# Numerical Aspects of Waveform Relaxation

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## 1. Introduction

Circuit simulation programs have proven to be most important CAD tools for the analysis of the electrical performance of integrated circuits. Depending on the number of modeled transistors, these simulators require the numerical solution of very large, sparse systems of differential (or even differential-algebraic) equations. For a survey see Bulirsch and Gilg (1986).

Kirchhoff's voltage and current laws and the branch equations of the various devices completely characterize the electrical behaviour of a circuit. They can be mathematically expressed as the following initial-value problem:

$$C(x(t),u(t))\dot{x}(t) = f(x(t),u(t)), \qquad x(0)=x_0, \tag{1}$$

where the time $t\in[0,T]$ and

$$C(x(t),u(t)) := \frac{\partial q}{\partial x}(x(t),u(t)) \qquad (\in \mathbb{R}^{n,n}),$$

$$f(x(t),u(t)) := i(x(t),u(t)) - \frac{\partial q}{\partial u}(x(t),u(t))\dot{u}(t),$$

$x(t)$ : vector of node voltages $(\in \mathbb{R}^n)$,

$u(t)$ : vector of input voltages $(\in \mathbb{R}^m)$,

$q(x(t),u(t))$ : vector of sums of capacitive charges at each node $(\in \mathbb{R}^n)$,

$i(x(t),u(t))$ : vector of sums of resistive currents at each node $(\in \mathbb{R}^n)$.

Cf. White and Sangiovanni-Vincentelli (1987). Under mild assumptions the capacitance matrix $C(x,u)$ of Equ. (1) is diagonally dominant. If, in addition, there exists a capacitor, whose capacitance is bounded away from zero, to ground or a voltage source at each node in the circuit, the matrix $C(x,u)$ is strictly diagonally dominant uniformly in $x,u$.

Waveform relaxation (WR) has been proposed for the numerical solution of initial-value problems of the type (1) by Lelarasmee (1982). Section 2 of the present note describes the basic idea. In Section 3 a new formulation of discretized WR is given in the linear case. This formulation allows the application of classical techniques (e.g. Varga, 1962; Young, 1971) for the determination of an optimal relaxation parameter.

## 2. Waveform Relaxation (WR)

Waveform relaxation applies the well-known Gauss-Seidel and Jacobi principles (e.g. Stoer and Bulirsch, 1980) to Equ. (1) on the function space level.

WR Gauss Seidel:

Initialize $x_2^0(t),\dots,x_n^0(t)$.

For k=1,2,3,...do (until stopping criterion is satisfied):

For i=1,...,n do: Solve

$$\sum_{j=1}^{i} c_{ij}(x_1^k,\dots,x_i^k,x_{i+1}^{k-1},\dots,x_n^{k-1},u)\cdot\dot{x}_j^k + \sum_{j=i+1}^{n} c_{ij}(x_1^k,\dots,x_i^k,x_{i+1}^{k-1},\dots,x_n^{k-1},u)\cdot\dot{x}_j^{k-1} = f_i(x_1^k,\dots,x_i^k,x_{i+1}^{k-1},\dots,x_n^{k-1},u)$$

for $x_i^k(t)$ in $[0,T]$ with initial condition $x_i^k(0)=x_{0i}$.

Here, $x=(x_1,\dots,x_n)^T$, $x_0=(x_{01},\dots,x_{0n})^T$, $f=(f_1,\dots,f_n)^T$, $C=(c_{ij})_{n,n}$.

WR Jacobi can be defined in a completely analogous manner. Lelarasmee (1982), Taubert (1986), Miekkala and Nevanlinna (1987), White and Sangiovanni-Vincentelli (1987) have given convergence theorems under various assumptions.

In practical applications the single differential equations in each cycle of a WR iteration step have to be solved numerically. For this purpose, let the interval [0,T] be subdivided by equidistant points

$$t_l = l \cdot h \quad (l=0,\dots,N), \qquad h = T/N \ . \tag{2}$$

For the sake of simplicity the same (uniform) stepsize is assumed for all the differential equations.

