Universität Regensburg
Mathematik

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Preprint Nr. 12/2008

ON THE EQUIVARIANT AND THE NON-EQUIVARIANT MAIN CONJECTURE FOR IMAGINARY QUADRATIC FIELDS

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Abstract. In this paper we first prove the main conjecture for imaginary quadratic fields for all prime numbers p, improving earlier results by Rubin. From this we deduce the equivariant main conjecture in the case that a certain μ-invariant vanishes. For prime numbers $p \nmid 6$ which split in K, this is a theorem by a result of Gillard.

Contents

Introduction

The Iwasawa main conjecture fields has been an important tool to study the arithmetic of special values of L-functions of Hecke characters of imaginary quadratic fields ([Ru1], [Ki], [Ts], [Bl], [JL]). To obtain the finest possible invariants it is important to know the main conjecture for all prime numbers p and also to have an equivariant version at disposal.

In this paper we address these questions and treat the main conjecture for imaginary quadratic fields K in the equivariant and the non-equivariant setting (i.e. for characters χ of finite order over K). Our results are twofold.

As a first theorem (see 5.1), we prove the traditional main conjecture first proven by Rubin [Ru1], [Ru2] for all prime numbers p. This improves the results by Rubin, who had to impose the condition that p does not divide the order of the abelian field defined by χ.

Date: April 17, 2008.

The second result of our paper treats the equivariant main conjecture. We reduce this conjecture to the vanishing of a certain μ-invariant (see 5.3 for the precise condition). A result of Gillard [Gi] implies that the equivariant main conjecture is a theorem for prime numbers $p \nmid 6$, which split in K.

It was Rubin's idea to prove the main conjecture with the techniques of Euler systems invented by Kolyvagin. Later, he (and also Kato and Perrin-Riou independently) developed the machinery in an abstract and conceptual way, which made it a very flexible and general tool.

Our approach to the main conjecture follows the scheme of proof developed by the second author with A. Huber in [HK]. Instead of decomposing the classical Iwasawa modules under characters (which is the main reason for getting primes where the procedure does not work), we use Galois cohomology with coefficients in the Galois representations defined by the character χ. Using this we reduce the main conjecture to the Tamagawa number conjecture for number fields at $s = 0$, which corresponds to the classical use of the class number formula. This approach was inspired by the Tamagawa number conjecture and in particular by the work of Kato.

To treat the equivariant main conjecture, Burns and Greither had the happy idea that the vanishing of certain μ-invariants had the consequence that the decisive Iwasawa modules vanish when localized at so called singular prime ideals (see 7.3). We essentially adopt this strategy but with a conceptual change in the strategy first explained by Witte [Wi]: we deduce the equivariant main conjecture from the characterwise one using the fact that the vanishing of the μ-invariant implies the vanishing of the localized H^2, which is the inverse limit of the class groups.

For the experts we like to point out one seemingly new technical feature in the proof. Kato had the idea that one should use the functor Det of Knudsen and Mumford [KM] instead of the more traditional characteristic ideal. We not only follow his suggestion, but we use also the functor Div in a systematic way. This allows to deal in an elegant way with the reduction of the main conjecture to the Tamagawa number conjecture.

The paper is organized as follows: after some notational preliminaries in the first section, we review the Tamagawa number conjecture for number fields at $s = 0$. The next section recalls the Euler system of elliptic units following the exposition by Kato in [Ka]. The third section introduces the basic Iwasawa modules and studies some of their properties. The technical part here is much simpler than in the corresponding case of the main conjecture for $\mathbb{Q}$, as we work here with the full $\mathbb{Z}_p$-extension of K unramified outside of p, which has Galois group $\mathbb{Z}_p^2$. The fourth section formulates the equivariant (here called Ω-main-conjecture) and the non-equivariant Iwasawa main conjecture (here called Λ-main-conjecture). The last two sections contain the proofs of these main conjectures.

1. Preliminaries

1.1. **General notations.** In this paper K always denotes an imaginary quadratic field with a fixed embedding $K \to \mathbb{C}$ and we fix an algebraic closure $\bar{K} \subset \mathbb{C}$. By $\mathcal{O}_K$ we denote the ring of integers. For each ideal $\mathfrak{q} \subset \mathcal{O}_K$ we consider the ray class field $K(\mathfrak{q})$ of conductor $\mathfrak{q}$ and we denote by

$$G(\mathfrak{q}) := \mathrm{Gal}(K(\mathfrak{q})/K)$$

its Galois group over K. Consider for an ideal $\mathfrak{f} \subset \mathcal{O}_K$ characters

$$\eta : G(\mathfrak{f}) \to \mathbb{C}^*.$$

The conductor of η will be denoted by $\mathfrak{f}_\eta$ and we let

$$\widehat{G}(\mathfrak{f}) := \{\eta : G(\mathfrak{f}) \to \mathbb{C}^*\}$$

be the dual group of $G(\mathfrak{f})$. For each character η, we let $E(\eta)$ be the smallest number field, which contains the values of η. If the character is clear from the context, we just write E. We denote by $\mathcal{O} := \mathcal{O}_E$ the ring of integers in E and we introduce the following conventions:

$$E_\infty := E \otimes_{\mathbb{Q}} \mathbb{R} \text{ and } E_p := E \otimes_{\mathbb{Q}} \mathbb{Q}_p.$$

In a similar way we let $\mathcal{O}_p := \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Note that this is a product of discrete valuation rings.

For each character $\eta : G(\mathfrak{f}) \to E^*$ and each embedding $\sigma : E \to \mathbb{C}$ we define the $E \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathrm{Hom}(E, \mathbb{C})$-valued L-function of η to be

$$L(\eta, s) := (\ldots, L(\sigma \circ \eta, s), \ldots),$$

where

$$L(\sigma \circ \eta, s) := \prod_{0 \neq \mathfrak{p} \subset \mathcal{O}_K} \frac{1}{1 - \frac{\sigma\eta(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^s}},$$

and the product is taken over all non-trivial prime ideals of $\mathcal{O}_K$. For each ideal $\mathfrak{n} \subset \mathcal{O}_K$ we define

$$L_{\mathfrak{n}}(\sigma\eta, s) := \prod_{\mathfrak{p} \nmid \mathfrak{n}} \frac{1}{1 - \frac{\sigma\eta(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^s}}.$$

These L-functions have a meromorphic continuation to $\mathbb{C}$ and satisfy a functional equation. If $\eta \neq 1$ is non-trivial, the functions $L(\sigma \circ \eta, s)$ have a zero of order 1 at $s = 0$. We write

$$L^*(\eta, 0)$$

for the leading term in the Laurent series at 0 of $L(\eta, s)$ as an $E \otimes \mathbb{C}$-valued function.

1.2. **The motive of a number field.** For each abelian Galois extension $K \subset F \subset \bar{K}$ with Galois group $G := \mathrm{Gal}(F/K)$, we denote by $h^0(F)$ its motive over K and

$$M(F) := h^0(F)_E$$

the motive with coefficients in E. Here we assume that E contains all the values of the characters in $\widehat{G}$. For each group G and a commutative ring R, we let

$$R[G]$$

be the group ring of G with coefficients in R. For a character $\eta : G \to E^*$ we let

$$p_{\eta^{-1}} := \frac{1}{\#G} \sum_{\sigma \in G} \eta(\sigma)\sigma \in E[G]$$

be the projector onto the η^{-1}-eigenspace. The projectors $p_{\eta^{-1}}$ decompose $M(F)$ into a direct sum

$$M(F) = \bigoplus_{\eta \in \widehat{G}} M(\eta),$$

where

$$M(\eta) := p_{\eta^{-1}} M(F).$$

The L-function of the motive $M(F)$ is the Dedekind zeta function of F,

$$L(M(F), s) = \zeta_F(s)$$

considered as $E \otimes_{\mathbb{Q}} \mathbb{C}$-valued function. Similarly,

$$L(M(\eta), s) = L(\eta, s)$$

for each character $\eta : G \to E^*$. We consider several realizations of the motives $M(F)$ and the dual motive $M(F)^\vee(1)$ with a Tate twist. In this case, since the dimension of the variety is 0, $M(F)^\vee = M(F)$. Note that the dual motive of $M(\eta)$ is

$$M(\eta)^\vee \simeq M(\eta^{-1}).$$

The *Betti realization* is the E-vector space

$$M(F)_B := H^0_B(\mathrm{Spec} F(\mathbb{C}), E) \simeq \bigoplus_{\tau : F \to_K \mathbb{C}} E \simeq E[G],$$

where we have used the fixed embedding of $F \subset \bar{K}$ into $\mathbb{C}$.

The *deRham realization*

$$M(F)_{\mathrm{dR}} := H^0_{\mathrm{dR}}(F/K) \otimes E \simeq F \otimes_{\mathbb{Q}} E$$

is a filtered $K \otimes_{\mathbb{Q}} E$-vector space, and in this case, $\mathrm{Fil}^0(M(F)_{\mathrm{dR}}) = M(F)_{\mathrm{dR}}$. The *étale realization* for any prime number p,

$$M(F)_p := H^0_{\mathrm{et}}(\mathrm{Spec} F \times_K \bar{K}, E_p) \simeq \bigoplus_{\tau : F \to_K \bar{K}} E_p \simeq E_p[G]$$

is an E_p-representation of $\mathrm{Gal}(\bar{K}/K)$.

The *motivic cohomology* groups are defined in terms of K-theory and we have $H^0_f(M(F)) = E$ and $H^1_f(M(F)) = 0$ while $H^0_f(M(F)^\vee(1)) = 0$ and

$$H^1_f(M(F)^\vee(1)) = K_1(\mathcal{O}_F) \otimes_{\mathbb{Z}} E \simeq \mathcal{O}_F^* \otimes_{\mathbb{Z}} E.$$

The realizations of the motives $M(\eta)$ are defined by applying the projector $p_{\eta^{-1}}$ to the realizations of $M(F)$. In particular, we have

$$H^1_f(M(\eta)^\vee(1)) = p_{\eta^{-1}}(\mathcal{O}_F^* \otimes_{\mathbb{Z}} E).$$

Definition 1.1. Using the identification $E[G] \simeq M(F)_B$ we define the *canonical lattice* to be $\mathcal{O}[G] \subset M(F)_B$. Similarly, we consider $\mathcal{O}_p[G] \subset M(F)_p$. This induces a *canonical lattice*

$$\mathcal{O}(\eta) := p_{\eta^{-1}}(\mathcal{O}[G]) \subset M(\eta)_B$$

with *canonical generator* $t_B(\eta) := p_{\eta^{-1}}(1)$ and Galois stable lattices

$$\mathcal{O}_p(\eta) := p_{\eta^{-1}}(\mathcal{O}_p[G]) \subset M(\eta)_p$$

with *canonical generator* $t_p(\eta) := p_{\eta^{-1}}(1)$. We also define

$$\mathcal{O}(\eta)^\vee := \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}(\eta), \mathcal{O})$$

and

$$\mathcal{O}_p(\eta)^\vee := \mathrm{Hom}_{\mathcal{O}_p}(\mathcal{O}_p(\eta), \mathcal{O}_p)$$

and denote by $t_B(\eta)^\vee$ and $t_p(\eta)^\vee$ the dual bases.

Note that the action of $\mathrm{Gal}(\bar{K}/K)$ on $M(F)_p$ factors through G but this action is contragredient to the canonical action of G on $M(F)_p$.

1.3. **The functors** Det **and** Div**.** We will use the graded determinant functor Det and the divisor functor Div of Knudsen and Mumford [KM]. Let R be a commutative ring, and

$$P^\cdot : \cdots \to P^{i-1} \to P^i \to P^{i+1} \to \cdots$$

a perfect complex of projective R-modules. One defines $\mathrm{Det}_R P^i := \bigwedge_R^{\mathrm{rk}_R P^i} P^i$ as a graded invertible R-module of (locally constant) degree $\mathrm{rk} P^i$. The determinant of the complex $P^\cdot$ is then the graded invertible R-module

$$\mathrm{Det}_R P^\cdot := \bigotimes_{i \in \mathbb{Z}} \mathrm{Det}_R^{(-1)^i} P^i.$$

Notice that the determinant depends only on the quasi-isomorphism class of $P^\cdot$. Indeed, one has

$$\mathrm{Det}_R P^\cdot = \bigotimes_{i \in \mathbb{Z}} \mathrm{Det}_R^{(-1)^i} H^i(P^\cdot).$$

This functor is closely related to the characteristic ideal. If P is a torsion R-module, R an integral domain and $Q(R)$ the total quotient ring of R, then $\mathrm{char}(P) = \mathrm{Det}_R^{-1} P$. Here we identify

$$\mathrm{Det}_R^{-1} P \subset (\mathrm{Det}_R^{-1} P) \otimes_R Q(R) = \mathrm{Det}_{Q(R)}^{-1} 0 = Q(R).$$

Assume now that R is noetherian and let

$$\lambda : \mathcal{F}^{\cdot} \to \mathcal{G}^{\cdot}$$

be a map of perfect complexes on $X := \operatorname{Spec} R$ in the derived category. Let $U(\lambda)$ be the open set of $x \in X$ such that λ is an isomorphism in a neighbourhood of x. The map λ is called good, if $U(\lambda)$ contains all points of depth 0. Knudsen and Mumford define for good λ a Cartier divisor $\operatorname{Div}(\lambda)$ on X, which has the property that the canonical map on $U(\lambda)$

$$\operatorname{Det}(\lambda) : \operatorname{Det}(\mathcal{F}^{\cdot}) \mid_{U(\lambda)} \simeq \operatorname{Det}(\mathcal{G}^{\cdot}) \mid_{U(\lambda)}$$

extends to an isomorphism on the whole of X

$$\operatorname{Det}(\lambda) : \operatorname{Det}(\mathcal{F}^{\cdot})(\operatorname{Div}(\lambda)) \simeq \operatorname{Det}(\mathcal{G}^{\cdot}) \tag{1}$$

In particular, one has an isomorphism $\mathcal{O}_X(\operatorname{Div}(\lambda)) \simeq \operatorname{Det}(\mathcal{G}^{\cdot}) \otimes \operatorname{Det}^{-1}(\mathcal{F}^{\cdot})$. One defines

$$\operatorname{Div}(\mathcal{F}^{\cdot}) := \operatorname{Div}(0 \to \mathcal{F}^{\cdot}),$$

if $0 \to \mathcal{F}^{\cdot}$ is good. The functor Div has among other the following properties (see [KM] Theorem 3): If

$$o \to \mathcal{F}^{\cdot} \xrightarrow{\lambda} \mathcal{G}^{\cdot} \xrightarrow{\mu} \mathcal{H}^{\cdot} \to 0$$

is a short exact sequence of perfect complexes such that λ is good, then $0 \to \mathcal{H}^{\cdot}$ is good and $\operatorname{Div}(\lambda) = \operatorname{Div}(\mathcal{H}^{\cdot})$.

