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SIMULATION OF INTEGRATED CIRCUITS: NUMERICAL TECHNIQUES

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Abstract. Relaxation-based techniques for the transient analysis of large-scale integrated circuits are promising candidates for a new generation of circuit simulators. The present note investigates the convergence of the discretized waveform relaxation. It is demonstrated that a SOR technique has the potential of beeing significantly faster than the conventional waveform relaxation Gauss-Seidel method. The determination of the optimal relaxation parameter is decisive and is discussed for the case of a general linear model problem.

Keywords. VLSI; waveform relaxation; successive overrelaxation; optimal relaxation parameter.

INTRODUCTION

Chip design involves simulation programs as CAD tools for the analysis of the electrical performance of integrated electronic circuits. These programs have almost completely replaced the traditional breadboard or fabrication and testing of integrated circuits as means of verifying design acceptability. In fact, a breadboard cannot accurately reproduce parasitic effects and element-matching characteristics between integrated devices and may give results which have small resemblance to the manufactured circuit. Fabrication and testing of a prototype integrated circuit for verifying a design is both expensive and time consuming.

Commonly used simulators for the transient analysis of dynamical circuits like SPICE (Nagel, 1975) were designed in the 1970's and are capable of the cost-effective analysis of circuits containing up to a few thousand transistors. Because of today's need to verify the performance of circuits even on the VLSI level with more than 100,000 transistors there is a strong demand for a new generation of circuit simulators. See Bulirsch and Gilg (1986); Sangiovanni-Vincentelli (1981); White and Sangiovanni-Vincentelli (1987).

Formulation of the Circuit Equations

The formulation of the circuit equations is based on Kirchhoff's voltage and current laws and on the branch equations of the various devices (see e.g. Desoer and Kuh, 1969). It is assumed that "nodal analysis" is applicable. The limitations of this technique are discussed by Chua and Lin (1975) and Sangiovanni- Vincentelli (1981).

The nodal equations can be expressed as

$$C(x(t),u(t))\dot{x}(t) = f(x(t),u(t)),\ x(0)=x_0, \quad (1)$$

where the time $t\in[0,T]$ and

$$C(x(t),u(t)) := \frac{\partial q}{\partial x}(x(t),u(t)) \qquad (\in \mathbb{R}^{n,n}),$$

$$f(x(t),u(t)) := i(x(t),u(t)) - \frac{\partial q}{\partial u}(x(t),u(t))\dot{u}(t),$$

$x(t)$: vector of node voltages $(\in \mathbb{R}^n)$,

$u(t)$: vector of input voltages $(\in \mathbb{R}^m)$,

$q(x(t),u(t))$: vector of sums of capacitive charges at each node $(\in \mathbb{R}^n)$,

$i(x(t),u(t))$: vector of sums of resistive currents at each node $(\in \mathbb{R}^n)$.

Cf. White and Sangiovanni-Vincentelli (1987).

These authors also present extensions of the nodal analysis technique that still allow to cast the circuit equations into the form of Equ. (1).

Under mild assumptions the capacitance matrix $C(x,u)$ of Equ. (1) is diagonally dominant. If, in addition, there exists a linear or nonlinear capacitor, whose capacitance is bounded away from zero, to ground or a voltage source at each node in the circuit, the matrix $C(x,u)$ is strictly diagonally dominant uniformly in x,u (White and Sangiovanni-Vincentelli, 1987).

If the matrix $C(x,u)$ is singular, Equ. (1) is a system of differential-algebraic equations. In the case of strict diagonal dominance the problem of stiffness still has to be faced.

Discretization

The use of backward differentiation formulas (BDF) has been advocated both for stiff differential equations and for differential-algebraic equations (Gear,1971; Lötstedt and Petzold, 1986):

$$Dx_\ell = \frac{1}{\Delta t} \sum_{k=0}^{s} \alpha_k x_{\ell-k} \qquad (s\le 6). \tag{2}$$

The x_j and Dx_ℓ are approximations of $x(t_j)$ and $\dot{x}(t_\ell)$ respectively. For the coefficients α_k see e.g. Gear (1971). It is well known that the BDF are distinguished among other multistep formulas by the property of stiff stability.