In VLSI applications the initial-value problem (1) is stiff. Consequently among all the methods available for the discretization of Equ. (1), this study considers the ones which are mostly used in circuit simulation, the so-called backward differentiation formulas

$$Dx_l = \frac{1}{h} \sum_{k=0}^{s} \alpha_k x_{l-k} \qquad (s \le 6). \tag{3}$$

They are stiffly stable (Gear, 1971). Here, the $x_j$ and $Dx_l$ are approximations of $x(t_j)$ and $\dot{x}(t_l)$ respectively.

## 3. The Linear Case as a Model Problem

Valuable insight can be gained by an investigation of the linear model problem

$$C\dot{x} = Bx + b, \qquad x(0) = x_0 \ . \tag{4}$$

Here, $B,C \in \mathbb{R}^{n,n}$ are constant matrices and $b \in \mathbb{R}^n$ is a constant vector. The matrix C is assumed to be strictly diagonally dominant.

Let Equ. (4) be discretized as described in Section 2 and let $x_{il}$ denote the approximation of $x_i(t_l)$. With the definitions

$$z := (x_{11},\dots,x_{1N}; \ \dots \ ; x_{n1},\dots,x_{nN})^T \in \mathbb{R}^{N \cdot n},$$

$$A := \begin{bmatrix} \alpha_0 & & & & 0 \\ \alpha_1 & \alpha_0 & & & \\ \vdots & \ddots & \ddots & & \\ \alpha_s & \cdots & \alpha_1 & \alpha_0 & \\ & \ddots & & \ddots & \ddots \\ 0 & & \alpha_s & \cdots \ \alpha_1 & \alpha_0 \end{bmatrix} \in \mathbb{R}^{N,N}, \qquad I_N \in \mathbb{R}^{N,N} \text{ identity matrix,}$$

$M_\pi(h,N) := C \otimes A - h \cdot B \otimes I_N$ ($\otimes$: direct product),

the discretized WR Gauss-Seidel (Jacobi) method is equivalent to the block Gauss-Seidel (Jacobi) method applied to the following system of linear algebraic equations:

$$M_\pi(h,N) \cdot z = \text{right hand side.} \tag{5}$$

The index $\pi$ denotes the natural blocking of $M_\pi$ into $n^2$ blocks of size $N \times N$. Please note that the enlargement of the size of matrices from n to $N \cdot n$ is for theoretical purposes only. In actual computations, the explicit formation of the matrix $M_\pi$ is unnecessary.

The iteration matrix for the block Jacobi method is

$$J_\pi(h,N) = D_\pi(h,N)^{-1} \cdot [D_\pi(h,N) - M_\pi(h,N)], \tag{6}$$

where $D_\pi(h,N)$ denotes the block diagonal part of $M_\pi(h,N)$.

3.1 Theorem

Let the matrix C be strictly diagonally dominant. Then there exists some $h_0$ such that the spectral radius

$\rho(J_\pi(h,N)) < 1$ for $0 < h \le h_0$ and all positive integers N.

Consequently the discretized WR Jacobi method converges to the solution of the discretized problem for sufficiently small stepsize h.

Proof: Let a vector norm for $x=(x_1,\dots,x_N)^T \in \mathbb{R}^N$ be defined by

$$\|x\|_\gamma := \max_{1\le l\le N} e^{-l\gamma}|x_l| \qquad (\gamma>0). \tag{7}$$

The subordinate matrix (operator) norm for $U=(u_{jl}) \in \mathbb{R}^{N,N}$ is given by

$$\|U\|_\gamma = \max_{1\le j\le N} \sum_{l=1}^{N} e^{-(j-l)\gamma}|u_{jl}|. \tag{8}$$

Correspondingly, let the vector norm $\|\cdot\|_\Gamma$ for $z = (y_1^T,\dots,y_n^T)^T \in \mathbb{R}^{N\cdot n}$, $y_k^T = (x_{k1},\dots,x_{kN})$ be defined as

$$\|z\|_\Gamma := \max_{1\le k\le n} \|y_k\|_\gamma . \tag{9}$$

The subordinate matrix norm for $W = (V_{ik})_{1\leq i,k\leq n}$ with

$V_{ik} = (v_{ik}^{(j,l)}) \in \mathbb{R}^{N,N}$ is given by

$$\| W \|_{\Gamma} = \max_{1\leq i\leq n} \max_{1\leq j\leq N} \sum_{k=1}^{n} \sum_{l=1}^{N} e^{-(j-l)\gamma} |v_{ik}^{(j,l)}|. \tag{10}$$

