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Generalization of a Theorem of Artin–Pfister to Arbitrary Semilocal Rings, and Related Topics

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1. INTRODUCTION

A. Pfister proved in 1966 the following theorem [14, Satz 21]: Let K be a field and let $a_1, \ldots, a_n$, b be elements of K^* such that b is positive with respect to every ordering of K in which all a_i are positive. Then

$$b = \sum_i c_i a_1^{i_1} \cdots a_n^{i_n} \qquad (*)$$

with $i = (i_1, \ldots, i_n)$ running through all multi-indices with coordinates 0 or 1 and with coefficients c_i which are sums of squares. In the special case $r = 1$, $a_1 = 1$, this is Artin's well known theorem that totally positive elements are sums of squares [1, Satz 1], and in fact Pfister deduces his theorem from Artin's theorem. He then uses this result in his study of the torsion elements of the Witt ring $W(K)$.

Throughout the present paper we work in the category of (not necessarily noetherian) semilocal rings with involution. We always denote by A such a ring and by J_A its involution. The image of an element λ of A under J_A will be denoted by $\bar{\lambda}$ and the norm $\lambda\bar{\lambda}$ will be denoted by $N(\lambda)$. Further the subring of all λ in A with $\lambda = \bar{\lambda}$ will be denoted by A_0. Of course A_0 will be equipped with the trivial involution, i.e., the identity. The case $A = A_0$ is allowed, and is in fact in the center of our interest.

In a recent paper [10] A. Rosenberg, R. Ware, and the present author introduced the notion of a *signature* of A. A signature σ of A is a homomorphism from the group A_0^* of units of A_0 to $\{\pm 1\}$ such that $\sigma(-1) = -1$ and the following holds true: If $a_1, \ldots, a_r$ are units of A_0 with $\sigma(a_1) = \cdots = \sigma(a_r) = 1$ then also $\sigma(b) = 1$ for every unit b of the form

$$b = N(\lambda_1) a_1 + \cdots + N(\lambda_r) a_r .$$

{It suffices to demand this for $r = 4$, and under very mild restrictions on A even for $r = 2$, cf. [10, Proposition 2.4]}. If A is a field then the signatures correspond uniquely to the orderings of A_0 for which all norms $N(\lambda)$ with λ in A^* are positive [10, 2.7], and for many problems in semilocal rings the signatures seem to be the right substitute for the orderings in the field case, cf. [10], [6], [7].

Now the following question is quite natural: Let $a_1, \ldots, a_n$ be units of A_0, and let b be a further unit of A_0 such that for every signature σ of A with $\sigma(a_1) = \cdots = \sigma(a_n) = 1$ also $\sigma(b) = 1$. Can then b be expressed by $a_1, \ldots, a_n$ in a similar way as in Pfister's theorem?

This problem (and in fact a slightly more general problem) has been solved in [10, Section 4] in the special case that the involution J_A is tracique, which means that there exists some μ in A with $\mu + \bar{\mu} = 1$. We have shown in this case that b has again a presentation (*) with sums of norms as coefficients c_i.

In Section 2 of the present paper the problem will be solved for—up to very mild restrictions—arbitrary semilocal rings.

We shall obtain for b an expression (*) with slightly more complicated coefficients c_i; just sums of norms will not suffice [Theorem 2.5]. Our result implies in particular that the ring A is nonreal, i.e., A has no signatures, if and only if -1 is a sum of norms [Corollary 2.7].

In Section 3 we prove for nonreal A in the case of trivial involution a result on the level of A, which by definition is the least number of norms (= squares) needed to represent -1, thus giving a first answer to a question posed in [5, p. 30].

In Section 4 we characterize the units b of A_0 of the form (*) with sums of norms as coefficients c_i. They are precisely the units b which lie in the subring of A_0 generated by $N(A)$ and $a_1, \ldots, a_n$, and which have value $\sigma(b) = 1$ for every signature σ of A with $\sigma(a_1) = \cdots = \sigma(a_n) = 1$ [Theorem 4.1].

In Section 5 an attack is made toward a characterization of the units

$$b = c_1 a_1 + \cdots + c_n a_n \tag{**}$$

with sums of norms c_i, where again the units $a_1, \ldots, a_n$ of A_0 are given. We introduce the "semisignatures" of A. These are maps from $A_0{}^*$ to $\{\pm 1\}$ which are no longer multiplicative but else fulfill similar conditions as the signatures. If 2 is a unit in A and all residue class fields of A_0 have more than 3 elements, then we prove that the units b of A_0 of the form (**) are precisely the units with value $\sigma(b) = 1$ for every semisignature σ of A with $\sigma(a_1) = \cdots = \sigma(a_n) = 1$ [Corollary 5.10]. In the case of fields with trivial involution the semisignatures correspond uniquely to the quadratic semiorderings studied by Prestel [15] and Bröcker [4], and then our result is already implicitely contained in [15] and explicitely in [4].

Our paper closes with a criterion involving semisignatures, that some multiple $n \times E$ of a given hermitian space E over A is isotropic ([Theorem 5.13], cf. [4, 2.12] in the field case).

We adopt throughout the notations of [10] with the following exceptions: Our semilocal ring with involution is denoted by A instead of (A, J_A). A *hermitian space* E over A is a free—instead of a projective—A-module of finite rank equipped with a nondegenerate hermitian form, and $W(A)$ denotes the Witt ring of these spaces, i.e., the ring denoted in [10] by $WF(A, J_A)$.

We call two hermitian spaces E, F over A *equivalent*, and write $E \sim F$, if they have the same image $[E] = [F]$ in $W(A)$. If A has trivial involution ($J_A = id.$) then the hermitian spaces over A are also called *bilinear* spaces. If again A has trivial involution a *quadratic space* over A is defined as a free module of finite rank over A equipped with a non-degenerate quadratic form, and $Wq(A)$ denotes the Witt group of these spaces, which is a module over $W(A)$. We call two quadratic spaces E, F over A equivalent, and write $E \sim F$, if they have the same image in $Wq(A)$. We usually denote the quadratic form of a quadratic space by q, and the associated bilinear form $q(x + y) - q(x) - q(y)$ by $B(x, y)$. The hermitian form of a hermitian space E will usually be denoted by Φ, and the values $\Phi(x, x)$ with x in E will be denoted by $n(x)$.

The approach chosen in the present paper is—with the exception of some parts of Section 5—in some sense an inversion of Pfister's procedure. We try to apply as much as possible known results about hermitian and quadratic forms and in particular about the Witt ring $W(A)$. We use the theory of Witt rings developed in [11], the elementary theory of signatures developed in Sections 2 and 3 of [10], and the theory of Pfister forms as far as contained in [5]. We further apply two cancellation theorems which will be stated now.

Assume that A has trivial involution.

PROPOSITION 1.1. *Let F_1, F_2, G be quadratic spaces over A with $F_1 \perp G \cong F_2 \perp G$. Then $F_1 \cong F_2$. Thus two quadratic spaces over A are isomorphic if they are equivalent and have the same rank.* {[8]; *cf. also* [12], *where a much stronger theorem is proved.*}

For a bilinear space E over A we denote by $\mathfrak{g}(E)$ the norm group of E. This is the additive subgroup of A generated by all values $n(x)$ with x in E.

