a + \underbrace{(a - c_0)}_{\geq 0} \in PO(\mathrm{Nat}(S))$, $g|_S \geq 0$ and $\deg(g) < \deg(f)$.

Case 2: S has a largest element a and $c > a$
Let c_0 be the largest root of f in $[a, c[$ then $f = (c_0 - X)g$ with
$c_0 - X = \underbrace{(c_0 - a)}_{\geq 0} + a - X \in PO(\mathrm{Nat}(S))$, $g|_S \geq 0$ and $\deg(g) < \deg(f)$.

Case 3: There are $a, b \in S$ with $a < b$, $]a, b[\cap S = \emptyset$ and $a < c < b$.
Let c_1 be the largest root of f in $[a, c[$ and c_2 the least root of f in $]c, b]$ then $f = (X - c_1)(X - c_2)g$ with $(X - c_1)(X - c_2) \in PO(\mathrm{Nat}(S))$ by the previous lemma, $g|_S \geq 0$ and $\deg(g) < \deg(f)$.

Thus in any of these three cases we can use the induction hypothesis to prove the claim.

Theorem 1.6 $\square$

Corollary 1.7
For $n = 1$, $P = PO(g_1, ..., g_s) \subseteq R[X]$ and $S = S(g_1, ..., g_s)$ we have:

$$P \text{ is saturated} \Leftrightarrow Nat(S) \subseteq P$$

Proof:
P is saturated if and only if $P = \mathcal{P}(S)$ which is by the previous theorem equal to $PO(\mathrm{Nat}(S))$. This immediately gives $\Rightarrow$. If otherwise $\mathrm{Nat}(S) \subseteq P$ then again by the previous theorem $\mathcal{P}(S) = PO(\mathrm{Nat}(S)) \subseteq P$. Hence $P = \mathcal{P}(S)$ because P is always a subset of $\mathcal{P}(S)$.

Corollary 1.7 □

We want to mention that if we additionally suppose that $S(g_1, ..., g_s) \subseteq R$ is not bounded then the natural generators of $\mathcal{P}(S)$ are in P if and only if they already appear in the set $\{g_1, ..., g_s\}$ ([K-M] Theorem 2.2).

Thus the membership in $P = PO(g_1, ..., g_s)$ of the finitely many natural generators of $\mathcal{P}(S(g_1, ..., g_s))$ decides in the case $n = 1$ whether P is saturated or not. In the case $n = 2$ there are examples of finitely generated preorderings which are not saturated - the probably most well known is the sums of squares itself - and it was long unknown whether there is an example of a saturated preordering in dimension two. Just recently Scheiderer presented such an example ([S5] Corollary 3.3).

Another important result of Scheiderer is that if the dimension of $S(g_1, .., g_s)$ is greater or equal to three we always get a negative answer what the saturation of the preordering $PO(g_1, ..., g_s)$ concerns ([S1] Proposition 6.1).

Stable quadratic modules

Now we come to the second class of weakly semialgebraic quadratic modules where we state an algorithm based on semidefinite programming in order to decide whether a polynomial lies in the quadratic module or not. The quadratic modules in consideration are the finitely generated stable quadratic modules.

The definition of stability we use is due to Powers and Scheiderer [P-S]. For a more general notion of stability with respect to filtrations and graduations and a lot of examples of stable quadratic modules we refer to Netzer [N].

For ease of notation let $g_0 := 1$.

Definition 1.8
$Q = QM(g_1, ..., g_s) \subseteq R[X] = R[X_1, ..., X_n]$ is stable if for every finite dimensional subspace U of $R[X]$ there is a finite dimensional subspace V of $R[X]$ such that

$$Q \cap U \subseteq \{\sum_{i=0}^{s} \sigma_i g_i \mid \sigma_i \in \sum V^2 \ (0 \leq i \leq s)\}.$$

If we consider $R[X]$ graduated by the total degree graduation then being stable means nothing else than the existence of a function $\varphi : \mathbb{N} \to \mathbb{N}$ such that every element f of Q has a representation in Q where the degree of the polynomials appearing in the sums of squares is less or equal to $\varphi(\deg(f)) =: \varphi(f)$. A bound $\varphi(\deg(f))$ on the degrees of the squares appearing in the representation also bounds the number of squares needed for a presentation of f by some number $N(\varphi(\deg(f))) =: N(f)$. This result together with the correspondence between sums of square representations and symmetric positive semidefinite matrices is the content of the next lemma. More about this well-known connection which forms the core of the Gram matrix method due to [P-W] can be found in [C-L-R]. We will state a generalization of the Gram matrix method later on. This method allows us to recognize whether a polynomial is a sum of squares via semidefinite programming which is a generalization of linear programming where the cone of nonnegative vectors is replaced by the cone of positive semidefinite matrices. For more information about the topic of semidefinite programming we refer ro [V-B] or [W-S-V].

Note that if $\Lambda(d) := \{\alpha \in \mathbb{N}_0^n \mid |\alpha| \leq d\}$ for some $d \in \mathbb{N}_0$ then $\{X^\alpha \mid \alpha \in \Lambda(d)\}$ forms a basis of the vector space $R[X]_{\leq d}$ of all polynomials of degree $\leq d$ with dimension $|\Lambda(d)| = \binom{n+d}{n}$.

Lemma 1.9
For $0 \not\equiv f \in R[X]_{\leq 2d}$ the following are equivalent:

i) $f \in \sum R[X]^2$

ii) $f = \sum_{\alpha,\beta \in \Lambda(d)} a_{\alpha\beta} X^{\alpha+\beta}$ *for some symmetric positive semidefinite matrix* $A = (a_{\alpha\beta})_{\alpha,\beta \in \Lambda(d)}$ *with entries from R (sometimes called Gram matrix of f)*

iii) $f = \sum_{i=1}^{N} h_i^2$ *for some* $h_i \in R[X]_{\leq d}$ $(1 \leq i \leq N)$ *with* $N = |\Lambda(d)| = \binom{n+d}{n}$

Proof:
$i) \Rightarrow ii)$: If $f = \sum_{i=1}^{k} h_i^2$ is a sum of squares of polynomials from $R[X]$ for some $k \in \mathbb{N}$ then we have $h_i \in R[X]_{\leq d}$ for every $1 \leq i \leq k$ because the homogeneous

part of f of highest degree is a sum of squares of forms where no cancelation can occur.

We define $a_{\alpha\beta} := \sum_{i=1}^{k} h_{i\alpha}h_{i\beta}$ for $\alpha, \beta \in \Lambda(d)$ where $(h_{i\gamma})_{\gamma\in\Lambda(d)}$ is the coefficient vector of h_i with respect to the basis $\{X^\gamma \mid \gamma \in \Lambda(d)\}$. Then we have

$$f = \sum_{i=1}^{k}\Big(\sum_{\alpha,\beta\in\Lambda(d)} h_{i\alpha}h_{i\beta}X^{\alpha+\beta}\Big) = \sum_{\alpha,\beta\in\Lambda(d)} a_{\alpha\beta}X^{\alpha+\beta}$$

and $A = (a_{\alpha\beta})_{\alpha,\beta\in\Lambda(d)}$ is certainly a symmetric matrix. It is even positive semidefinite since for any vector $z \in R^{|\Lambda(d)|}$ we have

$$z^T Az = \sum_{i=1}^{k}\sum_{\alpha,\beta\in\Lambda(d)} h_{i\alpha}h_{i\beta}z_\alpha z_\beta = \sum_{i=1}^{k}\Big(\sum_{\alpha\in\Lambda(d)} h_{i\alpha}z_\alpha\Big)^2 \geq 0.$$

$ii) \Rightarrow iii)$: Since $A = (a_{\alpha\beta})_{\alpha,\beta\in\Lambda(d)}$ is symmetric positive semidefinite we can write $A = B^T B$ for some matrix $B = (b_{\alpha\beta})_{\alpha,\beta\in\Lambda(d)}$. Thus

$$f = \sum_{\alpha,\beta\in\Lambda(d)}\sum_{\gamma\in\Lambda(d)} b_{\gamma\alpha}b_{\gamma\beta}X^{\alpha+\beta} = \sum_{\gamma\in\Lambda(d)}\Big(\sum_{\alpha\in\Lambda(d)} b_{\gamma\alpha}X^\alpha\Big)^2$$

which is the desired sum of squares representation.

$iii) \Rightarrow i)$: This is trivial.

Lemma 1.9 □

Now we can state the semialgebraic formula $\vartheta_f^{stab}(Y)$ which defines membership in the stable quadratic module $Q = QM(g_1, ..., g_s) \subseteq R[X]$ for $f(X, Y) \in \mathbb{Z}[X, Y]$.

$$\vartheta_f^{stab}(Y) := \exists W\left(\forall X\left(f(X,Y) = \sum_{i=0}^{s}\left(\sum_{j=1}^{N(f)} F_{\varphi(f)}(X, W_{ij})^2\right) g_i(X)\right)\right)$$

where $W = (W_{11}, ..., W_{1N(f)}, ..., W_{sN(f)})$, $F_{\varphi(f)}(X, W_{ij}) = \sum_{\alpha\in\Lambda(\varphi(f))} W_{ij,\alpha}X^\alpha$ denotes the general polynomial in $\mathbb{Z}[X, W_{ij}]$ of degree $\varphi(f)$ with respect to X, $N(f)$ is related to $\varphi(f)$ as in Lemma 1.9 iii) and $\varphi(f) = \varphi(\deg(f))$ with $\varphi : \mathbb{N} \to \mathbb{N}$ given by the stability of Q.

If $f(X, Y) \in \mathbb{Z}[X, Y], g_1(X, Z), ..., g_s(X, Z) \in \mathbb{Z}[X, Z]$ then the L-formula

$$\vartheta^{stab}(Y, Z) := \exists W\left(\forall X\left(f(X,Y) = \sum_{i=0}^{s}\left(\sum_{j=1}^{N(f)} F_{\varphi(f)}(X, W_{ij})^2\right) g_i(X, Z)\right)\right)$$

defines uniformly the membership in the subset of the quadratic module where the degrees of the sums of squares are bounded by $\varphi(f)$ which means that for every $c \in R^Y, b \in R^Z$

$$f(X,c) \in \sum_{i=0}^{s} \sum R[X]^2_{\leq \varphi(f)} g_i(X,b) \Leftrightarrow R \models \vartheta^{stab}(c,b).$$

If $Q = QM(g_1(X,b),...,g_s(X,b))$ is stable with stability function $\varphi : \mathbb{N} \to \mathbb{N}$ then this formula defines the membership in Q.

Now we describe explicitly an algorithm for testing membership in a stable quadratic module which is called the Gram matrix method in the case that the quadratic module in consideration is $\sum \mathbb{R}[X]^2$ (see [P-W]). This method can easily be generalized for finitely generated quadratic modules which are stable.

If $Q = QM(g_1,...,g_s) \subseteq \mathbb{R}[X]$ is a stable quadratic module then for some given polynomial $f(X) = \sum\limits_{\gamma \in \Lambda(\deg(f))} c_\gamma X^\gamma \in R[X]$ there are $d_i \in \mathbb{N}_0$ $(0 \leq i \leq s)$ just depending on the degree of f and the degree of the polynomials $g_0 := 1, g_1, ..., g_s$ such that deciding whether $f \in Q$ reduces to deciding whether

$$f \in \sum_{i=0}^{s} \sum R[X]^2_{\leq d_i} g_i.$$

By Lemma 1.9 this is equivalent to testing whether there are positive semidefinite matrices $A_i = (a^{(i)}_{\alpha\beta})_{\alpha,\beta \in \Lambda(d_i)}$ for $0 \leq i \leq s$ such that

$$f \in \sum_{i=0}^{s} \sum_{\alpha,\beta \in \Lambda(d_i)} a^{(i)}_{\alpha\beta} X^{\alpha+\beta} g_i.$$

We expand for every $i \in \{0,...,s\}$ and every $\alpha, \beta \in \Lambda(d_i)$ the polynomials

$$X^{\alpha+\beta} g_i = \sum_{\gamma \in \Lambda(D)} c^{(\gamma,i)}_{\alpha\beta} X^\gamma$$

with respect to the basis of $R[X]_{\leq D}$ where $D = \max\limits_{0 \leq i \leq s} \{2d_i + \deg(g_i)\}$. This gives symmetric matrices $C_{\gamma,i} = (c^{(\gamma,i))}_{\alpha\beta})_{\alpha,\beta \in \Lambda(d_i)}$ for every $\gamma \in \Lambda(D)$ and every $0 \leq i \leq s$. By comparing coefficients we see that $f \in Q$ if and only if

$$\sum_{i=0}^{s} \sum_{\alpha,\beta \in \Lambda(d_i)} c^{(\gamma,i)}_{\alpha\beta} a^{(i)}_{\alpha\beta} = c_\gamma \quad \forall \gamma \in \Lambda(D)$$

for some positive semidefinite matrices $A_i = (a^{(i)}_{\alpha\beta})_{\alpha,\beta\in\Lambda(d_i)}$ $0 \le i \le s$. This can be written as

$$(*) \begin{cases} \sum_{i=0}^{s} \operatorname{trace}(C_{\gamma,i}A_i^T) = c_\gamma \quad \forall\gamma \in \Lambda(D) \\ A_i \succeq 0 \quad (0 \le i \le s) \end{cases}$$

where $A_i \succeq 0$ denotes that A_i is positive semidefinite. The problem of finding matrices which solve $(*)$ is nothing else than a semidefinite programming feasibility problem.

Thus given the input data $c \in R^Y$ for some $f(X,Y) \in \mathbb{Z}[X,Y]$ the problem of deciding if $f(X,c)$ is in $QM(g_1,..,g_s)$ translates in the stable case by the above sketched generalization of the Gram-Matrix method into a semidefinite program of bounded size which can be solved efficiently by interior point methods (see e.g. [N-N]).

When does stability occur?

One important result in this context is that $Q = QM(g_1,...,g_s)$ is stable if the associated basic closed set $S(g_1,...,g_s) \subseteq R^n$ contains an n-dimensional affine cone.

Over the reals this was proved by Powers and Scheiderer ([P-S] Theorem 2.14) and by Kuhlmann and Marshall ([K-M] Theorem 3.5). For the case of a quadratic module over an arbitrary real closed field we use the ideas worked out by Julia Salzl in her diploma thesis ([Sa]).

Let $v : R(X) \to \mathbb{Z}$ denote the total degree valuation, i.e. $v(\frac{f}{g}) = \deg g - \deg f$ for $f, g \in R[X]$.

Lemma 1.10
Let $K := R(\frac{X_1}{X_n},...,\frac{X_{n-1}}{X_n}) \subseteq R(X)$. Then $K \subseteq \mathcal{O}$ and the composition $K \to \mathcal{O}$ with $\mathcal{O} \to \mathcal{O}/\mathfrak{m}$ is an isomorphism $K \to \mathcal{O}/\mathfrak{m}$. Hence K is a residue field of v.

Proof:
An arbitrary element of K is of the form $\frac{f}{g}$ where $f, g \in R[\frac{X_1}{X_n},...,\frac{X_{n-1}}{X_n}]$. Thus f and g are of the form

$$\sum_{\epsilon\in\mathbb{N}_0^{n-1}} c_\epsilon (\frac{X_1}{X_n})^{\epsilon_1} \cdots (\frac{X_{n-1}}{X_n})^{\epsilon_{n-1}} = X_n^{-d} \sum_{\epsilon\in\mathbb{N}_0^{n-1}} c_\epsilon X_1^{\epsilon_1} \cdots X_{n-1}^{\epsilon_{n-1}} X_n^{d-(\epsilon_1+...+\epsilon_{n-1})}$$

where only finitely many $c_\epsilon \neq 0$ and $d := \max\{\epsilon_1 + ... + \epsilon_{n-1} \mid \epsilon \in \mathbb{N}_0^{n-1}, c_\epsilon \neq 0\}$. Hence $\frac{f}{g} = \frac{f_d \cdot X_n^{d'}}{g_{d'} \cdot X_n^d}$ where the $f_d, g_{d'} \in R[X_1,...,X_n]$ are homogeneous of degree d, resp. d'. Thus $\deg f = \deg g = d + d'$ and therefore $v(\frac{f}{g}) = 0$ which gives $K \subseteq \mathcal{O}$.

In order to see that $K \to \mathcal{O}/\mathfrak{m}$ is surjective we must show that for every $\frac{f}{g} \in \mathcal{O}$ with $d := \deg f = \deg g \in \mathbb{N}_0$, there is some $H \in K$ and some $\mu \in \mathfrak{m}$ such that $\frac{f}{g} = H + \mu$.
Let $f = f_d + f'$ where $f_d \in R[X]$ is non zero, homogeneous of degree d and $f' \in R[X]$ is of degree $< d$. Then f_d can be written as $f_d = X_n^d \cdot f_*(\frac{X_1}{X_n}, ..., \frac{X_{n-1}}{X_n}, 1)$ and $\mu_f := \frac{f'}{X_n^d} \in \mathfrak{m}$. Thus $f = X_n^d \cdot (f_*(\frac{X_1}{X_n}, ..., \frac{X_{n-1}}{X_n}, 1) + \mu_f)$.
The same argument applied to g gives $g = X_n^d \cdot (g_*(\frac{X_1}{X_n}, ..., \frac{X_{n-1}}{X_n}, 1) + \mu_g)$ for some $\mu_g \in \mathfrak{m}$. Hence

$$\frac{f}{g} = \frac{f_*(\frac{X_1}{X_n}, ..., \frac{X_{n-1}}{X_n}, 1) + \mu_f}{g_*(\frac{X_1}{X_n}, ..., \frac{X_{n-1}}{X_n}, 1) + \mu_g}$$

and we can take $H := \frac{f_*(\frac{X_1}{X_n}, ..., \frac{X_{n-1}}{X_n}, 1)}{g_*(\frac{X_1}{X_n}, ..., \frac{X_{n-1}}{X_n}, 1)} \in K$. This gives us

$$\frac{f}{g} - H = \frac{\mu_f g_* - \mu_g f_*}{g_*^2 + g_* \mu_g} = \frac{\frac{f' g_d}{X_n^{2d}} - \frac{g' f_d}{X_n^{2d}}}{\frac{g_d^2}{X_n^{2d}} + \frac{g_d g'}{X_n^{2d}}} = \frac{f' g_d - g' f_d}{g_d^2 + g_d g'} \in \mathfrak{m}$$

because $\deg(f' g_d), \deg(g' f_d), \deg(g_d g') < 2d$ and $\deg(g_d^2) = 2d$.

Lemma 1.10 □

We remark that with $K = R(\frac{X_1}{X_n}, ..., \frac{X_{n-1}}{X_n})$ as in the previous lemma we clearly have $R(X) = K(X_1) = ... = K(X_n)$ and each X_i is transcendental over K.

Lemma 1.11

i) Let $1 \leq i \leq n$. For $F \in K[X_i]$, $F = F_d \cdot X_i^d + ... + F_0$ with $F_j \in K$, $F_d \neq 0$ we have $v(F) = -d$.

ii) If $f \in R[X]$ with homogeneous components $f_0, ..., f_d$, then

$$f = X_n^d \cdot f_d(\frac{X_1}{X_n}, ..., \frac{X_{n-1}}{X_n}, 1) + X_n^{d-1} \cdot f_{d-1}(\frac{X_1}{X_n}, ..., \frac{X_{n-1}}{X_n}, 1) + ... + f_0 \in K[X_n].$$

Proof:

i) : We proceed by induction on d. If $d = 0$, then $F \in K$ is of valuation 0.
If $d > 0$ then with $F = F_d \cdot X_i^d + ... + F_0$ with $F_j \in K$, $F_d \neq 0$, we have by induction hypothesis that the valuation of $F - F_d \cdot X_i^d$ is strictly bigger than $-d$, hence the valuation of F is the valuation of $F_d \cdot X_i^d$. But F_d has valuation 0 and X_i^d has degree d as desired.

ii) : Clearly holds since $f_j(\frac{X_1}{X_n}, ..., \frac{X_{n-1}}{X_n}, 1) \in K$ $(0 \leq j \leq d)$.

Lemma 1.11 □

Now we determine the orderings of $R(X)$ compatible with v in order to get a condition on the semialgebraic set $S = S(g_1, ..., g_s) \subseteq R^n$ which implies that $\widetilde{S} \subseteq \operatorname{Sper} R[X]$ contains an ordering of $R(X)$ compatible with v. This will finally imply that $Q = QM(g_1, ..., g_s)$ is stable. In this context we identify the orderings of $R(X)$ with the orderings of $R[X]$ that have support $\{0\}$.

We recall that a valuation v is compatible with $\alpha \in \operatorname{Sper} R(X)$ if and only if $1+\mathfrak{m} > 0$ (respectively $\mathfrak{m} \subseteq]-1, 1[$) in the ordered field $(R(X), \alpha)$ ([K-S] II.2 Theorem 3). This means that the total degree valuation v is compatible with α if and only if

$$\deg f < \deg g \ \Rightarrow |f| < |g| \ \text{ in } (R(X), \alpha) \ (f, g \in R[X])$$

Lemma 1.12
Let $i \in \{1, ..., n\}$ and let $\alpha \in \operatorname{Sper} R(X)$. Then α is compatible with v if and only if $|X_i| > K$ in $(R(X), \alpha)$.

Proof:
First suppose that α is compatible with v. As $v(X_i) = -1 < 0 = v(h)$ for every $h \in K$ the compatibility of v and α implies that $|X_i| > h$ in $(R(X), \alpha)$.

Conversely suppose that $|X_i| > K$ in $(R(X), \alpha)$. Then $|X_n| = |X_i \frac{X_n}{X_i}| > K$ as well. Let $f, g \in R[X]$ with $d = \deg f < \deg g = d + r$ for some $r > 0$. By Lemma 1.11 we can write $f = F_d \cdot X_n^d + ... + F_0$ with $F_j \in K$, $F_d \neq 0$ and $g = G_{d+r} \cdot X_n^{d+r} + ... + G_0$ with $G_j \in K$, $G_{d+r} \neq 0$. But then, as $|X_n| > K$ and $r > 0$, we see that $|g| > |f|$ in $(R(X), \alpha)$. Hence v is compatible with α.

Lemma 1.12 □

Theorem 1.13
Let $Q = QM(g_1, ..., g_s) \subseteq R[X]$ and $S = S(g_1, ..., g_s) \subseteq R^n$.
If there is an ordering $\alpha \in \widetilde{S} \cap \operatorname{Sper} R(X)$ which is compatible with the total degree valuation v then Q is stable.

Proof:
We take arbitrary $h_1, ..., h_r \in Q$ and show that $Q' := QM(h_1, ..., h_r)$ is stable.

Because of $\alpha \in \widetilde{S}$ we have in the ordered field $(R(X), \alpha)$ for $1 \leq i \leq s$ that $g_i \geq 0$ and therefore $g \geq 0$ for every $g \in Q$. In particular $h_i \geq 0$ $(1 \leq i \leq r)$.

Now we show that for every $d \in \mathbb{N}$ there is some $N \in \mathbb{N}$ such that

$$Q' \cap R[X]_{\leq d} \subseteq \{\sum_{i=0}^{r} \sigma_i h_i \mid \sigma_i \in \sum R[X]^2_{\leq N} \ (0 \leq i \leq r)\}$$

where $h_0 := 1$. Since $(R[X]_{\leq d})_{d\in\mathbb{N}}$ is a filtration of $R[X]$ into finite dimensional subspaces this will prove the stability of Q'.

Let $h = \sum_{i=0}^{r} \sigma_i h_i \in Q' \cap R[X]_{\leq d}$ with some $\sigma_i \in \sum R[X]^2$ $(0 \leq i \leq r)$. We want to show that there is a common bound on the degree of the polynomials appearing in the sums of squares σ_i $(0 \leq i \leq r)$.

More generally we have the following:
For $a_1, ..., a_m \in Q \setminus \{0\}$ with $a_1 + ... + a_m \in R[X]_{\leq d}$ we have $a_i \in R[X]_{\leq d}$ for every $1 \leq i \leq m$.
This is true because $\deg(a_1 + ... + a_m) \leq d$ means that for the total degree valuation $v(a_1 + ... + a_m) \geq -d$.
By the first remark above we have in $(R(X), \alpha)$ that $a_i \geq 0$ $(1 \leq i \leq m)$ which gives that $a_i > 0$ $(1 \leq i \leq m)$ because $\mathrm{supp}(\alpha) = \{0\}$. This implies that

$$v(a_1 + ... + a_m) = \min_{1\leq i\leq m} v(a_i)$$

because α is compatible with v.
Hence $\min_{1\leq i\leq m} v(a_i) \geq -d$ which means that $\deg(a_i) \leq d \quad (1 \leq i \leq m)$.

With this observation we see that $\deg(\sigma_i h_i) \leq d$ for every i appearing in the representation of h. From this we get by Lemma 1.9 some $N \in \mathbb{N}$ such that $\sigma_i \in R[X]^2_{\leq N}$ for $0 \leq i \leq r$ which proves that Q' is stable.

Theorem 1.13 □

Corollary 1.14
If $S = S(g_1, ..., g_s) \subseteq R^n$ contains an n-dimensional affine cone then the quadratic module $Q = QM(g_1, ..., g_s) \subseteq R[X]$ is stable.

Proof:
Let α be the ordering of $R(X)$ with the property that $R(X_1, ..., X_{i-1}) < X_i$ in $(R(X), \alpha)$ for every $1 \leq i \leq n$. This ordering fulfills $|X_n| > K$ in $(R(X), \alpha)$ which implies by Lemma 1.12 that α is compatible with v. By the previous theorem it remains to show that $\alpha \in \widetilde{S}$. Since the property of being stable remains unchanged under R-algebra isomorphisms we can without loss of generality suppose that $\{x \in R^n \mid x_i \geq 0 \ (1 \leq i \leq n)\}$ is a subset of $S(g_1, ..., g_s)$. This implies that $g_i \geq 0$ $(1 \leq i \leq s)$ in $(R(X), \alpha)$ and finally that $\alpha \in \widetilde{S}$.

Corollary 1.14 □

Actually the quadratic modules satisfying the conditions of Theorem 1.13 (resp. Corollary 1.14) show a stronger form of stability, namely the sums of squares appearing in every representation of an element $f \in QM(g_1, ..., g_s)$ can be bounded

by the degree of f which is called totally stable by Netzer. Netzer even shows that $Q = QM(g_1, ..., g_s)$ is totally stable if and only if $S(g_1, ..., g_s) \subseteq R^n$ contains an n-dimensional affine cone ([N] Corollary 5.2, Example 5.3(1)).

Furthermore in the proof of Theorem 1.13 we not just show that $Q = QM(g_1, ..., g_s)$ is stable but that every finitely generated quadratic module contained in Q is stable. This kind of stability is also useful for not finitely generated quadratic modules and we call such a quadratic module very stable. Very stable orderings have the following nice property with respect to the inverse topology on the real spectrum Sper$A$ of a commutative ring A which has $\{\overline{H}(a) \mid a \in A\}$ as a subbasis of open sets.

Proposition 1.15
Let A be an R-algebra. The set of all very stable orderings of A is an inverse closed subset of Sper A.

Proof:
Let α be in the inverse closure of the set of very stable orderings of A. We claim that α is very stable, too.
Let $g_1, ..., g_s \in \alpha$. Then $\alpha \in \overline{H}(g_1, ..., g_s)$ which is inverse open. Hence there is some very stable ordering β of A such that $\beta \in \overline{H}(g_1, ..., g_s)$. So $g_1, ..., g_s \in \beta$ and $QM(g_1, ..., g_s)$ is stable because β is very stable.

Prop. 1.15 □

The inverse closure of the set of all very stable orderings means that this set is closed with respect to the constructible topology on Sper$A$, i.e. a proconstructible subset of Sper$A$.

If the dimension of $S(g_1, .., g_s) \subset R^n$ is bigger there is similar to the case of saturation a negative result what stability concerns.
If $Q = QM(g_1, ..., g_s) \subseteq R[X]$ has the moment property, i.e. the set of positive semidefinite elements of $R[X]$ is contained in the closure of Q with respect to the natural linear topology, and $\dim(S(g_1, ..., g_s)) \geq 2$ then Q is not stable ([S3] Theorem 5.4). Thus in particular for dimension ≥ 2 the quadratic modules for which the Theorem of Putinar holds, i.e. every strictly positive polynomial on $S(g_1, ..., g_s) \subseteq \mathbb{R}^n$ is contained in the quadratic module $Q = QM(g_1, ..., g_s)$ ([Pu] Lemma 4.1), are not stable. Since the result of Putinar serves as the theoretical underpinning for the optimization algorithm of Lasserre [L] this is unpleasant from a practical point of view. In order to find the minimum of a polynomial on a basic closed semialgebraic set the algorithm of Lasserre constructs a sequence of semidefinite programming problems which can be solved efficiently by using interior point methods. The convergence of the sequence is ensured by the Theorem of Putinar. Lasserre's algorithm

is implemented in the optimization software GloptiPoly. We refer the reader who wants to know more about the extremely interesting interplay of real algebra and optimization to the articles of Laurent [La], Marshall [M2] and Schweighofer [Sw].

There is another way of characterizing stable quadratic modules by looking at real closed extension fields. We come back to this in Section 3.2 when we are dealing with heirs.

We will see later on that there are quadratic modules which are weakly semialgebraic without being saturated or stable namely the not stable and not saturated finitely generated quadratic modules over $\mathbb{R}$ in dimension 1 and that there are even not finitely generated weakly semialgebraic quadratic modules namely the orderings in arbitrary dimension over $\mathbb{R}$.

1.3 Solution for orderings

Before we investigate more closely the general case of quadratic modules we describe what is known for orderings.

If we consider not just a quadratic module but an ordering α of $R[X_1, ..., X_n]$ we have a corresponding type p_α which is determined by the set $\{f \geq 0 \mid f \in \alpha\}$. In this case the fact that α is weakly semialgebraic means exactly that the corresponding type p_α is definable. For a proof of this see the Appendix (Proposition A.5).

In the one-dimensional case there is furthermore the well-known correspondence between complete types and Dedekind cuts. In this case the following is true.

Proposition 1.16 (Marker-Steinhorn, [M-S] Lemma 2.3)
Let $\alpha \subseteq R[X] = R[X_1]$ be an ordering with corresponding Dedekind cut p.
Then α is weakly semialgebraic if and only if p is principal.

For orderings it is also in arbitrary dimension possible to completely determine when they are weakly semialgebraic. The following theorem is a special instance of the Marker-Steinhorn theorem where tame extensions play an important role.

Definition 1.17
Let $R' \supseteq R$ be real closed fields, $\mathcal{O}$ the convex hull of R in R' and $\mathfrak{m}$ the maximal ideal of $\mathcal{O}$. We say that $R' \supseteq R$ is a tame extension of R if for every $c' \in \mathcal{O}$ there is a (necessarily unique) $c \in R$ such that $c' = c + \mathfrak{m}$.
The element c is called the standard part of c in R.

Theorem 1.18 (Marker-Steinhorn, [M-S] Theorem 2.1)
Let $\alpha \subseteq R[X] = R[X_1, ..., X_n]$ be an ordering.
Then α is weakly semialgebraic if and only if $R \hookrightarrow k(\alpha)$ is tame where $k(\alpha)$ is the real closure of $Quot(R[X]/supp(\alpha))$.

Theorem 1.18 is a result over arbitrary real closed fields R which gives us in particular that every ordering α of $\mathbb{R}[X]$ is weakly semialgebraic because for $\mathbb{R}$ we always have that $\mathbb{R} \hookrightarrow k(\alpha)$ is tame. This shows how useful it is to consider a general real closed field instead of just looking at $\mathbb{R}$ even if one is just interested in the result over $\mathbb{R}$.

In [Tr1] the proof of this theorem is done by induction. The case $n = 1$ is the content of Proposition 1.16. For the induction step Tressl uses the characterization of definability with the help of the existence of unique heirs.

This approach structures the following chapters. Before we introduce heirs for arbitrary subsets of the polynomial ring over a real closed field we solve the definability question for quadratic modules in dimension 1 over $\mathbb{R}$ and in special cases over arbitrary real closed fields.

2 The Membership Problem for finitely generated quadratic modules of $R[X]$ in dimension 1

In this chapter X always denotes one variable unless explicitly stated otherwise.

2.1 Solution in the case $R = \mathbb{R}$

We consider a polynomial $g \in \mathbb{R}[X]$ and first solve the Membership Problem affirmatively for the special quadratic module $Q = QM(g)$ generated by that single polynomial. This quadratic module is actually a preordering since it is obviously closed under multiplication.
We solve the Membership Problem in the affirmative by looking at the possible quadratic modules generated by one single polynomial in formal power series rings. Since the characterization of the structure of these quadratic modules can be done more generally over an arbitrary real closed field R instead of $\mathbb{R}$ we work over R now.

For an arbitrary commutative ring A the completion $\widehat{A}_I$ with respect to an ideal $I \subseteq A$ is defined as the inverse limit

$$\widehat{A}_I := \varprojlim A/I^n.$$

With $\widehat{I}_n := \{(g_k \bmod I^k)_{k\in\mathbb{N}} \in \widehat{A}_I \mid g_k \bmod I^k = 0 \ (k \leq n)\}$ we have $A/I^n \cong \widehat{A}_I/\widehat{I}_n$ for every $n \in \mathbb{N}$ (see [E] 7.1).

For $a \in R$ the formal power series ring $R[[X-a]]$ is isomorphic to the completion $\widehat{R[X]}_{(X-a)R[X]}$ of $R[X]$ with respect to the maximal ideal $(X-a)R[X]$ ([E] Example in Section 7.1).
$R[[X-a]]$ is a local ring with maximal ideal $(X-a)R[[X-a]]$ and for every element $q \in (X-a)R[[X-a]]$ we have that $1+q$ is a unit and a square in $R[[X-a]]$. If we say that something is true locally at a then we mean that it is true in the formal power series ring $R[[X-a]]$.

Remark 2.1
Every element $f = \sum_{i=0}^{\infty} c_i(X-a)^i \in R[[X-a]]$ can be written in the form

$$f = c_d(X-a)^d(1+q)$$

for some uniquely determined $d \in \mathbb{N}_0, c_d \neq 0$ and $q \in (X-a)R[[X-a]]$.
Just let d be the minimal $i \in \mathbb{N}_0$ for which $c_i \neq 0$ then

$$f = c_d(X-a)^d\underbrace{(1 + \frac{c_{d+1}}{c_d}(X-a) + ...)}_{=:1+q}$$

Proposition 2.2
If $\sigma \in \sum R[[X-a]]^2$ is of the form $\sigma = c_d(X-a)^d(1+q)$ as in the previous remark then $c_d > 0$ and $d = 2k$ is even.
Furthermore $\sum R[[X-a]]^2 = R[[X-a]]^2$.

Proof:
Without loss of generality we take $a = 0$.
Let $\sigma = \sum\limits_{i=1}^{m} h_i^2$ with some $h_i \in R[[X]]$ $(1 \leq i \leq m)$. Then there are for $1 \leq i \leq m$ some $d_i \in \mathbb{N}_0$, $c_j^{(i)} \in R$ for $j \geq d_i$ and $c_{d_i}^{(i)} \neq 0$ such that

$$h_i = c_{d_i}^{(i)} X^{d_i} + c_{d_i+1}^{(i)} X^{d_i+1} + ... \quad (1 \leq i \leq m).$$

Hence for every $i \in \{1, ..., m\}$ we have

$$h_i^2 = \underbrace{(c_{d_i}^{(i)})^2}_{>0} X^{2d_i} + 2c_{d_i}^{(i)} c_{d_i+1}^{(i)} X^{2d_i+1} + ...$$

By defining $\widetilde{c}_i := (c_{d_i}^{(i)})^2 > 0$ for $1 \leq i \leq m$ we have

$$\sigma = \sum_{i=1}^{m} \widetilde{c}_i X^{2d_i} + 2c_{d_i}^{(i)} c_{d_i+1}^{(i)} X^{2d_i+1} + ...$$

Let $2k := \min\limits_{1 \leq i \leq m} 2d_i$ and without loss of generality $d_i = k$ for $1 \leq i \leq r$. Then we can sum up all the higher order terms to some $q \in XR[[X]]$ as in the previous remark and obtain

$$\sigma = (\widetilde{c}_1 + ... + \widetilde{c}_r) X^{2k}(1+q)$$

which proves the first claim.
In the power series ring $R[[X]]$ we have $\sum R[[X]]^2 = R[[X]]^2$ because for some $\sigma \in \sum R[[X]]^2$ we have just proved that $\sigma = c_{2k} X^{2k}(1+q)$ with some $c_{2k} > 0$ and some $q \in XR[[X]]$. Hence $\sigma = X^{2k} \cdot p^2 \in R[[X]]^2$ for some $p \in R[[X]]$.

Prop. 2.2 □

One can also see that d has to be even and c_d has to be positive by looking at the two orderings of the field $R((X-a))$. With respect to one ordering $X-a$ is positive, with respect to the other $X-a$ is negative. As σ is a sum of squares it has to be positive with respect to both orderings which implies the properties we want.

Theorem 2.3
In $R[[X-a]]$ we have the following structure of proper quadratic modules generated by one polynomial:

$$\begin{tikzcd}
QM(X-a) & QM(-(X-a)^2) & QM(-(X-a)) \\
QM((X-a)^3) \arrow[u, hook] \arrow[ur, hook] & QM(-(X-a)^4) \arrow[u, hook] & QM(-(X-a)^3) \arrow[u, hook] \arrow[ul, hook] \\
QM((X-a)^5) \arrow[u, hook] \arrow[ur, hook] & QM(-(X-a)^6) \arrow[u, hook] & QM(-(X-a)^5) \arrow[u, hook] \arrow[ul, hook] \\
 & R[[X-a]]^2 \arrow[ul, hook] \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd}$$

and all these quadratic modules are included in the ordering coming from evaluation in a which is given by $R^{\geq 0} + (X-a)R[[X-a]]$.

Proof:
Without loss of generality we take $a = 0$.
Let $g \in R[[X]]$. By Remark 2.1 we have $g = c_d X^d(1+q)$ for some $d \in \mathbb{N}_0, c_d \neq 0$ and $q \in XR[[X]]$. This implies that $QM(g) = QM(c_d X^d)$ because $1+q$ is a square and a unit in $R[[X]]$. This shows that all quadratic modules generated by one polynomial in $R[[X]]$ are of the form $\mathrm{QM}(\pm 1 \cdot X^d)$ for some $d \in \mathbb{N}_0$.

The quadratic module generated by -1 is equal to $R[[X]]$ which is not proper.
The quadratic modules of the form $\mathrm{QM}(X^{2k})$ for some $k \in \mathbb{N}_0$ are clearly equal to $\sum R[[X]]^2$ which is by the previous proposition equal to $R[[X]]^2$.
The inclusions

$$QM(X) \supset QM(X^3) \supset QM(X^5) \supset ...,$$

$$QM(-X) \supset QM(-X^3) \supset QM(-X^5) \supset ...$$

and

$$QM(-X^2) \supset QM(-X^4) \supset QM(-X^6) \supset ...$$

are clear where each inclusion is strict as for example $X^{2k+1} \notin QM(X^{2k+3})$.

First we examine the connection between the first and the third column. For arbitrary $k, l \in \mathbb{N}_0$ we have $X^{2k+1} \notin QM(-X^{2l+1})$. This will be proved by contradiction. So suppose that $X^{2k+1} = \sigma_0 + \sigma_1(-X^{2l+1})$ for some $\sigma_i \in R[[X]]^2$ $(i = 0, 1)$. Then by

the previous proposition $\sigma_i = c_i X^{2k_i}(1+q_i)$ for some $k_i \in \mathbb{N}_0, c_i > 0$ and $q_i \in XR[[X]]$ $(i = 0, 1)$. Thus

$$X^{2k+1} = c_0 X^{2k_0}(1 + q_0) + (c_1 X^{2k_1}(1 + q_1))(-X^{2l+1}).$$

Now we get a contradiction by comparing terms of lowest degree which on the right hand side is either even or odd but then with negative coefficient whereas the term of lowest degree on the left hand side is odd with positive coefficient.
The same proof works the other way around so that we have

$$QM(X^{2k+1}) \not\subseteq QM(-X^{2l+1})$$

and

$$QM(-X^{2k+1}) \not\subseteq QM(X^{2l+1})$$

for every $k, l \in \mathbb{N}_0$.

For the connection between the first and the second column we note that $X^{2k+1} \notin \mathrm{QM}(-X^{2l})$ if $2k + 1 < 2l$. For otherwise we would have

$$X^{2k+1} = \sigma_0 + \sigma_1(-X^{2l}) = s_0^2 + s_1^2(-X^{2l})$$

for some $s_i \in R[[X]]$. As the terms in $s_1^2(-X^{2l})$ are of degree at least $2l > 2k + 1$ X^{2k+1} must appear in s_0^2. Therefore the order of s_0^2 which is even has to be of the form $2k' < 2k + 1$. Hence on the right hand side there is a term of the form $c^2 X^{2k'}$ which cannot be killed by terms from $s_1^2(-X^{2l})$ because $2k' < 2k + 1 < 2l$. Thus $\mathrm{QM}(X^{2k+1}) \not\subseteq \mathrm{QM}(-X^{2l})$ for $2k + 1 < 2l$.
However if $2k+1 > 2l$ then $X^{2k+1} \in \mathrm{QM}(-X^{2l})$ and hence $\mathrm{QM}(X^{2k+1}) \subseteq \mathrm{QM}(-X^{2l})$. This is true because of the formula $X^{2k+1} = (\frac{X^s+X^l}{\sqrt{2}})^2 + (\frac{1+X^{2(s-l)}}{2})(-X^{2l})$ where we write $s = 2k + 1 - l \geq l$ (because $2k + 1 \geq 2l$).
Similar connections are true for the second and the third column which means that we proved the desired structure.
That the inclusions between the first resp. third column and the middle column are strict can be seen by the fact that $-X^{2l} \notin QM(\pm X^{2k+1})$ if $2k + 1 > 2l$ (which can be proved similarly to $X^{2k+1} \notin \mathrm{QM}(-X^{2l})$ if $2k + 1 < 2l$).
The inclusion in the stated ordering is clear.

Theorem 2.3 □

Now we return to the ring of polynomials and consider some $g \in R[X]$.

For $a \in R$ let $\mathrm{ord}_a(g) := \min\{k \in \mathbb{N}_0 \mid g^{(k)}(a) \neq 0\}$.
In a Taylor series expansion $g = \sum_{i=0}^{n} c_i(X - a)^i$ of g in a the order $\mathrm{ord}_a(g)$ appears as the minimal index $i \in \mathbb{N}_0$ such that $c_i \neq 0$.

We denote the sign of $c_{\mathrm{ord}_a(g)}$ by $\epsilon_a(g) \in \{\pm 1\}$.

Thus considered as an element of the formal power series ring $R[[X-a]]$ we can write

$$g = \epsilon_a(g)(X-a)^{\mathrm{ord}_a(g)} \underbrace{(\epsilon_a(g) c_{\mathrm{ord}_a(g)})}_{>0} \underbrace{(1 + \frac{c_2}{c_{\mathrm{ord}_a(g)}}(X-a) + ...)}_{=:1+q}$$

with some $q \in (X-a)R[[X-a]]$.

For the quadratic module generated by the images of $g_1, ..., g_s \in R[X]$ in the formal power series ring $R[[X-a]]$ we use the notation $\widehat{QM}_a(g_1, ..., g_s)$ or $\widehat{Q}_a$ in case that $Q = QM(g_1, ..., g_s)$.

If $Q = QM(g)$ then we have

$$\widehat{Q}_a = \widehat{QM}_a(\epsilon_a(g)(X-a)^{\mathrm{ord}_a(g)}).$$

The image of $g \in R[X]$ in the formal power series ring $R[[X-a]]$, which is nothing else as the Taylor series expansion of g in a, is denoted by $\widehat{g}_a$.

With this considerations Theorem 2.3 translates into conditions which ensure that a polynomial is locally in the quadratic module generated by another polynomial.

Corollary 2.4
Let $f, g \in R[X]$ and $a \in R$. Then the following is true:

1) *If $\mathrm{ord}_a(g)$ is even and $\epsilon_a(g) = 1$ then*

$$\widehat{f}_a \in \widehat{QM}_a(g) \Leftrightarrow \mathrm{ord}_a(f) \text{ even and } \epsilon_a(f) = 1.$$

2) *If $\mathrm{ord}_a(g) = 0$ and $\epsilon_a(g) = -1$ then*

$$\widehat{f}_a \in \widehat{QM}_a(g) \Leftrightarrow f \in R[X].$$

3) *If $\mathrm{ord}_a(g) > 0$ is even and $\epsilon_a(g) = -1$ then*

$$\widehat{f}_a \in \widehat{QM}_a(g) \Leftrightarrow \mathrm{ord}_a(f) \text{ even and } \epsilon_a(f) = 1 \text{ or } \mathrm{ord}_a(f) \geq \mathrm{ord}_a(g).$$

4) *If $\mathrm{ord}_a(g)$ is odd then*

$$\widehat{f}_a \in \widehat{QM}_a(g) \Leftrightarrow \begin{array}{l} \mathrm{ord}_a(f) \text{ even and } \epsilon_a(f) = 1 \text{ or} \\ \mathrm{ord}_a(f) - \mathrm{ord}_a(g) \in 2\mathbb{N}_0 \text{ and } \epsilon_a(f) = \epsilon_a(g). \end{array}$$

Proof:
This follows directly from the Theorem 2.3 because

$$\widehat{QM}_a(f) = \widehat{QM}_a(\epsilon_a(f)(X-a)^{\mathrm{ord}_a(f)}),$$

$$\widehat{QM}_a(g) = \widehat{QM}_a(\epsilon_a(g)(X-a)^{\mathrm{ord}_a(g)})$$

and $\widehat{f}_a \in \widehat{QM}_a(g)$ if and only if $\widehat{QM}_a(f) \subseteq \widehat{QM}_a(g)$.

Corollary 2.4 □

The first equivalence in particular means that f is locally a (sum of) square(s) if and only if $\mathrm{ord}_a(f)$ is even and $\epsilon_a(f) = 1$.

Since the conditions for the order and ϵ given in Corollary 2.4 can be formulated by semialgebraic formulas (see Remark 2.12) the result of Corollary 2.4 means nothing else than the definability of membership in the quadratic module in the formal power series ring.

Corollary 2.5
If $f(X,Y) \in \mathbb{Z}[X,Y]$ and $g(X,Z) \in \mathbb{Z}[X,Z]$ then there is some L-formula $\varphi(Y,Z)$ such that for every real closed field R, every $a \in R$ and any $c \in R^Y, b \in R^Z$

$$\widehat{f}_a(X,c) \in \widehat{QM}_a(g(X,b)) \Leftrightarrow R \models \varphi(c,b).$$

With the help of the local conditions from Corollary 2.4 we are now able to answer the question when a polynomial f is (globally) in the quadratic module generated by another polynomial g in the case that the basic closed semialgebraic set $S(g)$ is a compact subset of $\mathbb{R}$. In order to do so we apply a local-global principle due to Scheiderer which essentially uses the archimedean property of the quadratic module. This is the reason why we get our result for quadratic modules whose associated semialgebraic set is bounded.

For the convenience of the reader we include a proof of the local-global principle for the case $n = 1$ which is inspired by Marshall ([M3] Theorem 9.2.1).

The essential part of the proof are two lemmas which we will use in other situations later on.

With $Z(f)$ we denote the set of zeros of f.

Lemma 2.6
Let R be a real closed field, $f, g_1, ..., g_s \in R[X]$ and $Q = QM(g_1, ..., g_s)$ with associated semialgebraic set $S = S(g_1, ..., g_s)$.
If $\widehat{f}_a \in \widehat{Q}_a$ for every $a \in Z(f) \cap S$ then $f \in Q + f^2 R[X]$.

