Proceedings Conference La Tourbaille 1991,
to appear Springer, Lecture Notes Math.

56

Semialgebraic topology in the last ten years

Manfred Knebusch

Contents

§1 Brumfiel's program

Before discussing the subject named in the title it seems appropriate to outline the situation in semialgebraic topology in 1981, at the time of the first Rennes conference on real algebraic geometry.

Already in the seventies, in the long introduction to his book "Partially ordered rings and semialgebraic geometry" [B], G.W. Brumfiel had laid down a program for what we now call "semialgebraic topology". Here Brumfiel advocated a new way of handling topological problems which is closer to the spirit of algebraic geometry than traditional topology. Let me just quote the following passage:

"It thus seems to me that a true understanding of the relations between algebraic geometry and topology must stem from a deeper understanding of real algebraic geometry, or, actually, semi-algebraic geometry. Moreover, real algebraic geometry should not be studied by attempting to extend classical algebraic geometry to non-algebraically closed ground fields, nor by regarding the real field as a field with an added structure of a topology. Instead, the abstract algebraic treatment of inequalities originated by Artin and Schreier should be extended from fields to (partially ordered) algebras, with real closed fields replacing the algebraically closed fields as ground fields" [B, p.2].

In the main body of the book [B] Brumfiel develops a "real algebra" by studying partially ordered commutative rings and various sorts of convex ideals, with the perspective that this real algebra should perform a similar role in semialgebraic geometry as commutative algebra does in present day algebraic geometry. But the book does not go very far into semialgebraic topology.

§2 The two approaches

Even today not much semialgebraic topology has been done using Brumfiel's rather intricate real algebra from the seventies. Around 1979 two other approaches to semialgebraic topology emerged independently which turned out to be successful. These are the "abstract" approach by M. Coste and M.F. Roy, and the "geometric" approach by H. Delfs and M. Knebusch.

Before we get into this let me remind you of what are perhaps the two most serious difficulties which one encounters if one works over a real closed base field R different from $\mathbf{R}$.

a) R^n is totally disconnected in the strong topology (i.e. the topology coming from the ordering of R).

b) R^n has very few reasonable (i.e. geometrically relevant) compact subsets. In particular, the closed unit ball in R^n is not compact.

Let M be a semialgebraic subset of some R^n. In the abstract approach one adds to M "ideal points" which turn M into an honest (albeit not Hausdorff) topological space. More precisely, one passes from M to the corresponding constructible subset $\tilde{M}$ of the real spectrum $\operatorname{Sper} R[T_1, \ldots, T_n]$ of the polynomial ring $R[T_1, \ldots, T_n]$ (cf. [BCR, Chap. 7]). $\tilde{M}$ turns out to have only finitely many connected components, and $\tilde{M}$ is quasicompact. Thus in some sense the difficulties described above are overcome. The subspace topology of M in $\tilde{M}$ is the strong topology we started with.

One could also pass from M to the subspace $\tilde{M}^{\max}$ of closed points of $\tilde{M}$, which still contains M as a dense subset and is a compact Hausdorff space with only finitely many connected components. But although this compactification $\tilde{M}^{\max}$ of M has its merits (cf. [B$_1$]), the more interesting and more useful space is $\tilde{M}$ itself. The main reason for this is that $\tilde{M}$ is a spectral space, as defined by Hochster [Ho], and that the constructible subsets Y of $\tilde{M}$ correspond bijectively with the semialgebraic subsets N of M via the relation $Y = \tilde{N}$. A very nice consequence of this is that the "semialgebraic structure" of M is encoded to a large extent in the topology of $\tilde{M}$, since the lattice $\mathfrak{K}(\tilde{M})$ of constructible subsets of $\tilde{M}$ is by definition the boolean lattice generated by the lattice $\mathring{\mathfrak{K}}(\tilde{M})$ of quasicompact open subsets of $\tilde{M}$, and thus

$\mathfrak{K}(\tilde{M})$ is completely determined by the topology of $\tilde{M}$ (cf. [Ho]). We call the space $\tilde{M}$ the *abstraction* of the semialgebraic set M.

The wisdom of passing back and forth between the semialgebraic sets and their abstractions has been displayed well in the book [BCR] by Bochnak, Coste and Roy. Curiously another very important and fascinating aspect of the abstract approach is scarcely touched on in that book: One can study the constructible subsets of the real spectrum $\operatorname{Sper} A$ of *any* commutative ring A. Thus the abstract approach opens the door for an "abstract" semialgebraic topology where no base field (real closed or not) needs to be present. Coste and Roy were certainly well aware of this aspect at an early stage (cf. for example Roy's paper on abstract Nash functions [R]) but chose not to give much space to this in their book with Bochnak.

The geometric approach (cf. [DK]) relies on the following two ideas, the first one being very simple.

1° Don't consider any subset of a semialgebraic set $M \subset R^n$ which is not semialgebraic or any map $f: M \to N$ between semialgebraic sets which is not semialgebraic!

In this context a *map* f is called *semialgebraic* if the graph of f is a semialgebraic set and f is continuous with respect to the strong topologies of M and N.

2° Install on M a Grothendieck topology such that the semialgebraic functions, i.e., semialgebraic maps to R on the open semialgebraic subsets U of M (open with respect to the strong topology) form a sheaf $\mathcal{O}_M$ of R-algebras! Instead of studying M as a semialgebraic subset of R^n study the ringed space $(M, \mathcal{O}_M)$!

Let me give some comments and explanations on these ideas.

Ad 1°: The reason that this idea makes sense is Tarski's principle. It guarantees that many of the usual constructions of new sets and maps from given ones give us semialgebraic sets and maps if we start with such sets and maps. In particular, if $f: M \to N$ is a semialgebraic map between semialgebraic sets then the image $f(A)$ of a semialgebraic subset A of M is semialgebraic and the preimage $f^{-1}(B)$ of a semialgebraic subset B of N is semialgebraic. Continuity of f is not necessary for this but is appropriate since we want to do "topology".

Ad 2°: The Grothendieck topology on M is defined as follows. The underlying category is the category $\mathring{\mathfrak{S}}(M)$ of open semialgebraic subsets of M (i.e. semialgebraic subsets which are open in the strong topology), the morphisms being the inclusion mappings. An admissible open covering $(U_i | i \in I)$ of some $U \in \mathring{\mathfrak{S}}(M)$ is a family $(U_i | i \in I)$ in $\mathring{\mathfrak{S}}(M)$ with $U = \bigcup\limits_{i \in I} U_i$, such that there exists a finite subset J of I with $U = \bigcup\limits_{i \in J} U_i$. {Thus a property similar to quasicompactness is forced to hold.} Then the semialgebraic functions on the sets $U \in \mathring{\mathfrak{S}}(M)$ indeed form a sheaf $\mathcal{O}_M$. It turns out that a morphism from $(M, \mathcal{O}_M)$ to $(N, \mathcal{O}_N)$ is determined by the underlying map f from M to N, and that these maps f are just the semialgebraic maps from M to N as introduced above (cf. [DK, §7], by definition the morphism has to respect the R-algebra structures of the structure sheaves).

Replacing a semialgebraic set $M \subset R^n$ by the ringed space $(M, \mathcal{O}_M)$ allows us to forget the embedding $M \hookrightarrow R^n$. We call any ringed space of R-algebras which is isomorphic to such a space $(M, \mathcal{O}_M)$ an *affine semialgebraic space* over R. By abuse of notations we do not distinguish between a semialgebraic set M and the corresponding ringed space $(M, \mathcal{O}_M)$.

A *semialgebraic path* in M is a semialgebraic map from the unit interval [0,1] (which is a semialgebraic subset of R^1) to M. Having this notion of paths at hand one defines the path components of M in the obvious way. It turns out, that M has only finitely many path components $M_1, \ldots, M_r$ and that these are semialgebraic in M and closed, hence also open in the strong topology, cf. [DK]. Every M_i is "semialgebraically connected", i.e. M_i is not the union of two disjoint non empty open semialgebraic subsets, since this holds for [0,1], as is easily seen. Thus we have dealt with the first difficulty mentioned above, exploiting only idea N° 1. By the way, the abstractions $\tilde{M}_1, \ldots, \tilde{M}_r$ are the connected components of the topological space $\tilde{M}$.

In order to cope with the second difficulty one also needs idea N° 2. The category of affine semialgebraic spaces over R has fiber products. Thus we can define proper morphisms as in algebraic geometry. We call a semialgebraic map $f\colon M \to N$ *closed*, if the image $f(A)$ of a closed semialgebraic subset A of M is again closed. We call f *proper* if f is universally closed, i.e. for any semialgebraic map $g\colon N' \to N$ the cartesian square

$$\begin{tikzcd}
M \times_N N' \arrow[r, "f'"] \arrow[d, "g'"'] & N' \arrow[d, "g"] \\
M \arrow[r, "f"'] & N
\end{tikzcd}$$

gives us a closed semialgebraic map f'. We call an affine semialgebraic space M *complete* if the map from M to the one-point space is proper. Even more than in algebraic geometry over an algebraically closed field, it is true for many purposes, that complete spaces are the right substitute for compact spaces in topology. For example, a semialgebraic function on a complete space attains its maximum and minimum.

It turns out that there exist in abundance relevant complete affine semialgebraic spaces. Namely, the following analogue of the Heine-Borel theorem holds: A semialgebraic subset M of R^n is a complete space iff M is closed and bounded in R^n.

§3 The state of art in 1981

I give a rough sketch of the technical progress up till 1981. This is just to give an impression of the state of art at the first Rennes conference. It is not meant, of course, as a complete account of everything done up to that time.

In the geometric theory we have the following list.

1) Connected components
2) Complete affine semialgebraic spaces and the semialgebraic Heine-Borel theorem
3) Dimension theory
4) Existence of triangulations
5) Hardt's theorem
6) Semialgebraic homology

Here are some comments on these.
N^o 1 and N^o 2 have been described above. One may add to N^o 2 that in 1981 we also had a good insight into the nature of proper maps between affine semialgebraic spaces [DK, §9 and §12].

Ad 3: The dimension $\dim M$ of a semialgebraic set can be defined as the maximal integer d such that M contains a subspace N which is isomorphic to the unit ball in R^d {[DK, §8], there a different but equivalent definition had been given}. This notion of dimension behaves very well, better than in classical topology. For example, if a partition of M into finitely many semialgebraic subsets $A_1, \dots, A_r$ is given, then $\dim M$ is the maximum of the numbers $\dim A_1, \dots, \dim A_r$.

Ad 4: If M is an affine semialgebraic space and $A_1, \dots, A_r$ are finitely many semialgebraic subsets of M then there exists a finite simplicial complex X over R and an isomorphism of spaces $\varphi\colon X \xrightarrow{\sim} M$ such that, for every $i \in \{1, \dots, r\}$, the set $\varphi^{-1}(A_i)$ is a subcomplex of X [DK, §2]. Here the word "simplicial complex" is used in a non classical meaning: X is the union of finitely many open simplices $\sigma_1, \dots, \sigma_t$ in some R^N such that the intersection $\bar{\sigma}_i \cap \bar{\sigma}_j$ of the closures of any two simplices σ_i, σ_j is either a common face of them or empty. Thus the closure $\bar{X}$ of X is a classical finite simplicial complex ($\approx$ finite polyhedron), and X is obtained from $\bar{X}$ by omitting some open faces. Also "subcomplex" means just the union of some of the sets $\sigma_1, \dots, \sigma_t$. Clearly $X = \bar{X}$ iff M is complete.

In the case $R = \mathbb{R}$ the triangulation theorem has been well known since the sixties, even for semianalytic sets [L, Gi].

Ad 5: Hardt's theorem states that for every semialgebraic map $f: M \to N$ there exists a partition of N into finitely many semialgebraic subsets $N_1, \ldots, N_r$ such that f is trivial over each N_j, i.e. $f^{-1}(N_j)$ is isomorphic over N_j to a direct (= cartesian) product $N_j \times F_j$, cf [DK$_1$, §6]. The theorem had been proved for $R = \mathbb{R}$ by R. Hardt around 1978 [Ha].

Ad 6: In his thesis [D] Delfs constructed homology and cohomology groups with arbitrary constant coefficients for affine semialgebraic spaces over any real closed field R. In the case $R = \mathbb{R}$ these groups coincide with the singular groups known from classical topology.

Certainly Delfs' homology theory was the most profound achievement in semialgebraic topology up till 1981. But the proofs of the triangulation theorem and of Hardt's theorem also needed new ideas beyond the known proofs for $R = \mathbb{R}$.

The triangulation theorem is the main technical tool in developing semialgebraic homology (and also semialgebraic homotopy theory, cf. §10 below). Hardt's theorem is very useful if one wants to profit from semialgebraic homology. For a good example, cf. [DK$_1$, §7]. I will say more about semialgebraic homology in the next section §4.

