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HARMONIC SPINORS AND LOCAL DEFORMATIONS OF THE METRIC

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Abstract. Let (M, g) be a compact Riemannian spin manifold. The Atiyah-Singer index theorem yields a lower bound for the dimension of the kernel of the Dirac operator. We prove that this bound can be attained by changing the Riemannian metric g on an arbitrarily small open set.

1. Introduction and statement of results

Let M be a spin manifold, we assume that all spin manifolds come equipped with a choice of orientation and spin structure. We denote by $-M$ the same manifold M equipped with the opposite orientation. For a Riemannian manifold (M, g) we denote by $U_p(r)$ the set of points for which the distance to the point p is strictly less than r.

The Dirac operator D^g of (M, g) is a first order differential operator acting on sections of the spinor bundle associated to the spin structure on M. This is an elliptic, formally self-adjoint operator. If M is compact, then the spectrum of D^g is real, discrete, and the eigenvalues tend to plus and minus infinity. In this case the operator D^g is invertible if and only if 0 is not an eigenvalue, which is the same as vanishing of the kernel.

The Atiyah-Singer Index Theorem states that the index of the Dirac operator is equal to a topological invariant of the manifold,

$$\mathrm{ind}(D^g) = \alpha(M).$$

Depending on the dimension n of M this formula has slightly different interpretations. If n is even there is a $\pm$-grading of the spinor bundle and the Dirac operator D^g has a part $(D^g)^+$ which maps from positive to negative spinors. If $n \equiv 0, 4 \mod 8$ the index is integer valued and computed as the dimension of the kernel minus the dimension of the cokernel of $(D^g)^+$. If $n \equiv 1, 2 \mod 8$ the index is $\mathbb{Z}/2\mathbb{Z}$-valued and given by the dimension modulo 2 of the kernel of D^g (if $n \equiv 1 \mod 8$) resp. $(D^g)^+$ (if $n \equiv 2 \mod 8$). In other dimensions the index is zero. In all dimensions $\alpha(M)$ is a topological invariant depending only on the spin bordism class of M. In particular, $\alpha(M)$ does not depend on the metric, but it depends on the spin structure in dimension $n \equiv 1, 2 \mod 8$. For further details see [9, Chapter II, §7].

The index theorem implies a lower bound on the dimension of the kernel of D^g which we can write succinctly as

$$\dim \ker D^g \geq a(M), \tag{1}$$

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where

$$a(M) := \begin{cases} |\widehat{A}(M)|, & \text{if } n \equiv 0 \mod 4; \\ 1, & \text{if } n \equiv 1 \mod 8 \text{ and } \alpha(M) \neq 0; \\ 2, & \text{if } n \equiv 2 \mod 8 \text{ and } \alpha(M) \neq 0; \\ 0, & \text{otherwise.} \end{cases}.$$

If M is not connected, then this lower bound can be improved by studying each connected component of M. For this reason we restrict to connected manifolds from now on.

Metrics g for which equality holds in (1) are called D-minimal, see [3, Section 3]. The existence of D-minimal metrics on all connected compact spin manifolds was established in [1] following previous work in [10] and [3]. In this note we will strengthen this existence result by showing that one can find a D-minimal metric coinciding with a given metric outside a small open set. We will prove the following theorem.

Theorem 1.1. *Let (M, g) be a compact connected Riemannian spin manifold of dimension $n \geq 2$. Let $p \in M$ and $r > 0$. Then there is a D-minimal metric $\widetilde{g}$ on M with $\widetilde{g} = g$ on $M \setminus U_p(r)$.*

The new ingredient in the proof of this theorem is the use of the "invertible double" construction which gives a D-minimal metric on any spin manifold of the type $(-M)\#M$ where $\#$ denotes connected sum. For dimension $n \geq 5$ we can then use the surgery method from [3] with surgeries of codimension ≥ 3. For $n = 3, 4$ we need the stronger surgery result of [1] preserving D-minimality under surgeries of codimension ≥ 2. The case $n = 2$ follows from [1] and classical facts about Riemann surfaces.

1.1. **Generic metrics.** We denote by $\mathcal{R}(M, U_p(r), g)$ the set of all smooth Riemannian metrics on M which coincide with the metric g outside $U_p(r)$ and by $\mathcal{R}_{\min}(M, U_p(r), g)$ the subset of D-minimal metrics. From Theorem 1.1 it follows that a generic metric from $\mathcal{R}(M, U_p(r), g)$ is actually an element of $\mathcal{R}_{\min}(M, U_p(r), g)$, as made precise in the following corollary.