Proof:
The set $Z(f) \cap S$ is the basic closed semialgebraic set associated to the quadratic module $Q + f^2R[X] = QM(g_1, ..., g_s, -f^2)$. Thus if $Z(f) \cap S$ is empty then -1 is an element of $Q + f^2R[X]$ by the abstract Stellensatz for quadratic modules (Theorem 0.4) and Proposition 0.3. This implies that $Q + f^2R[X] = R[X]$ because every element of $R[X]$ can be written as the difference of two squares. The conclusion of the lemma is in this case trivially true.

From now on we suppose that $Z(f) \cap S \neq \emptyset$.
We factorize $f = \prod_{p|f} p^{k_p}$ with $p \in R[X]$ irreducible and $k_p \in \mathbb{N}$.
Then we have $f^2 = \prod_{p|f} p^{2k_p}$.
The Chinese remainder theorem gives us $R[X]/f^2R[X] \cong \prod_{p|f} R[X]/p^{2k_p}R[X]$. $\quad (*)$

In order to prove that $f \in Q + f^2R[X]$ we first show that for every irreducible $p|f$ the polynomial f lies in $Q + p^{2k_p}R[X]$ and then use the Chinese remainder theorem. By considering the possibilities for the irreducibles p we get the following three cases:

Case 1: $p(X) = X - a$ for some $a \in Z(f) \cap S$
Then by assumption $\widehat{f_a} \in \widehat{Q}_a$ which means that there are $h_i \in R[[X - a]]$ for $0 \leq i \leq s$ such that $\widehat{f_a} = \sum_{i=0}^{s} h_i^2 \widehat{(g_i)}_a$ where $g_0 := 1$.
If $h_i = \sum_{j=0}^{\infty} c_j^{(i)}(X-a)^j \in R[[X-a]]$ we define $\overline{h}_i := \sum_{j=0}^{2k_p} c_j^{(i)}(X-a)^j \in R[X]$ for every $i \in \{0, ..., s\}$. Then $\overline{f} := \sum_{i=0}^{s} \overline{h}_i^2 g_i \in Q$ and $f \equiv \overline{f} \mod (X-a)^{2k_p}R[X]$ which means that $f \in Q + p^{2k_p}R[X]$.

Case 2: $p(X) = X - a$ for some $a \in Z(f) \setminus S$
This means that there is some $g \in Q$ such that $g(a) < 0$. Hence we can write $g = -(\underbrace{-g(a)}_{>0} + q)$ for some $q \in (X - a)R[[X - a]]$. Thus locally $\widehat{g}_a = -h^2$ for some unit $h \in R[[X - a]]$ which implies that $-1 = (\frac{1}{h})^2\widehat{g}_a \in \widehat{Q}_a$. Hence $\widehat{Q}_a = R[[X - a]]$ so that clearly $\widehat{f_a} \in \widehat{Q}_a$ and again $f \in Q + p^{2k_p}R[X]$.

Case 3: $p(X) = (X - a)^2 + b^2$ for some $a, b \in R$ and $b \neq 0$
In the ring $R[X]/p^{2k_p}R[X]$ we have the identity $((X-a)^2+b^2)^{2k_p} = 0$ which means that $b^{4k_p} = -\sigma$ where σ is a sum of squares.
Hence $-1 \in \sum R[X]^2 + p^{2k_p}R[X]$ as b is a unit. Thus $Q + p^{2k_p}R[X] = R[X]$ so clearly $f \in Q + p^{2k_p}R[X]$.

Up to now we have constructed for every irreducible $p|f$ a polynomial $f_p \in Q$ such that $f \equiv f_p \mod p^{2k_p}R[X]$. By the Chinese Remainder theorem the system of finitely many congruences for irreducible polynomials $p, \tilde{p}|f$

$$q_p \equiv \begin{cases} 1 & \mod \tilde{p}^{2k_{\tilde{p}}}R[X] \quad \text{if } p = \tilde{p} \\ 0 & \mod \tilde{p}^{2k_{\tilde{p}}}R[X] \quad \text{if } p \neq \tilde{p} \end{cases}$$

can be solved. Now we define $g := \sum_{p|f} q_p^2 f_p \in Q$.
Then $g \mod p^{2k_p}R[X] \equiv f_p \mod p^{2k_p}R[X] \equiv f \mod p^{2k_p}R[X]$ for every irreducible p which divides f and therefore we have by $(*)$ $f \equiv g \mod f^2R[X]$, i.e. $f \in Q + f^2R[X]$.

Lemma 2.6 □

Lemma 2.7 (Scheiderer, [S2] Corollary 3.11)
Let A be a commutative ring with 1, $\frac{1}{2} \in A$, $Q \subseteq A$ an archimedean quadratic module and $f \in A$.
If $f \geq 0$ on $\overline{H}(Q)$ and $f \in Q + f^2A$ then $f \in Q$.

Proof:
Since $\frac{1}{2} \in A$ implies that every element of A can be written as the difference of two squares we have $Q + f^2A = Q - A^2f^2$. Thus

$$f(1 + pf) = q$$

for some $q \in Q$ and some $p \in A^2$. By assumption Q is archimedean such that for every $h \in A$ there is some $N \in \mathbb{N}$ with $N \pm h \in Q$ and therefore $N \pm h \geq 0$ on $\overline{H}(Q)$. With $t := 1$ and $s := -p \in A$ we have

$$sf + t(1 + pf) = 1$$

as well as $f \geq 0$, $1 + pf \geq 0$ on the bounded set $\overline{H}(Q) \subseteq \mathrm{Sper}(A)$. Hence we get by the Basic Lemma (Theorem 0.10) some $\sigma, \tau \in A$ with

$$\sigma f + \tau(1 + pf) = 1 \quad (*)$$

where $\sigma, \tau > 0$ on $\overline{H}(Q)$. Thus also $\sigma\tau > 0$ on $\overline{H}(Q)$ and Kadison-Dubois (Theorem 0.9) now implies that σ and τ as well as $\sigma\tau$ are elements of Q.
By multiplying $(*)$ with τf we get $\tau f = \sigma\tau f^2 + \tau^2 q \in Q$ and finally by multiplying $(*)$ with f we get $f = \sigma f^2 + \tau f + \tau p f^2 \in Q$.

Lemma 2.7 □

In arbitrary dimension the boundedness of the semialgebraic set $S(g_1, .., g_s)$ just gives us that the preordering $PO(g_1, ..., g_s) \subseteq \mathbb{R}[X_1, ..., X_n]$ is archimedean ([P-D] Theorem 5.1.17), it does not imply that the quadratic module $QM(g_1, ..., g_s)$ is archimedean. However in dimension 1 it does. We show this with the help of the result $\operatorname{SemiSper} R[X] = \operatorname{Sper} R[X]$ (Proposition 0.3). For another proof of this see [M3] Theorem 7.1.2.

Proposition 2.8
Let $g_1, ..., g_s \in \mathbb{R}[X]$.
If $S = S(g_1, ..., g_s) \subseteq \mathbb{R}$ is bounded then $Q = QM(g_1, ..., g_s)$ is archimedean.

Proof:
The boundedness of S implies that there is some $N_0 \in \mathbb{N}$ such that

$$N_0 - X^2 > 0 \text{ on } S.$$

Thus $N_0 - X^2$ is strictly positive on $\overline{H}(g_1, ..., g_s)$ which is by Proposition 0.3 equal to $\overline{H}_{semi}(g_1, ..., g_s) = \overline{H}_{semi}(Q)$. Hence the abstract Stellensatz for quadratic modules (Theorem 0.4) gives us some $p \in \sum \mathbb{R}[X]^2$ and some $q \in Q$ such that

$$p(N_0 - X^2) = 1 + q.$$

From this we get as in the proof of iii') $\Rightarrow$ ii') in [P-D] Theorem 5.1.18 some $N_1 \in \mathbb{N}$ such that

$$N_1 - X^2 \in Q.$$

This implies by [P-D] Corollary 5.1.14 that Q is archimedean.

Prop. 2.8 $\square$

This proposition together with the two previous lemmas now easily give the local-global principle of Scheiderer.

Theorem 2.9 (Scheiderer, [S2] Corollary 3.17)
Let $f, g_1, ..., g_s \in \mathbb{R}[X]$ and $Q = QM(g_1, ..., g_s)$ with $S = S(g_1, ..., g_s) \subseteq \mathbb{R}$ bounded.
If $\widehat{f}_a \in \widehat{Q}_a$ for every $a \in Z(f) \cap S$ and $f|_S \geq 0$ then $f \in Q$.

Proof:
By Lemma 2.6 we know that $f \in Q + f^2\mathbb{R}[X]$. Proposition 2.8 implies that Q is archimedean. Since $\overline{H}(Q) = \widetilde{S}$ we get by Lemma 2.7 that $f \in Q$.

Theorem 2.9 $\square$

Now we note that for isolated points $a \in S(g)_{isol}$ the order $\mathrm{ord}_a(g)$ is even and $\epsilon_a(g) = -1$.
For boundary points a of $S(g) \setminus S(g)_{isol}$ the order $\mathrm{ord}_a(g)$ is odd. If a is a left boundary point we have $\epsilon_a(g) = 1$ and if a is a right boundary point we have $\epsilon_a(g) = -1$.

In the case that $S(g)$ is a not degenerated interval the result of the next theorem can already be found in [P-R].

Theorem 2.10
Let $f, g \in \mathbb{R}[X]$ and $S = S(g) \subseteq \mathbb{R}$ bounded. Then $f \in Q = QM(g)$ if and only if $f|_S \geq 0$ and

i) for every boundary point a of $S \setminus S_{isol}$ we have $ord_a(f)$ is even or $ord_a(f) - ord_a(g) \in 2\mathbb{N}_0$

ii) for every isolated point a of S we have $ord_a(f)$ is even and $\epsilon_a(f) = 1$ or $ord_a(f) \geq ord_a(g)$.

Proof:

$\Rightarrow$: Since $f \in Q$ we clearly have that $f|_S \geq 0$ and $\widehat{f_a} \in \widehat{Q}_a$ for every $a \in \mathbb{R}$.
If a is a boundary point of $S \setminus S_{isol}$ then $\mathrm{ord}_a(g)$ is odd so we get by Corollary 2.4 4) the desired properties of f.
If a is an isolated point of S then $\mathrm{ord}_a(g)$ is even and $\epsilon_a(g) = -1$. Hence Corollary 2.4 3) gives us what we need.

$\Leftarrow$: Let a be a zero of f in S.
If a lies in the interior of S then $\mathrm{ord}_a(f)$ must be even and $\epsilon_a(f) = 1$ because of the nonnegativity condition on f. This implies that $\widehat{f_a}$ is a square in the formal power series ring at a.
If a is one of the boundary points of $S \setminus S_{isol}$ then $\mathrm{ord}_a(g)$ is odd and we have because of the nonnegativity condition for f on S that $\epsilon_a(f) = \epsilon_a(g)$. Thus we get by Corollary 2.4 4) under the additional assumption i) that $\widehat{f_a} \in \widehat{Q}_a$.
For isolated points a we finally have that $\mathrm{ord}_a(g)$ is even and $\epsilon_a(g) = -1$ so that we have together with assumption ii) by Corollary 2.4 3) that $\widehat{f_a} \in \widehat{Q}_a$.
Altogether we have shown that $\widehat{f_a}$ lies in the image of Q in the formal power series ring $\mathbb{R}[[X - a]]$ for every zero a of f in S. This gives by the local-global principle of Scheiderer (Theorem 2.9) that $f \in Q$.

Theorem 2.10 $\square$

Before expressing the conditions of the previous theorem by semialgebraic formulas and solving affirmatively the Membership Problem we remark that Theorem 2.10 can also be formulated for non-singular irreducible affine curves C over $\mathbb{R}$ as according to Scheiderer ([S2] Theorem 5.5) the local-global principle holds there and the completed local rings in non-singular points are nothing else but formal power series rings in one variable. Hence our way of reasoning works in a complete analogue for these objects just that compact is replaced by virtually compact which means the following. A closed semialgebraic subset S of $C(\mathbb{R})$ is virtually compact if (every irreducible component of) C has either a non-real point at infinity or a real point at infinity which does not lie in the closure of S.

Theorem 2.11
Let C be an affine curve over $\mathbb{R}$ which is non-singular and irreducible. Suppose that $f, g \in \mathbb{R}[C]$ and $S = S(g) \subseteq C(\mathbb{R})$ is virtually compact.
Then $f \in Q = QM(g)$ if and only if $f|_S \geq 0$ and

i) for every boundary point a of $S \setminus S_{isol}$ we have $ord_a(f)$ is even or $ord_a(f) - ord_a(g) \in 2\mathbb{N}_0$

ii) for every isolated point a of S we have $ord_a(f)$ is even and $\epsilon_a(f) = 1$ or $ord_a(f) \geq ord_a(g)$.

Remark 2.12
Since the order of g in a boundary point of $S(g) \setminus S(g)_{isol}$ is odd, condition i) of Theorem 2.10 can be rewritten as

$$ord_a(f) \text{ is even or } ord_a(f) \geq ord_a(g).$$

This shows that both conditions of Theorem 2.10 can be expressed by semialgebraic formulas. We state them explicitly now:
Let $f = f(X, c)$ and $g = g(X, b)$ for some $f(X, Y) \in \mathbb{Z}[X, Y]$, $g(X, Z) \in \mathbb{Z}[X, Z]$ and coefficients $c \in \mathbb{R}^k$ and $b \in \mathbb{R}^l$ where $k = |Y|$ and $l = |Z|$. Without loss of generality let $f(X, Y)$ be the general polynomial of degree d where d is the degree of f with respect to X and $g(X, Z)$ the general polynomial of degree e where e is the degree of g with respect to X.
This means that $k = d + 1$, $l = e + 1$, $f(X, Y) = Y_0 + Y_1X + ... + Y_dX^d$ and $g(X, Z) = Z_0 + Z_1X + ... + Z_eX^e$.
The largest even number less or equal to d (respectively e) will be denoted by $2D$ (respectively by $2E$).
Bit by bit we express now the sufficient and necessary conditions of theorem 2.10 by first order formulas in the language of ordered rings:

$$f|_{S(g)} \geq 0$$
can be expressed as
$$\forall X\ (g(X,b) \geq 0 \rightarrow f(X,c) \geq 0)$$
which is denoted by
$$\mathbb{R} \models \theta_{e,d,sat}(b,c)$$

a is a left boundary point of $S(g) \setminus S(g)_{isol}$
can be expressed as
$$g(a,b) = 0 \wedge \exists \delta > 0 [\forall \epsilon \in]0, \delta[(g(a-\epsilon, b) < 0 \wedge g(a+\epsilon, b) > 0)]$$
which is denoted by
$$\mathbb{R} \models \theta_{e,lend}(a,b)$$

a is a right boundary point of $S(g) \setminus S(g)_{isol}$
can be expressed as
$$g(a,b) = 0 \wedge \exists \delta > 0 [\forall \epsilon \in]0, \delta[(g(a-\epsilon, b) > 0 \wedge g(a+\epsilon, b) < 0)]$$
which is denoted by
$$\mathbb{R} \models \theta_{e,rend}(a,b)$$

a is an isolated point of $S(g)$
can be expressed as
$$\begin{aligned} g(a,b) = 0 \wedge \exists \delta > 0 [\forall \epsilon \in]0, \delta[\quad & (g(a-\epsilon, b) > 0 \wedge g(a+\epsilon, b) > 0) \\ \vee \quad & (g(a-\epsilon, b) < 0 \wedge g(a+\epsilon, b) < 0)] \end{aligned}$$
which is denoted by
$$\mathbb{R} \models \theta_{e,iso}(a,b)$$

To express the order of f or g in a point a we use the definition by derivatives and remark that $ord_a(f) \leq d$ and $ord_a(g) \leq e$:

$$k = ord_a(f)$$
can be expressed as
$$f(a,c) = 0 \wedge f'(a,c) = 0 \wedge ... \wedge f^{(k-1)}(a,c) = 0 \wedge f^{(k)}(a,c) \neq 0$$
or equivalently by
$$c_0 + c_1 a + ... + c_d a^d = 0 \wedge c_1 + 2c_2 a + ... + dc_d a^{d-1} = 0 \wedge ...$$
$$... \wedge k(k-1) \cdots 2 \cdot 1 \cdot c_k + (k+1)k \cdots 2 \cdot c_{k+1} a + ... + d(d-1) \cdots (d-k+1) c_d a^{d-k} \neq 0$$
which we denote by
$$\mathbb{R} \models \theta_{d,ord,k}(a,c)$$

If we in addition want to have that $f^{(k)}(a) > 0$ we denote the corresponding formula by $\theta_{d,ord,k,+}$.

Now condition i) of the Theorem 2.10 can be expressed as

$$\forall a[\ \ (\theta_{e,lend}(a,b) \vee \theta_{e,rend}(a,b)) \\ \rightarrow (\theta_{d,ord,0}(a,c) \vee \theta_{d,ord,2}(a,c) \vee ... \vee \theta_{d,ord,2D}(a,c) \vee \theta_{d,e,ord,\geq}(a,b,c))]$$

where $\theta_{d,e,ord,\geq}(a,b,c)$ *is the finite disjunction of the formulas*

$$\theta_{e,ord,0}(a,b) \rightarrow (\theta_{d,ord,0}(a,c) \vee ... \vee \theta_{d,ord,d}(a,c)),$$

$$\theta_{e,ord,1}(a,b) \rightarrow (\theta_{d,ord,1}(a,c) \vee ... \vee \theta_{d,ord,d}(a,c))$$

up to

$$\theta_{e,ord,d}(a,b) \rightarrow \theta_{d,ord,d}(a,c).$$

Condition ii) of Theorem 2.10 becomes

$$\forall a(\ \ \theta_{e,iso}(a,b) \\ \rightarrow (\theta_{d,ord,0,+}(a,c) \vee \theta_{d,ord,2,+}(a,c) \vee ... \vee \theta_{d,ord,2D,+}(a,c) \vee \theta_{d,e,ord,\geq}(a,b,c)))$$

This altogether even shows that for given general polynomials $f(X,Y) \in \mathbb{Z}[X,Y]$ *and* $g(X,Z) \in \mathbb{Z}[X,Y]$ *of degree* d *and* e *there is a semialgebraic formula* $\phi(Y,Z)$ *which has parameters just from* $\mathbb{Z}$ *such that for* $c \in \mathbb{R}^Y, b \in \mathbb{R}^Z$ *the following is true*

$$f(X,c) \in QM(g(X,b)) \Leftrightarrow \mathbb{R} \models \phi(c,b)$$

Now we are able to prove that the Membership Problem is solvable affirmatively for the special case that we are in dimension one over the reals and the quadratic module is generated by a single polynomial.

Theorem 2.13
For $g \in \mathbb{R}[X]$ *the quadratic module* $QM(g)$ *is weakly semialgebraic.*

Proof:
We consider the basic closed semialgebraic set $S(g)$.
If $S(g)$ is not bounded we know by the stability theorem of Kuhlmann/Marshall or Powers/Scheiderer (Corollary 1.14) that $QM(g)$ is stable and therefore weakly semialgebraic as explained in Section 1.2.
If $S(g)$ is bounded we know by Theorem 2.10 that $QM(g)$ is weakly semialgebraic.
Theorem 2.13 □

Corollary 2.14
If $g(X) \in \mathbb{R}[X]$ *and the input data is computable then the Membership Problem is solvable affirmatively for* $QM(g)$.

The consideration made above about the defining formulas show in particular the following uniform version of the positive solution of the Membership Problem.

Corollary 2.15
If $f(X,Y) \in \mathbb{Z}[X,Y]$ and $g(X,Z) \in \mathbb{Z}[X,Z]$ then there is an L-formula $\varphi(Y,Z)$ such that we have for every real closed subfield R of $\mathbb{R}$ and any $c \in R^Y, b \in R^Z$

$$f(X,c) \in QM_{R[X]}(g(X,b)) \Leftrightarrow R \models \varphi(c,b).$$

Proof:
With $\theta(Z) := \exists r[\forall X(g(X,Z) \geq 0 \to X^2 \leq r)]$ we define

$$\varphi(Y,Z) := (\theta(Z) \to \phi(Y,Z)) \vee (\neg\theta(Z) \to \vartheta^{stab}(Y,Z))$$

with $\phi(Y,Z)$ from Remark 2.12 and $\vartheta^{stab}(Y,Z)$ from the part about stable quadratic modules in Section 1.2. Then we have as in the proof of Theorem 2.13 that for $c \in \mathbb{R}^Y, b \in \mathbb{R}^Z$

$$f(X,c) \in QM_{\mathbb{R}[X]}(g(X,b)) \Leftrightarrow \mathbb{R} \models \varphi(c,b).$$

Let now R be an arbitrary real closed subfield of $\mathbb{R}$ and $c \in R^Y$ as well as $b \in R^Z$. If $f(X,c) \in QM_{R[X]}(g(X,b))$ then we have by $QM_{R[X]}(g(X,b)) \subseteq QM_{\mathbb{R}[X]}(g(X,b))$ that $\mathbb{R} \models \varphi(c,b)$. Since $\mathbb{R}$ is an elementary extension of R and b and c are from R we also have $R \models \varphi(c,b)$.
If on the other hand $R \models \varphi(c,b)$ then again by the property of being an elementary extension we know that $\mathbb{R} \models \varphi(c,b)$ and thus $f(X,c) \in QM_{\mathbb{R}[X]}(g(X,b)) \cap R[X]$. Thus we have $\mathbb{R} \models \exists W(\forall X(f(X,c) = \sum_{j=1}^{k_0} F_{d_0}(X,W)^2 + \sum_{j=1}^{k_1} F_{d_1}(X,W)^2 g(X,b)))$ for certain $k_i, d_i \in \mathbb{N}$ where $F_{d_i}(X,W)$ is the general polynomial of degree d_i with respect to X $(i = 0,1)$. By the Tarski transfer principle (Theorem 0.7) this formula is also true for R which finally implies that $f(X,c) \in QM_{R[X]}(g(X,c))$.

Corollary 2.15 □

Now we consider an arbitrary finitely generated quadratic module $Q = QM(G)$ of $R[X]$ for some $G = \{g_1, ..., g_s\} \subseteq R[X]$. The description of $\widehat{Q}_a \subseteq R[[X-a]]$ for some $a \in R$ will depend on the following values:

$$\begin{aligned} k_a(G) &:= \min_{1 \leq i \leq s} \{\mathrm{ord}_a(g_i) \mid \mathrm{ord}_a(g_i) \text{ even}, \epsilon_a(g_i) = -1\} \\ k_a^+(G) &:= \min_{1 \leq i \leq s} \{\mathrm{ord}_a(g_i) \mid \mathrm{ord}_a(g_i) \text{ odd}, \epsilon_a(g_i) = 1\} \\ k_a^-(G) &:= \min_{1 \leq i \leq s} \{\mathrm{ord}_a(g_i) \mid \mathrm{ord}_a(g_i) \text{ odd}, \epsilon_a(g_i) = -1\} \end{aligned}$$

In any of the three cases we define $k_a^+(G), k_a^-(G)$ and $k_a(G)$ to be ∞ if the corresponding set is empty.

How does $\widehat{Q}_a$ look like?
We have to distinguish the following cases.

Remark 2.16
Let $G = \{g_1, ..., g_s\} \subseteq R[X]$ and $Q = QM(G)$.

Case 1: $k_a(G) = k_a^+(G) = k_a^-(G) = \infty \Rightarrow \widehat{Q}_a = \mathbb{R}[[X-a]]^2$

Case 2: $k_a^+(G) < \infty, k_a(G) = k_a^-(G) = \infty \Rightarrow \widehat{Q}_a = \widehat{QM}_a((X-a)^{k_a^+(G)})$

Case 3: $k_a^-(G) < \infty, k_a(G) = k_a^+(G) = \infty \Rightarrow \widehat{Q}_a = \widehat{QM}_a(-(X-a)^{k_a^-(G)})$

Case 4: $k_a(G) < \infty, k_a^+(G) = k_a^-(G) = \infty \Rightarrow \widehat{Q}_a = \widehat{QM}_a(-(X-a)^{k_a(G)})$

Case 5: $k_a^+(G), k_a^-(G) < \infty, k_a(G) = \infty \Rightarrow \widehat{Q}_a = \widehat{QM}_a((X-a)^{k_a^+(G)}, -(X-a)^{k_a^-(G)})$

Case 6: $k_a(G), k_a^-(G) < \infty, k_a^+(G) = \infty$

Case 6a: $k_a(G) < k_a^-(G) \Rightarrow \widehat{Q}_a = \widehat{QM}_a(-(X-a)^{k_a(G)})$
Case 6b: $k_a(G) > k_a^-(G) \Rightarrow \widehat{Q}_a = \widehat{QM}_a(-(X-a)^{k_a^-(G)}, -(X-a)^{k_a(G)})$

Case 7: $k_a(G), k_a^+(G) < \infty, k_a^-(G) = \infty$

Case 7a: $k_a(G) < k_a^+(G) \Rightarrow \widehat{Q}_a = \widehat{QM}_a(-(X-a)^{k_a(G)})$
Case 7b: $k_a(G) > k_a^+(G) \Rightarrow \widehat{Q}_a = \widehat{QM}_a((X-a)^{k_a^+(G)}, -(X-a)^{k_a(G)})$

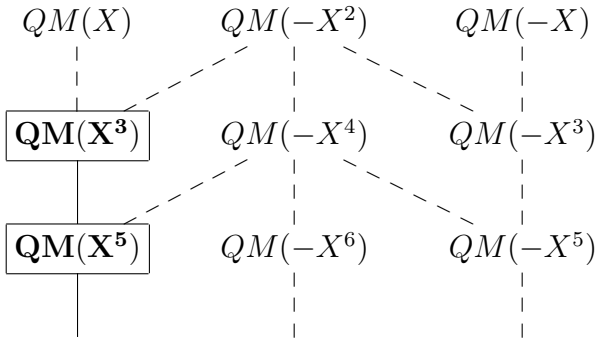
Case 8: $k_a(G), k_a^+(G), k_a^-(G) < \infty$

Case 8a: $k_a(G) < k_a^+(G), k_a^-(G) \Rightarrow \widehat{Q}_a = \widehat{QM}_a(-(X-a)^{k_a(G)})$
Case 8b: $k_a^+(G) < k_a(G) < k_a^-(G) \Rightarrow \widehat{Q}_a = \widehat{QM}_a((X-a)^{k_a^+(G)}, -(X-a)^{k_a(G)})$
Case 8c: $k_a^-(G) < k_a(G) < k_a^+(G) \Rightarrow \widehat{Q}_a = \widehat{QM}_a(-(X-a)^{k_a^-(G)}, -(X-a)^{k_a(G)})$
Case 8d: $k_a(G) > k_a^+(G), k_a^-(G) \Rightarrow \widehat{Q}_a = \widehat{QM}_a((X-a)^{k_a^+(G)}, -(X-a)^{k_a^-(G)})$

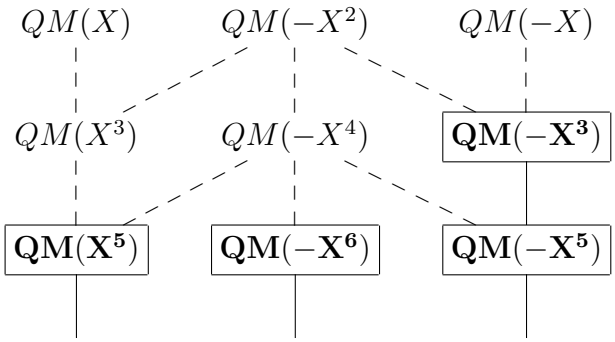
In every case the given representation of $\widehat{Q}_a$ follows immediately from Theorem 2.3 with the description of the inclusions between the three columns and the fact that $\pm(X-a)^l \in \widehat{Q}_a$ for some $l \in \mathbb{N}$ if and only if $\widehat{QM}_a(\pm(X-a)^l) \subseteq \widehat{Q}_a$.

We illustrate some cases by examples for $a = 0$ and indicate by bold letters the quadratic modules respectively elements lying in the given quadratic module:

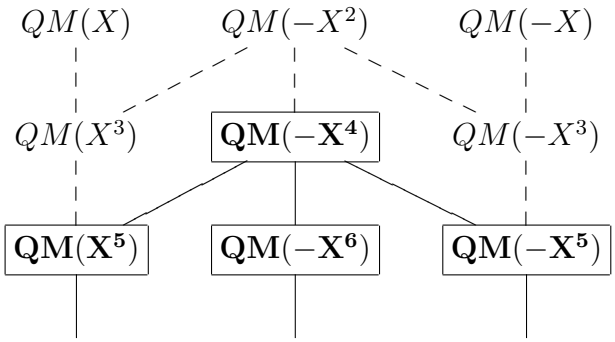
Case 2: $\widehat{QM}_0(X^3, X^7) = \widehat{QM}_0(X^3)$



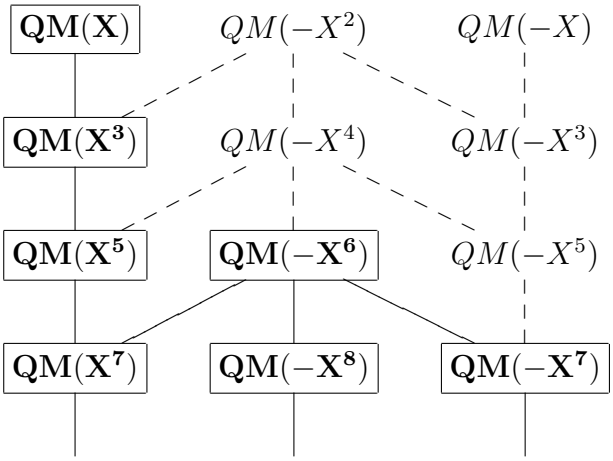
Case 5: $\widehat{QM}_0(X^5, -X^3, X^9, -X^5) = \widehat{QM}_0(X^5, -X^3)$



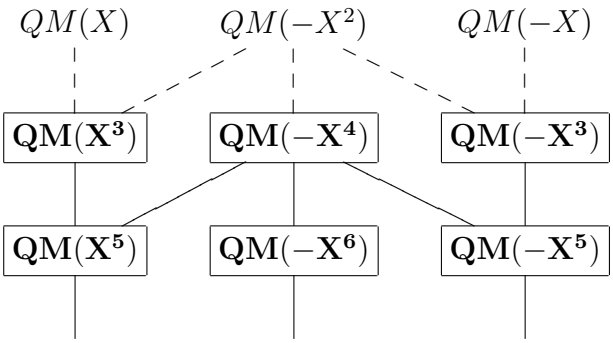
Case 8a: $\widehat{QM}_0(X^5, -X^4, -X^5) = \widehat{QM}_0(-X^4)$



Case 8b: $\widehat{QM}_0(X, -X^6, -X^7) = \widehat{QM}_0(X, -X^6)$



Case 8d: $\widehat{QM}_0(X^3, -X^4, -X^3) = \widehat{QM}_0(X^3, -X^3)$



Before we formulate the generalization of Theorem 2.10 for the finitely generated case we observe that the finitely generated quadratic modules in $R[[X-a]]$ listed above are all closed under multiplication which has the following nice consequence which has already been observed by Scheiderer ([S4] Corollary 4.4).

Theorem 2.17
Let $g_1, ..., g_s \in \mathbb{R}[X]$ with $S = S(g_1, ..., g_s) \subseteq \mathbb{R}$ bounded.
Then the quadratic module $Q = QM(g_1, ..., g_s)$ is closed under multiplication and thus $Q = PO(g_1, ..., g_s)$.

Proof:
For abbreviation we write $G := \{g_1, ..., g_s\}$.
We consider some $f_1, f_2 \in Q$ and show that $f_1 f_2 \in Q$.
Since f_1 and f_2 are elements of Q we certainly have that $f_i|_S \geq 0$ for $i = 1, 2$ and

thus also $f_1 f_2|_S \geq 0$.
Furthermore for every zero a of $f_1 f_2$ in S we have $\widehat{(f_i)}_a \in \widehat{Q}_a$ $(i = 1, 2)$.
$\widehat{Q}_a$ equals one of the quadratic modules given in Remark 2.16. Any of them is closed under multiplication.
For example if $\widehat{Q}_a$ equals $\widehat{QM}_a((X-a)^{k_a^+(G)}, (X-a)^{k_a^-(G)})$ from case 8d then we have $(X-a)^{k_a^+(G)}(X-a)^{k_a^-(G)} \in \widehat{QM}_a((X-a)^{k_a^+(G)}, (X-a)^{k_a^-(G)})$ because $\widehat{QM}_a((X-a)^{k_a^+(G)}(X-a)^{k_a^-(G)}) \subseteq \widehat{QM}_a((X-a)^{k_a^+(G)}, (X-a)^{k_a^-(G)})$ according to Theorem 2.3.
The closure of the quadratic modules in all the other cases also follows easily with the help of Theorem 2.3.
Hence we have $\widehat{(f_1 f_2)}_a \in \widehat{Q}_a$ for every $a \in Z(f_1 f_2) \cap S$ which now implies by the local-global principle of Scheiderer (Theorem 2.9) that $f_1 f_2 \in Q$.

Theorem 2.17 □

From now on we always keep in mind that whenever we deal with a finitely generated quadratic module $Q \subseteq \mathbb{R}[X]$ whose associated semialgebraic set is bounded then Q is in fact a preordering.

The next theorem characterizes the membership in such finitely generated quadratic modules.

Theorem 2.18
Let $f \in \mathbb{R}[X]$ and $G = \{g_1, ..., g_s\} \subseteq \mathbb{R}[X]$ with $S = S(G) \subseteq \mathbb{R}$ bounded.
Then $f \in Q = QM(G)$ if and only if $f|_S \geq 0$ and

i) for every left boundary point a of $S \setminus S_{isol}$ we have $ord_a(f)$ is even or $ord_a(f) - k_a^+(G) \in 2\mathbb{N}_0$

ii) for every right boundary point a of $S \setminus S_{isol}$ we have $ord_a(f)$ is even or $ord_a(f) - k_a^-(G) \in 2\mathbb{N}_0$

iii) for every isolated point a of S we have $ord_a(f)$ is even and $\epsilon_a(f) = 1$ or

Case 1: $ord_a(f) \geq k_a(G)$ if $k_a(G) < k_a^+(G)$ and $k_a(G) < k_a^-(G)$.

Case 2: $(ord_a(f) - k_a^+(G) \in 2\mathbb{N}_0$ and $\epsilon_a(f) = 1)$ or $ord_a(f) \geq \min(k_a(G), k_a^-(G))$ if $k_a^+(G) \leq \min(k_a(G), k_a^-(G))$.

Case 3: $(ord_a(f) - k_a^-(G) \in 2\mathbb{N}_0$ and $\epsilon_a(f) = -1)$ or $ord_a(f) \geq \min(k_a(G), k_a^+(G))$ if $k_a^-(G) \leq \min(k_a(G), k_a^+(G))$.

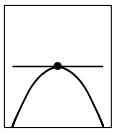
Proof:
This follows with a similar argument as in Theorem 2.10 from the description of $\widehat{QM}_a(g_1, ..., g_s)$ given in Remark 2.16 and the local-global principle of Scheiderer (Theorem 2.9) if we consider the following.
If a is a left boundary point of $S \setminus S_{isol}$ then there is some $i \in \{1, ..., s\}$ such that $\mathrm{ord}_a(g_i)$ is odd and $\epsilon_a(g_i) = 1$. For all other $j \neq i$ we must either also have $\mathrm{ord}_a(g_j)$ odd and $\epsilon_a(g_j) = 1$ or in the other case $\mathrm{ord}_a(g_j)$ even and $\epsilon_a(g_j) = 1$. In both cases we are in case 2 of Remark 2.16. Hence $\widehat{QM}_a(g_1, ..., g_s) = \widehat{QM}_a((X-a)^{k_a^+(G)})$.
Similar considerations show *ii*).
In the case of an isolated point there is either some $1 \leq i \leq s$ such that $\mathrm{ord}_a(g_i)$ is even and $\epsilon_a(g_i) = -1$ which means that $k_a(G) < \infty$ or at least two of the values $k_a(G), k_a^+(G)$ and $k_a^-(G)$ are less than infinity. Now depending on the relation between $k_a^+(G), k_a^-(G)$ and $k_a(G)$ we are in case 4, 5, 6, 7 or 8 of Remark 2.16 which covers the cases listed in *iii*).

Theorem 2.18 □

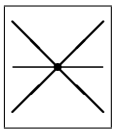
Now we distinguish the isolated points in the following way.

Let $G = \{g_1, ..., g_s\} \subseteq \mathbb{R}[X]$, $S = S(G)$ and $a \in S_{isol}$.
We say that a is an isolated point of type

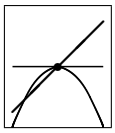
$\boxed{\text{A}}$ (for G) if $k_a(G) < k_a^+(G)$ and $k_a(G) < k_a^-(G)$



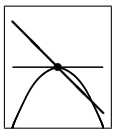
$\boxed{\text{B}}$ (for G) if $k_a(G) > k_a^+(G)$ and $k_a(G) > k_a^-(G)$



$\boxed{\text{C}}$ (for G) if $k_a^+(G) < k_a(G) < k_a^-(G)$



$\boxed{\text{D}}$ (for G) if $k_a^-(G) < k_a(G) < k_a^+(G)$



On the right hand side we illustrated for each type how typical generators of that type of an isolated point behave in a neighborhood of that point.
We note that the order conditions for the isolated points given in Theorem 2.18 just depend on one or two of the values $k_a(G), k_a(G)^+, k_a(G)^-$. The type of the isolated point decides which of the values are needed.

We give another formulation of Theorem 2.18 which makes the case differentiation in *iii*) according to the type of the isolated point.

Corollary 2.19
Let $f \in \mathbb{R}[X]$ and $G = \{g_1, ..., g_s\} \subseteq \mathbb{R}[X]$ with $S = S(G) \subseteq \mathbb{R}$ bounded. Then $f \in Q = QM(G)$ if and only if $f|_S \geq 0$ and

i) for every left boundary point a of $S \setminus S_{isol}$ we have $ord_a(f)$ is even or $ord_a(f) - k_a^+(G) \in 2\mathbb{N}_0$

ii) for every right boundary point a of $S \setminus S_{isol}$ we have $ord_a(f)$ is even or $ord_a(f) - k_a^-(G) \in 2\mathbb{N}_0$

iii) for every isolated point a of S we have $ord_a(f)$ is even and $\epsilon_a(f) = 1$ or

Type A: $ord_a(f) \geq k_a(G)$
if $k_a(G) < k_a^+(G)$ *and* $k_a(G) < k_a^-(G)$.

Type B1: $(ord_a(f) - k_a^+(G) \in 2\mathbb{N}_0$ *and* $\epsilon_a(f) = 1)$ *or* $ord_a(f) \geq k_a^-(G)$
if $k_a^+(G) \leq k_a^-(G) < k_a(G)$.

Type B2: $(ord_a(f) - k_a^-(G) \in 2\mathbb{N}_0$ *and* $\epsilon_a(f) = -1)$ *or* $ord_a(f) \geq k_a^+(G)$
if $k_a^-(G) < k_a^+(G) < k_a(G)$.

Type C: $(ord_a(f) - k_a^+(G) \in 2\mathbb{N}_0$ *and* $\epsilon_a(f) = 1)$ *or* $ord_a(f) \geq k_a(G)$
if $k_a^+(G) < k_a(G) < k_a^-(G)$.

Type D: $(ord_a(f) - k_a^-(G) \in 2\mathbb{N}_0$ *and* $\epsilon_a(f) = -1)$ *or* $ord_a(f) \geq k_a(G)$
if $k_a^-(G) < k_a(G) < k_a^+(G)$.

Theorem 2.18 immediately implies that $QM(g_1, ..., g_s) \subseteq \mathbb{R}[X]$ is weakly semialgebraic.

Theorem 2.20
For $g_1, ..., g_s \in \mathbb{R}[X]$ the quadratic module $QM(g_1, ..., g_s)$ is weakly semialgebraic.

Proof:
As in the proof of 2.13 the non bounded case is covered by the stability theorem of Kuhlmann/Marshall or Powers/Scheiderer (Corollary 1.14) and the bounded case by the previous theorem.

Theorem 2.20 □

Completely similar to the case of one generator we can deduce the following two corollaries.

Corollary 2.21
If $g_1, ..., g_s \in \mathbb{R}[X]$ and the input data is computable then the Membership Problem is solvable affirmatively for $QM(g_1, ..., g_s)$.

Corollary 2.22
For $f(X,Y) \in \mathbb{Z}[X,Y], g_1(X,Z), ..., g_s(X,Z) \in \mathbb{Z}[X,Z]$ there is some L-formula $\varphi(Y,Z)$ such that for every real closed subfield R of $\mathbb{R}$ and any $c \in R^Y, b \in R^Z$

$$f(X,c) \in QM_{R[X]}(g_1(X,b), ..., g_s(X,b)) \Leftrightarrow R \models \varphi(c,b).$$

Now we deduce other corollaries from Theorem 2.18.

One corollary characterizes when a finitely generated quadratic module in dimension 1 with bounded associated semialgebraic set is saturated which can be found in [K-M-S].

Corollary 2.23 (Kuhlmann, Marshall, Schwartz, [K-M-S] Theorem 3.2)
Let $g_1, ..., g_s \in \mathbb{R}[X]$ and $S = S(g_1, ..., g_s) \subseteq \mathbb{R}$ bounded.
Then $QM(g_1, ..., g_s)$ is saturated if and only if

i) for every boundary point a of a $S \setminus S_{isol}$ there is some $i \in \{1, ..., s\}$ with $ord_a(g_i) = 1$.

ii) for every isolated point a of S there is a pair $(i,j) \in \{1, ..., s\} \times \{1, ..., s\}$ with $ord_a(g_i) = ord_a(g_j) = 1$ and $\epsilon_a(g_i) \neq \epsilon_a(g_j)$.

Proof:
This is clear since in any other case we can by Theorem 2.18 construct a polynomial f which is nonnegative on S but not in $QM(g_1, ...g_s)$.

Corollary 2.23 □

Another corollary is the famous Theorem of Schmüdgen for the one dimensional case since strictly positive polynomials trivially satisfy the conditions of Theorem 2.18.

Corollary 2.24 (Schmüdgen, [Sm] Corollary 3)
Let $f, g_1, ..., g_s \in \mathbb{R}[X]$ and $S = S(g_1, ..., g_s) \subseteq \mathbb{R}$ bounded.
If $f|_S > 0$ then $f \in QM(g_1, ..., g_s) = PO(g_1, ..., g_s)$.

Theorem 2.18 furthermore implies that a finitely generated quadratic module $QM(G)$ of $\mathbb{R}[X]$ with nonempty bounded semialgebraic set $S(G)$ is completely determined by a natural number $m \in \mathbb{N}$ and two vectors $\vec{\sigma} \in \mathbb{R}^{2m}$ and $\vec{\omega} \in \mathbb{N}^{2m}$.

We will explain this after we have generalized the set of natural generators which we introduced in Section 1.2.

If the nonempty compact semialgebraic set $S \subseteq \mathbb{R}$ is written as $S = \bigcup_{i=1}^{m} [a_i, b_i]$ for some $a_i, b_i \in \mathbb{R}$ with $a_i \leq b_i$ for $1 \leq i \leq m$ and $b_i < a_{i+1}$ $(1 \leq i \leq m-1)$ then the set of natural generators of $\mathcal{P}(S)$ is given as

$$\mathrm{Nat}(S) = \{(X - b_i)(X - a_{i+1}) \mid 0 \leq i \leq m\}$$

where $b_0 := -\infty, a_{m+1} := \infty, X - (-\infty) := 1$ and $X - \infty := -1$.

In Theorem 1.6 we have shown that the finite set $\mathrm{Nat}(S)$ generates the saturation $\mathcal{P}(S) = \{f \in \mathbb{R}[X] \mid f|_S \geq 0\}$ of S. We will give a generalization of this by considering preorderings whose members satisfy in addition to the nonnegativity on S order conditions as in Theorem 2.18.

For $m \in \mathbb{N}$ we define $S_{vec}(m)$ as

$$\{(a_1, b_1, ..., a_m, b_m) \in \mathbb{R}^{2m} \mid a_i \leq b_i \ (1 \leq i \leq m) \text{ and } b_i < a_{i+1} \ (1 \leq i \leq m-1)\}$$

such that an element of $S_{vec}(m)$ is nothing else but the vector of endpoints of the intervals of some compact semialgebraic subset of $\mathbb{R}$ written in increasing order where isolated points are considered as degenerated intervals.

If $\vec{\sigma} \in S_{vec}(m)$ then we denote the corresponding semialgebraic subset of $\mathbb{R}$ by $S(\vec{\sigma})$. The other way round $S = \bigcup_{i=1}^{m} [a_i, b_i] \subseteq \mathbb{R}$ decomposed in its connected components in increasing order defines $\vec{\sigma}(S) := (a_1, b_1, ..., a_m, b_m) \in S_{vec}(m)$.

For $\vec{\sigma} \in S_{vec}(m)$ a vector

$$\vec{\omega} := (l_1, r_1, ..., l_m, r_m) \in \mathbb{N}^{2m}$$

lies in $\Omega_{vec}(\vec{\sigma})$ if and only if

$$l_i \text{ and } r_i \text{ are odd if } a_i < b_i$$

and

$$\min(r_i, l_i) \text{ is odd or } r_i = l_i \text{ is even if } a_i = b_i.$$

A vector $\vec{\omega} \in \Omega_{vec}(\vec{\sigma})$ represents order conditions attached to the elements of $\vec{\sigma}$.

For $(\vec{\sigma}, \vec{\omega}) \in S_{vec}(m) \times \Omega_{vec}(\vec{\sigma})$ we define the preordering $\mathcal{P}(\vec{\sigma}, \vec{\omega})$ as follows.

Let $f \in \mathbb{R}[X]$. Then

$$
\begin{aligned}
f \in \mathcal{P}(\vec{\sigma}, \vec{\omega}) :\Leftrightarrow \quad & f|_{S(\vec{\sigma})} \geq 0 \text{ and for every } 1 \leq i \leq m \\
& \text{if } a_i < b_i : \\
& \quad (\mathrm{ord}_{a_i}(f) \text{ even or } \mathrm{ord}_{a_i}(f) \geq l_i) \text{ and} \\
& \quad (\mathrm{ord}_{b_i}(f) \text{ even or } \mathrm{ord}_{b_i}(f) \geq r_i) \\
& \text{if } a_i = b_i : \\
& \quad \mathrm{ord}_{a_i}(f) \text{ even and } \epsilon_{a_i}(f) = 1 \text{ or} \\
& \qquad \text{if } l_i = r_i \text{ even:} \\
& \qquad\quad \mathrm{ord}_{a_i}(f) \geq l_i \\
& \qquad \text{if } \min(l_i, r_i) = l_i \text{ odd:} \\
& \qquad\quad (\mathrm{ord}_{a_i}(f) - l_i \in 2\mathbb{N}_0 \text{ and } \epsilon_{a_i}(f) = 1) \text{ or } \mathrm{ord}_{a_i}(f) \geq r_i \\
& \qquad \text{if } \min(l_i, r_i) = r_i \text{ odd:} \\
& \qquad\quad (\mathrm{ord}_{a_i}(f) - r_i \in 2\mathbb{N}_0 \text{ and } \epsilon_{a_i}(f) = -1) \text{ or } \mathrm{ord}_{a_i}(f) \geq l_i
\end{aligned}
$$

Lemma 2.25
If $(\vec{\sigma}, \vec{\omega}) \in S_{vec}(m) \times \Omega_{vec}(\vec{\sigma})$ for some $m \in \mathbb{N}$ then $\mathcal{P}(\vec{\sigma}, \vec{\omega})$ is a preordering.

