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Abstract

We give an explicit locally analytic cocycle which – composed with the Hurewicz map from algebraic K-theory to group homology – gives the p-adic Borel regulator defined by Annette Huber and Guido Kings for the K-theory of a p-adic field. Using this explicit cocycle we show that the p-adic Borel regulator equals Karoubi's regulator up to a constant.

Contents

Introduction

Let K be a finite extension of $\mathbf{Q}_p$ with valuation ring R. In [4] Annette Huber and Guido Kings introduce a regulator $b_p : K_{2n-1}(R) \to K$ and relate it via the Bloch-Kato exponential map to Soulé's regulator $K_{2n-1}(K) \to H^1(K, \mathbf{Q}_p(n))$. The definition of b_p parallels Borel's construction of his regulator $K_{2n-1}(\mathbf{C}) \to \mathbf{C}$, only that the van Est isomorphism is replaced by the Lazard isomorphism

$H^{2n-1}_{\mathrm{la}}(\mathrm{GL}_N(R), K) \simeq H^{2n-1}(\mathfrak{gl}_N, K)$ between locally analytic group cohomology and Lie algebra cohomology. Huber and Kings describe explicitely an element p_n in the Lie algebra cohomology and by definition the p-adic Borel regulator b_p is the composition of the Hurewicz map from K-theory to group homology and the preimage of p_n under Lazard's isomorphism.

There is another construction of a p-adic regulator from the relative K-theory of K to K due to Max Karoubi [5] which uses the Chern character $\mathrm{ch}^{\mathrm{rel}}_{2n-1} : K^{\mathrm{rel}}_{2n-1}(K) \to HC_{2n-2}(K)$ from relative K-theory to topological cyclic homology and the periodicity operator $HC_{2n-2}(K) \to HC_0(K) = K$. There is a canonical isomorphism $K^{\mathrm{rel}}_{2n-1}(K)_{\mathbf{Q}} \simeq K_{2n-1}(K)_{\mathbf{Q}}$ which allows us to compare both regulators. Karoubi's regulator also factors through the Hurewicz map to the rational homology of a certain simplicial set and Nadia Hamida [3] described an explicit simplicial cocycle which gives this regulator.

In the present paper we show that Hamida's cocycle induces a locally analytic cocycle for the group $\mathrm{GL}_N(R)$ and prove that this cocycle equals the one defining the p-adic Borel regulator up to a constant.

In the first section we shortly sketch Karoubi's construction and recall the relevant results of Hamida. In the second section we recall the construction of the p-adic Borel regulator and construct a locally analytic cocycle similar to Hamida's simplicial cocycle which gives the p-adic Borel regulator. In the third section we finally prove that both regulators agree up to a constant under the identifications $K^{\mathrm{rel}}_{2n-1}(K)_{\mathbf{Q}} \simeq K_{2n-1}(K)_{\mathbf{Q}} \simeq K_{2n-1}(R)_{\mathbf{Q}}$. Since the explicit cocycles involve the integration of a p-adic differential form over the standard simplex, we have included an appendix where the technical questions of integration are discussed.

1 Karoubi's p-adic regulator

1.1 Relative K-theory

Let A be an ultrametric Banach ring (cf. [1]), i.e. a ring A equipped with a ultrametric"quasi-norm" $\|.\| : A \to \mathbf{R}_+$ verifying $\|a\| = 0 \iff a = 0$, $\|a\| = \|-a\|$, $\|ab\| \leq \|a\|\|b\|$, $\|a+b\| \leq \max\{\|a\|, \|b\|\}$, for which it is complete. Let $A\langle x_0, \dots, x_n\rangle$ denote the ring of power series $\sum_{I \in \mathbf{N}_0^{n+1}} a_I x^I$ with $a_I \in A$, $x = (x_0, \dots, x_n)$ and $\|a_I\||I|^r \xrightarrow{|I|\to\infty} 0$ for every $r >\in \mathbf{N}_0$. We call it the ring of *indefinitely integrable power series with coefficients in A*. $A\langle x_0, \dots, x_n\rangle$ is an ultrametric Fréchet ring where the topology is given by the family of seminorms p_r, $r \in \mathbf{N}_0$, $p_r(\sum a_I x^I) = \sup_I \|a_I\| \cdot |I|^r$ (see the appendix for details). Let $I_n \subset A\langle x_0, \dots, x_n\rangle$ be the principal ideal generated by $x_0 + \dots + x_n - 1$ and define $A_n := A\langle x_0, \dots, x_n\rangle / I_n$.

A_* defines a simplicial ring with faces ∂_i and degeneracies s_i induced by

$$\partial_i(x_j) = \begin{cases} x_j & \text{if } j < i, \\ 0 & \text{if } j = i, \\ x_{j-1} & \text{if } j > i, \end{cases} \qquad s_i(x_j) = \begin{cases} x_j & \text{if } j < i, \\ x_i + x_{i+1} & \text{if } j = i, \\ x_{j+1} & \text{if } j > i. \end{cases}$$

The classifying space $\mathrm{BGL}(A_*)$ of the simplicial group $\mathrm{GL}(A_*)$ is an H-space and by [6] 6.17 the natural map $\mathrm{BGL}(A) \to \mathrm{BGL}(A_*)$ induces a homotopy fibre sequence

$$(\mathrm{GL}(A_*)/\mathrm{GL}(A))^+ \xrightarrow{\theta} \mathrm{BGL}(A)^+ \to \mathrm{BGL}(A_*)$$

where $(.)^+$ is Quillen's $+$-construction and θ is induced by the map $\mathrm{GL}(A_n) \ni \sigma \mapsto (\sigma(0)\sigma(1)^{-1}, \ldots, \sigma(n-1)\sigma(n)^{-1}) \in B_n\mathrm{GL}(A) = \mathrm{GL}(A)^{\times n}$. Here $\sigma(i) = (i)^*\sigma$ where $(i) : [0] \to [n]$ is the morphism in the simplicial category that sends $0 \in [0]$ to $i \in [n]$ ("the value of σ on the i^{th} vertex of the standard simplex").
The topological and relative K-groups of A are by definition (cf. [5], [7])

$$K_n^{\mathrm{top}}(A) := \pi_n(\mathrm{BGL}(A_*)) \quad \text{and} \quad K_n^{\mathrm{rel}}(A) := \pi_n\left((\mathrm{GL}(A_*)/\mathrm{GL}(A))^+\right).$$