PROPOSITION 1.2. *Let F_1, F_2, G be bilinear spaces over A with $F_1 \perp G \cong F_2 \perp G$. Assume further that $\mathfrak{g}(G)$ is contained in $\mathfrak{g}(F_1) \cap \mathfrak{g}(F_2)$, and that there exists at least one vector x in F_1 with $n(x) = 2a$, $a \in A^*$. Then $F_1 \cong F_2$* {[9, 6.1.3], *cf.* [13, 93:14a] *for A a discrete valuation ring*}.

We shall also need the following consequence of Proposition 1.2.

PROPOSITION 1.3 [cf. 9, 6.2.1]. *Let M and N be metabolic spaces over A, i.e., orthogonal sums of binary bilinear spaces of type $\left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Assume further that M represents a number $2a$ with a in A^*, and that M and N have the same rank and the same norm group. Then $M \cong N$.*

If the involution J_A is tracique nearly everything proved in the Sections 2 and 4 is already contained in [10, Section 4]. In fact Section 4 of [10] is a good introduction to the present paper, since in principle the same method is used but less machinery about forms is needed than in the general case.

2. DESCRIPTION OF THE SETS $\Gamma V(M)$

Throughout this paper A is a fixed semilocal ring with involution, and h denotes a fixed natural number (in particular $h \geqslant 1$) such that $4h - 1$ and $2h - 1$ are units in A. For example choose h as the product of all odd prime numbers which occur as characteristics of fields $A/\mathfrak{M}$ with $\mathfrak{M}$ a maximal ideal of A, if there are any such prime numbers, and else choose $h = 1$. That $2h - 1$ is a unit will not be needed before Section 4.

PROPOSITION 2.1. *Let $a_1, \dots, a_n$ and b be units of A_0. Assume there exists a natural number m such that mb lies in the semiring which is generated in A_0 by the set of norms $N(A)$, the set of elements $x^2 + xy + y^2h$ with x, y in A_0, and $a_1, \dots, a_n$, in other words*

$$mb = \sum_i c_i a_1^{i_1} \cdots a_n^{i_n} \tag{*}$$

with $i = (i_1, \dots, i_n)$ running through the multi-indices with coordinates 0 or 1 and coefficients c_i which are sums of elements $N(\lambda)(x^2 + xy + y^2h)$ with λ in A and x, y in A_0. Let σ be a signature of A with $\sigma(a_1) = \cdots = \sigma(a_n) = 1$. Then also $\sigma(b) = 1$.

Proof. For x and y in A_0 we have

$$2(x^2 + xy + y^2h) = (x + y)^2 + x^2 + (2h - 1)y^2.$$

Thus $2mb$ has the form

$$2mb = \sum_i d_i a_1^{i_1} \cdots a_n^{i_n}$$

with sums of norms d_i. Suppose σ is a signature with $\sigma(a_1) = \cdots = \sigma(a_n) = 1$ but $\sigma(b) = -1$.

Then the equation

$$b = (2m-1)(-b) + \sum_i d_i a_1^{i_1} \cdots a_n^{i_n}$$

yields $\sigma(b) = 1$, which is a contradiction. Q.E.D.

As in [10, Section 4] we use the following notations: For any signature σ of A we denote by $\Gamma(\sigma)$ the group of all a in $A_0{}^*$ with $\sigma(a) = 1$. For Y a set of signatures we denote by $\Gamma(Y)$ the intersection of all $\Gamma(\sigma)$ with σ in Y, with the convention $\Gamma(\phi) = A_0{}^*$. For M a nonempty subset of $A_0{}^*$ we denote by $V(M)$ the set of all signatures σ of A with $\sigma(M) = \{1\}$, and by $\mathfrak{a}(M)$ the ideal of $W(A)$ generated by the hermitian spaces $(1, -a)$ with a running through M. We want to describe for given M the set $\Gamma V(M)$ and the ideal $[\mathfrak{a}(M)]^{1/\infty}$ consisting of all z in $W(A)$ such that some power z^r lies in $\mathfrak{a}(M)$. These two problems are connected by part (iii) of the following

LEMMA 2.2. *Let M be a nonempty subset of $A_0{}^*$ and N denote the subgroup of $A_0{}^*$ generated by M and the set of unit norms $N(A^*)$. Let b denote a further unit of A_0.*

(i) $\mathfrak{a}(M) = \mathfrak{a}(N)$.

(ii) *If* $[(1, -b)] \in \mathfrak{a}(M)$, *then* $b \in N$.

(iii) *b lies in $\Gamma V(M)$ if and only if $[(1, -b)]$ lies in $[\mathfrak{a}(M)]^{1/\infty}$.*

Proof. The units a of $A_0{}^*$ with $[(1, -a)] \in \mathfrak{a}(M)$ form a group since

$$(1, -a_1) \perp (a_1) \otimes (1, -a_2) \sim (1, -a_1 a_2)$$

for a_1, a_2 in $A_0{}^*$. This implies the first assertion. Assume now $[(1, -b)] \in \mathfrak{a}(M)$. Then

$$(1, -b) \sim \mathop{\perp}_{i=1}^{r} (c_i) \otimes (1, -a_i)$$

with some a_i in M and c_i in $A_0{}^*$. Computing the signed determinant of both sides we obtain $(b) \cong (a_1 a_2 \cdots a_r)$. Thus b lies in N, and the second assertion is proved. Finally we obtain from the prime ideal theory of $W(A)$ developed in [11], that $[\mathfrak{a}(M)]^{1/\infty}$ is the set of all classes $[E]$ with $\dim E$ even and $\sigma(E) = 0$ for all signatures σ vanishing on $\mathfrak{a}(M)$. Since the set of these signatures is $V(M)$, assertion (iii) is now evident. Q.E.D.

DEFINITION 2.3.

(a) Let M be a nonempty subset of $A_0{}^*$. We call M *saturated* if M is a subgroup of $A_0{}^*$ and the following holds true: If $a_1, \ldots, a_r$ are elements of M

and $\lambda_1, \ldots, \lambda_r$ are elements of A such that

$$b := N(\lambda_1)\, a_1 + \cdots + N(\lambda_r)\, a_r$$

is a unit, then also $b \in M$. We call M *strictly saturated* if M again is a subgroup of $A_0{}^*$ and M contains with elements $a_1, \ldots, a_r$ also every unit b of the form

$$b = \sum_{i=1}^{r} N(\lambda_i)(x_i{}^2 + x_i y_i + y_i{}^2 h) a_i$$

$\{\lambda_i$ in A; x_i, y_i in $A_0\}$.

(b) For an arbitrary nonempty subset M we denote by $\hat{M}$ the smallest saturated subset of $A_0{}^*$ containing M and by $\tilde{M}$ the smallest strictly saturated subset of $A_0{}^*$ containing M. Clearly $\hat{M}$ is the intersection of the semiring generated by M and $N(A)$ with $A_0{}^*$, and $\tilde{M}$ is the intersection of the semiring generated by M, $N(A)$, and the set of elements $x^2 + xy + y^2 h$ with $A_0{}^*$. We have $\hat{M} \subset \tilde{M}$. We call $\hat{M}$ the *saturation* of M and $\tilde{M}$ the *strict saturation* of M. According to Proposition 2.1 (with $m = 1$) the set $\tilde{M}$ is contained in $\Gamma V(M)$.