Corollary 1.2. *Let (M, g) be a compact connected Riemannian spin manifold of dimension ≥ 3. Let $p \in M$ and $r > 0$. Then $\mathcal{R}_{\min}(M, U_p(r), g)$ is open in the C^1-topology on $\mathcal{R}(M, U_p(r), g)$ and it is dense in all C^k-topologies, $k \geq 1$.*

The proof follows [2, Theorem 1.2] or [10, Proposition 3.1]. The first observation of the argument is that the eigenvalues of D^g are continuous functions of g in the C^1-topology, from which the property of being open follows. The second observation is that spectral data of D^{g_t} for a linear family of metrics $g_t = (1-t)g_0 + tg_1$ depends real analytically on the parameter t. If $g_0 \in \mathcal{R}_{\min}(M, U_p(r), g)$ it follows that metrics arbitrarily close to g_1 are also in this set, from which we conclude the property of being dense.

1.2. **The invertible double.** Let N be a compact connected spin manifold with boundary. The double of N is formed by gluing N and $-N$ along the common boundary ∂N and is denoted by $(-N) \cup_{\partial N} N$. If N is equipped with a Riemannian metric which has product structure near the boundary, then this metric naturally gives a metric on $(-N) \cup_{\partial N} N$. The spin structures can be glued together to obtain a spin structure on $(-N) \cup_{\partial N} N$. The spinor bundle $(-N) \cup_{\partial N} N$ is obtained by gluing the spinor bundle of N with the spinor bundle of $-N$ along their common boundary ∂N. It is straightforward to check that the appropriate gluing map is the map used in [6, Chapter 9].

If a spinor field is in the kernel of the Dirac operator on $(-N) \cup_{\partial N} N$, then it restricts to a spinor field which is in the kernel of the Dirac operator on N and vanishes on ∂N. By the weak unique continuation property of the Dirac operator it follows that such a spinor field must vanish everywhere, and we conclude that the Dirac operator on $(-N) \cup_{\partial N} N$ is invertible. For more details on this argument see [6, Chapter 9] and [5, Proposition 1.4].

Proposition 1.3. *Let (M, g) be a compact connected Riemannian spin manifold. Let $p \in M$ and $r > 0$. Let $(-M)\#M$ be the connected sum formed at the points $p \in M$ and $p \in -M$. Then there is a metric on $(-M)\#M$ with invertible Dirac operator which coincides with g outside $U_p(r)$*

This Proposition is proved by applying the double construction to the manifold with boundary $N = M \setminus U_p(r/2)$, where N is equipped with a metric we get by deforming the metric g on $U_p(r) \setminus U_p(r/2)$ to become product near the boundary.

Metrics with invertible Dirac operator are obviously D-minimal, so the metric provided by Proposition 1.3 is D-minimal.

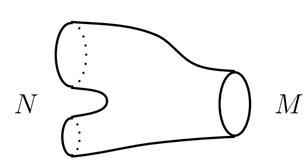
2. Proof of Theorem 1.1

Let M and N be compact spin manifolds of dimension n. Recall that a spin bordism from M to N is a manifold with boundary W of dimension $n+1$ together with a spin preserving diffeomorphism from $N \amalg (-M)$ to the boundary of W. The manifolds M and N are said to be spin bordant if such a bordism exists.

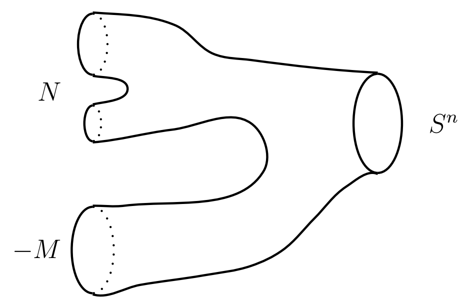
For the proof of Theorem 1.1 we have to distinguish several cases.