Proof:
Clearly $\mathbb{R}[X]^2 \subseteq \mathcal{P}(\vec{\sigma}, \vec{\omega})$ because squares are nonnegative on $\mathbb{R}$ and the order $\mathrm{ord}_a(f)$ of some square f is even and $\epsilon_a(f) = 1$ for every $a \in \mathbb{R}$.

In order to show that $\mathcal{P}(\vec{\sigma}, \vec{\omega})$ is closed under addition we restrict ourselves to the case that $f, g \in \mathcal{P}(\vec{\sigma}, \vec{\omega})$ and $\mathrm{ord}_{a_i}(f) = \mathrm{ord}_{a_i}(g)$ for some $1 \leq i \leq m$.
The considerations for b_i are similar and if the order is not equal then we have $\mathrm{ord}_{a_i}(f+g) = \min\{\mathrm{ord}_{a_i}(f), \mathrm{ord}_{a_i}(g)\}$ which easily gives that $f + g \in \mathcal{P}(\vec{\sigma}, \vec{\omega})$.
By going through the different cases from the definition of $\mathcal{P}(\vec{\sigma}, \vec{\omega})$ it is also for $\mathrm{ord}_{a_i}(f) = \mathrm{ord}_{a_i}(g)$ not hard to prove that $f + g \in \mathcal{P}(\vec{\sigma}, \vec{\omega})$.
Exemplarily we check the case $a_i = b_i$ and $\min(l_i, r_i) = l_i$ odd.
If $\mathrm{ord}_{a_i}(f) \geq r_i$ then also $\mathrm{ord}_{a_i}(f+g) \geq \mathrm{ord}_{a_i}(f) \geq r_i$.
Now suppose that $\mathrm{ord}_{a_i}(f) < r_i$.
If $\mathrm{ord}_{a_i}(f) - l_i \in 2\mathbb{N}_0$ and $\epsilon_{a_i}(f) = 1$ then the same is true for g which implies that also $\mathrm{ord}_{a_i}(f+g) - l_i \in 2\mathbb{N}_0$ and $\epsilon_{a_i}(f+g) = 1$ since no cancelation can occur.
If otherwise $\mathrm{ord}_a(f)$ is even and $\epsilon_a(f) = 1$ then also $\mathrm{ord}_a(g)$ even and $\epsilon_a(g) = 1$ which gives the same for $f + g$.

Also for the closure under multiplication one has to check that $f, g \in \mathcal{P}(\vec{\sigma}, \vec{\omega})$ implies that $fg \in \mathcal{P}(\vec{\sigma}, \vec{\omega})$ for the different possibilities of entries in $\vec{\sigma}$ and $\vec{\omega}$. Most of the cases are easy to check.
We show the closure for the case $a_i = b_i$ and $\min(l_i, r_i) = r_i$ odd.
If $\mathrm{ord}_{a_i}(f) \geq l_i$ then also $\mathrm{ord}_{a_i}(fg) = \mathrm{ord}_{a_i}(f) + \mathrm{ord}_{a_i}(g) \geq \mathrm{ord}_{a_i}(f) \geq l_i$.
Now we suppose that $\mathrm{ord}_{a_i}(f) < l_i$ and $\mathrm{ord}_{a_i}(f) - r_i \in 2\mathbb{N}_0$ as well as $\epsilon_{a_i}(f) = -1$.

If also $\mathrm{ord}_{a_i}(g) - r_i \in 2\mathbb{N}_0$ and $\epsilon_{a_i}(g) = -1$ then $\mathrm{ord}_{a_i}(fg)$ even and $\epsilon_{a_i}(fg) = 1$.
If $\mathrm{ord}_{a_i}(g)$ even and $\epsilon_{a_i}(g) = 1$ then $\mathrm{ord}_{a_i}(fg) - r_i \in 2\mathbb{N}_0$ and $\epsilon_{a_i}(fg) = -1$.
Finally if $\mathrm{ord}_{a_i}(g) \geq l_i$ then similar to above $\mathrm{ord}_{a_i}(fg) \geq l_i$.

Lemma 2.25 □

If $S = \bigcup_{i=1}^{m} [a_i, b_i]$ is equal to $S(\vec{\sigma})$ for some $\vec{\sigma} \in S_{vec}(m)$ then the saturation $\mathcal{P}(S(\vec{\sigma}))$ is equal to $\mathcal{P}(\vec{\sigma}, \vec{\omega})$ with $\vec{\omega} := (1, 1, ..., 1, 1)$.

For the definition of the set of generalized natural generators for $\mathcal{P}(\vec{\sigma}, \vec{\omega})$ we use the vector

$$\vec{\omega}_{\pm} := (\omega_1, \omega_1^+, \omega_1^-, ..., \omega_m, \omega_m^+, \omega_m^-)$$

which we call the complete vector of orders associated to $\vec{\omega}$. It is uniquely determined by $\vec{\omega}$ as follows.
For $1 \leq i \leq m$ we define

	$\omega_i :=$	$\omega_i^+ :=$	$\omega_i^- :=$	Type
if $a_i < b_i$:	∞	l_i	r_i	
if $a_i = b_i$:				
if $l_i = r_i$ even :	l_i	$l_i + 1$	$l_i + 1$	A
if l_i, r_i odd :	$\max(l_i, r_i) + 1$	l_i	r_i	B
if $l_i < r_i, l_i$ odd and r_i even :	r_i	l_i	$r_i + 1$	C
if $r_i < l_i, r_i$ odd and l_i even :	l_i	$l_i + 1$	r_i	D

In the last row we classify the isolated points in correspondence with the classification from Corollary 2.19.

We note that there is no additional information in $\vec{\omega}_{\pm}$ which is not yet in $\vec{\omega}$ and we can get back the vector $\vec{\omega}$ from $\vec{\omega}_{\pm}$.
The information necessary to define $\mathcal{P}(\vec{\sigma}, \vec{\omega})$ is given by $\vec{\sigma} \in S_{vec}(m)$ for some $m \in \mathbb{N}$ and $\vec{\omega} \in \Omega_{vec}(\vec{\sigma})$. However it is sometimes easier and clearer to work with $\vec{\omega}_{\pm}$ instead of $\vec{\omega}$.
The advantage of $\vec{\omega}_{\pm}$ is that for some isolated point a_i of $S(\vec{\sigma})$ the entry ω_i (resp. ω_i^+, resp. ω_i^-) is defined such that for some $f \in \mathbb{R}[X]$ with $\mathrm{ord}_{a_i}(f)$ even and $\epsilon_{a_i}(f) = -1$ (resp. $\mathrm{ord}_{a_i}(f)$ odd and $\epsilon_{a_i}(f) = 1$, resp. $\mathrm{ord}_{a_i}(f)$ odd and $\epsilon_{a_i}(f) = -1$) the condition for being an element of $\mathcal{P}(\vec{\sigma}, \vec{\omega})$ is given by $\mathrm{ord}_{a_i}(f) \geq \omega_i$ (resp. $\geq \omega_i^+$, resp. $\geq \omega_i^-$). This covers the conditions for all possible order behavior of f in a_i because we do not need a lower bound for the case that $\mathrm{ord}_{a_i}(f)$ is even and $\epsilon_{a_i}(f) = 1$.
In order to avoid case differentiations we will use $\vec{\omega}_{\pm}$ instead of $\vec{\omega}$ at certain points.

In the following table we state what kind of entries a vector of orders

$$\vec{\omega} = (l_1, r_1, ..., l_m, r_m) \in \Omega_{vec}(m)$$

and a complete vector of orders

$$\vec{\omega}_{\pm} = (\omega_1, \omega_1^+, \omega_1^-, ..., \omega_m, \omega_m^+, \omega_m^-)$$

associated to some vector from $\Omega_{vec}(m)$ can have.

Before we do so we note that in $\vec{\omega}_{\pm}$ the entry ω_i is even and ω_i^+, ω_i^- are odd for every $1 \leq i \leq m$.

		entry in $\vec{\omega}$	entry in $\vec{\omega}_{\pm}$
if $a_i < b_i$		l_i, r_i odd	$\omega_i = \infty$
if $a_i = b_i$	Type A	$l_i = r_i$ even	$\omega_i^+ = \omega_i^- = \omega_i + 1$
	Type B	l_i, r_i odd	$\omega_i = \max(\omega_i^+, \omega_i^-) + 1$
	Type C	$l_i < r_i, l_i$ odd, r_i even	$\omega_i^+ < \omega_i, \omega_i^- = \omega_i + 1$
	Type D	$l_i > r_i, l_i$ even, r_i odd	$\omega_i^- < \omega_i, \omega_i^+ = \omega_i + 1$

For some $(\vec{\sigma}, \vec{\omega}) \in S_{vec}(m) \times \Omega_{vec}(\sigma)$ we define now the set of generalized natural generators of $\mathcal{P}(\vec{\sigma}, \vec{\omega})$ by using the associated complete vector of orders $\vec{\omega}_{\pm}$.

$$\mathrm{Nat}(\vec{\sigma}, \vec{\omega})$$

is defined as the union of

$$\{(X - b_i)^{\omega_i^-}(X - a_{i+1})^{\omega_{i+1}^+} \mid 0 \leq i \leq m\}$$

and

$$\{(X - b_{i-1})^{\omega_{i-1}^-}(X - a_i)^{\omega_i}(X - a_{i+1})^{\omega_{i+1}^+} \mid 1 \leq i \leq m, a_i = b_i \text{ not of type B}\}$$

where $b_0 := -\infty, a_{m+1} := \infty, (X - (-\infty))^{\omega_0^-} := 1$ and $(X - \infty)^{\omega_{m+1}^+} := -1$.

The exclusion of isolated points of type B in the second set effects that $\mathrm{Nat}(S(\vec{\sigma}))$ is equal for $\mathrm{Nat}(\vec{\sigma}, \vec{\omega})$ with $\vec{\omega} = (1, 1, ..., 1, 1)$.

The finite set $\mathrm{Nat}(\vec{\sigma}, \vec{\omega})$ is in general not a minimal set of generators of $\mathcal{P}(\vec{\sigma}, \vec{\omega})$ regarding the number of elements but it is minimal what the degree of the elements concerns. We come back to the question of the minimal number of generators later on in this section.

Now we generalize the result $\mathcal{P}(S) = \mathrm{Nat}(S)$ of Theorem 1.6 and show that the set $\mathrm{Nat}(\vec{\sigma}, \vec{\omega})$ generates the preordering $\mathcal{P}(\vec{\sigma}, \vec{\omega})$.

Theorem 2.26
If $(\vec{\sigma}, \vec{\omega}) \in S_{vec}(m) \times \Omega_{vec}(\vec{\sigma})$ for some $m \in \mathbb{N}$ then

$$\mathcal{P}(\vec{\sigma}, \vec{\omega}) = PO(Nat(\vec{\sigma}, \vec{\omega})).$$

Proof:
This is an immediate consequence of Theorem 2.18 and the definition of $P(\vec{\sigma}, \vec{\omega})$.
With $G := \mathrm{Nat}(\vec{\sigma}, \vec{\omega})$ we have $S(G) = S(\vec{\sigma})$.
By definition of $\vec{\omega}_{\pm}$ and G the following is true.
For every left boundary point a_i of $S(\vec{\sigma}) \setminus S(\vec{\sigma})_{isol}$ we have

$$k^+_{a_i}(G) = \omega^+_i = l_i.$$

For every right boundary point b_i of $S(\vec{\sigma}) \setminus S(\vec{\sigma})_{isol}$ we have

$$k^-_{b_i}(G) = \omega^-_i = r_i.$$

For every isolated point a_i of $S(\vec{\sigma})$ we have

$$k^+_{a_i}(G) = \omega^+_i \text{ and } k^-_{a_i}(G) = \omega^-_i$$

and if a_i is not of type B then

$$k_{a_i}(G) = \omega_i.$$

For an isolated point a_i of type B we have

$$k_{a_i}(G) = \infty.$$

If $l_i = r_i$ even then

$$k_{a_i}(G) = l_i < l_i + 1 = k^+_{a_i}(G) \text{ and } k_{a_i}(G) = l_i < l_i + 1 = k^-_{a_i}(G).$$

If $\min(l_i, r_i) = l_i$ odd then

$$k^+_{a_i}(G) = l_i \leq r_i = \min(k^-_{a_i}(G), k_{a_i}(G)).$$

If $\min(l_i, r_i) = r_i$ odd then

$$k^-_{a_i}(G) = r_i \leq l_i = \min(k^+_{a_i}(G), k_{a_i}(G)).$$

By comparing the definition of $P(\vec{\sigma}, \vec{\omega})$ with Theorem 2.18 we see that $\mathcal{P}(\vec{\sigma}, \vec{\omega})$ is equal to $PO(\mathrm{Nat}(\vec{\sigma}, \vec{\omega}))$.

Theorem 2.26 □

Let $G = \{g_1, ..., g_s\} \subseteq \mathbb{R}[X]$ with $\emptyset \neq S(G) \subseteq \mathbb{R}$ bounded and

$$\vec{\sigma}(S(G)) = (a_1, b_1, ..., a_m, b_m) \in S_{vec}(m)$$

for some $m \in \mathbb{N}$.

Then we define the vector of orders

$$\vec{\omega}(G) = (l_1, r_1, ..., l_m, r_n) \in \Omega_{vec}(\vec{\sigma}(S(G)))$$

associated to G by

$$l_i := \begin{cases} k^+_{a_i}(G) & \text{if } a_i < b_i \text{ or } (a_i = b_i \text{ of type B or C}) \\ k_{a_i}(G) & \text{if } a_i = b_i \text{ of type A or D} \end{cases}$$

and

$$r_i := \begin{cases} k^-_{b_i}(G) & \text{if } a_i < b_i \text{ or } (a_i = b_i \text{ of type B or D}) \\ k_{b_i}(G) & \text{if } a_i = b_i \text{ of type A or C} \end{cases}$$

for $1 \leq i \leq m$.

Hence the complete vector of orders

$$\vec{\omega}_{\pm}(G) = (\omega_1, \omega_1^+, \omega_1^-, ..., \omega_m, \omega_m^+, \omega_m^-)$$

associated to G is given as follows.

For boundary points $a_i < b_i$ of $S(G) \setminus S(G)_{isol}$ we have

$$\omega_i = \infty,\ \omega_i^+ = k^+_{a_i}(G),\ \omega_i^- = k^-_{b_i}(G).$$

For isolated points $a_i = b_i$ of $S(G)$ we have

$\omega_i =$	$\omega_i^+ =$	$\omega_i^- =$	
$k_{a_i}(G)$	$k_{a_i}(G) + 1$	$k_{a_i}(G) + 1$	if a_i is of type A
$\max(k_{a_i}(G)^+, k_{a_i}(G)^-) + 1$	$k_{a_i}(G)^+$	$k_{a_i}(G)^-$	if a_i is of type B
$k_{a_i}(G)$	$k_{a_i}(G)^+$	$k_{a_i}(G) + 1$	if a_i is of type C
$k_{a_i}(G)$	$k_{a_i}(G) + 1$	$k_{a_i}(G)^-$	if a_i is of type D

With these definitions we can show that all the information needed to describe a finitely generated quadratic module of $\mathbb{R}[X]$ with nonempty bounded associated set is contained in two vectors.

Corollary 2.27
There is a bijection between

$$\{Q \subseteq \mathbb{R}[X] \mid Q = QM(G) \text{ for some finite set } G \text{ and } \emptyset \neq S(G) \text{ bounded}\}$$

and

$$\{(\vec{\sigma}, \vec{\omega}) \mid (\vec{\sigma}, \vec{\omega}) \in S_{vec}(m) \times \Omega_{vec}(\vec{\sigma}) \text{ for some } m \in \mathbb{N}\}.$$

Proof:
If $G \subseteq \mathbb{R}[X]$ is finite and $Q = QM(G)$ with $\emptyset \neq S(G)$ bounded then $\vec{\sigma}(S(G))$ is in $S_{vec}(m)$ for some $m \in \mathbb{N}$ and by definition $\vec{\omega}(G) \in \Omega_{vec}(\vec{\sigma}(S(G)))$.

If $(\vec{\sigma}, \vec{\omega}) \in S_{vec}(m) \times \Omega_{vec}(\vec{\sigma})$ for some $m \in \mathbb{N}$ then $S(\vec{\sigma})$ is nonempty and bounded. The associated finitely generated quadratic module which is in fact a preordering is given by $\mathcal{P}(\vec{\sigma}, \vec{\omega}) = PO(\mathrm{Nat}(\vec{\sigma}, \vec{\omega}))$ (Theorem 2.26). As $S(\mathrm{Nat}(\vec{\sigma}, \vec{\omega})) = S(\vec{\sigma})$ and $\mathrm{Nat}(\vec{\sigma}, \vec{\omega})$ is finite we know by Theorem 2.17 that $\mathcal{P}(\vec{\sigma}, \vec{\omega}) = QM(\mathrm{Nat}(\vec{\sigma}, \vec{\omega}))$.

The fact that these mappings are inverse to each other follows from Theorem 2.18 whose content is exactly that $\mathcal{P}(\vec{\sigma}(S(G)), \vec{\omega}(G)) = Q$ if $Q = QM(G)$ for some finite set $G \subseteq \mathbb{R}[X]$ with $\emptyset \neq S(G) \subseteq \mathbb{R}$ bounded. Furthermore we clearly have $\vec{\sigma}(S(\vec{\sigma})) = \vec{\sigma}$ and $\vec{\omega}(\mathrm{Nat}(\vec{\sigma}, \vec{\omega})) = \vec{\omega}$ if $(\vec{\sigma}, \vec{\omega}) \in S_{vec}(m) \times \Omega_{vec}(\vec{\sigma})$ for some $m \in \mathbb{N}$.

Corollary 2.27 □

This correspondence immediately implies that for every finitely generated quadratic module $Q = QM(G) \subseteq \mathbb{R}[X]$ with nonempty bounded set $S(G)$ there is a set of generalized natural generators of Q.

Corollary 2.28
Let G be a finite subset of $\mathbb{R}[X]$ such that $\emptyset \neq S(G) \subseteq \mathbb{R}$ is bounded.
Then the set of generalized natural generators $Nat(\vec{\sigma}(S(G)), \vec{\omega}(G))$ has the property that

$$QM(G) = PO(G) = PO(Nat(\vec{\sigma}(S(G)), \vec{\omega}(G))) = QM(Nat(\vec{\sigma}(S(G)), \vec{\omega}(G))).$$

Proof:
This follows with the correspondence described in the previous corollary from Theorem 2.26 and Theorem 2.17.

Corollary 2.28 □

Now we deal with the question how many generators we actually need to generate a finitely generated quadratic module $Q = QM(G)$ in $\mathbb{R}[X]$ if $S(G) \neq \emptyset$ is bounded. In some cases we just need one generator.

Corollary 2.29
Let $G \subseteq \mathbb{R}[X]$ be finite and $Q = QM(G)$ such that $S = S(G)$ is not empty and bounded.
Then the following are equivalent:

i) Q is generated by one polynomial.

ii) Every isolated point of S is of type A.

If one of the equivalent conditions is satisfied then

$$Q = QM(-\prod_{\substack{i=1\\a_i<b_i}}^{m}(X-a_i)^{k^+_{a_i}(G)}(X-b_i)^{k^-_{b_i}(G)}\prod_{\substack{i=1\\a_i=b_i}}^{m}(X-a_i)^{k_{a_i}(G)})$$

where $(a_1, b_1, ..., a_m, b_m) = \vec{\sigma}(S)$.

Proof:
If Q is generated by one polynomial then every $a \in S_{isol}$ must be of type A since for every other type at least two of the values $k_a(G), k^+_a(G)$ and $k^-_a(G)$ must be less than infinity. This cannot be realized by just one generator.

Now we suppose that every isolated point of S is of type A. Then the polynomial

$$g := -\prod_{\substack{i=1\\a_i<b_i}}^{m}(X-a_i)^{k^+_{a_i}(G)}(X-b_i)^{k^-_{b_i}(G)}\prod_{\substack{i=1\\a_i=b_i}}^{m}(X-a_i)^{k_{a_i}(G)}$$

has the same nonnegativity set as G and we have $\vec{\omega}(g) = \vec{\omega}(G)$. Thus by Corollary 2.27 $Q = \mathcal{P}(\vec{\sigma}(S(G)), \vec{\omega}(G)) = \mathcal{P}(\vec{\sigma}(S(g)), \vec{\omega}(g)) = QM(g)$.

Corollary 2.29 □

If there is at least one isolated point in $S = S(G)$ of type B, C or D then we need at least two generators for $Q = QM(G)$. For the case that S is an interval or a point Q can be generated by at most two generators which we state in the following remark.

Remark 2.30
Let $G \subseteq \mathbb{R}[X]$ be finite, $Q = QM(G)$ and $S = S(G)$.
Then

$$Q = QM(G_{min})$$

where G_{min} is given as follows.

If $S = [a, b]$ for some $a, b \in \mathbb{R}$ with $a < b$ then

$$G_{min} = \{-(X-a)^{k_a^+(G)}(X-b)^{k_b^-(G)}\}$$

If $S = \{a\}$ for some $a \in \mathbb{R}$ then we have the following cases according to the type of the isolated point a.

If a is of type A, i.e. $k_a(G) < k_a^+(G)$ and $k_a(G) < k_a^-(G)$ then

$$G_{min} = \{-(X-a)^{k_a(G)}\}.$$

If a is of type B, i.e. $k_a(G) > k_a^+(G)$ and $k_a(G) > k_a^-(G)$ then

$$G_{min} = \{(X-a)^{k_a^+(G)}, -(X-a)^{k_a^-(G)}\}.$$

If a is of type C, i.e. $k_a^+(G) < k_a(G) < k_a^-(G)$ then

$$G_{min} = \{(X-a)^{k_a^+(G)}, -(X-a)^{k_a(G)}\}.$$

If a is of type D, i.e. $k_a^-(G) < k_a(G) < k_a^+(G)$ then

$$G_{min} = \{-(X-a)^{k_a^-(G)}, -(X-a)^{k_a(G)}\}.$$

In every case the claim that $Q = QM(G_{min})$ follows directly from Theorem 2.18 or Corollary 2.19.

We note that these sets of generators which are minimal what the number of elements concerns do not always coincide with the set of generalized natural generators which is minimal with respect to the degree of its elements.

For example in the case $S = [a, b]$ we have

$$\mathrm{Nat}(\vec{\sigma}(S(G)), \vec{\omega}(G)) = \{(X-a)^{k_a^+(G)}, -(X-b)^{k_b^-(G)}\}.$$

Up to now we have seen examples where we need one or two generators for $QM(G)$. In general we need at most three generators. We show this by stating an algorithm which produces these three generators.

Before we state the algorithm we describe what the algorithm does.

Associated to some finite set $G \subseteq \mathbb{R}[X]$ with $\emptyset \neq S(G) \subseteq \mathbb{R}$ bounded we have

$$\vec{\sigma}(S(G)) = (a_1, b_1, ..., a_m, b_m) \in S_{vec}(m)$$

for some $m \in \mathbb{N}$ and the complete vector of orders

$$\vec{\omega}_{\pm}(G) = (\omega_1, \omega_1^+, \omega_1^-, ..., \omega_m, \omega_m^+, \omega_m^-).$$

These both vectors are the input data for the algorithm.

The output of the algorithm is the set of (coefficient vectors of) three polynomials $H = \{h_{neg}, h_{pos}, h_{var}\}$ which have the property that $QM(G) = QM(H)$.

The algorithm is composed of m steps.

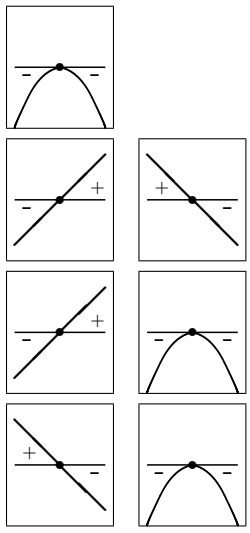
The index of the polynomials of H is chosen to indicate the sign of the polynomial on the right to the point b_i in step i of the algorithm. For every $1 \leq i \leq m$ the polynomial h_{neg} is negative and h_{pos} is positive to the right side of b_i whereas the sign of h_{var} depends on whether a_i or a_{i+1} is an isolated point of a certain type or not. The current sign of h_{var} is stored in the variable $sign_{var}$. To explain what happens in step i of the algorithm we recall that $a_i = b_i$ is an isolated point of type

$\boxed{\text{A}}$ (for G) if $\omega_i < \omega_i^+$ and $\omega_i < \omega_i^-$

$\boxed{\text{B}}$ (for G) if $\omega_i > \omega_i^+$ and $\omega_i > \omega_i^-$

$\boxed{\text{C}}$ (for G) if $\omega_i^+ < \omega_i < \omega_i^-$

$\boxed{\text{D}}$ (for G) if $\omega_i^- < \omega_i < \omega_i^+$



In step i the algorithm ensures that

$$\begin{array}{llll}
k^+_{a_i}(H) = \omega^+_i, & k^-_{b_i}(H) = \omega^-_i & & \text{if } a_i < b_i \\
k_{a_i}(H) = \omega_i, & k^+_{a_i}(H) \geq \omega^+_i, & k^-_{a_i}(H) \geq \omega^-_i & \text{if } a_i = b_i \text{ of type A} \\
k^+_{a_i}(H) = \omega^+_i, & k^-_{a_i}(H) = \omega^-_i, & k_{a_i}(H) \geq \omega_i & \text{if } a_i = b_i \text{ of type B} \\
k^+_{a_i}(H) = \omega^+_i, & k_{a_i}(H) = \omega_i, & k^-_{a_i}(H) \geq \omega^-_i & \text{if } a_i = b_i \text{ of type C} \\
k^-_{a_i}(H) = \omega^-_i, & k_{a_i}(H) = \omega_i, & k^+_{a_i}(H) \geq \omega^+_i & \text{if } a_i = b_i \text{ of type D}
\end{array}$$

In order to not change the sign behavior of h_{pos}, h_{neg} and h_{var} on the left side of a_i this will be achieved in step i by multiplying a polynomial from H with

$$\begin{array}{ll}
\text{negative sign to the left of } a_i \text{ with } -(X-a_i)^{\omega^+_i} & \text{which effects } k^+_{a_i}(H) = \omega^+_i, \\
\text{positive sign to the left of } b_i \text{ with } -(X-b_i)^{\omega^-_i} & \text{which effects } k^-_{b_i}(H) = \omega^-_i, \\
\text{negative sign to the left of } a_i \text{ with } (X-a_i)^{\omega_i} & \text{which effects } k_{a_i}(H) = \omega_i.
\end{array}$$

Now to the assignment of $sign_{var}$.
If a_i is an isolated point of type D then step i will produce two polynomials which are negative on the right side of $b_i = a_i$. If a_{i+1} is an isolated point of type C then step i has to produce two polynomials which are negative on the right side of b_i. In these two cases the assignment of $sign_{var}$ is -1.
In all other cases h_{var} we will have $sign_{var} = 1$.

If $sign_{var} = -1$ and a_i is not an isolated point of type C then h_{var} will be multiplied by $-(X-a_i)^{\omega^+_i}$ to ensure nonnegativity in a_i.

By construction we have $(\vec{\sigma}(S(H)), \vec{\omega}(H)) = (\vec{\sigma}(S(G)), \vec{\omega}(G))$ which implies by Corollary 2.28 that $QM(G) = QM(H)$.

Algorithm:

`INITIALIZE:`
$h_{neg} = -1$
$h_{pos} = 1$
$h_{var} = \begin{cases} -1 & \texttt{if } a_2 = b_2 \texttt{ of type C} \\ 1 & \texttt{else} \end{cases}$
$sign_{var} = \begin{cases} -1 & \texttt{if } a_2 = b_2 \texttt{ of type C} \\ 1 & \texttt{else} \end{cases}$

FOR $i = 1, ..., m$

IF $a_i < b_i$

$h^*_{neg} = h_{neg} \cdot (X - a_i)^{\omega_i^+} (X - b_i)^{\omega_i^-}$

$h^*_{pos} = h_{pos}$

IF $sign_{var} = -1$

$$h^*_{var} = \begin{cases} h_{var} \cdot (X - a_i)^{\omega_i^+} (X - b_i)^{\omega_i^-} & \texttt{if } a_{i+1} = b_{i+1} \texttt{ of type C} \\ h_{var} \cdot (-(X - a_i)^{\omega_i^+}) & \texttt{else} \end{cases}$$

ELSE IF $sign_{var} = 1$

$$h^*_{var} = \begin{cases} h_{var} \cdot (-(X - b_i)^{\omega_i^-}) & \texttt{if } a_{i+1} = b_{i+1} \texttt{ of type C} \\ h_{var} & \texttt{else} \end{cases}$$

ELSE IF $a_i = b_i$ of type A

$h^*_{neg} = h_{neg} \cdot (X - a_i)^{\omega_i}$

$h^*_{pos} = h_{pos}$

IF $sign_{var} = -1$

$$h^*_{var} = \begin{cases} h_{var} \cdot (X - a_i)^{\omega_i} & \texttt{if } a_{i+1} = b_{i+1} \texttt{ of type C} \\ h_{var} \cdot (-(X - a_i)^{\omega_i^+}) & \texttt{else} \end{cases}$$

ELSE IF $sign_{var} = 1$

$$h^*_{var} = \begin{cases} h_{var} \cdot (-(X - b_i)^{\omega_i^-}) & \texttt{if } a_{i+1} = b_{i+1} \texttt{ of type C} \\ h_{var} & \texttt{else} \end{cases}$$

ELSE IF $a_i = b_i$ of type B

$h^*_{neg} = h_{pos} \cdot (-(X - b_i)^{\omega_i^-})$

$h^*_{pos} = h_{neg} \cdot (-(X - a_i)^{\omega_i^+})$

IF $sign_{var} = -1$

$$h^*_{var} = \begin{cases} h_{var} \cdot (X - a_i)^{\omega_i} & \texttt{if } a_{i+1} = b_{i+1} \texttt{ of type C} \\ h_{var} \cdot (-(X - a_i)^{\omega_i^+}) & \texttt{else} \end{cases}$$

ELSE IF $sign_{var} = 1$

$$h^*_{var} = \begin{cases} h_{var} \cdot (-(X - b_i)^{\omega_i^-}) & \texttt{if } a_{i+1} = b_{i+1} \texttt{ of type C} \\ h_{var} & \texttt{else} \end{cases}$$

ELSE IF $a_i = b_i$ of type C

$h^*_{neg} = h_{neg} \cdot (X - a_i)^{\omega_i}$

$h^*_{pos} = h_{var} \cdot (-(X - a_i)^{\omega_i^+})$

$$h^*_{var} = \begin{cases} h_{pos} \cdot (-(X - b_i)^{\omega_i^-}) & \texttt{if } a_{i+1} = b_{i+1} \texttt{ of type C} \\ h_{pos} & \texttt{else} \end{cases}$$

ELSE IF $a_i = b_i$ of type D

$h^*_{neg} = h_{neg} \cdot (X - a_i)^{\omega_i}$

$$h^*_{pos} = \begin{cases} h_{var} \cdot (-(X - a_i)^{\omega_i^+}) & \texttt{if } sign_{var} = -1 \\ h_{var} & \texttt{else} \end{cases}$$

$h^*_{var} = h_{pos} \cdot (-(X - b_i)^{\omega_i^-})$

$$sign_{var} = \begin{cases} -1 & \texttt{if } a_{i+1} = b_{i+1} \texttt{ of type C or } a_i = b_i \texttt{ of type D} \\ 1 & \texttt{else} \end{cases}$$

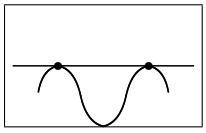
$i = i + 1, h_{neg} = h^*_{neg}, h_{pos} = h^*_{pos}, h_{var} = h^*_{var}$

DO;

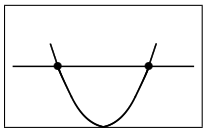
We include an example of a finitely generated quadratic module which needs the maximal number of three generators.

Example 2.31
Let $G := \{g_1, g_2, g_3\} \subseteq \mathbb{R}[X]$ *with* $g_1 := -X^3(X-1)^8$, $g_2 := -X^4(X-1)^6$ *and* $g_3 := -X^7(X-1)^5(X^2+1)$.
Then $S(G) = \{0, 1\}$, *i.e.* $\vec{\sigma}(S(G)) = (0, 0, 1, 1)$.
As $k_0(G) = 4, k_0^+(G) = 7, k_0^-(G) = 3, k_1(G) = 6, k_1^-(G) = 5$ *and* $k_1^+(G) = \infty$ *we have* $\vec{\omega}(G) = (4, 3, 6, 5)$ *and* $\vec{\omega}_\pm(G) = (4, 5, 3, 6, 7, 5)$. *Thus both isolated points are of type D.*
The three generators produced by the algorithm are

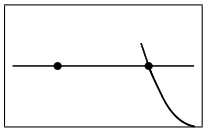
$h_{neg} = -X^4(X-1)^6$



$h_{pos} = X^3(X-1)^3$



$h_{var} = -(X-1)^5$



where the schematically behavior on the right hand side illustrates that all three generators are needed.

By representing a finitely generated quadratic module $Q = QM(G) \subseteq \mathbb{R}[X]$ with nonempty bounded set $S = S(G) \subseteq \mathbb{R}$ as $Q = \mathcal{P}(\vec{\sigma}(S), \vec{\omega}(G))$ (Corollary 2.27) we see that

$$Q = \bigcap_{i=1}^{m} \mathcal{P}(\vec{\sigma}(S)_i, \vec{\omega}(G)_i)$$

where the vectors $\vec{\sigma}(S)_i$ correspond to the connected components of S and the vectors $\vec{\omega}(G)_i$ are obtained by subdividing $\vec{\omega}(G)$ according to the subdivision of $\vec{\sigma}(S)$ by the $\vec{\sigma}(S)_i$ $(1 \leq i \leq m)$.
Since the connected components of S are intervals or points we know that every finitely generated quadratic module $QM(G)$ of $\mathbb{R}[X]$ with bounded $S(G) \neq \emptyset$ is the finite intersection of quadratic modules generated by one or two polynomials (Remark 2.30).

In general the intersection of two finitely generated quadratic modules does not have to be finitely generated any more. However if the associated semialgebraic set is a nonempty bounded subset of $\mathbb{R}$ then it is true.

Proposition 2.32
Let $G_1, G_2 \subseteq \mathbb{R}[X]$ be finite and $Q_i = QM(G_i)$ with nonempty bounded set $S(G_i)$ for $i = 1, 2$. Then $Q_1 \cap Q_2$ is again finitely generated.

Proof:
By decomposing $S(G_i)$ into its connected components we can without loss of generality suppose that $S_i := S(G_i) = [a_i, b_i]$ for some $a_i \leq b_i \in \mathbb{R}$ $(i = 1, 2)$.

Corollary 2.27 gives that $Q := Q_1 \cap Q_2 = \mathcal{P}(\vec{\sigma}(S_1), \vec{\omega}(G_1)) \cap \mathcal{P}(\vec{\sigma}(S_2), \vec{\omega}(G_2))$.

Let $S(Q) := \{x \in \mathbb{R} \mid g(x) \geq 0 \ \forall g \in Q\}$.

The quadratic module Q is by Theorem 2.26 finitely generated if the simultaneous fulfillment of the order conditions from $\vec{\omega}(G_1)$ and $\vec{\omega}(G_2)$ can be expressed by some order condition $\vec{\omega}(Q)$ such that $(\vec{\sigma}(S(Q)), \vec{\omega}(Q)) \in S_{vec}(m) \times \Omega_{vec}(\vec{\sigma}(S(Q)))$ for some $m \in \mathbb{N}$.

Clearly $S(Q) = S_1 \cup S_2$.

If $S_1 \cap S_2 = \emptyset$ we can simply join the two vectors $(\vec{\sigma}(S_1), \vec{\omega}(G_1))$ and $(\vec{\sigma}(S_2), \vec{\omega}(G_2))$ and get the vector $(\vec{\sigma}(Q), \vec{\omega}(Q)) \in S_{vec}(2) \times \Omega_{vec}(\vec{\sigma}(S(Q)))$.

Now we suppose that $S_1 \cap S_2 \neq \emptyset$.
Without loss of generality let $S_1 \leq S_2$.

In the following case differentiation we state for each case the vector of order conditions which describes $Q_1 \cap Q_2$ where we work now with $\vec{\omega}_\pm$ instead of $\vec{\omega}$.
Let $\vec{\omega}_\pm(G_i) := (\omega_i, \omega_i^+, \omega_i^-)$ for $i = 1, 2$.

Case 1: $a_1 < b_1$ and $a_2 < b_2$

Case 1.1: $a_1 < a_2 = b_1 < b_2$
Then $\vec{\sigma}(S(Q)) = (a_1, b_2)$ and $\vec{\omega}_\pm(Q) = (\infty, \omega_1^+, \omega_2^-)$.
In $a = a_2 = b_1$ we have for some $f \in Q_1 \cap Q_2$ that $\mathrm{ord}_a(f)$ even (and $\epsilon_a(f) = 1$) or $\mathrm{ord}_a(f) \geq k_1^-$ and $\epsilon_a(f) = -1$ because $f \in Q_1$. On the other hand $\mathrm{ord}_a(f)$ even (and $\epsilon_a(f) = 1$) or $\mathrm{ord}_a(f) \geq k_2^+$ and $\epsilon_a(f) = 1$ because $f \in Q_2$. Thus the order condition for $Q_1 \cap Q_2$ in a is given by $\mathrm{ord}_a(f)$ even (and $\epsilon_a(f) = 1$). This follows already from the fact that every $f \in Q$ is nonnegative on $S_1 \cup S_2$.

Case 1.2: $a_1 < a_2 < b_1 < b_2$
Then $\vec{\sigma}(S(Q)) = (a_1, b_2)$ and $\vec{\omega}_\pm(Q) = (\infty, \omega_1^+, \omega_2^-)$.
In $a = a_2$ we have for some $f \in Q_1 \cap Q_2$ that $\mathrm{ord}_a(f)$ even and $\epsilon_a(f) = 1$ because $f \in Q_1$ and $a_2 \in \mathrm{int}(S_1)$. On the other hand $\mathrm{ord}_a(f)$ is even (and $\epsilon_a(f) = 1$) or $\mathrm{ord}_a(f) \geq k_2^+$ and $\epsilon_a(f) = 1$ because $f \in Q_2$. Thus the order condition for $Q_1 \cap Q_2$ in a is given by $\mathrm{ord}_a(f)$ even (and $\epsilon_a(f) = 1$). This follows already from the fact that every $f \in Q$ is

nonnegative on $S_1 \cup S_2$. The same argument shows that we need no entry for b_1 in $\vec{\omega}_\pm(Q)$.

Case 1.3: $a_1 < a_2 < b_1 = b_2$
Then $\vec{\sigma}(S(Q)) = (a_1, b_1)$ and $\vec{\omega}_\pm(Q) = (\infty, \omega_1^+, \max(\omega_1^-, \omega_2^-))$.
The reason why ω_2^+ does not appear in $\vec{\omega}_\pm(Q)$ is similar to case 1.2.

Case 1.4: $a_1 = a_2 < b_1 = b_2$
Then $\vec{\sigma}(S(Q)) = (a_1, b_1)$ and $\vec{\omega}_\pm(Q) = (\infty, \max(\omega_1^+, \omega_2^+), \max(\omega_1^-, \omega_2^-))$.

Case 2: $a_1 < b_1$ and $a_2 = b_2$

Case 2.1: $a_1 < b_1 = a_2$
Then $\vec{\sigma}(S(Q)) = (a_1, b_1)$ and $\vec{\omega}_\pm(Q) = (\infty, \omega_1^+, \max(\omega_1^-, \omega_2^-))$.

Case 2.2: $a_1 < a_2 < b_1$
Then $\vec{\sigma}(S(Q)) = (a_1, b_1)$ and $\vec{\omega}_\pm(Q) = (\infty, \omega_1^+, \omega_1^-)$.
In $a = a_2$ we have for some $f \in Q_2$ that $\mathrm{ord}_a(f)$ even and $\epsilon_a(f) = 1$ or an order condition depending on the type of the isolated point a. Whatever this order condition is the fact that a lies in the interior of S_1 implies that $\mathrm{ord}_a(f)$ even and $\epsilon_a(f) = 1$ for some $f \in Q_1 \cap Q_2$. This is automatically fulfilled as every $f \in Q$ is nonnegative on $S_1 \cup S_2$. Hence there is no entry corresponding to a_2 in $\vec{\omega}_\pm(Q)$.

Case 2.3: $a_1 = a_2 < b_1$
Then $\vec{\sigma}(S(Q)) = (a_1, b_1)$ and $\vec{\omega}_\pm(Q) = (\infty, \max(\omega_1^+, \omega_2^+), \omega_1^-)$.

Case 3: $a_1 = b_1$ and $a_2 < b_2$
Completely similar to case 2.

Case 4: $a_1 = b_1 = a_2 = b_2$
Then $\vec{\sigma}(S(Q)) = (a_1, b_1)$ and for the complete vector of orders we have $\vec{\omega}_\pm(Q) = (\max(\omega_1, \omega_2), \max(\omega_1^+, \omega_2^+), \max(\omega_1^-, \omega_2^-))$.

The only case where it is not immediately clear that the obtained vector $\vec{\omega}_\pm(Q)$ is the complete vector of orders associated to some $\vec{\omega} \in \Omega_{vec}(\vec{\sigma}(S(Q)))$ is case 4.

We abbreviate $\omega := \max(\omega_1, \omega_2), \omega^+ := \max(\omega_1^+, \omega_2^+)$ and $\omega^- := \max(\omega_1^-, \omega_2^-)$.

If $\omega < \omega^+$ and $\omega < \omega^-$ then we have to show that $\omega^+ = \omega^- = \omega + 1$.
Without loss of generality we suppose that $\omega^+ = \omega_1^+$. Then $\omega_1 < \omega_1^+$ which implies that $\omega_1^+ = \omega_1 + 1$ since $\vec{\omega}_\pm(G_1)$ is a complete vector of orders. Thus $\omega = \omega_1$ and $\omega^+ = \omega + 1$. Similarly we get $\omega^- = \omega + 1$.

If $\omega^+ < \omega$ and $\omega^- < \omega$ then we have to show that $\omega = \max(\omega^+, \omega^-) + 1$.
Without loss of generality we suppose that $\omega = \omega_1$. Then $\omega_1^+ < \omega_1$ and $\omega_1^- < \omega_1$

which gives $\omega_1 = \max(\omega_1^+, \omega_1^-) + 1$ because $\vec{\omega}_\pm(G_1)$ is a complete vector of orders. Since $\omega_1^+ \leq \omega^+$ and $\omega_1^- \leq \omega^-$ we have $\max(\omega_1^+, \omega_1^-) \leq \max(\omega^+, \omega^-) < \omega = \omega_1$. Thus $\omega = \max(\omega^+, \omega^-) + 1$.

If $\omega^+ < \omega < \omega^-$ then we have to show that $\omega^- = \omega + 1$.
Without loss of generality we suppose that $\omega^- = \omega_1^-$. Then $\omega_1^+ < \omega_1 < \omega_1^-$ and thus $\omega_1^- = \omega_1 + 1$ because $\vec{\omega}_\pm(G_1)$ is a complete vector of orders. Thus $\omega = \omega_1$ and hence $\omega^- = \omega + 1$.

The case that $\omega^- < \omega < \omega^+$ implies $\omega^+ = \omega + 1$ has a similar reasoning.

With the vector $\vec{\omega}(Q)$ corresponding to $\vec{\omega}_\pm(Q)$ we have $Q = \mathcal{P}(\vec{\sigma}(S(Q)), \vec{\omega}(Q))$ where $(\vec{\sigma}(S(Q)), \vec{\omega}(Q)) \in S_{vec}(m) \times \Omega_{vec}(\vec{\sigma}(S(Q)))$ with $m = 1$ or $m = 2$. Theorem 2.26 now implies that $Q_1 \cap Q_2$ is finitely generated.

Prop. 2.32 □

Another result about being finitely generated in this context is the following.

Proposition 2.33
Let $Q \subseteq \mathbb{R}[X]$ be a quadratic module. If $S(Q) = \{x \in \mathbb{R} \mid g(x) \geq 0 \ \forall g \in Q\} \subseteq \mathbb{R}$ is a bounded semialgebraic set and there is a finitely generated quadratic module $\widetilde{Q} \subseteq Q$ with $S(\widetilde{Q}) = S(Q)$ then Q is finitely generated.

Proof:
Let $G \subseteq \mathbb{R}[X]$ be the finite set which generates $\widetilde{Q}$. Similar to the definition of $k_a(G), k_a^+(G)$ and $k_a^-(G)$ we define for every boundary point of $S := S(Q) = S(G)$ the values

$$k_a^+(Q) := \min\{\mathrm{ord}_a(q) \mid q \in Q,\ \mathrm{ord}_a(q) \text{ odd}, \epsilon_a(q) = 1\},$$

$$k_a^-(Q) := \min\{\mathrm{ord}_a(q) \mid q \in Q,\ \mathrm{ord}_a(q) \text{ odd}, \epsilon_a(q) = -1\}$$

and

$$k_a(Q) := \min\{\mathrm{ord}_a(q) \mid q \in Q,\ \mathrm{ord}_a(q) \text{ even}, \epsilon_a(q) = -1\}.$$

In any of the three cases we again define $k_a^+(Q), k_a^-(Q)$ and $k_a(Q)$ to be ∞ if the corresponding set is empty.

Because of $\widetilde{Q} \subseteq Q$ we have

$$k_a(Q) \leq k_a(G), k_a^+(Q) \leq k_a^+(G) \text{ and } k_a^-(Q) \leq k_a^-(G) \qquad (*)$$

for every boundary point of S. This implies that for some left (resp. right) boundary point a of $S \setminus S_{isol}$ we have $k_a^+(Q) < \infty$ (resp. $k_a^-(Q) < \infty$) whereas the other both values are ∞ because elements of Q are nonnegative on S. For isolated points a of

S we can conclude that at least one of the values $k_a(Q), k_a^+(Q)$ and $k_a^-(Q)$ is less than ∞. Now we can in complete analogy to $\vec{\omega}(G)$ define a vector of orders $\vec{\omega}(Q)$ corresponding to Q such that we have

$$Q \subseteq \mathcal{P}(\vec{\sigma}(S), \vec{\omega}(Q)).$$

By definition of the values $k_a(Q), k_a^+(Q)$ and $k_a^-(Q)$ and because of $(*)$ there is for every boundary point a of S some element of Q which has the order condition given by the element of $\vec{\omega}(Q)$. Let $\{q_1, ..., q_r\} \subseteq Q$ be the set of those polynomials. Then we have by Corollary 2.27

$$Q \subseteq \mathcal{P}(\vec{\sigma}(S), \vec{\omega}(Q)) = QM(q_1, ..., q_r) \subseteq Q$$

which proves the claim.

Prop. 2.33 □

2.2 Solution in the case of finite associated semialgebraic sets

The reason why we can solve the Membership Problem affirmatively over arbitrary real closed fields if the semialgebraic set S associated to the quadratic module is finite is that the local-global principle is in this case true not just over $\mathbb{R}$ but over arbitrary real closed fields. Instead of using the Basic Lemma and the Kadison-Dubois Theorem which forced us to work over $\mathbb{R}$ in Theorem 2.9 we now apply the abstract Stellensatz for quadratic modules.