Remark. Only recently (1989) I learned from Gert-Martin Greuel about the unpublished dissertation of Helmut Brakhage [Bra] (Heidelberg 1954, thesis advisor F.K. Schmidt). Here Brakhage studies semialgebraic topology over an arbitrary real closed field. He exploits idea N^o 1 of the geometric theory (cf. §2) to an enormous extent and obtains many of the results we had found up to 1981, in particular the triangulation theorem. The introduction to Brakhage's thesis reads very much like the talks Delfs and I used to give around 1980. He would have saved us a lot of work if we only would have known about his thesis. Brakhage is now a professor at Kaiserslautern, working mostly in applied mathematics.

It is difficult to give a good picture of the state of art in semialgebraic topology in 1981 on the abstract side, since in the abstract theory the main bias was towards algebraic problems. Topology seems to have been studied mainly as an aid for solving algebraic problems of current interest. I give the following list.

1) Connected components
2) Compactness of constructible sets
3) Specialization theory
4) Dimension theory
5) Abstract Nash functions
6) Separation of connected components by global quadratic forms

Here only N^o 1 - 4 truly belong to semialgebraic topology, but N^o 5 and 6 use topology in an essential way, and have also turned out to be stimulating for semialgebraic topology since 1981.

N° 1 has been discussed above, N° 2 alludes to the easily accessible but extremely important fact, that the real spectrum Sper A of any commutative ring A is compact in the constructible topology. This means that, if X is a constructible subset and $(Y_i | i \in I)$ is a family of constructible subsets of Sper A with $X \subset \bigcup_{i \in I} Y_i$, then there exists a finite subset J of I with $X \subset \bigcup_{i \in J} Y_i$. The quasicompactness of $\tilde{M}$ stated above is a rather special consequence of this.

Ad 3: If x and y are points of a topological space X then we say that y is a *specialization* of x (and x is generalization of y) if y lies in the closure of the set $\{x\}$. We write $x \succ y$ for this. N° 3 alludes to some – again simple but important – facts about specializations in a real spectrum Sper A, cf. [CR₂], [BCR, 7.1], [KS, III §3 and §7]. In particular, the specializations of a given point x in Sper A form a chain, i.e. if $x \succ y$ and $x \succ z$ then $y \succ z$ or $z \succ y$. Moreover if neither $x \succ y$ nor $y \succ x$ then there exist disjoint open subsets U, V in Sper A with $x \in U$ and $y \in V$.

Ad 4: The *dimension* $\dim X$ of a constructible subset X of Sper A is defined as the supremum of the lengths of the specialization chains in X. {Up till now it has been adequate to put $\dim X = \infty$ if the lengths do not have a finite bound.} The main result is that, if M is a semialgebraic set over some real closed field, then this "combinatorial" dimension $\dim \tilde{M}$ of the abstraction $\tilde{M}$ coincides with the semialgebraic dimension $\dim M$ from above, cf. [CR₂], [BCR].

Ad 5 and 6: One of the most important achievements in the early work of Coste and Roy is the construction of a sheaf of "abstract Nash functions" $\mathfrak{N}_A$ on the real spectrum of an arbitrary commutative ring A [R], which generalizes the sheaf of classical Nash functions for algebraic manifolds over $\mathbf{R}$. Indeed, right from the beginning they had the idea of constructing the real spectrum as a ringed space (Sper $A, \mathfrak{N}_A$) [CR], [CR₁], thus bringing semialgebraic geometry close to the spirit of abstract algebraic geometry in the sense of Grothendieck. The sheaf $\mathfrak{N}_A$ is more algebraic in nature than the sheaf of semialgebraic functions discussed in §2. It does not belong to semialgebraic topology, but nevertheless relies on the topological fact that every etale morphism $A \xrightarrow{\varphi} B$ induces a local homeomorphism Sper φ: Sper $B \to$ Sper A.

Building on this, Mahé was able to solve one of the main open problems of quadratic form theory from the seventies [K₁, Problem 16] affirmatively, namely the separation by global quadratic forms of the connected components of the set $V(\mathbf{R})$ of real points of an affine algebraic variety V, and later, together with Houdebine, also of a projective algebraic variety V over $\mathbf{R}$ [M], [HM]. In fact, they prove such a theorem over any real closed field R, and also for the real spectrum of any commutative ring.

Mahé's theorem in [M] is probably the first result which signaled to the outside world that something new in principle had happened in real algebraic geometry around 1980.

§4 Sheaves and homology

After 1981 semialgebraic topology has been dominated by two major new trends: A strong interaction between the geometric and the abstract theory, and the employment of new spaces. An important instance of the first trend is sheaf theory.

Let M be a semialgebraic set over some real closed field R. Then a (set valued) sheaf over M is essentially the same object as a sheaf over the abstraction $\tilde{M}$. Indeed, as was already known before 1981 [CR$_2$], [D], [De], a semialgebraic subset U of the affine semialgebraic space M is open iff the abstraction $\tilde{U}$ is open in $\tilde{M}$. Moreover, a family $(U_i|i \in I)$ of open semialgebraic subsets of M is an admissible open covering of U iff $(\tilde{U}_i|i \in I)$ is an open covering of $\tilde{U}$. The reason for this is the definition of the Grothendieck topology on M on the one hand, and the quasicompactness of $\tilde{U}$ on the other. Since the quasicompact open subsets of $\tilde{M}$ form a basis of the topology of $\tilde{M}$, all of this gives us a canonical isomorphism $\mathcal{F} \mapsto \tilde{\mathcal{F}}$ from the category of sheaves on M to the category of sheaves on $\tilde{M}$, via the rule $\mathcal{F}(U) = \tilde{\mathcal{F}}(\tilde{U})$.

Henceforth we only consider sheaves of abelian groups. Recall that M is dense in $\tilde{M}$. For $x \in M$ the stalks $\mathcal{F}_x$ and $\tilde{\mathcal{F}}_x$ are equal. It may well happen that all stalks $\mathcal{F}_x$, $x \in M$, are zero, but $\mathcal{F}$ is not zero. {An example is given in [D$_2$, I.1.7].} This is by no means astonishing: Of course, $\mathcal{F} \neq 0$ iff $\tilde{\mathcal{F}} \neq 0$. Then, since $\tilde{M}$ is an honest topological space, there exists some $\alpha \in \tilde{M}$ with $\tilde{\mathcal{F}}_\alpha \neq 0$. But it may happen that none of these points α lies in M.

This discussion makes it clear that most often sheaf theoretic techniques work better in the abstract setting than the geometric one. Only there one can argue "stalk by stalk" without further justification.

Now is a good moment to say something about the semialgebraic homology theory of Hans Delfs, since he has been able to simplify his theory greatly by using sheaves and the interplay back and forth between semialgebraic sets and their abstractions [D$_1$].

I first describe the main problem in defining homology groups $H_q(M, G)$ for a semialgebraic set M over some real closed field R and some abelian group of coefficients G. Let us assume for simplicity that M is complete. We choose a triangulation $\varphi: X \xrightarrow{\sim} M$. Here X is a finite simplicial complex in the classical sense but over R; X may be regarded as the realization $|K|_R$ over R of an abstract finite simplicial complex K, a purely combinatorial object (cf. [Spa, 3.1]; the realization is defined exactly as in the case $R = \mathbf{R}$).

It is intuitively clear that $H_q(M, G)$ should coincide, up to isomorphism, with the combinatorial homology group $H_q(K, G)$ from classical topology. To make an honest definition out of this, one has to verify that (up to natural isomorphism) the group $H_q(K,G)$ does not depend on the choice of the triangulation. The now traditional way to prove this is to define a complex $C.(M,G)$ of singular chains and to verify the seven Eilenberg-Steenrod axioms for the homology groups [ES, I §3]. Then one obtains, in a well known manner, that $H_q(C.(M,G)) \cong H_q(K,G)$ for the triangulation φ above. {One also has to consider noncomplete spaces M and the relative chain complex $C.(M,A;G)$ for A a semialgebraic subset of M. I omit these technicalities.}

We can indeed define singular chain groups $C_q(M,G)$ along classical lines, decreeing that a singular simplex is a semialgebraic map from the q-dimensional standard simplex Δ_q to M. Six of the seven Eilenberg-Steenrod axioms can be proved as in the classical theory, always using semialgebraic maps instead of continuous maps. But the excision axiom is difficult. The classical way to prove it is to make a given singular cycle Z "small" with respect to a given covering of M by two open (semialgebraic) sets U_1, U_2, by iterated barycentric subdivision of Z. This means that every singular simplex occuring in the subdivided cycle has its image either in U_1 or U_2. But if the base field R is not archimedian then this procedure completely breaks down, since then usually a given bounded semialgebraic set cannot be covered by finitely many semialgebraic sets all whose diameters are smaller than a given $\epsilon > 0$.

In his thesis [D] Delfs found the following way out of this difficulty. He defined cohomology groups $H^q(M,G)$ as the sheaf cohomology groups of the constant sheaf G_M on M, and similarly relative cohomology groups $H^q(M,A;G)$ as the sheaf cohomology groups of a suitable sheaf $G_{M,A}$ on M. {Recall that M is equipped with a Grothendieck topology.} For these groups $H^q(M,A;G)$ Delfs succeeded in verifying the Eilenberg-Steenrod axioms. Then he knew that $H^q(M,G)$ is isomorphic to the combinatorial group $H^q(K,G)$. Thus $H^q(K,G)$ is independent of the choice of the triangulation, up to natural isomorphism. From this Delfs concluded that also $H_q(K,G)$ is independent of the choice of triangulation [D].

The verification of six of the seven Eilenberg-Steenrod axioms for the groups $H^q(M,A,G)$ is straightforward, but this time the homotopy axiom causes difficulties. Delfs surmounted these difficulties in [D] by a complicated geometric procedure.

Later Delfs found an easier way [D_1]. He realized that the homotopy axiom follows from the statement that, for any sheaf $\mathcal{F}$ on M, the adjunction homomorphism $\mathcal{F} \longrightarrow \pi_*\pi^*\mathcal{F}$, with π the projection from $M \times [0,1]$ to M, is an isomorphism and $R^q\pi_*(\pi^*\mathcal{F}) = 0$ for $q \geq 1$. [D_1, Prop. 4.2 and 4.4]. This then could be deduced via a stalk by stalk argument from the fact that $H^q([0,1],G) = 0$ for $q \geq 1$ and any abelian group G, which in turn can be verified in an easy geometric way. The crucial point is that one needs the fact $H^q([0,1],G) = 0$ not just over R but over the residue class fields $k(x)$ of all points $x \in \tilde{M}$. Roughly one can summarize that Delfs reduced the verification of the homotopy axiom to an easy special case using sheaf theory, at the expense of enlarging the real closed base field in many ways.

§5 Locally semialgebraic spaces

Delfs and I had already introduced *"semialgebraic spaces"* over a real closed field R before 1981 by gluing together finitely many affine semialgebraic spaces over R along open subspaces [DK, §7]. What then was still missing was a handy criterion for a semialgebraic space $M = (M, \mathcal{O}_M)$ to be again affine. Such a criterion would allow the building of semialgebraic spaces M from semialgebraic sets in an "abstract" manner, i.e. without explicitly looking at polynomials, such that M eventually turns out to be an affine space, in other words, a semialgebraic set.

In 1982 R. Robson proved his imbedding theorem [Ro] which gives such a criterion. The theorem says that a semialgebraic space M over R is affine iff M is *regular*, i.e. a point x and a closed semialgebraic subset A of M with $x \notin A$ can be separated by open semialgebraic neighbourhoods. {A subset A of M is called closed semialgebraic if the complement $M - A$ is an open semialgebraic, i.e., an admissible open subset of M.}

Robson's theorem really paved the way for the trend of employing new spaces in the geometric theory. Before I go into details about this I should say some words about covering maps.

Having semialgebraic paths at hand we may define the fundamental group $\pi_1(M, x_0)$ for M a semialgebraic space over R and $x_0 \in M$, as in the classical theory, by considering homotopy classes of semialgebraic loops with base point x_0. Of course, homotopies also have to be defined in the semialgebraic sense, starting from the unit interval [0,1] in R, cf. §10 below. It turns out that for affine M the group $\pi_1(M, x_0)$ is very respectable. It is finitely presented and coincides with the topological fundamental group in the case $R = \mathbb{R}$. {These are consequences of the two comparison theorems on homotopy sets [DK$_2$, III §3 and §5], to be discussed in §10 below.}

Assume since now that M is affine and path connected. The question arises whether the subgroups of $\pi_1(M, x_0)$ classify "semialgebraic covering spaces" of M, as one might expect from classical topology.

It seems clear what a semialgebraic covering map $\pi: N \longrightarrow M$ has to be: N should be a semialgebraic space and π a semialgebraic map. Further there should exist an admissible open covering $(U_i | i \in I)$ of M such that π is trivial over each U_i with discrete fibers, i.e. $\pi^{-1}(U_i) \cong U_i \times F_i$ over U_i for a discrete semialgebraic space F_i. But what does it mean for a semialgebraic space F to be discrete? Reasonable answers, one can think of, are: $\dim F = 0$; the path components of F are one-point sets; the one-point sets in M are open in F; the one-point sets in M form an admissible open covering of M. – All of these properties mean the same thing, namely that the space F consists of finitely many points. We conclude that every semialgebraic covering map $\pi: N \longrightarrow M$ has finite degree.