In particular there are long exact sequences

$$\cdots \to K_n^{\mathrm{rel}}(A) \to K_n(A) \to K_n^{\mathrm{top}}(A) \to K_{n-1}^{\mathrm{rel}}(A) \to \ldots.$$

We are particularly interested in the case where $A = K$ is a finite extension of $\mathbf{Q}_p$ with residue field k. In this situation Adina Calvo ([1]) shows that one has exact sequences

$$0 \to K_n(k) \to K_n^{\mathrm{top}}(K) \to K_{n-1}(k) \to 0.$$

By Quillen's result on the K-groups of finite fields $K_{2i+1}^{\mathrm{top}}(K) = K_{2i+2}^{\mathrm{top}}(K) = K_{2i+1}(k)$ is finite for $i > 0$. In particular the canonical map $K_n^{\mathrm{rel}}(K)_{\mathbf{Q}} \to K_n(K)_{\mathbf{Q}}$ is an isomorphism for $n > 2$ (here we write $K_n^{\mathrm{rel}}(K)_{\mathbf{Q}}$ for $K_n^{\mathrm{rel}}(K) \otimes_{\mathbf{Z}} \mathbf{Q}$ etc.). Furthermore if R denotes the ring of interges in K there are canonical isomorphisms $K_n^{\mathrm{top}}(R) \xrightarrow{\simeq} K_n^{\mathrm{top}}(k) \simeq K_n(k)$. The localization sequence in algebraic K-theory yields isomorphisms $K_n(R)_{\mathbf{Q}} \to K_n(K)_{\mathbf{Q}}$ for $n > 2$. Thus we have canonical isomorphisms

$$\begin{tikzcd}
K_n^{\mathrm{rel}}(R)_{\mathbf{Q}} \arrow[r, "\simeq"] \arrow[d, "\simeq"] & K_n(R)_{\mathbf{Q}} \arrow[d, "\simeq"] \\
K_n^{\mathrm{rel}}(K)_{\mathbf{Q}} \arrow[r, "\simeq"] & K_n(K)_{\mathbf{Q}}.
\end{tikzcd}$$

1.2 The regulator

We sketch the construction of Karoubi's p-adic regulator as in [3]. More details may be found in [2].
Let K be a finite extension of $\mathbf{Q}_p$ with ring of integers R and uniformizing parameter π. Let $\left(C_*^{\lambda\mathrm{top}}(K), b\right)$ denote the complex defining the topological cyclic homology (with ground ring $\mathbf{Q}$) $HC_*^{\mathrm{top}}(K)$ of K (see e.g. [6], [5]). Karoubi has constructed a relative Chern character $\mathrm{ch}_{2n-1}^{\mathrm{rel}} : K_{2n-1}^{\mathrm{rel}}(K) \to C_{2n-2}^{\lambda\mathrm{top}}(K)/b\left(C_{2n-1}^{\lambda\mathrm{top}}(K)\right)$ and Hamida proves that its image is contained in the subgroup $HC_{2n-2}^{\mathrm{top}}(K)$ of $C_{2n-2}^{\lambda\mathrm{top}}(K)/b\left(C_{2n-1}^{\lambda\mathrm{top}}(K)\right)$ ([3] Def.-Prop. 2.1.2).

Definition. The p-adic regulator r_p is defined to be the composition

$$r_p : K^{\mathrm{rel}}_{2n-1}(K) \xrightarrow{\mathrm{ch}^{\mathrm{rel}}_{2n-1}} HC^{\mathrm{top}}_{2n-2}(K) \xrightarrow{S} HC^{\mathrm{top}}_0(K) = K,$$

where S is the $(n-1)$-fold iterate of Connes' periodicity operator.

In order to compare the above regulator with the p-adic Borel regulator we need the explicit description of r_p given by Hamida which uses Goodwillie's relative K-theory. We recall the relevant definitions and facts from [8] ch. 11. For a ring A and a two-sided ideal I in A denote by $\mathcal{K}(A,I)$ the connected component of the basepoint of the homotopy fiber of $\mathrm{BGL}(A)^+ \to \mathrm{BGL}(A/I)^+$. For $n \geq 1$, $K_n(A,I)$ is defined to be $\pi_n(\mathcal{K}(A,I))$. The space $\mathcal{K}(A,I)$ has a Volodin model $X(A,I)$ constructed as follows: For any ordering γ of $\{1,\dots,n\}$ define $T^\gamma_n(A,I)$ to be the subgroup $\{1+(a_{ij}) \in \mathrm{GL}_n(A) \mid a_{ij} \in I \text{ if } i \overset{\gamma}{\not<} j\}$ of $\mathrm{GL}_n(A)$. Then $X(A,I)$ is the union of classifying spaces $\bigcup_{n,\gamma} BT^\gamma_n(A,I)$ in $\mathrm{BGL}(A)$. We consider $X(A,I)$ also as a simplicial subset of $\mathrm{BGL}(A)$.