Theorem 2.34
Let $f, g_1, ..., g_s \in R[X]$ and $Q = QM(g_1, ..., g_s)$ with $S = S(g_1, ..., g_s) \subseteq R$ finite. If $\widehat{f}_a \in \widehat{Q}_a$ for every $a \in Z(f) \cap S$ and $f|_S \geq 0$ then $f \in Q$.

Proof:
An argument similar to the one used in the proof of Lemma 2.6 shows that $S = \emptyset$ gives $Q = R[X]$ such that $f \in Q$ is trivially true.

Now we suppose that $\emptyset \neq S = \{a_1, ..., a_m\}$ for some $a_i \in R$ $(1 \leq i \leq m)$.
Then the elements $\alpha \in \mathrm{Sper}R[X]$ with $Q \subseteq \alpha$ are the orderings α_{a_i} $(1 \leq i \leq m)$ which correspond to evaluation in a_i.
We consider the polynomial $p := \prod_{i=1}^{m}(X - a_i)$ which is clearly in the support of α_{a_i} for every $1 \in \{1, ..., m\}$. By Proposition 0.3 this means that $p \in \mathrm{supp}(\beta)$ for every

semiordering $\beta \in \overline{H}_{Semi}(Q)$. Thus the abstract Stellensatz for quadratic modules (Theorem 0.4 iii)) gives some $N \in \mathbb{N}$ such that

$$-p^{2N} = -\prod_{i=1}^{m}(X-a_i)^{2N} \in Q.$$

Now let $f \in R[X]$ with $f|_S \geq 0$, $\widehat{f}_a \in \widehat{Q}_a$ for every zero a of f in S and $i \in \{1,..,m\}$ arbitrary but fixed.
The task is to find a polynomial $\overline{f}_i \in Q$ with $\overline{f}_i \equiv f \mod (X-a_i)^{2N}R[X]$.

Case 1: $f(a_i) = 0$

By assumption we have with $g_0 := 1$ a representation $\widehat{f}_{a_i} = \sum_{l=0}^{s} h_l^2 \widehat{(g_l)}_a$ for some $h_l \in R[[X-a_i]]$. If $h_l = \sum_{j=0}^{\infty} c_j^{(l)}(X-a_i)^j \in R[[X-a_i]]$ we define the polynomial $\overline{h}_l := \sum_{j=0}^{2N} c_j^{(l)}(X-a_i)^j \in R[X]$.

Then $\overline{f}_i := \sum_{l=0}^{s} \overline{h}_l^2 g_l \in Q \subseteq R[X]$ and $f \equiv \overline{f}_i \mod (X-a_i)^{2N}R[X]$.

Case 2: $f(a_i) > 0$

Then $\widehat{f}_{a_i}$ is a square in $R[[X-a_i]]$, i.e. there is some $h_i \in R[[X-a_i]]$ such that $\widehat{f}_{a_i} = h_i^2$. Again by truncating we get an element $\overline{h}_i \in R[X]$ such that $f \equiv \overline{f}_i \mod (X-a_i)^{2N}R[X]$ with $\overline{f}_i := \overline{h}_i^2 \in Q$.

By the Chinese Remainder Theorem we can now find for every $1 \leq i \leq m$ a polynomial $q_i \in R[X]$ with $h_i \equiv \delta_{ij} \mod (X-a_j)^{2N}$ where δ_{ij} denotes the Kronecker symbol $(1 \leq j \leq m)$.
We put these ingredients together and define $q := \sum_{i=1}^{m} q_i^2 \overline{f}_i \in Q$.
Since for every $i \in \{1,...,m\}$ we have $f \equiv \overline{f}_i \equiv q_i^2 \overline{f}_i \equiv q \mod (X-a_i)^{2N}R[X]$ we can conclude that $f \equiv q \mod (\prod_{i=1}^{m}(X-a_i)^{2N})R[X]$.
Thus there is some $v \in R[X]$ such that $f = q + v\prod_{i=1}^{m}(X-a_i)^{2N}$ which can be written as $f = q + (\frac{v+1}{2})^2 \prod_{i=1}^{m}(X-a_i)^{2N} + (\frac{v-1}{2})^2(-\prod_{i=1}^{m}(X-a_i)^{2N})$. This is an element from the quadratic module Q because $-\prod_{i=1}^{m}(X-a_i)^{2N} \in Q$ and $q \in Q$.

Theorem 2.34 □

Since the description of the structure of the quadratic modules in the local power series rings in Section 2.1 is given for arbitrary real closed fields this local-global principle gives us by looking at the proof of Theorem 2.10 or 2.18 the following characterization of membership in $QM(g_1, ..., g_s) \subseteq R[X]$ if $S(g_1, ..., g_s)$ is finite.

Theorem 2.35
Let $f \in R[X]$ and $G = \{g_1, ..., g_s\} \subseteq R[X]$ with $S(g_1, ..., g_s) = \{a_1, ..., a_m\} \subseteq R$. Then $f \in QM(g_1, ..., g_s)$ if and only if $f(a_i) \geq 0$ $(1 \leq i \leq m)$ and for every $i \in \{1, ..., m\}$ we have $ord_{a_i}(f)$ even and $\epsilon_{a_i}(f) = 1$ or

Case 1: $ord_{a_i}(f) \geq k_{a_i}(G)$ if $k_{a_i}(G) < k^+_{a_i}(G)$ and $k_{a_i}(G) < k^-_{a_i}(G)$.

Case 2: $(ord_{a_i}(f) - k^+_{a_i}(G) \in 2\mathbb{N}_0$ and $\epsilon_{a_i}(f) = 1)$ or $ord_{a_i}(f) \geq \min(k_{a_i}(G), k^-_{a_i}(G))$ if $k^+_{a_i}(G) \leq \min(k_{a_i}(G), k^-_{a_i}(G))$.

Case 3: $(ord_{a_i}(f) - k^-_{a_i}(G) \in 2\mathbb{N}_0$ and $\epsilon_{a_i}(f) = -1)$ or $ord_{a_i}(f) \geq \min(k_{a_i}(G), k^+_{a_i}(G))$ if $k^-_{a_i}(G) \leq \min(k_{a_i}(G), k^+_{a_i}(G))$.

Proof:
In analogy to the proof of Theorem 2.18 just using Theorem 2.34 instead of 2.9.

Theorem 2.35 □

Similar to Corollary 2.19 we give another formulation of the theorem where we do distinguish the different types of isolated points.

Corollary 2.36
Let $f \in R[X]$ and $G = \{g_1, ..., g_s\} \subseteq R[X]$ with $S(g_1, ..., g_s) = \{a_1, ..., a_m\} \subseteq R$. Then $f \in QM(g_1, ..., g_s)$ if and only if $f(a_i) \geq 0$ $(1 \leq i \leq m)$ and for every $i \in \{1, ..., m\}$ we have $ord_{a_i}(f)$ even and $\epsilon_{a_i}(f) = 1$ or

Type A: $ord_{a_i}(f) \geq k_{a_i}(G)$ if $k_{a_i}(G) < k^+_{a_i}(G)$ and $k_{a_i}(G) < k^-_{a_i}(G)$.

Type B1: $(ord_{a_i}(f) - k^+_{a_i}(G) \in 2\mathbb{N}_0$ and $\epsilon_{a_i}(f) = 1)$ or $ord_{a_i}(f) \geq k^-_{a_i}(G)$ if $k^+_{a_i}(G) \leq k^-_{a_i}(G) < k_{a_i}(G)$.

Type B2: $(ord_{a_i}(f) - k^-_{a_i}(G) \in 2\mathbb{N}_0$ and $\epsilon_{a_i}(f) = -1)$ or $ord_{a_i}(f) \geq k^+_{a_i}(G)$ if $k^-_{a_i}(G) < k^+_{a_i}(G) < k_{a_i}(G)$.

Type C: $(ord_{a_i}(f) - k^+_{a_i}(G) \in 2\mathbb{N}_0$ and $\epsilon_{a_i}(f) = 1)$ or $ord_{a_i}(f) \geq k_{a_i}(G)$ if $k^+_{a_i}(G) < k_{a_i}(G) < k^-_{a_i}(G)$.

Type D: $(ord_{a_i}(f) - k^-_{a_i}(G) \in 2\mathbb{N}_0$ and $\epsilon_{a_i}(f) = -1)$ or $ord_{a_i}(f) \geq k_{a_i}(G)$ if $k^-_{a_i}(G) < k_{a_i}(G) < k^+_{a_i}(G)$.

The fact that we now have a characterization over arbitrary real closed fields does in fact imply stability of this kind of quadratic modules. We come back to this in the chapter about heirs.

Theorem 2.37
For $g_1, ..., g_s \in R[X]$ with $S(g_1, ..., g_s) \subseteq R$ finite the quadratic module $QM(g_1, ..., g_s)$ is weakly semialgebraic.

Proof:
This is clear by the previous theorem.

Theorem 2.37 □

As in Section 2.1 this gives the following two corollaries.

Corollary 2.38
If $g_1, ..., g_s \in R[X]$ and the input data is computable then the Membership Problem is solvable affirmatively for $QM(g_1, ..., g_s)$ if $S(g_1, ..., g_s)$ is finite.

Corollary 2.39
For $f(X, Y) \in \mathbb{Z}[X, Y], g_1(X, Z), ..., g_s(X, Z) \in \mathbb{Z}[X, Z]$ there are L-formulas $\psi(Z)$ and $\varphi(Y, Z)$ such that we have for every real closed field R and any $c \in R^Y, b \in R^Z$: If

$$R \models \psi(b)$$

then

$$f(X, c) \in QM(g_1(X, b), ..., g_s(X, b)) \Leftrightarrow R \models \varphi(c, b).$$

Proof:
The formula

$$\psi(Z) := \forall X \left(\bigwedge_{i=1}^{s} g_i(X, Z) \geq 0 \rightarrow \bigvee_{i=1}^{s} g_i(X, Z) = 0 \right)$$

expresses the finiteness of the basic closed set $S(g_1, ..., g_s)$ and the formula $\varphi(Y, Z)$ can be obtained with the considerations from Remark 2.12.

Corollary 2.39 □

Theorem 2.35 together with Lemma 2.6 imply that for $f, g_1, ..., g_s \in R[X]$ the membership of f in the quadratic module generated by $g_1, ..., g_s, -f^2$ is given by finitely many local conditions which can be expressed by a semialgebraic formula as explained in Section 2.1.

Corollary 2.40
Let $f, g_1, ..., g_s \in R[X]$.
Then we have with $Q = QM(g_1, ..., g_s)$ and $S = S(g_1, ..., g_s)$

$$\widehat{f}_a \in \widehat{Q}_a \ \forall a \in Z(f) \cap S \Leftrightarrow f \in Q + f^2 R[X] = QM(g_1, ..., g_s, -f^2)$$

Proof:
The implication $\Rightarrow$ is just Lemma 2.6.
For the other inclusion let $\widetilde{Q} := QM(g_1, ..., g_s, -f^2) = Q + f^2 R[X]$.
Then $S(\widetilde{Q}) = Z(f) \cap S$ which is empty or finite.

If $S(\widetilde{Q})$ is empty then $\overline{H}(g_1, ...g_s, -f^2) = \emptyset$ and thus by Proposition 0.3 we also have $\emptyset = \overline{H}_{Semi}(g_1, ...g_s, -f^2) = \overline{H}_{semi}(\widetilde{Q})$ which implies by the abstract Stellensatz for quadratic modules (Theorem 0.4 iv)) that $-1 \in \widetilde{Q}$ and thus $\widetilde{Q} = R[X]$. Hence the assumption as well as the conclusion is in this case true for every $f \in R[X]$.

Now we suppose that $S(\widetilde{Q}) \neq \emptyset$. As $f \in \widetilde{Q}$ we know by Theorem 2.35 that f fulfills for every $a \in S(\widetilde{Q}) = Z(f) \cap S$ the order conditions determined by the values $k_a(\widetilde{G}), k_a^+(\widetilde{G})$ and $k_a^-(\widetilde{G})$ where $\widetilde{G} := \{g_1, ..., g_s, -f^2\}$. Since $\mathrm{ord}_a(-f^2) > \mathrm{ord}_a(f)$ for every zero of f this means that f also satisfies the order conditions from Theorem 2.35 for every $a \in Z(f) \cap S$ with respect to the values $k_a(G), k_a^+(G)$ and $k_a^-(G)$ where $G := \{g_1, ..., g_s\}$. This means nothing else than $\widehat{f}_a \in \widehat{Q}_a \ \forall a \in Z(f) \cap S$.

Corollary 2.40 □

As in Section 2.1 we transfer with the help of the local-global principle the multiplicative closure of the quadratic modules in the formal power series ring to the ring of polynomials.

Theorem 2.41
Let $g_1, ..., g_s \in R[X]$ with $S(g_1, ..., g_s) \subseteq R$ finite.
Then the quadratic module $Q = QM(g_1, ..., g_s)$ is closed under multiplication and thus $Q = PO(g_1, ..., g_s)$.

Proof:
We use exactly the same argument as in the proof of Theorem 2.17 with Theorem 2.9 replaced by Theorem 2.34.

Theorem 2.41 □

Now we describe the support of finitely generated quadratic modules $Q = QM(G)$ of $R[X]$ with finite $S = S(G)$. In general it is hard to determine the support $Q \cap -Q$ of a quadratic module Q. With the help of our explicit characterization of membership

we can do better. Actually what the support concerns the case where S consists of finitely many points is the interesting one. Since in the other case $\text{int}(S) \neq \emptyset$ which forces the support of Q to be $\{0\}$.

Corollary 2.42
Let $G = \{g_1, ..., g_s\} \subseteq R[X]$ and $\emptyset \neq S(G) = \{a_1, ..., a_m\} \subseteq R$. Then the support of the quadratic module $Q = QM(G)$ is given by

$$supp(Q) = \left(\prod_{i=1}^{m} (X - a_i)^{k_i} \right) R[X]$$

where for every $1 \leq i \leq m$

$$k_i = \begin{cases} k_{a_i}(G) & \text{if } k_{a_i}(G) < k^+_{a_i}(G), k^-_{a_i}(G) \\ \min(k_{a_i}(G), k^+_{a_i}(G)) & \text{if } k^-_{a_i}(G) \leq \min(k_{a_i}(G), k^+_{a_i}(G)) \\ \min(k_{a_i}(G), k^-_{a_i}(G)) & \text{if } k^+_{a_i}(G) \leq \min(k_{a_i}(G), k^-_{a_i}(G)) \end{cases}$$

Proof:
The condition that f and $-f$ has to be in Q implies that f has to fulfill the order conditions from Theorem 2.35 which do not depend on $\epsilon_{a_i}(f)$ in every a_i $(1 \leq i \leq m)$. Thus $\prod_{i=1}^{m} (X - a_i)^{k_i}$ divides every $f \in \text{supp}(Q)$ where the exponents k_i are defined as in the statement of the Corollary. This gives $\subseteq$.
The other inclusion is true because $\prod_{i=1}^{m} (X - a_i)^{k_i} \in \text{supp}(Q)$ by Theorem 2.35.

Corollary 2.42 □

With the help of this characterization of the support we can derive the following.

Corollary 2.43
Let $Q = QM(g_1, ..., g_s) \subseteq R[X]$ and $S = S(g_1, ..., g_s) \neq \emptyset$. Then

$$supp(Q) = \{0\} \Leftrightarrow int(S) \neq \emptyset.$$

Proof:
If the interior of S is not empty then every polynomial $f \in \text{supp}(Q)$ has to be zero because it must fulfill $f|_S = 0$.
For the other implication we use the preceding corollary. Suppose that the interior of S is empty. Then S is of the form $S = \{a_1, ..., a_m\}$ for some $a_i \in R$ $(1 \leq i \leq m)$. By Corollary 2.42 there is some polynomial f which is not identically zero and belongs to the support of Q.

Corollary 2.43 □

Theorem 2.35 and Corollary 2.42 also allow us to describe the finitely generated quadratic modules $Q \subseteq R[X]$ with finite associated semialgebraic set in the form $Q = Q' + \mathrm{supp}(Q)$ where Q' is a definable subset of a finite dimensional vector space $R[X]_{\leq D}$ for some $D \in \mathbb{N}$.

Corollary 2.44
Let $G = \{g_1, ..., g_s\} \subseteq R[X]$ and $\emptyset \neq S(G) = \{a_1, ..., a_m\} \subseteq R$.
If $supp(Q) = \left(\prod_{i=1}^{m}(X - a_i)^{k_i}\right) R[X]$ for some $k_1, ..., k_m \in 2\mathbb{N}$ is the support of the quadratic module $Q := QM(G)$ and $D := k_1 + ... + k_m - 1$ then we have

$$Q = Q' + supp(Q)$$

with $Q' \subseteq R[X]_{\leq D}$ characterized by $p \in Q'$ if and only if for every $1 \leq i \leq m$ $ord_{a_i}(p)$ even and $\epsilon_{a_i}(p) = 1$ or

Case 1: $ord_{a_i}(p) \geq k_{a_i}(G)$ if $k_{a_i}(G) < k^+_{a_i}(G)$ and $k_{a_i}(G) < k^-_{a_i}(G)$.

Case 2: $(ord_{a_i}(p) - k^+_{a_i}(G) \in 2\mathbb{N}_0$ and $\epsilon_{a_i}(p) = 1)$ or $ord_{a_i}(p) \geq \min(k_{a_i}(G), k^-_{a_i}(G))$ if $k^+_{a_i}(G) \leq \min(k_{a_i}(G), k^-_{a_i}(G))$.

Case 3: $(ord_{a_i}(p) - k^-_{a_i}(G) \in 2\mathbb{N}_0$ and $\epsilon_{a_i}(p) = -1)$ or $ord_{a_i}(p) \geq \min(k_{a_i}(G), k^+_{a_i}(G))$ if $k^-_{a_i}(G) \leq \min(k_{a_i}(G), k^+_{a_i}(G))$.

Proof:
For abbreviation we write $g(X) := \prod_{i=1}^{m}(X - a_i)^{k_i}$.
We consider some $f \in Q$ and get by division through g that

$$f = qg + r$$

with unique $q, r \in R[X]$ such that either $r = 0$ or $\deg(r) \leq D$.
If $r = 0$ then $f \in \mathrm{supp}(Q)$.
In the other case let $a = a_i$ be some point of S with $k = k_i$. We have

$$\mathrm{ord}_a(r) = \mathrm{ord}_a(f - qg) \geq \min\{\mathrm{ord}_a(q) + \mathrm{ord}_a(g), \mathrm{ord}_a(f)\} \qquad (*)$$

with equality if $\mathrm{ord}_a(q) + \mathrm{ord}_a(g) \neq \mathrm{ord}_a(f)$.
We note that $\mathrm{ord}_a(g) = k$ and distinguish the different possibilities for k according to Corollary 2.42.
First let $k = k_a(G)$ (i.e. $k_a(G) < k^+_a(G), k^-_a(G)$)
If $\mathrm{ord}_a(f) \geq k$ then we also have $\mathrm{ord}_a(r) \geq k$ because of $(*)$.
If otherwise $\nu = \mathrm{ord}_a(f) < k$ even and $\epsilon_a(f) = 1$ then we have equality in $(*)$ and hence $\mathrm{ord}_a(r) = \mathrm{ord}_a(f)$ also even. From $f = (X - a)^\nu \widetilde{f} = (X - a)^\nu (q\widetilde{g} + \widetilde{r})$ we see

that $\tilde{r}(a) = \tilde{f}(a) > 0$ because of $\tilde{g}(a) = 0$ as $\nu < k$. Hence $\epsilon_a(r) = \epsilon_a(f) = 1$.
Now suppose that $k = \min(k_a(G), k_a^+(G))$ (i.e. $k_a^-(G) \leq \min(k_a(G), k_a^+(G))$)
If $\mathrm{ord}_a(f) \geq k$ then we also have $\mathrm{ord}_a(r) \geq k$ because of $(*)$.
If otherwise $\nu = \mathrm{ord}_a(f) < k$ we have equality in $(*)$ and hence $\mathrm{ord}_a(r) = \mathrm{ord}_a(f)$.
Similar to the first case we also see that $\epsilon_a(f) = \epsilon_a(r)$ which means that the conditions for $\mathrm{ord}_a(r)$ and $\epsilon_a(r)$ are the same as the conditions for $\mathrm{ord}_a(f)$ and $\epsilon_a(f)$.
The final case $k = \min(k_a(G), k_a^-(G))$ (i.e. $k_a^+(G) \leq \min(k_a(G), k_a^-(G))$)
can be done similarly.
Altogether we have proved that $r \in Q'$,i.e. $Q \subseteq Q' + \mathrm{supp}(Q)$.
The other inclusion is clear.

Corollary 2.44 □

An important class of quadratic modules with finite associated semialgebraic set are preorderings of the form $\sum R[X]^2 + I$ where $I \subseteq R[X]$ is an ideal. If I is generated by $g_1, ..., g_l$ then $\sum R[X]^2 + I = PO(g_1, ..., g_l, -g_1, ..., -g_l) = QM(g_1, ..., g_l, -g_1, ..., -g_l)$ with associated semialgebraic set $Z(I) = \{x \in R \mid g_i(x) = 0 \ (1 \leq i \leq l)\}$. This kind of preordering which is composed of the two stable parts sums of squares and ideal is itself again stable as we will see in the chapter about heirs. Although one might guess that the support of $\sum R[X]^2 + I$ is I this is not true in general, the support can strictly contain I, which we show with the following example.

Example 2.45
The support of the preordering

$$P = QM(X(X^2+1), -X(X^2+1)) = \sum R[X]^2 + \big(X(X^2+1)\big) R[X] \subseteq R[X]$$

is by Corollary 2.42 given by

$$supp(P) = XR[X] \supset \big(X(X^2+1)\big) R[X].$$

The fact that $X \in supp(P)$ can be seen explicitly as follows:
Since $\pm X(X^2+1) \in P$ we have

$$pX(X^2+1) = (\frac{p+1}{2})^2 X(X^2+1) + (\frac{p-1}{2})^2(-X(X^2+1)) \in P$$

for every $p \in R[X]$. Thus in particular $(-X)X(X^2+1) = -X^4 - X^2 \in P$ which gives that $-X^2 \in P$. Hence

$$X = X(X^2+1) - X^3 = X(X^2+1) + (\frac{X+1}{2})^2(-X^2) + (\frac{X-1}{2})^2 X^2 \in P.$$

Similarly

$$-X = -X(X^2+1) + X^3 = -X(X^2+1) + (\frac{X+1}{2})^2(X^2) + (\frac{X-1}{2})^2(-X^2) \in P$$

which implies that $X \in supp(P)$.

2.3 Positivity and convexity divisors

In Section 2.1 we solved the Membership Problem in the affirmative for finitely generated quadratic modules in $\mathbb{R}[X]$ where X denotes one indeterminate. If in addition the associated semialgebraic set is bounded Theorem 2.18 states the conditions for membership in the quadratic module which is in fact a preordering (Theorem 2.17). With the help of this explicit description we can again explicitly describe the positivity and convexity divisors in such preordered rings.

In this section A denotes an integral domain.

Definition 2.46
Let (A, P) be a preordered ring and $h \in A$. We say that h is a

i) positivity divisor of A if $h \in P$ and for every $f \in A$ we have

$$hf \in P \Rightarrow f \in P.$$

ii) convexity divisor of A if h is a positivity divisor and the principal ideal hA is convex.

We note that for a positivity divisor h the property of being a convexity divisor means the following:

$$\text{if } f, g \in A \text{ with } g \in P, f - g \in P \text{ and } h|f \text{ then } h|g.$$

For more details on this topic we refer to [K2].

The positivity divisors of a ring A play a crucial role in answering the following question:
Let $P \subseteq A$ be a preordering. When does the preordering

$$\widetilde{P} := \{\frac{f}{h} \mid f \in P, h \in P \setminus \{0\}\}$$

of the quotient field $\mathrm{Quot}(A)$ has the property that $\widetilde{P} \cap A = P$?

Even if P is proper and $\mathrm{supp}(P) = \{0\}$, which implies that $\tilde{P}$ is a proper preordering, $\widetilde{P} \cap A$ may actually strictly contain P as the following example shows.

Example 2.47
Let $A = \mathbb{R}[X]$ and $P = QM((4-X^2)^3)$.
By Theorem 2.10 the preordering which has support $\{0\}$ since $int[-2,2] \neq \emptyset$ can be described as

$$P = \{f \in \mathbb{R}[X] \mid f|_{[-2,2]} \geq 0, ord_{\pm 2} \neq 1\}.$$

P is not saturated because the natural generators $2-X, 2+X$ of the interval $[-2,2]$ are not in P (Corollary 1.7).
However $\widetilde{P} \cap \mathbb{R}[X] = \mathcal{P}(S(P)) = \mathcal{P}([-2,2])$ and hence P is a proper subset of $\widetilde{P} \cap \mathbb{R}[X]$. That the intersection is the saturation follows again with Corollary 1.7 from the fact that $2 \pm X = \frac{2 \pm X}{1} = \frac{(2 \pm X)^3}{(2 \pm X)^2} \in \widetilde{P} \cap \mathbb{R}[X]$ where we used the result of Proposition 2.33 which implies that $\widetilde{P} \cap \mathbb{R}[X]$ is finitely generated.

We note that the denominator $(2-X)^2$ appearing in the example is not a positivity divisor of $(\mathbb{R}[X], QM((4-X^2)^3))$ because $(2-X)^3 \in QM((4-X^2)^3)$ but $(2-X) \notin QM((4-X^2)^3)$.

If we localize A with respect to the multiplicative set $\Sigma_+(A)$ of all positivity divisors of A then we get a ring extension

$$\mathrm{Quot}_+(A) := \Sigma_+(A)^{-1}A$$

of A (called the total preordered ring of quotients of A) with preordering

$$\mathrm{Quot}_+(A)^+ = \{\frac{f}{h} \mid f \in P, h \in \Sigma_+(A)\}$$

such that $\mathrm{Quot}_+(A)^+ \cap A = P$ ([K2] Proposition 6.2b)).
We note that $-1 \notin P$ implies that $-1 \notin \mathrm{Quot}_+(A)^+$ because no element of $\mathrm{supp}(P)$ is a positivity divisor.

If we want to have in addition that A is convex in the ring extension we have to restrict ourselves to convexity divisors. By [K2] Proposition 6.10 A is convex in

$$\mathrm{Quot}_c(A) := \Sigma_c(A)^{-1}A$$

where $\Sigma_c(A)$ denotes the multiplicative set of convexity divisors of A. $\mathrm{Quot}_c(A)$ is preordered by

$$\mathrm{Quot}_c(A)^+ := \mathrm{Quot}_+(A)^+ \cap \mathrm{Quot}_c(A).$$

Therefore it is interesting to determine the set of positivity and convexity divisors for a preordered ring (A, P). We do this in some special cases.

We consider a finitely generated preordering $P \subseteq \mathbb{R}[X]$ with bounded associated semialgebraic set $S = S(g_1, ..., g_s) \subseteq \mathbb{R}$. To shorten the notation we say inspired by Corollary 2.23 that the condition for saturation is satisfied

- in a boundary point a of $S \setminus S_{isol}$ if $\min\limits_{1 \leq i \leq s} (\mathrm{ord}_a(g_i)) = 1$
- in an isolated point a of S if there is a pair g_i, g_j such that $\mathrm{ord}_a(g_i) = \mathrm{ord}_a(g_j) = 1$ and $\epsilon_a(g_i) = -\epsilon_a(g_j)$

This means that with $G := \{g_1, ..., g_s\}$ the condition for saturation is not satisfied in some boundary point a of $S \setminus S_{isol}$ if the corresponding entry in $\vec{\omega}(G)$ is not 1. Similarly the condition for saturation is not satisfied in some isolated point a of S if the corresponding entries in $\vec{\omega}(G)$ are not $(1, 1)$.

Proposition 2.48
Let $g_1, ..., g_s \in \mathbb{R}[X]$ such that $S = S(g_1, ..., g_s) \subseteq \mathbb{R}$ is bounded.
Then the set of positivity divisors $\Sigma_+(\mathbb{R}[X])$ for the preordered ring $(\mathbb{R}[X], P)$ with $P = QM(g_1, ..., g_s)$ is given by

$$\Sigma_+(\mathbb{R}[X]) = \{h \in \mathbb{R}[X] \mid \begin{array}{l} h|_S \geq 0 \text{ and } h(a) \neq 0 \text{ for all boundary points } a \text{ of } S \\ \text{in which the condition for saturation is not satisfied } \} \end{array}$$

Proof:
If P is saturated then by Corollary 2.23 the condition for saturation is satisfied in every boundary point a of S. Hence the set on the right hand side consists of the polynomials which are nonnegative on S, which means that the claim is in this case that $\Sigma_+(\mathbb{R}[X]) = \mathcal{P}(S) = P$. This is clear because for any $h \in \mathbb{R}[X]$ with $h|_S \geq 0$ we have $hf|_S \geq 0$ if and only if $f|_S \geq 0$.

Now we suppose that P is not saturated. This means that in at least one boundary point a of S the condition for saturation fails (Corollary 2.23).

The inclusion $\supseteq$ is clear because in all those points a of S in which an element of P has to fulfill order conditions according to Theorem 2.18 we have $\mathrm{ord}_a(h) = 0$ for some h which lies in the set on the right hand side. This means first that $h \in P$ and secondly that for some $f \in \mathbb{R}[X]$ we have $\mathrm{ord}_a(fh) = \mathrm{ord}_a(f)$. Hence if $hf \in P$ the nonnegativity and the necessary order conditions transfer from hf to f and we can conclude that $f \in P$.

For the other inclusion we take some $h \in \Sigma_+(\mathbb{R}[X])$. We suppose that $h(a) = 0$ for some boundary point a of S where the condition for saturation is not satisfied. We distinguish the types of the point a and define in every case a polynomial $f \in \mathbb{R}[X]$

with $hf \in P$ but $f \notin P$. This will prove the claim.

We abbreviate $G := \{g_1, ..., g_s\}$ and write $S = \bigcup_{i=1}^{m} [a_i, b_i]$ such that $\vec{\sigma}(S) \in S_{vec}(m)$ for some $m \in \mathbb{N}$. We will use the vector $\vec{\omega}_{\pm}(G) = (\omega_1, \omega_1^+, \omega_1^-, ..., \omega_m, \omega_m^+, \omega_m^-)$ which stores the order conditions for P as explained in Section 2.1.
As in the definition of the generalized natural generators

$$b_0 := -\infty, a_{m+1} := \infty, (X - (-\infty))^{\omega_0^-} := 1 \text{ and } (X - \infty)^{\omega_{m+1}^+} := -1.$$

First we suppose that there is some $i \in \{1, ..., m\}$ such that $a_i < b_i$ and the condition for saturation fails in a_i, i.e. $\omega_i^+ = k_{a_i}^+(G) > 1$.
As $h(a_i) = 0$ and $h \in P$ we have $\text{ord}_{a_i}(h) \geq 2$. Then

$$f := (X - b_{i-1})^{\omega_{i-1}^-}(X - a_i)^{\omega_i^+ - 2}$$

has the property that $f \notin P$ but $hf \in P$.
The case that the condition of saturation fails in some b_i with $a_i < b_i$ is similar.

Now we suppose that there is some $1 \leq i \leq m$ with $a_i = b_i$. If a_i is an isolated point of type B we know by assumption that $\omega_i^+ = k_{a_i}^+(G) > 1$ or $\omega_i^- = k_{a_i}^-(G) > 1$.
As $h(a_i) = 0$ we have $\text{ord}_{a_i}(h) \geq 1$.

If $a_i = b_i$ is of type A then we have similar to the case $a_i < b_i$ that $\text{ord}_{a_i}(h) \geq 2$ and can again define

$$f := (X - b_{i-1})^{\omega_{i-1}^-}(X - a_i)^{\omega_i^+ - 2}.$$

If $a_i = b_i$ is of type B then we suppose without loss of generality that $\omega_i^+ > 1$. With

$$f := (X - b_{i-1})^{\omega_{i-1}^-}(X - a_i)^{\omega_i^+ - 1}(X - a_{i+1})^{\omega_{i+1}^+}$$

we have $hf \in P$ but $f \notin P$.

If $a_i = b_i$ is of type C then we define

$$f := (X - b_i)^{\omega_i - 1}(X - a_{i+1})^{\omega_{i+1}^+}$$

and finally if $a_i = b_i$ is of type D then the polynomial

$$f := (X - b_{i-1})^{\omega_{i-1}^-}(X - a_i)^{\omega_i - 1}$$

is not in P but $hf \in P$.

In each case $f \notin P$ but $hf \in P$ follows from Theorem 2.18 and the definition of $\vec{\omega}_{\pm}(G)$ as the minimal order values of elements of P.

Prop. 2.48 □

For the preordering of Example 2.47 we have by the result of the proposition

$$\Sigma_+(\mathbb{R}[X]) = \{h \in \mathbb{R}[X] \mid h|_{[-2,2]} \geq 0, h(\pm 2) > 0\}.$$

Before we describe the set of convexity divisors we deduce a consequence from Proposition 2.48 which concerns a second local-global principle due to Scheiderer.

Theorem 2.49 (Scheiderer, [S5] Theorem 2.8)
Let A be a commutative ring with 1, $\frac{1}{2} \in A$, $f \in A$ and $P \subseteq A$ an archimedean preordering.
If f lies in the preordering $P_{\mathfrak{m}}$ generated by P in the localization $A_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of A with supp$(P) \subseteq \mathfrak{m}$ then $f \in P$.

We note that $P_{\mathfrak{m}} = \{\frac{p}{h^2} \mid p \in P, h \in A \setminus \mathfrak{m}\}$.

In general one has to check infinitely many local conditions and the maximal ideals with non-real residue field can not be excluded. However in the situation of Proposition 2.48 where we have the concrete description of the positivity divisors of $(\mathbb{R}[X], P)$ one has to check only finitely many conditions. We show that the local-global principle can be derived from Proposition 2.48 in this special situation.

Proposition 2.50
Let $f, g_1, ..., g_s \in \mathbb{R}[X]$ and $P = QM(g_1, ..., g_s)$ with $S = S(g_1, ..., g_s) \subseteq \mathbb{R}$ bounded.
If P is saturated then $f \in P$ if f lies in the preordering $P_{\mathfrak{m}}$ generated by P in $\mathbb{R}[X]_{\mathfrak{m}}$ for one maximal ideal $\mathfrak{m} \subseteq \mathbb{R}[X]$.
For the non saturated case the following is true:
Suppose that f lies in the preordering $P_{\mathfrak{m}}$ generated by P in $\mathbb{R}[X]_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of the form $(X - a)\mathbb{R}[X]$ where a is a boundary point of S in which the condition for saturation is not satisfied. Then $f \in P$.

Proof:
If P is saturated then the assumption implies that $h^2 f \in P$ for some $h \in \mathbb{R}[X]$. As the set of positivity divisors is in this case equal to $\mathcal{P}(S)$ we get $f \in P$.
Now suppose that P is not saturated.
Let $\mathfrak{m}_1, ..., \mathfrak{m}_r$ denote the maximal ideals of $\mathbb{R}[X]$ corresponding to the finitely many boundary points $a_1, ..., a_r$ of S in which the condition for saturation fails. By assumption we have for every $1 \leq i \leq r$ an element $h_i \in \mathbb{R}[X] \setminus \mathfrak{m}_i$, i.e. $h_i(a_i) \neq 0$, with $h_i^2 f \in P$. Thus $h := h_1^2 + ... + h_r^2$ is an element which is nonnegative on all of $\mathbb{R}$ and has the property that $h(a_i) > 0$ $(1 \leq i \leq r)$. By Proposition 2.48 h is a positivity divisor of $(\mathbb{R}[X], P)$. Since $hf = \sum_{i=1}^{r} h_i^2 f \in P$ this implies that $f \in P$.
Prop. 2.50 □

We include here a few remarks on the conditions appearing in the first and second local-global principle of Scheiderer for the following situation:

Let $A = R[X]$ for an arbitrary real closed field R, X denotes one indeterminate as always in this section and $P = PO(G) \subseteq A$ is a finitely generated preordering with associated basic closed set $S = S(G)$.

As the ideal $\mathfrak{m} := (X-a)R[X]$ for some $a \in R$ is a maximal ideal of $R[X]$ we have (see e.g. [E] 7.1)

$$\widehat{R[X]}_{\mathfrak{m}} \cong (\widehat{R[X]_{\mathfrak{m}}})_{\mathfrak{m}R[X]_{\mathfrak{m}}}.$$

Thus $R[X]_{\mathfrak{m}} \subseteq \widehat{R[X]}_{\mathfrak{m}} \cong R[[X-a]]$ and therefore

$$P_{(X-a)R[X]} \subseteq \widehat{P}_a.$$

This means that if $R = \mathbb{R}$ and S is bounded then by the (first) local-global principle of Scheiderer (Theorem 2.9) the conditions $f \in P_{(X-a)\mathbb{R}[X]}$ for the finitely many zeros of f in S together with the nonnegativity of f on S imply that $f \in P$.

If R is again an arbitrary real closed field but $S = \{a_1, ..., a_m\}$ is finite then we even have

$$P_{(X-a)R[X]} = \widehat{P}_a$$

for every $a \in S$.

This can be seen as follows:

Let $f \in \widehat{P}_a$ for some $a \in S$. Then we have for every $N \in \mathbb{N}$ some $p_N \in P$ and some $h_N \in R[X]$ such that $f = p_N + h_N(X-a)^{2N}$. By the Stellensatz for preorderings (Theorem 0.6 iii)) there is some $N_0 \in \mathbb{N}$ such that $q := -\prod_{i=1}^{m}(X-a_i)^{2N_0} \in P$. Now we define $h := \prod_{a_i \neq a}(X-a_i)^{N_0}$ and get

$$h^2 f = h^2 p_{N_0} + (\frac{h_{N_0}+1}{2})^2 h^2 (X-a)^{2N_0} + (\frac{h_{N_0}-1}{2})^2 \underbrace{(-h^2(X-a)^{2N_0})}_{=q \in P} \in P.$$

As $h \notin (X-a)R[X]$ this means that $f \in P_{(X-a)R[X]}$.

In this case the conditions appearing in the first and the second local global-principle are equivalent because:

$$\begin{aligned}
& \widehat{f_a} \in \widehat{P}_a \text{ for every } a \in Z(f) \cap S \text{ and } f|_S \geq 0 \\
\Leftrightarrow\ & \widehat{f_a} \in \widehat{P}_a \text{ for every } a \in S \\
\Leftrightarrow\ & f \in P_{(X-a)R[X]} \text{ for every } a \in S \\
\Leftrightarrow\ & f \in P_{\mathfrak{m}} \text{ for every maximal ideal } \mathfrak{m} \text{ with } \operatorname{supp}(P) \subseteq \mathfrak{m}
\end{aligned}$$

Hence we can formulate Theorem 2.34 in the following form which uses localizations instead of formal power series rings and gives a version of the second local-global

principle where the assumption that $QM(g_1, ..., g_s)$ is archimedean is replaced by the assumption that $S(g_1, ..., g_s)$ is finite.

Corollary 2.51
Let $f, g_1, ..., g_s \in R[X]$ and $P = QM(g_1, ..., g_s)$ with $S = S(g_1, ..., g_s) \subseteq R$ finite. If $f \in P_{(X-a)R[X]}$ for every $a \in S$ then $f \in P$.

Now we characterize the set of convexity divisors for the case that $P \subseteq \mathbb{R}[X]$ is a finitely generated preordering whose associated semialgebraic set is not empty and bounded.

Proposition 2.52
Let $G = \{g_1, ..., g_s\} \subseteq \mathbb{R}[X]$ such that $\emptyset \neq S = S(G) = \bigcup_{i=1}^{m} [a_i, b_i] \subseteq \mathbb{R}$ is bounded and $(a_1, b_1, ..., a_m, b_m) \in S_{vec}(m)$. Then a polynomial $h \in \mathbb{R}[X]$ is a convexity divisor of $(\mathbb{R}[X], P)$ where $P = QM(G)$ if and only if h is nonnegative on S and of the form

$$h = c \prod_{\substack{i=1 \\ a_i<b_i \\ k^+_{a_i}(G)=1}}^{m} (X - a_i)^{\mu_i} \prod_{\substack{i=1 \\ a_i<b_i \\ k^-_{b_i}(G)=1}}^{m} (X - b_i)^{\nu_i} \prod_{\substack{i=1 \\ a_i=b_i \\ k^+_{a_i}(G)=k^-_{b_i}(G)=1}}^{m} (X - a_i)^{\epsilon_i} \prod_{i=1}^{N} (X - \alpha_i)^{2\lambda_i}$$

for some $c \in \mathbb{R} \setminus \{0\}, \mu_i, \nu_i \in \mathbb{N}_0$ $(1 \leq i \leq m), \epsilon_i \in \{0, 1\}$ $(1 \leq i \leq m)$, $N \in \mathbb{N}_0, \lambda_i \in \mathbb{N}$ and $\alpha_i \in int(S)$ $(1 \leq i \leq N)$.

Proof:
For the implication $\Leftarrow$ we take some polynomial h of the prescribed form and write for abbreviation $h = c \prod_{i=1}^{r} (X - c_i)^{k_i}$ where $c_1, ..., c_r$ are the distinct zeros of h with $\text{ord}_{c_i}(h) = k_i \in \mathbb{N}$ $(1 \leq i \leq r)$.
By Proposition 2.48 we know that h is a positivity divisor. In order to show that h is a convexity divisor we consider some $f, g \in \mathbb{R}[X]$ with $g, f - g \in P$ and $h|f$. For showing that $h|g$ it is enough to prove that $(X - c_i)^{k_i}|g$ for $1 \leq i \leq r$. Since $f - g \in P$ and $g \in P$ we have in particular that $f \geq g \geq 0$ on S. $(*)$
As h divides f we have for every $1 \leq i \leq r$ that $f(c_i) = 0$ and thus by $(*)$ also $g(c_i) = 0$.
For an isolated point this is enough for showing that $(X - c_i)^{k_i}|g$ since we have by assumption $k_i = 1$.
Now we consider some zero c_i of g which is not an isolated point of S. The fact that there is an interval to the left or to the right of c_i where $(*)$ has to be fulfilled forces the multiplicity of the zero c_i of g to be at least as big as k_i. Thus also in this case $(X - c_i)^{k_i}|g$.

Altogether this proves that h is a convexity divisor and hence $\Leftarrow$.

For the other implication we already know by Proposition 2.48 that $h|_S \geq 0$ and

$$h = h_1 \prod_{\substack{i=1 \\ a_i<b_i \\ k^+_{a_i}(G)=1}}^{m} (X-a_i)^{\mu_i} \prod_{\substack{i=1 \\ a_i<b_i \\ k^-_{b_i}(G)=1}}^{m} (X-b_i)^{\nu_i} \prod_{\substack{i=1 \\ a_i=b_i \\ k^+_{a_i}(G)=k^-_{b_i}(G)=1}}^{m} (X-a_i)^{\epsilon_i} \prod_{i=1}^{N} (X-\alpha_i)^{2\lambda_i}$$

for some $h_1 \in \mathbb{R}[X]$ with $h_1(x) \neq 0 \ \forall x \in S, \mu_i, \nu_i, \epsilon_i \in \mathbb{N}_0 \ (1 \leq i \leq m)$, $N \in \mathbb{N}_0, \lambda_i \in \mathbb{N}$ and $\alpha_i \in \operatorname{int}(S) \ (1 \leq i \leq N)$ because h is a positivity divisor.

It remains to show that $\epsilon_i \in \{0,1\}$ for every $1 \leq i \leq m$ and $h_1 \equiv c$ for some $c \in \mathbb{R} \setminus \{0\}$.

For abbreviation we write again $h = h_1 \prod_{i=1}^{r} (X-c_i)^{k_i}$ where $c_1, ..., c_r$ are the distinct zeros of h in S with multiplicities $k_1, ..., k_r$.

We first suppose that h_1 is not constant and lead this to a contradiction.
Then we can write $h_1 = \delta H_1 \prod_{i=1}^{\nu} (X-\gamma_j)^{l_j}$ for some $0 \neq \delta \in \mathbb{R}, H_1 \in \mathbb{R}[X]$ with $H_1(x) \neq 0 \ \forall x \in \mathbb{R}, \nu \in \mathbb{N}_0, \gamma_j \in \mathbb{R} \setminus S$ and $l_j \in \mathbb{N} \ (1 \leq j \leq \nu)$. The assumption that h_1 is not constant means that either H_1 is not constant or $\nu \in \mathbb{N}$.
As in [Sc] Example 30 we can find $\widetilde{\gamma}_j \in \mathbb{R} \setminus S$ close to γ_j but $\widetilde{\gamma}_j \neq \gamma_j$ for $1 \leq j \leq \nu$ and $\epsilon > 0$ small such that $\widetilde{h}_1 := \epsilon \prod_{i=1}^{\nu} (X-\widetilde{\gamma}_j)^{l_j}$ has the following properties. On every connected component of S the polynomials h_1 and $\widetilde{h}_1$ have the same sign and $0 < |\widetilde{h}_1(x)| < |h_1(x)|$ for every $x \in S$. This implies that the polynomials

$$\widetilde{h} := \widetilde{h}_1 \prod_{i=1}^{r} (X-c_i)^{k_i}$$

and

$$h - \widetilde{h} := (h_1 - \widetilde{h}_1) \prod_{i=1}^{r} (X-c_i)^{k_i}$$

are both nonnegative on the set S and for every boundary point a of S we have $\operatorname{ord}_a(h) = \operatorname{ord}_a(\widetilde{h}) = \operatorname{ord}_a(h-\widetilde{h})$ together with $\epsilon_a(h) = \epsilon_a(\widetilde{h}) = \epsilon_a(h-\widetilde{h})$. Thus $h \in P$ implies by Theorem 2.18 also $\widetilde{h} \in P$ and $h - \widetilde{h} \in P$. Since h is a convexity divisor this means that $h|\widetilde{h}$, i.e. there is some $p \in \mathbb{R}[X]$ with

$$h_1 \prod_{i=1}^{r} (X-c_i)^{k_i} \cdot p = \widetilde{h}_1 \prod_{i=1}^{r} (X-c_i)^{k_i}.$$

Thus

$$h_1 \cdot p = \widetilde{h}_1$$

which is a contradiction since h_1 and $\widetilde{h}_1$ are by definition relatively prime. Hence h_1 is constant. Furthermore h_1 cannot be identically zero because 0 is not a positivity divisor.

Now we suppose that there is some isolated point $a = a_i$ of S with the property that $k_a(G)^+ = k_a^-(G) = 1$ and $\mathrm{ord}_a(h) \geq 2$.
We write $h = H_1(X-a)^{ord_a(h)}$ with $H_1 \in \mathbb{R}[X]$ and $H_1(a) \neq 0$. As a is an isolated point there is some $\epsilon > 0$ such that $]a-\epsilon, a+\epsilon[\cap S = \emptyset$. We choose some $\alpha \in]a-\epsilon, a+\epsilon[\setminus\{a\}$ and consider the polynomial $(X-\alpha)(X-a)$.
As $(X-a)^2$ and $(X-\alpha)(X-a)$ are strict positive on the compact set $S\setminus\{a\}$ there is some $\mu \in \mathbb{N}$ such that $0 < \mu(x-\alpha)(x-a) < (x-a)^2$ for every $x \in S\setminus\{a\}$.
Thus the polynomials

$$\widetilde{g} := H_1(X-a)^{ord_a(h)-2}\mu(X-a)(X-\alpha)$$

and

$$h - \widetilde{g} = H_1(X-a)^{ord_a(h)-2}((X-a)^2 - \mu(X-a)(X-\alpha))$$

have the following properties. They are both nonnegative on S and for every boundary point b of S with $b \neq a$ we have $\mathrm{ord}_b(h) = \mathrm{ord}_b(\widetilde{g}) = \mathrm{ord}_b(h-\widetilde{g})$ and $\epsilon_b(h) = \epsilon_b(\widetilde{g}) = \epsilon_b(h-\widetilde{g})$. For a we have $\mathrm{ord}_a(\widetilde{g}) = \mathrm{ord}_a(h-\widetilde{g}) = \mathrm{ord}_a(h) - 1 \geq 1$. Hence $\widetilde{g}$ and $h-\widetilde{g}$ are according to Theorem 2.18 elements of P because $h \in P$ and a is an isolated point with $k_a(G)^+ = k_a^-(G) = 1$. Since h is a convexity divisor we conclude that $h|\widetilde{g}$ which is not possible because $\mathrm{ord}_a(h) > \mathrm{ord}_a(\widetilde{g})$. Hence $\mathrm{ord}_a(h) \leq 1$.