Working with path lifting techniques one verifies that the semialgebraic coverings $\pi: N \rightarrow M$ of M are indeed classified by the conjugacy classes of subgroups of $\pi_1(M, x_0)$ of finite index [K3]. Using Robson's embedding theorem one also sees that N is again affine.

Having verified this in 1982 [DK3], Delfs and I realized that the category of semialgebraic spaces is not broad enough. There should exist some sort of covering space N of M corresponding to any given subgroup H of $\pi_1(M, x_0)$, in particular a "universal covering", corresponding to $H = \{1\}$. This led us to introduce locally semialgebraic spaces. A *locally semialgebraic space M over R* is obtained by gluing together (maybe infinitely many) affine semialgebraic spaces over R along open semialgebraic subspaces. Of course, the gluing is meant in the sense of ringed spaces with Grothendieck topologies, cf. [DK2, I §1].

The nice locally semialgebraic spaces are those which are *regular* (defined in the same way as above) and *paracompact*, as defined in [DK2, I §4]. The category $LSA(R)$ of regular paracompact locally semialgebraic spaces over R contains the category of affine semialgebraic spaces over R as a full subcategory. In $LSA(R)$ we have a fully satisfactory theory of covering spaces. In particular every space $M \in LSA(R)$ has a universal covering (cf. [DK3, §5]; a full treatment of this topic still awaits publication [K3]).

In $LSA(R)$ there exist fibre products. There is also a good notion of subspaces. Namely, if M is a locally semialgebraic space and $(M_i | i \in I)$ is an admissible open covering of M, such that every M_i is an affine semialgebraic space, then a subset A of M is called a *subspace* if $A \cap M_i$ is semialgebraic in M_i for every $i \in I$. Indeed, collecting the affine semialgebraic spaces $A \cap M_i$ we obtain on A the structure of a locally semialgebraic space over R, which is independent of the choice of the covering $(M_i | i \in I)$. This space A is regular and paracompact if M has these properties [DK2, I, §3 and §4].

Up to now $LSA(R)$ has proved to be the appropriate basic category for all geometric studies over R, as long as one does not pass to abstract spaces. In particular the triangulation theorem for semialgebraic sets (cf. §3 above) extends to a triangulation theorem of equal strength for these spaces (simultaneous triangulation of M and a locally finite family of subspaces of M, cf. [DK2, Chap.II]). Also the homology theory of Delfs discussed above extends to these spaces [DK2, Chap.III]. And we have a fairly good homotopy theory in $LSA(R)$ at hand, to be discussed below.

§6 Abstract semialgebraic functions and real closed spaces

We come back to the relationships between a semialgebraic set M over R and its abstraction $\tilde{M}$. Recall from §4 that the sheaves on the affine semialgebraic space M correspond uniquely with the sheaves on $\tilde{M}$. In particular we have a sheaf of rings

$\tilde{\mathcal{O}}_M$ on $\tilde{M}$ which corresponds to the sheaf $\mathcal{O}_M$ of semialgebraic functions on M. The question arises whether $\tilde{\mathcal{O}}_M$ generalizes in a natural way to a sheaf of rings $\mathcal{O}_X$ on any constructible subset X of any real spectrum $\operatorname{Sper} A$, which then can be regarded as a sheaf of "abstract" semialgebraic functions on X.

This is indeed the case. Around 1983 G. Brumfiel [B4] and N. Schwartz [S] gave two solutions of this problem. A (slightly "corrected", cf. [D, 1.7], [S, Example 58]) version of Brumfiel's definition runs as follows. Let $p\colon \operatorname{Sper} A[T] \to \operatorname{Sper} A$ be the natural map from the real spectrum of the polynomial ring $A[T]$ in one variable over A to $\operatorname{Sper} A$, induced by the inclusion $A \hookrightarrow A[T]$. For any quasicompact open subset U of the space X the elements of $\mathcal{O}_X(U)$ are the continuous sections s of $p|p^{-1}(U)\colon p^{-1}(U) \to U$ such that $s(U)$ is a closed constructible subset of $p^{-1}(U)$.

What does this mean? For any $x \in A$ we may identify $p^{-1}(x)$ with the real spectrum $\operatorname{Sper} k(x)[T]$, where $k(x)$ denotes the residue class field of $\operatorname{Sper} A$ at x, a real closed field. This real spectrum is the abstraction of the real affine line over $k(x)$. Thus $k(x)$ injects into $p^{-1}(x)$ as a dense subset (cf. §2). For a section s as above, $s(x)$ lies in this subset and hence corresponds to an element $f(x)$ of $k(x)$, which should be regarded as the value of the abstract semialgebraic function f given by s. The section s is completely determined by the values $f(x)$ and should be regarded as the graph of f.

N. Schwartz defined an abstract semialgebraic function f on U directly as a family $(f(x)|x \in U) \in \prod_{x\in U} k(x)$ with compatibility relations between the values $f(x)$ coming from canonical valuations $\lambda_{x,y}\colon k(x) \to \kappa(x,y) \cup \infty$. For any pair (x,y) with $x \in U$ and y a specialization of x in U, $\kappa(x,y)$ is an overfield of $k(y)$, and $\lambda_{x,y}$ has to map $f(x)$ to $f(y) \in k(y) \subset \kappa(x,y)$. The definition of Schwartz has the advantage that here it is immediately clear that $\mathcal{O}_X(U)$ is a ring, while in Brumfiel's definition one has to work for this.

Then Delfs proved that the definitions of Brumfiel and Schwartz give the same sheaf $\mathcal{O}_X$ [D1, §1]. The stalks of $\mathcal{O}_X$ are local rings. In the geometric case, i.e., if $A = R[T_1,\ldots,T_n]$ and $X = \tilde{M}$ with $M \subset R^n$ a semialgebraic set, we indeed have $\mathcal{O}_X = \tilde{\mathcal{O}}_M$. From now on we call the ringed space $(\tilde{M}, \tilde{\mathcal{O}}_M)$ – instead of just the topological space $\tilde{M}$ – the abstraction of the affine semialgebraic space $(M, \mathcal{O}_M)$.

In the paper [B4] cited above Brumfiel introduced abstract semialgebraic functions as a tool to prove a vast generalization of Mahé's theorem on the separation of connected components by global quadratic forms. For every commutative ring A there is a natural homomorphism from the Witt ring $W(A)$ to the orthogonal K-group $KO_0(\operatorname{Sper} A)$ of the real spectrum of A. Brumfiel proves that both the kernel and the cokernel of this homomorphism are 2-primary torsion groups. Thus, from our viewpoint, the localization $2^{-\infty}W(A)$ of $W(A)$ at the prime 2 is a purely topological object.

Brumfiel's paper is a bold step into the realm of abstract semialgebraic topology. A full understanding of it is a challenge even today, since some arguments are only sketched. For a discussion cf. [K, §6], and for a treatment in the geometric case cf. [BCR, 15.3].

N. Schwartz studied in [S] the spaces $(X, \mathcal{O}_X)$, with X a constructible subset of some real spectrum $\operatorname{Sper} A$, for their own sake. The ring $\mathcal{O}(X)$ of global sections of $\mathcal{O}_X$ is a sort of "real closure" of the ring A. Schwartz describes how to obtain $\mathcal{O}(X)$ from the ring A in a constructive way. He further makes the important discovery that the natural map from X to $\operatorname{Sper} \mathcal{O}(X)$ is an embedding which makes X a dense subspace of $\operatorname{Sper} \mathcal{O}(X)$. Even more is true: the closed points and also the minimal points of $\operatorname{Sper} \mathcal{O}(X)$ all lie in X. In the special case that X is convex in $\operatorname{Sper} A$ with respect to specialization, it turns out that the ringed spaces $(X, \mathcal{O}_X)$ and $\operatorname{Spec} \mathcal{O}(X)$ are equal. In the geometric case $X = \tilde{M}$ this happens iff the semialgebraic set $M \subset R^n$ is locally closed in R^n.

Later Schwartz realized that all we have said above about $(X, \mathcal{O}_X)$ remains true if X is a *proconstructible* subset of $\operatorname{Sper} A$, i.e., the intersection of an arbitrary family of constructible subsets of X [S_1], [S_2]. He called any ringed space isomorphic to such a space $(X, \mathcal{O}_X)$ an *affine real closed space.* He then introduced the category $\mathcal{R}$ of *real closed spaces* as a full subcategory of the category of all locally ringed spaces. The definition of a real closed space is simple: a ringed space $(X, \mathcal{O}_X)$ - always with X a genuine topological space, no Grothendieck topology - is called real closed if every point $x \in X$ has an open neighbourhood U such that $(U, \mathcal{O}_X|U)$ is an affine real closed space.

The books [S_1], [S_2] are both versions of Schwartz's Habilitationsschrift [S]. For the insiders they constitute a sort of bible of abstract semialgebraic topology - an incomplete bible, I should add, since more can and should be written down with the methods developed there. The shorter version [S_1] is easier to read, while [S_2] is closer to the original Habilitationsschrift and contains much more material.

In [S] Schwartz defined real closed spaces using as building blocks only constructible subsets of real spectra, instead of proconstructible ones. I will call these more special ringed spaces here *"abstract locally semialgebraic spaces"* and denote their category by $\mathcal{R}_0$. The analogy with locally semialgebraic spaces over a real closed field R is striking. But there is more than analogy. One can attach to any locally semialgebraic space $(M, \mathcal{O}_M)$ over R an abstract locally semialgebraic space $(\tilde{M}, \tilde{\mathcal{O}}_M)$ in a rather obvious way, starting from the abstractions of affine semialgebraic spaces discussed above. Schwartz proves that this gives an embedding of $LSA(R)$ into the category $\mathcal{R}_0$, making $LSA(R)$ a full subcategory of the category of abstract locally semialgebraic spaces over $\operatorname{Sper} R$ [S], [S_1], [S_2]. A good thing about $\mathcal{R}$ is that here more constructions - in particular more quotients - are possible than in $LSA(R)$.

In view of what has been said above about affine real closed spaces, it is clear that real closed spaces are close to schemes and many of them are in fact schemes. This constitutes a rather thorough algebraization of semialgebraic topology, since the notion of schemes originates from polynomial equalities and non-equalities ($f = 0, f \neq 0$) instead of inequalities ($f \geq 0, f > 0$).

The books [S1], [S2] give ample evidence that scheme theoretic notions and techniques work well in the category $\mathcal{R}$. The books are close in spirit to the foundations of Grothendieck's abstract algebraic geometry [EGA I], [EGA I*]. In particular, the transition from a locally semialgebraic space to its abstraction is fully analogous to the transition from an algebraic variety over an algebraically closed field to the associated scheme. Of course, in many respects [S1] and [S2] are simpler than Grothendieck's theory, since here only "topological" phenomena have to be captured. This is already reflected by the fact that no nilpotent elements occur in the structure sheaves.

It also pays well to pass from the category $\mathcal{R}_0$ to $\mathcal{R}$. For example, a finite subset of a real spectrum $\operatorname{Sper} A$ is always proconstructible but only rarely constructible. This implies that for a real closed space $(X, \mathcal{O}_X)$ every finite subset S of X gives us a "subspace" $(S, \mathcal{O}_S)$ of $(X, \mathcal{O}_X)$ which is again real closed. Especially useful are the two-point spaces $S = \{\xi, x\}$ coming from the real spectrum $\operatorname{Sper} \mathfrak{o}$ of a convex subring $\mathfrak{o}$ of a real closed field K with ξ the minimal point ("general point") and x the closed point ("special point") of $\operatorname{Sper} \mathfrak{o}$. {Recall that $\mathfrak{o}$ is a valuation ring.} These two-point spaces occur in valuative criteria for various properties of morphism between real closed spaces, cf. §13 A below. By the way, $\operatorname{Sper} \mathfrak{o} = \operatorname{Spec} \mathfrak{o}$ as a ringed space.

My student Michael Prechtel has given an interesting classification of all real closed spaces which contain only finitely many points [P].

§7 Cohomology with supports

Starting with this section, a *geometric space* means a regular paracompact locally semialgebraic space over some real closed field R, and an *abstract space* means a regular paracompact abstract locally semialgebraic space.

A word of explanation for these terms: A real closed space X is called *regular* if the specializations of a point $x \in X$ form a chain. X is called *paracompact* if X has a locally finite covering $(X_i | i \in I)$ by affine open subsets X_i.

There are several reasons why this terminology makes sense. On the one hand the abstraction $\tilde{M}$ of a locally semialgebraic space M over some real closed field R is regular if M is regular and is paracompact if M is paracompact. Thus the abstraction of a geometric space is an abstract space. On the other hand, if X is an abstract space then the subspace $X^{\max}$ of closed points of X is a paracompact topological space [D2, Chap. I].