If $\lambda : \mathcal{F}^{\cdot} \to \mathcal{G}^{\cdot}$ and $\mu : \mathcal{H}^{\cdot} \to \mathcal{I}^{\cdot}$ are good, then $\operatorname{Div}(\lambda \oplus \mu) = \operatorname{Div}(\lambda) + \operatorname{Div}(\mu)$.

Proposition 1.2 ([KM] Theorem 3)**.** *If $f : Y \to X$ is a morphism of noetherian schemes, $\lambda : \mathcal{F}^{\cdot} \to \mathcal{G}^{\cdot}$ a good map on X and for all $y \in Y$ of depth 0 one has $f(y) \in U(\lambda)$, then*

$$Lf^*(\lambda) : Lf^*\mathcal{F}^{\cdot} \to Lf^*\mathcal{G}^{\cdot}$$

is good on Y and one has

$$\operatorname{Div}(Lf^*(\lambda)) = f^*\operatorname{Div}(\lambda).$$

For more details on these functors, see [KM].

2. The Tamagawa number conjecture for the motive $M(F)$

In this section we review the Tamagawa number conjecture for number fields in the case $s = 0$, which is essentially a reformulation of the class number formula. As in the classical case we will reduce the main conjecture to the case of the Tamagawa number conjecture. The extension of the Tamagawa number conjecture of Bloch and Kato to coefficients is due to Kato, Fontaine-Perrin-Riou and Burns-Flach.

2.1. **Étale cohomology.** In this section M is one of the motives $M(F)$ or $M(\eta)$. As usual, using our fixed algebraic closure $\bar{K}$, we identify continuous Galois cohomology and continuous étale cohomology.

In the formulation of the Tamagawa number conjecture, as well as in the sequel, we have need of several complexes of Galois cohomology, which we define following Fontaine [Fo]. Fix a rational prime p, and for every finite place v of K, define the local unramified cohomology of M_p to be the complex

$$R\Gamma_f(K_v, M_p) = \begin{cases} M_p^{I_v} \overset{1-\mathrm{Frob}_v}{\rightarrow} M_p^{I_v} & v \nmid p \\ D_{\mathrm{cris}}(M_p) \overset{1-\phi}{\rightarrow} D_{\mathrm{cris}}(M_p) & v \mid p \end{cases}$$

where I_v is the inertia group at v. Recall that $D_{\mathrm{cris}}(M_p) := (B_{\mathrm{cris}} \otimes M_p)^{G_{K_v}}$ carries an action of the Frobenius of B_{cris}, which is denoted by ϕ. Moreover, the tangent space $(B_{\mathrm{dR}}/\mathrm{Fil}^0 \otimes M_p)^{G_{K_v}} = 0$ for our motive. This unramified cohomology is necessary to keep track of the Euler factors that arise when removing primes. We further define

$$R\Gamma_{/f}(K_v, M_p) := \mathrm{Cone}\left(R\Gamma_f(K_v, M_p) \to R\Gamma(K_v, M_p)\right).$$

Definition 2.1. Let S be a finite set of primes such that M_p is unramified outside of S and let $j : \mathrm{Spec}(\mathcal{O}_K[1/pS]) \hookrightarrow \mathrm{Spec}(\mathcal{O}_K[1/p])$, then the étale sheaf j_*M_p (resp. $j_*\mathcal{O}_p(\eta)$ as defined in 1.1) on $\mathcal{O}_K[1/p]$ will be denoted by M_p (resp. $\mathcal{O}_p(\eta)$), i.e., we omit j_* from the notation.

Using the convention in 2.1, the compact support cohomology is defined for any Galois stable lattice $T_p \subset M_p$ as

$$R\Gamma_c(\mathcal{O}_K[1/p], T_p) := \mathrm{Cone}\left(R\Gamma(\mathcal{O}_K[1/p], T_p) \to \bigoplus_{v|p} R\Gamma(K_v, T_p) \oplus T_p\right)[-1].$$

Note that as K is imaginary quadratic, $R\Gamma(\mathcal{O}_K[1/p], T_p)$, $R\Gamma(K_v, T_p)$ and $R\Gamma_c(\mathcal{O}_K[1/p], T_p)$ are perfect complexes and that $R\Gamma(\mathcal{O}_K[1/p], T_p)$ and $R\Gamma(K_v, T_p)$ have cohomological dimension two. The global unramified cohomology is defined similarly as a mapping cone

$$R\Gamma_f(\mathcal{O}_K[1/p], M_p) := \mathrm{Cone}\left(R\Gamma(\mathcal{O}_K[1/p], M_p) \to \bigoplus_{v|p} R\Gamma_{/f}(K_v, M_p)\right)[-1].$$

We have isomorphisms $H^0_f(M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq H^0_f(\mathcal{O}_K[1/p], M_p)$ and thanks to results of Soulé an isomorphism

$$H^1_f(M(1)) \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq H^1_f(\mathcal{O}_K[1/p], M_p(1))$$

given by the regulator map. Further, by Artin-Verdier duality, we have that

$$H^i_f(\mathcal{O}_K[1/p], M_p) \simeq H^{3-i}_f(\mathcal{O}_K[1/p], M_p^\vee(1))^\vee,$$

where $^\vee$ denotes the E_p-dual. Thus, we can compute $R\Gamma_f(\mathcal{O}_K[1/p], M_p)$ in all degrees and get for our motives the triangle

$$(2)\quad H^0_f(\mathcal{O}_K[1/p], M_p) \to R\Gamma_f(\mathcal{O}_K[1/p], M_p) \to H^1_f(\mathcal{O}_K[1/p], M_p^\vee(1))^\vee[-2].$$

From the above, we deduce a fourth exact triangle (note that $M_B \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong M_p$):

$$(3)\qquad R\Gamma_c(\mathcal{O}_K[1/p], M_p) \to R\Gamma_f(\mathcal{O}_K[1/p], M_p) \to \bigoplus_{v|p} R\Gamma_f(K_v, M_p) \oplus M_p.$$

For later use, we note the behaviour of $R\Gamma_c(\mathcal{O}_K[1/p], M_p)$ under addition of a finite set of places S.

Lemma 2.2. *Let S be a finite set of places of $\mathcal{O}_K$ not dividing p, then one has a localization sequence for any $\mathcal{O}_p$-lattice $T_p \subset M_p$*

$$R\Gamma_c(\mathcal{O}_K[1/pS], T_p) \to R\Gamma_c(\mathcal{O}_K[1/p], T_p) \to \bigoplus_{v \in S} R\Gamma_f(K_v, T_p).$$

Proof. This follows from the localization sequence for cohomology with compact support (see [Mi] II.2.3(d)) and the isomorphism

$$R\Gamma(\kappa(v), M_p^{I_v}) \cong R\Gamma_f(K_v, M_p)$$

where $\kappa(v)$ is the residue class field and I_v the inertia group at v. □

2.2. **Review of the Tamagawa number conjecture for M.** In this section we formulate the Tamagawa number conjecture for the motives $M(F)$ and $M(\eta)$. Let M be one of the motives $M(F)$ or $M(\eta)$.

Beilinson's regulator r_∞ sits in a short exact sequence

$$(4)\qquad 0 \to H^0_f(M) \otimes_{\mathbb{Q}} \mathbb{R} \to M_B \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{r_\infty^\vee} H^1_f(M^\vee(1))^\vee \otimes_{\mathbb{Q}} \mathbb{R} \to 0.$$

We need the precise normalization of the regulator to treat the prime 2 correctly. Recall that $H^1_f(M(F)^\vee(1)) \cong \mathcal{O}_F^* \otimes_{\mathbb{Q}} E$, and that

$$M(F)_B \otimes_{\mathbb{Q}} \mathbb{R} = \bigoplus_{\tau: F \to_K \mathbb{C}} E_\infty \simeq E_\infty[G].$$

Then r_∞ is given by

$$(5)\qquad \begin{aligned} \mathcal{O}_F^* \otimes_{\mathbb{Z}} E &\xrightarrow{r_\infty} \bigoplus_{\tau: F \to_K \mathbb{C}} E_\infty \\ u &\mapsto \sum_{\tau \in G} (\log|\tau(u)|)\tau, \end{aligned}$$

where $|\tau(u)| := (\tau(u)\overline{\tau(u)})^{1/2}$ is the usual complex norm. We define the *fundamental line* to be the E-vector space

$$(6)\qquad \Xi(M) := \mathrm{Det}_E(H^0_f(M)) \otimes_E \mathrm{Det}_E^{-1}(H^1_f(M^\vee(1))) \otimes_E \mathrm{Det}_E^{-1}(M_B)$$

By the exact sequence (4), we have an isomorphism

$$\vartheta_\infty : E_\infty \simeq \Xi(M) \otimes_{\mathbb{Q}} \mathbb{R}$$

The leading term of the L-function at $s = 0$, $L(M,0)^*$ considered as $E \otimes_{\mathbb{Q}} \mathbb{C}$-valued function is in E_∞^*, so we can consider its image under the isomorphism above.

Conjecture 2.3 (Rational Conjecture)**.**

$$\vartheta_\infty(L^*(M,0)^{-1}) \in \Xi(M) \otimes_{\mathbb{Q}} 1.$$

The triangle in (3) induces an isomorphism

$$\vartheta_p : \Xi(M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\simeq} \mathrm{Det}_{E_p} R\Gamma_c(\mathcal{O}_K[1/p], M_p), \tag{7}$$

where one identifies $\mathrm{Det}_{E_p} R\Gamma_f(K_v, M_p) = E_p$. Let T_p be any $\mathrm{Gal}(\bar{K}/K)$-stable $\mathcal{O}_p$-lattice inside of M_p. In the application to $M(F)$ we will use

$$T_p = \bigoplus_{\eta \in \widehat{G}} \mathcal{O}_p(\eta).$$

Conjecture 2.4 (Tamagawa Number Conjecture)**.** For all rational primes p, there is an equality of lattices

$$\mathcal{O}_p \cdot \vartheta_p \vartheta_\infty(L^*(M,0)^{-1}) = \mathrm{Det}_{\mathcal{O}_p} R\Gamma_c(\mathcal{O}_K[1/p], T_p)$$

inside of $\mathrm{Det}_{E_p} R\Gamma_c(\mathcal{O}_K[1/p], M_p)$, which is independent of the choice of T_p.

For the independence of T_p, see [BF]. This conjecture is compatible with enlarging p to any finite set of primes S by lemma 2.2 and hence coincides with the usual formulation, where one uses $R\Gamma_c(\mathcal{O}_K[1/pS], T_p)$. Both conjectures hold for number fields:

Theorem 2.5. *Let F be a number field, then the conjectures 2.3 and 2.4 hold for $M(F)$ and all primes p.*

Proof. This is actually a consequence of the analytic class number formula. For the proof of 2.3 we refer to [HK] Proposition 2.3.1. There are some differences in notation, in particular V is used for the motive called M in this text, and the fundamental line is denoted by $\Delta_f(V)$. The conjecture 2.4 is proved in [HK] Proposition 2.3.1 if $p \neq 2$. A proof of the case $p = 2$ is given in [It] 3.1. □

Remark 2.6. Note that for the motives $M(\eta)$ the conjecture 2.3 is equivalent to Stark's conjecture and is not known in general.

2.3. **A reformulation of the Tamagawa number conjecture.** In our proof of the equivariant main conjecture, we will not use the Tamagawa number conjecture for the motives $M(F)$ but for certain quotients.

Consider an abelian Galois extension L/K, with $K \subset F \subset L \subset \bar{K}$ and write $G_L := \mathrm{Gal}(L/K)$ and $G_F := \mathrm{Gal}(F/K)$. Then we have a decomposition

$$M(L)/M(F) \simeq \bigoplus_{\eta \in \widehat{G}_L \setminus \widehat{G}_F} M(\eta).$$

Here we assume that E contains all values of $\eta \in \widehat{G}_L \smallsetminus \widehat{G}_F$. As the Tamagawa number conjecture holds for $M(L)$ and $M(F)$ it also holds for the quotient motive $M(L)/M(F)$ and we get from theorem 2.5:

Corollary 2.7. *For all rational primes p, there is an equality of lattices inside of $\bigotimes_{\eta \in \widehat{G}_L \smallsetminus \widehat{G}_F} \mathrm{Det}_{\mathcal{O}_p} R\Gamma_c(\mathcal{O}_K[1/p], M(\eta)_p)$:*

$$\bigotimes_{\eta \in \widehat{G}_L \smallsetminus \widehat{G}_F} \mathrm{Det}_{\mathcal{O}_p} \mathcal{O}_p \vartheta_p \vartheta_\infty (L^*(\eta, 0)^{-1}) = \bigotimes_{\eta \in \widehat{G}_L \smallsetminus \widehat{G}_F} \mathrm{Det}_{\mathcal{O}_p} R\Gamma_c(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta)).$$

We now give a reformulation of this corollary without using cohomology with compact support. This is necessary as the classical formulation of the Iwasawa main conjecture also does not mention cohomology with compact support. We first need to identify the lattice given by $\mathrm{Det}_{\mathcal{O}_p} R\Gamma_c(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta))$.

Let $\eta \in \widehat{G}_L \smallsetminus \widehat{G}_F$ and $\mathcal{O}_p(\eta)$ be our standard $\mathcal{O}_p$-lattice inside of $M_p(\eta)$ defined in (1.1). Recall that $\mathcal{O}_p(\eta)^\vee$ is the $\mathcal{O}_p$-dual of $\mathcal{O}_p(\eta)$.

Proposition 2.8 ([HK] 1.2.10, [It] 1.15)**.** *Consider the Artin-Verdier duality isomorphism (see 4.5)*

$$\mathrm{Det}_{E_p} R\Gamma_c(\mathcal{O}_K[1/p], M(\eta)_p) \otimes \mathrm{Det}_{E_p} M(\eta)_p \cong \mathrm{Det}_{E_p} R\Gamma(\mathcal{O}_K[1/p], M(\eta)_p^\vee(1))$$

and the lattice $\mathcal{O}_p(\eta^{-1})^\vee \subset M(\eta)_p$. Then, for all p the $\mathcal{O}_p$-structures given by

$$\mathrm{Det}_{\mathcal{O}_p} R\Gamma_c(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta^{-1})^\vee) \otimes \mathrm{Det}_{\mathcal{O}_p} \mathcal{O}_p(\eta^{-1})^\vee$$

on the left hand side and by

$$\mathrm{Det}_{\mathcal{O}_p} R\Gamma(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta^{-1})(1))$$

on the right hand side agree under this duality isomorphism.