Discretization of (1) by formula (2) yields a nonlinear system of equations $F(x_\ell)=0$ at every time step $t=t_\ell$. The already mentioned routine SPICE solves these nonlinear systems of equations for a mesh of time steps

$$0 = t_0 < t_1 < \ldots < t_\ell < \ldots < t_N = T \tag{3}$$

using Newton's method and a sparse matrix technique for the resulting systems of linear equations.

However, large parts of digital circuits are often inactive or "latent". Consequently, numerical schemes are desireable that avoid the unnecessary computations involved in the re-evaluation of the voltage at nodes that are latent. Obviously, relaxation-based algorithms have the capability to exploit latency much more efficiently than SPICE-like routines. Indeed, speed improvements up to two orders of magnitude have been reported. For a detailed discussion see Newton and Sangiovanni-Vincentelli (1983). A family of promising relaxation techniques, called "waveform relaxation" (WR), has been proposed by Lelarasmee (1982).

Waveform Relaxation (WR)

Both the Gauss-Seidel principle and the Jacobi principle for the iterative solution of large, sparse linear systems of equations are described in most textbooks on numerical analysis (e.g. Stoer and Bulirsch, 1980). The extension to nonlinear systems is straightforward (Ortega and Rheinboldt, 1970). Waveform relaxation applies these principles to Equ. (1) on the function space level.

WR Gauss-Seidel:

Initialize $x_2^0(t),\ldots,x_n^0(t)$.

For $k=1,2,3,\ldots$ do (until stopping criterion is satisfied):

For $i=1,\ldots,n$ do: Solve

$$\sum_{j=1}^{i} c_{ij}(x_1^k,\ldots,x_i^k,x_{i+1}^{k-1},\ldots,x_n^{k-1},u)\cdot\dot{x}_j^k + \sum_{j=i+1}^{n} c_{ij}(x_1^k,\ldots,x_i^k,x_{i+1}^{k-1},\ldots,x_n^{k-1},u)\cdot\dot{x}_j^{k-1} = f_i(x_1^k,\ldots,x_i^k,x_{i+1}^{k-1},\ldots,x_n^{k-1},u)$$

for $x_i^k(t)$ in the interval $[0,T]$ with initial condition $x_i^k(0)=x_{0i}$.

Here, $x=(x_1,\ldots,x_n)^T$, $x_0=(x_{01},\ldots,x_{0n})^T$, $f=(f_1,\ldots,f_n)^T$, $C=(c_{ij})_{n,n}$.

The definition of WR Jacobi is completely analogous. Lelarasmee (1982), Taubert (1986), White (1985), White and Sangiovanni-Vincentelli (1987) have given convergence theorems on the function space level. The routines RELAX (Lelarasmee and Sangiovanni-Vincentelli,1982) and SISAL (Ingenbleek et al.,1986) are circuit simulators that are based on waveform relaxation.

DISCRETIZED WAVEFORM RELAXATION

Waveform relaxation is a "multirate" integration method, i.e. it allows to pick different discretization points, or time steps, for each differential equation in the system. Nevertheless, equal time steps are assumed in the present note. As a first approach, this assumption is reasonable and gives interesting theoretical insights.