Abstract spaces are very amenable to sheaf cohomology with supports, as defined in classical topology (cf. e.g. [Bre]). This has been amply demonstrated by Delfs in Chapter II of his Habilitationsschrift [D_2]. Recall that if Φ is a family of supports on X (= antifilter of closed subsets), and $\mathcal{F}$ is a sheaf on X (as always, with values in abelian groups), then $\Gamma_\Phi(X,\mathcal{F})$ denotes the group of sections $s \in \mathcal{F}(X)$ with support $\operatorname{supp} s \in \Phi$, and $\mathcal{F} \mapsto H^q_\Phi(X,\mathcal{F})$ is the q-th derived functor of the left exact functor $\mathcal{F} \mapsto \Gamma_\Phi(X,\mathcal{F})$.

There is a close connection with the sheaf cohomology on the subspace $X^{\max}$, provided $X^{\max}$ is dense in X and every point of X has a specialization in $X^{\max}$ (which holds in many applications). Then we have a canonical continuous retraction $r\colon X \to X^{\max}$. The direct image functor r_* from the category of sheaves on X to the category of sheaves on $X^{\max}$ is easily seen to be equal to the inverse image functor i^* for i the inclusion $X^{\max} \hookrightarrow X$. For every sheaf $\mathcal{F}$ on X this gives us canonical isomorphisms $H^q_\Phi(X,\mathcal{F}) \cong H^q_\Psi(X^{\max}, i^*\mathcal{F})$, and for every sheaf $\mathcal{G}$ on X, canonical isomorphisms $H^q_\Psi(X^{\max},\mathcal{G}) \cong H^q_\Phi(X, i_*\mathcal{G})$, with Ψ denoting the set of intersections $\{A \cap X^{\max} \mid A \in \Phi\}$. In the case that X is affine (or, more generally, a "normal spectral space") this important observation goes back to Carral and Coste [CC].

If M is a geometric space then we have a canonical isomorphism $\mathcal{F} \mapsto \tilde{\mathcal{F}}$ from the category of sheaves on M to the category of sheaves over $\tilde{M}$, as explained above in the special case that M is affine (§4). A *family of supports* Φ *on* M is by definition an antifilter of closed subspaces of M. We can define cohomology groups $H^q_\Phi(M,\mathcal{F})$ in much the same way as in the classical theory. $\{\Gamma_\Phi(M,\mathcal{F})$ is the group of all $s \in \mathcal{F}(M)$ such that $s|M - A = 0$ for some $A \in \Phi$.$\}$ Let $\tilde{\Phi}$ denote the antifilter of closed subsets of $\tilde{M}$ generated by the abstractions $\tilde{A}$ of all $A \in \Phi$. Then it is evident that

$$H^q_\Phi(M,\mathcal{F}) = H^q_{\tilde{\Phi}}(\tilde{M},\tilde{\mathcal{F}})$$

for all $q \geq 0$.

There are many useful families of supports Φ, even more than in classical topology. In particular we can choose for Φ the family of all complete subspaces of M, "complete" being defined as in §2. We denote this family by c, suppressing its dependence on M in the notation. In the abstract setting we also have the notion of complete spaces. Here, for X an abstract space, we denote by c the antifilter of closed subsets generated by the complete subspaces of X. The groups $H^q_c(M,\mathcal{F})$ resp. $H^q_c(X,\mathcal{F})$ are the analogues of the cohomology groups with compact support in classical topology. For M a geometric space we have $H^q_c(M,\mathcal{F}) = H^q_c(\tilde{M},\tilde{\mathcal{F}})$.

In Chapter II of [D_4] Delfs develops the formal theory of sheaf cohomology with supports for geometric and abstract spaces virtually to the same extent as the classical theory (e.g. [Bre]). Some important topics are omitted, in particular cup products, but it is evident that these topics can be dealt with by the same methods. Delfs has often talked to me about such things in very clear terms.

As a sample of his theory I will now say something about the cohomology of fibers [D2, II §8]. Let $f: M \to N$ be a morphism between geometric spaces and $\mathcal{F}$ a sheaf on M. Let $\tilde{f}: \tilde{M} \to \tilde{N}$ denote the abstraction of f. For every $\alpha \in \tilde{N}$ the fibre $\tilde{f}^{-1}(\alpha)$ is the abstraction of a geometric space over $k(\alpha)$. In particular, if $y \in N$, then $\tilde{f}^{-1}(y) = f^{-1}(y)^{\sim}$. Fix some $q \in \mathbb{N}_0$. For every $\alpha \in \tilde{N}$ we consider the cohomology group $H^q_c(\tilde{f}^{-1}(\alpha), \tilde{\mathcal{F}}|\tilde{f}^{-1}(\alpha))$ which we denote more briefly by $H^q_c(\tilde{f}^{-1}(\alpha), \tilde{\mathcal{F}})$. The question arises as to how these groups are related for varying α. A first answer is that there exists a sheaf $\mathfrak{G}$ on N, namely $\mathfrak{G} = R^q f_! \mathcal{F}$, such that, for every $\alpha \in \tilde{N}$, $H^q_c(\tilde{f}^{-1}(\alpha), \tilde{\mathcal{F}})$ is the stalk of $\tilde{\mathfrak{G}}$ at α,

$$(*) \qquad (R^q f_! \mathcal{F})^{\sim}_{\alpha} = H^q_c(\tilde{f}^{-1}(\alpha), \tilde{\mathcal{F}}).$$

Here $\mathcal{F} \mapsto R^q f_! \mathcal{F}$ is the q-th derived functor of a left exact functor $\mathcal{F} \mapsto f_! \mathcal{F}$ defined as follows. For U an admissible open subset of N the group $(f_! \mathcal{F})(U)$ consists of all sections// $\mathcal{F}(f^{-1}(U))$ such that there exists some closed subspace A of $f^{-1}(U)$ such that s vanishes on $M - A$ and the restriction $f|A: A \to N$ is proper. It follows from $(*)$ that, for $y \in N$,

// s of

$$(**) \qquad (R^q f_! \mathcal{F})_y = H^q_c(f^{-1}(y), \mathcal{F}).$$

The equations $(*)$ and $(**)$ throw some light on the relationship between the geometric and the abstract setting. In the geometric theory we are interested in the groups $H^q_c(f^{-1}(y), \mathcal{F})$ for $y \in N$. But it is evident from $(*)$ and $(**)$ that the groups $H^q_c(\tilde{f}^{-1}(\alpha), \tilde{\mathcal{F}})$ with $\alpha \in \tilde{N}$, not necessarily $\alpha \in N$, have an influence on these groups.

§8 Borel-Moore homology

Again, let M be a geometric space and Φ a family of supports on M. Let G_M be the constant sheaf on M attached to some abelian group G. We denote the groups $H^q_\Phi(M, G_M)$ more briefly by $H^q_\Phi(M, G)$. If Φ is the set of all closed subspaces of M, henceforth denoted by cld (= "closed"), then these groups are just the semialgebraic cohomology groups $H^q(M, G)$ of Delfs discussed in §4.

It would also be desirable to have at hand homology groups $H^\Phi_q(M, G)$ with support in Φ. This is a problem even for $\Phi = $ cld. It is intuitively clear that the group $H_q(M, G)$, as described in §5, should coincide with $H^c_q(M, G)$, and not with $H^{\mathrm{cld}}_q(M, G)$, since a simplicial chain, as used there, clearly has support in a complete subspace of M. {By the way, every complete subspace of M is affine semialgebraic, cf. §13 A }.

In classical topology homology groups with closed support have been defined by Borel and Moore for locally compact spaces [BM], [Bre, Chap.V]. They used a rather complicated sheaf theoretic procedure, but their groups turned out to be very useful.

Delfs found out that, for M a locally complete geometric space, such Borel-Moore groups ${}^{\mathrm{BM}}H_q(M,G)$ can be defined in a down-to-earth combinatorial manner as follows [D₂, Chap. III], [D₃]. We choose a triangulation $\varphi: X \xrightarrow{\sim} M$. Then X is a locally finite and locally closed simplicial complex over the base field R. This means that every open simplex of X is the face of only finitely many open simplices of X, and that X is open in $\bar{X}$. {Recall that we use the notion "simplicial complex" in a non-classical sense (§3). $\bar{X}$ is a simplicial complex in the classical sense.}

All the combinatorial technique is based on open simplices instead of closed ones. We choose a (say) total ordering of the vertices of $\bar{X}$ and then can talk about oriented open simplices. For pairwise different vertices $x_0,\dots,x_q$ of $\bar{X}$ let $\langle x_0,\dots,x_q\rangle$ denote the open oriented q-simplex with these vertices and the orientation given by the sequence $x_0,\dots,x_q$. We define a chain complex ${}^{\mathrm{BM}}C.(X,G)$ as follows. ${}^{\mathrm{BM}}C_q(X,G)$ consists of all formal sums $\sum_\sigma m_\sigma \sigma$ with σ running through the set of positively oriented open simplices of X and $m_\sigma \in G$. The boundary operator is given by the classical rule

$$\partial(\langle x_0,\dots,x_q\rangle) = \sum_{i=0}^{q}{}' (-1)^i \langle x_0,\dots,\widehat{x_i},\dots x_q\rangle,$$

but omitting on the right hand side all open simplices which do not lie in X. The fact that X is locally closed guarantees that $\partial \circ \partial = 0$. ${}^{\mathrm{BM}}H_q(M,G)$ is defined as the q-th homology group of the chain complex ${}^{\mathrm{BM}}C.(X,G)$.

Delfs verifies that these "semialgebraic Borel-Moore homology groups" do not depend on the choice of the triangulation [D₂, III.2.1]. They have the formal properties one is used to from topological Borel-Moore homology, and in the case $R = \mathbb{R}$ they coincide with the classical Borel-Moore groups.

From now on we replace G by a principal ideal domain Λ. If M is an n-dimensional Λ-oriented paracompact locally semialgebraic manifold over R (this defined in a rather obvious way [D₂, III §3]), and $\varphi: X \xrightarrow{\sim} M$ is a triangulation, then the sum Z of all n-dimensional open suitably oriented simplices of X is an element of ${}^{\mathrm{BM}}C_n(M,\Lambda)$ with boundary $\partial Z = 0$. The cap product with the "fundamental class" $[Z] \in {}^{\mathrm{BM}}H_n(M,\Lambda)$ gives an isomorphism from $H^q(M,\Lambda)$ onto ${}^{\mathrm{BM}}H_q(M,\Lambda)$ for every q, as in classical Poincaré duality theory [D, III §9].

If M is the space of real points $V(R)$ of an n-dimensional algebraic variety V over R then Delfs defines again a fundamental class $\zeta_M \in {}^{\mathrm{BM}}H_n(M,\mathbb{Z}/2)$ in a similar way [D₃]. ζ_M is characterized by the property that its restriction to any connected component N_i of the open subset $V_{\mathrm{reg}}(R)$ of regular points is the unique non-zero element of ${}^{\mathrm{BM}}H_n(N_i,\mathbb{Z}/2)$. {In the case that M is complete this property had been verified before by other methods in [DK₁].} Delfs also constructs a fundamental class for $\Lambda = \mathbb{Z}$ and $M = V = V(R(\sqrt{-1}))$ with V an algebraic variety (more generally, an "isoalgebraic space") over $R(\sqrt{-1})$. Such fundamental classes have been introduced for real and complex analytic spaces by Borel and Haefliger, and used by them to establish a topological intersection theory on these spaces [BH].

Let again M be a locally complete space and Φ be a family of supports on M. In order to define homology groups with supports in Φ and, for M an oriented manifold, to establish Poincaré-duality with the cohomology groups with supports in Φ, one probably can use still combinatorial methods since we have a strong triangulation theorem at hand, but this would be more complicated than in the case $\Phi = \mathrm{cld}$. Instead of this Delfs returns to sheaf theoretic methods similar to those used in the classical theory [BM], [Bre]. Sheafifying the Borel-Moore chain complex ${}^{\mathrm{BM}}C.(M,\Lambda)$ he obtains a complex $\Delta.$ of sheaves of simplicial chains. For $\mathcal{F}$ a sheaf of Λ-modules he defines $H_q^{\Phi}(M,\mathcal{F})$ as the q-th homology groups of $\Gamma_\Phi(M,\Delta.\otimes_\Lambda\mathcal{F})$ [D_2, Chap. III]. For G a Λ-module and $\mathcal{F}$ the constant sheaf G_M this group coincides with ${}^{\mathrm{BM}}H_q(M,G)$ if $\Phi = \mathrm{cld}$, and with the semialgebraic homology group $H_q(M,G)$ if $\Phi = c$.

Being able to vary Φ and $\mathcal{F}$, Delfs has enough flexibility to establish Poincaré-duality and semialgebraic intersection theory, the latter for $M = V(R)$, $\Lambda = \mathbb{Z}/2$, resp. $M = V(R(\sqrt{-1}))$, $\Lambda = \mathbb{Z}$, with V an algebraic manifold over R or $R(\sqrt{-1})$ respectively (even an isoalgebraic manifold in the latter case) [D_2].