Proof. The statement for $p \neq 2$ is [HK] 1.2.10 applied to $T_p = \mathcal{O}_p(\eta^{-1})^\vee$. The statement for $p = 2$ follows from [It] 1.15 using that $R\Gamma(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta^{-1})^\vee(1))$ is concentrated in degrees ≤ 2 and that $\widehat{H}^0(\mathbb{R}, \mathcal{O}_2(\eta^{-1})^\vee) = 0$, which gives $\mathrm{Det}_{\mathcal{O}_2} \widehat{H}^0(\mathbb{R}, \mathcal{O}_2(\eta^{-1})^\vee) = \mathcal{O}_2$. □

Definition 2.9. Let η be non-trivial, so that $H^0_f(M(\eta)) = 0$, and consider the lattice $\mathcal{O}(\eta^{-1})^\vee \subset M(\eta)$ with generator $t_B(\eta^{-1})^\vee$ from 1.1. Then there is a unique $z(\eta) \in H^1_f(M(\eta)^\vee(1))$, the *zeta element* of $M(\eta)$, such that

$$\vartheta_\infty(L^*(\eta, 0)^{-1}) = z(\eta)^{-1} \otimes (t_B(\eta^{-1})^\vee)^{-1}$$

in $\mathrm{Det}_E^{-1} H^1_f(M(\eta)^\vee(1)) \otimes \mathrm{Det}_E^{-1} M(\eta)_B$. Note that $z(\eta)$ depends on the choice of $t_B(\eta^{-1})^\vee$. Let

$$z_p(\eta) := \prod_{v|p} (1 - \eta(v)) z(\eta)$$

be the zeta element with the Euler factors above p at $s = 0$ removed (here we use the convention that $\eta(v) = 0$, if η is ramified at v).

Consider the regulator map for $\eta \in \widehat{G}_L \smallsetminus \widehat{G}_F$

$$r_p : H^1_f(M(\eta)^\vee(1)) \otimes_E E_p \simeq R\Gamma(\mathcal{O}_K[1/p], M(\eta)^\vee_p(1))[1].$$

Corollary 2.10. *The element*

$$\bigotimes_\eta r_p(z_p(\eta)) \in \bigotimes_\eta \mathrm{Det}^{-1}_{E_p} R\Gamma(\mathcal{O}_K[1/p], M(\eta)^\vee_p(1)),$$

where the tensor product is taken over all $\eta \in \widehat{G}_L \smallsetminus \widehat{G}_F$, generates the $\mathcal{O}_p$-lattice

$$\bigotimes_\eta \mathrm{Det}^{-1}_{\mathcal{O}_p} R\Gamma(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta^{-1})(1)).$$

Proof. By proposition 2.8 the statement in corollary 2.7 is equivalent to the statement that under the isomorphism ϑ_p the lattice

$$\bigotimes_{\eta \in \widehat{G}_L \smallsetminus \widehat{G}_F} \mathrm{Det}^{-1}_{\mathcal{O}_p}(\mathcal{O}_p r_p(z(\eta)) \otimes \mathcal{O}_p(\eta^{-1})^\vee) \otimes \bigotimes_{v|p} \mathrm{Det}^{-1}_{\mathcal{O}_p} R\Gamma_f(K_v, \mathcal{O}_p(\eta^{-1})^\vee)$$

coincides with

$$\bigotimes_{\eta \in \widehat{G}_L \smallsetminus \widehat{G}_F} \mathrm{Det}^{-1}_{\mathcal{O}_p} R\Gamma(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta^{-1})(1)) \otimes \mathrm{Det}^{-1}_{\mathcal{O}_p} \mathcal{O}_p(\eta^{-1})^\vee$$

The claim follows from the fact that

$$\mathrm{Det}_{\mathcal{O}_p} R\Gamma_f(K_v, \mathcal{O}_p(\eta^{-1})^\vee) = \prod_{v|p}(1 - \eta(v))$$

(see [HK] 1.2.5). $\square$

3. Review of the Euler System of elliptic units

In the proof of the Iwasawa main conjecture, the machinery of Euler systems is an essential tool. In this section, we construct an Euler system by twisting the elliptic units by a finite order character. The general theory of Euler systems, invented by Kolyvagin, was further developed by Kato, Perrin-Riou and Rubin (alphabetical order). We follow Rubin as his approach is closest to our setting.

3.1. **Euler systems.** Rubin gives a general definition for an Euler system in [Ru3]. We recall this definition using much of his notation. Fix a prime p and let T_p be a p-adic representation of the absolute Galois group of K with coefficients in $\mathcal{O}_p$, and let $\mathcal{N}$ denote an ideal of $\mathcal{O}_K$ divisible by p and the primes at which T_p is ramified. Denote by $\mathcal{K} := \bigcup_{\mathfrak{q} \nmid \mathcal{N}_0} K(\mathfrak{q})$ the union of the ray class fields of conductor not dividing the prime to p-part $\mathcal{N}_0$ of $\mathcal{N}$. We denote by K_∞ the maximal abelian $\mathbb{Z}_p$-extension of K unramified outside of p. Note that no finite prime of $\mathcal{O}_K$ splits completely in K_∞ and $\mathrm{Gal}(K_\infty/K) \simeq \mathbb{Z}_p^2$.

Definition 3.1 ([Ru3] Definition 2.1.1 and 2.1.3)**.** A collection of Galois cohomology classes $c_{\mathfrak{m}} \in H^1(K(\mathfrak{m}) \cap \mathcal{K}, T_p)$ for all ideals $\mathfrak{m}$ of $\mathcal{O}_K$ is called an *Euler system* for $(\mathcal{K}, T_p, \mathcal{N})$ if for every prime ideal $\mathfrak{q}$

$$\mathrm{Cor}_{K(\mathfrak{mq})\cap\mathcal{K}/K(\mathfrak{m})\cap\mathcal{K}}(c_{\mathfrak{mq}}) = \begin{cases} P(\mathrm{Frob}_{\mathfrak{q}}^{-1}|T_p^{\vee}(1);\mathrm{Frob}_{\mathfrak{q}}^{-1})c_{\mathfrak{m}} & \mathfrak{q} \nmid \mathfrak{m}\mathcal{N} \\ c_{\mathfrak{m}} & \mathfrak{q} \mid \mathfrak{m}\mathcal{N}. \end{cases}$$

Here the Euler factors are given by the characteristic polynomial

$$P(\mathrm{Frob}_{\mathfrak{q}}^{-1}|T_p^{\vee}(1);x) = \det(1-\mathrm{Frob}_{\mathfrak{q}}^{-1}x|\mathrm{Hom}_{\mathcal{O}_p}(T_p,\mathcal{O}_p(1))) \in \mathcal{O}_p[x].$$

3.2. **Elliptic Units.** We recall the definition of the Euler system of elliptic units, following the treatment of Kato [Ka] section 15.

First we recall Kato's definition of a CM-pair (E,α) of modulus $\mathfrak{m}$. Fix a non-zero ideal $\mathfrak{m}$ of $\mathcal{O}_K$, such that $\mathcal{O}_K^* \to (\mathcal{O}_K/\mathfrak{m})^*$ is injective. Then a CM-pair (E,α) consists of an elliptic curve E/K', where K'/K is a field extension together with an isomorphism $\mathcal{O}_K \simeq \mathrm{End}(E)$, such that the composition $\mathcal{O}_K \simeq \mathrm{End}(E) \to \mathrm{End}_{K'}(\mathrm{Lie}(E)) = K'$ is the canonical inclusion, and $\alpha \in E(K')$ is a torsion point, such that the annihilator of α in $\mathcal{O}_K$ coincides with $\mathfrak{m}$. Any isomorphism between two CM-pairs of modulus $\mathfrak{m}$ over K' is unique, because $\mathcal{O}_K^* \to (\mathcal{O}_K/\mathfrak{m})^*$ is injective.

The main theorem of complex multiplication implies that there exists a CM-pair (unique up to unique isomorphism) of modulus $\mathfrak{m}$ over the ray class field $K(\mathfrak{m})$, which is isomorphic to $(\mathbb{C}/\mathfrak{m}, 1 \bmod \mathfrak{m})$ over $\mathbb{C}$.

Kato constructs in [Ka] 15.4 for each $\mathfrak{a} \subset \mathcal{O}_K$, which is prime to 6 a function ${}_{\mathfrak{a}}\theta_E \in \mathcal{O}(E \smallsetminus E[\mathfrak{a}])^*$, which is characterized uniquely by the following two properties (denote by $E[\mathfrak{a}]$ the kernel of the $[\mathfrak{a}]$-multiplication):

- The divisor of ${}_{\mathfrak{a}}\theta_E$ is $\mathrm{N}(\mathfrak{a})(0) - E[\mathfrak{a}]$
- For each integer b, which is prime to $\mathfrak{a}$, one has $[b]_*{}_{\mathfrak{a}}\theta_E = {}_{\mathfrak{a}}\theta_E$.

We can now define elliptic units following Kato:

Definition 3.2. Fix a prime p and choose an integer $r \geq 1$, such that $\mathcal{O}_K^* \hookrightarrow (\mathcal{O}_K/p^r)^*$ is injective. Let $\mathfrak{a}$ be prime to $6p$. For any non zero ideal $\mathfrak{m}$ of $\mathcal{O}_K$ prime to $\mathfrak{a}$ we define

$$\zeta_{\mathfrak{m}} := {}_{\mathfrak{a}}\zeta_{\mathfrak{m}} := \mathrm{N}_{K(p^r\mathfrak{m})/K(\mathfrak{m})}{}_{\mathfrak{a}}\theta_E(\alpha)^{-1} \in K(\mathfrak{m})^*,$$

where (E,α) is "the" CM-pair of modulus $p^r\mathfrak{m}$ defined over $K(p^r\mathfrak{m})$. Note that this is independent of the chosen $r \geq 1$. We omit the auxiliary ideal $\mathfrak{a}$ from the notation, whenever no confusion is possible.

These elements have the following properties.

Proposition 3.3. *Let p^r and $\mathfrak{a}$ be as in definition 3.2, then:*

(1) *(Integrality) $\zeta_{\mathfrak{m}} \in \mathcal{O}_{K(\mathfrak{m})}^*$ if $p^r\mathfrak{m}$ is divisible by two different primes and $\zeta_{\mathfrak{p}^n} \in \mathcal{O}_{K(\mathfrak{p}^n)}[1/\mathfrak{p}]^*$ if $p^r\mathfrak{m}$ is a power of $\mathfrak{p}$.*

(2) *(Euler system property) For a prime ideal $\mathfrak{q} \subset \mathcal{O}_K$ such that $\mathfrak{m}\mathfrak{q}$ is prime to $\mathfrak{a}$ one has*

$$\mathrm{N}_{K(\mathfrak{q}\mathfrak{m})/K(\mathfrak{m})}(\zeta_{\mathfrak{q}\mathfrak{m}}) = \begin{cases} \zeta_{\mathfrak{m}}^{1-\mathrm{Frob}_{\mathfrak{q}}^{-1}} & \mathfrak{q} \nmid p\mathfrak{m} \\ \zeta_{\mathfrak{m}} & \mathfrak{q} \mid p\mathfrak{m} \end{cases}$$

(3) *(Independence from $\mathfrak{a}$) If $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}_K$ are prime to $6p$ and $\mathfrak{m}$ is prime to $\mathfrak{a}\mathfrak{b}$ let $\sigma_{\mathfrak{a}} = (\mathfrak{a}, K(\mathfrak{m})/K)$ and $\sigma_{\mathfrak{b}} = (\mathfrak{b}, K(\mathfrak{m})/K)$ be the Artin symbols in $G(\mathfrak{m})$, then*

$${}_{\mathfrak{b}}\zeta_{\mathfrak{m}}^{\sigma_{\mathfrak{a}}-\mathrm{N}(\mathfrak{a})} = {}_{\mathfrak{a}}\zeta_{\mathfrak{m}}^{\sigma_{\mathfrak{b}}-\mathrm{N}(\mathfrak{b})}.$$

(4) *(Relation to L-values) For any non-trivial character $\eta : G(\mathfrak{m}) \to \mathbb{C}^*$ (not necessarily proper) we have*

$$\sum_{\tau \in G(\mathfrak{m})} \eta(\tau) \log |\tau({}_{\mathfrak{a}}\zeta_{\mathfrak{m}})| = (\mathrm{N}(\mathfrak{a}) - \eta(\mathfrak{a})^{-1}) \lim_{s \to 0} s^{-1} L_{p\mathfrak{m}}(\eta, s),$$

where $|z| = (z\bar{z})^{1/2}$.

Proof. Observe first that the function ${}_{\mathfrak{a}}\theta_E$ is uniquely determined by the norm compatibility and its divisor. Then it is clear that ${}_{\mathfrak{a}}\theta_E$ is a twelfth root of the function used by [dS] in II. Property 1) follows immediately from [dS] II. 2.4. and property 2) follows in the same way as II. 2.5. i) in [dS], if one observes that $w_{p^r\mathfrak{m}} = w_{p^r\mathfrak{m}\mathfrak{q}} = 1$ in our case. Property 3) is [Ka] 15.4.4 and property 4) is [Ka] (15.5.1). $\square$

Corollary 3.4. *Choose $\mathfrak{a}$ prime to $6p$ and let $\mathcal{K} := \bigcup_{\mathfrak{q} \nmid \mathfrak{a}} K(\mathfrak{q})$. Then the ${}_{\mathfrak{a}}\zeta_{\mathfrak{m}} \in \mathcal{O}_{K(\mathfrak{m})}[1/p]^* \subset H^1(\mathcal{O}_{K(\mathfrak{m})}[1/p], \mathbb{Z}_p(1))$ for all $\mathfrak{m}$ prime to $\mathfrak{a}$ form an Euler system for $(\mathcal{K}, \mathbb{Z}_p(1), p\mathfrak{a})$ in the sense of definition 3.1.*

3.3. **The twisted Euler system.** Consider a character

$$\eta : G(\mathfrak{f}_\eta) \to E^*$$

of conductor $\mathfrak{f}_\eta$. Let $\mathcal{K}$ be the field extension defined in 3.4 and assume that $\mathfrak{a}$ is chosen prime to $\mathfrak{f}_\eta$. We wish to study a twist of the Euler system of elliptic units by η.