THEOREM 2.4. *Assume that A_0 has no maximal ideal $\mathfrak{m}$ with $A_0/\mathfrak{m} \cong \mathbf{F}_2$. Assume further in the case of nontrivial involution ($J_A \neq id$) that A has no maximal ideal $\mathfrak{M}$ with $A/\mathfrak{M} \cong \mathbf{F}_3$. Then for an arbitrary nonempty subset M of $A_0{}^*$ we have $[\mathfrak{a}(M)]^{1/\infty} = \mathfrak{a}(\tilde{M})$.*

Proof. $\tilde{M} \subset \Gamma V(M)$ and thus $\mathfrak{a}(\tilde{M}) \subset [\mathfrak{a}(M)]^{1/\infty}$ by Lemma 2.2. We now shall prove the opposite inclusion $\mathfrak{a}(\tilde{M}) \supset [\mathfrak{a}(M)]^{1/\infty}$. We first consider the case $J_A = id$. Let z be an arbitrary element of $[\mathfrak{a}(M)]^{1/\infty}$. Since $W(A)/\mathfrak{a}(M)$ is an abstract Witt ring for an abelian group of exponent 2 [cf. 10, 4.15], there exists some natural number m with $2^m z \in \mathfrak{a}(M)$ [11]. Thus

$$2^m z = \sum_{i=1}^{n} u_i[(1, -a_i)]$$

with elements a_i of M and u_i of $W(A)$. We now consider the quadratic Pfister space

$$F := 2^m \times (1, a_1) \otimes \cdots \otimes (1, a_n) \otimes \begin{bmatrix} 1 & 1 \\ 1 & h \end{bmatrix}.$$

Here $\left[\begin{smallmatrix} 1 & 1 \\ 1 & h \end{smallmatrix}\right]$ denotes the quadratic space of rank 2 with basis e_1, e_2 and $q(e_1) = 1$, $q(e_2) = h$, $B(e_1, e_2) = 1$. Notice that $\left[\begin{smallmatrix} 1 & 1 \\ 1 & h \end{smallmatrix}\right]$ is indeed a space since $4h - 1 \in A^*$. Clearly z lies in the annullator $\mathfrak{b}$ in $W(A)$ of the element $[F]$ of $Wq(A)$ represented by F. Now it has been proved in [5, Section 4] that under our assumption about A the ideal $\mathfrak{b}$ of $W(A)$ is generated by the classes $[(1, -c)]$ with c running through the set $D^*(F)$ of units represented by F. Clearly $D^*(F) \subset \tilde{M}$, and thus $z \in \mathfrak{a}(\tilde{M})$.

We now consider the case $J_A \neq id$. For any nonempty subset S of $A_0{}^*$ we denote by $\mathfrak{a}_0(S)$ the ideal of $W(A_0)$ generated by the bilinear spaces $(1, -s)$ with s in S and by S' the strict saturation of S with respect to A_0 instead of A. The natural map from $W(A_0)$ to $W(A)$ is surjective and has under our assumptions about A the kernel $\mathfrak{a}_0(N(A^*))$ [10, Proposition 2.5]. Thus the pre-image of $\mathfrak{a}(M)$ in $W(A_0)$ is $\mathfrak{a}_0(T)$ with $T := M \cup N(A^*)$. Now $[\mathfrak{a}_0(T)]^{1/\infty} = \mathfrak{a}_0(T')$, as we have already proved. Applying the natural map from $W(A_0)$ onto $W(A)$ we obtain the equation $[\mathfrak{a}(M)]^{1/\infty} = \mathfrak{a}(T')$. Clearly $T' \subset \tilde{M}$, hence $[\mathfrak{a}(M)]^{1/\infty} \subset \mathfrak{a}(\tilde{M})$. Q.E.D.

Theorem 2.4 gives in the case $M = \{1\}$ a description of the nil radical of $W(A)$, which by [11] coincides with the set $I(A)_t$ of torsion elements of even dimension.

By Lemma 2.2 we obtain from Theorem 2.4 that the sets $\Gamma V(M)$ and $\tilde{M}$ coincide. We shall now give a second proof of this fact, which works under weaker assumptions about A (cf. proof of Theorem 4.8 in [10]).

THEOREM 2.5. *Assume either A has trivial involution or that A has no maximal ideal $\mathfrak{M}$ such that one of the following exceptional cases occurs:*

(a) $A/\mathfrak{M} \cong \mathbf{F}_2$ *or* $\mathbf{F}_3$,

(b) $\mathfrak{M}$ *is stable under* J_A, $A/\mathfrak{M} \cong \mathbf{F}_4$, $A_0/\mathfrak{M} \cap A_0 \cong \mathbf{F}_2$.

Let M be a nonempty subset of $A_0{}^*$, and b be a further unit of A_0. Then the following are equivalent

(i) $\sigma(b) = 1$ *for every signature σ of A with* $\sigma(M) = 1$.

(ii) *b has a presentation*

$$b = \sum_{i=1}^{t} a_i N(\lambda_i)(x_i{}^2 + x_i y_i + y_i{}^2 h)$$

with elements λ_i in A; x_i, y_i in A_0, and products a_i of elements of M.

(iii) *$2b$ has a presentation*

$$2b = \sum_{i=1}^{t} a_i N(\lambda_i)$$

with elements λ_i in A, and products a_i of elements of M.

(iv) *There exists a natural number m such that mb has a presentation as in* (iii).

Proof. The implication (ii) ⇒ (iii) is clear from the equation

$$2(x^2 + xy + hy^2) = (x + y)^2 + x^2 + (2h - 1)y^2.$$

The implication (iii) ⇒ (iv) is trivial, and (iv) ⇒ (i) follows from Proposition 2.1. We now prove (i) ⇒ (ii). We first consider the case of trivial involution. Let b be an element of $\Gamma V(M)$. Then there exists a finite subset $M_1 = \{a_1, \ldots, a_n\}$ of M such that b is already contained in $\Gamma V(M_1)$, as has been shown in [10] on p. 231 above by a simple topological argument. All signatures of A vanish on the bilinear space $(1, -b) \otimes (1, a_1) \otimes \cdots \otimes (1, a_n)$, thus, as proved in [11], there exists a natural number r with

$$2^r \times (1, -b) \otimes (1, a_1) \otimes \cdots \otimes (1, a_n) \sim 0.$$

Then also the quadratic space

$$2^r \times (1, -b) \otimes (1, a_1) \otimes \cdots \otimes (1, a_n) \otimes \begin{bmatrix} 1 & 1 \\ 1 & h \end{bmatrix}$$

is equivalent to zero. Consider the Pfister space

$$F := 2^r \times (1, a_1) \otimes \cdots \otimes (1, a_n) \otimes \begin{bmatrix} 1 & 1 \\ 1 & h \end{bmatrix}.$$

We have $(1, -b) \otimes F \sim 0$, hence $F \sim (b) \otimes F$, and by Proposition 1.1 even $F \cong (b) \otimes F$. Thus b is represented by F, and b fulfills an equation as indicated in (ii).