Proposition 1.1 ([8] Prop. 11.3.6, Cor. 11.3.8). *There is a natural homotopy equivalence $X(A,I)^+ \approx \mathcal{K}(A,I)$. In particular $K_n(A,I) = \pi_n(X(A,I)^+)$. Moreover the direct sum of matrices induces an H-space structure on $X(A,I)^+$ so that $K_n(A,I)_{\mathbf{Q}}$ is isomorphic to the primitive part* $\mathrm{Prim}H_n(X(A,I),\mathbf{Q})$ *of the rational homology of $X(A,I)$ via the Hurewicz homomorphism.*

Now let $A = R$ and $I = \pi R$ the maximal ideal of R. Hamida proves the following

Proposition 1.2 ([3] Thm. 1.3). *There exists an isomorphism*

$$K_n(R,\pi R) \xrightarrow{\simeq} K^{\mathrm{rel}}_n(R).$$

It is induced by the simplicial map $\varphi : X(R,\pi R) \to \mathrm{GL}(R_*)/\mathrm{GL}(R)$ that sends $(g_1,\dots,g_r) \in T^\gamma(R,\pi R)^{\times r} \subset X_r(R,\pi R)$ to $\sum_{i=0}^r x_i g_{i+1}\cdots g_r \in \mathrm{GL}(R_r)$.

It is not a priori clear that $\sum_{i=0}^r x_i g_{i+1}\cdots g_r$ is invertible in $\mathrm{Mat}(R_r)$. For a proof of a similar statement see lemma 2.3. Now the explicit description of the regulator r_p is the following

Proposition 1.3 ([3] Prop 2.1.3). *The composition*

$$K_{2n-1}(R,\pi R) \underset{\simeq}{\xrightarrow{\varphi}} K^{\mathrm{rel}}_{2n-1}(R) \to K^{\mathrm{rel}}_{2n-1}(K) \xrightarrow{r_p} K$$

is equal to the composition

$$K_{2n-1}(R,\pi R) \to H_{2n-1}(X(R,\pi R)) \xrightarrow{\phi} K,$$

where the first arrow is the Hurewicz map and ϕ is given by the simplicial cocycle that sends $(g_1,\dots,g_{2n-1}) \in T^\gamma(R,\pi R)^{\times(2n-1)}$ to

$$\frac{(-1)^n(n-1)!}{(2n-1)!(2n-2)!}\mathrm{Tr}\int_{\Delta^{2n-1}} (d\nu\cdot\nu^{-1})^{2n-1} \in K$$

with $\nu = \sum_{i=0}^{2n-1} x_i g_{i+1}\cdots g_{2n-1} \in \mathrm{GL}(R_{2n-1})$.

See the appendix for the definition of the integral.

2 An explicit description of the p-adic Borel regulator

2.1 The construction of the p-adic Borel regulator

We recall the construction of the p-adic Borel regulator by Annette Huber and Guido Kings in [4]. Let $K/\mathbf{Q}_p$ be a finite extension with ring of integers R and uniformizing parameter π. Define $U_N(R) = 1 + \pi \mathrm{Mat}_N(R) \subset \mathrm{GL}_N(R)$ and let $\mathfrak{gl}_N$ denote the K-Lie algebra of GL_N. For $n \leq N$ the map $\bigwedge^{2n-1} \mathfrak{gl}_N \to K$,

$$(X_1, \dots, X_{2n-1}) \mapsto \frac{((n-1)!)^2}{(2n-1)!} \sum_{\sigma \in \mathfrak{S}_{2n-1}} \mathrm{sgn}(\sigma) \mathrm{Tr}(X_{\sigma(1)} \cdots X_{\sigma(2n-1)}),$$

where $\mathfrak{S}_{2n-1}$ is the symmetric group, Tr ist the trace map $\mathfrak{gl}_N \to K$ and $\cdot$ is matrix multiplication, defines the primitive element $p_n \in H^{2n-1}(\mathfrak{gl}_N, K)$.
In [4] Huber and Kings prove the following version of Lazard's theorem:

Theorem 2.1. *There are natural isomorphisms*

$$H^{2n-1}_{\mathrm{la}}(\mathrm{GL}_N(R), K) \xrightarrow{\simeq} H^{2n-1}_{\mathrm{la}}(U_N(R), K) \xrightarrow{\simeq} H^{2n-1}(\mathfrak{gl}_N, K)$$

between the locally analytic group cohomology and the Lie algebra cohomology.

They also give an explicit description of this isomorphism which will be recalled in the next section.
Let b_p be the image of p_n under the composition

$$H^{2n-1}(\mathfrak{gl}_N, K) \xleftarrow{\simeq} H^{2n-1}_{\mathrm{la}}(\mathrm{GL}_N(R), K) \xrightarrow{\text{“forget la”}} H^{2n-1}(\mathrm{GL}_N(R), K).$$

Definition. The p-adic Borel regulator

$$K_{2n-1}(R) \to K$$

is defined to be the composition

$$K_{2n-1}(R) \xrightarrow{\text{Hurewicz}} H_{2n-1}(\mathrm{GL}(R), \mathbf{Q}) \simeq H_{2n-1}(\mathrm{GL}_N(R), \mathbf{Q}) \xrightarrow{b_p} K.$$

Here one uses the fact that the canonical homomorphism $H_{2n-1}(\mathrm{GL}_N(R), \mathbf{Q}) \to H_{2n-1}(\mathrm{GL}(R), \mathbf{Q})$ is an isomorphism if N is big enough (depending on n).

2.2 The Lazard isomorphism

Let $G = \mathrm{GL}_N(R)$ considered as a K-Lie group with unit element e and Lie algebra $\mathfrak{gl}_N$. By [4], section 5, the Lazard isomorphism $H^n_{\mathrm{la}}(G, K) \xrightarrow{\simeq} H^n(\mathfrak{gl}_N, K)$ is induced by the map

$$\Phi : \mathcal{O}^{\mathrm{la}}(G^{\times n}) \simeq \mathcal{O}^{\mathrm{la}}(G)^{\hat{\otimes} n} \to \bigwedge^{n} \mathfrak{gl}_N^{\vee}, \quad f_1 \otimes \cdots \otimes f_n \mapsto df_1(e) \wedge \cdots \wedge df_n(e),$$

where $\mathcal{O}^{\mathrm{la}}$ denotes the ring of locally analytic functions and $df(e)$ is the differential of f at e.