Consider the composition of the following two maps (8) and (9)

$$(8) \qquad H^1(\mathcal{O}_{K(\mathfrak{f}_\eta\mathfrak{m})}[1/p], \mathcal{O}_p(1)) \xrightarrow{\otimes t_p(\eta)} H^1(\mathcal{O}_{K(\mathfrak{f}_\eta\mathfrak{m})}[1/p], \mathcal{O}_p(\eta)(1)),$$

where we have identified

$$H^1(\mathcal{O}_{K(\mathfrak{f}_\eta\mathfrak{m})}[1/p], \mathcal{O}_p(1)) \otimes \mathcal{O}_p(\eta) \simeq H^1(\mathcal{O}_{K(\mathfrak{f}_\eta\mathfrak{m})}[1/p], \mathcal{O}_p(\eta)(1)),$$

and of the trace map (for $\mathcal{O}_{K(\mathfrak{f}_\eta\mathfrak{m})}[1/p] \to \mathcal{O}_{K(\mathfrak{m})}[1/p]$)

$$(9) \quad H^1(\mathcal{O}_{K(\mathfrak{f}_\eta\mathfrak{m})}[1/p], \mathcal{O}_p(\eta)(1)) \xrightarrow{\mathrm{tr}_{K(\mathfrak{f}_\eta\mathfrak{m})/K(\mathfrak{m})}} H^1(\mathcal{O}_{K(\mathfrak{m})}[1/p], \mathcal{O}_p(\eta)(1)).$$

Definition 3.5. For all $\mathfrak{m}$ prime to $\mathfrak{a}$ define

$$\zeta_{\mathfrak{m}}(\eta) := {}_{\mathfrak{a}}\zeta_{\mathfrak{m}}(\eta) := \mathrm{tr}_{K(\mathfrak{f}_\eta\mathfrak{m})/K(\mathfrak{m})}(\zeta_{\mathfrak{f}_\eta\mathfrak{m}} \otimes t_p(\eta)) \in H^1(\mathcal{O}_{K(\mathfrak{m})}[1/p], \mathcal{O}_p(\eta)(1)).$$

For any field $K \subset F \subset K(\mathfrak{m})$ we define

$$\zeta_F(\eta) := \mathrm{tr}_{K(\mathfrak{m})/F}\zeta_{\mathfrak{m}}(\eta).$$

Note that $\zeta_F(\eta)$ depends on $t_p(\eta)$.

The following proposition is shown in Rubin [Ru3]

Proposition 3.6 ([Ru3] 2.4.2)**.** *Let $\mathcal{K}$ be as above and $\mathfrak{a}$ prime to $6p$. The collection*

$${}_{\mathfrak{a}}\zeta_{\mathfrak{m}}(\eta) \in H^1(\mathcal{O}_{K(\mathfrak{m})}[1/p], \mathcal{O}_p(\eta)(1))$$

for all ideals $\mathfrak{m}$ prime to $\mathfrak{a}$ is an Euler system for $(\mathcal{K}, \mathcal{O}_p(\eta)(1), p\mathfrak{f}_\eta\mathfrak{a})$.

3.4. **A compatibility.** For later use we need the following compatibility: Let $G := \mathrm{Gal}(K(\mathfrak{f}_\eta\mathfrak{m})/K(\mathfrak{m}))$. Consider the map induced by $p_{\eta^{-1}}$ from

$$H^1(\mathcal{O}_{K(\mathfrak{f}_\eta\mathfrak{m})}[1/p], \mathcal{O}_p(1)) \simeq H^1(\mathcal{O}_{K(\mathfrak{m})}[1/p], \mathcal{O}_p[G](1))$$

to

$$H^1(\mathcal{O}_{K(\mathfrak{m})}[1/p], \mathcal{O}_p(\eta)(1)).$$

Lemma 3.7. *The image of $\zeta_{\mathfrak{f}_\eta\mathfrak{m}} \in H^1(\mathcal{O}_{K(\mathfrak{f}_\eta\mathfrak{m})}[1/p], \mathcal{O}_p(1))$ under the above map $p_{\eta^{-1}}$ coincides with $\zeta_{\mathfrak{m}}(\eta)$ and is given by*

$$\left(\sum_{\sigma\in G} \eta(\sigma)\sigma(\zeta_{\mathfrak{f}_\eta\mathfrak{m}})\right) t_p(\eta).$$

Proof. This follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{Map}(G, E_p) & \xrightarrow{p_{\eta^{-1}}} & \mathrm{Map}(G, E_p) \\ {\scriptstyle \otimes t_p(\eta)}\downarrow & & \uparrow{\scriptstyle \cup} \\ \mathrm{Map}(G, T_p(\eta)) & \longrightarrow & T_p(\eta) \end{array}$$

where $t_p(\eta)$ is the image of the delta function δ_e at the identity in G under $p_{\eta^{-1}}$ and the lower horizontal map is given by

$$f \mapsto \sum_{\sigma\in G} \sigma f(\sigma^{-1}).$$

□

3.5. **Relation to zeta elements.** In this section we make the relation between the Euler system and the zeta elements precise. This is crucial for the reduction of the main conjecture to the Tamagawa number conjecture.

Let $K_\infty = \bigcup_{n\geq 0} K_n$ be the $\mathbb{Z}_p^2$-extension of K, which is unramified outside of p and where $K \subset K_n \subset K_\infty$ is the unique subextension with Galois group $(\mathbb{Z}/p^n\mathbb{Z})^2$.

Definition 3.8. Let $\eta : G(\mathfrak{f}_\eta) \to E^*$ be a character of conductor $\mathfrak{f}_\eta$. The biggest $n \geq 0$, such that $K_n \subset K(\mathfrak{f}_\eta)$ is called the *level* of η.

Observe that if the level of η is big enough, then η is ramified at all primes above p.

Our aim in this section is to show that for characters of big enough level that $\zeta_K(\eta)$ coincides with the zeta element $z_p(\eta)$.

Theorem 3.9. *Let η be a character of conductor $\mathfrak{f}_\eta$ and level n such that η is ramified at the primes above p. Choose $\mathfrak{a}$ prime to $6p\mathfrak{f}_\eta$, such that $\mathrm{N}\mathfrak{a} - \eta(\mathfrak{a})$ is a unit in $\mathcal{O}_p$. Then the element $\zeta_K(\eta^{-1}) \in H^1(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta^{-1})(1))$ from 3.5 and the zeta element defined in 2.9 agree under the regulator map*

$$\zeta_K(\eta^{-1}) = r_p(z_p(\eta))$$

inside $H^1(\mathcal{O}_K[1/p], M(\eta^{-1})_p(1))$. In particular, the element $\zeta_K(\eta^{-1})$ is not torsion in $H^1(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta^{-1})(1))$.

Proof. Recall that $z(\eta) \in H^1_f(M(\eta)^\vee(1)) \cong H^1_f(M(\eta^{-1})(1))$. By definition the element $z(\eta)\otimes t_B(\eta^{-1})^\vee$ is the one which maps to $L^*(\eta, 0)$ under ϑ_∞^{-1}. We consider $\zeta_K(\eta^{-1})$ with proposition 3.7 as an element in $p_{\eta^{-1}}(\mathcal{O}_{K(\mathfrak{f}_\eta)}[1/p]^* \otimes_{\mathbb{Z}} E)$, i.e., $\zeta_K(\eta^{-1}) = p_{\eta^{-1}}(\zeta_{\mathfrak{f}_\eta})$. We show first that

$$p_{\eta^{-1}}(\mathcal{O}_{K(\mathfrak{f}_\eta)}[1/p]^* \otimes_{\mathbb{Z}} E) \cong p_{\eta^{-1}}(\mathcal{O}^*_{K(\mathfrak{f}_\eta)} \otimes_{\mathbb{Z}} E)$$

if η is ramified at all places above p. Thus, we assume $p^k \mid \mathfrak{f}_\eta$ for some $k \geq 1$. As $\mathfrak{f}_\eta$ is divisible by p^k, we see by 3.3 that the element $\zeta_K(\eta^{-1})$ is a unit, if p is split in K. If $\mathfrak{p}$ is inert or ramified, we must have $\mathfrak{f}_\eta = \mathfrak{p}^l$ for $\mathfrak{p}$ the only prime above p. Consider the exact sequence

$$1 \to \mathcal{O}^*_{K(\mathfrak{p}^l)} \otimes_{\mathbb{Z}} E \to \mathcal{O}_{K(\mathfrak{p}^l)}[1/\mathfrak{p}]^* \otimes_{\mathbb{Z}} E \to \prod_{v|\mathfrak{p}} E.$$

As $K(\mathfrak{p}^l)/K(1)$ is totally ramified at $\mathfrak{p}$, the decomposition group at v contains $\mathrm{Gal}(K(\mathfrak{p}^l)/K(1))$ and hence

$$p_\eta \prod_{v|\mathfrak{p}} E = 0$$

except if $\mathfrak{f}_\eta = \mathcal{O}_K$. But this can not happen because $\mathfrak{f}_\eta$ is divisible by p^k with $k \geq 1$. This shows that $\zeta_K(\eta^{-1})$ is represented by a unit.

With the explicit form of the regulator r_∞ in (4) we get using 3.3 (4):

$$\begin{aligned} r_\infty(\zeta_K(\eta^{-1})) &= p_\eta \sum_{\tau \in G(\mathfrak{f}_\eta)} (\log \mid \tau(\zeta_{\mathfrak{f}_\eta})) \mid)\tau \\ &= \sum_{\tau \in G(\mathfrak{f}_\eta)} \eta^{-1}(\tau)(\log \mid \tau(\zeta_{\mathfrak{f}_\eta})) \mid) t_B(\eta^{-1}) \\ &= (\mathrm{N}\mathfrak{a} - \eta(\mathfrak{a})) \lim_{s \to 0} s^{-1} L_{p\mathfrak{f}_\eta}(\eta^{-1}, s) t_B(\eta^{-1}). \end{aligned}$$

As p divides $\mathfrak{f}_\eta$ we get $L_{p\mathfrak{f}_\eta}(\eta^{-1}, s) = L(\eta^{-1}, s)$ and $z(\eta) = z_p(\eta)$. By our choice of $\mathfrak{a}$, $\mathrm{N}\mathfrak{a} - \eta(\mathfrak{a})$ is a unit in $\mathcal{O}_p$ and we get $r_\infty(\zeta_K(\eta^{-1})) = L^*(\eta^{-1}, 0)t_B(\eta^{-1})$. On the other hand

$$r_\infty(z(\eta)) = L^*(\eta^{-1}, 0)t_B(\eta^{-1}),$$

so that $\zeta_K(\eta^{-1}) = z(\eta) = z_p(\eta)$. □

Let $K_n(\mathfrak{f}_\chi) := K_n K(\mathfrak{f}_\chi)$ be the compositum of K_n and $K(\mathfrak{f}_\chi)$ and write $G_n(\mathfrak{f}) := \mathrm{Gal}(K_n(\mathfrak{f})/K)$.

Combining the above theorem 3.9 with 2.10 for $L = K_n(\mathfrak{f}_\chi)$ and $F = K_{n-1}(\mathfrak{f}_\chi)$ one gets:

Corollary 3.10. *Let η be of level n and n so big that η is ramified at all primes above p, then the element*

$$\bigotimes_\eta \zeta_K(\eta^{-1}) \in \bigotimes_\eta \mathrm{Det}^{-1}_{E_p} R\Gamma(\mathcal{O}_K[1/p], M(\eta^{-1})_p(1)),$$

where the tensor product is taken over all $\eta \in \widehat{G}_n(\mathfrak{f}_\eta) \smallsetminus \widehat{G}_{n-1}(\mathfrak{f}_\eta)$, generates the $\mathcal{O}_p$-lattice

$$\bigotimes_\eta \mathrm{Det}^{-1}_{\mathcal{O}_p} R\Gamma(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta^{-1})(1)).$$

4. Iwasawa modules

In this section we introduce the basic Iwasawa modules we want to study and state some of their properties used later.

4.1. The Iwasawa algebras Λ and Ω.

Consider inside $K(p^\infty) := \bigcup_{n\geq 1} K(p^n)$ the maximal $\mathbb{Z}_p^2$-extension K_∞, which is unramified outside of p, so that

$$\Gamma := \mathrm{Gal}(K_\infty/K) \simeq \mathbb{Z}_p^2.$$

We denote by $K \subset K_n \subset K_\infty$ the unique subextension with Galois group $G_n := \mathbb{Z}_p^2/p^n\mathbb{Z}_p^2$. For an ideal $\mathfrak{f} \subset \mathcal{O}_K$ we define

$$\mathcal{G}_\mathfrak{f} := \mathrm{Gal}(K(\mathfrak{f}p^\infty)/K). \tag{10}$$

We denote by $\Delta \subset \mathcal{G}_\mathfrak{f}$ the torsion subgroup and fix once for all a splitting

$$\mathcal{G}_\mathfrak{f} \simeq \Delta \times \Gamma.$$

For each profinite group $\mathcal{G} = \varprojlim \mathcal{G}/\mathcal{H}$ we define its *Iwasawa algebra* to be the inverse limit

$$\Lambda(\mathcal{G}) := \varprojlim_{\mathcal{H}\subset\mathcal{G}} \mathbb{Z}_p[\mathcal{G}/\mathcal{H}].$$

Two Iwasawa algebras are especially important in the sequel:

Definition 4.1. The *Iwasawa algebra* for Γ is denoted by

$$\Lambda := \Lambda(\Gamma),$$

which is (non-canonically) isomorphic to $\mathbb{Z}_p[[T,S]]$. The Iwasawa algebra for $\mathcal{G}_{\mathfrak{f}}$ is denoted by

$$\Omega := \Lambda(\mathcal{G}_{\mathfrak{f}}),$$

which is (non-canonically) isomorphic to

$$\Omega \cong \mathbb{Z}_p[\Delta][[T,S]].$$

We also let $\Lambda_{\mathcal{O}} := \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_p$ and $\Omega_{\mathcal{O}} := \Omega \otimes_{\mathbb{Z}_p} \mathcal{O}_p$ be the Iwasawa algebras with coefficients in $\mathcal{O}_p$.