We finally deal with the case $J_A \neq id$. Our assumptions about A then imply that the signatures $\sigma\colon A_0^* \to \{\pm 1\}$ of A are precisely the signatures σ of A_0 with $\sigma(N(A^*)) = \{1\}$, cf. [11, Corollary 2.6]. Thus denoting the operations Γ, V with respect to A_0 instead of A by Γ_0, V_0, we have

$$\Gamma V(M) = \Gamma_0 V_0(M \cup N(A^*))$$

and applying (i) ⇒ (ii) to A_0 and the set $M \cup N(A^*)$ we obtain the implication (i) ⇒ (ii) in the general case. Q.E.D.

Remarks 2.6. Our proof shows that—under the assumptions of Theorem 2.5 about A—every b in $\tilde{M}$ has a presentation

$$b = \sum_{i=1}^{t} N(\lambda_i)\, a_i(x_i^2 + x_i y_i + y_i^2 h)$$

with *units* λ_i of A, elements x_i, y_i of A_0 and products a_i of elements of M. It also should be noticed that by Theorem 2.5 the set $\tilde{M}$ does not depend on our choice of h. If 2 is a unit in A then 2 and $\frac{1}{2}$ are sums of squares in A_0

and then the equivalence of the statements (ii) and (iii) in Theorem 2.5 means $\tilde{M} = \hat{M}$.

We call the ring A with involution *nonreal,* if A has no signatures. Theorem 2.5 implies the following criterion for A to be nonreal:

COROLLARY 2.7. *Let A fulfill the assumptions made in Theorem 2.5. Then the following statements are equivalent:*

(i) *A is nonreal.*

(ii) *There exists an equation*

$$-1 = N(\lambda_1) + \cdots + N(\lambda_r)$$

with λ_i in A.

Proof. (ii) ⇒ (i) is evident. Assume now that A is nonreal. Then applying Theorem 2.5 with $M = \{1\}$ and $b = -1$ we obtain an equation

$$-2 = \sum_{i=1}^{t} N(\lambda_i)$$

and then

$$-1 = 1 + \sum_{i=1}^{t} N(\lambda_i). \qquad \text{Q.E.D.}$$

Another consequence of Theorem 2.5 is the following:

COROLLARY 2.8. *Let C be a semilocal ring with involution containing A as a subring and with J_C extending J_A. Assume that the assumptions made about A in Theorem 2.5 are now fulfilled for C. Let σ be a signature of A. Then σ can* not *be extended to C if and only if there exists an equation*

$$-1 = N(\lambda_1)\, a_1 + \cdots + N(\lambda_r)\, a_r$$

with elements λ_i of C and units a_i of A_0 fulfilling $\sigma(a_1) = \cdots = \sigma(a_r) = 1$.

Proof. Apply Theorem 2.5 to the ring C, the set M of all units of A_0 with $\sigma(a) = 1$, and $b = -1$, proceeding as in the proof of Corollary 2.7.

3. A REMARK ON THE LEVEL OF A NONREAL SEMILOCAL RING

In this section we have to assume that the involution of our semilocal ring A is trivial. This restriction seems to be necessary at this moment since up to now no cancellation theorem analogous to Proposition 1.2 has been established in the unitary case. Assume that A is nonreal. Then we know from [11] that $W(A)$ is a 2-torsion group. We further know from Corollary 1.7 that -1 is a sum of squares in A. As usual we call the smallest number of squares needed to represent -1 the *level* $s(A)$ of A. The goal of this section is to prove.

PROPOSITION 3.1. *Let 2^t denote the order of the unit element of $W(A)$. Then*

$$2^{t-2} < s(A) \leqslant 2^t.$$

Remarks. If 2 is a unit in A then $s(A) = 2^{t-1}$, as has been shown by Baeza in [2]. If A is a local ring and 2 is not a unit in A then it is known from [5, Section 3] that $s(A)$ is a number of the form 2^r or $2^r - 1$. Thus Proposition 2.1 implies in this case that $s(A)$ has one of the four values 2^t, $2^t - 1$, 2^{t-1}, $2^{t-1} - 1$.

The inequality $s(A) > 2^{t-2}$ in Proposition 3.1 is easily established. Nothing has to be proved for $t = 1$. Let $t \geqslant 2$ and suppose $s(A) \leqslant 2^{t-2}$. Then the space $2^{t-2} \times (1)$ represents -1 and thus -1 is a norm of similarity of this Pfister space [5, Theorem 1.5.d]. We obtain

$$2^{t-1} \times (1) \cong 2^{t-2} \times (1, 1) \sim 0,$$

which contradicts the definition of t. Thus $s(A) > 2^{t-2}$.

To prove the other inequality in Proposition 3.1 we need the following lemma, which deserves independent interest.

LEMMA 3.2. *Let A be a semilocal ring with trivial involution and F be a bilinear space over A with $2^n \times F \sim 0$ for some $n \geqslant 2$. Then $2^{n+1} \times F$ is isomorphic to $2^n \times F \otimes (1, -1)$ and in particular metabolic.*

Proof. Let E denote the space $2^n \times F$. The quadratic space $E \otimes \left[\begin{smallmatrix}1 & 1\\ 1 & h\end{smallmatrix}\right]$ is equivalent to zero, and thus by Proposition 1.1.

$$E \otimes \begin{bmatrix}1 & 1\\ 1 & h\end{bmatrix} \cong r \times \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

with $r := \dim E$ and $\left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$ the quadratic hyperbolic plane. Passing to the associated bilinear spaces we obtain

$$E \otimes \begin{pmatrix}2 & 1\\ 1 & 2h\end{pmatrix} \cong r \times \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Now the following isomorphy is easily verified:

$$\begin{pmatrix}2 & 1\\ 1 & 2h\end{pmatrix} \perp (-1) \cong (1, 2h - 1, (2h - 1)(1 - 4h)).$$

{Pass from a basis x, y, z of the left hand side corresponding to the indicated matrices to an orthogonal basis $x + z, y + z, w$. Notice that $2h - 1$ is a unit.} Thus

$$2 \times (-1) \perp \begin{pmatrix} 2 & 1 \\ 1 & 2h \end{pmatrix} \cong (1, -1, 2h - 1, (2h - 1)(1 - 4h)).$$

Multiplying by $-E$ we obtain

$$2 \times E \perp r \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong E \otimes (1, -1, 1 - 2h, (2h - 1)(4h - 1)).$$

Now the natural numbers $2h - 1$ and $4h - 1$ are sums of four squares by Lagrange's theorem, and are units of A. Since E has as a factor the Pfister space $4 \times (1)$, the elements $2h - 1$ and $4h - 1$ are norms of similarity of E, and we obtain

$$2 \times E \perp r \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong E \otimes (1, -1, 1, -1).$$

Now

$$(1, -1, 1, -1) \cong (1, -1) \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

as is easily seen {pass from a basis x, y, z, w corresponding to the left hand side to a basis $x', y, z - w, w - x$, with x' orthogonal to the other three vectors}. Thus finally

$$2 \times E \perp r \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong E \otimes (1, -1) \perp r \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We now show that the space $E \otimes (1, -1)$ represents the number 2. Then we obtain from Proposition 1.2

$$2 \times E \cong E \otimes (1, -1)$$

which is the desired assertion.