2.1. Dimension $n \geq 5$.

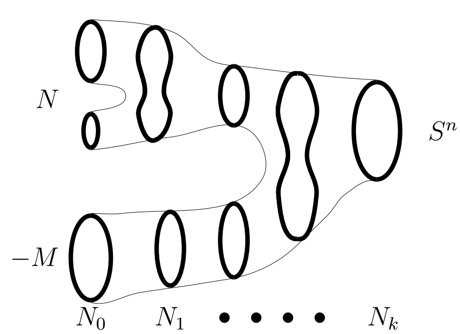
Proof of Theorem 1.1 in the case $n \geq 5$. To prove the Gromov-Lawson conjecture, Stolz [11] showed that any compact spin manifold with vanishing index is spin bordant to a manifold of positive scalar curvature. Using this we see that M is spin bordant to a manifold N which has a D-minimal metric h, where the manifold N is not necessarily connected. For details see [3, Proposition 3.9].



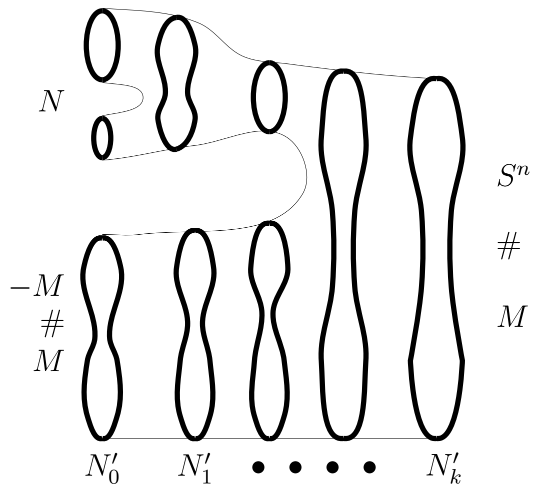
By removing an open ball from the interior of a spin bordism from M to N we get that $N \amalg (-M)$ is spin bordant to the sphere S^n.



Since S^n is simply connected and $n \geq 5$ it follows from [9, Proof of Theorem 4.4, page 300] that S^n can be obtained from $N \amalg (-M)$ by a sequence of surgeries of codimension at least 3. By making r smaller and possibly move the surgery spheres slightly we may assume that no surgery hits $U_p(r) \subset M$. We obtain a sequence of manifolds $N_0, N_1, \ldots, N_k$, where $N_0 = N \amalg (-M)$, $N_k = S^n$, and N_{i+1} is obtained from N_i by a surgery of codimension at least 3.



Since the surgeries do not hit $U_p(r) \subset M \subset N \amalg (-M) = N_0$ we can consider $U_p(r)$ as a subset of every N_i. We define the sequence of manifolds $N_0', N_1', \ldots, N_k'$ by forming the connected sum $N_i' = M \# N_i$ at the points p. Then $N_0' = N \amalg (-M) \# M$, $N_k' = S^n \# M = M$, and N_{i+1}' is obtained from N_i' by a surgery of codimension at least 3 which does not hit $M \setminus U_p(r)$.



We now equip N'_0 with a Riemannian metric. On N we choose a D-minimal metric. The manifold $(-M)\#M$ has vanishing index, so a D-minimal metric is a metric with invertible Dirac operator. From Proposition 1.3 we know that there exists such a metric on $(-M)\#M$ which coincides with g outside $U_p(r)$. Note that we here use the assumption that M is connected. Together we get a D-minimal metric g'_0 on N'_0.

From [3, Proposition 3.6] we know that the property of being D-minimal is preserved under surgery of codimension at least 3. We apply the surgery procedure to g'_0 to produce a sequence of D-minimal metrics g'_i on N'_i. Since the surgery procedure of [3, Theorem 1.2] does not affect the Riemannian metrics outside arbitrarily small neighborhoods of the surgery spheres we may assume that all g'_i coincide with g on $M \setminus U_p(r)$. The Theorem is proved by choosing $\tilde{g} = g'_k$ on $N'_k = M$. □

2.2. **Dimensions $n = 3$ and $n = 4$.**

Proof of Theorem 1.1 in the case $n \in \{3, 4\}$. In these cases the argument works almost the same, except that we can only conclude that S^n is obtained from $N \amalg (-M)$ by surgeries of codimension at least 2, see [7, VII, Theorem 3] for $n = 3$ and [8, VIII, Proposition 3.1] for $n = 4$. To take care of surgeries of codimension 2 we use [1, Theorem 1.2]. Since this surgery construction affects the Riemannian metric only in a small neighborhood of the surgery sphere we can finish the proof as described in the case $n \geq 5$. □

Alternatively, it is straight-forward to adapt the perturbation proof by Maier [10] to prove Theorem 1.1 in dimensions 3 and 4.