Now let exp be the exponential map of G defined on a neighbourhood of zero in $\mathfrak{gl}_N$. For a locally analytic function $f \in \mathcal{O}^{\mathrm{la}}(G^{\times n})$ we define $\Delta f \in \bigwedge^n \mathfrak{gl}_N^\vee$ by

$$\Delta f(X_1, \ldots, X_n) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \frac{d^n}{dt_1 \ldots dt_n} f(\exp(t_1 X_{\sigma 1}), \ldots, \exp(t_n X_{\sigma n}))\Big|_{t=0}$$

If f is of the special form $f = f_1 \otimes \cdots \otimes f_n$ one has

$$\begin{aligned} &\frac{d}{dt_i} f(\exp(t_1 X_{\sigma 1}), \ldots, \exp(t_n X_{\sigma n}))\Big|_{t_i=0} = \\ &\qquad = f_1(\exp(t_1 X_{\sigma 1})) \ldots df_i(e)(X_{\sigma i}) \ldots f_n(\exp(t_n X_{\sigma n})) \end{aligned}$$

and therefore

$$\begin{aligned} \Delta f(X_1, \ldots, X_n) &= \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) df_1(e)(X_{\sigma 1}) \cdots df_n(e)(X_{\sigma n}) \\ &= df_1(e) \wedge \cdots \wedge df_n(e)(X_1, \ldots, X_n) = \Phi(f)(X_1, \ldots, X_n). \end{aligned}$$

Since the functions of the form $f_1 \otimes \cdots \otimes f_n$ are topological generators of $\mathcal{O}^{\mathrm{la}}(G^{\times n})$ and Φ and Δ are continuous for the Fréchet topology on $\mathcal{O}^{\mathrm{la}}(G^{\times n})$ we have proven

Proposition 2.2. *The Lazard isomorphism* $H^n_{\mathrm{la}}(\mathrm{GL}_N(R), K) \xrightarrow{\simeq} H^n(\mathfrak{gl}_N, K)$ *is induced by* $\Delta : \mathcal{O}^{\mathrm{la}}(G^{\times n}) \to \bigwedge^n \mathfrak{gl}_N^\vee$.

2.3 An explicit cocycle

Recall the ring $R_n = R\langle x_0, \ldots, x_n\rangle / (\sum x_i - 1)$ from section 1.1.

Lemma 2.3. *Let* $g_1, \ldots, g_n$ *be elements of* $U_N(R)$*. Then* $\nu = \sum_{i=0}^n x_i g_{i+1} \cdots g_n$ *is invertible, i.e. lies in* $\mathrm{GL}_N(R_n)$.

Proof. Write $g_{i+1} \cdots g_n = 1 - h_i$ with $h_i \in \pi \mathrm{Mat}_N(R)$. Then $\nu = \sum_{i=0}^n x_i - \sum_{i=0}^n x_i h_i = 1 - \sum_{i=0}^n x_i h_i =: 1 - h$. We show that $\sum_{k \in \mathbf{N}} h^k$ converges in $\mathrm{Mat}_N(R\langle x_0, \ldots, x_n\rangle)$. Its image in $\mathrm{Mat}_N(R_n)$ will be an inverse of ν. Let $p_r, r \in \mathbf{N}_0$, be the family of seminorms defining the Fréchet topology on $R\langle x_0, \ldots, x_n\rangle$ and extend p_r in the obvious way to matrices. Since h^k is homogeneous of degree k we have

$$p_r(h^k) \leq k^r \max_{|I|=k} \|h_0\|^{i_0} \cdots \|h_n\|^{i_n} \leq k^r c^k$$

where $c := \max_{i=0,\ldots,n} \|h_i\| < 1$ and I runs through multiindices in $\mathbf{N}_0^{n+1}$. But $k^r c^k$ tends to zero as k tends to infinity and so $\sum_{k \in \mathbf{N}} h^k$ indeed converges to an element of $\mathrm{Mat}_N(R\langle x_0, \ldots, x_n\rangle)$. ☐

Since the p-adic Borel regulator $b_p \in H^{2n-1}(\mathrm{GL}(R), K)$ is the image of a locally analytic cocycle and $H^{2n-1}_{\mathrm{la}}(\mathrm{GL}_N(R), K) \simeq H^{2n-1}_{\mathrm{la}}(U_N(R), K)$, b_p is determined by a locally analytic cocycle $U_N(R)^{\times(2n-1)} \to K$.

Theorem 2.4. *The p-adic Borel regulator b_p is given by the locally analytic cocycle $f : U_N(R)^{\times(2n-1)} \to K$,*

$$f(g_1, \dots, g_{2n-1}) = -\frac{((n-1)!)^2}{(2n-1)!} \operatorname{Tr} \int_{\Delta^{2n-1}} (d\nu \cdot \nu^{-1})^{2n-1}$$

where $\nu = \nu(g_1, \dots, g_{2n-1}) = \sum_{i=0}^{2n-1} x_i g_{i+1} \cdots g_{2n-1} \in \mathrm{GL}_N(R_{2n-1})$.