In particular Delfs obtains an important result about algebraic intersection numbers. Let C be an algebraically closed field of characteristic zero. Let V be a smooth algebraic variety over C, and let X_1, X_2 be irreducible subvarieties of V. Let Y be an irreducible component of $X_1 \cap X_2$ with $\operatorname{codim} Y = \operatorname{codim} X_1 + \operatorname{codim} X_2$. Then the algebraic intersection number $i(X_1 \cdot X_2, Y)$ is a well defined natural number. Now choose a real closed field R with $C = R(\sqrt{-1})$, which can of course usually be done in very many different ways. Then $V = V(C)$ is also a geometric space over R, and Borel-Moore homology with $\Lambda = \mathbb{Z}$ gives us a semialgebraic intersection number $i_R(X_1 \cdot X_2, Y)$. Delfs proves that this number coincides with $i(X_1 \cdot X_2, Y)$ for every choice of R.

§9 Base field extension and comparison theorems

We fix a real closed base field R and a real closed field extension K of R. Let M be a semialgebraic subset of R^n. Then we obtain from M a semialgebraic subset $M(K)$ of K^n by "base field extension" as follows. We write

$$M = \bigcup_{i=1}^{r} \{x \in R^n \mid f_{ij}(x) > 0,\ j = 1,\dots,s(i);\ g_{ik}(x) \geq 0,\ k = 1,\dots,t(i)\}$$

with some polynomials f_{ij}, g_{ik} in n variables over R. Then we define

$$M(K) := \bigcup_{i=1}^{r} \{x \in K^n \mid f_{ij}(x) > 0,\ j = 1,\dots,s(i);\ g_{ik}(x) \geq 0,\ k = 1,\dots,t(i)\}.$$

It is an immediate consequence of Tarski's principle that $M(K)$ does not depend on the choice of the description of M above.

If $f\colon M \to N$ is a semialgebraic map between semialgebraic sets over R then, again by Tarski's principle, there exists a well defined semialgebraic map $f_K\colon M(K) \to N(K)$ whose graph $\Gamma(f_K)$ is obtained from $\Gamma(f)$ by base field extension, $\Gamma(f_K) = \Gamma(f)(K)$.

Thus we have a functor $M \mapsto M(K)$ from the category of affine semialgebraic spaces over R to the analogous category over K. This functor extends in a natural way to a functor $M \mapsto M(K)$ from the category $LSA(R)$ of geometric (= regular paracompact locally semialgebraic) spaces over R to $LSA(K)$ [DK$_2$, Chap. I].

This functor "base field extension from R to K" has an illuminating interpretation in the abstract setting. Let $\mathfrak{S}$ denote the category of abstract spaces (a full subcategory of the category $\mathcal{R}_0$ considered in §6). Recall that the abstraction functor $M \mapsto \tilde{M}$ embeds $LSA(R)$ into $\mathfrak{S}$. In the same way, $LSA(K)$ is a subcategory of $\mathfrak{S}$. In particular $\mathfrak{S}$ contains the one-point ringed spaces $\operatorname{Sper} R$ and $\operatorname{Sper} K$. The inclusion $R \hookrightarrow K$ gives us a morphism $\operatorname{Sper} K \to \operatorname{Sper} R$. {By the way, $\operatorname{Sper} R = \operatorname{Spec} R$ and $\operatorname{Sper} K = \operatorname{Spec} K$.} In $\mathfrak{S}$ we have fibre products [S$_1$], [S$_2$], and

$$\widetilde{M(K)} = \tilde{M} \times_{\operatorname{Sper} R} \operatorname{Sper} K.$$

For semialgebraic homology $H_*(-,G)$ with values in some abelian group G, the following comparison theorems are rather evident from the simplicial definition of $H_*(-,G)$ (cf. [DK$_2$, III §7] for a more precise version).

First Comparison Theorem. There is a canonical isomorphism

$$H_q(M,G) \xrightarrow{\sim} H_q(M(K),G).$$

Second Comparison Theorem. If $R = \mathbb{R}$ there is a canonical isomorphism

$$H_q(M,G) \xrightarrow{\sim} H_q(M_{\mathrm{top}},G).$$

Here M_{top} denotes the set M equipped with the topology which has the admissible open subsets of M as a basis, the so called "strong topology". Notice that this is the subspace topology of M in $\tilde{M}$. The groups $H_q(M_{\mathrm{top}},G)$ are the singular homology groups. There is an important case, namely if M is "partially complete" (cf. §13 A below for this), in which M_{top} coincides with the space $M^{\max}$ of closed points in $\tilde{M}$.

We have similar comparison theorems in sheaf cohomology due to Delfs [D$_2$, Chap. II]. Let Φ be a family of supports on M. It gives us a family of supports $\Phi(K)$ on $M(K)$, namely the antifilter of closed subspaces generated by $\{A(K) | A \in \Phi\}$. In the case $R = \mathbb{R}$ it also gives us a family of supports Φ_{top} on M_{top}, namely the antifilter of closed subsets generated by Φ.

Let $\mathcal{F}$ be a sheaf on M. It gives us a sheaf $\mathcal{F}(K)$ on $M(K)$ by the rule $\mathcal{F}(K)^{\sim} = \pi^*(\tilde{\mathcal{F}})$, with π the natural projection from $M(S)^{\sim} = \tilde{M} \times_{\operatorname{Sper} R} \operatorname{Sper} K$ to $\tilde{M}$. In the case $R = \mathbb{R}$ we also obtain a sheaf $\mathcal{F}_{\text{top}}$ on M_{top} by the rule $\mathcal{F}_{\text{top}} = i^*(\tilde{\mathcal{F}})$ with i the inclusion $M_{\text{top}} \hookrightarrow \tilde{M}$.

First Comparison Theorem [D$_2$, II.6.1]. There are canonical isomorphisms

$$H^q_\Phi(M, \mathcal{F}) \xrightarrow{\sim} H^q_{\Phi(K)}(M(K), \mathcal{F}(K)).$$

Second Comparison Theorem [D$_2$, II §5]. Let $R = \mathbb{R}$. Assume either that $\mathcal{F}$ is locally constant or that M is partially complete. Then there are canonical isomorphisms

$$H^q_\Phi(M, \mathcal{F}) \xrightarrow{\sim} H^q_{\Phi_{\text{top}}}(M_{\text{top}}, \mathcal{F}_{\text{top}}).$$

Delfs points out that some restrictive hypothesis in the second theorem is indeed necessary [D$_2$, II.5.6]. One should add that the first theorem also does not always give the result one would like to have. For example, if Φ is the set of all complete subspaces of M, denoted previously by c, then $\Phi(K)$ may be smaller than the set of all complete subspaces of $M(K)$. Nevertheless Delfs proves that there exist canonical isomorphisms

$$H^q_c(M, \mathcal{F}) \xrightarrow{\sim} H^q_c(M(K), \mathcal{F}(K))$$

if M is locally complete [D$_2$, II.6.10].

If M is locally complete then $M(K)$ is locally complete, and, in the case $R = \mathbb{R}$, the space M_{top} is locally compact. Thus one may also look for comparison theorems in Borel-Moore homology with supports and with sheaves as coefficients. Delfs proves such theorems, again with a necessary restriction in the assumptions of the second comparison theorem [D$_2$, III §10 and §11].

§10 Homotopy sets

We fix a real closed base field R. A pair of geometric spaces (M, A) over R consists of a geometric space M and a subspace A of M. A morphism f from (M, A) to another pair (N, B) is a morphism $f: M \to N$ with $f(A) \subset B$. {Then the restriction $f|A: A \to B$ is again a morphism [DK$_2$, I.3.2].}

It is now clear how to define a homotopy F between two morphisms $f, g: (M, A) \rightrightarrows (N, B)$; namely, F is a morphism from $(M \times [0,1], A \times [0,1])$ to (N, B) with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for every $x \in M$. More generally all the basic notions of the elementary classical homotopy theory make sense in the category $LSA(R)$. But notice that we do not have something like spaces of morphisms at our disposal.

Let $[(M,A),(N,B)]$ denote the set of homotopy classes of morphisms from (M,A) to (N,B). One would like to know as much as possible about it. In the case that $B=\{x\}$ is a one-point set and $(M,A)=(S^q(R),\{\infty\})$, with $S^q(R)$ the unit sphere in R^{q+1} and ∞ its north pole, we equip this set with a group structure, in the classical way which is of semialgebraic nature. This gives us the homotopy group $\pi_q(N,x)$. It is abelian for $q \geq 2$.

For homotopy sets we have the following two comparison theorems.

First Comparison Theorem. If K is a real closed overfield of R then the natural map $[f] \to [f_K]$ from $[(M,A),(N,B)]$ to $[(M(K),A(K)),(N(K),B(K))]$ is bijective.

Second Comparison Theorem. If $R = \mathbf{R}$ then the natural map from $[(M,A),(N,B)]$ to the set of homotopy classes of continuous maps $[(M_{\text{top}},A_{\text{top}}),(N_{\text{top}},B_{\text{top}})]$ is bijective.

All this can be proved by using Tarski's principle, triangulations, and simplicial approximation techniques [DK$_2$, Chap. III]. More generally such theorems hold for relative homotopy classes, where a subspace of M has to be fixed pointwise by the homotopies [loc. cit.].

If $B=\{x\}$ and $(M,A)=(S^q(R),\{\infty\})$ then the bijections in the theorems are, of course, group isomorphisms. The theorems now tell us what the homotopy groups of a pointed geometric space (N,x) over R look like "in principle". To be more precise, we choose a triangulation $\varphi: X \xrightarrow{\sim} N$ such that $\varphi^{-1}(x)$ is a vertex e of X. Let us assume for simplicity that N is "partially complete" (cf. §13 A below), a property which forces the simplicial complex to coincide with its closure $\bar{X}$. Then X is the realization $|L|_R$ of a locally finite abstract simplicial complex L (in the classical sense [Spa, Chap. 3]). Let R_0 denote the real closure of $\mathbf{Q}$ with respect to its unique ordering. This field has a unique embedding into any other real closed field. Of course, $|L|_R = |L|_{R_0}(R)$. Now the isomorphism φ and the two comparison theorems give us group isomorphisms

$$\pi_q(N,x) \cong \pi_q(|L|_R, e) \cong \pi_q(|L|_{R_0}, e) \cong \pi_q(|L|_{\mathbf{R}}, e) \cong \pi_q(|L|_{\mathbf{R},\text{top}}, e).$$

The last group, a classical homotopy group, can be written - somewhat tautologically - as the homotopy group $\pi_q(L,e)$ of the abstract simplicial complex L.

§11 Weakly semialgebraic spaces, and quotients

The comparison theorems stated in the last section, and analogous theorems for triples, quadruples, etc. of geometric spaces instead of pairs, allow us to transfer a considerable amount of classical homotopy theory to the category of geometric spaces $LSA(R)$, cf. [DK$_2$, III §6] for examples.

Nevertheless, homotopy theory in $LSA(R)$ is hampered by the fact that many of the constructions used in the classical theory do not have counterparts here. In particular, quotients by seemingly innocent equivalence relations can cause problems. For example, CW-complexes exist in $LSA(R)$ only to a limited extent. Even the (reduced) suspension SM of a pointed geometric space (M, x_0) usually does not exist, since it may be impossible to contract the closed subspace $M \times \{0\} \cup M \times \{1\} \cup \{x_0\} \times [0,1]$ of $M \times [0,1]$ to a point in some universal way. Thus we cannot start stable homotopy theory in $LSA(R)$.

Fortunately there is a way out of many of these difficulties. As has been shown in [K$_2$], we can embed $LSA(R)$ as full subcategory in the category $WSA(R)$ of "weakly semialgebraic spaces" over R, where homotopy theory is more pleasant.

A *weakly semialgebraic space* $M = (M, \mathcal{O}_M)$ over R is a set M equipped with a Grothendieck topology, consisting of admissible open subsets and admissible open coverings, and a structure sheaf $\mathcal{O}_M$, consisting of R-valued functions, such that M arises from affine semialgebraic spaces by gluing along *closed* subspaces carefully observing certain rules, cf. [K$_2$, p.3 f]. As for locally semialgebraic spaces, a morphism $(M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ between such ringed spaces is determined by the underlying map $f: M \to N$, and will henceforth be identified with this map (cf. [K$_2$, IV §1 for the definition of morphisms).

In $WSA(R)$ there exist fibre products. Thus notions like proper morphisms and complete spaces make sense here. It turns out that every complete weakly semialgebraic space is a complete affine semialgebraic space, hence isomorphic to a polytope (= finite polyhedron), [K$_2$, p. 44] and [DK$_2$, p. 59 f].

Of particular importance for homotopy theory are the weakly semialgebraic spaces which can be obtained by gluing *complete* affine spaces along closed subspaces. They are called *weak polytopes*. Every weakly semialgebraic space is homotopy equivalent to a weak polytope [K$_2$, V §4].