Let $e_1, \ldots, e_s$ be an arbitrary basis of F and let a_i denote the value $n(e_i)$. Applying some results about metabolic spaces proved in [9, Section 3] we see

$$F \otimes (1, -1) \cong \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} a_s & 1 \\ 1 & 0 \end{pmatrix}$$

[9, 3.1.3 and 3.1.1], and then

$$E \otimes (1, -1) \cong \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} a_s & 1 \\ 1 & 0 \end{pmatrix} \perp (r - s) \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

[9, 3.4.1]. Thus $E \otimes (1, -1)$ contains a hyperbolic plane $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ which represents the number 2. Q.E.D.

Now we are able to prove the remaining inequality $s(A) \leqslant 2^t$ of Proposition 3.1. If $t = 1$, then $2 \times (1) \sim 0$, and considering the signed determinant of $2 \times (1)$ we see $(1) \cong (-1)$. Thus $s(A) = 1 = 2^{t-1}$. Assume $t \geqslant 2$. Then we can apply Lemma 3.2 to the space $F = (1)$ and obtain

$$2^{t+1} \times (1) \cong 2^t \times (1) \perp 2^t \times (-1).$$

Proposition 1.2 yields

$$2^t \times (1) \cong 2^t \times (-1).$$

Thus -1 is a sum of 2^t squares.

4. Description of the Saturations $\hat{M}$

Up to the end of the paper the subring of A_0 generated by the set $N(A)$ of norms is denoted by B, and for any nonempty subset M of $A_0{}^*$ the ring generated in A_0 by $N(A)$ and M is denoted by $B[M]$. Clearly the saturation $\hat{M}$ (cf. 2.3) is contained in $B[M]$, and $\hat{M}$ is also contained in $\Gamma V(M)$. Throughout this section we assume that *in the case* $J_A \neq id$ *all fields* $A_0/\mathfrak{m}$ with $\mathfrak{m}$ a maximal ideal of A_0 *have at least four elements.*

THEOREM 4.1. *Let M be a nonempty subset of $A_0{}^*$. Then*

$$\hat{M} = \Gamma V(M) \cap B[M].$$

In other terms, for every unit b of A_0 the following properties are equivalent:

(i) $b = \sum_{i=1}^{r} a_i N(\lambda_i)$

with elements λ_i of A and products a_i of elements of M.

(ii) *b lies in $B[M]$ and $\sigma(b) = 1$ for every signature σ of A with $\sigma(M) = \{1\}$.*

Remark. If J_A is tracique then $B = A_0$ and we essentially obtain Theorem 4.8 of [10]. {In [10] no assumptions about the fields $A_0/\mathfrak{m}$ are needed at this place.}

Proof of Theorem 4.1. Let b be an element lying in $\Gamma V(M)$ and $B[M]$. We have to show that b lies in $\hat{M}$. We first consider the case $J_A = id$. There exists a finite subset $M_1 = \{a_1, ..., a_r\}$ of M such that b lies already in

$\Gamma V(M_1) \cap B[M_1]$, cf. [10, p. 231]. The bilinear space $(1, -b) \otimes (1, a_1) \otimes \cdots \otimes (1, a_n)$ is nilpotent in $W(A)$, since all signatures vanish on this space. According to [11] and Lemma 3.2 there exists a ntaural number n such that

$$2^n \times (1, -b) \otimes (1, a_1) \otimes \cdots \otimes (1, a_n)$$

is metabolic. Let F denote the Pfister space $2^n \times (1, a_1) \otimes \cdots \otimes (1, a_n)$. This space has the norm group $B[M_1]$. Now b lies in $B[M_1]$, and b^2 is a unit of $B[M_1]$. Thus b is a unit of $B[M_1]$, and the space $(b) \otimes F$ has also the norm group $B[M_1]$. We obtain from Proposition 1.3

$$(b) \otimes F \perp (-1) \otimes F \cong F \perp (-F),$$

since both spaces are metabolic, have the same rank and norm group, and represent the number -2. Then Proposition 1.2 yields $(b) \otimes F \cong F$. Thus b is represented by F and lies in $\hat{M}$.

To prove $b \in \hat{M}$ in the case $J_A \neq id$ we need the following lemma, which will be proved afterwards.

Lemma 4.2. *The ring B is already generated by $N(A^*)$ and the set $A_0{}^2$ of squares in A_0.*

Let P denote the set $M \cup N(A^*)$ and let Γ_0, V_0 denote the operations Γ, V with respect to A_0 instead of A. We have $\Gamma V(M) = \Gamma_0 V_0(P)$ by [10, Corollary 2.6]. Furthermore $B[M]$ coincides with the ring generated in A_0 by P and $A_0{}^2$ by Lemma 4.2. Thus, as we have already proved, b lies in the saturation of P with respect to A_0. A fortiori b lies in $\hat{M}$.

It remains to prove Lemma 4.2. Let B' denote the subring of A_0 generated by $A_0{}^2$ and $N(A^*)$. We have to show $N(\lambda) \in B'$ for a given element λ of A. Since all fields $A_0/\mathfrak{m}$ have more than 3 elements, there exists a unit a of A_0 with $a \not\equiv 1$ and $a \not\equiv -1 \bmod \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of A_0, and then there exists a unit ϵ of A with $\epsilon \not\equiv -\lambda$ and $\epsilon \not\equiv -a^2\lambda \bmod \mathfrak{M}$ for all maximal ideals $\mathfrak{M}$ of A. Thus $\lambda + \epsilon$ and $a\lambda + a^{-1}\epsilon$ are units, and

$$(a^2 - 1) N(\lambda) = N(a\lambda + a^{-1}\epsilon) - N(\lambda + \epsilon) - N(a^{-1}\epsilon) + N(\epsilon)$$

lies in B'. Now $a^2 - 1$ is a unit in A_0, and thus by an argument already used $a^2 - 1$ is a unit of B. We see that $N(\lambda)$ lies in B'. This completes the proof of Lemma 4.2 and Theorem 4.1.

Remark 4.3. Our proof yields also that every element b of $\hat{M}$ has a presentation

$$b = \sum_{i=1}^{r} a_i N(\epsilon_i) c_i{}^2$$

with *units* ϵ_i of A, elements c_i of A_0, and products a_i of elements of M.

As an application of Theorem 3.1 we describe the saturated sets which contain -1.

COROLLARY 4.4. *The saturated subsets M of $A_0{}^*$ which contain -1 corresponds uniquely to the subrings D of A_0 which contain B by the relations*

$$M = D^*, \qquad D = B[M].$$

Proof. (i) Let D be a subring of A_0 containing B. Clearly D^* contains -1 and is saturated. We now prove $B[D^*] = D$. Let d be an arbitrary element of D. We can find some a in A_0 such that for every maximal ideal $\mathfrak{m}$ of A_0 we have $a \equiv 1 \bmod \mathfrak{m}$ if $d \equiv 0 \bmod \mathfrak{m}$ and $a \equiv 0 \bmod \mathfrak{m}$ if $d \equiv 1 \bmod \mathfrak{m}$. The element $d + a^2$ is a unit of A_0, and thus also a unit of D, since A_0 is integral over D. Thus $d = (d + a^2) - a^2$ lies in $B[D^*]$.