Prop. 2.52 □

For the preordering of Example 2.47 we find

$$\Sigma_c(\mathbb{R}[X]) = \{h \in \mathbb{R}[X] \quad | \quad h = c\prod_{i=1}^{N}(X-\alpha_i)^{2\lambda_i} \text{ with } c \in \mathbb{R}\setminus\{0\}, N \in \mathbb{N}_0, \\ \lambda_i \in \mathbb{N}, -2 < \alpha_1 < ... < \alpha_N < 2\}$$

Now we suppose that the preordering is a partial preordering, i.e. $\mathrm{supp}(P) = \{0\}$. Then the quotient field $\mathrm{Quot}(\mathbb{R}[X])$ for the integral domain $\mathbb{R}[X]$ coincides with complete ring of quotients $Q(\mathbb{R}[X])$. For an arbitrary ring A the complete ring of quotients $Q(A)$ is defined as the inductive limit $\varinjlim_{I\in D(A)} \mathrm{Hom}(I, A)$ where $I \in D(A)$ if and only if I is a dense ideal of A, i.e. no element $f \in A\setminus\{0\}$ is annulated by the ideal I.
In the more general setting of the complete ring of quotients the concept of positivity divisors has to be replaced by positively dense subsets. Again we refer to [K2] for more details.

Definition 2.53
Let (A, P) be a preordered ring. We say that $M \subseteq P$ is positively dense if for every $f \in A$ we have

$$Mf \in P \Rightarrow f \in P.$$

Positively dense subsets are used to define the complete partially ordered ring of quotients $Q_+(A)$ of a partially ordered ring (A, P).

Knebusch showed that $Q_+(A)$ the set of all $F \in Q(A)$ such that there is a set $M \subseteq P$ with $MF \subseteq A$ and M positively dense in A ([K2] Proposition 2.14).

Even in the situation where $Q(A) = \mathrm{Quot}(A)$ one can ask whether it is true that $Q_+(A) = \mathrm{Quot}_+(A)$. We do this and answer the question for the preordered rings for which we determined the positivity divisors (Proposition 2.48) under the additional assumption that $\mathrm{supp}(P) = \{0\}$. First we describe the positively dense subsets.

Proposition 2.54
Let $g_1, ..., g_s \in \mathbb{R}[X]$ and $P = QM(g_1, ..., g_s)$ with $S = S(g_1, ..., g_s) \subseteq \mathbb{R}$ bounded. In the preordered ring $(\mathbb{R}[X], P)$ a subset $M \subseteq P$ is positively dense if and only if for every boundary point a of S in which the condition for saturation is not satisfied there is some $h_a \in M$ with $h_a(a) \neq 0$.

Proof:
We suppose that there is some boundary point a of S where the condition for saturation fails and $\mathrm{ord}_a(h) \geq 1$ for every $h \in M$. Then one can construct as in the proof of Proposition 2.48 a polynomial $f \in \mathbb{R}[X]$ with $Mf \in P$ and $f \notin P$.

Conversely if for every boundary point a of S in which the condition for saturation fails there is some $h_a \in M$ with $\mathrm{ord}_a(h_a) = 0$ then the condition $Mf \subseteq P$ implies that in all points where an order condition according to Theorem 2.18 has to be fulfilled, f already satisfies these conditions because of $M \subseteq P$. Also the nonnegativity transfers from M to f and hence $f \in P$.

Prop. 2.54 □

In the situation of 2.47 for example $M = \{(2 - X)^2, (2 + X)^2\}$ is a positively dense subset of $\mathbb{R}[X]$. We note that the elements of M itself are no positivity divisors.

Proposition 2.54 now immediately gives the answer to the question from above whether $Q_+(A) = \mathrm{Quot}_+(A)$.

Proposition 2.55
Let $g_1, ..., g_s \in \mathbb{R}[X] = \mathbb{R}[X_1]$ with $P = QM(g_1, ..., g_s)$ such that $supp(P) = \{0\}$ and $S = S(g_1, ..., g_s) \subseteq \mathbb{R}$ bounded. Then we have for the preordered ring $(\mathbb{R}[X], P)$

$$Quot_+(\mathbb{R}[X]) = Q_+(\mathbb{R}[X]).$$

Proof:
The inclusion $\subseteq$ is clear because for $F = \frac{f}{h} \in \mathrm{Quot}_+(\mathbb{R}[X])$ we have $MF \subseteq \mathbb{R}[X]$ for $M = \{h\} \subseteq P$. Since h is a positivity divisor the set M is positively dense in A.

For the other inclusion we consider some $F \in Q_+(\mathbb{R}[X])$. Hence there is a positively dense subset $M \subseteq P$ with $MF \subseteq \mathbb{R}[X]$. We have to show that there is some positivity divisor $h \in \Sigma_+(\mathbb{R}[X])$ such that $hF \in \mathbb{R}[X]$. If the set of boundary points of S where the condition for saturation fail is denoted by $\{a_1, ..., a_r\}$ then we have by Proposition 2.54 polynomials $h_{a_i} \in M$ with $h_{a_i}(a_i) > 0$ and $h_{a_i}(a_j) \geq 0$ for $1 \leq i, j \leq r, i \neq j$. By defining $h := \sum_{i=1}^{r} h_{a_i}$ we get an element of P with the property that $h(a_i) \neq 0$ for every $i \in \{1, ..., r\}$. Thus by Proposition 2.48 h is a positivity divisor which satisfies $hF \in \mathbb{R}[X]$ as $MF \subseteq \mathbb{R}[X]$.

Prop. 2.55 $\square$

3 Heirs of subsets of $R[X]$

3.1 Definition of heirs

As already mentioned before the definability question for orderings in $R[X_1, ..., X_n]$ could be solved (Theorem 1.18). One crucial step in the proof of this uses the fact that an ordering is definable if and only if for every real closed extension $R' \supseteq R$ it has a unique heir. The notion of heirs originates in model theory and can be used for orderings because of the correspondence between orderings and types as explained in Section 1.3 and in the Appendix. The aim of this section is to generalize heirs of types to heirs of subsets of $R[X_1, ..., X_n]$ such that one can speak of an heir of a quadratic module or a preordering.

First we motivate the definition of an heir of a general subset of $R[X_1, ..., X_n]$.

As in Chapter 1, R is a real closed field, $X = (X_1, ..., X_n)$ and Y, Z finite tuples of variables (of variable length) as well as $L = L_{or} = \{+, -, \cdot, 0, 1, <\}$ the language of ordered rings. In this setting we know that the notion of being weakly semialgebraic and of being definable is equivalent for some $Q \subseteq R[X]$. However we use the notion of definability when developing the concept of heirs because everything is also true if L is an arbitrary first-order language and M some L-structure.

If $Q \subseteq R[X]$ is definable and $R' \supseteq R$ is a real closed extension field then we can define in a canonical way a set associated to Q, namely

$$Q' := \{f(X, c') \mid f(X, Y) \in \mathbb{Z}[X, Y], c' \in R'^Y \text{ and } R' \models \vartheta_f(c')\}$$

where $\vartheta_f(Y)$ is an $L(R)$-formula defining $D_R(f, Q)$.
Recall that because of the definability of Q we have for every $f(X, Y) \in \mathbb{Z}[X, Y]$ a formula $\vartheta_f(Y) \in \mathrm{Fml}L(R)$ such that the set $D_R(f, Q) = \{c \in R^Y \mid f(X, c) \in Q\}$ is defined by the formula $\vartheta_f(Y)$. We note that Q' does not depend on the particular formula $\vartheta_f(Y)$ defining $D_R(f, Q)$, it only depends on $D_R(f, Q)$.

What properties does the set Q' have?

One property is the following:

(H^+) For all $f(X, Y) \in \mathbb{Z}[X, Y]$ and every $\varphi(Y) \in \mathrm{Fml}L(R)$ we have:
$D_{R'}(f, Q') \cap \varphi(R'^Y) \neq \emptyset \Rightarrow D_R(f, Q) \cap \varphi(R^Y) \neq \emptyset$

Another one is:

(H^-) For all $f(X, Y) \in \mathbb{Z}[X, Y]$ and every $\varphi(Y) \in \mathrm{Fml}L(R)$ we have:
$D_{R'}(f, R'[X] \setminus Q') \cap \varphi(R'^Y) \neq \emptyset \Rightarrow D_R(f, R[X] \setminus Q) \cap \varphi(R^Y) \neq \emptyset$

That the properties (H^+) and (H^-) are fulfilled for Q' as defined above is clear because in this case $D_{R'}(f, Q') = \vartheta_f(R'^Y)$, $D_{R'}(f, R'[X] \setminus Q') = \neg\vartheta_f(R'^Y)$ and R' is an elementary extension of R.

Now we consider an arbitrary subset $Q' \subseteq R'[X]$.

It is easy to see that if (H^+) is satisfied for Q and Q' then (H^+) is also satisfied for Q and every subset of Q'. Similarly if (H^-) is satisfied for Q and Q' then (H^-) is also satisfied for Q and every set containing Q'.

Therefore it is interesting to look at the smallest (resp. largest) subset of $R'[X]$ such that (H^-) (resp. (H^+)) is satisfied for Q and this set.

Lemma 3.1
Suppose $R' \supseteq R$ is real closed and $Q \subseteq R[X]$.
The set

$$h(Q, R') := \{f(X, c') \mid f(X, Y) \in \mathbb{Z}[X, Y] \text{ such that there is a } \varphi(Y) \in \mathrm{Fml}L(R) \text{ with } c' \in \varphi(R'^Y) \text{ and } \varphi(R^Y) \subseteq D_R(f, Q)\}$$

is the smallest subset of $R'[X]$ such that (H^-) is satisfied for Q and this set.
The set

$$H(Q, R') := \{f(X, c') \mid f(X, Y) \in \mathbb{Z}[X, Y] \text{ such that for every } \varphi(Y) \in \mathrm{Fml}L(R) \text{ with } c' \in \varphi(R'^Y) \text{ we have } \varphi(R^Y) \cap D_R(f, Q) \neq \emptyset\}$$

is the largest subset of $R'[X]$ such that (H^+) is satisfied for Q and this set.

Proof:
We first note that if (H^-) is satisfied for Q and Q' then $h(Q, R')$ must be a subset of Q' by definition of $h(Q, R')$. For suppose to the contrary that there is some $f(X, c') \in h(Q, R')$ which is not in Q'. Then we would have by (H^-) for Q and Q' that for every $\varphi(Y) \in \mathrm{Fml}L(R)$ with $c' \in \varphi(R'^Y)$ there is some $c \in \varphi(R^Y)$ with $f(X, c) \notin Q$. This contradicts the definition of $h(Q, R')$.

It remains to show that (H^-) is satisfied for Q and $h(Q, R')$.
Therefore we take some $f(X, Y) \in \mathbb{Z}[X, Y]$ and some $\varphi(Y) \in \mathrm{Fml}L(R)$ with $D_{R'}(f, R'[X] \setminus h(Q, R')) \cap \varphi(R'^Y) \neq \emptyset$, i.e. there is some $c' \in \varphi(R'^Y)$ such that $f(X, c')$ is not in $h(Q, R')$. Being not in $h(Q, R')$ implies that there is some element $c \in \varphi(R^Y)$ with $f(X, c) \notin Q$ which means that $D_R(f, R[X] \setminus Q) \cap \varphi(R^Y) \neq \emptyset$. This gives the claim for the first part of the lemma.

The result for $H(Q, R')$ follows from this since $H(Q, R') = R'[X] \setminus h(R[X] \setminus Q, R')$ and (H^+) is satisfied for Q and Q' if and only if (H^-) is satisfied for $R[X] \setminus Q$ and $R'[X] \setminus Q'$.

Lemma 3.1 □

The sets $h(Q,R')$ (resp. $H(Q,R')$) do not just satisfy (H^-) (resp. (H^+)) with respect to Q. They satisfy even the following properties as we will see in the next lemma.

For all $k \in \mathbb{N}_0$, all $f_1(X,Y),...,f_k(X,Y),f^-(X,Y) \in \mathbb{Z}[X,Y]$ and every $\varphi(Y) \in \mathrm{Fml}L(R)$ we have:

$$(H_w) \quad \bigcap_{i=1}^{k} D_{R'}(f_i,Q') \cap D_{R'}(f^-,R'[X]\setminus Q') \cap \varphi(R'^Y) \neq \emptyset$$
$$\Rightarrow \bigcap_{i=1}^{k} D_R(f_i,Q) \cap D_R(f^-,R[X]\setminus Q) \cap \varphi(R^Y) \neq \emptyset$$

Q' is called a weak heir of Q on R' if (H_w) is satisfied for Q and Q'.

For all $k \in \mathbb{N}_0$, all $f_1^-(X,Y),...,f_k^-(X,Y),f(X,Y) \in \mathbb{Z}[X,Y]$ and every $\varphi(Y) \in \mathrm{Fml}L(R)$ we have:

$$(H_{dw}) \quad \bigcap_{i=1}^{k} D_{R'}(f_i^-,R'[X]\setminus Q') \cap D_{R'}(f,Q') \cap \varphi(R'^Y) \neq \emptyset$$
$$\Rightarrow \bigcap_{i=1}^{k} D_R(f_i^-,R[X]\setminus Q) \cap D_R(f,Q) \cap \varphi(R^Y) \neq \emptyset$$

Q' is called a dual weak heir of Q on R' if (H_{dw}) is satisfied for Q and Q'.

Clearly (H_w) and (H_{dw}) imply both conditions (H^+) and (H^-).

Proposition 3.2
Suppose $R' \supseteq R$ is real closed and $Q \subseteq R[X]$.
Then $h(Q,R')$ is a weak heir of Q on R' and $H(Q,R')$ a dual weak heir of Q on R'.

Proof:
We take $f_1(X,Y),...,f_k(X,Y),f^-(X,Y) \in \mathbb{Z}[X,Y]$ and some $\varphi(Y) \in \mathrm{Fml}L(R)$ with $\bigcap_{i=1}^{k} D_{R'}(f_i,h(Q,R')) \cap D_{R'}(f^-,R'[X]\setminus h(Q,R')) \cap \varphi(R'^Y) \neq \emptyset$. Hence there is some $c' \in R'^Y$ with $f_1(X,c'),...,f_k(X,c') \in h(Q,R')$, $f^-(X,c') \notin h(Q,R')$ and $c' \in \varphi(R'^Y)$. By definition of $h(Q,R')$ there is for every $i \in \{1,...,k\}$ a formula $\varphi_i(Y) \in \mathrm{Fml}L(R)$ with $c' \in \varphi_i(R'^Y)$ and $\varphi_i(R^Y) \subseteq D_R(f_i,Q)$. $(*)$
Since $f^-(X,c') \notin h(Q,R')$ and $R' \models \varphi(c') \wedge \bigwedge_{i=1}^{k} \varphi_i(c')$ there is by the fact that (H^-) is satisfied for Q and $h(Q,R')$ (Lemma 3.1) some $c \in R^Y$ with $R \models \varphi(c) \wedge \bigwedge_{i=1}^{k} \varphi_i(c)$ and $f^-(X,c) \notin Q$. By $(*)$ we have $f_1(X,c),...,f_k(X,c) \in Q$ which proves the claim for $h(Q,R')$.
The claim for $H(Q,R')$ follows again by $H(Q,R') = R'[X] \setminus h(R[X]\setminus Q,R')$.
Prop. 3.2 $\square$

Having Lemma 3.1 in mind this actually means that $h(Q, R')$ is the smallest weak heir of Q on R' and $H(Q, R')$ the largest dual weak heir of Q on R'.

As the adjective weak already indicates we are close to the definition of an heir. The stronger property of an heir will be the one which allows to show that $Q \subseteq R[X]$ is definable if and only if it has a unique heir on every real closed field $R' \supseteq R$. This will give us another possibility to prove definability of a quadratic module or a preordering.

Definition 3.3
Q' is called an heir of Q on R' if (H) is satisfied for Q and Q' where property (H) is given by the following:

$$(H) \quad \begin{array}{l} \text{For all } k, l \in \mathbb{N}_0, f_1(X,Y), ..., f_k(X,Y), f_1^-(X,Y), ..., f_l^-(X,Y) \in \mathbb{Z}[X,Y] \\ \text{and every } \varphi(Y) \in FmlL(R) \text{ we have:} \\ \bigcap_{i=1}^{k} D_{R'}(f_i, Q') \cap \bigcap_{i=1}^{l} D_{R'}(f_i^-, R'[X] \setminus Q') \cap \varphi(R'^Y) \neq \emptyset \\ \Rightarrow \bigcap_{i=1}^{k} D_R(f_i, Q) \cap \bigcap_{i=1}^{l} D_R(f_i^-, R[X] \setminus Q) \cap \varphi(R^Y) \neq \emptyset \end{array}$$

In general $h(Q, R')$ is not an heir but it is obtained from heirs as follows.

Proposition 3.4
Let $R' \supseteq R$ be a real closed field and $Q \subseteq R[X]$. Then

$$h(Q, R') = \bigcap_{Q' \text{heir of } Q \text{on } R'} Q'$$

and

$$H(Q, R') = \bigcup_{Q' \text{heir of } Q \text{on } R'} Q'$$

Proof:
Appendix Proposition A.11

Prop. 3.4 □

Corollary 3.5
If $R' \supseteq R$ is real closed and $Q \subseteq R[X]$ is a quadratic module (resp. a prerodering) then $h(Q, R')$ is also a quadratic module (resp. a preordering).
The same is in general not true for $H(Q, R')$.

Proof:
We show that the property of being a quadratic module or a preordering transfers from Q to every weak heir Q' of Q.

First we show that Q' is closed under addition. In order to do so we consider $f_1(X,Y), f_2(X,Y) \in \mathbb{Z}[X,Y]$ and $f^-(X,Y) := f_1(X,Y) + f_2(X,Y) \in \mathbb{Z}[X,Y]$. If there would be some $c' \in R'^Y$ with $f_1(X,c'), f_2(X,c') \in Q'$ but $f^-(X,c') \notin Q'$ then the property (H_w) would give us some $c \in R^Y$ such that $f_1(X,c), f_2(X,c) \in Q$ but $f^-(X,c) = f_1(X,c) + f_2(X,c) \notin Q$. This is a contradiction to the fact that Q is closed under addition.

Similarly the closure under multiplication transfers from Q to Q'.

The fact that $1 \in Q'$ follows by (H_w) with $f^-(X,Y) := 1$.

Now we prove that $R'[X]^2Q' \subseteq Q'$. With $F_d(X,Z) \in \mathbb{Z}[X,Z]$ we denote the general polynomial of degree d with respect to X.
We suppose that there is $f_1(X,Y), f^-(X,Y) \in \mathbb{Z}[X,Y]$ and $c' \in R'^Y$ such that $f_1(X,c') \in Q'$, $f^-(X,c') \notin Q'$ and $R' \models \varphi(c')$ where the L-formula $\varphi(Y)$ is defined as $\varphi(Y) := \exists Z(\forall X(f^-(X,Z) = F_d(X,Z)^2 f_1(X,Y)))$. Then (H_w) implies that there is some $c \in R^Y$ with $f_1(X,c) \in Q, f^-(X,c) \notin Q$ and $R \models \varphi(c)$. This contradicts $R[X]^2Q \subseteq Q$.

Thus in particular $h(Q,R')$ and every heir of Q is a quadratic module (resp. a preordering) if Q is a quadratic module (resp. a preordering).

This is in general not true for $H(Q,R')$ because the union of quadratic modules (resp. preorderings) is in general not a quadratic module (resp. a preordering).

Corollary 3.5 □

With the help of heirs as defined above we can now characterize the definability of subsets of $R[X]$ similar as for types.

Theorem 3.6
A set $Q \subseteq R[X]$ is definable if and only if it has a unique heir on R' for every real closed extension field $R' \supseteq R$.

Proof:
Appendix Theorem A.13

Theorem 3.6 □

If we look once more at the canonical set

$$Q' := \{f(X,c') \mid f(X,Y) \in \mathbb{Z}[X,Y], c' \in R'^Y \text{ and } R' \models \vartheta_f(c')\}$$

where $\vartheta_f(Y)$ is an $L(R)$-formula defining $D_R(f,Q)$ which we defined at the beginning of this section for a definable set $Q \subseteq R[X]$ to motivate the notion of heirs then it turns out that in the case where Q is definable, $Q' = h(Q,R') = H(Q,R')$ is the unique heir of Q on R'.

Proposition 3.4 together with Theorem 3.6 gives another way of showing definability of a set $Q \subseteq R[X]$, namely to show that $h(Q,R') = H(Q,R')$ for every real closed field $R' \supseteq R$.

We use this to give an example of a finitely generated quadratic module on some real closed field which is not definable, i.e. not weakly semialgebraic.

We are considering the preordering $QM_{R[X]}((1-X^2)^3)$ in the ring of polynomials with one indeterminate over a real closed field R for which Stengle showed in [St2] that it is not stable over $\mathbb{R}$. We deduce this result later on as a consequence of an explicit description of heirs (Corollary 3.13). Now we are going to show that this preordering is not weakly semialgebraic if $R \supset \mathbb{R}$ contains infinitesimal elements.

First note that $1 - X^2 \notin QM_{R[X]}((1-X^2)^3)$.
For suppose that $1 - X^2 = \sigma_0 + \sigma_1(1-X^2)^3$ for some $\sigma_i \in \sum R[X]^2$ $(i = 0,1)$. Evaluation of this expression in 1 shows that $\sigma_0(1) = 0$. Thus $(1-X)|\sigma_0$ and since this is a sum of squares even $(1-X)^2|\sigma_0$. Hence $(1-X)^2$ divides the right hand side of the above expression whereas it does not divide the left hand side.
A similar argument shows that also the natural generators $1-X$ and $1+X$ are not in $QM_{R[X]}((1-X^2)^3)$.

Proposition 3.7
Let $R \supseteq \mathbb{R}$ be a real closed field and let $n = 1, \epsilon \in R$. Then

$$f(X,\epsilon) := 1 - X^2 + \epsilon \in P := QM_{R[X]}((1-X^2)^3) \Leftrightarrow \epsilon > 0 \text{ not infinitesimal.}$$

Proof:

$\Rightarrow$: If $\epsilon < 0$ then f is clearly not in P because it is not nonnegative on $[-1,1]$. We suppose now that $\epsilon > 0$ is an infinitesimal element of R or $\epsilon = 0$. As an element of P the polynomial f has a representation $f = \sigma_0 + \sigma_1(1-X^2)^3$ for some $\sigma_i \in \sum R[X]^2$ $(i = 0,1)$. If all coefficients of σ_0 and σ_1 lie in the convex hull $\mathcal{O}$ of $\mathbb{R}$ in R then we get by applying λ a representation of $1-X^2$ as an element of $QM_{\overline{\mathcal{O}}[X]}((1-X^2)^3)$. This is not possible because the residue field $\overline{\mathcal{O}}$ is a subfield of $\mathbb{R}$ and we have seen above that $1-X^2$ does not lie in this preordering. (We actually just need that $\overline{\mathcal{O}}$ is a real closed field).
If one of the coefficients of σ_0 or σ_1 does not lie in $\mathcal{O}$, i.e. has negative value with respect to the corresponding valuation v, then we take the coefficient c

with the most negative value and divide the representation of f by c^2. Again by applying λ we get in the residue field $0 = \widetilde{\sigma}_0 + \widetilde{\sigma}_1(1-X^2)^3$ for some $\widetilde{\sigma}_i \in \sum \overline{\mathcal{O}}[X]^2$ $(i = 0, 1)$. Since at least one of the coefficients of the polynomials appearing in $\widetilde{\sigma}_i$ is 1 we get a nontrivial representation of 0 in the residue field and hence a contradiction as the interior of $[-1, 1]$ is not empty and therefore $\mathrm{supp}(QM_{\overline{\mathcal{O}}[X]}((1-X^2)^3)) = \{0\}$.

$\Leftarrow$: Because nonnegative elements from R are in the additively closed set P we can suppose without loss of generality that $\epsilon \in \mathcal{O} \setminus \mathfrak{m}$. Since λ is order preserving we get $\lambda(\epsilon) > 0$ and we find some $q \in \mathbb{Q}$ with $\frac{\lambda(\epsilon)}{2} < q < \lambda(\epsilon)$ because $\overline{\mathcal{O}}$ is archimedean. As $\overline{\mathcal{O}} \subseteq \mathbb{R}$ we get by Schmüdgen (Corollary 2.24) that $1 - X^2 + q \in QM_{\overline{\mathcal{O}}[X]}((1-X^2)^3)$. The order preserving residue map $\lambda : \mathcal{O} \to \mathcal{O}/\mathfrak{m}$ admits an order preserving section $\rho : \mathcal{O}/\mathfrak{m} \to \mathcal{O}$ which is the identity on $\mathbb{R}$ because $\mathbb{R} \subseteq R$. Hence we get a representation of $1 - X^2 + q$ as an element of $QM_{R[X]}((1-X^2)^3)$ by applying ρ. We have $q < \epsilon$ because ρ is order preserving and thus we get what we want.

Prop. 3.7 □

The proposition enables us to give the promised example of a not weakly semialgebraic finitely generated preordering on some real closed field.

Example 3.8
The preordering $P := QM_{R[X]}((1-X^2)^3) \subseteq R[X] = R[X_1]$ is not weakly semialgebraic if $R \supseteq \mathbb{R}$ contains infinitesimal elements.

For the proof of this we show that the weak heir and the dual weak heir of P do not coincide in some suitable real closed extension field R' of R.

We consider therefore the polynomial $f(X, Y) := 1 - X^2 + Y \in \mathbb{Z}[X, Y]$. Proposition 3.7 shows that

$$f(X, c) \in P \Leftrightarrow c > 0, c \text{ not infinitesimal}$$

Let now $\mathfrak{m}^+$ be the cut corresponding to the upper edge of the maximal ideal $\mathfrak{m}$, i.e. $\mathfrak{m}^+ = ((\mathfrak{m}^+)^L, (\mathfrak{m}^+)^R)$ with $(\mathfrak{m}^+)^R := \{x \in R \mid x > \mathfrak{m}\}$, and β be a realization of $\mathfrak{m}^+$ in some real closed field $R' \supseteq R$.

Then $f(X, \beta) \in H(P, R') \setminus h(P, R')$ because for every formula $\varphi(Y) \in FmlL(R)$ with $R' \models \varphi(\beta)$ (i.e. for every formula from the type of β over R) there is some $c_1 \in R$ with $R \models \varphi(c_1)$ and $c_1 > 0$ not infinitesimal, i.e. $f(X, c_1) \in P$, but there is also some $c_2 \in R$ with $R \models \varphi(c_2)$ and $c_2 > 0$ infinitesimal, i.e. $f(X, c_2) \notin P$.

3.2 Heirs and stability of quadratic modules

Before we give explicit descriptions of heirs we show how the stability of a quadratic module can be expressed with the help of heirs.

Stable quadratic modules were one of the examples for finitely generated weakly semialgebraic quadratic modules given in Section 1.2. As we developed up to now definability means that there is a unique heir on every real closed extension field. A theorem of Scheiderer translated in the language of heirs now says that if in addition the unique heir of $QM(g_1, ..., g_s)$ equals the quadratic module generated by $g_1, ..., g_s$ in $R'[X] = R'[X_1, ..., X_n]$ then the quadratic module is stable. For convenience of the reader we give the proof of this theorem which can be found in [S3].

Theorem 3.9 (Scheiderer, [S3] Proposition 3.6)
If $g_1, ..., g_s \in R[X]$ and $Q = QM(g_1, ..., g_s) \subseteq R[X]$ then the following are equivalent:

i) Q is stable.

ii) $Q \subseteq R[X]$ is definable and the unique heir $h(Q, R')$ equals $QM_{R'[X]}(g_1, ..., g_s)$ for every real closed field $R' \supseteq R$.

Proof:
For ease of notation we define $g_0 := 1$.

$i) \Rightarrow ii)$: It is clear that Q is definable (see Section 1.2) and therefore has a unique heir on every real closed field $R' \supseteq R$ (Theorem 3.6).

Even if Q is not definable we have $QM_{R'[X]}(g_1, ..., g_s) \subseteq h(Q, R')$ because $g_1, ..., g_s \in h(Q, R')$ and $h(Q, R')$ is by Corollary 3.5 a quadratic module.

The other inclusion follows from the fact that the stability of Q implies that the unique heir is given by

$$h(Q, R') = \{f(X, c') \mid f(X, Y) \in \mathbb{Z}[X, Y], c' \in R'^Y, R' \models \vartheta_f^{stab}(c')\}$$

with the formula ϑ_f^{stab} from Section 1.2. By looking at the definition of the formula ϑ_f^{stab} one immediately sees that some $f(X, c')$ with $R' \models \vartheta_f^{stab}(c')$ lies in $QM_{R'[X]}(g_1, ..., g_s)$.

$ii) \Rightarrow i)$: The finite dimensional subspaces of polynomials up to degree d form an increasing sequence $R[X]_{\leq 0} \subseteq R[X]_{\leq 1} \subseteq ...$ with $R[X] = \bigcup_{d=0}^{\infty} R[X]_{\leq d}$.

We have $Q = \bigcup_{d=0}^{\infty} \vartheta_d(R^{Y_d})$ where $\vartheta_d(R^{Y_d})$ denotes the definable set

$$\{\sum_{i=0}^{s} \sigma_i g_i \mid \sigma_i \in \sum R[X]^2_{\leq d} \ (0 \leq i \leq s)\}.$$

Let $R' \supseteq R$ be real closed, then we clearly have

$$QM_{R'[X]}(g_1, ..., g_s) = \bigcup_{d=0}^{\infty} \vartheta_d(R'^{Y_d}).$$

By assumption $QM_{R'[X]}(g_1, ..., g_s) = h(Q, R')$ is definable. Thus for every $N \in \mathbb{N}_0$ the set $QM_{R'[X]}(g_1, ..., g_s) \cap R'[X]_{\leq N}$ is semialgebraic. We now consider in particular some real closed field $R' \supseteq R$ which is $\aleph_1$-saturated (e.g. a non principal ultrapower of R). Then the semialgebraic covering of the semialgebraic set $QM_{R'[X]}(g_1, ..., g_s) \cap R'[X]_{\leq N}$ has a finite subcovering. This means that there is some $d \in \mathbb{N}$ such that

$$QM_{R'[X]}(g_1, ..., g_s) \cap R'[X]_{\leq N} = \vartheta_d(R'^{Y_d}) \cap R'[X]_{\leq N}.$$

Since we have by assumption that $h(Q, R') = QM_{R'[X]}(g_1, ..., g_s)$ and $h(Q, R') \cap R[X] = Q$ we get

$$Q \cap R[X]_{\leq N} = \vartheta_d(R^{Y_d}) \cap R[X]_{\leq N}.$$

As $(R[X]_{\leq N})_{N \in \mathbb{N}_0}$ forms a filtration of $R[X]$ into finite dimensional subspaces this proves the stability of Q.

Theorem 3.9 □

In Section 2.2 we solved in the univariate case the Membership Problem affirmatively for finitely generated quadratic modules for the case that the associated semialgebraic set consists just of finitely many points over arbitrary real closed fields. This will now with the help of Theorem 3.9 imply that such kind of quadratic modules are in fact stable.

Corollary 3.10
If $n = 1$, $g_1, ..., g_s \subseteq R[X]$ with $S = S(g_1, ..., g_s) \subseteq R$ finite then the quadratic module $Q = QM(g_1, ..., g_s)$ is stable.

Proof:
Let $S = \{a_1, ..., a_m\}$ for some $a_i \in R$ $(1 \leq i \leq m)$.
By Theorem 2.37 we know that Q is weakly semialgebraic.
If R' is an arbitrary real closed extension field of R and $Q_{R'} := QM_{R'[X]}(g_1, ..., g_s)$ then by Tarski $S(Q_{R'}) = \{x \in R' \mid g_i(x) \geq 0 \ (1 \leq i \leq s)\}$ and also the order of the polynomials g_i in the points a_j $(1 \leq i \leq s, 1 \leq j \leq m)$ stays the same when the polynomials are considered as elements of $R'[X]$. Thus the defining formula for Q and $Q_{R'}$ is by Theorem 2.35 the same which shows that $h(Q, R') = Q_{R'}$. Hence Theorem 3.9 implies that Q is stable.

Corollary 3.10 □

The following question arises:

Is there a quadratic module or a preordering which is weakly semialgebraic but not stable?

Actually we will see a class of examples very soon. This shows that the notion of stability is strictly stronger than the notion of definability.

We work out the example by describing the heir of a finitely generated quadratic module of $\mathbb{R}[X]$ with nonempty compact semialgebraic set in dimension 1.

We will use the correspondence between finitely generated quadratic modules of $\mathbb{R}[X] = \mathbb{R}[X_1]$ with nonempty bounded basic closed set and tuples $(\vec{\sigma}, \vec{\omega})$ where $\vec{\sigma} \in S_{vec}(m)$ for some $m \in \mathbb{N}$ and $\vec{\omega} \in \Omega_{vec}(\vec{\sigma})$ as explained in Corollary 2.27.

As in the definition of the generalized natural generators in Section 2.1 the complete vector of orders $\vec{\omega}_{\pm}(G) = (\omega_1, \omega_1^+, \omega_1^-, ..., \omega_m, \omega_m^+, \omega_m^-)$ associated to the finite set $G \subseteq \mathbb{R}[X]$ plays an important role.

We recall that for $S = S(G) = \bigcup_{i=1}^{m} [a_i, b_i]$ with $\vec{\sigma}(S) = (a_1, b_1, ..., a_m, b_m) \in S_{vec}(m)$ the entries of $\vec{\omega}_{\pm}(G)$ are

$$\omega_i = \infty,\ \omega_i^+ = k_{a_i}^+(G),\ \omega_i^- = k_{b_i}^-(G)$$

if $a_i < b_i$ are boundary points of $S \setminus S_{isol}$ whereas for isolated points $a_i = b_i$

$$\omega_i = k_{a_i}(G),\ \omega_i^+ = k_{a_i}(G) + 1,\ \omega_i^- = k_{a_i}(G) + 1 \qquad \text{if } a_i \text{ is of type A}$$

$$\omega_i = \max(k_{a_i}^+(G), k_{a_i}^-(G)) + 1,\ \omega_i^+ = k_{a_i}^+(G),\ \omega_i^- = k_{a_i}^-(G) \qquad \text{if } a_i \text{ is of type B}$$

$$\omega_i = k_{a_i}(G),\ \omega_i^+ = k_{a_i}^+(G),\ \omega_i^- = k_{a_i}(G) + 1 \qquad \text{if } a_i \text{ is of type C}$$

$$\omega_i = k_{a_i}(G),\ \omega_i^+ = k_{a_i}(G) + 1,\ \omega_i^- = k_{a_i}^-(G) \qquad \text{if } a_i \text{ is of type D}$$

For $0 \leq j < l \leq m + 1$ we define the polynomial

$$\pi_{j,l}(X) := \prod_{\substack{b_j < a_i < a_l \\ a_i = b_i}} (X - a_i)^{\omega_i} \in \mathbb{R}[X]$$

where $b_0 := -\infty, a_{m+1} := \infty$ and $\pi_{j,l}(X) = 1$ if there is no isolated point a_i with $b_j < a_i < a_l$.

Theorem 3.11
Let $n = 1$, $G = \{g_1, ..., g_s\} \subseteq \mathbb{R}[X]$ such that $\emptyset \neq S = S(G) = \bigcup_{i=1}^{m} [a_i, b_i]$ is a bounded subset of $\mathbb{R}$ and $\vec{\sigma}(S) = (a_1, b_1, ..., a_m, b_m) \in S_{vec}(m)$.
For $R \supseteq \mathbb{R}$ the heir of $Q = QM(g_1, ..., g_s) = PO(g_1, ..., g_s) \subseteq \mathbb{R}[X]$ is given by

$$h(Q, R) = PO_{R[X]}(H)$$

where

$$H = \{g_1(X), ..., g_s(X)\}$$

together with the following polynomials

- *for the least boundary point a of $S \setminus S_{isol}$, for every $l \in \{1, ..., m\}$ with $a_l \leq a$ and every positive infinitesimal $\mu \in R$*

$$\pi_{0,l}(X)(X - a_l + \mu)$$

- *for the largest boundary point b of $S \setminus S_{isol}$, for every $j \in \{1, ..., m\}$ with $b_j \geq b$ and every positive infinitesimal $\mu \in R$*

$$-(X - b_j - \mu)\pi_{j,m+1}(X)$$

- *for every pair of successive boundary points $b < a$ of $S \setminus S_{isol}$ - where we also treat $-\infty$ and ∞ as boundary points, for every pair $1 \leq j, l \leq m$ with $b \leq b_j < a_l \leq a$ but $b_j \neq -\infty$, $a_l \neq \infty$ and every positive infinitesimal $\mu \in R$*

$$(X - b_j - \mu)\pi_{j,l}(X)(X - a_l + \mu)$$

$$(X - b_j)^{\omega_j^-}\pi_{j,l}(X)(X - a_l + \mu)$$

$$(X - b_j - \mu)\pi_{j,l}(X)(X - a_l)^{\omega_l^+}$$

Proof:
By Theorem 2.20 Q is a weakly semialgebraic subset of $\mathbb{R}[X]$ where for some polynomial $f(X, Y) \in \mathbb{Z}[X, Y]$ and some $c \in \mathbb{R}^Y$ the truth of the defining formula $\vartheta_f(c)$ for $D(f, Q)$ expresses that $f(X, c) \geq 0$ on S and for all boundary points a of S the order $\mathrm{ord}_a(f(X, c))$ fulfills the conditions described in Theorem 2.18. The heir of Q on R is therefore given by

$$h(Q, R) = \{f(X, c) \mid f(X, Y) \in \mathbb{Z}[X, Y], c \in R^Y, R \models \vartheta_f(c)\}.$$

We clearly have the inclusion $h(Q, R) \supseteq PO_{R[X]}(H)$ since any of the listed polynomials $h \in H$ fulfills the conditions given by the formula $\vartheta_h(Y)$.

For the other inclusion we consider some polynomial $f = f(X, c) \in R[X]$ with $R \models \vartheta_f(c)$ and $c \in R^Y$. Similar to the proof of Theorem 1.6 we proceed by induction on the degree of $f = f(X, c)$.

If $\deg(f) = 0$ then we trivially have $f \in PO_{R[X]}(H)$ because then f is just some nonnegative element of R.

Let now $\deg(f) > 0$.
If $f \geq 0$ on R then $f \in \sum R[X]^2 \subseteq PO_{R[X]}(H)$.
Thus we suppose now that there is some $\gamma \in R$ with $f(\gamma) < 0$.
In order to use the induction hypothesis we will factorize $f = q \cdot p$ for some $q \in PO_{R[X]}(H)$ and some $p \in h(Q, R)$ which has lower degree than f. The properties of the polynomial p and the polynomial p itself will in any of the following cases follow from the fact that q divides f, $\deg(q) > 0$, $q|_S \geq 0$ and for every boundary point a of S either $\mathrm{ord}_a(q) = \mathrm{ord}_a(f)$ or $\mathrm{ord}_a(q) = 0$.

We denote for $1 \leq i \leq m$ the order of f in a_i with λ_i and the order of f in b_i with ρ_i where $\rho_i = \lambda_i$ if $a_i = b_i$.

If there is no zero of odd order of f in $]-\infty, \gamma[\cap[a_1, \infty[$ then either $\gamma < a_1$ or there are just isolated points of S on the left side of γ. By interpreting the empty product as 1 we know in both cases that $\prod\limits_{\substack{a_i<\gamma \\ a_i=b_i}} (X - a_i)^{\lambda_i} | f$ where the appearing even exponents λ_i are $\geq \omega_i$ because $f \in h(Q, R)$.
If there is no zero of odd order of f to the right of γ which is $\leq b_m$ then the same argumentation gives that $\prod\limits_{\substack{a_i>\gamma \\ a_i=b_i}} (X - a_i)^{\lambda_i} | f$.
If there is no zero of odd order of f in $]-\infty, \gamma[\cap[a_1, \infty[$ and $]\gamma, \infty[\cap]-\infty, b_m]$ which is just possible if $S = S_{isol}$ then we define

$$q := \prod_{\substack{1\leq i\leq m \\ a_i=b_i}} (X - a_i)^{\lambda_i} \in Q \subseteq PO_{R[X]}(H)$$

where $q \in Q$ because of Theorem 2.18 and $f \in h(Q, R)$.

Now we suppose that there is no zero of odd order of f in $]-\infty, \gamma[\cap[a_1, \infty[$ but a least zero α of odd order f in $]\gamma, \infty[\cap]-\infty, b_m]$.
We suppose that $b_{l-1} < \alpha \leq a_l$ for some $l \in \{1, ..., m\}$.
The nonnegativity of f on S implies that $a_l \leq a$ where a is the least boundary point of $S \setminus S_{isol}$.

If there are isolated points between γ and α then as before $\prod\limits_{\substack{\gamma<a_i<\alpha \\ a_i=b_i}} (X - a_i)^{\lambda_i} | f$ where $\lambda_i \geq \omega_i$ is even because of $f \in h(Q, R)$.

If $\alpha = a_l$ then we define

$$q := \left(\prod_{\substack{a_i < a_l \\ a_i = b_i}} (X - a_i)^{\lambda_i} \right) (X - a_l)^{\lambda_l} \in Q \subseteq PO_{R[X]}(H)$$

where $q \in Q$ again follows from $f \in h(Q, R)$ and Theorem 2.18.

If $\alpha = a_l - \epsilon$ for some positive $\epsilon \in R$ then

$$q := \left(\prod_{\substack{a_i < a_l \\ a_i = b_i}} (X - a_i)^{\lambda_i} \right) (X - a_l + \epsilon).$$

There is some positive infinitesimal $\mu \in R$ with $0 < \mu \leq \epsilon$.
By definition of H the polynomial $\pi_{0,l}(X)(X - a_l + \mu)$ is in H which gives that $\left(\prod_{\substack{a_i < a_l \\ a_i = b_i}} (X - a_i)^{\lambda_i} \right) (X - a_l + \mu) \in PO_{R[X]}(H)$ as it is $\pi_{0,l}(X)(X - a_l + \mu)$ multiplied by a square. This implies that $\left(\prod_{\substack{a_i < a_l \\ a_i = b_i}} (X - a_i)^{\lambda_i} \right) (X - a_l + \epsilon) \in PO_{R[X]}(H)$ because it can be written as $\left(\prod_{\substack{a_i < a_l \\ a_i = b_i}} (X - a_i)^{\lambda_i} \right) (X - a_l + \mu) + \underbrace{(\epsilon - \mu) \prod_{\substack{a_i < a_l \\ a_i = b_i}} (X - a_i)^{\lambda_i}}_{\in R[X]^2}$.

The case that there is a largest zero β of odd order of f in $]-\infty, \gamma[\cap [a_1, \infty[$ and no zero of odd order of f in $]\gamma, \infty[\cap]-\infty, b_m]$ can be solved similarly.

Now we consider the case that there is a largest zero β of odd order of f in $]-\infty, \gamma[\cap [a_1, \infty[$ and a least zero α of odd order of f in $]\gamma, \infty[\cap]-\infty, b_m]$.
We suppose that $b_j \leq \beta < a_{j+1}$ and $b_{l-1} < \alpha \leq a_l$ for some $0 \leq j < l \leq m + 1$.
The fact that $f|_S \geq 0$ implies the following. If there is a boundary point of $S \setminus S_{isol}$ in $]-\infty, \gamma[$ then $b \leq b_j$ where b is the largest boundary point of $S \setminus S_{isol}$. Similarly if there is a boundary point of $S \setminus S_{isol}$ to the right of γ then $a_l \leq a$ where a is the least boundary point of $S \setminus S_{isol}$ in $]\gamma, \infty[$.

If there are isolated points between β and α then we know because of $f \in h(Q, R)$ that $\prod_{\substack{\beta < a_i < \alpha \\ a_i = b_i}} (X - a_i)^{\lambda_i} | f$ where $\lambda_i \geq \omega_i$ is even.

If $\beta = b_j$ and $\alpha = a_l$ then we define

$$q := (X - b_j)^{\rho_j} \left(\prod_{\substack{b_j < a_i < a_l \\ a_i = b_i}} (X - a_i)^{\lambda_i} \right) (X - a_l)^{\lambda_l} \in Q \subseteq PO_{R[X]}(H)$$

which as above follows from $f \in h(Q, R)$ and Theorem 2.18.

If $\beta = b_j$ and $\alpha = a_l - \epsilon$ for some positive $\epsilon \in R$ then we define

$$q := (X - b_j)^{\rho_j} \left(\prod_{\substack{b_j < a_i < a_l \\ a_i = b_i}} (X - a_i)^{\lambda_i} \right) (X - a_l + \epsilon).$$

We can find some infinitesimal $\mu \in R$ with $0 < \mu \leq \epsilon$ such that by Lemma 1.5 $(X - b_j)(X - a_l + \epsilon) \in PO_{R[X]}((X - b_j)(X - a_l + \mu))$. Hence by multiplying this with the square $(X - b_j)^{\omega_j^- - 1} \pi_{j,l}(X)$ we also have that $(X - b_j)^{\omega_j^-} \pi_{j,l}(X)(X - a_l + \epsilon)$ is in $PO_{R[X]}((X - b_j)^{\omega_j^-} \pi_{j,l}(X)(X - a_l + \mu))$ which is a subset of $PO_{R[X]}(H)$ by definition of H. Since q is obtained from $(X - b_j)^{\omega_j^-} \pi_{j,l}(X)(X - a_l + \epsilon)$ by multiplication with a square we conclude that $q \in PO_{R[X]}(H)$.

If $\beta = b_j + \epsilon$ and $\alpha = a_l$ for some positive $\epsilon \in R$ then we similarly define

$$q := (X - b_j - \epsilon) \left(\prod_{\substack{b_j < a_i < a_l \\ a_i = b_i}} (X - a_i)^{\lambda_i} \right) (X - a_l)^{\lambda_l}$$

and get $q \in PO_{R[X]}(H)$.