Let T be an equivalence relation on a weakly semialgebraic space M. Under which conditions can we expect that the quotient M/T exists in a reasonable sense? Certainly we should assume that T is a closed subspace of $M \times M$. Let $p_T: T \to M$ denote the natural projection from T to the first factor in $M \times M$. For M affine semialgebraic and p_T proper ("proper equivalence relation") Brumfiel has proved the important theorem that a "strong" quotient M/T exists [B_5]: the set theoretic quotient M/T can be equipped in a unique way with the structure of an affine semialgebraic space such that the projection map $\pi_T: M \to M/T$ is "identifying" in a strong sense. This implies, in particular, that π_T is the categorical quotient of M by T in the category of affine semialgebraic spaces.

In $WSA(R)$ we can do even better. As an easy consequence of Brumfiel's theorem such a strong quotient $\pi_T: M \to M/T$ exists if $p: T \to M$ is *partially proper*, i.e. the restriction of p to any closed semialgebraic subspace of T is proper [K_2, IV §11].

At first glance, partially proper equivalence relations might not look much more general than proper ones. But in fact quotients by partially proper equivalence relations suffice for many important topics in homotopy theory. For instance, if A is a closed subspace of a weakly semialgebraic space M, and if $f: A \to N$ is a partially proper map, then we have a space $M \cup_{A,f} N$ at our disposal, obtained by gluing M to N along A via f.

In particular take M to be a direct sum of closed balls, and A the boundary ∂M, which is a direct sum of spheres. This is the essential step to build CW-complexes. Notice that now A is a weak polytope, and every morphism from a weak polytope to a weakly semialgebraic space is partially proper.

As a result we can construct CW-complexes and, more generally, relative CW-complexes in $WSA(R)$ in much the same way and for much the same purposes (e.g. killing of homotopy groups) as one is used to in classical topology.

If (M, x_0) is a pointed weak polytope then we obtain the suspension SM by choosing $A = M \times \{0\} \cup M \times \{1\} \cup \{x_0\} \times [0,1]$ and f as the morphism from A to the one point space $*$. Thus we can start stable homotopy in the full subcategory $\mathcal{P}(R)$ of $WSA(R)$ whose objects are the weak polytopes over R.

If M is not a weak polytope then also A is not a weak polytope, and this means that the morphism $A \to *$ is not partially proper. It turns out, that then SM does not exist as a strong quotient of $M \times [0,1]$ in the sense indicated above. One can conclude from [Sch, §3] that SM does not exist even as a categorical quotient, at least if M is a geometric space.

The existence of strong quotients by partially proper equivalence relations seems to be a best possible result. Indeed, C. Scheiderer has proved a "converse" to Brumfiel's theorem [Sch]: If T is a closed semialgebraic equivalence relation on a locally complete semialgebraic space M, and the strong quotient M/T exists, then, up to obvious modifications, T is proper. Scheiderer's method is very interesting. He makes essential use of abstract spaces and applies an extension of Tarski's principle to real closed fields with convex valuation rings [loc. cit.].

Weakly semialgebraic spaces often cannot be triangulated, but "patch techniques" have been developed to work with them nearly as easily as with triangulable spaces, cf. [K_2, Chap. V].

For every real closed overfield K of R we have a natural base field extension functor from $WSA(R)$ to $WSA(K)$ which extends the base field extension functor from $LSA(R)$ to $LSA(K)$ discussed above (§9). It turns out that the two comparison theorems for homotopy sets from above (§10) remain true for weakly semialgebraic spaces.

Thus nothing is lost and much is gained in homotopy theory by passing from geometric spaces to weakly semialgebraic spaces.

In classical homotopy theory one often works in the category of topological spaces which are homotopy equivalent to CW-complexes, or variants of this category, which all look a little artificial, since a topological space may carry many structures of a CW-complex (cell decompositions), none of which is given in an intrinsic way. A nice fact about $WSA(R)$ is that every weak polytope is homotopy equivalent to a CW-complex [K_2, V §7]. Thus here the category $\mathcal{P}(R)$ is a satisfactory basic category for homotopy theory, the notion of a weak polytope being intrinsic and quite natural, cf. §13 A below.

What are the "abstractions" of weakly semialgebraic spaces? Do they exist? Recently N. Schwartz gave a fascinating solution to this problem by inventing his *"inverse real closed spaces"* [S_3]. In some sense the approach of Schwartz to inverse real closed spaces is simpler than my approach to weakly semialgebraic spaces in [K_2], and thus throws new light on them. Inverse real closed spaces have passed early tests of usefulness successfully. For example they are instrumental in the homotopy classification of vector bundles and the definition of Stiefel-Whitney classes on real closed spaces (which helps to understand the old paper [B_4] of Brumfiel, in particular) [Schwartz, talk at the Ragsquad seminar in Berkeley, October 1990]. I refrain here from a discussion of inverse real closed spaces, since this subject is in a less mature state than the other topics dealt with in this article. Instead I refer the reader to the paper [S_3].

§12 Generalized homology

Let $\mathcal{P}^*(R)$ denote the category of weak polytopes with base points over R. In this category the direct sum of an arbitrary family of objects $(M_\lambda, \lambda \in \Lambda)$ is the wedge $\bigvee\limits_{\lambda\in\Lambda} M_\lambda$, obtained from the disjoint union $\bigsqcup\limits_{\lambda\in\Lambda} M_\lambda$ by identifying the base points of all the M_λ. We also have the suspension functor $S: \mathcal{P}^*(R) \to \mathcal{P}^*(R)$. We pass from $\mathcal{P}^*(R)$ to the associated homotopy category $H\mathcal{P}^*(R)$. It has the same objects as $\mathcal{P}^*(R)$, but the morphisms are the homotopy classes of morphisms in $\mathcal{P}^*(R)$. Now we have all the tools we need to define generalized homology theories as one does in classical topology. There one uses the homotopy category $H\mathfrak{W}^*$ of pointed topological CW-complexes instead of $H\mathcal{P}^*(R)$.

Definition (cf. [Sw, p. 109 f]). A *reduced semialgebraic homology theory* over R is a family $(k_n | n \in \mathbb{Z})$ of covariant functors k_n from $H\mathcal{P}^*(R)$ to the category Ab of abelian groups, together with a family $(\sigma_n | n \in \mathbb{Z})$ of natural equivalences $\sigma_n \colon k_n \xrightarrow{\sim} k_{n+1} \circ S$, such that the following two axioms hold:

Exactness Axiom. For every pair (M, A) of pointed weak polytopes over R, the sequence

$$k_n(A) \longrightarrow k_n(M) \longrightarrow k_n(M/A),$$

induced by the inclusion $A \hookrightarrow M$ and the projection $M \to M/A$, is exact.

Wedge Axiom. For every family $(M_\lambda | \lambda \in \Lambda)$ of pointed weak polytopes, and every $n \in \mathbb{Z}$, the map

$$\bigoplus_{\lambda\in\Lambda} k_n(M_\lambda) \longrightarrow k_n(\bigvee_{\lambda\in\Lambda} M_\lambda),$$

induced by the inclusions $M_\lambda \hookrightarrow \bigvee\limits_{\mu\in\Lambda} M_\mu$, is an isomorphism.

Applying the homotopy theory developed in [K$_2$] one proves that every reduced homology over R "extends" in an unique way to a non reduced homology theory $(h_n | n \in \mathbb{Z})$ on the category of all pairs of weakly semialgebraic spaces over R. Such unreduced homology theories are again defined in an axiomatic way analogous to classical topology, in other words, the Eilenberg-Steenrod axioms [ES, I §3] - with the exception of the dimension axiom - have to hold, cf. [K$_2$, VI §4]. The relationship between h_n and k_n is given by $h_n(M, A) = k_n(M/A)$ for any pair (M/A) of weak polytopes, the space M/A being equipped with its natural base point A/A. {Here I suppress a consideration of the boundary maps $\partial_n \colon h_n(M, A) \to h_{n-1}(A, \emptyset)$, which are present in the axioms of an unreduced homology theory.} It turns out that the functors h_n satisfy a strong excision property [K$_2$, VI.6.10], which is even better than excision in classical topology.

A main result in [K$_2$] about generalized homology is the following: Given a reduced homology theory $(k_n^{\mathrm{top}} | n \in \mathbb{Z})$ on the homotopy category $H\mathfrak{W}^*$ of topological CW-complexes, there exists, for every real closed field R, a reduced homology theory $(k_n^R | n \in \mathbb{Z})$ on $H\mathcal{P}^*(R)$ such that two comparison theorems analogous to those stated in §9 are true (cf. [K$_2$, VI §3] for a more precise statement). One then also has comparison theorems for the associated unreduced homology theories [K$_2$, VI.5.4].

Running parallel to all of this, there is a theory of reduced and unreduced cohomology with results analogous to those just stated.

To appreciate the content of these results let us choose for (k_n^{top}) the reduced singular homology theory with coefficients in some abelian group G. Then we obtain a homology theory (h_n^R) for pairs of weakly semialgebraic spaces over R which satisfies all the Eilenberg-Steenrod axioms. Given, say, a complete geometric space M over R and a triangulation $|L|_R \xrightarrow{\sim} M$, with L a finite abstract simplicial complex, one verifies in the classical way that $h_q^R(M) \cong H_q(L,G)$. Thus we have obtained anew the result of Delfs that (up to canonical isomorphism) the combinatorial homology group $H_q(L,G)$ does not depend on the choice of the triangulation.

In some sense, this new proof of Delfs' result is considerably easier than the two original proofs discussed in §4. Once one knows that weakly semialgebraic spaces exist with suitable formal properties, in particular the two comparison theorems for homotopy sets, the proof works by straightforward homotopy methods, cf. [K2]. But notice that the new proof gives something less than the old ones: a connection with the sheaf cohomology groups $H^q(M, G_M)$ is missing.

From now on I call the groups $H_q(M,A;G)$ which come from the topological singular homology theory *ordinary homology groups.* They coincide with the semialgebraic homology groups of Delfs discussed in §4 and later.

There exist many prominent homology and cohomology theories in topology. For example, let $(h^n | n \in \mathbb{Z})$ be one of the classical cobordism theories [Sw, 12.24]. Then, by the results above, we have associated groups $h^n(M,A)$ for every pair of weakly semialgebraic spaces (M,A) over some real closed field. {From here on I omit the subscripts R and "top".} Many real algebraic geometers will be interested in these groups only for (M,A) a pair of semialgebraic sets. But it seems to be difficult to understand the groups $h^n(M,A)$ working only with semialgebraic sets or, more generally, with geometric spaces. So weakly semialgebraic spaces serve us well, even if we have no geometric interest in them.

There remains the task of giving a *geometric* interpretation of the groups $h^n(M,A)$ for, say, semialgebraic sets. They can probably be described using some semialgebraic differential topology, of the sort that appears in the *definition* of the topological cobordism groups. The theory in [K2] does not give such an interpretation. To the best of my knowledge, nobody has tackled this task up to now.

In the category $\mathcal{P}(R)$ of weak polytopes over R there exists the realization $|L|_R$ of any simplicial set L (= "semisimplicial set" in older terminology) [K2, Chap. VII], with the formal properties one expects from classical topology [Mi], [May]. This makes it possible to solve a problem left open in Delfs' semialgebraic homology theory discussed in §4, namely the question of whether the semialgebraic singular chain complexes give us the ordinary homology groups. For M a weakly semialgebraic space over R let Sin$M$ denote the singular set of M. This is the simplicial set consisting of the semialgebraic maps from the standard simplices over R to M. As in topology one has an obvious morphism $j_M\colon |\mathrm{Sin}M|_R \longrightarrow M$. One can prove directly that j_M is a homotopy equivalence [K$_2$, VII §7]. {In topology the analogous map is only a "weak" homotopy equivalence [Mi].} From this it follows immediately that the homology groups of the singular chain complex of M with coefficients in G are the ordinary homology groups $H_q(M, G)$ [loc. cit.].

In particular, we now know that semialgebraic singular homology obeys the excision axiom. It would be desirable to have a more elementary proof of this, say, by nonlinear subdivision of singular chains. Certainly we would learn a lot from this about the geometry behind ordinary homology.

§13 Novel features of semialgebraic topology: three examples.

Much of what has been said up to now gives the impression that semialgebraic topology, at least in the geometric setting, is very similar to traditional topology. The definition of spaces, morphisms, etc. may look a little exotic to the classical topologist, but then the results in homology and homotopy are as he or she is used to.

In fact semialgebraic topology has also features which may be unexpected from the classical viewpoint, even if the base field R is the field $\mathbb{R}$ of real numbers. In the following I give three examples.

A. Proper and partially proper maps

I start with some definitions. Let M be a locally semialgebraic space over R. A subset A of M can carry at most one structure of a subspace of M (cf. §5 above and [DK$_2$, I §3] for the definition of subspace). If it does we call A a *locally semialgebraic subset* of M. Locally semialgebraic subsets are the only subsets of M which have a geometric meaning. If A carries the structure of a subspace which is even a semialgebraic space we call A a *semialgebraic subset* of M.