An Introductory Example

Consider the following two-node circuit:

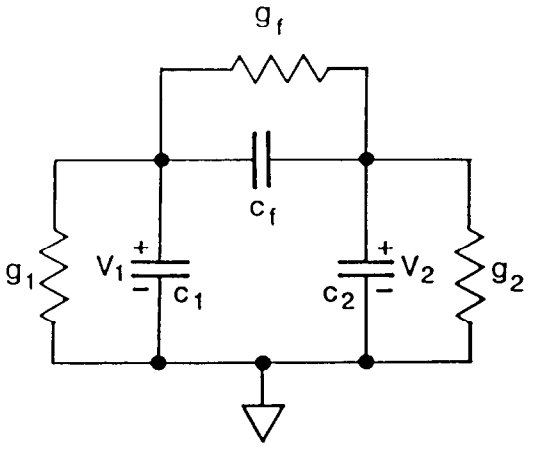

FIG. 1. A two-node circuit with a floating capacitor (adapted from White and Sangiovanni-Vincentelli, 1987).

Applying the nodal analysis technique yields the following differential equations:

$$(c_1 + c_f)\dot{x} - c_f\dot{y} = -(g_1 + g_f)x + g_f y \ , \qquad (4a)$$

$$-c_f\dot{x} + (c_2 + c_f)\dot{y} = g_f x - (g_2 + g_f)y \ . \qquad (4b)$$

Of course, nontrivial initial conditions $(x_0,y_0)^T$ are assumed. Let the mesh (3) of time steps be defined by $t_l := l \cdot h$ ($l=0,1,\ldots,N$) with stepsize $h:=T/N$, and let the differential equations (4) be discretized by the backward Euler formula (simplest backward differentiation formula):

$$(c_1 + c_f)\frac{x_l - x_{l-1}}{h} - c_f\frac{y_l - y_{l-1}}{h} =$$

$$-(g_1 + g_f)x_l + g_f y_l \ , \qquad (5a)$$

$$-c_f\frac{x_l - x_{l-1}}{h} + (c_2 + c_f)\frac{y_l - y_{l-1}}{h} =$$

$$g_f x_l - (g_2 + g_f)y_l \ . \qquad (5b)$$

Applying the discretized WR Gauss-Seidel method to Equ. (5) yields the following procedure:

Initialize $y_0^0, y_1^0, \ldots, y_N^0$ $\quad (y_0^0 = y_0)$.

For k=1,2,3,... do:

$x_0^k := x_0$,

for l=1,...,N do: Solve for x_l^k

$$-(c_1 + c_f)x_{l-1}^k + [c_1 + c_f + h(g_1 + g_f)]x_l^k + c_f y_{l-1}^{k-1} - (c_f + hg_f)y_l^{k-1} = 0 \ ,$$

$y_0^k := y_0$,

for l=1,...,N do: Solve for y_l^k

$$-(c_2 + c_f)y_{l-1}^k + [c_2 + c_f + h(g_2 + g_f)]y_l^k + c_f x_{l-1}^k - (c_f + hg_f)x_l^k = 0 \ .$$

This is nothing else than the application of the block Gauss-Seidel (or group Gauss-Seidel) method (Varga,1962; Young,1971) to the linear system

$$M_\pi z = r \qquad (6)$$

where

$z = (x_1,\ldots,x_N;y_1,\ldots,y_N)^T$,

$r = (\ (c_1+c_f)x_0 - c_f y_0,\ 0,\ldots,0;$
$\qquad -c_f x_0 + (c_2+c_f)y_0,\ 0,\ldots,0\ )^T$,

$M_\pi = \left[\begin{array}{c:c} D_1 & E \\ \hdashline E & D_2 \end{array}\right]$. (The dotted lines define the blocking π.)