Both Iwasawa algebras Λ and Ω carry a natural action of $\mathrm{Gal}(\bar{K}/K)$, which acts through its quotient Γ (resp. $\mathcal{G}_{\mathfrak{f}}$) by the canonical inclusions $\Gamma \subset \Lambda^*$ (resp. $\mathcal{G}_{\mathfrak{f}} \subset \Omega^*$). The $\mathrm{Gal}(\bar{K}/K)$-module Λ is unramified outside of p and Ω is unramified outside of $\mathfrak{f}p$. Note that Λ and Ω are products of local rings so that we can apply the Nakayama lemma to each component of Λ and Ω.

4.2. **The basic Iwasawa modules.** Let η be a character of conductor $\mathfrak{f}_\eta$ and $\mathcal{O}_p(\eta)$ the associated $\mathcal{O}_p$-module with $\mathrm{Gal}(\bar{K}/K)$-action by η as defined in 1.1. The action on $\mathcal{O}_p(\eta)$ is unramified outside of $p\mathfrak{f}_\eta$ and factors through $\mathcal{G}_{\mathfrak{f}_\eta}$. In particular, $\mathcal{O}_p(\eta)$ is an $\Omega_{\mathcal{O}}$-module and by restriction also a $\Lambda_{\mathcal{O}}$-module.

Recall that we consider $\mathcal{O}_p(\eta)$ as étale sheaf on $\mathcal{O}_K[1/p]$ via the map $j : \mathrm{Spec}(\mathcal{O}_K[1/p\mathfrak{f}_\eta]) \hookrightarrow \mathrm{Spec}(\mathcal{O}_K[1/p])$ and that we omit j_* from the notation.

Definition 4.2. For the Iwasawa algebra Λ (resp. Ω) let

$$\Lambda(\eta) := \mathcal{O}_p(\eta) \otimes_{\mathbb{Z}_p} \Lambda,$$

resp.

$$\Omega(\eta) := \mathcal{O}_p(\eta) \otimes_{\mathbb{Z}_p} \Omega$$

considered as ètale sheaves (of $\Lambda_{\mathcal{O}}$ resp. $\Omega_{\mathcal{O}}$-modules) on $\mathrm{Spec}(\mathcal{O}_K[1/p])$. We also use the notation $\Lambda(\eta)(1) := \Lambda(\eta) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$ etc.

We have

$$H^i(\mathcal{O}_K[1/p], \Lambda(\eta)) = \varprojlim_{K \subset K_n \subset K_\infty} H^i(\mathcal{O}_{K_n}[1/p], \mathcal{O}_p(\eta)) \tag{11}$$

and

$$H^i(\mathcal{O}_K[1/p], \Omega(\eta)) = \varprojlim_{K \subset F \subset K(p^\infty \mathfrak{f})} H^i(\mathcal{O}_{K_n}[1/p], \mathcal{O}_p(\eta)). \tag{12}$$

In particular,

$$H^0(\mathcal{O}_K[1/p], \Omega(1)) = H^0(\mathcal{O}_K[1/p], \Lambda(\eta)(1)) = 0. \tag{13}$$

Here the (left) $\Lambda_{\mathcal{O}}$-module structure on $H^i(\mathcal{O}_K[1/p],\Lambda(\eta))$ is induced by multiplication with the inverse on Λ (see [HoKi] Appendix for details). We consider also the cohomology with compact support

$$H^i_c(\mathcal{O}_K[1/p],\Lambda(\eta)) \tag{14}$$

and the local cohomology groups

$$H^i(K_v,\Lambda(\eta)) \tag{15}$$

and similarly for $\Omega(\eta)$. These $\Lambda_{\mathcal{O}}$-modules (resp. $\Omega_{\mathcal{O}}$-modules) are the basic Iwasawa modules, which are involved in the formulation of the main conjecture.

We collect some information about these Iwasawa modules. The following lemma will be often used without further comment.

Lemma 4.3. *Let M be a compact Λ-module, which is of finite* Tor*-dimension, then*

$$R\Gamma(\mathcal{O}_K[1/p],\Lambda(\eta)(1))\otimes^{\mathbb{L}}_{\Lambda} M \cong R\Gamma(\mathcal{O}_K[1/p],\Lambda(\eta)(1)\otimes_{\Lambda} M).$$

In particular, one has a spectral sequence

$$\operatorname{Tor}^{\Lambda}_r\left(H^s(\mathcal{O}_K[1/p],\Lambda(\eta)(1)),M\right)\Rightarrow H^{s-r}(\mathcal{O}_K[1/p],\Lambda(\eta)(1)\otimes_{\Lambda} M).$$

Proof. This is clear if M is a free Λ-module and follows by the usual arguments using a free resolution of finite length. $\square$

Lemma 4.4. *Let $\Lambda_{\mathcal{O}}$ and $\Omega_{\mathcal{O}}$ be the two basic Iwasawa algebras with coefficients in $\mathcal{O}_p$. Then the $\Lambda_{\mathcal{O}}$-modules*

$$H^i(\mathcal{O}_K[1/p],\Lambda(\eta)(1)), H^i_c(\mathcal{O}_K[1/p],\Lambda(\eta)(1)) \text{ and } H^i(K_v,\Lambda(\eta)(1))$$

are finitely generated. The same statement holds with $\Lambda_{\mathcal{O}}$ replaced by $\Omega_{\mathcal{O}}$.

Proof. As the $H^i(\mathcal{O}_K[1/p],\mathcal{O}_p(\eta)(1))$ and the $H^i(K_v,\mathcal{O}_p(\eta)(1))$ are finitely generated $\mathcal{O}_p$-modules and $\Lambda_{\mathcal{O}}$ and $\Omega_{\mathcal{O}}$ are product of local rings, this follows from the topological Nakayama lemma (see [NSW] 5.2.18.) and the above spectral sequence

$$\operatorname{Tor}^{\Lambda_{\mathcal{O}}}_r\left(H^s(\mathcal{O}_K[1/p],\Lambda(\eta)(1)),\Lambda_{\mathcal{O}}\right)\Rightarrow H^{s-r}(\mathcal{O}_K[1/p],\mathcal{O}_p(\eta)(1)).$$

(resp. the analogous spectral sequence for $H^{s-r}(K_v,\mathcal{O}_p(\eta)(1))$, resp. for $\Omega_{\mathcal{O}}$). For $H^i_c(\mathcal{O}_K[1/p],\Lambda(\eta)(1))$ (resp. for $\Omega_{\mathcal{O}}$) the finite generation follows then from the definition of the cohomology with compact support. $\square$

Consider the triangle for cohomology with compact support

$$R\Gamma_c(\mathcal{O}_K[1/p],\Lambda(\eta)(1))\to R\Gamma(\mathcal{O}_K[1/p],\Lambda(\eta)(1))\to\bigoplus_{v|p}R\Gamma(K_v,\Lambda(\eta)(1))\oplus\Lambda(\eta)(1).$$

For the computations of some Iwasawa modules, we need an Artin-Verdier duality result for $H^i_c(\mathcal{O}_K[1/p],\Lambda(\eta)(1))$.

Proposition 4.5. *One has a perfect pairing*

$$H^i(\mathcal{O}_{K_\infty}[1/p], \mathcal{O}_p(\eta)^*) \times H_c^{3-i}(\mathcal{O}_K[1/p], \Lambda(\eta)(1)) \to E_p/\mathcal{O}_p,$$

where $\mathcal{O}_p(\eta)^* := \underline{\mathrm{Hom}}(\mathcal{O}_p(\eta), E_p/\mathcal{O}_p)$ *is the Pontryagin dual of* $\mathcal{O}_p(\eta)$.

Proof. Recall from [Mi] II.1.8 b) that for each number field $K \subset F \subset K_\infty$ one has a perfect pairing

$$\mathrm{Ext}^i_{\mathcal{O}_F[1/p]}(j_*\mathcal{O}_p(\eta)(1), \mathbb{G}_m) \times H_c^{3-i}(\mathcal{O}_F[1/p], \mathcal{O}_p(\eta)(1)) \to E_p/\mathcal{O}_p.$$

Taking $\varinjlim$ of the Ext^i and $\varprojlim$ of the H_c^{3-i} one gets still a perfect pairing and because

$$\varprojlim_{K\subset F\subset K_\infty} H_c^{3-i}(\mathcal{O}_F[1/p], \mathcal{O}_p(\eta)(1)) = H_c^{3-i}(\mathcal{O}_K[1/p], \Lambda(\eta)(1))$$

it suffices to show that

$$\mathrm{Ext}^i_{\mathcal{O}_F[1/p]}(j_*\mathcal{O}_p(\eta)(1), \mathbb{G}_m) \cong H^i(\mathcal{O}_F[1/p], \mathcal{O}_p(\eta)^*).$$

Using the local to global Ext-spectral sequence

$$H^r(\mathcal{O}_F[1/p], \mathcal{E}xt^s(j_*\mathcal{O}_p(\eta)(1), \mathbb{G}_m)) \Longrightarrow \mathrm{Ext}^{r+s}_{\mathcal{O}_F[1/p]}(j_*\mathcal{O}_p(\eta)(1), \mathbb{G}_m)$$

we see that it suffices to see $\mathcal{E}xt^s(j_*\mathcal{O}_p(\eta)(1), \mathbb{G}_m) \cong j_*\mathcal{E}xt^s(\mathcal{O}_p(\eta)(1), \mathbb{G}_m)$ and that $\mathcal{E}xt^s(j_*\mathcal{O}_p(\eta)(1), \mathbb{G}_m) = 0$ for $s > 0$. Both statements follow from the proof of II. 1.10 b) in [Mi]. $\square$

Lemma 4.6. *The modules*

$$H_c^3(\mathcal{O}_K[1/p], \Lambda(\eta)(1)), H^2(K_v, \Lambda(\eta)(1)) \text{ and } H^2(K_v, \Omega(1))$$

for $v \mid p$ *are finitely generated* $\mathcal{O}_p$*-modules. In particular, they are* $\Lambda_{\mathcal{O}}$*-pseudo-null (resp.* Ω*-pseudo-null).*

Proof. Let us consider $H_c^3(\mathcal{O}_K[1/p], \Lambda(\eta)(1))$. By Artin-Verdier duality 4.5 we have

$$H_c^3(\mathcal{O}_K[1/p], \Lambda(\eta)(1))^* \cong H^0(\mathcal{O}_{K_\infty}[1/p], \mathcal{O}_p(\eta)^*) \subset \mathcal{O}_p(\eta)^*,$$

which is obviously a finitely generated $\mathcal{O}_p$-module. Similarly,

$$H^2(K_v, \Lambda(\eta)(1))^* \cong H^0(K_\infty \otimes K_v, \mathcal{O}_p(\eta)^*) \subset \mathcal{O}_p(\eta)^*$$

and

$$H^2(K_v, \Omega(1))^* \cong H^0(K(p^\infty\mathfrak{f}) \otimes K_v, \mathcal{O}_p(\eta)^*) = \mathcal{O}_p(\eta)^*$$

are finitely generated $\mathcal{O}_p$-modules. $\square$

We finally study the operation of twisting with a character $\varrho : \Gamma \to \mathcal{O}_p^*$.

Lemma 4.7. *Let* $\varrho : \Gamma \to \mathcal{O}_p^*$ *be a character and consider* $\Lambda(\varrho)$*. Then there is an isomorphism of* $\mathrm{Gal}(\bar{K}/K)$*-modules depending on the generator* $t_p(\varrho)$ *of* $\mathcal{O}_p(\varrho)$

$$\Lambda \simeq \Lambda(\varrho)$$

given by $\gamma \mapsto 1 \otimes \varrho^{-1}(\gamma)t_p(\varrho)$*. In particular, one has isomorphisms*

$$H^i(\mathcal{O}_K[1/p], \Lambda(\eta)) \simeq H^i(\mathcal{O}_K[1/p], \Lambda(\eta\varrho))$$

for all $i \geq 0$.

Proof. As $\Lambda \simeq \Lambda(\varrho)$ is obviously an isomorphism of $\mathrm{Gal}(\bar{K}/K)$-modules, the statement follows. $\square$

5. Statement of the two main conjectures

Recall the definition of the Iwasawa algebras Λ and Ω from 4.1. We will formulate in this section two main-conjectures. One for the ring Λ, which corresponds to the statement of the *main conjecture* decomposed into characters, and another for the ring Ω, which is elsewhere called the *equivariant main conjecture*.

The Ω-main-conjecture is apparently stronger because it is an equivariant statement, which does not involve any characters. Nevertheless, we will deduce the Ω-main-conjecture from the Λ-main-conjecture for all $\mathcal{O}_p(\eta)$ by a simple observation, which is inspired by the work of Burns-Greither [BG] for the cyclotomic case and was first explained by Witte in [Wi].

5.1. **The Λ-main-conjecture.** Consider a character $\chi : G(\mathfrak{f}_\chi) \to E^*$ of conductor $\mathfrak{f}_\chi$ and fix $\mathfrak{a}$ prime to $6p\mathfrak{f}_\chi$. In 3.3 we have defined elements

$$\zeta_{K_n}(\chi) \in H^1(\mathcal{O}_{K_n}[1/p], \mathcal{O}_p(\chi)(1)),$$

which are part of an Euler system in the sense of 3.1. In particular, these elements are norm compatible in the K_∞-direction and we can define

$$\zeta(\chi) := \varprojlim_n \zeta_{K_n}(\chi) \in H^1(\mathcal{O}_K[1/p], \Lambda(\chi)(1)).$$

We consider the submodule $\Lambda_{\mathcal{O}}\zeta(\chi) \subset H^1(\mathcal{O}_K[1/p], \Lambda(\chi)(1))$ generated by $\zeta(\chi)$. Recall that $\zeta(\chi)$ depends on our choice of a generator $t_p(\chi)$ of the lattice $\mathcal{O}_p(\chi)$.

Theorem 5.1 (Main-Conjecture)**.** *Denote by $Q(\Lambda_{\mathcal{O}})$ the total quotient ring of $\Lambda_{\mathcal{O}}$. Then, for each character $\chi : G(\mathfrak{f}_\chi) \to E^*$ of conductor $\mathfrak{f}_\chi$*

$H^0(\mathcal{O}_K[1/p], \Lambda(\chi)(1)) = 0$
$H^1(\mathcal{O}_K[1/p], \Lambda(\chi)(1))$ *has $\Lambda_{\mathcal{O}}$ rank* 1
$H^2(\mathcal{O}_K[1/p], \Lambda(\chi)(1))$ *is a $\Lambda_{\mathcal{O}}$-torsion module*

The isomorphism

$$Q(\Lambda_{\mathcal{O}})\zeta(\chi) \simeq H^1(\mathcal{O}_K[1/p], \Lambda(\chi)(1)) \otimes_{\Lambda_{\mathcal{O}}} Q(\Lambda_{\mathcal{O}})$$

induces an equality of lattices

$$\Lambda_{\mathcal{O}}\zeta(\chi)^{-1} = \mathrm{Det}_{\Lambda_{\mathcal{O}}} R\Gamma(\mathcal{O}_K[1/p], \Lambda(\chi)(1)).$$

This theorem will be proved in section 6. Note that the statement is for all primes p with no exceptions.