If A and B are locally semialgebraic subsets of M with $B \subset A$ and A is semialgebraic then also B is semialgebraic. The image $f(A)$ of a semialgebraic subset A of M under any morphism $f\colon M \to N$ is a semialgebraic subset of N. All this justifies the idea that "semialgebraic" is something like "small". This notion of smallness is alien to classical topology.

If $f\colon M \to N$ is a morphism and B is a locally semialgebraic subset of N, then the preimage $f^{-1}(B)$ is a locally semialgebraic subset of M. {It is not true in general that images of locally semialgebraic subsets of M are locally semialgebraic in N}. We call the morphism f *semialgebraic* if the preimages of semialgebraic subsets of

N are semialgebraic, and we call f *affine semialgebraic*, if the preimages of affine semialgebraic subsets of N are affine semialgebraic.

Theorem 1. Every proper morphism $f\colon M \to N$ is affine semialgebraic.

This theorem has an interesting history. Delfs and I gave an argument which proves that f is semialgebraic [DK$_2$, p. 59 ff]. This reduces the proof of the theorem to the case that N is affine semialgebraic and M is semialgebraic. In this case the theorem is due to N. Schwartz, who proved it by transition to abstract spaces [S], [S$_1$], [S$_2$]. It was one of the really remarkable early applications of his theory of real closed spaces. Then Robson gave a proof within the geometric setting in the special case that N is the one-point space, applying his embedding theorem, cf. [DK$_2$, p. 60]. It was not until much later that R. Huber found a proof in general which stays in the geometric setting. This proof is contained in a joint paper by Huber and Scheiderer [HuS], which contains related results about "locally proper maps", and which, in my opinion, also gives a good feeling for the interplay between the geometric and the abstract setting for such questions. In addition it contains a deepening of Schwartz's result on abstract spaces [S$_1$, p. 83], [S$_2$, V.5.7] which is behind his proof of Theorem 1 for M and N semialgebraic.

If one views a morphism $f\colon M \to N$ as a "family of spaces" then Theorem 1 may be regarded as a negative result. Properness is commonly regarded as a condition which is often needed to ensure good behaviour in families of spaces. Theorem 1 tells us that we do not get anything new in essence if we study proper morphisms between geometric spaces instead of just affine semialgebraic spaces.

Fortunately there is another notion, alien to classical topology, but related to properness, which still gives us good families of geometric spaces which are not necessarily semialgebraic. A morphism $f\colon M \to N$ is called *partially proper* if the restriction $f|A\colon A \to N$ to any closed semialgebraic subset A of M is a proper morphism. A geometric space M is called *partially complete* if the morphism from M to the one-point space is partially proper.

These notions are quite natural, as is illustrated by the following "relative path completion criterion" [DK$_2$, I §6]:

Theorem 2. A morphism $f\colon M \to N$ between geometric spaces is partially proper iff for every commuting square of morphisms (solid arrows)

$$\begin{tikzcd}
{]0,1]} \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & M \arrow[d, "f"] \\
{[0,1]} \arrow[r, "\beta"'] \arrow[ur, dashed, "\bar{\alpha}"] & N
\end{tikzcd}$$

with i the inclusion map, there exists a path $\bar{\alpha}$ (dotted arrow) which extends α. {N.B. Since geometric spaces are "separated" there can exist at most one such path $\bar{\alpha}$, and $\bar{\alpha}$ lies over β, $f \circ \bar{\alpha} = \beta$.}

In the special case that N is the one-point space the theorem tells us that a geometric space M is partially complete iff every "incomplete path" $\alpha\colon]0,1] \to M$ can be completed. Here the classical notion of sequential compactness comes to our mind, but sequences are a blunt instrument in semialgebraic topology. {There even exist real closed fields which do not contain any non-trivial zero sequences.} Incomplete paths may be regarded as the right substitute of sequences in the geometric theory. From this viewpoint partial completeness, and not completeness, is the right analogue of compactness.

By the way, all the definitions in this subsection A make sense more generally for weakly semialgebraic spaces, and both theorems 1 and 2 remain true for them. The partially complete weakly semialgebraic spaces are precisely the weak polytopes discussed in §11.

In the abstract setting the counterparts of path completion and path lifting criteria like Theorem 2 are *valuative criteria* similar to the ones in algebraic geometry. The books of Schwartz [S$_1$], [S$_2$], his paper [S$_4$], and the paper [HuS] by Huber and Scheiderer contain several such criteria. Here the unit interval $[0,1]$ is replaced by the two point subspace $\{\xi, x\}$ of the real spectrum $\operatorname{Sper} \mathfrak{o} = \operatorname{Spec} \mathfrak{o}$ of a convex subring $\mathfrak{o}$ of some real closed field, with ξ the generic and x the closed point of $\operatorname{Sper} \mathfrak{o}$, while $]0,1]$ is replaced by $\{\xi\}$.

B. Different semialgebraic structures on the same classical space

Let M be a locally closed semialgebraic subset of $\mathbf{R}^n$. Then we have on the set M the structure of a locally complete affine semialgebraic space, which we denote again by M.

Let $c(M)$ denote the direct system of all complete semialgebraic subsets K of M. For every $K \in c(M)$ the interior $\overset{\circ}{K}$ is again semialgebraic, thus both K and $\overset{\circ}{K}$ are affine semialgebraic spaces. We can form the direct limit

$$M_{\mathrm{loc}} = \varinjlim_{K \in c(M)} \overset{\circ}{K}$$

in the category $LSA(\mathbf{R})$ [DK$_2$, I.2.6]. M_{loc} is also the direct limit of the spaces K [DK$_2$, I.7.8]. It is a partially complete geometric space over $\mathbf{R}$ which has the same underlying set as M. Even $M_{\mathrm{top}} = (M_{\mathrm{loc}})_{\mathrm{top}}$.

Assume that M is not complete. The identity map $M_{\mathrm{loc}} \to M$ is a morphism but by no means an isomorphism. Indeed, a subset A of M is semialgebraic in M_{loc} iff A is semialgebraic and bounded in $\mathbf{R}^n$ and A has a positive distance from the boundary $\bar{M} - M$ of M, provided this boundary is not empty. In particular the space M_{loc} itself is not semialgebraic.

A function $f: M \to \mathbf{R}$ is a global section of the structure sheaf of M_{loc} iff f is continuous and, for every $x \in M$, the restriction of f to a suitable semialgebraic neighbourhood of x in M is semialgebraic ("locally semialgebraic functions"). Thus $\mathcal{O}(M_{\text{loc}})$ is a bigger ring than $\mathcal{O}(M)$.

The different nature of M and M_{loc} is perhaps best illustrated by looking at triangulations. For instance, if the set M is an open n-simplex in $\mathbf{R}^n$, then the space M has the tautological triangulation id: $M \to M$, while we obtain a triangulation of M_{loc} by subdividing the set M into infinitely many closed simplices.

If M is an *arbitrary* semialgebraic subset of $\mathbf{R}^n$ then we can still form the inductive limit of the spaces K with $K \in c(M)$ in the category $WSA(\mathbf{R})$ of weakly semialgebraic spaces, and we obtain a weak polytope $P(M)$. If M is locally closed in $\mathbf{R}^n$ the ringed space $P(M)$ equals M_{loc}.

The space $P(M)$ has the same underlying set as M, and the identity map $j_M: P(M) \to M$ is a morphism in $WSA(\mathbf{R})$. It has the following universal property: any morphism from a weak polytope to M factors through j_M in a unique way [K2, IV §9].

It is evident from this that j_M induces isomorphisms between the homotopy groups of $P(M)$ and M. By a semialgebraic version of the Whitehead theorem [K2, V.6.10] it follows that j_M is a homotopy equivalence.

Intuitively, replacing M by $P(M)$ means forgetting what happens in M "at infinity".

Everything said in this subsection remains true if we replace $\mathbf{R}$ by a *"sequential"* real closed base field R, i.e. a real closed field which contains at least one non trivial zero sequence. Fortunately, most real closed fields occuring in practice are sequential.

C. Fixed points

Let R be an *arbitrary* real closed field and M a locally closed semialgebraic subset of R^n. Then the direct limit M_{loc} of the spaces $\mathring{K}$ with $K \in c(M)$ still exists in the category of all locally semialgebraic spaces over R. $\{M_{\text{loc}}$ is regular but perhaps not paracompact.$\}$

It turns out that the abstraction $\tilde{M}_{\text{loc}}$ of M_{loc}, as a locally ringed space, is an open subspace of $\tilde{M}$, namely the union of the open subspaces $(\mathring{K})^{\sim}$ of $\tilde{M}$ with K running through $c(M)$. It follows that $\tilde{M}_{\text{loc}}$ (as a set) is also the union of the closed subsets $\tilde{K}$ of $\tilde{M}$.

A good understanding of $\tilde{M}_{\text{loc}}$ as a subset of $\tilde{M}$ is possible using ultrafilters of semialgebraic sets. Let $\mathfrak{S}(M)$ denote the boolean lattice of all semialgebraic subsets of M. As is well known, the points of $\tilde{M}$ can be identified with the ultrafilters in $\mathfrak{S}(M)$ [Br, §4], [BCR, 7.2.4], [KS, III §5], namely the ultrafilter $F(\alpha)$ corresponding to a point $\alpha \in \tilde{M}$ consists of all $S \in \mathfrak{S}(M)$ with $\alpha \in \tilde{S}$. In this interpretation $\tilde{M}_{\text{loc}}$ is the set of all $\alpha \in \tilde{M}$ with $K \in F(\alpha)$ for at least one $K \in c(M)$.

Definition. The *fringe* $\text{Frin}(M)$ of M is the set $\tilde{M} - \tilde{M}_{\text{loc}}$.

Intuitively, $\text{Frin}(M)$ consists of the points in $\tilde{M}$ "at infinity".

We now turn to fixed points. Let $f: M \to M$ be a semialgebraic map. It induces $\mathbf{Q}$-linear maps $H_q(f): H_q(M, \mathbf{Q}) \to H_q(M, \mathbf{Q})$ in ordinary (= semialgebraic) homology with coefficients in $\mathbf{Q}$. The trace $TrH_q(f)$ of $H_q(f)$ is well defined and an integer, since $H_q(f)$ comes from an endomorphism of the finitely generated abelian group $H_q(M, \mathbf{Z})$. We define the *Lefschetz number* $\Lambda(f)$ as usual,

$$\Lambda(f) := \sum_{i=0}^{n} (-1)^i \, TrH_i(f).$$

Brumfiel has proved the following remarkable theorem ([B3], cf. also [B2]):

Theorem. Assume that $\Lambda(f) \neq 0$. Then the abstraction $\tilde{f}: \tilde{M} \to \tilde{M}$ of f has a fixed point. More precisely, either $f: M \to M$ has a fixed point or $\tilde{f}$ has a closed fixed point in $\text{Frin}(M)$.

Example. Let $R = \mathbf{R}$, $M = \mathbf{R}^2$, $f(x, y) = (x + 1, xy)$. Certainly $f: M \to M$ has no fixed point. But $\Lambda(f) = 1$, since M is contractible. Thus $\tilde{f}$ must have at least one closed fixed point in $\text{Frin}(\mathbf{R}^2)$. Such a fixed point is provided by the graph of the gamma function $\Gamma(x)$. Let F denote the set of all $A \in \mathfrak{S}(M)$ such that A contains the set $\{(x, \Gamma(x)) | x \geq c\}$ for some $c > 0$. This is an ultrafilter which gives us a closed point $\alpha \in \text{Frin}\,(M)$ fixed under $\tilde{f}$.

Brumfiel's fixed point theorem is important for his "real spectrum compactification" $\widehat{M_g}$ of the Teichmüller space M_g of compact Riemann surfaces of genus g [B1], since it implies that every element of the Teichmüller modular group has a fixed point in $\widehat{M_g}$.

§14 An outlook

What can be said about the situation in semialgebraic topology now, in the year 1991, and about prospects for the future, without too much speculation?

Certainly abstract spaces contain a rich potential for further research. They should be "superior" to geometric spaces. But up to now nearly all the deeper results in homology and homotopy theory rely heavily on arguments in the geometric category, using triangulations, simplicial approximations, the homotopy extension property and the like.

It would be highly desirable to have a homotopy theory for real closed spaces. There can be no doubt that we have found the "right" homotopy groups in the geometric setting. But all this does not tell us how to proceed in the abstract setting. It is only clear that an abstract homotopy theory should give us the geometric theory if we apply it to the abstractions of geometric spaces.

For abstract cohomology the analogous problem seems to be easier. There exists an approach by N. Schwartz using inverse real closed spaces which looks promising. Schwartz talked about this at Luminy in October 1989, and then in the Ragsquad seminar at Berkeley in October 1990.

In abstract homotopy one could proceed along the lines given in chapter II of Baues' book [Ba], where the homotopy theory of a "cofibration category" has been developed. Or one could try to associate with a real closed space X (perhaps fulfilling some conditions like regularity or "tautness" [D_2, I §3]) a simplicial set or, perhaps better, something like a simplicial space VX (in the classical sense, with Hausdorff topology), such that the homotopy type of VX encodes the homotopy information about X.