Proof. The fact that f is locally analytic is proven in the appendix (proposition A.8). For the proof that f indeed defines a cocycle cf. [2], Prop. II 3.3.1.
We have to show that f is mapped to the primitive element p_n of section 2.1 under the Lazard isomorphism, i.e. $\Delta(f) = p_n$. Write ∂_i instead of $\frac{d}{dt_i}$. We have

$$\Delta(f)(X_1, \dots, X_{2n-1}) = -\frac{((n-1)!)^2}{(2n-1)!} \sum_{\sigma \in \mathfrak{S}_{2n-1}} \operatorname{sgn}(\sigma) \partial_1 \dots \partial_{2n-1} \Big|_{t_1 = \dots = 0}$$
$$\operatorname{Tr} \int_{\Delta^{2n-1}} (d\nu \cdot \nu^{-1})^{2n-1} (\exp(t_1 X_{\sigma 1}), \dots, \exp(t_{2n-1} X_{\sigma(2n-1)})).$$

By proposition A.7 we may interchange differentiation and integration. Let us first consider the $\sigma = 1$ summand. Write

$$\begin{aligned} \omega &:= \left[\sum_{i=0}^{2n-1} dx_i \exp(t_{i+1} X_{i+1}) \cdots \exp(t_{2n-1} X_{2n-1}) \right], \\ \omega' &:= \left[\sum_{i=0}^{2n-1} x_i \exp(t_{i+1} X_{i+1}) \cdots \exp(t_{2n-1} X_{2n-1}) \right]. \end{aligned}$$

Then

$$(d\nu \cdot \nu^{-1})^{2n-1} (\exp(t_1 X_1), \dots, \exp(t_{2n-1} X_{2n-1})) = (\omega \cdot \omega'^{-1})^{2n-1}$$

and

$$\partial_1 \dots \partial_{2n-1} (d\nu \cdot \nu^{-1})^{2n-1} (\exp(t_1 X_1), \dots, \exp(t_{2n-1} X_{2n-1})) \Big|_{t=0} =$$
$$= \partial_1 \dots \partial_{2n-1} (\omega \cdot \omega'^{-1})^{2n-1} \Big|_{t=0} = \sum_{i_1=1}^{2n-1} \cdots \sum_{i_{2n-1}=1}^{2n-1} \cdots \partial_j \left(\omega \cdot \omega'^{-1} \right) \cdots \Big|_{t=0}.$$

The last product in the sum is a product of $2n-1$ factors $(\omega \cdot \omega'^{-1})$ with ∂_j in front of the i_j^{th} factor (of course there may be several ∂'s in front of one factor). Note that $\omega\big|_{t=0} = \sum_{i=0}^{2n-1} dx_i = 0$, so in the last sum all summands with $(i_1, \dots, i_{2n-1})$ not a permutation of $(1, \dots, 2n-1)$ vanish. On the other hand using $\omega'\big|_{t=0} = \sum_{i=0}^{2n-1} x_i = 1$ we get

$$\partial_j \left(\omega \cdot \omega'^{-1} \right) \big|_{t=0} = (\partial_j \omega)\big|_{t=0} \cdot \omega'^{-1}\big|_{t=0} + \omega\big|_{t=0} \cdot (\partial_j \omega'^{-1})\big|_{t=0} =$$
$$= (\partial_j \omega)\big|_{t=0} = \sum_{i=0}^{j-1} dx_i \cdot X_j.$$

Alltogether we obtain

$$\begin{aligned}
&\partial_1 \dots \partial_{2n-1}(d\nu \cdot \nu^{-1})^{2n-1}(\exp(t_1X_1), \dots, \exp(t_{2n-1}X_{2n-1}))\Big|_{t=0} = \\
&\qquad = \sum_{\tau \in \mathfrak{S}_{2n-1}} \left(\sum_{i=0}^{\tau(1)-1} dx_i \cdot X_{\tau(1)} \right) \cdots \left(\sum_{i=0}^{\tau(2n-1)-1} dx_i \cdot X_{\tau(2n-1)} \right) \\
&\qquad = \sum_{\tau \in \mathfrak{S}_{2n-1}} X_{\tau(1)} \cdots X_{\tau(2n-1)} \left(\sum_{i=0}^{\tau(1)-1} dx_i \right) \cdots \left(\sum_{i=0}^{\tau(2n-1)-1} dx_i \right) \\
&\qquad\qquad = \sum_{\tau \in \mathfrak{S}_{2n-1}} \operatorname{sgn}(\tau) X_{\tau(1)} \cdots X_{\tau(2n-1)} dx_0 dx_1 \dots dx_{2n-2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{\sigma \in \mathfrak{S}_{2n-1}} \operatorname{sgn}(\sigma) \partial_1 \dots \partial_{2n-1} \\
&\qquad (d\nu \cdot \nu^{-1})^{2n-1}(\exp(t_1X_{\sigma 1}), \dots, \exp(t_{2n-1}X_{\sigma(2n-1)}))\Big|_{t=0} = \\
&= \sum_{\sigma \in \mathfrak{S}_{2n-1}} \operatorname{sgn}(\sigma) \sum_{\tau \in \mathfrak{S}_{2n-1}} \operatorname{sgn}(\tau) X_{\sigma\tau(1)} \cdots X_{\sigma\tau(2n-1)} dx_0 dx_1 \dots dx_{2n-2} = \\
&\qquad = (2n-1)! \sum_{\sigma \in \mathfrak{S}_{2n-1}} \operatorname{sgn}(\sigma) X_{\sigma(1)} \cdots X_{\sigma(2n-1)} dx_0 dx_1 \dots dx_{2n-2}.
\end{aligned}$$

Because

$$\begin{aligned}
\int_{\Delta^{2n-1}} dx_0 \dots dx_{2n-2} &= -\int_{\Delta^{2n-1}} dx_{2n-1} dx_1 \dots dx_{2n-2} \\
&= -\int_{\Delta^{2n-1}} dx_1 \dots dx_{2n-1} = -\frac{1}{(2n-1)!}
\end{aligned}$$

(cf. the explicit formula in the proof of proposition A.4) we finally obtain

$$\Delta(f)(X_1, \dots, X_{2n-1}) = \frac{((n-1)!)^2}{(2n-1)!} \sum_{\sigma \in \mathfrak{S}_{2n-1}} \operatorname{sgn}(\sigma) \operatorname{Tr}(X_{\sigma 1} \cdots X_{\sigma(2n-1)}),$$

that is $\Delta(f) = p_n$. □

3 Comparison of the two regulators

Theorem 3.1. *For $n > 1$, the diagram*

$$\begin{tikzcd}
K^{\mathrm{rel}}_{2n-1}(K)_{\mathbf{Q}} \arrow[r, "\simeq"] \arrow[dr, "r_p"'] & K_{2n-1}(K)_{\mathbf{Q}} & K_{2n-1}(R)_{\mathbf{Q}} \arrow[l, "\simeq"'] \arrow[dl, "\frac{(-1)^{n-1}}{(n-1)!(2n-2)!} b_p"] \\
& K &
\end{tikzcd}$$

is commutative.