Remark 5.2. Observe that our formulation here follows [HK] and is different from the classical approach by Rubin. Rubin decomposes the Iwasawa modules into χ-eigenspaces, we use instead cohomology with coefficients in

$\mathcal{O}_p(\chi)$. This approach avoids many problems with the χ-eigenspaces and is very close to the spirit of the Tamagawa number conjecture.

This theorem can also conveniently formulated with the functor Div from section 1.3 and proposition 1.2: Consider the morphism of perfect complexes

$$\kappa_\chi : \Lambda_{\mathcal{O}}\zeta(\chi) \to R\Gamma(\mathcal{O}_K[1/p], \Lambda(\chi)(1))[1].$$

Then this morphism is good and

$$\mathrm{Div}(\kappa_\chi) = 0.$$

5.2. **The Ω-main-conjecture.** We are ultimately interested in an equivariant version of the Λ-main-conjecture. To this end, we admit the following hypothesis.

Conjecture 5.3. Let $\mathfrak{q}$ be a height one prime ideal of Ω containing p, then

$$H^2(\mathcal{O}_K[1/p], \Omega(1))_{\mathfrak{p}} = 0.$$

This conjecture is essentially equivalent to the vanishing of the μ-invariant for the maximal abelian $\mathbb{Z}_p$-extension of K_∞. Using results of Gillard, we show in 5.10 that this conjecture holds for primes $p \nmid 6$, which are split in K:

Theorem 5.4 (see 5.10)**.** *In the case that $p \nmid 6$ splits in $K/\mathbb{Q}$, Conjecture 5.3 is true.*

Recall the Euler system of elliptic units presented in section 3.2. As in the Λ-main-conjecture, we can consider the Euler system to be an element of the Iwasawa cohomology

$$\zeta := \varprojlim_n \zeta_{\mathfrak{f}p^n} \in H^1(\mathcal{O}_K[1/p], \Omega(1)).$$

Theorem 5.5 (Equivariant Main Conjecture)**.** *Fix an non-zero ideal $\mathfrak{f} \subset \mathcal{O}_K$. Let $Q(\Omega)$ be the total quotient ring of Ω. Then*

$H^0(\mathcal{O}_K[1/p], \Omega(1)) = 0$
$H^1(\mathcal{O}_K[1/p], \Omega(1))$ *has Ω rank* 1
$H^2(\mathcal{O}_K[1/p], \Omega(1))$ *is an Ω-torsion module.*

Assume conjecture 5.3, i.e., $H^2(\mathcal{O}_K[1/p], \Omega(1))_{\mathfrak{p}} = 0$ for all height one prime ideals with $p \in \mathfrak{q}$, then the isomorphism

$$Q(\Omega)\zeta \simeq H^1(\mathcal{O}_K[1/p], \Omega(1)) \otimes_\Omega Q(\Omega)$$

induces an equality of lattices

$$\Omega\zeta^{-1} = \mathrm{Det}_\Omega R\Gamma(\mathcal{O}_K[1/p], \Omega(1)).$$

In particular, this statement holds for all prime numbers $p \nmid 6$, which split in K.

This theorem will be proved in section 7.

Remark 5.6. As with theorem 5.1, this theorem can also be formulated with the functor Div: The morphism of perfect complexes

$$\kappa : \Omega\zeta \to R\Gamma(\mathcal{O}_K[1/p], \Omega)(1))[1].$$

is good and

$$\mathrm{Div}(\kappa) = 0.$$

5.3. **Conjecture 5.3 and the vanishing of the μ-invariant.** In this section we show that the results of Gillard [Gi] imply the conjecture 5.3 for $p \nmid 6$, which are split in K.

Assume that $p = \mathfrak{p}\mathfrak{p}'$ in K and let $K \subset K_\infty^{\mathfrak{p}} \subset K_\infty$ (resp. $K_\infty^{\mathfrak{p}'}$) be the $\mathbb{Z}_p$-extension of K, which is unramified outside of $\mathfrak{p}$ (resp. $\mathfrak{p}'$). Recall from (10) that we fixed a splitting $\mathcal{G}_\mathfrak{f} \simeq \Delta \times \Gamma$ and define L/K such that $\mathrm{Gal}(L/K) \simeq \Delta$. Let $F_\infty := LK_\infty^{\mathfrak{p}}$ be the compositum, then $\mathrm{Gal}(F_\infty/K) \cong \Delta \times \mathrm{Gal}(K_\infty^{\mathfrak{p}}/K)$ and

$$\mathcal{G}_\mathfrak{f} \cong \mathrm{Gal}(K_\infty^{\mathfrak{p}'}/K) \times \mathrm{Gal}(F_\infty/K).$$

Define $\mathcal{H} := \mathrm{Gal}(K_\infty^{\mathfrak{p}'}/K) \cong \mathbb{Z}_p$, so that

$$0 \to \mathcal{H} \to \mathcal{G}_\mathfrak{f} \to \mathrm{Gal}(F_\infty/K) \to 0 \tag{16}$$

is exact.

Let M_∞ be the maximal abelian $\mathbb{Z}_p$-extension of F_∞, which is unramified outside of $\mathfrak{p}$. Gillard proves:

Theorem 5.7 (Gillard [Gi] 3.4.)**.** *Let $p \nmid 6$ be split in K. The group $\mathrm{Gal}(M_\infty/F_\infty)$ has no $\mathbb{Z}_p$-torsion. In particular, it is a finitely generated $\mathbb{Z}_p$-module.*

We want to apply this theorem to prove conjecture 5.3, i.e., we want to show that $H^2(\mathcal{O}_K[1/p], \Omega(1))_\mathfrak{q} = 0$ for all height one prime ideals with $p \in \mathfrak{q}$. We first study $H^2(\mathcal{O}_K[1/p], \Lambda_\mathcal{O}(\mathrm{Gal}(F_\infty/K))(1))$, where $\Lambda_\mathcal{O}(\mathrm{Gal}(F_\infty/K))$ is the Iwasawa algebra of $\mathrm{Gal}(F_\infty/K)$.

Let

$$\mathcal{A}(F_\infty) := \varprojlim_n (Cl(F_n) \otimes_\mathbb{Z} \mathcal{O}_p)$$

be the inverse limits of the class groups of the fields $F_n := K_n L$ so that $F_\infty = \bigcup_n F_n$. Then $\mathcal{A}(F_\infty)$ is a $\mathcal{O}_p$-module, which is a quotient of $\mathrm{Gal}(M_\infty/F_\infty) \otimes_{\mathbb{Z}_p} \mathcal{O}_p$. The above theorem implies that $\mathcal{A}(F_\infty)$ is a finitely generated $\mathcal{O}_p$-module.

Corollary 5.8. *With the above notations*

$$H^2(\mathcal{O}_K[1/p], \Lambda_\mathcal{O}(\mathrm{Gal}(F_\infty/K))(1))$$

is a finitely generated $\mathcal{O}_p$-module. In particular, $H^2(\mathcal{O}_K[1/p], \Omega(1))$ is a finitely generated $\Lambda_\mathcal{O}(\mathcal{H})$-module.

Proof. Recall (from [HK] Prop. A.3 and passing to the limit) that one has an exact sequence

$$\mathcal{A}(F_\infty) \to H^2(\mathcal{O}_K[1/p], \Lambda_{\mathcal{O}}(\mathrm{Gal}(F_\infty/K))(1)) \to \bigoplus_{v|p} H^2(K_v, \Lambda_{\mathcal{O}}(\mathrm{Gal}(F_\infty/K))(1))$$

According to lemma 4.6 the groups $H^2(K_v, \Lambda_{\mathcal{O}}(\mathrm{Gal}(F_\infty/K))(1))$ are finitely generated $\mathcal{O}_p$-modules.

Consider $H^2(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1))$ as compact $\Lambda_{\mathcal{O}}(\mathcal{H})$-module. Using lemma 4.3 one sees that

$$H^2(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1)) \otimes_{\Lambda_{\mathcal{O}}(\mathcal{H})} \mathcal{O}_p \simeq H^2(\mathcal{O}_K[1/p], \Lambda_{\mathcal{O}}(\mathrm{Gal}(F_\infty/K))(1)).$$

It follows from Nakayama's lemma and the above corollary that $H^2(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1))$ is a finitely generated $\Lambda_{\mathcal{O}}(\mathcal{H})$-module. □

We conclude with following general structure result.

Lemma 5.9. *Let M be an $\Omega_{\mathcal{O}}$-module, which is finitely generated as $\Lambda_{\mathcal{O}}(\mathcal{H})$-module. Then for any height one prime ideal $\mathfrak{q} \subset \Omega_{\mathcal{O}}$ with $p \in \mathfrak{q}$, one has*

$$M_{\mathfrak{q}} = 0.$$

Proof. Let $\widetilde{M} := M/\mathfrak{q}M$, $\widetilde{\Omega} := \Omega_{\mathcal{O}}/\mathfrak{q}\Omega_{\mathcal{O}}$. We denote by $\kappa(\mathfrak{q})$ the residue class field of $\mathfrak{q}$. By Nakayama's lemma it suffices to show that

$$M_{\mathfrak{q}}/\mathfrak{q}M_{\mathfrak{q}} = \widetilde{M} \otimes_{\widetilde{\Omega}} \kappa(\mathfrak{q}) = 0,$$

By the exact sequence (16), we have $\Omega_{\mathcal{O}} \otimes_{\Lambda_{\mathcal{O}}(\mathcal{H})} \mathcal{O}_p \simeq \Lambda_{\mathcal{O}}(\mathrm{Gal}(F_\infty/K))$ and we let

$$I := \ker(\Omega_{\mathcal{O}} \to \Lambda_{\mathcal{O}}(\mathrm{Gal}(F_\infty/K))).$$

By our assumption, M/IM is a finitely generated $\mathcal{O}_p$-module. Identify $\Lambda_{\mathcal{O}}(\mathrm{Gal}(F_\infty/K)) \simeq \mathcal{O}_p[\Delta][[u]]$ and choose $\widetilde{u} \in \Omega_{\mathcal{O}}$ mapping to $u \in \mathcal{O}_p[\Delta][[u]]$. We show that

$$\widetilde{M} \otimes_{\widetilde{\Omega}} \widetilde{\Omega}[\widetilde{u}^{-1}] = 0,$$

which gives the desired result, as $\widetilde{\Omega}[\widetilde{u}^{-1}] \subset \kappa(\mathfrak{q})$. Let $\widetilde{I} := I/\mathfrak{q}I \subset \widetilde{\Omega}$. As $p \in \mathfrak{q}$ the $\widetilde{\Omega}$-module $\widetilde{M}/\widetilde{I}\widetilde{M}$ is finitely generated $\mathcal{O}_p/p\mathcal{O}_p$-module, hence a finite group. This implies that there is an integer k such that $\widetilde{u}^k(\widetilde{M}/\widetilde{I}\widetilde{M}) = \widetilde{u}^{k+1}(\widetilde{M}/\widetilde{I}\widetilde{M})$. As $\widetilde{u}$ is in the radical of $\widetilde{\Omega}$, Nakayama's lemma shows that $\widetilde{u}^k(\widetilde{M}/\widetilde{I}\widetilde{M}) = 0$. This shows $(\widetilde{M}/\widetilde{I}\widetilde{M}) \otimes_{\widetilde{\Omega}/\widetilde{I}\widetilde{\Omega}} \widetilde{\Omega}/\widetilde{I}\widetilde{\Omega}[\widetilde{u}^{-1}] = 0$. As $\widetilde{I}$ is in the radical of $\widetilde{\Omega}[\widetilde{u}^{-1}]$ Nakayama's lemma implies that $\widetilde{M} \otimes_{\widetilde{\Omega}} \widetilde{\Omega}[\widetilde{u}^{-1}] = 0$. □

Corollary 5.10 (Conjecture 5.3 for split primes)**.** *Let p be a prime, which splits in K and assume that $p \nmid 6$. Then, for any height one prime ideal $\mathfrak{q} \subset \Omega_{\mathcal{O}}$ with $p \in \mathfrak{q}$, one has*

$$H^2(\mathcal{O}[1/p], \Omega(1))_{\mathfrak{q}} = 0.$$

6. Proof of the Λ-main-conjecture

In this section we prove the Λ-main-conjecture as formulated in theorem 5.1.

6.1. **Reduction to characters of big enough level.** Let $\chi : G(\mathfrak{f}_\chi) \to E^*$ be a character of conductor $\mathfrak{f}_\chi$. Consider the submodule $\Lambda_\mathcal{O}\zeta(\chi) \subset H^1(\mathcal{O}_K[1/p], \Lambda(\chi)(1))$.

Lemma 6.1. *Consider a character $\varrho : \Gamma \to \mathcal{O}_p^*$. Then the twisting map of lemma 4.7 maps $\zeta(\chi)$ to*

$$\zeta(\chi\varrho) \in H^1(\mathcal{O}_K[1/p], \Lambda(\chi\varrho)(1)).$$

In particular, the Λ-main-conjecture is compatible with twists.

Proof. This is a direct consequence of the construction of $\zeta(\chi)$ in 3.5 (see also [HoKi] section 1.2). As $\Lambda(\chi) \cong \Lambda(\chi\varrho)$ as $\mathrm{Gal}(\bar{K}/K)$-modules, it is clear that

$$R\Gamma(\mathcal{O}_K[1/p], \Lambda(\chi)(1)) \cong R\Gamma(\mathcal{O}_K[1/p], \Lambda(\chi\varrho)(1)).$$

□

This lemma allows us to reduce the Λ-main-conjecture for χ to the one for $\eta := \chi\varrho$ using the isomorphisms in 4.7. Choose ϱ such that the level of $\eta = \chi\varrho$ is big enough. This gives:

Corollary 6.2. *To prove the Λ-main-conjecture, it suffices to consider characters η of level big enough.*

6.2. **Divisibility obtained from the Euler system.** In this section we use the Euler system defined by the elliptic units to prove one divisibility in the statement of the Λ-main-conjecture. We consider characters η of level big enough.

Let us define a subgroup of $H^2(\mathcal{O}_K[1/p], \Lambda(\eta)(1))$, which plays the role of the Selmer group.