(ii) Let now M be a saturated subset of $A_0{}^*$ containing -1. Then $\Gamma V(M) = A_0{}^*$, and applying Theorem 4.1 we obtain

$$M = \hat{M} = A_0{}^* \cap B[M] = B[M]^*. \qquad \text{Q.E.D.}$$

We close this section with two characterizations of the subgroups $\Gamma(\sigma)$ of $A_0{}^*$ corresponding to the signatures σ of A.

COROLLARY 4.5. (i) *Let M be a saturated subset of $A_0{}^*$ which does not contain -1 and is maximal with respect to these properties. Then there exists a signature σ of A with $M = \Gamma(\sigma)$.*

(ii) *Assume in the case $J_A = \mathrm{id}$ that all fields $A_0/\mathfrak{m}$ have at least 3 elements and—as always in this section—in the case $J_A \neq \mathrm{id}$ that all $A_0/\mathfrak{m}$ have at least 4 elements. Let M be a saturated subgroup of $A_0{}^*$ with $(A_0{}^* : M) = 2$. Then again there exists a signature σ with $M = \Gamma(\sigma)$.*

Proof. (i) It suffices to show that there exists a signature σ with $M \subset \Gamma(\sigma)$. Suppose this is not true. Then $\Gamma V(M) = A_0{}^*$, and Theorem 4.1 implies

$$M = A_0{}^* \cap B[M] = B[M]^*.$$

This is a contradiction, since $-1 \notin M$.

(ii) We choose a unit a in $A_0{}^*$ which does not lie in M. Then $A_0{}^* = M \cup aM$. By part (i) of our corollary, which has already been proved, it suffices to show that -1 does not lie in M. Suppose $-1 \in M$. By our assumption about the fields $A_0/\mathfrak{m}$ and the Chinese remainder theorem it is possible to find elements b_1, b_2 of A_0 such that $c := b_1{}^2 + b_2{}^2$ is a unit and also $a + c$ is a unit. Now either $a + c = m$ or $a + c = am$ with some element m of M. In the first case we obtain $a = m - c \in M$, which is a contradiction. In the second case we obtain $a(m - 1) = c \in M$. This implies that $m - 1$ is a unit and then $m - 1 \in M$. We again arrive at the contradiction $a \in M$. Thus -1 does not lie in M. Q.E.D.

5. SEMISIGNATURES

Let M be a subset of $A_0{}^*$. How can we characterize the set of units of A_0 which are finite sums $N(\lambda_1) a_1 + \cdots + N(\lambda_r) a_r$ with a_i in M and λ_i in A? This problem has been solved in the previous section in the special case that MA_0^{*2} is a subgroup of $A_0{}^*$. An analogous characterization should be possible in the general case using "semisignatures" instead of signatures (see definition below).

We denote by $Q(A)$ the group $A_0{}^*/N(A^*)$ which can be interpreted as the group of isomorphy classes of hermitian spaces of rank one over A. The canonical map from $Q(A)$ to $W(A)$ is injective, as can be seen by use of the signed determinant. We regard $Q(A)$ as a subgroup of $W(A)^*$.

DEFINITION 5.1. A *semisignature* of A is an additive map from $W(A)$ to $\mathbf{Z}$ which on $Q(A)$ only takes values ± 1.

Since $Q(A)$ generates the ring $W(A)$ additively, a semisignature σ is uniquely determined by its restriction to $Q(A)$. For a space (a) of rank one we shortly write $\sigma(a)$ instead of $\sigma([a])$. We usually identify a semisignature σ with the corresponding map $a \mapsto \sigma(a)$ from $A_0{}^*$ to $\{\pm 1\}$. We have $\sigma(-a) = -\sigma(a)$ for every unit a of A_0, since $(a) \perp (-a) \sim 0$.

LEMMA 5.2. *Every semisignature σ of A has the following property (S_r) for every $r \geqslant 2$. (S_r): If $a_1, \ldots, a_r$ are units of A_0 with $\sigma(a_1) = \cdots = \sigma(a_r) = 1$, and $\lambda_1, \ldots, \lambda_r$ are elements of A such that*

$$b := N(\lambda_1) a_1 + \cdots + N(\lambda_r) a_r$$

is again a unit, then $\sigma(b) = 1$.

The proof is the same as in the special case of signatures, cf. [10, p. 211–212].

PROPOSITION 5.3. *Assume A_0 is field or assume that A has* no *maximal ideal $\mathfrak{M}$ such that either $A/\mathfrak{M} \cong \mathbf{F}_2$ or $A/\mathfrak{M} \cong \mathbf{F}_4$ and $A_0/\mathfrak{M} \cap A_0 \cong \mathbf{F}_2$. Let σ be a map from $A_0{}^*$ to $\{\pm 1\}$ with $\sigma(-a) = -\sigma(a)$ for all a in $A_0{}^*$.*

(i) *If σ fulfills (S_2), then σ is a semisignature of A.*

(ii) *Assume in addition that A has no maximal ideal $\mathfrak{M}$ with $A/\mathfrak{M} \cong \mathbf{F}_3$.*

Let $\sigma\colon A_0{}^* \to \{\pm 1\}$ *be a semisignature of* A_0, *and assume* $\sigma(a) = \sigma(aN(\epsilon))$ *for every* a *in* $A_0{}^*$ *and* ϵ *in* A^*. *Then* σ *is a semisignature of* A.

Proof. The first assertion follows immediately from the description of $W(A)$ as a quotient of the group ring $\mathbf{Z}[Q(A)]$ in [11, Theorem 1.16(iii)]. The second one follows from the fact, that the natural map from $W(A_0)$ to $W(A)$ is surjective and that under our assumptions the kernel of this map is additively generated by the spaces $(a, -aN(\epsilon))$ with a in $A_0{}^*$ and ϵ in A^* [10, Proposition 2.5]. Q.E.D.

Remark 5.4. Assume A is a field with trivial involution. A *quadratic semiordering* of A is by definition a total ordering $<$ of the additive group of A such that $a < b$ implies $ac^2 < bc^2$ for a, b in A and c in A^* [4, 15]. By Proposition 5.3(i) the quadratic semiorderings $<$ of A correspond uniquely to the semisignatures σ of A, the correspondence being given by

$$\sigma(a) = 1 \Leftrightarrow a > 0 \qquad (a \in A^*).$$

Remark 5.5. Let $A_1, \ldots, A_t$ be semilocal rings with involution, let C denote their product $A_1 \times \cdots \times A_t$, and let $p_i\colon C \to A_i$ denote the canonical projections. Then it is not difficult to show that for every semisignature $\sigma\colon C_0{}^* \to \{\pm 1\}$ of C there exists a unique index j with $1 \leqslant j \leqslant t$ and a unique semisignature $\tau\colon A_{j0}^* \to \{\pm 1\}$ of A_j such that $\sigma(a) = \tau(p_j(a))$ for every a in $C_0{}^*$. According to this fact the reader may assume in all the following proofs without loss of generality that the semilocal ring A with involution is connected.