Proof. We have a commutative diagram

$$\begin{tikzcd}
 & K^{\mathrm{rel}}_{2n-1}(K)_{\mathbf{Q}} \arrow[r, "\simeq"] & K_{2n-1}(K)_{\mathbf{Q}} \\
K_{2n-1}(R,\pi R)_{\mathbf{Q}} \arrow[r, "\simeq"] \arrow[d, "\simeq"] & K^{\mathrm{rel}}_{2n-1}(R)_{\mathbf{Q}} \arrow[r, "\simeq"] \arrow[u, "\simeq"'] & K_{2n-1}(R)_{\mathbf{Q}} \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\
\mathrm{Prim}H_{2n-1}(X(R,\pi R),\mathbf{Q}) \arrow[rr, "\beta"] & & \mathrm{Prim}H_{2n-1}(\mathrm{BGL}(R),\mathbf{Q})
\end{tikzcd}$$

where β is induced by the composition

$$X(R,\pi R) \xrightarrow{\varphi} \mathrm{GL}(R_*)/\mathrm{GL}(R) \xrightarrow{\theta} \mathrm{BGL}(R)$$

(see section 1.1 for the definition of θ and proposition 1.2 for that of φ) which is just the natural inclusion $X(R,\pi R) \subset \mathrm{BGL}(R)$ as one easily checks. We also denote by β the induced map on homology. To prove the theorem it suffices to show that

$$\begin{tikzcd}
H_{2n-1}(X(R,\pi R),\mathbf{Q}) \arrow[rr, "\beta"] \arrow[dr, "r_p"'] & & H_{2n-1}(\mathrm{BGL}(R),\mathbf{Q}) \arrow[dl, "\frac{(-1)^{n-1}}{(n-1)!(2n-2)!}b_p"] \\
 & K &
\end{tikzcd}$$

commutes.

It follows from the long exact sequence of relative K-theory and the finiteness of $K_i(R/\pi R)$ for $i > 0$ that $K_i(R,\pi R)_{\mathbf{Q}} \to K_i(R)_{\mathbf{Q}}$ is an isomorphism for all $i > 0$. Thus we know that $X(R,\pi R)^+ \to \mathrm{BGL}(R)^+$ induces an isomorphism on the subspaces of primitive elements in rational homology and it follows from the theorem of Cartan-Milnor-Moore that it induces in fact an isomorphism $H_*(X(R,\pi R),\mathbf{Q}) \underset{\simeq}{\xrightarrow{\beta}} H_*(\mathrm{BGL}(R),\mathbf{Q})$.

Since for each N the subgroup $U_N(R)$ has finite index in $\mathrm{GL}_N(R)$ it follows that $H_{2n-1}(BU(R),\mathbf{Q}) \to H_{2n-1}(\mathrm{BGL}(R),\mathbf{Q})$ is surjective where $U(R) = \varinjlim U_N(R)$. Next $BU(R)$ is actually contained in $X(R,\pi R)$ and thus we have a commutative diagram

$$\begin{tikzcd}
H_{2n-1}(BU(R),\mathbf{Q}) \arrow[d, "\alpha"'] \arrow[dr, "\gamma"] & \\
H_{2n-1}(X(R,\pi R),\mathbf{Q}) \arrow[r, "\beta", "\simeq"'] & H_{2n-1}(\mathrm{BGL}(R),\mathbf{Q})
\end{tikzcd}$$

with γ and hence also α surjective. Now r_p is given by the cocycle ϕ of proposition 1.3 and $b_p \circ \gamma$ is given by the cocycle f of theorem 2.4. From the explicit formulae for ϕ and f it is clear that $\phi \circ \alpha = \frac{(-1)^{n-1}}{(n-1)!(2n-2)!} f \circ \gamma$ which proves the theorem. $\square$

A Integration on the standard simplex

A.1 The ring $A\langle x_0, \dots, x_n \rangle$

Let A be an ultrametric Banach ring. For simplicity write $A\langle x\rangle$ for $A\langle x_0, \dots, x_n\rangle$ (cf. section 1.1). Recall also the family of seminorms p_r, $r \in \mathbf{N}_0$, $p_r(\sum a_I x^I) = \sup_I \|a_I\| \cdot |I|^r$. We write also $\|.\|_r$ for p_r.

Proposition A.1. *$A\langle x\rangle$ is a sub-A-algebra of the algebra of formal power series with coefficients in A. Its underlying module is an ultrametric Fréchet module. Furthermore*

$$\|f \cdot g\|_r \leq \sum_{s=0}^{r} \binom{r}{s} \|f\|_s \|g\|_{s-r}.$$

Proof. Let $f = \sum_I a_I x^I$ and $g = \sum b_I x^I$ be in $A\langle x\rangle$. We want to show that $f \cdot g = \sum_I c_I x^I$ with $c_I = \sum_{K+L=I} a_K b_L$ is also in $A\langle x\rangle$. Fix a non negative integer r and let $A := \sup_K(\sum_{s=0}^{r} \binom{r}{s} |K|^s \|a_K\|)$, $B := \sup_K(\sum_{s=0}^{r} \binom{r}{s} |K|^{r-s} \|b_K\|)$. Given $\varepsilon > 0$ choose $N > 0$ such that $|K|^s\|a_K\| < \frac{\varepsilon}{B}$, $|K|^{r-s}\|b_K\| < \frac{\varepsilon}{A}$ for all $s = 0, \dots, r$ and $|K| \geq N$. Then for every $|I| \geq 2N$ we have

$$\begin{aligned} |I|^r \|c_I\| &\leq \max_{K+L=I} \left((|K| + |L|)^r \|a_K\| \|b_L\| \right) \\ &= \max_{K+L=I} \left(\sum_{s=0}^{r} \binom{r}{s} |K|^s \|a_K\| |L|^{r-s} \|b_L\| \right) < \varepsilon. \end{aligned}$$