Definition 6.3. Let

$$H^2_0(\mathcal{O}_K[1/p], \Lambda(\eta)(1)) := \ker\left(H^2(\mathcal{O}_K[1/p], \Lambda(\eta)(1)) \to \bigoplus_{v|p} H^2(K_v, \Lambda(\eta(1))\right).$$

Kolyvagin's theory of Euler systems as developed by Kato, Perrin-Riou and Rubin (alphabetical order), gives:

Theorem 6.4. *Let η be a character of conductor $\mathfrak{f}_\eta$ and level n, chosen so big that $\mathcal{O}_p(\eta)$ is ramified at all places $v \mid p$, then:*

1) *$H^2(\mathcal{O}_K[1/p], \Lambda(\eta)(1))$ is $\Lambda_\mathcal{O}$-torsion.*
2) *$H^1(\mathcal{O}_K[1/p], \Lambda(\eta)(1))$ has $\Lambda_\mathcal{O}$-rank one.*
3) *$H^1(\mathcal{O}_K[1/p], \Lambda(\eta)(1))/\Lambda_\mathcal{O}\zeta(\eta)$ is $\Lambda_\mathcal{O}$-torsion.*

4) *Identify the $\Lambda_{\mathcal{O}}$-determinants of the torsion modules*

$$H^2_0(\mathcal{O}_K[1/p], \Lambda(\eta)(1)) \text{ and } H^1(\mathcal{O}_K[1/p], \Lambda(\eta)(1))/\Lambda_{\mathcal{O}}\zeta(\eta)$$

with invertible submodules of the total quotient ring $Q(\Lambda_{\mathcal{O}})$. Then:

$$\mathrm{Det}_{\Lambda_{\mathcal{O}}}\left(H^2_0(\mathcal{O}_K[1/p], \Lambda(\eta)(1))\right) \subset \mathrm{Det}_{\Lambda_{\mathcal{O}}}\left(H^1(\mathcal{O}_K[1/p], \Lambda(\eta)(1))/\Lambda_{\mathcal{O}}\zeta(\eta)\right).$$

Proof. This is a consequence of the theory of Euler systems. We follow the exposition in Rubin, as this is closest to our setting. Let us begin by checking the hypothesis $\mathrm{Hyp}(K_\infty/K)$ and $\mathrm{Hyp}(K_\infty, \mathcal{O}_p(\eta)(1))$ in Rubin [Ru3] 2.3.3. This is clear for $\mathrm{Hyp}(K_\infty/K)$ as K is imaginary quadratic. For $\mathrm{Hyp}(K_\infty, \mathcal{O}_p(\eta)(1))$ we take $\tau = \mathrm{id}$. We also remark that it is clear from the definition that $H^2_0(\mathcal{O}_K[1/p], \Lambda(\eta)(1)) = X_\infty$ in Rubin's notation. As our element $\zeta(\eta)$ is non-torsion by 3.9, the Theorem 2.3.2 in [Ru3] implies that $H^2_0(\mathcal{O}_K[1/p], \Lambda(\eta)(1))$ is $\Lambda_{\mathcal{O}}$-torsion. As $H^2(K_v, \Lambda(\eta)(1))$ for $v \mid p$ is $\Lambda_{\mathcal{O}}$-torsion by lemma 4.6, it follows that $H^2(\mathcal{O}_K[1/p], \Lambda(\eta)(1))$ is $\Lambda_{\mathcal{O}}$-torsion as well, which shows 1). To show 2) note that $H^0(\mathcal{O}_K[1/p], \Lambda(\eta)(1)) = 0$ by (13). Then, 2) follows from the formula

$$\begin{aligned}\mathrm{rk}_{\Lambda_{\mathcal{O}}} H^1(\mathcal{O}_K[1/p], \Lambda(\eta)(1)) - \mathrm{rk}_{\Lambda_{\mathcal{O}}} H^2(\mathcal{O}_K[1/p], \Lambda(\eta)(1)) = \\ \mathrm{rk}_{\mathcal{O}_p} H^2(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta)(1)) - \mathrm{rk}_{\mathcal{O}_p} H^1(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta)(1)).\end{aligned}$$

(see [Ts] prop. 9.2. (3) for example) and

$$\mathrm{rk}_{\mathcal{O}_p} H^2(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta)(1)) - \mathrm{rk}_{\mathcal{O}_p} H^1(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta)(1)) = \mathrm{rk}_{\mathcal{O}_p} \mathcal{O}_p(\eta) = 1$$

(see [Ja] Lemma 2). As $\Lambda_{\mathcal{O}}(\zeta(\eta)) \subset H^1(\mathcal{O}_K[1/p], \Lambda(\eta)(1))$ is a non-torsion submodule by 3.9, we get 3). For the statement in 4) we have to consider

$$\mathrm{ind}_{\Lambda_{\mathcal{O}}}(\zeta(\eta)) := \{\phi(\zeta(\eta)) | \phi \in \mathrm{Hom}_{\Lambda_{\mathcal{O}}}(H^1(\mathcal{O}_K[1/p], \Lambda(\eta)(1)), \Lambda_{\mathcal{O}})$$

as defined by Rubin in [Ru3] p. 41. Our $H^1(\mathcal{O}_K[1/p], \Lambda(\eta)(1))$ is Rubin's $H^1_\infty(K, \mathcal{O}_p(\eta)(1))$ by [Ru3] Corollary B.3.5. By the structure theory of $\Lambda_{\mathcal{O}}$-modules, we can find a pseudo-isomorphism

$$H^1(\mathcal{O}_K[1/p], \Lambda(\eta)(1)) \to \Lambda_{\mathcal{O}} \oplus H^1(\mathcal{O}_K[1/p], \Lambda(\eta)(1))_{\mathrm{tors}}.$$

Let $\phi : H^1(\mathcal{O}_K[1/p], \Lambda(\eta)(1)) \to \Lambda_{\mathcal{O}}$ be the projection onto $\Lambda_{\mathcal{O}}$, then the kernel of ϕ is torsion and one has an exact sequence of $\Lambda_{\mathcal{O}}$-torsion modules

$$0 \to \ker\phi \to H^1(\mathcal{O}_K[1/p], \Lambda(\eta)(1))/\Lambda_{\mathcal{O}}\zeta(\eta) \to \Lambda_{\mathcal{O}}/\Lambda_{\mathcal{O}}\phi(\zeta(\eta)) \to 0.$$

This gives inside $Q(\Lambda_{\mathcal{O}})$, using $\mathrm{Det}^{-1}_{\Lambda_{\mathcal{O}}} \ker\phi \subset \Lambda_{\mathcal{O}}$,

$$\mathrm{Det}^{-1}_{\Lambda_{\mathcal{O}}}\left(H^1(\mathcal{O}_K[1/p], \Lambda(\eta)(1))/\Lambda_{\mathcal{O}}\zeta(\eta)\right) \subset \phi(\zeta(\eta))\Lambda_{\mathcal{O}} \subset \mathrm{ind}_{\Lambda_{\mathcal{O}}}(\zeta(\eta)).$$

Finally, theorem 2.3.3 in [Ru3] shows

$$\mathrm{ind}_{\Lambda}(\zeta(\eta)) \subset \mathrm{Det}^{-1}_{\Lambda_{\mathcal{O}}} H^2_0(\mathcal{O}_K[1/p], \Lambda(\eta)(1)),$$

which gives statement 4). □

Next, we strengthen the divisibility of theorem 6.4. For this, we need a lemma:

Corollary 6.5. *Let η be as in theorem 6.4. Under the isomorphism*

$$Q(\Lambda_{\mathcal{O}})\zeta(\eta) \simeq H^1(\mathcal{O}_K[1/p], \Lambda(\eta)(1)) \otimes_{\Lambda_{\mathcal{O}}} Q(\Lambda_{\mathcal{O}})$$

one has an inclusion of $\Lambda_{\mathcal{O}}$-modules

$$\mathrm{Det}_{\Lambda_{\mathcal{O}}}\left(H^2(\mathcal{O}_K[1/p], \Lambda(\eta)(1))\right) \subset \mathrm{Det}_{\Lambda_{\mathcal{O}}}\left(H^1(\mathcal{O}_K[1/p], \Lambda(\eta)(1))/\Lambda_{\mathcal{O}}\zeta(\eta)\right).$$

Proof. By definition of $H^2_0(\mathcal{O}_K[1/p], \Lambda(\eta)(1))$ and the Poitou-Tate sequence we have an exact sequence

$$\begin{aligned} 0 \to H^2_0(\mathcal{O}_K[1/p], \Lambda(\eta)(1)) \to H^2(\mathcal{O}_K[1/p], \Lambda(\eta)(1)) \to \\ \to \bigoplus_{v|p} H^2(K_v, \Lambda(\eta)(1)) \to H^3_c(\mathcal{O}_K[1/p], \Lambda(\eta)(1)) \to 0 \end{aligned}$$

By lemma 4.6 the modules $H^2(K_v, \Lambda(\eta)(1))$ and $H^3_c(\mathcal{O}_K[1/p], \Lambda(\eta)(1))$ are pseudo-null. It follows that inside $Q(\Lambda_{\mathcal{O}})$

$$\mathrm{Det}^{-1}_{\Lambda_{\mathcal{O}}} H^2_0(\mathcal{O}_K[1/p], \Lambda(\eta)(1)) = \mathrm{Det}^{-1}_{\Lambda_{\mathcal{O}}} H^2(\mathcal{O}_K[1/p], \Lambda(\eta)(1)).$$

This, together with the vanishing of $H^0(\mathcal{O}_K[1/p], \Lambda(\eta)(1))$ by (13) and the divisibility in theorem 6.4 gives the result. □

6.3. **Reduction to the Tamagawa number conjecture.** In this section we reduce the Λ-main-conjecture 5.1 to the Tamagawa number conjecture. In this section η is a character of conductor $\mathfrak{f}_\eta$ and level n, chosen so that $\mathcal{O}_p(\eta)$ is ramified at all $v \mid p$.

Observe that $\Lambda_{\mathcal{O}}$ is a product of regular local noetherian rings, so that we can use the functor Div from 1.3. Consider the inclusion map of perfect complexes

$$\kappa_\eta : \Lambda_{\mathcal{O}}\zeta(\eta) \to R\Gamma(\mathcal{O}_K[1/p], \Lambda(\eta)(1))[1], \tag{17}$$

induced by $\zeta(\eta) \in H^1(\mathcal{O}_K[1/p], \Lambda(\eta)(1))$. By theorem 6.4, this is an isomorphism after tensoring with $Q(\Lambda_{\mathcal{O}})$, hence κ_η is good as defined in 1.3 and we can consider $\mathrm{Div}(\kappa_\eta)$ on $\mathrm{Spec}\Lambda_{\mathcal{O}}$. Again by 6.4 the divisors $\mathrm{Div}(\kappa_\eta)$ are effective. The statement of the Λ-main-conjecture is that $\mathrm{Div}(\kappa_\eta) = 0$. Consider the augmentation map

$$\iota : \Lambda_{\mathcal{O}} \to \mathcal{O}_p,$$

We denote also by ι the induced map $\iota : \mathrm{Spec}\mathcal{O}_p \to \mathrm{Spec}\Lambda_{\mathcal{O}}$.

Lemma 6.6. *Let ι be as above, then $L\iota^*(\kappa_\eta)$ is the map induced by the inclusion $\zeta_K(\eta) \in H^1(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta)(1))$*

$$L\iota^*(\kappa_\eta) : \mathcal{O}_p\zeta_K(\eta) \to R\Gamma(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta)(1))[1].$$

To show that $\mathrm{Div}(\kappa_\eta) = 0$ it is sufficient to show that the divisor $\mathrm{Div}(L\iota^*(\kappa_\eta)) = 0$, i.e., that $\zeta_K(\eta)$ generates

$$\mathrm{Det}^{-1}_{\mathcal{O}_p} R\Gamma(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta)(1))$$

inside $\mathrm{Det}^{-1}_{E_p} R\Gamma(\mathcal{O}_K[1/p], M(\eta)_p(1))$.

Proof. The map $\iota : \Lambda_{\mathcal{O}} \to \mathcal{O}_p$ induces a map of $\mathrm{Gal}(\bar{K}/K)$-modules

$$\Lambda(\eta) \to \mathcal{O}_p(\eta)$$

and hence an isomorphism

$$L\iota^* R\Gamma(\mathcal{O}_K[1/p], \Lambda(\eta)(1)) \cong R\Gamma(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta)(1)).$$

Using the definition of $\zeta(\eta)$, we see that $L\iota^*(\kappa_\eta)$ is the map induced by the inclusion $\zeta_K(\eta) \in H^1(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta)(1))$

$$L\iota^*(\kappa_\eta) : \mathcal{O}_p\zeta_K(\eta) \to R\Gamma(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta)(1))[1].$$

Applying (1) to κ_η one gets

$$\mathrm{Det}_{\Lambda_{\mathcal{O}}}(\Lambda_{\mathcal{O}}\zeta(\eta))(\mathrm{Div}(\kappa_\eta)) = \mathrm{Det}_{\Lambda_{\mathcal{O}}}\left(R\Gamma(\mathcal{O}_K[1/p], \Lambda(\eta)(1))[1]\right).$$

With corollary 6.5 we get inside $Q(\Lambda_{\mathcal{O}})$

$$\Lambda_{\mathcal{O}}(\mathrm{Div}(\kappa_\eta)) \subset \Lambda_{\mathcal{O}},$$

where $\Lambda_{\mathcal{O}}(\mathrm{Div}(\kappa_\eta))$ is the line bundle associated to the divisor $\mathrm{Div}(\kappa_\eta)$. To show that $\mathrm{Div}(\kappa_\eta) = 0$ we have to show that this inclusion is an equality. By Nakayama's lemma this is the case, if $L\iota^*(\kappa_\eta)$ is an isomorphism. Combining this with the formula $\iota^*\mathrm{Div}(\kappa_\eta) = \mathrm{Div}(L\iota^*(\kappa_\eta)) = 0$ gives the desired result. □

6.4. **End of proof.** Recall that $K_n(\mathfrak{f}_\eta)$ is the compositum $K_nK(\mathfrak{f}_\eta)$ and that we defined $G_n(\mathfrak{f}) := \mathrm{Gal}(K_n(\mathfrak{f})/K)$. Application of 3.10 to $L = K_n(\mathfrak{f}_\eta)$ and $F = K_{n-1}(\mathfrak{f}_\eta)$ gives that

$$\bigotimes_\eta \zeta_K(\eta) \text{ generates } \bigotimes_\eta \mathrm{Det}^{-1}_{\mathcal{O}_p} R\Gamma(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta)^\vee(1)),$$

where the sums is taken over all $\eta \in \widehat{G}_n(\mathfrak{f}_\eta) \smallsetminus \widehat{G}_{n-1}(\mathfrak{f}_\eta)$. This implies that

$$\mathrm{Div}(\bigotimes_\eta L\iota^*(\kappa_\eta)) = \sum_\eta \mathrm{Div}(L\iota^*(\kappa_\eta)) = 0.$$

As all divisors in this sum are effective, they have to be all 0, that is $\mathrm{Div}(L\iota^*(\kappa_\eta)) = 0$, which means that $\zeta_K(\eta)$ generates

$$\mathrm{Det}^{-1}_{\mathcal{O}_p} R\Gamma(\mathcal{O}_K[1/p], \mathcal{O}_p(\eta)(1)).$$

This proves the Λ-main-conjecture.