PROPOSITION 5.6. *Let* $a_1, \ldots, a_n$, *and* b *be units of* A_0, *and let* m *be a natural number such that*

$$mb = a_1c_1 + \cdots + a_nc_n$$

with coefficients c_i, *which are sums of products* $N(\lambda)(x^2 + xy + y^2h)$ *with* λ *in* A *and* x, y *in* A_0. *Then for every semisignature* σ *of* A *with* $\sigma(a_1) = \cdots = \sigma(a_n) = 1$ *also* $\sigma(b) = 1$.

This can be proved as Proposition 2.1. We now state the main result of this section.

THEOREM 5.7. *Assume that* A_0 *has no maximal ideal* $\mathfrak{m}$ *with* $A_0/\mathfrak{m} \cong \mathbf{F}_2$. *Assume further in the case of nontrivial involution, that* A *has no maximal ideal* $\mathfrak{M}$ *with* $A/\mathfrak{M} \cong \mathbf{F}_3$. *Let* M *be a nonempty subset of* $A_0{}^*$ *and* b *be a unit of* A_0. *Then the following are equivalent*:

(i) *For every semisignature* σ *of* A *with* $\sigma(M) = \{1\}$ *also* $\sigma(b) = 1$.

(ii) b *has a presentation*

$$b = \sum_{i=1}^{r} N(\lambda_i)(x_i^2 + x_i y_i + y_i^2 h) a_i$$

with some $r \geqslant 1$ *and elements* λ_i *of* A, x_i *and* y_i *of* A_0, *and* a_i *of* M.

(iii) $2b$ *has a presentation*

$$2b = \sum_{i=1}^{r} N(\lambda_i) a_i$$

with some $r \geqslant 1$ *and* λ_i *in* A, a_i *in* M.

(iv) *There exists a natural number* m *such that* mb *has a presentation as in* (iii).

In this theorem the implication (ii) ⇒ (iii) is easy to prove (cf. proof of Theorem 2.5), (iii) ⇒ (iv) is trivial, and (iv) ⇒ (i) follows from Proposition 5.6. It remains to prove (i) ⇒ (ii). For this we need the following lemma, which is a consequence of a "transversality theorem" proved in [3].

LEMMA 5.8. *Assume that all fields* $A_0/\mathfrak{m}$ *have at least* 3 *elements. Let* E *be a quadratic space over* A_0, *and let* $E = F_1 \perp F_2$ *be an orthogonal decomposition with summands* F_i *of rank* $\geqslant 2$. *Further let* x *be a primitive vector of* E, *i.e.,* $x \notin \mathfrak{m}E$ *for every maximal ideal* $\mathfrak{m}$ *of* E. *Then there exist vectors* y_1 *in* F_1 *and* y_2 *in* F_2 *with* $q(x) = q(y_1) + q(y_2)$ *and* $q(y_1)$, $q(y_2)$ *both units.*

The idea in the proof of this lemma is to move the vector x by a proper automorphism $\sigma \in O^+(E)$ such that $\sigma(x)$ becomes transversal to our decomposition of E, i.e., $\sigma(x) = y_1 + y_2$ with each y_i in F_i and $q(y_i)$ a unit. Since the canonical map

$$O^+(E) \to \prod_{\mathfrak{m}} O^+(E/\mathfrak{m}E)$$

with $\mathfrak{m}$ running through the finitely many maximal ideals of A_0 is surjective [8], it suffices to solve this problem in the field case. See [3] for the details, as soon as this paper has appeared.

For any nonempty subset T of $A_0{}^*$ we denote by $\check{T}$ the set of all units of A_0 which are finite sums of elements $N(\epsilon)(x^2 + xy + y^2 h)a$ with a in T, ϵ in A^*, and x, y in A_0. Let M be our nonempty subset of $A_0{}^*$ in Theorem 5.7, and assume that b is a unit of A_0 *not* contained in $\check{M}$. We want to find a semisignature σ of A with $\sigma(M) = \{1\}$ and $\sigma(b) = -1$. Then the remaining part (i) ⇒ (ii) of Theorem 5.7 will be proved.

We show that there exists a subset S of A_0^* with

$$M \subset S, \quad b \notin S, \quad \check{S} = S, \quad S \cap (-S) = \varnothing, \quad S \cup (-S) = A_0^*.$$

Then the map $\sigma\colon A_0^* \rightarrow \{\pm 1\}$ defined by $\sigma(a) = 1$ if $a \in S$, and $\sigma(a) = -1$ if $a \in -S$, is a semisignature of A with the desired property. Indeed, for elements a_1, a_2 of S also any unit $c_1{}^2a_1 + c_2{}^2a_2$ with c_1, c_2 in A_0 lies in S. Thus by part (i) of Proposition 5.3 σ is a semisignature of A_0, and then by part (ii) of the same proposition σ is even a semisignature of A.

Let N denote the set $(M \cup \{-b\})^\vee$, and let U denote the set $\{1\}^\vee$, i.e., the set of all units in A_0, which are sums of elements $N(\epsilon)(x^2 + xy + y^2h)$ with ϵ in A^* and x, y in A_0. It follows from Lemma 5.8 that every element of N has the form $x - by$ with x in $\check{M}$ and y in U. We want to show $N \cap (-N) = \varnothing$. Suppose this is not true. Then we have an equation

$$x_1 - by_1 = -x_2 + by_2$$

with x_1, x_2 in $\check{M}$, and y_1, y_2 in U. From

$$(x_1 + x_2) - b(y_1 + y_2) = 0$$

we obtain by Lemma 5.8 an equation $x_3 - by_3 = 0$ with x_3 in $\check{M}$ and y_3 in U. This implies that b lies in $\check{M}$ contrary to our assumption. Thus $N \cap (-N) = \varnothing$.

By Zorn's lemma there exists a maximal subset S of A_0^* fulfilling the following conditions:

$$N \subset S, \qquad S = \check{S}, \qquad S \cap (-S) = \varnothing.$$

We now show that in addition $S \cup (-S) = A_0^*$, and thus S fulfills all our requests. Assume a is a unit of A_0 not contained in $-S$. Let T denote the set $(S \cup \{a\})^\vee$, which by Lemma 5.8 consists of units $x + ay$ with x in S and y in U. Suppose $T \cap (-T) \neq \varnothing$. Then we see as above that $-a$ lies in S, which is a contradiction. Thus $T \cap (-T) = \varnothing$, and by the maximality of S we have $T = S$, i.e., $a \in S$. This finishes the proof of Theorem 5.7.

As in previous sections we may add the following remark, which is evident by our proof of Theorem 5.7.

Remark 5.9. Let A fulfill the assumptions made in Theorem 5.7, and let b be a unit of A_0 having the equivalent properties (i)–(iv) of Theorem 5.7. Then the presentations of b respectively $2b$ indicated there can be chosen in such a way that in addition the λ_i are units of A.

If 2 is a unit of A then the implication (i) $\Rightarrow$ (iii) of Theorem 5.7 reads as follows.