The lower bidiagonal (N,N)-matrices D_1, D_2, E are given by

$$D_j = \begin{bmatrix} c_j+c_f+h(g_j+g_f) & & 0 \\ -(c_j+c_f) & \ddots & \\ & \ddots & \ddots \\ 0 & & -(c_j+c_f),\ c_j+c_f+h(g_j+g_f) \end{bmatrix},$$

$$E = \begin{bmatrix} -(c_f+hg_f) & & & 0 \\ c_f & \ddots & & \\ & \ddots & \ddots & \\ 0 & & c_f, & -(c_f+hg_f) \end{bmatrix} .$$

The iteration matrix J_π for the block Jacobi method is given by

$$J_\pi = D_\pi^{-1}(D_\pi - M_\pi) \quad \text{where} \quad D_\pi = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} .$$

Consequently,

$$J_\pi = \begin{bmatrix} 0 & -D_1^{-1}E \\ -D_2^{-1}E & 0 \end{bmatrix} .$$

The matrices $-D_j^{-1}E$ (j=1,2) are lower triangular matrices with diagonal entries

$$(c_f + hg_f)/[c_j + c_f + h(g_j + g_f)].$$

The matrix J_π^2 is lower triangular with diagonal entries

$$\mu^2 = \frac{(c_f + hg_f)^2}{[c_1+c_f + h(g_1+g_f)][c_2+c_f + h(g_2+g_f)]} . \quad (7)$$

As the matrix M_π is π-consistently ordered (Young,1971; Hageman and Young,1981), the eigenvalues of J_π are $\pm\mu$ and the spectral radius of the iteration matrix H_π for the block Gauss-Seidel method is

$$\rho(H_\pi) = \mu^2 \quad \text{with } \mu^2 \text{ given in (7).}$$

Let for example

$$c_f = g_f = 60, \quad c_1 = c_2 = g_1 = g_2 = 1. \quad (8)$$

Then $\rho(H_\pi) = 60^2/61^2 \doteq 0.967$, and the asymptotic rate of convergence (Varga,1962; Young, 1971) is

$$R_\infty(H_\pi) = 0.0331 . \quad (9)$$

It is well known that the convergence of the Gauss-Seidel method can be significantly speeded up by introducing an appropriate relaxation parameter ω (successive overrelaxation or SOR method). For details see Varga (1962), Young (1971), Hageman and Young (1981). Let the block SOR method be applied to (6). As the matrix M_π is π-consistently ordered and the eigenvalues of J_π are real, the optimal relaxation parameter for the parameter set (8) can be computed as

$$\omega_{opt} = 2/[1 + (1-\mu^2)^{1/2}] = 61/36 \doteq 1.694 .$$

The spectral radius of the corresponding iteration matrix is given by

$$\rho(H_\pi(\omega_{opt})) = \omega_{opt} - 1 \doteq 0.694 ,$$

and the asymptotic rate of convergence is

$$R_\infty(H_\pi(\omega_{opt})) \doteq 0.365 . \quad (10)$$

A comparison of (9) and (10) shows that there is an order-of-magnitude improvement of the SOR method over the Gauss-Seidel method.

In complete analogy to the definition of the WR Gauss-Seidel method in the Introduction, a WR SOR method can be established. The above investigations demonstrate that the discretized version of this method converges significantly faster than the discretized WR Gauss-Seidel method, if the relaxation parameter is chosen properly.

General Linear Case

This section investigates how the attractive results for the introductory example (4) can be extended to the general linear model problem

$$C\dot{x} = Bx + b, \quad x(0)=x_0, \quad (11)$$

where the matrix C is strictly diagonally dominant ($B,C\in\mathbb{R}^{n,n}$; $b,x\in\mathbb{R}^n$).

Let Equ. (11) be discretized by a backward differentiation formula (2) on a mesh (3) with $t_\ell=\ell\cdot h$. Then the discretized WR SOR method is equivalent to the block SOR method applied to

$$M_\pi z = \text{right hand side} , \quad (12)$$

where $M_\pi = M_\pi(h,N) = C\otimes A - h\cdot B\otimes I_N$,

$z = (x_{11},\ldots,x_{1N}; \ \ldots \ ; x_{n1},\ldots,x_{nN})^T$,

$$A = \begin{bmatrix} \alpha_0 & & & 0 \\ \alpha_1 & \alpha_0 & & \\ \vdots & \ddots & \ddots & \\ \alpha_s & \cdots & \alpha_1 & \alpha_0 \\ & \ddots & \ddots & \ddots \\ 0 & & \alpha_s & \cdots \alpha_1 \ \alpha_0 \end{bmatrix} \in \mathbb{R}^{n,n} ,$$