7. Proof of the Ω-main-conjecture

The proof of the Ω-main-conjecture essentially reduces, using an observation of Burns and Greither, to the Λ-main-conjecture plus the conjecture 5.3, which, as we stress again, is a theorem in the case where p is split in K and p does not divide 6.

7.1. **Preliminary reductions.** Recall that $\Omega \cong \mathbb{Z}_p[\Delta][[S,T]]$.

Lemma 7.1. *Let $\mathcal{O}_p$ contain the values of all characters of Δ, then it suffices to prove 5.5 for $\Omega_{\mathcal{O}}$.*

Proof. Inside $\mathrm{Det}_\Omega H^1(\mathcal{O}_K[1/p], \Omega(1)) \otimes Q(\Omega)$ we have two Ω-modules $\mathrm{Det}_\Omega \Omega\zeta$ and $\mathrm{Det}_\Omega R\Gamma(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1))[1]$. To check that they are equal, we can make the faithfully flat base extension $\Omega \to \Omega_{\mathcal{O}}$. □

We assume now that $\mathcal{O}_p$ contains the values of the characters of Δ and collect some results about the ring $\Omega_{\mathcal{O}} \cong \mathcal{O}_p[\Delta][[S,T]]$.

Lemma 7.2. *The normalization $\widetilde{\Omega}_{\mathcal{O}}$ of $\Omega_{\mathcal{O}}$ inside of $Q(\Omega_{\mathcal{O}})$ is given by*

$$\widetilde{\Omega}_{\mathcal{O}} \cong \prod_{\chi \in \widehat{\Delta}} \Lambda(\chi).$$

In particular, $\Omega_{\mathcal{O}} \otimes_{\mathcal{O}_p} E_p \cong \widetilde{\Omega}_{\mathcal{O}} \otimes_{\mathcal{O}_p} E_p$.

Proof. This follows from the fact that $\Omega_{\mathcal{O}} \subset \prod_{\chi \in \widehat{\Delta}} \Lambda(\chi)$ and that the later ring is normal. □

We can now prove the first part of the equivariant main conjecture 5.5

Corollary 7.3. *The module $H^2(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1))$ is an $\Omega_{\mathcal{O}}$-torsion module and $H^1(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1))$ has $\Omega_{\mathcal{O}}$-rank one.*

Proof. It follows from

$$R\Gamma(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1)) \otimes^{\mathbb{L}}_{\Omega_{\mathcal{O}}} \widetilde{\Omega}_{\mathcal{O}} \cong R\Gamma(\mathcal{O}_K[1/p], \widetilde{\Omega}_{\mathcal{O}}(1)) \tag{18}$$

and the fact $H^0(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1)) = 0$ (by (13)) that

$$H^2(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1)) \otimes_{\Omega_{\mathcal{O}}} \widetilde{\Omega}_{\mathcal{O}} \cong H^2(\mathcal{O}_K[1/p], \widetilde{\Omega}_{\mathcal{O}}(1)) \cong \prod_{\chi \in \widehat{\Delta}} H^2(\mathcal{O}_K[1/p], \Lambda(\chi)(1)).$$

is a torsion $\widetilde{\Omega}_{\mathcal{O}}$-module by 5.1. As $Q(\Omega_{\mathcal{O}}) = Q(\widetilde{\Omega}_{\mathcal{O}})$, it follows that

$$H^2(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1)) \otimes_{\Omega_{\mathcal{O}}} Q(\Omega_{\mathcal{O}}) = 0,$$

which proves that $H^2(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1))$ is $\Omega_{\mathcal{O}}$-torsion. Moreover, one gets from (18) an exact sequence

$$\mathrm{Tor}_2^{\Omega_{\mathcal{O}}}(H^2(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1)), \widetilde{\Omega}_{\mathcal{O}}) \to H^1(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1)) \otimes_{\Omega_{\mathcal{O}}} \widetilde{\Omega}_{\mathcal{O}} \to H^1(\mathcal{O}_K[1/p], \widetilde{\Omega}_{\mathcal{O}}(1)).$$

As $H^2(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1))$ is a torsion module and $H^1(\mathcal{O}_K[1/p], \widetilde{\Omega}_{\mathcal{O}}(1))$ has $Q(\Omega_{\mathcal{O}})$-rank one by 7.2 and 5.1, we get the result. □

To prove the rest of the equivariant main conjecture 5.5 we want to use the following lemma taken from Flach [Fl] (recall that $\Lambda_{\mathcal{O}}$ and $\Omega_{\mathcal{O}}$ are products of local rings):

Lemma 7.4 ([Fl] 5.3)**.** *Let $R = \Lambda_{\mathcal{O}}$ or $R = \Omega_{\mathcal{O}}$ and $Q(R)$ be the total quotient ring. Let M and N be two invertible R-submodules of some invertible $Q(R)$-module D, then $M = N$ if and only if for all height 1 prime ideals $\mathfrak{q}$ of R one has $M_{\mathfrak{q}} = N_{\mathfrak{q}}$ inside $D_{\mathfrak{q}}$.*

Inside $H^1(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1)) \otimes_{\Omega_{\mathcal{O}}} Q(\Omega_{\mathcal{O}})$ we have two rank one $\Omega_{\mathcal{O}}$-modules:

$$\mathrm{Det}_{\Omega_{\mathcal{O}}} \Omega_{\mathcal{O}} \zeta$$

and

$$\mathrm{Det}^{-1}_{\Omega_{\mathcal{O}}} R\Gamma(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1)).$$

To show that these are equal we can by 7.4 localize at all height one primes of $\Omega_{\mathcal{O}}$. We distinguish two cases following Burns and Greither:

Definition 7.5. A prime ideal $\mathfrak{q} \subset \Omega_{\mathcal{O}}$ of height one is called *regular* if $p \notin \mathfrak{q}$. If $p \in \mathfrak{q}$, the prime ideal is called *singular.*

The proof of the Ω-main-conjecture in these two cases is given in the next two sections.

7.2. **Proof for regular prime ideals.** First note the following consequence of lemma 7.2:

Lemma 7.6. *Let $\mathfrak{q} \subset \Omega_{\mathcal{O}}$ be a regular prime ideal of height one, then*

$$(\Omega_{\mathcal{O}})_{\mathfrak{q}} \cong (\widetilde{\Omega}_{\mathcal{O}})_{\mathfrak{q}} \cong \prod_{\chi \in \widehat{\Delta}} \Lambda(\chi)_{\mathfrak{q}}.$$

Proof. As p is invertible in $(\Omega_{\mathcal{O}})_{\mathfrak{q}}$ both rings are localizations of $\Omega_{\mathcal{O}} \otimes_{\mathcal{O}_p} E_p$ resp. $\widetilde{\Omega}_{\mathcal{O}} \otimes_{\mathcal{O}_p} E_p$, which agree by lemma 7.2. □

It follows that for regular $\mathfrak{q}$

$$R\Gamma(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1))_{\mathfrak{q}} \cong R\Gamma(\mathcal{O}_K[1/p], \widetilde{\Omega}_{\mathcal{O}}(1))_{\mathfrak{q}} \cong \prod_{\chi \in \widehat{\Delta}} R\Gamma(\mathcal{O}_K[1/p], \Lambda(\chi)(1))_{\mathfrak{q}}.$$

Using lemma 3.7 and 7.2 we have

$$(\Omega_{\mathcal{O}})_{\mathfrak{q}} \zeta \cong \prod_{\chi \in \widehat{\Delta}} (\Lambda_{\mathcal{O}})_{\mathfrak{q}} \zeta(\chi).$$

Taking determinants, theorem 5.1 implies that

$$\mathrm{Det}_{(\Omega_{\mathcal{O}})_{\mathfrak{q}}} (\Omega_{\mathcal{O}})_{\mathfrak{q}} \zeta = \mathrm{Det}^{-1}_{(\Omega_{\mathcal{O}})_{\mathfrak{q}}} R\Gamma(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1))_{\mathfrak{q}}$$

inside of $H^1(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1)) \otimes_{\Omega_{\mathcal{O}}} Q(\Omega_{\mathcal{O}})$. This proves the Ω-main-conjecture for regular prime ideals.

7.3. **Proof for singular prime ideals.** Let $\mathfrak{q} \subset \Omega_{\mathcal{O}}$ be a singular prime ideal (i.e., $p \in \mathfrak{q}$). Then by our assumption $H^2(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1))_{\mathfrak{q}} = 0$ and we get

$$\operatorname{Det}^{-1}_{(\Omega_{\mathcal{O}})_{\mathfrak{q}}} R\Gamma(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1))_{\mathfrak{q}} = \operatorname{Det}_{(\Omega_{\mathcal{O}})_{\mathfrak{q}}} H^1(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1))_{\mathfrak{q}}.$$

As $(\Omega_{\mathcal{O}})_{\mathfrak{q}}\zeta \subset H^1(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1))_{\mathfrak{q}}$ we get inside $H^1(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1)) \otimes_{\Omega_{\mathcal{O}}} Q(\Omega_{\mathcal{O}})$ an inclusion of two free $(\Omega_{\mathcal{O}})_{\mathfrak{q}}$-modules of rank one:

$$\operatorname{Det}_{(\Omega_{\mathcal{O}})_{\mathfrak{q}}} (\Omega_{\mathcal{O}})_{\mathfrak{q}}\zeta \subset \operatorname{Det}_{(\Omega_{\mathcal{O}})_{\mathfrak{q}}} H^1(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1))_{\mathfrak{q}}. \tag{19}$$

We now use an idea of Witte. Choosing generators for both modules, we see that there is an element $u \in (\Omega_{\mathcal{O}})_{\mathfrak{q}}$ such that

$$\operatorname{Det}_{(\Omega_{\mathcal{O}})_{\mathfrak{q}}} (\Omega_{\mathcal{O}})_{\mathfrak{q}}\zeta = u\operatorname{Det}_{(\Omega_{\mathcal{O}})_{\mathfrak{q}}} H^1(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1))_{\mathfrak{q}}.$$

We want to show that u is a unit in $(\Omega_{\mathcal{O}})_{\mathfrak{q}}$. Consider the normal ring homomorphism $(\Omega_{\mathcal{O}})_{\mathfrak{q}} \to (\widetilde{\Omega}_{\mathcal{O}})_{\mathfrak{q}}$. An element $u \in (\Omega_{\mathcal{O}})_{\mathfrak{q}}$ is a unit if and only if it is a unit in $(\widetilde{\Omega}_{\mathcal{O}})_{\mathfrak{q}}$. Thus it suffices to show that after extending scalars in (19) we get an equality. For this consider $\rho : \Omega_{\mathcal{O}} \to \widetilde{\Omega}_{\mathcal{O}}$ and the map of perfect complexes

$$\kappa : \Omega_{\mathcal{O}}\zeta \to R\Gamma(\mathcal{O}_K[1/p], \Omega_{\mathcal{O}}(1))[1].$$

Then κ is good in the sense of section 1.3 and for all $y \in \operatorname{Spec}\widetilde{\Omega}_{\mathcal{O}}$ of depth 0 the map $\kappa_{\rho(y)}$ is an isomorphism, so that we can apply the results in 1.3.

Lemma 7.7. *Let $\rho : \operatorname{Spec}\widetilde{\Omega}_{\mathcal{O}} \to \operatorname{Spec}\Omega_{\mathcal{O}}$, then the map*

$$L\rho^*(\kappa) : \widetilde{\Omega}_{\mathcal{O}}\zeta \to R\Gamma(\mathcal{O}_K[1/p], \widetilde{\Omega}_{\mathcal{O}}(1))[1]$$

induces an isomorphism

$$\operatorname{Det}_{\widetilde{\Omega}_{\mathcal{O}}} \widetilde{\Omega}_{\mathcal{O}}\zeta \cong \operatorname{Det}^{-1}_{\widetilde{\Omega}_{\mathcal{O}}} R\Gamma(\mathcal{O}_K[1/p], \widetilde{\Omega}_{\mathcal{O}}(1)).$$

Proof. By 3.7 and 7.2, we have

$$(\widetilde{\Omega}_{\mathcal{O}})_{\mathfrak{q}}\zeta \cong \prod_{\chi \in \widehat{\Delta}} (\Lambda_{\mathcal{O}})_{\mathfrak{q}}\zeta(\chi).$$

Moreover,

$$R\Gamma(\mathcal{O}_K[1/p], \widetilde{\Omega}_{\mathcal{O}}(1)) \cong \prod_{\chi \in \widehat{\Delta}} R\Gamma(\mathcal{O}_K[1/p], \Lambda_{\mathcal{O}}(\chi)(1))$$

and the claim follows from theorem 5.1. □

This lemma shows that for singular prime ideals $\mathfrak{q}$ the extension of coefficients by ρ in (19) gives an equality

$$\operatorname{Det}^{-1}_{(\widetilde{\Omega}_{\mathcal{O}})_{\mathfrak{q}}} (\widetilde{\Omega}_{\mathcal{O}})_{\mathfrak{q}}\zeta = \operatorname{Det}_{(\widetilde{\Omega}_{\mathcal{O}})_{\mathfrak{q}}} H^1(\mathcal{O}_K[1/p], \widetilde{\Omega}_{\mathcal{O}}(1))_{\mathfrak{q}}.$$

This shows that the element $u \in (\Omega_{\mathcal{O}})_{\mathfrak{q}}$ becomes a unit in $(\widetilde{\Omega}_{\mathcal{O}})_{\mathfrak{q}}$. Thus u is already a unit in $(\Omega_{\mathcal{O}})_{\mathfrak{q}}$ and we get equality in (19), which proves the Ω-main-conjecture for singular prime ideals.

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