With these definitions

$$\| M_{\pi}(h,N) - M_{\pi}(0,N) \|_{\Gamma} = \| h\cdot B \circledast I_N \|_{\Gamma} = h\cdot \max_{1\leq i\leq n} \sum_{k=1}^{n} |b_{ik}|. \tag{11}$$

Similarly,

$$\| D_{\pi}(h,N) - D_{\pi}(0,N) \|_{\Gamma} = h\cdot\beta \quad \text{with} \quad \beta := \max_{1\leq i\leq n} |b_{ii}|. \tag{12}$$

It is easy to see that

$$\| M_{\pi}(0,N) \|_{\Gamma} = \| C \circledast A \|_{\Gamma} \leq K_1 \tag{13}$$

for a constant $K_1$, which is independent on N. As a consequence of (11) and (13)

$$\| M_{\pi}(h,N) \|_{\Gamma} \leq K \quad \text{for} \quad h \leq h_0, \tag{14}$$

where the bound K is independent on N.

Let diag C denote the diagonal matrix with the same diagonal elements as C. Since diag C and A are nonsingular

$$D_{\pi}(0,N)^{-1} = (\operatorname{diag} C)^{-1} \circledast A^{-1}. \tag{15}$$

The matrix $A^{-1}$ is lower triangular. The elements in each of its columns satisfy a recurrence relation with the $\alpha_k$'s as coefficients. As the discretization methods under consideration satisfy the stability condition (e.g. Stoer and Bulirsch,1980), there exists an upper bound $L_1$ for the absolute values of the entries in $A^{-1}$. This bound is independent on N. Consequently

$$\| A^{-1} \|_{\gamma} \leq L_1/(1-e^{-\gamma}) \tag{16}$$

and

$$\| D_\pi(0,N)^{-1} \|_\Gamma \leq L \tag{17}$$

for a constant L, which is independent on N. As a consequence of (12) and the Perturbation Lemma (e.g. Collatz, 1964), $D_\pi(h,N)^{-1}$ exists for $0 < h \leq h_0$ and

$$\| D_\pi(h,N)^{-1} - D_\pi(0,N)^{-1} \|_\Gamma \leq \frac{h \cdot \beta \cdot L^2}{1 - h \cdot \beta \cdot L} \, . \tag{18}$$

$J_\pi(0,N) = [I_n - (\text{diag}\, C)^{-1} C] \otimes I_N$ .

Because of the strict diagonal dominance of C

$$\| J_\pi(0,N) \|_\Gamma =: \nu < 1 \quad (\nu \text{ is independent on N}). \tag{19}$$

Finally,

$$\begin{aligned} J_\pi(h,N) - J_\pi(0,N) &= -D_\pi(h,N)^{-1} M_\pi(h,N) + D_\pi(0,N)^{-1} M_\pi(0,N) \\ &= -[D_\pi(h,N)^{-1} - D_\pi(0,N)^{-1}] \cdot M_\pi(h,N) \\ &\qquad - D_\pi(0,N)^{-1} \cdot [M_\pi(h,N) - M_\pi(0,N)]. \end{aligned} \tag{20}$$

As a consequence of (11), (14), (17), (18), (19), (20)

$$\| J_\pi(h,N) \|_\Gamma \leq \nu' < 1 \quad \text{for} \quad 0 < h \leq h_0,$$

where $\nu'$ is independent on N. This proves Theorem 3.1.

It is well-known textbook material that the iterative solution of linear algebraic equations can be substantially speeded up by the introduction of a relaxation parameter (SOR method). The proper selection of this parameter is decisive and has been the subject of extensive research (see Varga, 1962 and Young, 1971). The formulation given in Equ. (5) allows to carry over this theory to discretized WR. Let $M_\pi$ be $\pi$-consistently ordered. Then the result of Theorem 3.1 guarantees that $|\text{Re}\, \mu| < 1$ for all eigenvalues $\mu$ of $J_\pi$. Hence, the efficient determination of the optimal relaxation parameter is possible by the method of Huang (1973; see also Young and Huang, 1983). In general, π-consistent ordering cannot be expected in VLSI applications. However, a proper blocking can be generated by the algorithm of Cuthill and McKee (1969).

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