2.3. **Dimension $n = 2$.**

Proof of Theorem 1.1 in the case $n = 2$. The argument in the case $n = 2$ is different. Assume that a metric g on a compact surface with chosen spin structure is given. In [1, Theorem 1.1] it is shown that for any $\varepsilon > 0$ there is a D-minimal metric $\hat{g}$ with $\|g - \hat{g}\|_{C^1} < \varepsilon$. Using the following Lemma 2.1, we see that for $\varepsilon > 0$ sufficiently small, there is a spin-preserving diffeomorphism $\psi : M \to M$ such that $\tilde{g} := \psi^* \hat{g}$ is conformal to g on $M \setminus U_p(r)$. As the dimension of the kernel of the

Dirac operator is preserved under spin-preserving conformal diffeomorphisms, $\tilde{g}$ is D-minimal as well. □

Lemma 2.1. *Let M be a compact surface with a Riemannian metric g and a spin structure. Then for any $r > 0$ there is an $\varepsilon > 0$ with the following property: For any $\hat{g}$ with $\|g - \hat{g}\|_{C^1} < \varepsilon$ there is a spin-preserving diffeomorphism $\psi : M \to M$ such that $\tilde{g} := \psi^* \hat{g}$ is conformal to g on $M \setminus U_p(r)$.*

To prove the lemma one has to show that a certain differential is surjective. This proof can be carried out in different mathematical languages. One alternative is via Teichmüller theory and quadratic differentials. We will follow a different way of presentation and notation.

Sketch of Proof of Lemma 2.1. If g_1 and g_2 are metrics on M, then we say that g_1 is spin-conformal to g_2 if there is a spin-preserving diffeomorphism $\psi : M \to M$ such that $\psi^* g_2 = g_1$. This is an equivalence relation on the set of metrics on M, and the equivalence class of g_1 is denoted by $\Phi(g_1)$. Let $\mathcal{M}$ be the set of equivalence classes. Showing the lemma is equivalent to showing that $\Phi(\mathcal{R}(M, U_p(r), g))$ is a neighborhood of g in $\mathcal{M}$.

Variations of metrics are given by symmetric $(2,0)$-tensors, that is by sections of S^2T^*M. The tangent space of $\mathcal{M}$ can be identified with the space of transverse (= divergence free) traceless sections,

$$S^{TT} := \{h \in \Gamma(S^2T^*M) \,|\, \mathrm{div}^g h = 0, \mathrm{tr}^g h = 0\},$$

see for example [4, Lemma 4.57] and [12].

The two-dimensional manifold M has a complex structure which is denoted by J. The map $H : T^*M \to S^2T^*M$ defined by $H(\alpha) := \alpha \otimes \alpha - \alpha \circ J \otimes \alpha \circ J$ is quadratic, it is 2-to-1 outside the zero section, and its image are the trace free symmetric tensors. Furthermore $H(\alpha \circ J) = -H(\alpha)$. Hence by polarization we obtain an isomorphism of real vector bundles from $T^*M \otimes_{\mathbb{C}} T^*M$ to the trace free part of S^2T^*M. Here the complex tensor product is used when T^*M is considered as a complex line bundle using J. A trace free section of S^2T^*M is divergence free if and only if the corresponding section $T^*M \otimes_{\mathbb{C}} T^*M$ is holomorphic, see [12, pages 45-46]. We get that S^{TT} is finite-dimensional, and it follows that $\mathcal{M}$ is finite dimensional.

In order to show that $\Phi(\mathcal{R}(M, U_p(r), g))$ is a neighborhood of g in $\mathcal{M}$ we show that the differential $d\Phi : T\mathcal{R}(M, U_p(r), g) \to T\mathcal{M}$ is surjective at g. Using the above identification $T\mathcal{M} = S^{TT}$, $d\Phi$ is just orthogonal projection from $\Gamma(S^2T^*M)$ to S^{TT}.

Assume that $h_0 \in S^{TT}$ is orthogonal to $d\Phi(T\mathcal{R}(M, U_p(r), g))$. Then h_0 is L^2-orthogonal to $T\mathcal{R}(M, U_p(r), g)$. As $T\mathcal{R}(M, U_p(r), g))$ consists of all sections of S^2T^*M with support in $U_p(r)$ we conclude that h_0 vanishes on $U_p(r)$. Since h_0 can be identified with a holomorphic section of $T^*M \otimes_{\mathbb{C}} T^*M$ we see that h_0 vanishes everywhere on M. The surjectivity of $d\Phi$ and the lemma follow. □

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