Thus $f \cdot g$ in fact belongs to $A\langle x\rangle$. The assertion on $\|f \cdot g\|_r$ follows immediately from the above computation.
It remains to show that $A\langle x\rangle$ is complete. This is easy. $\square$

Remark. Let $|f|_s = \sum_{r=0}^{s} \frac{1}{r!} \|f\|_r$. Then it follows from the above proposition that $(|.|_s)_{s \in \mathbf{N}}$ is a family of seminorms which defines the same topology on $A\langle x\rangle$ and satisfies $|f \cdot g|_s \leq |f|_s |g|_s$.

Now let $\phi : [n] \to [m]$ be a monotone map (a morphism in the simplicial category Δ). We want to define $\phi^* : A\langle x_0, \dots, x_m\rangle \to A\langle x_0, \dots, x_n\rangle$ by $x_i \mapsto \sum_{\phi(j)=i} x_j$, $i = 0, \dots m$. We have to show that ϕ^* is well defined and continuous. Slightly more generally we have:

Lemma A.2. *Let $g = (g_0, \dots, g_m)$ be a tuple of polynomials of degree 1 in $A\langle y_0, \dots, y_n\rangle$ with integral coefficients. Then for $f = \sum_I a_I x^I \in A\langle x_0, \dots, x_m\rangle$ the formal composition $f \circ g = \sum_I a_I g^I$ lies in $A\langle y_0, \dots, y_n\rangle$ and $f \mapsto f \circ g$ is continuous.*

Proof. g^I is a polynomial of degree $\leq |I|$ with integral coefficients and thus $\|g^I\|_r \leq \|1\| \cdot |I|^r$. Since $\|a_I\| |I|^r$ tends to zero whren $|I|$ tends to infinity it follows that $(\sum_{|I| \leq n} a_I g^I)_{n \in \mathbf{N}}$ is a Cauchy sequence. Since $A\langle y_0, \dots, y_n\rangle$ is complete this Cauchy sequence converges and the limit is $f \circ g$.
Moreover $\|f \circ g\|_r = \|\sum_I a_I g^I\|_r \leq \sup_I \|a_I g^I\|_r \leq \sup_I \|a_I\| \cdot \|1\| \cdot |I|^r = \|1\| \cdot \|f\|_r$ and thus $f \mapsto f \circ g$ is continuous. $\square$

Recall that $I_n \subset A\langle x_0, \dots, x_n\rangle$ is the principal ideal generated by $x_0+\cdots+x_n-1$ and $A_n := A\langle x_0, \dots, x_n\rangle / I_n$.

Lemma A.3. *The homomorphism $\eta : A\langle x_0, \dots, x_n\rangle \to A\langle t_1, \dots, t_n\rangle$ that sends x_i to t_i for $i > 0$ and x_0 to $1 - t_1 - \cdots - t_n$ induces an isomorphism $A_n \to A\langle t_1, \dots, t_n\rangle$.*

Proof. We have the obvious continuous section $\iota : t_i \mapsto x_i$, $i > 0$, so that η is surjective. Assume $f = \sum_I a_I x^I$ is in the kernel of η. Then

$$\begin{aligned} f &= f - \iota(\eta(f)) = \\ &= \sum_I a_I \left(x^I - (1 - x_1 - \cdots - x_n)^{i_0} x_1^{i_1} \cdots x_n^{i_n}\right) \\ &= \sum_I a_I \cdot g_I \cdot (x_0 + \cdots + x_n - 1) \end{aligned}$$

where the g_I are polynomials with integral coefficients of total degree $\leq |I|$. In particular $\|g_I\|_r \leq \|1\| \cdot |I|^r$ and thus $\sum_I a_I g^I$ is an element of $A\langle x_0, \dots, x_n\rangle$ which satisfies $(\sum_I a_I g^I) \cdot (x_0 + \cdots + x_n - 1) = f$. $\square$

If $f \in I_m$ then clearly $\phi^*(f) \in I_n$ and thus there are induced continuous homomorphisms $\phi^* : A_m \to A_n$ for every $\phi : [n] \to [m]$ which make $[n] \mapsto A_n$ a simplicial Fréchet ring.

A.2 Integration of differential forms

Fix a non archimedean field $(K, |.|)$ of characteristic 0. We want to define the integral of a n-form with values in K over the standard simplex Δ^n. Since integration produces denominators we assume that there are constants $C > 0, s \in \mathbf{N}$ such that $|\frac{1}{k}| \leq Ck^s$ for all $k \in \mathbf{N}$.
We define $\Omega^0(\Delta^n) = K_n$, $\Omega^1(\Delta^n) = (\bigoplus_{i=0}^n K_n dx_i)/(\sum_{i=0}^n dx_i)$ and $\Omega^r(\Delta^n) = \bigwedge_{K_n}^r \Omega^1(\Delta^n)$ with the obvious differential $d : \Omega^i(\Delta^n) \to \Omega^{i+1}(\Delta^n)$.
Since $\Omega^n(\Delta^n) = K_n dx_1 \dots dx_n$ every n-form ω can be written uniquely as $\omega = f dx_1 \dots dx_n$ with $f \in K_n$. We also denote by f the image $\sum_I a_I x^I$ of f in $K\langle x_1, \dots, x_n\rangle$. We want to define

$$\int_{\Delta^n} \omega := \int_{\Delta^n} f dx_1 \dots dx_n := \sum_I a_I \int_{\Delta^n} x^I dx_1 \dots dx_n \in K,$$

where the integral on the right hand side is the usual integral of the n-form $x^I dx_1 \dots dx_n$ over the geometric standard simplex $\Delta^n \subset \mathbf{R}^{n+1}$ where the orientation of Δ^n is given by $dx_1 \dots dx_n$.