If $\beta = b_j + \epsilon_1$ and $\alpha = a_l - \epsilon_2$ for some positive $\epsilon_1, \epsilon_2 \in R$ then we define

$$q := (X - b_j - \epsilon_1) \left(\prod_{\substack{b_j < a_i < a_l \\ a_i = b_i}} (X - a_i)^{\lambda_i} \right) (X - a_l + \epsilon_2).$$

We can find infinitesimals $\mu_1, \mu_2 \in R$ such that $0 < \mu_i \leq \epsilon_i$ for $i = 1, 2$. By Lemma 1.5 we have $(X - b_j - \epsilon_1)(X - a_l + \epsilon_2) \in PO_{R[X]}((X - b_j - \mu)(X - a_l + \mu))$ with $\mu := \min(\mu_1, \mu_2)$. Hence $(X - b_j - \epsilon_1)\pi_{j,l}(X)(X - a_l + \epsilon_2)$ is an element of $PO_{R[X]}((X - b_j - \mu)\pi_{j,l}(X)(X - a_l + \mu))$ which is a subset of $PO_{R[X]}(H)$ because $\pi_{j,l}(X) \in R[X]^2$ and $(X - b_j - \mu)\pi_{j,l}(X)(X - a_l + \mu) \in H$. Thus $q \in PO_{R[X]}(H)$ as it is equal to $(X - b_j - \epsilon_1)\pi_{j,l}(X)(X - a_l + \epsilon_2)$ multiplied by a square.

Theorem 3.11 □

If the set $S = S(G) \subseteq \mathbb{R}$ is bounded and does not have isolated points then we know by the results of Section 2.1 that the set of generalized natural generators $\mathrm{Nat}(\vec{\sigma}(S), \vec{\omega}(G))$ for which we by Corollary 2.28 have

$$QM_{\mathbb{R}[X]}(G) = QM_{\mathbb{R}[X]}(\mathrm{Nat}(\vec{\sigma}(S), \vec{\omega}(G))) = PO_{\mathbb{R}[X]}(\mathrm{Nat}(\vec{\sigma}(S), \vec{\omega}(G)))$$

are given by

$$\mathrm{Nat}(\vec{\sigma}(S), \vec{\omega}(G)) = \{(X - b_i)^{\omega_i^-}(X - a_{i+1})^{\omega_{i+1}^+} \mid 0 \leq i \leq m\}$$

with $b_0 := -\infty, a_{m+1} := \infty, (X - (-\infty))^{\omega_0^-} := 1$ and $(X - \infty)^{\omega_{m+1}^+} := -1$.
In this case the polynomials needed in addition to $\mathrm{Nat}(\vec{\sigma}(S), \vec{\omega}(G))$ in order to generate the heir of $QM_{\mathbb{R}[X]}(G)$ are obtained from the generalized natural generators by varying one or both factors infinitesimally in the following way.

Corollary 3.12
Let $n = 1$, $G = \{g_1, ..., g_s\} \subseteq \mathbb{R}[X]$ such that $\emptyset \neq S = S(G) = \bigcup_{i=1}^{m} [a_i, b_i]$ is a bounded subset of $\mathbb{R}$ without isolated points.
Let $\vec{\sigma}(S) = (a_1, b_1, ..., a_m, b_m) \in S_{vec}(m)$ and $\vec{\omega}_{\pm}(G) = (\omega_1, \omega_1^+, \omega_1^-, ..., \omega_m, \omega_m^+, \omega_m^-)$ associated to $\vec{\omega}(G) \in \Omega_{vec}(\vec{\sigma}(S))$.
For a real closed field $R \supseteq \mathbb{R}$ we have

$$h(QM_{\mathbb{R}[X]}(G), R) = PO_{R[X]}(H)$$

with

$$\begin{aligned} H = \quad & \{(X - b_i)^{\omega_i^-}(X - a_{i+1})^{\omega_{i+1}^+} \mid 0 \leq i \leq m\} \\ \cup \quad & \{(X - b_i - \mu)(X - a_{i+1} + \mu) \mid 1 \leq i \leq m-1, 0 < \mu \in R \text{ infinitesimal}\} \\ \cup \quad & \{(X - b_i)^{\omega_i^-}(X - a_{i+1} + \mu) \mid 0 \leq i \leq m-1, 0 < \mu \in R \text{ infinitesimal}\} \\ \cup \quad & \{(X - b_i - \mu)(X - a_{i+1})^{\omega_{i+1}^+} \mid 1 \leq i \leq m, 0 < \mu \in R \text{ infinitesimal}\} \end{aligned}$$

where $b_0 := -\infty, a_{m+1} := \infty, (X - (-\infty))^{\omega_0^-} := 1$ and $(X - \infty)^{\omega_{m+1}^+} := -1$.

As a consequence of the explicit description of heirs in Theorem 3.11 we prove the following equivalence of stability and saturation.

Corollary 3.13
Let $n = 1$, $g_1, ..., g_s \in \mathbb{R}[X]$ such that $S = S(g_1, ..., g_s)$ is a bounded subset of $\mathbb{R}$ with nonempty interior.
Then the following are equivalent for $Q = QM(g_1, ..., g_s) = PO(g_1, ..., g_s)$:

i) Q is stable.

ii) Q is saturated.

Proof:

$i) \Rightarrow ii)$: We suppose that Q is not saturated. By Corollary 1.7 there is at least one natural generator g of $\mathcal{P}(S)$ which is not contained in Q. Since the natural generators of $\mathcal{P}(S)$ have coefficients in $\mathbb{R}$ and the interior of S is not empty we can proceed as in the proof of the implication $\Rightarrow$ in Proposition 3.7 and get that $g + \mu \notin QM_{R[X]}(g_1, ..., g_s)$ if $R \supset \mathbb{R}$ real closed and $\mu > 0$ some infinitesimal element of R.
Without loss of generality let $g = (X - b)(X - a)$ for some $a, b \in S$ with $b < a$ and $]b, a[\cap S = \emptyset$. Since $g + \mu$ is of the form $(X - b - \widetilde{\mu})(X - a + \widetilde{\mu})$ for some positive infinitesimal element $\widetilde{\mu}$ of R this proves by Theorem 3.11 that for every real closed extension R of $\mathbb{R}$ with $R \neq \mathbb{R}$ we have $h(Q, R) \neq QM_{R[X]}(g_1, ..., g_s)$. By Theorem 3.9 this implies that Q is not stable and we get a contradiction to the assumption.
Hence Q has to be saturated.

$ii) \Rightarrow i)$: If Q is saturated then by Corollary 1.7 all natural generators of $\mathcal{P}(S)$ are in Q. Let R be a real closed extension of $\mathbb{R}$. With the help of Lemma 1.5 and multiplication by appropriate squares we see that every element of the set H which generates $h(Q, R)$ according to Theorem 3.11 is in $PO_{R[X]}(g_1, ..., g_s)$. Thus $h(Q, R) = PO_{R[X]}(g_1, ..., g_s)$ by Theorem 3.11. Since R was an arbitrary real closed extension of $\mathbb{R}$ we get by Theorem 3.9 that Q is stable.

Corollary 3.13 □

We note that the equivalence of the corollary is no longer true if we are working in higher dimensions. Scheiderer gave an example of a finitely generated saturated preordering in dimension 2 ([S5] Corollary 3.3) which has compact semialgebraic set and is thus by another result of Scheiderer ([S3] Theorem 5.4) not stable.

Since by Theorem 2.20 every finitely generated quadratic module over $\mathbb{R}$ in the one dimensional case is weakly semialgebraic Corollary 3.13 gives us a lot of examples of definable but not stable quadratic modules.

In particular the Stengle preordering $P := QM((1 - X^2)^3) \subseteq \mathbb{R}[X]$ is not stable since it does not contain the polynomial $1 - X^2$ as explained before Proposition 3.7. Gilbert Stengle proved in his paper [St2] that P is not stable by giving explicit lower bounds for the degrees of the sums of squares appearing in a representation of $1 - X^2 + \epsilon$ ($\epsilon \in \mathbb{R}, \epsilon > 0$) as an element of P.

Before we use these explicit bounds to prove that the heir of P on some real closed extension field is not finitely generated we state a general result which tells us how we can get information about degree bounds by looking at real closed extension fields. This once more shows that it is important to consider arbitrary real closed fields even if one is just interested in the case $R = \mathbb{R}$.

Proposition 3.14
We consider polynomials $g_1(X,Y), ..., g_s(X,Y) \in R[X,Y]$ where R is a real closed field, $X = (X_1, ..., X_n)$ and Y a finite tuple of variables.
Then the following are equivalent for $f(X,Y) \in R[X,Y]$, $R' \supseteq R$ real closed and $c' \in R'^Y$:

i) $f(X,c') \in QM_{R'[X]}(g_1(X,c'), ..., g_s(X,c'))$

ii) There is a semialgebraic set $A = \psi(R^Y) \subset R^Y$ with $\psi(Y) \in tp(c'/R)$ such that there is some $d \in \mathbb{N}$ with the property that for every $a \in A$ the polynomial $f(X,a) \in QM_{R[X]}(g_1(X,a), ..., g_s(X,a))$ has a representation where the sums of squares are of degree at most d .

Proof:
Without loss of generality we suppose that d is even. We describe a general representation of an element of $QM_{R[X]}(g_1(X,Y), ..., g_s(X,Y))$ where the appearing sums of squares have degree at most d with respect to X by

$$t_d(X,Y,Z) = \sum_{i=0}^{s} \left(\sum_{j=1}^{|\Lambda(\frac{d}{2})|} F_{\frac{d}{2}}(X, Z_{ij})^2 \right) g_i(X,Y)$$

where $g_0 := 1$, $F_{\frac{d}{2}}(X, Z_{ij})$ denotes the general polynomial of degree $\frac{d}{2}$ with respect to X and $Z = (Z_{01}, ..., Z_{s|\Lambda(d)|})$ (see Lemma 1.9).

ii) $\Rightarrow$ *i)* : Let $\varphi(Y)$ be the $L(R)$-formula $\exists Z(\forall X f(X,Y) = t_d(X,Y,Z))$.
By assumption $R \models \forall Y(\psi(Y) \to \varphi(Y))$ which gives by Tarski that also $R' \models \forall Y(\psi(Y) \to \varphi(Y))$. Since $A = \psi(R^Y)$ with $\psi(Y) \in tp(c'/R)$ we have $R' \models \psi(c')$ and thus $R' \models \varphi(c')$ which gives the desired representation.

i) $\Rightarrow$ *ii)* : By assumption we have $f(X,c') = t_d(X,c',b')$ for some $d \in \mathbb{N}$ and some $b' \in R'^Z$.
Thus $R' \models \underbrace{(\exists Z(\forall X f(X,Y) = t_d(X,Y,Z)))}_{=:\psi(Y) \in \mathrm{Fml}L(R)}[c']$. Then $A := \psi(R^Y) \subseteq R^Y$ is semialgebraic and $\psi(Y) \in tp(c'/R)$. For elements a of A we have the desired representations for $f(X,a)$ as $R \models \psi(a)$.

Prop. 3.14 $\square$

Stengle used approximation theory to get estimates for the degrees of the sums of squares needed in some representation of $f(X,\epsilon) = 1 - X^2 + \epsilon$ as an element of $QM_{\mathbb{R}[X]}((1-X^2)^3)$. For the convenience of the reader we give this reasoning in the slightly more general case of $QM_{\mathbb{R}[X]}((a-X^2)^3)$ for some given $0 < a \leq 1$.

Proposition 3.15 (Stengle, [St2] Theorem 4)
Let $n = 1$, $a, \epsilon \in \mathbb{R}$ with $0 < a \leq 1$, $0 < \epsilon < 1$ and $N(a,\epsilon)$ be the least integer which bounds the degree of some sum of squares appearing in a representation of $f(X,\epsilon) = a - X^2 + \epsilon$ as an element of $QM_{\mathbb{R}[X]}((a-X^2)^3)$.
Then there is a constant $C > 0$ such that $N(a,\epsilon) \geq \sqrt{\frac{a}{\epsilon}} \cdot C$

Proof:
For abbreviation we define $g(X) := (a - X^2)^3$.
Since $S(g) = [-\sqrt{a}, \sqrt{a}] \subseteq \mathbb{R}$ is compact and $f|_{S(g)} > 0$ we get by Schmüdgens Theorem (Corollary 2.24) that $f \in QM_{\mathbb{R}[X]}(g) = PO_{\mathbb{R}[X]}(g)$.
Thus there are some $\sigma_0, \sigma_1 \in \sum \mathbb{R}[X]^2$ such that

$$a - X^2 + \epsilon = \sigma_0 + \sigma_1(a - X^2)^3 \tag{1}$$

We give a lower bound for the degree N of σ_1 which is also a lower bound for $N(a,\epsilon)$ since $N(a,\epsilon) \geq N$.
Because the sums of squares σ_0 and σ_1 are nonnegative on $\mathbb{R}$ we get from (1)

$$\sigma_1(x)(a-x^2)^3 = a - x^2 + \epsilon \underbrace{-\sigma_0(x)}_{\leq 0 \text{ on } \mathbb{R}} \leq a - x^2 + \epsilon \quad \forall x \in \mathbb{R} \tag{2}$$

This implies that $\sigma_1(x) \leq \frac{1}{(a-x^2)^2} + \frac{\epsilon}{(a-x^2)^3} \quad \forall x \in]-\sqrt{a}, \sqrt{a}[$.
If we take some $r \in \mathbb{R}$ with $0 < r < a$ (this element will be chosen more properly later on) then $[-\sqrt{a-r}, \sqrt{a-r}] \subseteq]-\sqrt{a}, \sqrt{a}[$. Thus

$$\max_{|x| \leq \sqrt{a-r}} \sigma_1(x) \leq \frac{1}{r^2} + \frac{\epsilon}{r^3} \tag{3}$$

From (2) we can conclude

$$\epsilon \geq (x^2 - a) - (x^2 - a)^3 \sigma_1(x) \quad \forall x \in \mathbb{R} \tag{4}$$

We suppose now that $\sigma_1(X)$ is a polynomial of degree N and use the extremal property of the Tschebyscheff polynomial $T_N(X)$ of degree N ([R] Theorem 1.10) which says that for some $\delta \geq 1$

$$\max_{|x| \leq \delta} |\sigma_1(x)| \leq T_N(\delta) \max_{|x| \leq 1} |\sigma_1(x)| \tag{5}$$

We can estimate the maximal value of $|\sigma_1| = \sigma_1$ on the interval $[-\frac{\sqrt{a}}{\sqrt{1-\frac{r}{a}}}, \frac{\sqrt{a}}{\sqrt{1-\frac{r}{a}}}]$ as follows:

$$\begin{aligned}\max_{x^2\le\frac{a}{1-\frac{r}{a}}}\sigma_1(x) &= \max_{x^2\le\frac{1}{1-\frac{r}{a}}}\sigma_1(\sqrt{a}x) = \max_{x^2\le\frac{1}{(1-\frac{r}{a})^2}}\sigma_1(\sqrt{a(1-\tfrac{r}{a})}x)\\ &\overset{(5)}{\le} T_N(\tfrac{1}{1-\frac{r}{a}})\max_{x^2\le 1}\sigma_1(\sqrt{a(1-\tfrac{r}{a})}x) = T_N(\tfrac{1}{1-\frac{r}{a}})\max_{x^2\le a-r}\sigma_1(x)\end{aligned}$$

With (3) this gives

$$\max_{x^2\le\frac{a}{1-\frac{r}{a}}}\sigma_1(x)\le T_N(\frac{1}{1-\frac{r}{a}})(\frac{1}{r^2}+\frac{\epsilon}{r^3})$$

If $x^2\le\frac{a}{1-\frac{r}{a}}$ this implies that

$$-\sigma_1(x)\ge -T_N(\frac{1}{1-\frac{r}{a}})(\frac{1}{r^2}+\frac{\epsilon}{r^3})$$

and by (4)

$$\epsilon\ge(x^2-a)-(x^2-a)^3T_N(\frac{1}{1-\frac{r}{a}})(\frac{1}{r^2}+\frac{\epsilon}{r^3}).$$

With the help of the expression $T_N(X)=\frac{1}{2}((X+\sqrt{X^2-1})^N+(X-\sqrt{X^2-1})^N)$ ([V-L] p.76 1.) we get $-T_N(\frac{1}{1-\frac{r}{a}})\ge-(\frac{1+\sqrt{\frac{r}{a}}}{\sqrt{1-\frac{r}{a}}})^N$. This gives

$$\epsilon\ge(x^2-a)-(x^2-a)^3(\frac{1}{r^2}+\frac{\epsilon}{r^3})(\frac{1+\sqrt{\frac{r}{a}}}{\sqrt{1-\frac{r}{a}}})^N$$

for all x with $x^2\le\frac{a}{1-\frac{r}{a}}$.
Now we choose in particular $r=\frac{a}{N^2}$ (which fulfills $0<r<a$) and let $x^2=a+\frac{a}{2N^2}$ (such that $x^2\le\frac{a}{1-\frac{r}{a}}$) and get

$$\epsilon\ge\frac{a}{2N^2}-\frac{a^3}{8N^6}(\frac{N^4}{a^2}+\frac{\epsilon N^6}{a^3})(\frac{1+\frac{1}{N}}{\sqrt{1-\frac{1}{N^2}}})^N=\frac{a}{2N^2}-\frac{1}{8}(\frac{a}{N^2}+\epsilon)(\frac{1+\frac{1}{N}}{\sqrt{1-\frac{1}{N^2}}})^N$$

Hence

$$N^2\epsilon\ge\frac{a}{2}-\frac{1}{8}(a+N^2\epsilon)(e+o(1))$$

where the last bracket is independent from a. This can equivalently be written as

$$(1-\frac{e}{8}+o(1))N^2\epsilon\ge a(\frac{1}{2}-\frac{1}{8}(e+o(1)))$$

Therefore we get for N big enough that $N^2\epsilon\ge a\cdot C$ for some constant $C>0$ and hence $N\ge\sqrt{\frac{a}{\epsilon}}\cdot C$ which is the desired bound.

Prop. 3.15 □

The advantage of proving a bound in dependence of a is that it shows the following. If we consider the polynomial

$$f(X,Y) := Y - X^2 + Y^p \in \mathbb{Z}[X,Y]$$

for some fixed $p \in \mathbb{N}$, $p > 1$, then for some $0 < a \leq 1$ and $\epsilon = a^p$ the lower bound from the previous proposition becomes $N(a, a^p) \geq \sqrt{a^{1-p}} \cdot C$ which tends to zero for $a \to 0$. This means that for given $\delta > 0$ there is no global bound d such that for all $a \in]0, \delta[$ the polynomial $f(X, a)$ has a representation as an element of $QM_{\mathbb{R}[X]}((a - X^2)^3)$ with sums of squares of degree at most d. This non-existence of the degree bound in turn means by Proposition 3.14 that

$$\mu - X^2 + \mu^p \notin QM_{R[X]}((\mu - X^2)^3)$$

where $R \supset \mathbb{R}$ a real closed field and $0 < \mu \in R$ infinitesimal.

Now we use the lower bound from Proposition 3.15 together with the upper bound from [St2] to show that the heir of the Stengle preordering is not finitely generated.

Proposition 3.16
Let $n = 1$, $P = QM((1 - X^2)^3) \subseteq \mathbb{R}[X]$ and $R \supseteq \mathbb{R}$ a real closed extension field which contains infinitesimal elements.
Then the heir $h(P, R)$ of P on R is not finitely generated.

Proof:
Corollary 3.12 tells us that the heir of P on some real closed extension $R \supseteq \mathbb{R}$ is given by $h(P, R) = PO_{R[X]}(H)$ with

$$\begin{aligned} H = \quad & \{(1 - X^2)^3\} \\ \cup \quad & \{1 + \mu - X \mid \mu \in R, \mu > 0 \text{ infinitesimal } \} \\ \cup \quad & \{1 + \mu + X \mid \mu \in R, \mu > 0 \text{ infinitesimal } \}. \end{aligned}$$

In order to show that the heir is not finitely generated we suppose that we would just need finitely many of the polynomials of H.

Let $\mu > 0$ be the smallest infinitesimal element of R such that $1 + \mu - X \in H$ or $1 + \mu + X \in H$. Without loss of generality we suppose that

$$h(P, R) = PO_{R[X]}((1 - X^2)^3, 1 + \mu - X, 1 + \mu + X).$$

We show that this is not possible by showing that $1 - X^2 + \mu^k$ which is clearly in $h(P, R)$ is not in $PO_{R[X]}((1 - X^2)^3, 1 + \mu - X, 1 + \mu + X)$ for $k \in \mathbb{N}$ big enough.

For suppose that $1 - X^2 + \mu^k \in PO_{R[X]}((1 - X^2)^3, 1 + \mu - X, 1 + \mu + X)$ then there would exist some $\delta_0 \in \mathbb{R}$ such that $\forall \delta \in]0, \delta_0[$ the polynomial $1 - X^2 + \delta^k$ would have

a representation as an element of $PO_{\mathbb{R}[X]}((1-X^2)^3, 1+\delta-X, 1+\delta+X)$ with the property that the degree of the sums of squares appearing is fixed by some $d \in \mathbb{N}$ (Proposition 3.14). This means that

$$\begin{aligned} 1-X^2+\delta^k = \; & s_0 + s_1(1-X^2)^3 + s_2(1+\delta-X) + s_3(1+\delta+X) + \\ & + s_4(1-X^2)^3(1+\delta+X) + s_5(1-X^2)^3(1+\delta-X) + \\ & + s_6(1+\delta-X)(1+\delta+X) + s_7(1-X^2)^3(1+\delta-X)(1+\delta+X)(*) \end{aligned}$$

where $s_i \in \sum \mathbb{R}[X]^2$ $(0 \leq i \leq 7)$ of degree less or equal to d.
Now we use a representation

$$1+\delta-X = \sigma_0 + \sigma_1(1-X^2)^3$$

and

$$1+\delta+X = \widetilde{\sigma}_0 + \widetilde{\sigma}_1(1-X^2)^3$$

of these both generators lying itself in the preordering $QM_{\mathbb{R}[X]}((1-X^2)^3)$. By a result of Stengle ([St2] Theorem 5) we have an upper bound on the degree of the $\sigma_i, \widetilde{\sigma}_i$ $(i = 0, 1)$ which is given by $C_1\sqrt{1+\frac{2}{\delta}}\log(1+\frac{2}{\delta})$ for some constant C_1. By substituting these representations into $(*)$ we obtain

$$1-X^2+\delta^k = \tau_0 + \tau_1(1-X^2)^3$$

with $\tau_0, \tau_1 \in \sum \mathbb{R}[X]^2$ and more exactly

$$\tau_1 = s_1 + s_2\sigma_1 + s_3\tilde{\sigma}_1 + s_4\tilde{\sigma}_0 + s_5\sigma_0 + s_6\sigma_1\tilde{\sigma}_0 + s_6\sigma_0\tilde{\sigma}_1 + s_7\sigma_0\tilde{\sigma}_0 + s_7\sigma_1\tilde{\sigma}_1(1-X^2)^6.$$

This implies that $\deg(\tau_1) \leq d + 12 + 2C_1\sqrt{1+\frac{2}{\delta}}\log(1+\frac{2}{\delta})$.
If δ is small enough (the degree d stays fixed!) then this degree is less or equal than

$$3C_1\sqrt{1+\frac{2}{\delta}}\log(1+\frac{2}{\delta}) \leq \underbrace{3C_1}_{=:\widetilde{C}}(1+\frac{2}{\delta}).$$

On the other hand we know by the result about the lower bound (Proposition 3.15) that the degree of τ_1 has to be greater than or equal to $C\frac{1}{\sqrt{\delta^k}}$.
Hence

$$C\frac{1}{\sqrt{\delta^k}} \leq \deg(\tau_1) \leq \widetilde{C}(1+\frac{2}{\delta}).$$

By choosing k big enough such that $\widetilde{C}(1+\frac{2}{\delta}) < C\frac{1}{\sqrt{\delta^k}}$ we get a contradiction which proves that $h(P,R)$ is not generated by just finitely many elements of H.

If we would have other finitely many generators than those given by the set H, say $h(P,R) = PO_{R[X]}(q_1, ..., q_r)$ then we could express the polynomials q_i by finitely many of the elements of H so we could as above deduce a contradiction. This shows that $h(P,R)$ is not finitely generated.

Prop. 3.16 □

3.3 Traces of heirs

In this section we do not consider arbitrary extensions of real closed fields. We concentrate on tame extensions $R' \supseteq R$ as defined in Section 1.3 (Definition 1.17).

If the extension $R' \supseteq R$ is tame then the embedding $R \hookrightarrow \mathcal{O} \to \mathcal{O}/\mathfrak{m}$ is onto and the place $\lambda : \mathcal{O} \to R$ is called the standard part map.

We deal with the question: What do we get when we look at the images of $h(Q, R')$ and $H(Q, R')$ under λ if Q is a quadratic module of $R[X] = R[X_1, ..., X_n]$?

Interestingly we get

$$Q^{(\ddagger)} := \{f \in R[X] \mid \exists q \in R[X]\ \forall \epsilon > 0\ f + \epsilon q \in Q\}$$

which is an object introduced by Kuhlmann, Marshall and Schwartz in their paper [K-M-S] inspired by the (‡)-condition of [K-M].

$Q^{(\ddagger)}$ plays an important role in the solution of the moment problem because it approximates the closure $\overline{Q}$ of Q with respect to the natural linear topology. If $R = \mathbb{R}$ we know by Haviland ([H1], [H2]) that Q solves the moment problem for S if and only if $\mathcal{P}(S) = \overline{Q}$. By the fact that $Q^{(\ddagger)} \subseteq \overline{Q}$ the result $\mathcal{P}(S) = Q^{(\ddagger)}$ also implies that Q solves the moment problem for S.

For our result we need the following lemma which is from [K-M-S].

Lemma 3.17 (Kuhlmann,Marshall,Schwartz, [K-M-S] Prop. 1.4)
Let $Q \subseteq R[X]$ be a quadratic module. Then

$$Q^{(\ddagger)} = \bigcup_{d \in \mathbb{N}_0} \overline{Q \cap R[X]_{\leq d}}$$

where the closure is taken in the euclidian norm.

Proof:
For the inclusion $\subseteq$ we take some $f \in Q^{(\ddagger)}$ which means that there is some $q \in R[X]$ such that for every $\epsilon > 0$ $f + \epsilon q \in Q$. With $d := \max\{\deg f, \deg q\}$ this means that $f + \epsilon q \in Q \cap R[X]_{\leq d}$ for every $\epsilon > 0$ and thus $f \in \overline{Q \cap R[X]_{\leq d}}$.

For the other inclusion we consider an element $f \in \overline{Q \cap R[X]_{\leq d}}$ for some $d \in \mathbb{N}_0$. Since the identity $f = (\frac{f+1}{2})^2 - (\frac{f-1}{2})^2$ for every polynomial $f \in R[X]_{\leq d}$ implies that $R[X]_{\leq d} = (Q \cap R[X]_{\leq d}) - (Q \cap R[X]_{\leq d})$ we know that $\mathrm{int}(Q \cap R[X]_{\leq d}) \neq \emptyset$ because $Q \cap R[X]_{\leq d}$ contains a basis of $R[X]_{\leq d}$. By choosing some $q \in \mathrm{int}(Q \cap R[X]_{\leq d})$ we get that $\lambda f + (1 - \lambda)q \in Q \cap R[X]_{\leq d}$ for every $0 < \lambda < 1$ which means nothing else than $f + \epsilon q \in Q \cap R[X]_{\leq d} \subseteq Q$ for every $\epsilon > 0$.

Lemma 3.17 □

Theorem 3.18
For every quadratic module $Q \subseteq R[X]$ and any tame extension $R' \supseteq R$ we have

$$\lambda(h(Q,R') \cap \mathcal{O}[X]) = \lambda(H(Q,R') \cap \mathcal{O}[X]) = Q^{(\ddagger)}.$$

Proof:
We will show the following inclusions

$$Q^{(\ddagger)} \subseteq \lambda(h(Q,R') \cap \mathcal{O}[X])$$

and

$$\lambda(H(Q,R') \cap \mathcal{O}[X]) \subseteq Q^{(\ddagger)}$$

which will prove the claim as $\lambda(h(Q,R') \cap \mathcal{O}[X]) \subseteq \lambda(H(Q,R') \cap \mathcal{O}[X])$.

Proof of the first inclusion:
For $f = f(X,c) \in Q^{(\ddagger)}$ there is some $q = q(X,b) \in R[X]$ such that $f + \epsilon q \in Q$ for every $\epsilon > 0$. We define $g(X,T,Y,Z) := f(X,Y) + Tq(X,Z) \in \mathbb{Z}[X,T,Y,Z]$. Then for every $\mu \in \mathcal{O}$, $\mu > 0$ we have $g(X,\mu,c,b) \in h(Q,R') \cap \mathcal{O}[X]$ because with $\varphi_g(T,Y,Z) \in \mathrm{Fml}L(R)$ defined as

$$T > 0 \wedge \forall X(g(X,T,Y,Z) = f(X,Y) + Tq(X,Z)) \wedge Y = c \wedge Z = b$$

$R' \models \varphi_g(\mu,c,b)$ and $\varphi_g(R^{T,Y,Z}) \subseteq D_R(g,Q)$.
If $0 < \mu \in \mathfrak{m}$ then $\lambda(g(X,\mu,c,b)) = \lambda(f(X,c)) + \lambda(\mu)\lambda(q(X,b)) = f(X,c)$.

Proof of the second inclusion:
We take some polynomial $f(X,Y) \in \mathbb{Z}[X,Y]$ and some $c' \in R'^Y$ such that

$$\lambda(f(X,c')) \notin \overline{Q \cap R[X]_{\leq d}}$$

where d is the degree of $f(X,Y)$ with respect to X and show that $f(X,c')$ is not in $H(Q,R')$. This will by the previous lemma prove the second inclusion.
We can suppose that $c' \in \mathcal{O}^Y$. The fact that $\lambda(f(X,c'))$ is not in the closure of $Q \cap R[X]_{\leq d}$ with respect to the euclidian norm implies that there is a continuous semialgebraic function $s : R[X]_{\leq d} \to R$ such that $s(\lambda(f(X,c'))) = 1$ and $s(Z) = 0$ where Z is some semialgebraic set that contains $\overline{Q \cap R[X]_{\leq d}}$.
Because of the continuity of s we have $s \circ \lambda = \lambda \circ s_{R'}$ on $\mathcal{O}^Y$ where $s_{R'} : R'[X]_{\leq d} \to R'$ is the semialgebraic function defined by the same formula as s.
Thus $1 = s(\lambda(f(X,c'))) = \lambda(s_{R'}(f(X,c')))$. Hence we have $s_{R'}(f(X,c')) \geq \frac{1}{2}$ which means that $R' \models \varphi(c')$ with $\varphi(Y) := \forall X(s(f(X,Y)) \geq \frac{1}{2})$. But in R we have by definition of s that $\varphi(R^Y) \cap D_R(f,Q) = \emptyset$ which shows that $f(X,c') \notin H(Q,R')$.

We note that this second inclusion can also be proved if we just consider some subset K of $R[X]$ instead of a quadratic module Q.

Theorem 3.18 □

This result together with the explicit description of heirs which we gave in the last section implies that $Q^{(\ddagger)}$ is equal to the saturation of Q in the one dimensional case for a finitely generated quadratic module $Q \subseteq \mathbb{R}[X]$ whose associated semialgebraic set is bounded.

Corollary 3.19
Let $n = 1$, $g_1, ..., g_s \in \mathbb{R}[X]$ such that $S = S(g_1, ..., g_s)$ is a bounded subset of $\mathbb{R}$. Then we have for $Q = QM(g_1, ..., g_s) = PO(g_1, ..., g_s) \subseteq \mathbb{R}[X]$

$$Q^{(\ddagger)} = \mathcal{P}(S).$$

Proof:
Let R' be an arbitrary real closed extension of $R = \mathbb{R}$. Since $R = \mathbb{R}$ the extension is tame. Therefore we know by Theorem 3.18 that $Q^{(\ddagger)} = \lambda(h(Q, R') \cap \mathcal{O}[X])$. By Theorem 3.11 we know how the heir looks like and see that the natural generators of S are in $\lambda(h(Q, R') \cap \mathcal{O}[X])$. Thus by Corollary 1.7 $Q^{(\ddagger)} = \mathcal{P}(S)$ where we use Proposition 2.33 which ensures that $Q^{(\ddagger)}$ is finitely generated.

Corollary 3.19 □

The fact that $\mathcal{P}(S) = Q^{(\ddagger)}$ can also be deduced from the Theorem of Schmüdgen (Corollary 2.24) which translates to

$$\mathcal{P}(S) = \{f \in \mathbb{R}[X] \mid \forall \epsilon > 0 \ f + \epsilon \in Q\}.$$

On the other side we clearly have

$$\{f \in \mathbb{R}[X] \mid \forall \epsilon > 0 \ f + \epsilon \in Q\} \subseteq Q^{(\ddagger)} \subseteq \mathcal{P}(S).$$

In the special case that the quadratic module Q is stable and the support of Q is zero we can deduce that $Q = Q^{(\ddagger)}$ which together with the previous Corollary gives another proof of the Implication $i) \Rightarrow ii)$ in Corollary 3.13.

Proposition 3.20
If $Q \subseteq R[X] = R[X_1, ..., X_n]$ is a finitely generated stable quadratic module and $supp(Q) = \{0\}$ then

$$Q = Q^{(\ddagger)}.$$

Proof:
Let $Q = QM(g_1, ..., g_s)$ for some $\{g_1, ..., g_s\} \subseteq R[X]$.
By definition of $Q^{(\ddagger)}$ we have $Q \subseteq Q^{(\ddagger)}$.
For the other inclusion let $R' \supseteq R$ be the real closure of the field $R(\mu)$ obtained

from R by adjoining a transcendental element μ provided with $0 < \mu << 1$. This means R' is a real closed field which contains an element μ which is infinitesimal with respect to R, i.e. $\frac{1}{\mu} > R$.
We consider an arbitrary element $f \in Q^{(\ddagger)}$ and show that $f \in Q$.
By definition of $Q^{(\ddagger)}$ there is some $q \in R[X]$ such that $f + \epsilon q \in Q$ for every $\epsilon > 0$. Since Q is stable this implies by Proposition 3.14 that $f + \mu q$ lies in the quadratic module $QM_{R'[X]}(G)$. Thus we have a representation of the form

$$f + \mu q = \sum_{i=0}^{r} \sigma_i g_i \qquad (*)$$

with $g_0 := 1$ and $\sigma_i \in \sum R'[X]^2$ $(0 \leq i \leq s)$. Let $\sigma_i = \sum_{j=1}^{m_i} h_{ij}(X)^2$ for some $m_i \in \mathbb{N}_0$ and $h_{ij} \in R'[X]$ $(1 \leq j \leq m_i, 0 \leq i \leq s)$.
If some of the coefficients of the sums of squares σ_i are not in $\mathcal{O}$, let c be the coefficient with the lowest valuation $v(c) < 0$. By dividing through c^2 we get an equation of the form $\frac{1}{c^2}(f + \mu q) = \sum_{i=0}^{s} \sum_{j=1}^{m_i} (\frac{1}{c} h_{ij}(X))^2 g_i$. We apply λ to both sides of this equation and get 0 on the left hand side but at least one of the coefficients on the right hand side is 1. This gives us a nontrivial representation of 0 in the residue field which is isomorphic to R. This is not possible as the support of Q is $\{0\}$.
Thus all the coefficients of the sums of squares σ_i $(0 \leq i \leq s)$ lie in the valuation ring $\mathcal{O}$. Hence again by applying the residue map to $(*)$ we get $f = \sum_{i=0}^{s} \tilde{\sigma}_i g_i$ with $\tilde{\sigma}_i \in \sum R[X]^2$ $(0 \leq i \leq s)$ which is a representation of f as an element of Q.

Prop. 3.20 □

4 Towards the solution of the Membership Problem over $R = \mathbb{R}((t^{\mathbb{R}}))$

In this chapter R is always a real closed field which contains $\mathbb{R}$.

For ease of notation we often write λf instead of $\lambda(f)$ and λx instead of $\lambda(x)$ if $f \in \mathcal{O}[X]$, $x \in \mathcal{O}$ and λ denotes the residue map.

Since for every extension field R of $\mathbb{R}$ the extension $R \supseteq \mathbb{R}$ is tame as defined in 1.17 we are dealing with tame extensions in this chapter. Later on we will specialize to the field of formal series $R = \mathbb{R}((t^{\mathbb{R}}))$ in order to have $\operatorname{Spec} \mathcal{O} = \{\{0\}, \mathfrak{m}\}$ but now the real closed field $R \supseteq \mathbb{R}$ is still arbitrary.

In Section 2.1 we solved the Membership Problem affirmatively for finitely generated quadratic modules of $\mathbb{R}[X]$ in dimension 1. A direct copy of this method for arbitrary real closed fields R instead of $\mathbb{R}$ is not possible because the local global principle as stated in 2.9 is not valid any more. The fact that the semialgebraic set $S = S(G)$ associated to the finitely generated quadratic module $Q = QM(G) \subseteq R[X]$ is bounded is not enough to give: $\widehat{f}_a \in \widehat{Q}_a$ for all $a \in Z(f) \cap S \Rightarrow f \in Q$.

Exemplarily the Stengle preordering illustrates this observation.
If $R \supseteq \mathbb{R}$ contains infinitesimal elements then the polynomial $f(X) := 1 - X^2 + \mu$ where $\mu > 0$ is some infinitesimal element of R is strictly positive on $S := [-1, 1]$. The interval $[-1, 1]$ is the basic closed semialgebraic set associated to the preordering $P := QM_{R[X]}((1 - X^2)^3)$. Thus $\widehat{f}_a \in \widehat{P}_a$ for every zero a of f in S is trivially true but by Proposition 3.7 we know that f is not in P.

In the two extreme cases that the interior of the semialgebraic set $S \subseteq R$ associated to the finitely generated quadratic module $Q \subseteq R[X]$ is empty or that S is not bounded we already have worked out that the Membership Problem is solvable affirmatively over arbitrary real closed fields since we even have stability in those cases (Theorem 2.35, Corollary 1.14).

The remaining case for $n = 1$ is the case where S is bounded and $\operatorname{int}(S) \neq \emptyset$ which implies that $\operatorname{supp}(Q) = \{0\}$.

As in Chapter 2, X denotes one indeterminate from now on.

Let $f, g_1, ..., g_s \in R[X]$ and

$$Q_R := QM_{R[X]}(g_1, ..., g_s) \subseteq R[X]$$

as well as

$$S_R := S(g_1, ..., g_s) = \{x \in R \mid g_i(x) \geq 0 \ (1 \leq i \leq s)\}.$$

We proved in Section 2.2 that the local conditions $\widehat{f_a} \in \widehat{(Q_R)}_a$ for the finitely many zeros a of f in S_R are equivalent to $f \in QM_{R[X]}(g_1, ..., g_s, -f^2) = Q_R + f^2R[X]$ (Corollary 2.40).

However for arbitrary real closed fields R it is not clear under which assumptions the fact that $f|_{S_R} \geq 0$ and $f \in Q_R + f^2R[X]$ implies that $f \in Q_R$.

If $R = \mathbb{R}$ the boundedness of S_R is enough because this implies that Q_R is archimedean (Theorem 2.9).

This motivates the following strategy.

We consider $f, g_1, ..., g_s \in \mathcal{O}[X]$ and impose first the necessary assumptions such that

$$Q_{\mathcal{O}} := QM_{\mathcal{O}[X]}(g_1, ..., g_s) \subseteq \mathcal{O}[X]$$

is archimedean because we can then conclude by Lemma 2.7 from $f \geq 0$ on $\overline{H}(Q_{\mathcal{O}})$ and $f \in Q_{\mathcal{O}} + f^2\mathcal{O}[X]$ that $f \in Q_{\mathcal{O}} \subseteq Q_R \cap \mathcal{O}[X]$. Then we prove under some additional assumptions for $R = \mathbb{R}((t^{\mathbb{R}}))$ a suitable local-global principle which allows us to express $f \in Q_{\mathcal{O}}$ by finitely many conditions in formal power series rings.

We note that a positive solution of the Membership Problem for $Q_R \cap \mathcal{O}[X]$ would also give a positive solution of the Membership Problem for Q_R.

The reason is that for $f(X, Y) \in \mathbb{Z}[X, Y]$ there is a function $\mu : \mathcal{O}^Y \to \mathcal{O}$ which can chosen to be semialgebraic such that for every $c \in \mathcal{O}^Y$

$$f(X, c) \in Q_R \Leftrightarrow \mu(c)^2 f(X, c) \in Q_R \cap \mathcal{O}[X]$$

From now on let

$$f, g_1, ...g_s \in \mathcal{O}[X].$$

Since $\overline{\mathcal{O}}$ is an archimedean real closed field it can be viewed as a subfield of $\mathbb{R}$. Because we assume that $\mathbb{R} \subseteq R$ it is actually isomorphic to $\mathbb{R}$. We denote the section of the place λ that selects $\mathbb{R}$ by $\rho : \overline{\mathcal{O}} \to \mathbb{R}$. One can think of $\overline{\mathcal{O}}$ being a copy of $\mathbb{R}$ but we will not identify them (yet).

In the following not just the quadratic modules Q_R and $Q_{\mathcal{O}}$ will be important but also

$$Q^{\lambda}_{\overline{\mathcal{O}}} := QM_{\overline{\mathcal{O}}[X]}(\lambda g_1, ..., \lambda g_s) \subseteq \overline{\mathcal{O}}[X].$$

together with

$$S^{\lambda}_{\overline{\mathcal{O}}} := S_{\overline{\mathcal{O}}}(\lambda g_1, ..., \lambda g_s) = \{x \in \overline{\mathcal{O}} \mid \lambda g_i(x) \geq 0 \ (1 \leq i \leq s)\}.$$

We illustrate the quadratic modules and basic closed semialgebraic sets appearing in this chapter with the following picture where i, i_1 and i_2 are inclusions.

$$\begin{tikzcd}
 & \begin{array}{c} R[X] \\ Q_R, S_R \end{array} & \\
\mathbb{R}[X] \arrow[ur, "i"] \arrow[r, "i_1"] \arrow[rr, bend right, dotted, "\cong"'] & \begin{array}{c} \mathcal{O}[X] \\ Q_{\mathcal{O}} \end{array} \arrow[u, "i_2"] \arrow[r, "\lambda"] & \begin{array}{c} \overline{\mathcal{O}}[X] \\ Q^{\lambda}_{\overline{\mathcal{O}}}, S^{\lambda}_{\overline{\mathcal{O}}} \end{array}
\end{tikzcd}$$

The reason why in this picture no associated semialgebraic set appears for $Q_{\mathcal{O}}$ is that this finitely generated quadratic module is not a quadratic module of a polynomial ring over a real closed field. However if we are working in the real spectrum then there is an associated set for $Q_{\mathcal{O}}$ namely

$$\overline{H}(Q_{\mathcal{O}}) = \overline{H}(g_1, ..., g_s) = \{\alpha \in \mathrm{Sper}\mathcal{O}[X] \mid g_i(\alpha) \geq 0 \ (1 \leq i \leq s)\}.$$

For the quadratic modules Q_R and $Q^{\lambda}_{\overline{\mathcal{O}}}$ we have $\overline{H}(Q_R) = \widetilde{S_R} \subseteq \mathrm{Sper}\, R[X]$ and $\overline{H}(Q^{\lambda}_{\overline{\mathcal{O}}}) = \widetilde{S^{\lambda}_{\overline{\mathcal{O}}}} \subseteq \mathrm{Sper}\, \overline{\mathcal{O}}[X]$.

By applying the real spectrum functor we get the following picture.

$$\begin{tikzcd}
 & \begin{array}{c} \mathrm{Sper}\, R[X] \\ \widetilde{S_R} \end{array} \arrow[dl, "\mathrm{Sper}\, i"'] \arrow[d, "\mathrm{Sper}\, i_2"] & \\
\mathrm{Sper}\, \mathbb{R}[X] \arrow[rr, bend right, dotted, "\cong"'] & \begin{array}{c} \mathrm{Sper}\, \mathcal{O}[X] \\ \overline{H}(Q_{\mathcal{O}}) \end{array} \arrow[l, "\mathrm{Sper}\, i_1"'] & \begin{array}{c} \mathrm{Sper}\, \overline{\mathcal{O}}[X] \\ \widetilde{S^{\lambda}_{\overline{\mathcal{O}}}} \end{array} \arrow[l, "\mathrm{Sper}\, \lambda"']
\end{tikzcd}$$

We note that the appropriate space for quadratic modules would normally be the semi-real spectrum. If $n = 1$ we know however by Proposition 0.3 that SemiSper $R[X]$ is equal to Sper $R[X]$ for every real closed field R.
Thus

$$\overline{H}_{semi}(Q_R) = \overline{H}_{semi}(g_1, ..., g_s) = \overline{H}(g_1, ..., g_s) = \widetilde{S_R} \subseteq \mathrm{Sper}\, R[X]$$

and also

$$\overline{H}_{semi}(Q^{\lambda}_{\overline{\mathcal{O}}}) = \widetilde{S^{\lambda}_{\overline{\mathcal{O}}}} \subseteq \mathrm{Sper}\, \overline{\mathcal{O}}[X].$$

4.1 Description of $\operatorname{Sper}\mathcal{O}[X]$ and $\operatorname{Sper}\mathcal{O}[X]^{max}$

In the first proposition we do not yet specialize to $R = \mathbb{R}((t^{\mathbb{R}}))$ and consider an arbitrary real closed extension $R \supseteq \mathbb{R}$.

Proposition 4.1
If $R \supseteq \mathbb{R}$ then the following is true:

i) $Q_{\mathcal{O}} = \lambda^{-1}(Q^{\lambda}_{\overline{\mathcal{O}}})$ *if and only if* $\mathfrak{m}[X] \subseteq supp(Q_{\mathcal{O}})$.

ii) *If α is a semiordering of $\mathcal{O}[X]$ then $\alpha = \lambda^{-1}(\gamma)$ for some $\gamma \in \operatorname{SemiSper}\overline{\mathcal{O}}[X]$ if and only if $supp(\alpha) \cap \mathcal{O} = \mathfrak{m}$.*

iii) *If $g_1, ..., g_s \in \mathbb{R}[X]$ and the support $supp(QM_{\mathbb{R}[X]}(g_1, ..., g_s))$ is radical then we have $Q_R \cap \mathcal{O}[X] = Q_{\mathcal{O}}$.*

iv) *If α is a semiordering of $\mathcal{O}[X]$ then $\alpha = \beta \cap \mathcal{O}[X]$ for some $\beta \in \operatorname{SemiSper} R[X]$ if and only if $supp(\alpha) \cap \mathcal{O} = \{0\}$.*

Proof:

i) : If $Q_{\mathcal{O}} = \lambda^{-1}(Q^{\lambda}_{\overline{\mathcal{O}}})$ then for some $f(X) \in \mathfrak{m}[X]$ the fact that $\lambda(f) = 0 \in Q^{\lambda}_{\overline{\mathcal{O}}}$ implies that $f \in \lambda^{-1}(Q^{\lambda}_{\overline{\mathcal{O}}}) = Q_{\mathcal{O}}$. Since for $f \in \mathfrak{m}[X]$ also $-f \in \mathfrak{m}[X]$ we actually have shown that $\mathfrak{m}[X] \subseteq \operatorname{supp}(Q_{\mathcal{O}})$.

In the proof of the other implication the inclusion $\subseteq$ is even true without our assumption.

To show this we take some $f = \sum_{i=0}^{s} \sigma_i g_i \in Q_{\mathcal{O}}$ where $\sigma_i \in \sum \mathcal{O}[X]^2, 0 \leq i \leq s$.

As $\lambda(\sigma_i) \in \sum \overline{\mathcal{O}}[X]^2$ for every $0 \leq i \leq s$ we have $\lambda(f) = \sum_{i=0}^{s} \lambda(\sigma_i)\lambda g_i \in Q^{\lambda}_{\overline{\mathcal{O}}}$ which proves the first inclusion.

For the inclusion $\lambda^{-1}(Q^{\lambda}_{\overline{\mathcal{O}}}) \subseteq Q_{\mathcal{O}}$ we consider some polynomial $f \in \mathcal{O}[X]$ with $\lambda(f) \in Q^{\lambda}_{\overline{\mathcal{O}}}$ and show that $f \in Q_{\mathcal{O}}$.