COROLLARY 5.10. *Assume 2 is a unit of A. Assume further in the case of nontrivial involution that no residue class field $A/\mathfrak{M}$ has only 3 elements. Let M be a nonempty subset of $A_0{}^*$ and b be a unit of A_0 such that for every semisignature σ of A with $\sigma(M) = \{1\}$ also $\sigma(b) = 1$. Then*

$$b = \sum_{i=1}^{r} N(\lambda_i)a_i \tag{*}$$

with some $r \geqslant 1$, a_i in M, and λ_i in A (even in A^*). {cf. [4, Satz 2.10] in the case that A is a field with trivial involution.}

For a general semilocal ring A with involution it seems to be reasonable to conjecture that a unit b of A_0 has a presentation (*) if $\sigma(b) = 1$ for every semisignature σ with $\sigma(M) = \{1\}$ and in addition b lies in the B-submodule of A_0 generated by M. (Recall that B is the subring of A_0 generated by $N(A)$). Up to now I have not been able to prove this.

We call an hermitian space E over A *weakly isotropic*, if $m \times E$ is isotropic for some natural number m, i.e., $m \times E$ contains a primitive vector x with $n(x) = 0$, and we call E strongly anisotropic if E is not weakly isotropic. The goal of the last part of this section is to develop a criterion for spaces to be weakly isotropic using semisignatures {cf. [4, 2.14] in the case of fields}.

LEMMA 5.11. *Let σ be a semisignature of A and E be an hermitian space over A. Then $|\sigma(E)| \leqslant \dim E$. If E is weakly isotropic, then $|\sigma(E)| \leqslant \dim E - 2$.*

Proof. We may assume $\sigma(1) = 1$, replacing σ by $-\sigma$ if this is not the case. The space $E \perp (1)$ is proper, i.e., represents units. Thus

$$E \perp (1) \cong (a_1, \ldots, a_{n+1})$$

with some units a_i of $A_0{}^*$ and $n = \dim E$, cf. [11, Lemma 1.12]. We have

$$\sigma(E) + 1 = \sigma(a_1) + \cdots + \sigma(a_{n+1}) \leqslant n + 1, \tag{*}$$

and thus $\sigma(E) \leqslant n$. Applying this to the space $-E$ we obtain $\sigma(E) \geqslant -n$.

Assume now that a multiple $r \times E$ is isotropic. Then $r \times E$ contains a metabolic plane and thus $r \times E \sim F$ with $\dim F < nr$. We obtain

$$r\,|\sigma(E)| = |\sigma(F)| \leqslant \dim F < nr$$

and thus $|\sigma(E)| < n$. We also see from (*) that $\sigma(E) \equiv n \bmod 2$. This implies $|\sigma(E)| \leqslant n - 2$. Q.E.D.

We call E *positive definite* with respect to a semisignature σ if $\sigma(E) = \dim E$, *negative definite* if $\sigma(E) = -\dim E$, and *indefinite* if $|\sigma(E)| < \dim E$.

Clearly a proper space is positive definite if and only if $\sigma(b) = 1$ for every unit b represented by E. For arbitrary spaces we have

Lemma 5.12. *Let σ be a semisignature of A and let η denote the value $\sigma(1)$, $\eta = \pm 1$. A hermitian space over A is positive definite with respect to σ if and only if $\sigma(b) = 1$ for every unit b represented by the space $E \perp (\eta)$.*

Proof. Let F denote the space $E \perp (\eta)$, which is proper. We have $\sigma(E) = \dim E$ if and only if $\sigma(F) = \dim F$, and this holds true if and only if $\sigma(b) = 1$ for every unit b represented by F. Q.E.D.

A weakly isotropic space is by Lemma 5.11 indefinite with respect to every semisignature of A.

Theorem 5.13. *Assume all residue class fields $A_0/\mathfrak{m}$ have at least 3 elements. Assume further in the case of nontrivial involution that all residue class fields $A/\mathfrak{M}$ have at least 4 elements. Let E be an hermitian space over A which is indefinite with respect to all semisignatures of A. Then E is weakly isotropic.*

For the proof we need the following:

Lemma 5.14. *Let F be a bilinear space over A_0 such that the quadratic space $F \otimes \left[\begin{smallmatrix}1&1\\1&h\end{smallmatrix}\right]$ over A_0 is isotropic. Then also the space $6 \times F$ is isotropic.*

Proof. $\left[\begin{smallmatrix}1&1\\1&h\end{smallmatrix}\right]$ has a free basis e_1, e_2 with $q(e_1) = B(e_1, e_2) = 1$ and $q(e_2) = h$. Let $x \otimes e_1 + y \otimes e_2$ be a primitive isotropic vector of $F \otimes \left[\begin{smallmatrix}1&1\\1&h\end{smallmatrix}\right]$. Then

$$n(x) + \Phi(x, y) + n(y)h = 0.$$

$\{\Phi$ is the bilinear form of F, and $n(z) = \Phi(z, z).\}$ Multiplying the equation by 2 we obtain

$$n(x + y) + n(x) + (2h - 1)\, n(y) = 0.$$

Since $2h - 1$ is a sum of 4 squares, this proves $6 \times F$ to be isotropic. Q.E.D.

Let E be a strongly anisotropic hermitian space over A. We now show that there exists a semisignature σ of A such that E is positive definite with respect to σ. Then Theorem 5.13 will be proved.

We first choose a bilinear space F over A_0 such that $E \cong F \otimes_{A_0} A$. This is always possible: If E is proper then E is an orthogonal sum of spaces (a), with a in $A_0{}^*$, and if E is improper then E is an orthogonal sum of spaces $\left(\begin{smallmatrix}a&1\\1&b\end{smallmatrix}\right)$ with a, b in A_0 and $1 - 4ab \in A_0{}^*$. Since E is strongly anisotropic, every bilinear space $F \otimes (N\epsilon_1, \ldots, N\epsilon_r)$ with $r \geqslant 1$ and units ϵ_i of A is anisotropic over A_0, and we see from Lemma 5.14 that also every quadratic space

$$F \otimes (N\epsilon_1, \ldots, N\epsilon_r) \otimes \begin{bmatrix} 1 & 1 \\ 1 & h \end{bmatrix}$$

is anisotropic. Let N be the union of the sets of units represented by these quadratic spaces. Clearly $N \cap (-N) = \varnothing$ and $N = \check{N}$ with $\check{N}$ having the same meaning as in the proof of Theorem 5.7. We see from Remark 5.9—or directly from the proof of Theorem 5.7, that there exists at least one semisignature σ of A with $\sigma(N) = \{1\}$. We claim that E is positive definite with respect to σ. Regarding σ also as a semisignature of A_0 we have $\sigma(E) = \sigma(F)$, and thus it suffices to show that F is positive definite with respect to σ. Let η denote the value $\sigma(1)$. By Lemma 5.12 we have to consider the units of A_0 represented by $F \perp (\eta)$. Let

$$b = n(x) + \eta c^2$$

be such a unit with x in F and c in A_0. By Lemma 5.8 there exists a unit u represented by $F \otimes \left[\begin{smallmatrix} 1 & 1 \\ 1 & h \end{smallmatrix}\right]$ and a unit v represented by $\left[\begin{smallmatrix} 1 & 1 \\ 1 & h \end{smallmatrix}\right]$ such that

$$b = u + \eta v.$$

Since u lies in N we have $\sigma(u) = 1$, and by Proposition 5.6 we also have $\sigma(\eta v) = 1$. Thus $\sigma(b) = 1$, and F is positive definite with respect to σ according to Lemma 5.12. This completes the proof of Theorem 5.13.

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