$I_N \in \mathbb{R}^{N,N}$ identity matrix. For the definition of the direct product $\otimes$ see e.g. Lancaster (1969). The matrix M_π is partitioned in a natural way into n^2 blocks of size N×N (blocking π). The iteration matrix for the block Jacobi method is

$$J_\pi(h,N) = D_\pi(h,N)^{-1}[D_\pi(h,N) - M_\pi(h,N)], \quad (13)$$

where $D_\pi(h,N)$ denotes the block diagonal part of $M_\pi(h,N)$.

For h=0,

$$M_\pi(0,N) = C\otimes A \quad \text{and} \quad D_\pi(0,N) = (\operatorname{diag} C)\otimes A,$$

where diag C denotes the diagonal matrix with the same diagonal elements as C. Consequently,

$$J_\pi(0,N) = I_n\otimes I_N - [(\operatorname{diag} C)\otimes A]^{-1}\cdot(C\otimes A)$$
$$= [I_n - (\operatorname{diag} C)^{-1}C]\otimes I_N .$$

Because of the strict diagonal dominance of C

$$\| J_\pi(0,N)\| =: \nu < 1 \quad (\nu \text{ is independent on } N). \quad (14)$$

Here, $\|.\|$ denotes the matrix (operator) norm, which is induced by the maximum norm. It is easy to see that

$$\| D_\pi(h,N) - D_\pi(0,N)\| \le |h|\cdot \max_{1\le i\le n} |b_{ii}| ,$$
$$\| M_\pi(h,N) - M_\pi(0,N)\| \le |h|\cdot\|B\| .$$

Because of Equ. (13) there exists a constant h_0, which is independent on N, such that

$$\| J_\pi(h,N)\| \le (\nu+1)/2 < 1 \quad \text{for} \quad 0 \le h \le h_0 .$$

Consequently, the discretized WR Jacobi method converges for sufficiently small h to the solution of the discrete problem. This confirms a result of White and Sangiovanni-Vincentelli (1987). However, the present approach can be extended to the SOR method. Indeed, a check on the π-consistency of the matrix M_π is possible by the following Theorem, which can be proven in a staightforward manner.

Theorem.

Let $B=(b_{ij})_{n,n}$ and $C=(c_{ij})_{n,n}$ and let the matrix $T=(t_{ij})_{n,n}$ be defined by

$$t_{ij} := \begin{cases} 1 & \text{if } b_{ij}\neq 0 \text{ or } c_{ij}\neq 0 , \\ 0 & \text{otherwise} . \end{cases}$$

Then, the matrix M_π is π-consistently ordered if and only if T is consistently oerdered.

If M_π is π-consistently ordered, the eigenvalues λ of the block SOR iteration matrix $H_\pi(\omega)$ and the eigenvalues μ of the block Jacobi iteration matrix J_π are related through

$$(\lambda + \omega - 1)^2 = \lambda\omega^2\mu^2 .$$

This relation is the basis for the algorithm of Young and Eidson (1970; see also Young,1971), which allows the efficient determination of the optimal relaxation parameter ω_{opt}. In general, the matrix M_π will not be π-consistently ordered. In this case, the algorithm of Cuthill and McKee (1969) should be applied to the matrix T (as defined in the above Theorem).

CONCLUDING REMARKS

The above example of a two-node circuit shows that the use of the WR SOR method instead of the conventional WR Gauss-Seidel method significantly improves the convergence rate, if the relaxation parameter ω is chosen properly. Proposals are made, how to find the optimal relaxation parameter in the case of a general linear problem. Practical experience with the suggested method will be reported in a subsequent paper.

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