Let $\lambda(f) = \sum_{i=0}^{s} \sigma_i \lambda g_i$ for some $\sigma_i \in \sum \overline{\mathcal{O}}[X]^2$ $(0 \leq i \leq s)$. Then we define

$$\widetilde{f} := \sum_{i=0}^{s} i_1(\rho(\sigma_i))g_i.$$

Hence $\widetilde{f} \in Q_{\mathcal{O}}$ because $i_1(\rho(\sigma_i)) \in \sum \mathbb{R}[X]^2 \subseteq \sum \mathcal{O}[X]^2$ for $0 \leq i \leq s$. On the other hand we have

$$\begin{aligned} \lambda(f - \widetilde{f}) &= \sum_{i=0}^{s} \sigma_i \lambda g_i - \lambda(\sum_{i=0}^{s} i_1(\rho(\sigma_i))g_i) \\ &\overset{\lambda \circ i_1 = \rho^{-1}}{=} \sum_{i=0}^{s} \sigma_i \lambda g_i - \sum_{i=0}^{s} \sigma_i \lambda g_i = 0 \end{aligned}$$

Hence there is some $m \in \mathfrak{m}[X]$ with $f = \widetilde{f} + m$ which implies that $f \in Q_{\mathcal{O}}$ because $\tilde{f} \in Q_{\mathcal{O}}$ and $\mathfrak{m}[X] \subseteq \mathrm{supp}(Q_{\mathcal{O}})$.

ii) : If $\alpha = \lambda^{-1}(\gamma)$ for some $\gamma \in \mathrm{SemiSper}\,\overline{\mathcal{O}}[X]$ then we have for the support of α that $\mathrm{supp}(\alpha) \cap \mathcal{O} = \mathrm{supp}(\lambda^{-1}(\gamma)) \cap \mathcal{O}$. Since γ is a semiordering of $\overline{\mathcal{O}}[X]$ the intersection of the prime ideal $\mathrm{supp}(\gamma)$ with the field $\overline{\mathcal{O}}$ is $\{0\}$. Now we have $a \in \mathrm{supp}(\lambda^{-1}(\gamma)) \cap \mathcal{O} \Leftrightarrow \lambda(a) \in \mathrm{supp}(\gamma) \cap \overline{\mathcal{O}} = \{0\}$ for every $a \in R$. This means that $\mathrm{supp}(\alpha) \cap \mathcal{O} = \ker(\lambda) = \mathfrak{m}$ as λ is the canonical homomorphism $\mathcal{O}$ onto $\mathcal{O}/\mathfrak{m} = \overline{\mathcal{O}}$.

Now we suppose that $\alpha \in \mathrm{SemiSper}\,\mathcal{O}[X]$ with $\mathrm{supp}(\alpha) \cap \mathcal{O} = \mathfrak{m}$. Then we consider the ring homomorphism $\Phi : \mathcal{O}[X] \to \mathcal{O}[X]/\mathrm{supp}(\alpha) \hookrightarrow k(\alpha)$ where $k(\alpha)$ denotes the quotient field of $\mathcal{O}[X]/\mathrm{supp}(\alpha)$ which carries a semiordering T induced by α. As we have by assumption $\ker(\Phi|_{\mathcal{O}}) = \mathfrak{m}$ the homomorphism theorem gives a ring homomorphism $\Phi_1 : \overline{\mathcal{O}} = \mathcal{O}/\mathfrak{m} \to k(\alpha)$ such that $\Phi|_{\mathcal{O}} = \Phi_1 \circ \lambda|_{\mathcal{O}}$. We extend Φ_1 to a ring homomorphism $\widetilde{\Phi}_1 : \overline{\mathcal{O}}[X] \to k(\alpha)$ with $\Phi = \widetilde{\Phi}_1 \circ \lambda$ and define $\gamma := \widetilde{\Phi}_1^{-1}(T)$. Since T is a semiordering of $k(\alpha)$ and $\widetilde{\Phi}_1$ is a ring homomorphism γ is an element of $\mathrm{SemiSper}\,\overline{\mathcal{O}}[X]$.
Furthermore $\lambda^{-1}(\gamma) = \lambda^{-1}(\widetilde{\Phi}_1^{-1}(T)) = (\widetilde{\Phi}_1 \circ \lambda)^{-1}(T) = \Phi^{-1}(T) = \alpha$ by definition of Φ.

iii) : By [M-R] Corollary 4.2 we may assume that R is a truncation closed subfield of $\mathbb{R}((t^{\Gamma}))$ and $\mathcal{O} = R \cap \mathbb{R}((t^{\Gamma^{\geq 0}}))$ where Γ is the value group of v.

The inclusion $Q_R \cap \mathcal{O}[X] \supseteq Q_{\mathcal{O}}$ is clear.

Thus we consider now some $f \in Q_R \cap \mathcal{O}[X]$ and show that $f \in Q_{\mathcal{O}}$. This means we have $\sigma_i \in \sum R[X]^2$ for $0 \leq i \leq s$ such that $f(X) = \sum\limits_{i=0}^{s} \sigma_i(X) g_i(X)$. There are $r, d \in \mathbb{N}$ and $h_{i,j}(X) \in R[X]$ for $i \in \{0, ..., s\}$ and $1 \leq j \leq r$ such that $\sigma_i(X) = \sum\limits_{j=1}^{r} h_{i,j}(X)^2$ and $\deg g_i$, $\deg h_{i,j} \leq d$. Thus we have

$$f(X) = \sum_{i=0}^{s} \sum_{j=1}^{r} h_{i,j}(X)^2 g_i(X).$$

As we consider R as a subfield of $\mathbb{R}((t^{\Gamma}))$ there are $h_{i,j,\gamma}(X) \in \mathbb{R}[X]$ of degree $\leq d$ such that

$$h_{i,j}(X) = \sum_{\gamma} h_{i,j,\gamma}(X) t^{\gamma}$$

which implies that

$$f(X) = \sum_{\gamma} \left(\sum_{i=0}^{s} \sum_{j=1}^{r} \sum_{\gamma_1 + \gamma_2 = \gamma} h_{i,j,\gamma_1}(X) h_{i,j,\gamma_2}(X) g_i(X) \right) t^{\gamma} \qquad (*).$$

We abbreviate $Q_{\mathbb{R}} := QM_{\mathbb{R}[X]}(g_1, ..., g_s)$ and define

$$\widehat{\gamma} := \min\{\gamma \in \Gamma \mid h_{i,j,\gamma}(X)g_i(X) \notin \operatorname{supp}(Q_{\mathbb{R}}) \text{ for some } 0 \leq i \leq s, 1 \leq j \leq r\}$$

We claim that

$$\widehat{\gamma} \geq 0$$

and suppose to the contrary that $\widehat{\gamma} < 0$ which also means that $2\widehat{\gamma} < 0$. Then the coefficient of $t^{2\widehat{\gamma}}$ in the representation $(*)$ must be zero because $f(X) \in \mathcal{O}[X]$. This means that

$$\sum_{i=0}^{s}\sum_{j=1}^{r}\sum_{\gamma_1+\gamma_2=2\widehat{\gamma}} h_{i,j,\gamma_1}(X)h_{i,j,\gamma_2}(X)g_i(X) = 0$$

or

$$h(X) + \sum_{i=0}^{s}\sum_{j=1}^{r} h_{i,j,\widehat{\gamma}}(X)^2 g_i(X) = 0 \qquad (**)$$

where we have defined

$$h(X) := \sum_{i=0}^{s}\sum_{j=1}^{r}\sum_{\substack{\gamma_1+\gamma_2=2\widehat{\gamma} \\ \gamma_1<\widehat{\gamma} \text{or } \gamma_2<\widehat{\gamma}}} h_{i,j,\gamma_1}(X)h_{i,j,\gamma_2}(X)g_i(X).$$

The definition of $\widehat{\gamma}$ implies that $h(X) \in \operatorname{supp}(Q_{\mathbb{R}})$ because every summand is an element of the support of $Q_{\mathbb{R}}$. Again by definition of $\widehat{\gamma}$ now together with the radicality of $\operatorname{supp}(Q_{\mathbb{R}})$ we have some $\widehat{i}$ and some $\widehat{j}$ such that $h_{\widehat{i},\widehat{j},\widehat{\gamma}}(X)^2 g_{\widehat{i}}(X)$ is not in $\operatorname{supp}(Q_{\mathbb{R}})$ which means that $-h_{\widehat{i},\widehat{j},\widehat{\gamma}}(X)^2 g_{\widehat{i}}(X) \notin Q_{\mathbb{R}}$. This contradicts $(**)$ and proves the claim that $\widehat{\gamma} \geq 0$.

Now we define for every $0 \leq i \leq s$ and every $1 \leq j \leq r$

$$\widehat{h}_{i,j,\gamma}(X) := \begin{cases} h_{i,j,\gamma}(X) & \text{if } \gamma \geq \widehat{\gamma} \\ 0 & \text{if } \gamma < \widehat{\gamma} \end{cases}$$

and

$$\widehat{h}_{i,j}(X) = \sum_{\gamma} \widehat{h}_{i,j,\gamma}(X)t^{\gamma-\widehat{\gamma}} \in \mathcal{O}[X].$$

Since $f(X) \in \mathcal{O}[X]$ we have

$$f(X) = \sum_{\gamma<0}\left(\sum_{i=0}^{s}\sum_{j=1}^{r}\sum_{\gamma_1+\gamma_2=\gamma} h_{i,j,\gamma_1}(X)h_{i,j,\gamma_2}(X)g_i(X)\right)t^{\gamma} = 0$$

such that

$$f(X) = f_1(X) + t^{2\widehat{\gamma}}\sum_{i=0}^{s}\sum_{j=1}^{r}\widehat{h}_{i,j}(X)^2 g_i(X)$$

with

$$f_1(X) := \sum_{\gamma \geq 0} \left(\sum_{i=0}^{s} \sum_{j=1}^{r} \sum_{\substack{\gamma_1+\gamma_2=\gamma \\ \gamma_1<\widehat{\gamma} \text{or } \gamma_2<\widehat{\gamma}}} h_{i,j,\gamma_1}(X) h_{i,j,\gamma_2}(X) g_i(X) \right) t^{\gamma}.$$

By definition of $\widehat{\gamma}$ the element $f_1(X)$ is in the ideal generated by $\mathrm{supp}(Q_{\mathbb{R}})$ in $\mathcal{O}[X]$ and hence $f_1 \in Q_{\mathcal{O}}$. Since $\widehat{\gamma} \geq 0$ and $\widehat{h}_{i,j}(X) \in \mathcal{O}[X]$ we have altogether shown that $f \in Q_{\mathcal{O}}$.

iv) : If $\alpha = \beta \cap \mathcal{O}[X]$ for some semiordering β from $\mathrm{SemiSper}\, R[X]$ then we clearly have that $\mathrm{supp}(\alpha) \cap \mathcal{O} = (\mathrm{supp}(\beta) \cap R) \cap \mathcal{O} = \{0\}$ because $\mathrm{supp}(\beta)$ is a prime ideal and R is a field.

Now we suppose that $\alpha \in \mathrm{SemiSper}\, \mathcal{O}[X]$ with $\mathrm{supp}(\alpha) \cap \mathcal{O} = 0$. We have the ring homomorphism $\Phi : \mathcal{O}[X] \to \mathcal{O}[X]/\mathrm{supp}(\alpha) \hookrightarrow k(\alpha)$ where $k(\alpha)$ denotes the quotient field of $\mathcal{O}[X]/\mathrm{supp}(\alpha)$ semiordered by the semiordering T induced by α. Since by assumption $\ker(\Phi|_{\mathcal{O}}) = \{0\}$ the map $\Phi|_{\mathcal{O}}$ is injective. Since $R = \mathrm{Quot}(\mathcal{O})$ we have a ring homomorphism $\Phi_1 : R \to k(\alpha)$ which sends $\frac{a}{b} \in R$ to $\frac{\Phi(a)}{\Phi(b)}$. The map Φ has the property that $\Phi|_{\mathcal{O}} = \Phi_1 \circ i$ where $i : \mathcal{O} \to R$ is the canonical inclusion. We extend Φ_1 to a ring homomorphism $\widetilde{\Phi}_1 : R[X] \to k(\alpha)$ which also satisfies $\widetilde{\Phi}_1 \circ i = \Phi$ and define $\beta := \widetilde{\Phi}_1^{-1}(T)$. Since T is a semiordering of $k(\alpha)$ and $\widetilde{\Phi}_1$ is a ring homomorphism $\beta \in \mathrm{SemiSper}(R[X])$. Furthermore $\beta \cap \mathcal{O}[X] = \Phi^{-1}(T) = \alpha$.

Prop. 4.1 □

Now we specialize to the case that $R = \mathbb{R}((t^{\mathbb{R}}))$ where $\mathrm{Spec}\, \mathcal{O} = \{\{0\}, \mathfrak{m}\}$ and identify $\overline{\mathcal{O}}$ with $\mathbb{R}$. Then $\mathrm{SemiSper}\, \mathcal{O}[X]$ is according to the previous proposition given as

$$\{\lambda^{-1}(\gamma) \mid \gamma \in \mathrm{SemiSper}\, \mathbb{R}[X]\} \dot{\bigcup} \{\beta \cap \mathcal{O}[X] \mid \beta \in \mathrm{SemiSper}\, R[X]\}.$$

Proposition 0.3 implies that the elements of $\mathrm{SemiSper}\, \mathcal{O}[X]$ are in fact orderings and we have

$$\begin{aligned} \mathrm{SemiSper}\, \mathcal{O}[X] = & \ \mathrm{Sper}\, \mathcal{O}[X] = \\ = & \ \{\lambda^{-1}(\gamma) \mid \gamma \in \mathrm{Sper}\, \mathbb{R}[X]\} \dot{\bigcup} \{\beta \cap \mathcal{O}[X] \mid \beta \in \mathrm{Sper}\, R[X]\}. \end{aligned}$$

This implies that

$$\begin{aligned} \overline{H}_{semi}(Q_{\mathcal{O}}) &= \overline{H}_{semi}(g_1, ..., g_s) = \\ &= \overline{H}(g_1, ..., s) = \\ &= \{\lambda^{-1}(\gamma) \mid \gamma \in \widetilde{S_{\mathbb{R}}^{\lambda}}\} \dot{\bigcup} \{\beta \cap \mathcal{O}[X] \mid \beta \in \widetilde{S_R}\}. \end{aligned}$$

In order to see that $\overline{H}(Q_{\mathcal{O}})$ is made up of the two sets as stated above we consider some $\alpha \in \mathrm{SemiSper}\,\mathcal{O}[X] = \mathrm{Sper}\,\mathcal{O}[X]$ with $\alpha \supseteq Q_{\mathcal{O}}$, i.e. $g_i \in \alpha$ for $1 \leq i \leq s$.

If $\mathrm{supp}(\alpha) \cap \mathcal{O} = \mathfrak{m}$ then by Proposition 4.1 ii) $\alpha = \lambda^{-1}(\gamma)$ for some $\gamma \in \mathrm{Sper}\,\mathbb{R}[X]$. Since $g_i \in \alpha = \lambda^{-1}(\gamma)$ implies that $\lambda g_i \in \gamma$ for every $0 \leq i \leq s$ we have $\gamma \in \widetilde{S^{\lambda}_{\mathbb{R}}}$.

If otherwise $\mathrm{supp}(\alpha) \cap \mathcal{O} = \{0\}$ then there is some $\beta \in \mathrm{Sper}\,R[X]$ such that $\alpha = \beta \cap \mathcal{O}[X]$ (Proposition 4.1 iv)). As $\beta \supseteq \alpha$ we have $g_i \in \beta$ for $1 \leq i \leq s$ and thus $\beta \in \widetilde{S_R}$.

If we look at the maximal spectrum then $\mathrm{Sper}\mathcal{O}[X]^{max}$ clearly is a subset of

$$(\mathrm{Sper}\,\lambda)(\mathrm{Sper}\mathbb{R}[X]^{max}) \dot{\bigcup} (\mathrm{Sper}\,i_2)(\mathrm{Sper}R[X]^{max})$$

by the description of $\mathrm{Sper}\,\mathcal{O}[X]$ given above .

As the support of some ordering in the image of $\mathrm{Sper}\lambda$ intersected with $\mathcal{O}$ is $\mathfrak{m}$ whereas the intersection is $\{0\}$ if the ordering is in the image of $\mathrm{Sper}\,i_2$ we just have to check the following:
Given some ordering $\beta \in \mathrm{Sper}\,R[X]^{max}$ is there some ordering $\gamma \in \mathrm{Sper}\,\mathbb{R}[X]^{max}$ such that $\beta \cap \mathcal{O}[X] \subseteq \lambda^{-1}(\gamma)$?

Before we go through the possible cases for β the following observation is useful which can be made for arbitrary real closed fields $R \supseteq \mathbb{R}$.

Lemma 4.2
Let $R \supseteq \mathbb{R}$ and $\beta \in \mathrm{Sper}R[X]$. Then $-1 \in \lambda(\beta \cap \mathcal{O}[X])$ if and only if there is some semialgebraic set $S \subseteq R$ with $\beta \in \widetilde{S}$ and $S \cap \mathcal{O} = \emptyset$.

Proof:
$\Rightarrow$: Since $-1 \in \lambda(\beta \cap \mathcal{O}[X])$ there is some $m(X) \in \mathfrak{m}[X]$ such that the element $f(X) := -1 + m(X)$ is in β. Thus the set $S := \{x \in R \mid f(x) \geq 0\}$ is a semialgebraic subset of R which satisfies $\beta \in \widetilde{S}$. If we take some $x \in \mathcal{O}$ then $m(x) \in \mathfrak{m}$ and therefore $f(x) < 0$. Thus $S \cap \mathcal{O} = \emptyset$.

$\Leftarrow$: Let $S \subseteq R$ be semialgebraic with $\beta \in \widetilde{S}$ and $S \cap \mathcal{O} = \emptyset$. By o-minimality there is some $r > \mathcal{O}$ such that $[-r, r] \cap S = \emptyset$. The fact that $\beta \in \widetilde{S}$ implies that $X^2 - r^2 \in \beta$ and hence also $(\frac{X}{r})^2 - 1 \in \beta$. As $\frac{X}{r} \in \mathfrak{m}[X]$ we get that $-1 = \lambda((\frac{X}{r})^2 - 1) \in \lambda(\beta \cap \mathcal{O}[X])$.

Lemma 4.2 $\square$

Now we consider some $\beta \in \operatorname{Sper} R[X]^{max}$ for $R = \mathbb{R}((t^{\mathbb{R}}))$.

Case 1: $\beta = a$ is the ordering corresponding to evaluation at a for some $a \in R$.
If $a \in \mathcal{O}$ then we have $\beta \cap \mathcal{O}[X] \subseteq \lambda^{-1}(\gamma)$ for $\gamma := \lambda a \in \operatorname{Sper}\mathbb{R}[X]^{max}$ because λ is order preserving.
If $a \notin \mathcal{O}$ then there is some positive $\mu \in \mathfrak{m}$ such that $\beta > \frac{1}{\mu}$ or $\beta < -\frac{1}{\mu}$. Thus there is a semialgebraic set $S \subseteq R$ with $\beta \in \widetilde{S}$ and $S \cap \mathcal{O} = \emptyset$. By Lemma 4.2 we have $-1 \in \lambda(\beta \cap \mathcal{O}[X])$. Thus $\beta \cap \mathcal{O}[X]$ does not specialize to any element of the image of Sperλ.

Case 2: $\beta = \pm\infty_R$
By definition of $+\infty_R$ sets of the form $S =]r, \infty[$ with $r \in R$ and $r > \mathcal{O}$ have the property that $\beta \in \widetilde{S}$ and also $S \cap \mathcal{O} = \emptyset$. Hence by Lemma 4.2 β does not specialize to any element of the image of Sperλ.
Similar for $\beta = -\infty$.

Case 3: β is an ordering corresponding to a free Dedekind cut.
If $\beta > \mathcal{O}^+$ or $\beta < \mathcal{O}^-$ then again by Lemma 4.2 $\lambda(\beta \cap \mathcal{O}[X])$ is not proper and therefore β does not specialize to any element of Sperλ(Sper$\mathbb{R}[X]$).
If $\beta = \mathcal{O}^+$ then we have $\beta \cap \mathcal{O}[X] \subseteq \lambda^{-1}(\gamma)$ with $\gamma := +\infty_{\mathbb{R}}$. In order to see this we take some $f(X) \in \beta \cap \mathcal{O}[X]$ and write $f(X) = p(X) + m(X)$ with $p(X) \in \mathbb{R}[X]$ and $m(X) \in \mathfrak{m}[X]$. Without loss of generality we suppose that $p \neq 0$. By definition of $\mathcal{O}^+$ the polynomial f has to be positive for all points close to $\mathcal{O}^+$. Since for every $x \in \mathbb{R}$ we have $|m(x)| < |p(x)|$ this means that $p(x) > 0$ for all $x \in \mathbb{R}$ close to $\mathcal{O}^+$. Thus $p \in +\infty_{\mathbb{R}}$ which proves the inclusion above.
Similarly we have for $\beta = \mathcal{O}^-$ that $\beta \cap \mathcal{O}[X] \subseteq \lambda^{-1}(\gamma)$ with $\gamma := -\infty_{\mathbb{R}}$.
It remains the case that $\mathcal{O}^- < \beta < \mathcal{O}^+$. By [Tr2] Theorem 2.12 β is of the form $a + b\mathfrak{m}^+$ for some $a, b \in \mathbb{R}$ and $b \neq 0$. This implies that if $S \subseteq R$ is some semialgebraic set with $\beta \in \widetilde{S}$ then $S \cap \mathcal{O} \neq \emptyset$ and therefore by Lemma 4.2 $-1 \notin \lambda(\beta \cap \mathcal{O}[X])$. Thus there is some ordering $\gamma \in$Sper$\mathbb{R}[X]^{max}$ with $\lambda(\beta \cap \mathcal{O}[X]) \subseteq \gamma$ and therefore $\beta \cap \mathcal{O}[X] \subseteq \lambda^{-1}(\gamma)$.

The case differentiation shows that $\operatorname{Sper}\mathcal{O}[X]^{max}$ is given as

$$\{\lambda^{-1}(\gamma) \mid \gamma \in \operatorname{Sper}\mathbb{R}[X]^{max}\} \dot{\bigcup} \{\beta \cap \mathcal{O}[X] \mid \beta \in \operatorname{Sper} R[X]^{max}, \beta > \mathcal{O}^+ \text{ or } \beta < \mathcal{O}^-\}$$

and hence $\overline{H}(Q_{\mathcal{O}})^{max} = \{\alpha \in \operatorname{Sper}\mathcal{O}[X]^{max} \mid g_i(\alpha) \geq 0 \ (1 \leq i \leq s)\}$ decomposes as

$$\{\lambda^{-1}(\gamma) \mid \gamma \in \widetilde{S_{\mathbb{R}}^{\lambda}}^{max}\} \dot{\bigcup} \{\beta \cap \mathcal{O}[X] \mid \beta \in \widetilde{S_R}^{max}, \beta > \mathcal{O}^+ \text{ or } \beta < \mathcal{O}^-\}$$

4.2 Reduction to the formal power series ring over $\mathcal{O}$

We recall that our aim is to characterize when $f(X) \in \mathcal{O}[X]$ lies in the quadratic module $Q_{\mathcal{O}} = QM_{\mathcal{O}[X]}(g_1, ..., g_s)$ for some $g_1, ..., g_s \in \mathcal{O}[X]$ and $R = \mathbb{R}((t^{\mathbb{R}}))$.

First we state two necessary conditions.
If $f \in Q_{\mathcal{O}}$ then

1) $f \in Q_R$ which implies that $f|_{S_R} \geq 0$

2) $\lambda f \in Q_{\mathbb{R}}^{\lambda}$ which implies that $\lambda f|_{S_{\mathbb{R}}^{\lambda}} \geq 0$

Since λ is an order preserving map we always have $\lambda(S_R \cap \mathcal{O}) \subseteq S_{\mathbb{R}}^{\lambda}$.
The other inclusion is not true in general. If $g_1(X) = (1 - X^2)t \in \mathbb{R}((t^{\mathbb{R}}))[X]$ for example then $\lambda g_1(X) = 0$ such that $S_{\mathbb{R}}^{\lambda} = \mathbb{R}$ whereas $S_R \cap \mathcal{O} = [-1, 1]$.
This shows that the fact that $f|_{S_R \cap \mathcal{O}} \geq 0$ does not imply $\lambda f|_{S_{\mathbb{R}}^{\lambda}} \geq 0$ for some $f(X) \in \mathcal{O}[X]$. From $f|_{S_R \cap \mathcal{O}} \geq 0$ it follows that $\lambda f|_{\lambda(S_R \cap \mathcal{O})} \geq 0$ but it can happen that $\lambda(S_R \cap \mathcal{O})$ is a strict subset of $S_{\mathbb{R}}^{\lambda}$ as in the example above. For that example the polynomial $f(X) = 1 - X^2 = \lambda f(X)$ is nonnegative on $S_R \cap \mathcal{O} = [-1, 1]$ but not nonnegative on $S_{\mathbb{R}}^{\lambda} = \mathbb{R}$.

The same example also shows that boundedness of S_R (by some element from $\mathbb{N}$) does not imply the boundedness of $S_{\mathbb{R}}^{\lambda}$. The reverse conclusion also fails. Take for example $g_1(X) := (1 - X^2)(1 - tX) \in \mathbb{R}((t^{\mathbb{R}}))[X]$ then $S_R = [-1, 1] \cup [\frac{1}{t}, \infty[$ and $S_{\mathbb{R}}^{\lambda} = [-1, 1]$.

The boundedness of the sets S_R and $S_{\mathbb{R}}^{\lambda}$ is for $R = \mathbb{R}((t^{\mathbb{R}}))$ enough to ensure that the preordering $PO_{\mathcal{O}[X]}(g_1, ..., g_s)$ is archimedean if $-1 \notin PO_{\mathcal{O}[X]}(g_1, ..., g_s)$ ([P-D] Lemma 8.3.1 b)). We will show that in dimension 1 this also gives that $Q_{\mathcal{O}}$ is archimedean.

We recall that a quadratic module in $\mathcal{O}[X]$ is already archimedean if there is some $N \in \mathbb{N}$ such that $N - X^2 \in Q$.
This follows as in the case of $\mathbb{R}[X]$ (see e.g. [P-D] Corollary 5.1.14) from the fact that the set $H_Q := \{f \in \mathcal{O}[X] \mid \exists N \in \mathbb{N}\ N \pm f \in Q\}$ is a subring of $\mathcal{O}[X]$ with $\mathcal{O} \subseteq H_Q$ and by assumption $X \in H_Q$.

The properness of the preordering $PO_{\mathcal{O}[X]}(g_1, ..., g_s)$ is assured by the fact that either S_R or $S_{\mathbb{R}}^{\lambda}$ is not empty ([P-D] Lemma 8.3.1 a)). We claim that

$$-1 \in Q_{\mathcal{O}} \Leftrightarrow -1 \in PO_{\mathcal{O}[X]}(g_1, ..., g_s).$$

This can be seen as follows. We have by the abstract Stellensatz for quadratic modules (Theorem 0.4 iv)) $-1 \in Q_{\mathcal{O}} \Leftrightarrow \overline{H}_{Semi}(Q_{\mathcal{O}}) = \emptyset$.

With Proposition 0.3 we obtained as a consequence of Proposition 4.1 that

$$\emptyset = \overline{H}_{Semi}(Q_{\mathcal{O}}) = \overline{H}_{semi}(g_1, ..., g_s) = \overline{H}(g_1, ..., g_s) = \overline{H}(PO_{\mathcal{O}[X]}(g_1, ..., g_s))$$

which is by the abstract Stellensatz for preorderings (Theorem 0.5 iv)) equivalent to $-1 \in PO_{\mathcal{O}[X]}(g_1, ..., g_s)$.
Thus as in the case for preorderings we have $Q_{\mathcal{O}}$ is proper if and only if $S_R \neq \emptyset$ or $S_{\mathbb{R}}^{\lambda} \neq \emptyset$.
We recall that $Q_{\mathcal{O}} = \mathcal{O}[X]$ if $-1 \in Q_{\mathcal{O}}$ because every element in $\mathcal{O}[X]$ can be written as the difference of two squares.

Proposition 4.3
Let $R \supseteq \mathbb{R}$ with $\operatorname{Spec} \mathcal{O} = \{\{0\}, \mathfrak{m}\}$. Suppose that $g_1, ..., g_s \in \mathcal{O}[X]$ with $S_{\mathbb{R}}^{\lambda} \neq \emptyset$ and $\|S_R\| \leq N$ as well as $\|S_{\mathbb{R}}^{\lambda}\| \leq N$ for some $N \in \mathbb{N}$.
Then $Q_{\mathcal{O}} = QM_{\mathcal{O}[X]}(g_1, ..., g_s)$ is archimedean.

Proof:
The assumption $S_{\mathbb{R}}^{\lambda} \neq \emptyset$ implies that $Q_{\mathcal{O}}$ is proper as explained above.
As in the proof of [P-D] Lemma 8.3.1 b) the boundedness of S_R and $S_{\mathbb{R}}^{\lambda}$ imply that there is some $N_0 \in \mathbb{N}$ such that $N_0 - X^2 > 0$ on $\overline{H}(g_1, ..., g_s)$. By the consequence of Proposition 4.1 we know that $\overline{H}(g_1, ..., g_s) = \overline{H}_{semi}(Q_{\mathcal{O}})$. Hence we have by the abstract Stellensatz for quadratic modules some $p \in \sum \mathcal{O}[X]^2$ and some $p \in Q_{\mathcal{O}}$ with $p(N_0 - X^2) = 1 + q$. From this we can construct as in the proof of iii') $\Rightarrow$ ii') in [P-D] Theorem 5.1.18 some $N_1 \in \mathbb{N}$ such that $N_1 - X^2 \in Q_{\mathcal{O}}$. This implies that $Q_{\mathcal{O}}$ is archimedean.

Prop. 4.3 □

Theorem 4.4
Let $R \supseteq \mathbb{R}$ with $\operatorname{Spec} \mathcal{O} = \{\{0\}, \mathfrak{m}\}$. Suppose that $f, g_1, ..., g_s \in \mathcal{O}[X]$ with $S_{\mathbb{R}}^{\lambda} \neq \emptyset$ and $\|S_R\| \leq N$ as well as $\|S_{\mathbb{R}}^{\lambda}\| \leq N$ for some $N \in \mathbb{N}$.
If $f|_{S_R} \geq 0$, $\lambda f|_{S_{\mathbb{R}}^{\lambda}} \geq 0$ and $f \in Q_{\mathcal{O}} + f^2\mathcal{O}[X]$ then $f \in Q_{\mathcal{O}}$.

Proof:
The assumptions for $S_{\mathbb{R}}^{\lambda}$ and S_R imply by the previous proposition that $Q_{\mathcal{O}}$ is archimedean.
By the description of $\overline{H}(Q_{\mathcal{O}})$ which we worked out after Proposition 4.1 $f|_{S_R} \geq 0$ and $\lambda f|_{S_{\mathbb{R}}^{\lambda}} \geq 0$ imply that $f \geq 0$ on $\overline{H}(Q_{\mathcal{O}})$. Thus Lemma 2.7 gives that $f \in Q_{\mathcal{O}}$.

Theorem 4.4 □

Now we have to find out when we can achieve that $f \in Q_{\mathcal{O}} + f^2\mathcal{O}[X]$.

We recall that the first part of the proof of the local global principle in Section 2.1 (Lemma 2.6) is valid over arbitrary real closed fields. This means that in the case that $\widehat{f_a}$ is in the quadratic module generated by the images of $g_1, ..., g_s$ in $R[[X-a]]$ for every zero a of f in S_R (which is fulfilled if finitely many order conditions are true) then $f \in Q_R + f^2R[X] = QM_{R[X]}(g_1, ..., g_s, -f^2)$. But note that this is not exactly what we want because the quadratic module and the ideal are formed in $R[X]$ not in $\mathcal{O}[X]$.

A similar conclusion gives that in the case that $\widehat{\lambda f}_a$ is in the quadratic module generated by the images of $\lambda g_1, ..., \lambda g_s$ in $\mathbb{R}[[X-a]]$ for every zero a of λf in $S_{\mathbb{R}}^{\lambda}$ we get $\lambda f \in Q_{\mathbb{R}}^{\lambda} + \lambda f^2\mathbb{R}[X] = QM_{\mathbb{R}[X]}(\lambda g_1, ..., \lambda g_s, -\lambda f^2)$.

The both conditions $f \in Q_R + f^2R[X]$ and $\lambda f \in Q_{\mathbb{R}}^{\lambda} + \mathbb{R}[X]\lambda f^2$ are not enough to give $f \in Q_{\mathcal{O}} + f^2\mathcal{O}[X]$.

We illustrate this with an example from Chapter 3. In the proof of Proposition 3.16 we showed with the help of the explicit upper and lower bounds from Stengle that for some $k \in \mathbb{N}$ big enough $1 - X^2 + \mu^k \notin PO_{R[X]}((1-X^2)^3, 1-X+\mu, 1+X+\mu)$. Using an isomorphism $R \to R$ which sends an infinitesimal to another this actually gives us that $f := 1 - X^2 + \mu^2 \notin PO_{R[X]}((1-X^2)^3, 1-X+\mu, 1+X+\mu)$ and thus in particular $f \notin PO_{\mathcal{O}[X]}((1-X^2)^3, 1-X+\mu, 1+X+\mu)$.

In this example we have $S_R = [-1, 1]$ and since f does not have zeros in S_R we get $f \in PO_{R[X]}((1-X^2)^3, 1-X+\mu, 1+X+\mu) + f^2R[X]$. Furthermore $\lambda f = 1 - X^2$ is in the saturated preordering $PO_{\mathbb{R}[X]}((1-X^2)^3, 1-X, 1+X) = PO_{\mathbb{R}[X]}(1-X, 1+X)$. However $f \notin PO_{\mathcal{O}[X]}((1-X^2)^3, 1-X+\mu, 1+X+\mu) + \mathcal{O}[X]f^2$ since this would by Theorem 4.4 imply that $f \in PO_{\mathcal{O}[X]}((1-X^2)^3, 1-X+\mu, 1+X+\mu)$ which is a contradiction to what we have proved in Proposition 3.16.

The advantage of the quadratic module $Q_{\mathcal{O}} + f^2\mathcal{O}[X] = QM_{\mathcal{O}[X]}(g_1, ..., g_s, -f^2)$ in comparison to $Q_{\mathcal{O}} = QM_{\mathcal{O}[X]}(g_1, ..., g_s)$ is that the associated semialgebraic sets $S(g_1, ..., g_s, -f^2) \subseteq R$ and $S(\lambda g_1, ..., \lambda g_s, -\lambda f^2)$ are finite. We have already seen in Section 2.2 that for the case of finite semialgebraic sets we can prove a local-global principle over arbitrary real closed fields (Proposition 2.34). We will see in the following that in this situation with an additional assumption we can prove a local-global principle over $\mathcal{O}$.

The completion of $\mathcal{O}[X]$ with respect to the ideal $(X-a)\mathcal{O}[X]$ for some $a \in \mathcal{O}$ can as in the case of the ring of polynomials over a real closed field be considered as the formal power series ring $\mathcal{O}[[X-a]]$ ([E] Example in Section 7.1).

It follows a local-global principle valid for $\mathcal{O}[X]$.

Proposition 4.5
Let $R \supseteq \mathbb{R}$ with $\operatorname{Spec}\mathcal{O} = \{\{0\}, \mathfrak{m}\}$ and $f, g_1, ..., g_s \in \mathcal{O}[X]$.
We suppose that $S_R = \{a_1, ..., a_m\} \subset \mathcal{O}$ and $a_i - a_j \notin \mathfrak{m}$ for all $i \neq j$ as well as $S_{\mathbb{R}}^{\lambda} = \{\lambda a_1, ..., \lambda a_m, b_1, ..., b_t\} \subseteq \mathbb{R}$ where $b_i - a_j \notin \mathfrak{m}$ for all $i \neq j$.
Then the following are equivalent:

i) $f \in Q_{\mathcal{O}} = QM_{\mathcal{O}[X]}(g_1, ..., g_s)$

ii) *for every $b \in S_{\mathbb{R}}^{\lambda} \setminus \lambda(S_R)$ there is some $\mu_b \in \mathfrak{m}$ such that*
$\widehat{f_a} \in \widehat{(Q_{\mathcal{O}})}_a$ *for every $a \in S_R$ and*
$\widehat{f}_{b+\mu_b} \in \widehat{(Q_{\mathcal{O}})}_{b+\mu_b}$ *for every $b \in S_{\mathbb{R}}^{\lambda} \setminus \lambda(S_R)$*

Proof:
The implication $i) \Rightarrow ii)$ is clear.

For the other implication we show as a first step that for all $\mu_1, ..., \mu_t \in \mathfrak{m}$ there are $N = N_{\mu_1,...,\mu_t} \in \mathbb{N}$ such that we have for the polynomial

$$p_{\mu_1,...,\mu_t} := \prod_{i=1}^{m}(X - a_i)\prod_{j=1}^{t}(X - b_j - \mu_j)$$

that $-p_{\mu_1,...,\mu_t}^{2N} \in Q_{\mathcal{O}}$.

To see this we consider some $\mu_1, ..., \mu_t \in \mathfrak{m}$ and show for $p := p_{\mu_1,...,\mu_t}$ that $p = 0$ on $\overline{H}_{semi}(Q_{\mathcal{O}})$ and then we get what we want by the abstract Stellensatz for quadratic modules (Theorem 0.4).

As explained after Proposition 4.1 we have

$$\begin{aligned} \overline{H}_{semi}(Q_{\mathcal{O}}) &= \overline{H}(Q_{\mathcal{O}}) = \\ &= \{\lambda^{-1}(\gamma) \mid \gamma \in \widetilde{S_{\mathbb{R}}^{\lambda}}\} \dot{\bigcup} \{\beta \cap \mathcal{O}[X] \mid \beta \in \widetilde{S_R}\}. \end{aligned}$$

Let first $\alpha = \lambda^{-1}(\gamma)$ for some $\gamma \in \widetilde{S_{\mathbb{R}}^{\lambda}}$. Since by definition of p we have $\lambda p|_{S_{\mathbb{R}}^{\lambda}} = 0$ as $S_{\mathbb{R}}^{\lambda} = \{\lambda a_1, ..., \lambda a_m, b_1, ..., b_t\}$ it follows that $\lambda p \in \operatorname{supp}(\gamma)$. Thus $p \in \operatorname{supp}(\alpha)$, i.e. $p(\alpha) = 0$.

Now suppose that $\alpha = \beta \cap \mathcal{O}[X]$ for some $\beta \in \widetilde{S_R}$. Since $S_R = \{a_1, ..., a_m\}$ the element p is in the support of β and thus $p \in \operatorname{supp}(\alpha)$, i.e. $p(\alpha) = 0$.

If S_R and $S_{\mathbb{R}}^{\lambda}$ are empty then we have by the description of $\overline{H}_{semi}(Q_{\mathcal{O}})$ as given above and the abstract Stellensatz for quadratic modules $-1 \in Q_{\mathcal{O}}$ and hence $Q_{\mathcal{O}} = \mathcal{O}[X]$

such that i) is trivially true.

Now we suppose that $S_R \neq \emptyset$ or $S_{\mathbb{R}}^{\lambda} \neq \emptyset$.
By assumption there are $\mu_{b_i} \in \mathfrak{m}$ such that $\widehat{f}_{b_i+\mu_{b_i}} \in \widehat{(Q_{\mathcal{O}})}_{b_i+\mu_{b_i}} \subseteq \mathcal{O}[[X-b_i-\mu_{b_i}]]$ for every $1 \leq i \leq t$ and $\widehat{f}_{a_i} \in \widehat{(Q_{\mathcal{O}})}_{a_i} \subseteq \mathcal{O}[[X-a_i]]$ for every $1 \leq i \leq m$.

As in the proof of Theorem 2.34 we use for $a \in \mathcal{O}$ and N big enough the projection $\mathcal{O}[[X-a]] \to \mathcal{O}[X]/(X-a)^{2N}\mathcal{O}[X]$ to obtain $\overline{f}_i \in Q_{\mathcal{O}}$ and $\overline{h}_i \in \mathcal{O}[X]$ $(1 \leq i \leq m)$ with

$$f = \overline{f}_i + \overline{h}_i(X-a_i)^{2N}$$

as well as $\widetilde{f}_j \in Q_{\mathcal{O}}$ and $\widetilde{h}_j \in \mathcal{O}[X]$ $(1 \leq j \leq t)$ with

$$f = \widetilde{f}_j + \widetilde{h}_j(X-b_j-\mu_{b_j})^{2N}.$$

In order to use the Chinese remainder theorem we observe the following.
The fact that $a_i - a_j \notin \mathfrak{m}$ for $1 \leq i,j \leq m$, $i \neq j$ implies that

$$(X-a_j)\mathcal{O}[X] + (X-a_i)\mathcal{O}[X] = \mathcal{O}[X]$$

because $X - a_j - (X-a_i) = a_i - a_j$ is a unit in $\mathcal{O}[X]$ and hence 1 is an element of $(X-a_j)\mathcal{O}[X] + (X-a_i)\mathcal{O}[X]$.
Similarly we have

$$(X-a_j)\mathcal{O}[X] + (X-b_i-\mu_{b_i})\mathcal{O}[X] = \mathcal{O}[X]$$

for $1 \leq j \leq m$ and $1 \leq i \leq t$ as well as

$$(X-b_j-\mu_{b_j})\mathcal{O}[X] + (X-b_i-\mu_{b_i})\mathcal{O}[X] = \mathcal{O}[X]$$

for $1 \leq i,j \leq t, i \neq j$.

The Chinese remainder theorem gives completely similar to the proof of Proposition 2.34 some element $q \in Q_{\mathcal{O}}$ and some $v \in \mathcal{O}[X]$ such that

$$f = q + (\frac{v+1}{2})^2 p^{2N} + (\frac{v-1}{2})^2 \underbrace{(-p^{2N})}_{\in Q_{\mathcal{O}}} \in Q_{\mathcal{O}}$$

where $p = p_{\mu_{b_1},\ldots,\mu_{b_t}}$.

Prop. 4.5 □

These proposition together with the local-global principle (Theorem 4.5) gives the following reduction of the Membership Problem to the formal power series ring.

Theorem 4.6
Let $R \supseteq \mathbb{R}$ with $\operatorname{Spec} \mathcal{O} = \{\{0\}, \mathfrak{m}\}$, $f, g_1, ..., g_s \in \mathcal{O}[X]$.
We suppose that $S_{\mathbb{R}}^{\lambda} \neq \emptyset$ and $\|S_R\| \leq N$ as well as $\|S_{\mathbb{R}}^{\lambda}\| \leq N$ for some $N \in \mathbb{N}$.
If $f|_{S_R} \geq 0$, $\lambda f|_{S_{\mathbb{R}}^{\lambda}} \geq 0$ and $a - a' \notin \mathfrak{m}$ for any $a, a' \in Z(f) \cap S_R, a \neq a'$ as well as $b - a \notin \mathfrak{m}$ for all $a \in Z(f) \cap S_R, b \in (Z(\lambda f) \cap S_{\mathbb{R}}^{\lambda}) \setminus \lambda(Z(f) \cap S_R)$.
Then the following are equivalent:

i) $f \in Q_{\mathcal{O}} = QM_{\mathcal{O}[X]}(g_1, ..., g_s)$

ii) *for every $b \in (Z(\lambda f) \cap S_{\mathbb{R}}^{\lambda}) \setminus \lambda(Z(f) \cap S_R)$ there is some $\mu_b \in \mathfrak{m}$ such that*
$\widehat{f}_a \in \widehat{(Q_{\mathcal{O}})}_a$ *for every $a \in Z(f) \cap S_R$ and*
$\widehat{f}_{b+\mu_b} \in \widehat{(Q_{\mathcal{O}})}_{b+\mu_b}$ *for every $b \in (Z(\lambda f) \cap S_{\mathbb{R}}^{\lambda}) \setminus \lambda(Z(f) \cap S_R)$*

Proof:
The implication $i) \Rightarrow ii)$ is clear.

For the implication $ii) \Rightarrow i)$ we define

$$Q := Q_{\mathcal{O}} + f^2\mathcal{O}[X] = QM_{\mathcal{O}[X]}(g_1, ..., g_s, -f^2).$$

Then the associated basic closed semialgebraic set of $QM_{R[X]}(g_1, ..., g_s, -f^2)$ is equal to $Z(f) \cap S_R$ and the associated basic closed semialgebraic set of the quadratic module $QM_{\mathbb{R}[X]}(\lambda g_1, ..., \lambda g_s, -\lambda f^2)$ is $Z(\lambda f) \cap S_{\mathbb{R}}^{\lambda}$. Since both sets are finite and the fact that $Q_{\mathcal{O}} \subseteq Q$ implies that $\widehat{(Q_{\mathcal{O}})}_a \subseteq \widehat{Q}_a$ for every $a \in R$ the assumptions made about $Z(f) \cap S_R$ and $Z(\lambda f) \cap S_{\mathbb{R}}^{\lambda}$ together with $ii)$ give by Proposition 4.5 that $f \in Q = Q_{\mathcal{O}} + f^2\mathcal{O}[X]$.

With the additional assumptions made in the statement of the theorem we can now deduce by Theorem 4.4 that $f \in Q_{\mathcal{O}}$.

Theorem 4.6 □

We include some remarks which are useful for working in the formal power series ring $\mathcal{O}[[X - a]]$ for some $a \in \mathcal{O}$.
For ease of notation we take $a = 0$ now.

Let $f = \sum_{i=0}^{\infty} a_i X^i \in \mathcal{O}[[X]]$. Then f is a unit in $\mathcal{O}[[X]]$ if and only if a_0 is a unit in $\mathcal{O}$. This means in our case if and only if $a_0 \notin \mathfrak{m}$.

Since a factorization of the form $f = a(1 + q)$ with $a \in \mathcal{O}$ and $q \in X\mathcal{O}[[X]]$ is just possible if $a_0 \neq 0$ and $v(a_0) = \min_{i \in \mathbb{N}}\{v(a_i)\}$ we see that the elements of $\mathcal{O}[[X]]$ are not as simple to describe as the elements of $R[[X]]$.

What the squares in $\mathcal{O}[[X]]$ concerns this observation about the factorization implies that if $a_0 > 0$ and $v(a_0) = \min_{i \in \mathbb{N}}\{v(a_i)\}$ then f is a square in $\mathcal{O}[[X]]$. However

it is already very hard to give a necessary and sufficient condition for some element of $\mathcal{O}[[X]]$ to be a square or a sum of squares.
For more information about formal power series rings we refer to [Br].

Now we want to describe some observations about the support of $Q_{\mathcal{O}}$ in the case that S_R is a finite subset of $\mathcal{O}$.

Remark 4.7
We have $supp(Q_{\mathcal{O}})R[X] = supp(Q_R)$.

The inclusion $supp(Q_{\mathcal{O}})R[X] \subseteq supp(Q_R)$ is true because $Q_{\mathcal{O}} \subseteq Q_R$ and thus $supp(Q_{\mathcal{O}}) \subseteq supp(Q_R)$.