In his talk at Oberwolfach in June 1990 C. Scheiderer proposed a simplicial space VX (together with a "quasiaugmentation" $VX \to X$) which looks promising. The definition of VX employs chains of real valuations of commutative rings. Scheiderer was able to verify that, for any sheaf $\mathcal{F}$ on X, the cohomology groups $H^q(X, \mathcal{F})$ can be computed from VX. Also, for X the abstraction of a geometric space M, VX represents the homotopy type of M. These are hints that VX gives the right homotopy type.

In the last years the interest in abstract spaces also has been enhanced by the discovery of the Spanish school (Andradas, Ruiz, ...), that questions in semianalytic geometry can be treated by considering real spectra. I just mention two basic observations: If A is a local ring of real analytic functions then the constructible subsets of Sper A are in natural one-to-one correspondence with the germs of semianalytic sets [Rz]. If A is the ring of global analytic functions on a compact real analytic manifold M then the constructible subsets of Sper A correspond bijectively with the globally semianalytic subsets of M [Rz_1]. The homotopy or homology of such a constructible set (which is a real closed space) should have a close relation to the homotopy resp. homology of the corresponding semianalytic object.

Concerning the two trends in the last ten years mentioned above (§4) we can safely say that the first one has run its course. Interaction between the geometric and the abstract theory is no longer a "trend" but a well established and widely used technique. However the first trend may continue in a wider context if we allow other meanings for the word "geometric".

Geometric spaces might be built, for instance, from semianalytic sets, or perhaps even from subanalytic sets. J. Denef and L. van den Dries have found a new approach to subanalytic sets [DvdD] which gives much hope that subanalytic sets are amenable to techniques similar to the ones we now have in the geometric semialgebraic topology. Subanalytic sets (more precisely, subsets of $\mathbf{R}^n$ which are subanalytic in $\mathbf{P}^n(\mathbf{R})$) and semialgebraic sets can both be subsumed under *o-minimal structures*, a notion stemming from model theory, cf. [PiS]. In his talk in the Ragsquad seminar at Berkeley, April 1991, van den Dries has outlined a "tame topology of o-minimal structures" with many of the features we are used to for semialgebraic sets, in particular N^o 1 - N^o 5 of the list in §3 for the geometric theory (cf. his forthcoming book [vdD], as soon as it appears). Thus o-minimal structures seem to provide a good framework for highly interesting new geometric spaces in the years to come.

In 1990 R. Huber introduced "semirigid functions" [Hu$_1$] as an offspring of his abstract approach to rigid analytic geometry [Hu]. They give us semirigid sets, which are vaguely analoguous to semianalytic sets in real analytic geometry. To understand these sets well, one definitely needs abstract spaces which are derivates of the *real valuation spectra* of commutative rings [Hu$_1$].

This, and the frequent occurrence of valuations in the theory of real closed spaces, are hints that valuation spectra will play a role in a further development of semialgebraic topology, for which the word "semialgebraic" is probably no longer appropriate. An introduction to valuation spectra has been given in [Hu, Chap. I] and [HuK].

From all of this it is pretty clear that the second trend mentioned in §4, i.e., the employment of new spaces - both geometric and abstract, will persist.

Acknowledgements. I thank Roland Huber, Claus Scheiderer, Colm Mulcahy, and Victoria Powers for helpful criticism, both in mathematics (R.H., C.S.) and in language (C.M., V.P.). I further thank Michel Coste, Louis Mahé, and Marie-Francoise Roy for organizing the conference and thus providing - among many other things - a very pleasant opportunity to give a talk, of which this article is an expanded version.

References

[Ba] **H.J. Baues**, "Algebraic homotopy". Cambridge studies advanced math. 15, Cambridge Univ. Press 1988.

[BCR] **J. Bochnak, M. Coste, M.-F. Roy**, "Géométrie algébrique réelle". Ergebnisse Math. Grenzgeb. 3. Folge Band 12, Springer 1987.

[BH] **A. Borel, A. Haefliger**, La classe d'homologie fondamentale d'un espace analytique, Bull. Soc. Math. France 89, 461 - 513 (1961).

[BM] **A. Borel, J.C. Moore**, Homology for locally compact spaces. Mich. Math. J. 7, 137 - 159 (1960).

[Bra] **H. Brakhage**, "Topologische Eigenschaften algebraischer Gebilde über einem beliebigen reell-abgeschlossenen Grundkörper". Dissertation Heidelberg 1954.

[Bre] **G. Bredon**, "Sheaf theory". McGraw Hill 1967.

[Br] **L. Bröcker**, Real spectra and distribution of signatures. In: "Géométrie algébrique réelle et formes quadratiques" (Ed. J.-L. Colliot-Thélène, M. Coste, L. Mahé, M.F. Roy), pp. 249 - 272. Lecture Notes Math. 959, Springer 1982.

[B] **G.W. Brumfiel**, "Partially ordered rings and semialgebraic geometry". London Math. Soc. Lecture Notes 37, Cambridge Univ. Press 1979.

[B_1] ——, The real spectrum compactification of Teichmüller spaces. In: "Geometry of group representations" (Ed. W.M. Goldman, A.R. Magid), Contemporary Math. 74, A.M.S. 1988, pp 51 - 75.

[B_2] ——, A semi-algebraic Brouwer fixed point theorem for real affine spaces, ibid., pp 77 - 82.

[B_3] ——, A Hopf fixed point theorem for semialgebraic maps. Preprint Stanford University.

[B_4] ——, Wittrings and K-theory. Rocky Mountain J. Math. 14, 733 - 765 (1984).

[B_5] ——, Quotient spaces for semialgebraic equivalence relations. Math. Z. 195, 69 - 78 (1987).

[CC] **M. Carral, M. Coste**, Normal spaces and their dimensions. J. Pure Appl. Algebra 30, 227 - 235 (1983).

[CR] **M. Coste, M.-F. Coste-Roy**, Le spectre réel d'un anneau est spatial. C.R.A.S. Paris 290, 91 - 94 (1980).

[CR_1] ——, ——, Le topos étale réel d'un anneau. In: 3^e colloque sur les catégories dédier a Charles Ehresmann, Cahiers de topologie et géom. diff. 22-1, 19 - 24 (1981).

[CR_2] ——, ——, La topologie du spectre réel. In: "Ordered fields and real algebraic geometry" (Ed. D.W. Dubois, T. Recio), Contemp. Math. 8, 27 - 59 (1982).

[D] **H. Delfs**, Kohomologie affiner semialgebraischer Räume. Dissertation Regensburg 1980.

[D_1] ——, The homotopy axiom in semialgebraic cohomology. J. reine u. angew. Math. 355, 108 - 128 (1985).

[D_2] ——, Homology of locally semialgebraic spaces. Lecture Notes in Math. 1484. Springer, to appear ($\approx$ Habilitationsschrift Regensburg 1984).

[D_3] ——, Semialgebraic Borel-Moore homology, Rocky Mountain J. Math. 14, 987 - 990 (1984).

[DK] **H. Delfs, M. Knebusch**, Semialgebraic topology over a real closed field II: Basic theory of semialgebraic spaces. Math. Z. 178, 175 - 213 (1981).

[DK_1] ——, ——, On the homology of algebraic varieties over real closed fields. J. reine u. angew. Math. 335, 122 - 163 (1982).

[DK_2] ——, ——, "Locally semialgebraic spaces". Lecture Notes Math. 1173 (1985).

[DK_3] ——, ——, An introduction to locally semialgebraic spaces. Rocky Mountain J. Math. 14, 945 - 963 (1984).

[De] **C.N. Delzell**, A finiteness theorem for open semialgebraic sets, with applications to Hilbert's 17th problem. In: "Ordered fields and real algebraic geometry" (Ed. D.W. Dubois, T. Recio), Contemporary Math. 8, 79 - 97 (1982).

[DvdD] **J. Denef, L. van den Dries**, P-adic and real subanalytic sets. Ann. Math. 128, 79 - 138 (1988).

[ES] **S. Eilenberg, N. Steenrod**, "Foundations of algebraic topology". Princeton Univ. Press 1952.

[Gi] **B. Giesecke**, Simpliziale Zerlegung abzählbarer analytischer Räume. Math. Z. 83, 177 - 213 (1964).

[EGA I] **A. Grothendieck, J. Dieudonné**, "Élements de géométrie algébrique I: Le language des schémas. Publ. Math. N° 4, Inst. Hautes Etudes Sci. 1960.

[EGA I*] ——, ——, "Élements de géométrie algébrique I". Grundlehren math. Wiss. 166, Springer 1971.

[Ha] **R. Hardt**, Semialgebraic local triviality in semi-algebraic mappings. Amer. J. Math. 102, 291 - 302 (1980).

[Ho] **M. Hochster**, Prime ideal structure in commutative rings. Trans. Amer. Math. Soc. 142, 43 - 60 (1969).

[HM] **J. Houdebine, L. Mahé**, Séparation des composantes connexes réelles dans le cas des variétés projectives. In: "Géometrie algébrique réelle et formes quadratiques (Ed. J.-L. Colliot-Thélène, M. Coste, L. Mahé, M.F. Roy), pp 358 - 370. Lecture Notes Math. 959, Springer 1982.

[Hu] **R. Huber**, "Bewertungsspektrum und rigide Geometrie". Habilitationsschrift, Regensburg 1990.

[Hu_1] ——, Semirigide Funktionen. Preprint Regensburg 1990.

[HuK] **R. Huber, M. Knebusch**, On valuation spectra. To appear in the proceedings of the RAGSQUAD-seminar Berkeley 1990/91, Amer. Math. Soc.

[HuS] **R. Huber, C. Scheiderer**, A relative notion of local completeness in semialgebraic geometry. Arch. Math. 53, 571 - 584 (1989).

[K] **M. Knebusch**, An invitation to real spectra. In: "Quadratic and hermitian forms" (Ed. C.R. Riehm, I. Hambleton), Canadian Math. Soc. Conference Proceedings 4, 51 - 105 (1984).

[K_1] ——, Some open problems. In: "Conference on quadratic forms - 1976" (Ed. G. Orzech). Queen's Papers Pure Appl. Math. 46, 361 - 370 (1977).

[K_2] ——, "Weakly semialgebraic spaces". Lecture Notes Math. 1367, Springer 1989.

[K3] ——, "Semialgebraic fibrations and covering maps". In preparation.

[KS] **M. Knebusch, C. Scheiderer**, "Einführung in die reelle Algebra", Vieweg, Braunschweig-Wiesbaden 1989.

[L] **S. Lojasiewicz**, Triangulation of semi-analytic sets. Ann. Scuola Norm. Sup. Pisa (3) 18, 449 - 474 (1964).

[M] **L. Mahé**, Signatures et composantes connexes. Math. Ann. 260, 191 - 210 (1982).

[May] **J.P. May**, "Simplicial objects in algebraic topology". Van Nostrand Math. Studies 11, Van Nostrand 1967.

[Mi] **J. Milnor**, The geometric realization of a semi-simplicial complex. Annals of Math. 65, 357 - 362 (1957).

[P] **M. Prechtel**, "Endliche semialgebraische Räume". Diplomarbeit Regensburg 1988.

[PiS] **A. Pillay, C. Steinhorn**, Definable sets in ordered structures, I. Trans. Amer. Math. Soc. 295, 565 - 592 (1986).

[Ro] **R. Robson**, Embedding semialgebraic spaces. Math. Z. 183, 365 - 370 (1983).

[R] **M.-F. Roy**, Faisceau structural sur le spectre réel et fonctions de Nash. In: "Géométrie algébrique réelle et formes quadratiques", pp. 406 - 432. Lecture Notes Math. 959, Springer 1982.

[Rz] **J.M. Ruiz**, Basic properties of real analytic and semianalytic germs, Publ. Inst. Recherche Math. Rennes 4, 29 - 51 (1986).

[Rz1] ——, On the real spectrum of a ring of global analytic functions. ibid., 84 - 95.

[Sch] **C. Scheiderer**, Quotients of semi-algebraic spaces. Math. Z. 201, 249 - 271 (1989).

[S] **N. Schwartz**, "Real closed spaces". Habilitationsschrift Ludwig-Maximilians-Universität München 1984.

[S1] ——, "The basic theory of real closed spaces". Memoirs Amer. Math. Soc. 77, N° 397 (1989).

[S2] ——, "The basic theory of real closed spaces". Regensburger Mathematische Schriften 15, Fakultät für Mathematik der Universität Regensburg 1987.

[S3] ——, Inverse real closed spaces. Illinois J. Math., to appear soon.

[S4] ——, Open morphisms of real closed spaces. Rocky Mountain J. Math. 19, 913 - 939 (1989).

[Spa] **E.H. Spanier**, "Algebraic topology". McGraw-Hill 1966.

[Sw] **R.M. Switzer**, "Algebraic topology - Homotopy and homology". Grundlehren math. Wiss. 212, Springer 1975.

Address: Fakultät für Mathematik, Universität Regensburg,
Universitätsstr. 31, D-8400 Regensburg