For the other inclusion we consider some $f \in supp(Q_R)$, i.e. $f \in R[X]$ with $f = \sum_{i=0}^{s} \sigma_i g_i$ and $-f = \sum_{i=0}^{s} \tau_i g_i$ for some $\sigma_i, \tau_i \in \sum R[X]^2$. Let c be a coefficient in σ_i or τ_i with minimal valuation. Then $\frac{1}{c^2} f \in supp(Q_{\mathcal{O}}) \subseteq \mathcal{O}[X]$.
Thus $supp(Q_R) \subseteq supp(Q_{\mathcal{O}})R \subseteq supp(Q_{\mathcal{O}})R[X]$.

Furthermore if d is the minimal positive degree of a polynomial in $supp(Q_R)$ then d is also the minimal positive degree of a polynomial in $supp(Q_{\mathcal{O}})$ ([K-Y] Remark 1).

Proposition 4.8
Let $R \supseteq \mathbb{R}$ with $\operatorname{Spec} \mathcal{O} = \{\{0\}, \mathfrak{m}\}$ and $f, g_1, ..., g_s \in \mathcal{O}[X]$.
If $\emptyset \neq S_R = \{a_1, ..., a_m\} \subseteq \mathcal{O}$ and $supp(Q_R) = \left(\prod_{i=1}^{m} (X - a_i)^{k_i}\right) R[X]$ then

$$\left(\mu \prod_{i=1}^{m} (X - a_i)^{k_i}\right) \mathcal{O}[X] \subseteq supp(Q_{\mathcal{O}}) \subseteq \left(\prod_{i=1}^{m} (X - a_i)^{k_i}\right) \mathcal{O}[X]$$

for every $\mu \in \mathcal{O}$ with $\mu \prod_{i=1}^{m} (X - a_i)^{k_i} \in supp(Q_{\mathcal{O}})$

Proof:
By Corollary 2.42 $\operatorname{supp}(Q_R) = \left(\prod_{i=1}^{m} (X - a_i)^{k_i}\right) R[X]$ for some $k_i \in \mathbb{N}$ $(1 \leq i \leq m)$.
This implies that $\operatorname{supp}(Q_R) \cap \mathcal{O} = \{0\}$ and therefore also $\operatorname{supp}(Q_{\mathcal{O}}) \cap \mathcal{O} = \{0\}$ because $\operatorname{supp}(Q_{\mathcal{O}}) \subseteq \operatorname{supp}(Q_R)$.

Similar as in the previous remark there is some $\mu \in \mathfrak{m}$ such that $\mu \prod_{i=1}^{m}(X-a_i)^{k_i}$ is in $\mathrm{supp}(Q_{\mathcal{O}})$. For such an element μ we clearly have

$$\left(\mu \prod_{i=1}^{m}(X-a_i)^{k_i}\right)\mathcal{O}[X] \subseteq \ \mathrm{supp}(Q_{\mathcal{O}}).$$

Let f be an arbitrary element of $\mathrm{supp}(Q_{\mathcal{O}})$. Since $\prod_{i=1}^{m}(X-a_i)^{k_i} \in \mathcal{O}[X]$ is monic and hence in particular has an invertible leading coefficient there are unique elements $q(X), r(X) \in \mathcal{O}[X]$ such that

$$f(X) = q(X)\prod_{i=1}^{m}(X-a_i)^{k_i} + r(X)$$

and either $r(X) \equiv 0$ or $\deg(r) < d := \sum_{i=1}^{m} k_i$. Hence

$$\mu r(X) = \mu f(X) - q(X)\mu \prod_{i=1}^{m}(X-a_i)^{k_i} \in \mathrm{supp}(Q_{\mathcal{O}})$$

Thus if $r \not\equiv 0$ then $\deg(r) = 0$ because the minimal positive degree of a polynomial in $\mathrm{supp}(Q_{\mathcal{O}})$ is by the previous remark d.
This means that $\mu r \in \mathrm{supp}(Q_{\mathcal{O}}) \cap \mathcal{O} = \{0\}$ and consequently

$$f(X) = q(X)\prod_{i=1}^{m}(X-a_i)^{k_i}.$$

Altogether we have shown that

$$\left(\mu \prod_{i=1}^{m}(X-a_i)^{k_i}\right)\mathcal{O}[X] \subseteq \mathrm{supp}(Q_{\mathcal{O}}) \subseteq \left(\prod_{i=1}^{m}(X-a_i)^{k_i}\right)\mathcal{O}[X]$$

Prop. 4.8 □

In the situation of the Proposition we determine in a particular example how the support exactly looks like.

Let $Q_{\mathcal{O}} := QM_{\mathcal{O}[X]}(\mu X(X^2+1), -\mu X(X^2+1)) \subseteq \mathcal{O}[X]$ for some $\mu > 0$ infinitesimal with $v(\mu) =: \delta > 0$. Then $\mathrm{supp}(Q_{\mathcal{O}}) = \mu X \mathcal{O}[X]$.

This can be seen as follows.
We have $S = \{0\}$ and $\mathrm{supp}(Q_R) = XR[X]$.
Since

$$-\mu X^2 = (\frac{-X+1}{2})^2 \mu X(X^2+1) + (\frac{-X-1}{2})^2(\mu X(X^2+1)) + \mu X^4 \in Q_{\mathcal{O}}$$

we get

$$\mu X = \mu X(X^2+1) + (\frac{X+1}{2})^2(-\mu X^2) + (\frac{X-1}{2})^2 \mu X^2 \in Q_{\mathcal{O}}.$$

Similarly $-\mu X \in Q_{\mathcal{O}}$ and hence with the considerations above

$$\mu X \mathcal{O}[X] \subseteq \mathrm{supp}(Q_{\mathcal{O}}) \subseteq X\mathcal{O}[X].$$

We show now that every element $f \in \mathrm{supp}(Q_{\mathcal{O}})$ satisfies $v(f) \geq v(\mu) = \delta$.
We suppose to the contrary that there is some $f \in \mathrm{supp}(Q_{\mathcal{O}})$ with $v(f) < \delta$. We have a representation

$$f = \sigma_0 + \sigma_1 \mu X(X^2+1) + \sigma_2(-\mu X(X^2+1))$$

with some $\sigma_i \in \sum \mathcal{O}[X]^2$ $(i = 0, ..., 2)$. Let $\gamma := \min\{v(f), v(\sigma_0)\}$ and a some element of $\mathcal{O}$ with $v(a^2) = \gamma$. Then we have

$$\frac{1}{a^2} f = \frac{1}{a^2}\sigma_0 + \sigma_1 \frac{\mu}{a^2} X(X^2+1) + \sigma_2(-\frac{\mu}{a^2} X(X^2+1))$$

Because of the fact that $v(\mu^2) > v(\mu) > v(f) \geq v(a^2)$ we get by applying the residue map $\lambda : \mathcal{O}[X] \to \mathcal{O}/\mathfrak{m}[X]$ that

$$\lambda(\sigma_1 \frac{\mu}{a^2} X(X^2+1) + \sigma_2(-\frac{\mu}{a^2} X(X^2+1))) = 0$$

which means that we have

$$\lambda(\frac{1}{a^2} f) = \lambda(\frac{1}{a^2}\sigma_0)$$

in $\mathcal{O}/\mathfrak{m}[X]$.

Case a: $v(f) < v(\sigma_0)$
Then $\lambda(\frac{1}{a^2} f) \equiv 0$ in $\mathcal{O}/\mathfrak{m}[X]$ which is a contradiction as at least one coefficients of $\frac{1}{a^2} f$ has valuation 0.

Case b: $v(f) > v(\sigma_0)$
Then $\lambda(\frac{1}{a^2}\sigma_0) \equiv 0$ in $\mathcal{O}/\mathfrak{m}[X]$ which is a contradiction as $\lambda(\frac{1}{a^2}\sigma_0)$ is a sum of squares in $\mathcal{O}/\mathfrak{m}[X]$ with at least one coefficient not equal to zero.

Case c: $v(f) = v(\sigma_0)$

Then $\lambda(\frac{1}{a^2}f)$ is a sum of squares in $\mathcal{O}/\mathfrak{m}[X]$. Now we use the fact that f is not just in $Q_{\mathcal{O}}$ but in $\mathrm{supp}(Q_{\mathcal{O}})$. Similar considerations as in Case a) and b) either lead to a contradiction or to the fact that $\lambda(\frac{1}{a^2}(-f))$ is a sum of squares in $\mathcal{O}/\mathfrak{m}[X]$. Note that we can take the same a for f and $-f$ in this case because $\gamma = v(f) = v(-f)$. Thus we have $\widetilde{\sigma}_0 = -\widetilde{\tau}_0$ for some $\widetilde{\sigma}_0, \widetilde{\tau}_0 \in \sum \mathcal{O}/\mathfrak{m}[X]^2$ which are not equal to zero. This is a contradiction.

Thus we have shown

$$f \in \mathrm{supp}(Q_{\mathcal{O}}) \Rightarrow v(f) \geq v(\mu).$$

Since every $f \in \mathrm{supp}(Q_{\mathcal{O}})$ can be written as $f = h \cdot X$ for some $h \in \mathcal{O}[X]$ this implies that f is divisible by μX because $f = \frac{1}{\mu}h \cdot \mu X$ and $\frac{1}{\mu}h \in \mathcal{O}[X]$ because $v(f) = v(h) \geq v(\mu)$. This gives

$$\mathrm{supp}(Q_{\mathcal{O}}) = \mu X \mathcal{O}[X].$$

Open questions:

In the end we want to list some open questions which are topics for future research.

1. In $R[X]$ where R is an arbitrary real closed field and X denotes one indeterminate there are examples of finitely generated quadratic modules which are weakly semialgebraic (e.g. the stable or saturated ones) and which are not weakly semialgebraic (Example 3.8). For orderings $\alpha \subseteq R[X]$ we know that α is weakly semialgebraic if and only if the Dedekind cut corresponding to α is principal (Proposition 1.16). Which property of a finitely generated quadratic module Q of $R[X]$ is equivalent to the fact that Q is weakly semialgebraic?

2. Is in the situation of Theorem 4.6 the membership in $QM_{\mathcal{O}[X]}(g_1, ..., g_s)$ definable by some $L(R)$-formula where L is the language of ordered rings extended by one predicate $\mathcal{O}$ which stands for the valuation ring?

3. Corollary 2.44 reduces the membership problem for finitely generated quadratic modules of $R[X]$ with finite associated semialgebraic set to the Membership Problem for ideals. Is this also possible in more general situations?

4. In dimension one we proved that every finitely generated quadratic module of $\mathbb{R}[X]$ is weakly semialgebraic (Corollary 2.20). Is this also true for higher dimensions?

Regarding the last open question we want to include an example of a preordering in $\mathbb{R}[X_1, X_2]$ which is not finitely generated and not weakly semialgebraic.

Example 4.9
The preordering

$$P = \{f \in \mathbb{R}[X_1, X_2] \mid f \geq 0 \text{ on } \{(x_1, x_2) \in \mathbb{R}^2 | x_2 \geq e^{x_1}\}\} \subseteq \mathbb{R}[X_1, X_2]$$

is not weakly semialgebraic. This can be seen by considering the polynomial

$$g(X_1, X_2, Y_1, Y_2) := X_2 - Y_1 X_1 - Y_2 \in \mathbb{Z}[X_1, X_2, Y_1, Y_2].$$

Since the tangents to the curve $X_2 = e^{X_1}$ are given by $X_2 = e^a X_1 + e^a(1-a)$ for some $a \in \mathbb{R}$ we have for $c = (c_1, c_2) \in \mathbb{R}^2$ that $g(X_1, X_2, c_1, c_2) \in P$ if and only if ($c_1 > 0$ and $c_2 \leq c_1(1 - \log c_1)$) or ($c_1 = 0$ and $c_2 \leq 0$). This means that

$$\begin{aligned} D(g, P) = \{(c_1, c_2) \in \mathbb{R}^2 \mid \; & (c_1 > 0 \text{ and } c_2 \leq c_1(1 - \log c_1)) \\ & \text{or } (c_1 = 0 \text{ and } c_2 \leq 0)\} \end{aligned}$$

As the logarithm appears in the description of $D(g, P)$ this is not a semialgebraic subset of $\mathbb{R}^2$, i.e. P is not definable with respect to L_{or} which means that P is not weakly semialgebraic.
A direct proof of the fact that P is not finitely generated can be seen for example with [S1] 6.7.

A Appendix

A.1 Definability of term sets and types

For a review of the basic concepts of first order logic we refer to [P].

Let L be some first order language, M an L-structure, $X = (X_1, ..., X_n)$ and Y some finite tuple of variables of variable length. $M[X]$ is the set of maps $f : M^n \to M$ such that there is some L-term $t(X, Y)$ and some $m \in M^Y$ with $f(x) = t(x, m)$ for every $x \in M^n$. If $L = L_{or}$ and $M = R$ a real closed field then $M[X]$ is nothing else but the polynomial ring over R in n indeterminates. Completely similar to Definition 1.1 we can say when a term set $Q \subseteq M[X]$ is definable.

Definition A.1
$Q \subseteq M[X]$ is definable if and only if for every L-term $t(X, Y)$ there is a formula $\vartheta_t(Y) \in FmlL(M)$ such that for all $c \in M^Y$

$$t(X, c) \in Q \Leftrightarrow R \models \vartheta_t(c),$$

i.e. if and only if the set

$$D(t, Q) = \{c \in M^Y \mid t(X, c) \in Q\}$$

is definable (by the formula ϑ_f).

In the literature there is also the notion of being weakly semialgebraic given in Definition 1.2. We will show that in the case that $L = L_{or}$ and $M = R$ is a real closed field definability is equivalent to the property of being weakly semialgebraic.

Proposition A.2
If $L = L_{or}, R$ a real closed field and $Q \subseteq R[X]$.
Then Q is weakly semialgebraic if and only if Q is definable.

Proof:
$\Rightarrow$: Let $f(X, Y) \in \mathbb{Z}[X, Y]$. We consider the finite dimensional R-vector space $U = R[X]_{\leq d}$ of polynomials up to degree d where $d = \deg(f)$ is the degree of f with respect to X. By assumption $Q \cap U$ is semialgebraic in U where the dimension of U is $\binom{n+d}{d}$. Let $V_d : R^Y \to R^{\binom{n+d}{d}}$ be the semialgebraic function which gives for a general polynomial $h(X, Y) \in \mathbb{Z}[X, Y]$ of degree d in X the coordinate vector of h with respect to the monomial basis $\{X^\alpha \mid |\alpha| \leq d\}$. With this map we get

$$\{c \in R^Y \mid f(X, c) \in Q\} = \{c \in R^Y \mid V_d(c) \in Q \cap U\}$$

which is semialgebraic by assumption.
$\Leftarrow$: It is enough to consider finite dimensional subspaces of $R[X]$ of the form $R[X]_{\leq d}$ for some $d \in \mathbb{N}$. Then we get similarly as in the proof of $\Rightarrow$ the description of the following form $\{c \in R^Y \mid f(X,c) \in Q\} = \{c \in R^Y \mid V_d(c) \in Q \cap U\}$ with the map V_d as above. Since Q is definable the set on the left is semialgebraic and therefore $Q \cap U$ is semialgebraic. Hence Q is weakly semialgebraic.

Prop. A.2 $\square$

Furthermore we show that there is a connection between the fact that a quadratic module Q in $A := R[X]/I$ is weakly semialgebraic and this property of the preimage of Q under the canonical epimorphism in $R[X]$.

Proposition A.3
Let R be a real closed field and $I \subseteq R[X]$ an ideal.
If $Q \subseteq A := R[X]/I$ is weakly semialgebraic then $\psi^{-1}(Q) \subseteq R[X]$ is weakly semialgebraic where $\psi : R[X] \to A$ is the canonical epimorphism.

Proof:
For $d \in \mathbb{N}$ we define $U_d := \psi(R[X]_{\leq d})$. This is a finite dimensional R-vector space for which we fix a basis $\{\psi(X^\epsilon) \mid \epsilon \in J\} = \{\overline{X}^\epsilon \mid \epsilon \in J\}$ for some finite subset $J \subseteq \{\epsilon \in \mathbb{N}_0^n : |\epsilon| \leq d\}$ where we abbreviate $\psi(X_1)^{\epsilon_1} \cdots \psi(X_n)^{\epsilon_n}$ with $\overline{X}^\epsilon$.
The ideal I is of the form $(h_1, ..., h_l) = \sum_{i=1}^{l} h_i R[X]$ for some $h_i \in R[X]$ $(1 \leq i \leq l)$. By results about Gröbner bases there is an algebraic formula $\theta((u_\epsilon)_{\epsilon \in J}, (v_\epsilon)_{|\epsilon| \leq d})$ which says for $(a_\epsilon)_{\epsilon \in J} \in R^{|J|}$ and $(b_\epsilon)_{|\epsilon| \leq d} \in R^{|\Lambda(d)|}$

$$R \models \theta((a_\epsilon)_{\epsilon \in J}, (b_\epsilon)_{|\epsilon| \leq d}) \Leftrightarrow \sum_{\epsilon \in J} a_\epsilon X^\epsilon - \sum_{|\epsilon| \leq d} b_\epsilon X^\epsilon \in (h_1, ..., h_l).$$

Now we can prove our claim.
Therefore we consider some $d \in \mathbb{N}$. By assumption there is a semialgebraic formula $\phi((u_\epsilon)_{\epsilon \in J})$ which defines $Q \cap U_d$ in U_d, i.e. for every $(a_\epsilon)_{\epsilon \in J} \in R^{|J|}$ we have

$$R \models \phi((a_\epsilon)_{\epsilon \in J}) \Leftrightarrow \sum_{\epsilon \in J} a_\epsilon \overline{X}^\epsilon \in Q$$

Then we can define $\psi^{-1}(Q) \cap R[X]_{\leq d}$ by the semialgebraic formula

$$\rho((v_\epsilon)_{|\epsilon| \leq d}) := \exists (u_\epsilon)_{\epsilon \in J} \left(\phi(u_\epsilon)_{\epsilon \in J} \wedge \theta((u_\epsilon)_{\epsilon \in J}, (v_\epsilon)_{|\epsilon| \leq d}) \right).$$

To see this we consider some $G = \sum_{|\epsilon| \leq d} b_\epsilon X^\epsilon \in R[X]_{\leq d}$. Then $G \in \psi^{-1}(Q)$ if and only if $\psi(G) = \sum_{|\epsilon| \leq d} b_\epsilon \overline{X}^\epsilon \in Q \cap U_d$. Since $\{\overline{X}^\epsilon \mid \epsilon \in J\}$ is a basis of

U_d there is exactly one tuple $(a_\epsilon)_{\epsilon\in J} \in R^{|J|}$ such that $\sum_{|\epsilon|\leq d} b_\epsilon \overline{X}^\epsilon = \sum_{\epsilon\in J} a_\epsilon \overline{X}^\epsilon$ and therefore $\sum_{|\epsilon|\leq d} b_\epsilon \overline{X}^\epsilon - \sum_{\epsilon\in J} a_\epsilon \overline{X}^\epsilon \in I$. Thus $R \models \phi(a_\epsilon)_{\epsilon\in J} \wedge \theta((a_\epsilon)_{\epsilon\in J}, (b_\epsilon)_{|\epsilon|\leq d})$, i.e. $R \models \rho((b_\epsilon)_{|\epsilon|\leq d})$.

Prop. A.3 □

In Section 1.3 we mentioned the relationship between orderings and types.

An n-type p of some L-structure M is a set of $L(M)$-formulas with n free variables which is maximal consistent with the theory $Th(M, M)$. This means that for every finite subset $\{\varphi_1(X), ..., \varphi_k(X)\}$ of p we have $M \models \exists X(\bigvee_{i=1}^{k} \varphi_i(X))$ and for every $L(M)$-formula $\varphi(X)$ we have $\varphi(X) \in p$ or $\neg\varphi(X) \in p$.

By compactness every n-type p is realized in some elementary extension $N \succ M$ of M which means that there is some $b \in N^n$ such that $N \models \varphi(b)$ for every $\varphi(X) \in p$.

If $A \subseteq M$ and $a \in M^n$ then the type $tp(a/A)$ of a over A is given by

$$tp(a/A) := \{\varphi(X) \in \mathrm{Fml}L(A) \mid M \models \varphi(a)\}.$$

With $S_n(M)$ we denote the set of all n-types of M.

If $L = L_{or}$ and $M = R$ some real closed field then we have a bijection

$$\begin{array}{rcl} \Psi : S_n(R) & \to & \mathrm{Sper}\, R[X] \\ p & \mapsto & \{f \in R[X] \mid f(X) \geq 0 \in p\} \end{array}$$

which gives the correspondence between types and orderings.

We recall the notion of definable types and show afterwards that an ordering is weakly semialgebraic if and only if the corresponding type is definable.

Definition A.4
If M is an L-structure and $p \in S_n(M)$ some n-type then p is definable if and only if for all formulas $\phi(X, Y, m)$ with some $m \in M^Z$ the set

$$\{c \in M^Y \mid \phi(X, c, m) \in p\}$$

is definable by some $L(M)$-formula.

Equivalently if $b \in N^n$ with $N \succ M$ is a realization of p then p is definable if and only if for all formulas $\phi(X, Y, m)$ with $m \in M^Z$ the set

$$\{c \in M^Y \mid N \models \phi(b, c, m)\}$$

is definable by some $L(M)$-formula.

Proposition A.5
Let $L = L_{or}, R$ a real closed field and $\alpha \subseteq R[X]$ an ordering.
Then α is weakly semialgebraic if and only if the corresponding type p_α is definable.

Proof:
$\Rightarrow$: We pick some $\phi(X, Y, Z)$ and show that $\{(c, m) \in R^Y \times R^Z \mid \phi(X, c, m) \in p_\alpha\}$ is semialgebraic.
By quantifier elimination we have finitely many polynomials $g_{ij}, \widetilde{g}_{ij} \in \mathbb{Z}[X, Y, Z]$ such that $\phi(X, Y, Z)$ is equivalent to $\bigvee_i \bigwedge_j (g_{ij} \geq 0 \wedge \widetilde{g}_{ij} \not\geq 0)$ modulo the theory of R. Let $\psi_{ij}(Y, Z)$ be an $L(R)$-formula such that

$$\psi_{ij}(R^Y \times R^Z) = \{(c, m) \in R^Y \times R^Z \mid g_{ij}(X, c, m) \in \alpha\}$$

which exists by assumption. We do the same for $\widetilde{g}_{ij}$ and get $\widetilde{\psi}_{ij}$.
Then $\phi(X, c, m) \in p_\alpha$ if and only if $R \models \bigvee_i \bigwedge_j (\psi_{ij}(c, m) \wedge \neg\widetilde{\psi}_{ij}(c, m))$. This shows that p_α is definable.

$\Leftarrow$: Without loss of generality we take $U = R[X]_{\leq d}$. Proposition A.2 implies that we have to show that $\alpha \cap U$ is semialgebraic. By using the general polynomial $F_d(X, c) = \sum_{|\alpha| \leq d} c_\alpha X^\alpha$ in n variables of degree d this means that we have to show that the set $\{c \in R^{\binom{n+d}{d}} \mid F_d(X, c) \in \alpha\}$ is semialgebraic. But $F_d(X, c) \in \alpha$ means that $F_d(X, c) \geq 0 \in p_\alpha$. Since p_α is definable by assumption we get that $\alpha \cap U$ is semialgebraic.

Prop. A.5 □

A.2 Properties of heirs

In the following we prove some statements about heirs and weak heirs given in Section 3.1.

We recall the setting from that section: R, R' denote real closed fields which are model theoretically an example of L-structures where $L = L_{or} = \{+, -, \cdot, 0, 1, <\}$ is the first order language of ordered rings. Y and Z will denote finite tuples of variables (of variable length) whereas $X = (X_1, ..., X_n)$ for some fixed $n \in \mathbb{N}$.

First we show that the property of being an heir can also be expressed by using the notion of being existentially closed relative L for certain L^*-structures in a particular extended language L^*.

Definition A.6
Let $L^* \supseteq L$ be first order languages and $M^* \subseteq N^*$ an extension of L^*-structures with $M := M^*|L \prec N^*|L =: N$.
Then M^* is existentially closed in N^* relative L if for every $L(M)$-formula $\varphi(Y)$ and every quantifier free $L^*(M)$-formula $\chi(Y)$ we have

$$N^* \models \exists Y(\varphi(Y) \wedge \chi(Y)) \Rightarrow M^* \models \exists Y(\varphi(Y) \wedge \chi(Y))$$

If a set $Q \subseteq R[X]$ is definable we can express the fact that for $f(X,Y) \in \mathbb{Z}[X,Y]$ and $c \in R^Y$ we have $f(X,c) \in Q$ because of

$$f(X,c) \in Q \Leftrightarrow R \models \vartheta_f(c)$$

with help of the $L(R)$-formula $\vartheta_f(Y)$.

If Q is not definable then there is some $f(X,Y)$ such that we can not express $f(X,c) \in Q$ with the trueness of some $L(R)$-formula.

Since expressions like $f(X,c) \in Q$ appear in the definition of heirs the idea is now to extend the language L by adding an Y-ary predicate $\mathfrak{D}_f$ for every $f(X,Y) \in \mathbb{Z}[X,Y]$ to get the language

$$L^* = L(\mathfrak{D}_f | f \in \mathbb{Z}[X,Y]).$$

If we expand now R to an L^*-structure

$$M = (R, (D_R(f,Q) | f \in \mathbb{Z}[X,Y]))$$

by interpreting $\mathfrak{D}_f$ in the way that for some $c \in R^Y$

$$f(X,c) \in Q \Leftrightarrow M \models \mathfrak{D}_f(c)$$

we can now formulate the membership in Q by saying that a certain formula in the extended language L^* is true in the expanded structure M.

This way of reasoning gives us an equivalent characterization of heirs which the following theorem shows.

Theorem A.7
Let $L(\mathfrak{D}_f | f \in \mathbb{Z}[X,Y])$ be the language extending L, having an Y-ary predicate $\mathfrak{D}_f$ for every $f = f(X,Y) \in \mathbb{Z}[X,Y]$. For some $Q \subseteq R[X]$ we define the $L(\mathfrak{D}_f | f \in \mathbb{Z}[X,Y])$-structure M by

$$M := (R, (D_R(f,Q) | f \in \mathbb{Z}[X,Y])).$$

Let $R' \supseteq R$ be a real closed field.

i) If M' is an expansion of R' to an $L(\mathfrak{D}_f | f \in \mathbb{Z}[X,Y])$-structure such that M is existentially closed in M' relative L then the following is true for

$$Q' := \{f(X,c') \mid f(X,Y) \in \mathbb{Z}[X,Y], c' \in R'^Y, M' \models \mathfrak{D}_f(c')\} :$$

for every $f(X,Y) \in \mathbb{Z}[X,Y]$ and every $c' \in R'^Y$ we have

$$f(X,c') \in Q' \Leftrightarrow M' \models \mathfrak{D}_f(c')$$

ii) A subset $Q' \subseteq R'[X]$ is an heir of Q if and only if M is existentially closed in

$$M' := (R', (D_{R'}(f,Q') | f \in \mathbb{Z}[X,Y]))$$

relative L.

Proof:

$i)$: The implication $\Leftarrow$ holds by definition.

For the other implication we take some $f(X,Y) \in \mathbb{Z}[X,Y]$ and some $c' \in R'^Y$ with $f(X,c') \in Q'$. This means that there is some $g(X,Z) \in \mathbb{Z}[X,Z]$ and some $d' \in R'^Z$ with $f(X,c') = g(X,d')$ and $M' \models \mathfrak{D}_g(d')$. In M the sentence

$$\forall Y, Z[\forall X f(X,Y) = g(X,Z)] \wedge \mathfrak{D}_g(Z) \rightarrow \mathfrak{D}_f(Y)$$

is true. This sentence is also true in M' because M is existentially closed in M' relative L and $\forall X f(X,Y) = g(X,Z)$ is an L-formula. Hence we get from $R' \models \forall X f(X,c') = g(X,d')$ and $M' \models \mathfrak{D}_g(d')$ that we also have $M' \models \mathfrak{D}_f(c')$.

$ii)$: Let M be existentially closed in M' relative L. We show that this means that Q' is an heir of Q on R'.
Therefore we take $f_1(X,Y), ..., f_k(X,Y), f_1^-(X,Y), ..., f_l^-(X,Y) \in \mathbb{Z}[X,Y]$, some $\varphi(Y) \in \mathrm{Fml}L(R)$ and some $c' \in R'^Y$ such that

$$c' \in \bigcap_{i=1}^{k} D_{R'}(f_i, Q') \cap \bigcap_{i=1}^{l} D_{R'}(f_i^-, R'[X] \setminus Q') \cap \varphi(R'^Y).$$

This means that

$$f_1(X,c'),...,f_k(X,c') \in Q', f_1^-(X,c'),...,f_l^-(X,c') \notin Q', R' \models \varphi(c')$$

which can be formulated in the $L(\mathfrak{D}_f | f \in \mathbb{Z}[X,Y])$-structure M' as

$$M' \models \mathfrak{D}_{f_1}(c') \wedge ... \wedge \mathfrak{D}_{f_k}(c') \wedge \neg\mathfrak{D}_{f_1^-}(c') \wedge ... \wedge \neg\mathfrak{D}_{f_l^-}(c') \wedge \varphi(c').$$

Now the existentially closure of M in M' implies that there is some $c \in R^Y$ such that

$$M \models \mathfrak{D}_{f_1}(c) \wedge ... \wedge \mathfrak{D}_{f_k}(c) \wedge \neg\mathfrak{D}_{f_1^-}(c) \wedge ... \wedge \neg\mathfrak{D}_{f_l^-}(c) \wedge \varphi(c)$$

which means that

$$f_1(X,c),...,f_k(X,c) \in Q, f_1^-(X,c),...,f_l^-(X,c) \notin Q, R \models \varphi(c)$$

in other words

$$c \in \bigcap_{i=1}^{k} D_R(f_i, Q) \cap \bigcap_{i=1}^{l} D_R(f_i^-, R[X] \setminus Q) \cap \varphi(R^Y)$$

which proves by Definition 3.3 that Q' is an heir of Q on R'.

Now we suppose that Q' is an heir of Q on R' and show that M is existentially closed in M' relative L.
Therefore we take some $\varphi(Y) \in \mathrm{Fml}L(R)$ and some quantifier free formula $\chi(Y) \in \mathrm{Fml}L^*(R)$ where $L^* = L(\mathfrak{D}_f | f \in \mathbb{Z}[X,Y])$. By definition of L^* we can assume that $\chi(Y)$ is a finite disjunction of formulas of the form

$$\mathfrak{D}_{f_i}(g_1(Y,a),...,g_{l(i)}(Y,a))$$

and

$$\neg\mathfrak{D}_{f_j}(g_1(Y,a),...,g_{l(j)}(Y,a))$$

where $g_k(Y,Z) \in \mathbb{Z}[Y,Z]$ and $a \in R^Z$.
We suppose that

$$M' \models \exists Y(\varphi(Y) \wedge \chi(Y)),$$

i.e. there is some $d' \in R'^Y$ such that

$$M' \models \varphi(d') \wedge \chi(d').$$

By defining $c'_k := g_k(d',a)$ for every k appearing in the finite disjunction of $\chi(Y)$ we have in particular

$$M' \models \mathfrak{D}_{f_i}(c'_1,...,c'_{l(i)})$$

and

$$M' \models \neg\mathfrak{D}_{f_j}(c'_1, ..., c'_{l(j)})$$

for every i, j appearing in the finite disjunction of $\chi(Y)$. This means that for all i, j we have

$$f_i(X, c'_1, ..., c'_{l(i)}) \in Q', f_j(X, c'_1, ..., c'_{l(j)}) \notin Q'$$

and

$$R' \models \exists(Y\varphi(Y) \wedge \bigwedge_k c'_k = g_k(Y, a)).$$

Because Q' is an heir of Q on R' there are $c_k \in R$ such that

$$f_i(X, c_1, ..., c_{l(i)}) \in Q, f_j(X, c_1, ..., c_{l(j)}) \notin Q$$

for every i, j and

$$R \models \exists Y(\varphi(Y) \wedge \bigwedge_k c_k = g_k(Y, a)).$$

If $d \in R^Y$ with $R \models \varphi(d) \wedge \bigwedge_k c_k = g_k(d, a)$ then we have $M \models \varphi(d) \wedge \chi(d)$, i.e.

$$M \models \exists Y(\varphi(Y) \wedge \chi(Y)),$$

which proves the claim.

Theorem A.7 □

In this model theoretic setting we will prove the existence of heirs and later on give a proof of Proposition 3.4 by using a theorem about resplendent structures.

Definition A.8
Let L be a first-order language and κ a cardinal.
An L-structure M is κ-resplendent, if the following is true:
Given $A \subseteq M, |A| < \kappa$,
$\mathfrak{R}$ a set of new relation symbols, $|\mathfrak{R}| < \kappa$,
$\mathfrak{F}$ a set of new function symbols, $|\mathfrak{F}| < \kappa$,
$\mathfrak{C}$ a set of new constant symbols, $|\mathfrak{C}| < \kappa$.
If for some set of sentences $\theta \subseteq SenL(\underline{A} \cup \mathfrak{R} \cup \mathfrak{F} \cup \mathfrak{C})$

$$Th(M, \underline{M}) \cup \theta \subseteq SenL(\underline{M} \cup \mathfrak{R} \cup \mathfrak{F} \cup \mathfrak{C})$$

is consistent then there is an expansion of $(M, \underline{M})$ to an $L(\underline{M} \cup \mathfrak{R} \cup \mathfrak{F} \cup \mathfrak{C})$-structure which satisfies this set.

This means that if we have a set of sentences θ in a language which extends L by strictly fewer than κ new constant symbols taken from M and strictly fewer than κ new symbols and $Th(M, \underline{M}) \cup \theta$ has a model then we can interpret the new symbols on the domain of M in such a way as to have a model of θ.

Important for us is that for every L-structure M there is an elementary extension M' which is $|M|^+$-resplendent, where $|M|^+$ denotes the smallest cardinal strictly greater than the cardinality of M.

Theorem A.9 (Poizat, [P] Theorem 9.14)
For every L-structure M there is some elementary extension $M' \succ M$ such that M' is $|M|^+$-resplendent.

Proposition A.10
Let $R' \supseteq R$ be real closed fields and $Q \subseteq R[X]$. Then Q has an heir on R'.

Proof:
By theorem A.9 we have an elementary extension $R'' \succ R'$ which is $|R|^+$-resplendent, i.e. a real closed overfield R'' of R' which is $|R|^+$-resplendent.

As in Theorem A.7 we denote by

$$M := (R, (D_R(f, Q) | f \in \mathbb{Z}[X, Y]))$$

the $L^* := L(\mathfrak{D}_f | f \in \mathbb{Z}[X, Y])$-structure expanding R.

The $|R|^+$-resplendency of R'' implies that there is an expansion M'' of R'' to an $L(\mathfrak{D}_f | f \in \mathbb{Z}[X, Y])$-structure such that M'' is an elementary extension of M.

Let M' be the restriction of M'' to R'. Then M is existentially closed in M' relative L. This can be seen as follows:
Let $\varphi(Y)$ be some $L(R)$-formula and $\chi(y)$ some quantifier-free $L^*(R)$-formula and

$$M' \models \exists Y(\varphi(Y) \wedge \chi(Y)).$$

As M' is a substructure of M'' we also have

$$M'' \models \exists Y(\varphi(Y) \wedge \chi(Y))$$

which implies by the fact that M'' is an elementary extension of M that

$$M \models \exists Y(\varphi(Y) \wedge \chi(Y))$$

as desired. We define

$$Q' := \{f(X, c') \mid f(X, Y) \in \mathbb{Z}[X, Y], c' \in R'^Y, M' \models \mathfrak{D}_f(c')\}$$

and get by Theorem A.7 i) that $M' = (R', (D_{R'}(f, Q') | f \in \mathbb{Z}[X, Y]))$. This means by Theorem A.7 ii) that Q' is an heir of Q on R'.

Prop. A.10 $\square$

Proposition A.11
Let $R' \supseteq R$ be a real closed field and $Q \subseteq R[X]$. Then

$$h(Q, R') = \bigcap_{Q' \text{heir of } Q \text{on } R'} Q'$$

and

$$H(Q, R') = \bigcup_{Q' \text{heir of } Q \text{on } R'} Q'$$

Proof:
First we prove the claim for $h(Q, R')$.

The inclusion $\subseteq$ is clear because every heir of Q on R' is in particular a weak heir and $h(Q, R')$ is the smallest weak heir of Q on R' (Remark after Proposition 3.2).

Now we prove $\supseteq$.
Therefore we suppose that $g(X, d') \notin h(Q, R')$ for some $g(X, Z) \in \mathbb{Z}[X, Z]$ and some $d' \in R'^Z$.

Claim: There is some heir Q' of Q on R' such that $g(X, d') \notin Q'$.

Because $g(X, d') \notin h(Q, R')$ we have for every $L(R)$-formula $\varphi(Z)$ with $R' \models \varphi(d')$ some $d \in \varphi(R^Z)$ with $g(X, d) \notin Q$.
The $L(R)$-formulas $\varphi(Z)$ with $R' \models \varphi(d')$ are exactly the elements of the type $p \in S_Z(R)$ of d' over R.
By Theorem A.9 we may extend R' if necessary and assume that R' is $|R|^+$-resplendent. (If $R'' \supseteq R'$ real closed and $|R|^+$-resplendent and Q'' an heir of Q on R'' then $Q' := Q'' \cap \{f(X, c') \mid f(X, Y) \in \mathbb{Z}[X, Y], c' \in R'^Y\}$ is an heir of Q on R').
As in Theorem A.7 we consider the $L(\mathfrak{D}_f | f \in \mathbb{Z}[X, Y])$-structure

$$M := (R, (D_R(f, Q) | f \in \mathbb{Z}[X, Y]))$$

and denote by $\underline{a}$ a Z-tuple of new constants.

We prove in the following that

$$Th(R', \underline{R'}) \cup Th(M, \underline{R}) \cup p(\underline{a}) \cup \{\neg D_g(\underline{a})\}$$

is consistent.
Since $g(X, d') \notin h(Q, R')$ we have for every $\varphi(\underline{a}) \in p(\underline{a})$ some $d \in R^Z$ with $R \models \varphi(d)$ and $M \models \neg\mathfrak{D}_g(d)$. Thus $(M, \underline{R}, d) \models Th(M, \underline{R}) \cup \varphi(\underline{a}) \cup \neg\mathfrak{D}_g(\underline{a})$ which proves that every finite subset of $Th(M, \underline{R}) \cup p(\underline{a}) \cup \neg D_g(\underline{a})$ is consistent.

By Robinsons consistency theorem ([Ho] Corollary 8.5.11) applied to the theory $Th(M, \underline{R}) \cup p(\underline{a}) \cup \neg D_g(\underline{a})$ in the language $L(R, (\mathfrak{D}_f | f \in \mathbb{Z}[X,Y]), \underline{a})$ and the theory $Th(R', \underline{R'})$ in the language $L(R')$ we can conclude that also $Th(R', \underline{R'}) \cup Th(M, \underline{R}) \cup p(\underline{a}) \cup \{\neg D_g(\underline{a})\}$ is consistent because the intersection of both theories with the common sublanguage is $Th(R, \underline{R})$ which is consistent.

The consistency of $Th(R', \underline{R'}) \cup Th(M, \underline{R}) \cup p(\underline{a}) \cup \{\neg D_g(\underline{a})\}$ and the $|R|^+$-resplendence of R' imply the following:
R' can be expanded to an $L(R, (\mathfrak{D}_f | f \in \mathbb{Z}[X,Y]), \underline{a})$-structure (M', b') which satisfies $Th(M, \underline{R}) \cup p(\underline{a}) \cup \{\neg D_g(\underline{a})\}$, i.e. $M' \succ M$, $M' \models p(b')$ and $M' \models \neg D_g(b')$.
As $M' \models p(b')$ the type of b' over R is the same as the type of d' over R. Since the $|R|^+$-resplendence of R' implies that R' is strong $|R|^+$-homogeneous there is an R-automorphism σ of R' with $\sigma(b') = d'$.
Now we define for $f(X,Y) \in \mathbb{Z}[X,Y]$

$$D_{R'}(f) := \{\sigma(c') \mid c' \in R'^Y, M' \models \mathfrak{D}_f(c')\}$$

and

$$M'' := (R', (D_{R'}(f) | f \in \mathbb{Z}[X,Y])).$$

Since σ is an R-isomorphism $M' \to M''$ we have $M'' \succ M$. Furthermore we have $M'' \models \neg\mathfrak{D}_g(d')$ because $M' \models \neg\mathfrak{D}_g(b')$ and $\sigma(b') = d'$.
Now we are able to define the desired heir of Q on R':

$$Q' := \{f(X, c') \mid f(X,Y) \in \mathbb{Z}[X,Y], c' \in R'^Y, M'' \models \mathfrak{D}_f(c')\}$$

The fact that M is an elementary substructure of M'' implies as shown in the proof of A.10 that M is existentially closed in M'' relative L. Hence by Theorem A.7 Q' is an heir of Q on R' with $D_{R'}(f, Q') = D_{R'}(f)$ for every $f \in \mathbb{Z}[X,Y]$. Furthermore $g(X, d') \notin Q'$ as desired because $M'' \models \neg\mathfrak{D}_g(d')$.

The statement for $H(Q, R')$ follows from $H(Q, R') = R'[X] \setminus h(R[X] \setminus Q, R')$. If we apply the result proved above for $R[X] \setminus Q$ then we get that $h(R[X] \setminus Q, R')$ is the intersection of all heirs of $R[X] \setminus Q$ on R'. By definition of an heir we have the following: if Q' is an heir of $R[X] \setminus Q$ on R' then $R'[X] \setminus Q'$ is an heir of Q on R'. Thus

$$h(R[X] \setminus Q, R') = \bigcap_{Q' \text{heir of } Q \text{on } R'} R'[X] \setminus Q'$$

which implies that

$$H(Q, R') = \bigcup_{Q' \text{heir of } Q \text{on } R'} Q'. \qquad \textit{Prop. A.11} \ \square$$

Before we prove that $Q \subseteq R[X]$ is definable if and only if it has a unique heir on every real closed extension of R (Theorem 3.3) we give a topological reformulation of the membership in $h(Q, R')$ and $H(Q, R')$.

We denote the set of all types of length $|Y|$ with $S_Y(R)$ and provide $S_Y(R)$ with the topology which is generated by the basic open sets

$$<\varphi(Y)>:= \{p \in S_Y(R) \mid \varphi(Y) \in p\}$$

for $\varphi(Y) \in \mathrm{Fml}L(R)$. Then $S_Y(R)$ is a Stone space.

Proposition A.12
Let $R \subseteq R'$ be real closed fields, $f(X, Y) \in \mathbb{Z}[X, Y]$ and $c' \in R'^Y$. Then the following is true:

i) $f(X, c') \in h(Q, R')$ if and only if the type $tp(c'/R)$ lies in

$$\bigcup\{<\varphi(Y)>\mid \varphi(Y) \in FmlL(R) \text{ and } \varphi(R^Y) \subseteq D_R(f, Q)\}$$

ii) $f(X, c') \in H(Q, R')$ if and only if the type $tp(c'/R)$ lies in the closure of $D_R(f, Q)$ viewed as a subset of $S_Y(R)$.

Proof:

$i)$: By definition $f(X, c') \in h(Q, R')$ if and only if there is some $\varphi(Y) \in \mathrm{Fml}L(R)$ with $R' \models \varphi(c')$ and $\varphi(R^Y) \subseteq D_R(f, Q)$. This exactly means that the type $tp(c'/R)$ lies in $<\varphi(Y)>$ for some $\varphi(Y) \in \mathrm{Fml}L(R)$ with $\varphi(R^Y) \subseteq D_R(f, Q)$ as desired.

$ii)$: By definition $f(X, c') \in H(Q, R')$ if and only if for every $\varphi(Y) \in \mathrm{Fml}L(R)$ with $R' \models \varphi(c')$ there is some $c \in R^Y$ such that $R \models \varphi(c)$ and $f(X, c) \in Q$. This means that for every basic open set $<\varphi(Y)>$ with $tp(c'/R) \in <\varphi(Y)>$ the intersection of $<\varphi(Y)>$ with the set $\{c \in R^Y \mid f(X, c) \in Q\} = D_R(f, Q)$ is not empty. This proves the claim.

Prop. A.12 □

Finally we give a proof of Theorem 3.6.

Theorem A.13
A set $Q \subseteq R[X]$ is definable if and only if it has a unique heir on R' for every real closed extension field $R' \supseteq R$.

Proof:

$\Rightarrow$: Let $R' \supseteq R$ be some real closed extension. As explained in Section 3.1 we have in the definable case the canonical set

$$Q' := \{f(X,c') \mid f(X,Y) \in \mathbb{Z}[X,Y], c' \in R'^Y, R' \models \vartheta_f(c'), D_R(f,Q) = \vartheta_f(R^Y)\}$$

We show that $h(Q,R') = H(Q,R') = Q'$ in this case which proves by Theorem 3.4 that there is a unique heir of Q on R' given by Q'.

Since (H^+) and (H^-) is fulfilled for Q' and Q we have by Lemma 3.1 that $h(Q,R') \subseteq Q' \subseteq H(Q,R')$.

The inclusion $H(Q,R') \subseteq Q'$ can be seen as follows:
Suppose that $f(X,c') \notin Q'$ then $R' \models \neg\vartheta_f(c')$ where $\vartheta_f(Y)$ is an $L(R)$-formula defining $D(f,Q)$. Thus $\neg\vartheta_f(Y) \in \mathrm{Fml}L(R)$ with $c' \in \neg\vartheta_f(R'^Y)$ but obviously $\neg\vartheta(R^Y) \cap D_R(f,Q) = \emptyset$ which means by definition of $H(Q,R')$ that $f(X,c')$ is not in $H(Q,R')$

For the inclusion $Q' \subseteq h(Q,R')$ we take some $f(X,c') \in Q'$. Then we have for $\vartheta_f(Y) \in \mathrm{Fml}L(R)$ that $c' \in \vartheta_f(R'^Y)$ and $\vartheta_f(R^Y) = D_R(f,Q)$. Hence by definition $f(X,c') \in h(Q,R')$.

Altogether we have shown that $h(Q,R') = Q' = H(Q,R')$.

$\Leftarrow$: By Theorem A.9 there is a real closed overfield R' of R which is $|R|^+$-resplendent and by assumption there is a unique heir on R'.
As in Theorem A.7 we consider the $L(\mathfrak{D}_f | f \in \mathbb{Z}[X,Y])$-structure

$$M := (R, (D_R(f,Q) | f \in \mathbb{Z}[X,Y])).$$

With the help of the definability theorem of Svenonius ([P] Theorem 9.2) we want to show that M is a definable expansion of R (with parameters).

In order to do so we take an expansion M' of R' to an $L(\mathfrak{D}_f | f \in \mathbb{Z}[X,Y])$-structure such that $M' \succ M$ and an automorphism σ of R' which fixes R pointwise. We have to show that σ is an automorphism of M'.
We define

$$Q' := \{f(X,c') \mid f(X,Y) \in \mathbb{Z}[X,Y], M' \models \mathfrak{D}_f(c')\}.$$

By Theorem A.7 $D_{R'}(f,Q')$ is the interpretation of $\mathfrak{D}_f$ in M' and Q' is an heir of Q on R'. The set

$$Q'' := \{f(X,\sigma(c')) \mid f(X,c') \in Q'\}$$

is again an heir of Q on R'. By assumption there is only one heir which implies that $Q' = Q''$. Then σ also fixes $D(f,Q')$ setwise and because of the fact that $D(f,Q')$ is the interpretation of $\mathfrak{D}_f$ in M', σ is an automorphism of M'.

Theorem A.13 $\square$

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