Proposition A.4. *The above integral is well defined and $\omega \mapsto \int_{\Delta^n} \omega$ gives a continuous homomorphism $\Omega^n(\Delta^n) \to K$.*

Proof. We have topological isomorphisms $\Omega^n(\Delta^n) = K_n dx_1 \dots dx_n \xrightarrow{\simeq} K_n \xrightarrow{\simeq} K\langle x_1, \dots, x_n\rangle$ and by construction the integral $\int_{\Delta^n}$ factors through this isomorphism. Thus we have to show that $K\langle x_1, \dots, x_n\rangle \ni f \mapsto \int_{\Delta^n} f dx_1 \dots dx_n \in K$ is well defined and continuous.

For simplicity we write $\int_{\Delta^n} f$ for $\int_{\Delta^n} f dx_1 \dots dx_n$.
For $I = (i_1, \dots, i_n)$ one computes

$$\int_{\Delta^n} x^I = \prod_{j=1}^{n} \left(\sum_{l=0}^{i_j} \binom{i_j}{l} (-1)^l \frac{1}{n-j+1+l+\sum_{k=j+1}^{n} i_k} \right).$$

By our general assumption on K we have

$$\left| \frac{1}{n-j+1+l+\sum_{k=j+1}^{n-1} i_k} \right| \le C \cdot \left(n-j+1+l+\sum_{k=j+1}^{n} i_k \right)^s \le C(n+|I|)^s,$$

thus

$$\left| \int_{\Delta^n} x^I \right| \le C^n (n+|I|)^{sn} \le \tilde{C} |I|^{sn}$$

if $|I| \ge 1$ with a constant $\tilde{C}$ depending only on C, n and s.
Now, for $f = \sum_I a_I x^I \in K\langle x_1, \dots, x_n \rangle$, $|a_I| \cdot |I|^{sn}$ tends to zero when $|I|$ tends to infinity and thus $\int_{\Delta^n} f = \sum_I a_I \int_{\Delta^n} x^I$ converges in A. Furthermore $|\int_{\Delta^n} f| \le \sup_I |a_I \int_{\Delta^n} x^I| \le \max\{\tilde{C} \cdot \sup_I |a_I| \cdot |I|^{sn}, |\int_{\Delta^n} 1| \cdot |a_0|)\} \le (\text{const}) \cdot \max\{\|f\|_{sn}, \|f\|_0\}$. It follows that $f \mapsto \int_{\Delta^n} f$ is continuous. □

Remark. In the definition of the integral $\int_{\Delta^n} \omega$ we could take any representative of f in $K\langle x_0, \dots, x_n \rangle$. The resulting value of the integral would be the same.

A.3 Dependence on parameters

For any K-Banach algebra A we denote by $\mathcal{F}_\varepsilon(K^r, A)$ the Banach algebra of ε-convergent power series in r variables, i.e. power series $f = \sum_J a_J y^J$, $a_J \in A$, $y = (y_1, \dots, y_r)$, $J \in \mathbf{N}_0^r$ such that $\lim_{|J| \to \infty} \|a_J\| \varepsilon^{|J|} = 0$, with norm $\|f\|_\varepsilon = \sup_J \|a_J\| \varepsilon^{|J|}$.

Definition. Let M be a locally analytic r-dimensional K-manifold and $f : M \to K\langle x_0, \dots, x_n \rangle$ a function. We say that f is *locally analytic* if for every $u \in M$ there exists an $\varepsilon > 0$ and a chart $M \supset V \xrightarrow{\psi} B_\varepsilon(0) = \{|.| \le \varepsilon\} \subset K^r$ with $\psi(u) = 0$ such that $f \circ \psi^{-1}$ is given by $\sum_I a_I x^I$ where the a_I are in $\mathcal{F}_\varepsilon(K^r, K)$ and satisfy $\|a_I\|_\varepsilon \cdot |I|^t \to 0$ as $|I| \to \infty$ for every $t \in \mathbf{N}_0$.

Note that if the condition is satisfied for $u \in M$ with chart $\psi : V \to B_\varepsilon(0)$ then it is also satisfied for all $u' \in V$ with chart $\psi - \psi(u')$.
It follows from the next Proposition that if the condition of the definition is satisfied by one chart $\psi : V \to B_\varepsilon(0)$ with $u \in V$ and $\psi(u) = 0$ then it is also satisfied by any other chart $\psi' : V' \to B_{\varepsilon'}(0)$ with $u \in V'$ and $\psi'(u) = 0$ after possibly shrinking V'.

Remark. If we embed $K\langle x_0, \dots, x_n \rangle$ in the Banach algebra $\mathcal{F}_1(K^{n+1}, K)$ then $f : M \to \mathcal{F}_1(K^{n+1}, K)$ as above is locally analytic in the ordinary sense but not vice versa.

Proposition A.5. *(a) Let $f, g : M \to K\langle x_0, \dots, x_n\rangle$ be locally analytic. Then also $f + g$ and $f \cdot g$ are locally analytic.*

(b) If $\varphi : M' \to M$ is locally analytic then $f \circ \varphi : M' \to K\langle x_0, \dots, x_n\rangle$ is locally analytic.

Proof. (a) is easy (cf. the proof of proposition A.1). (b) Let $u' \in M'$, $u := \varphi(u')$ and choose a chart $u \in V \xrightarrow{\psi} B_\varepsilon(0)$ with $\psi(v) = 0$ such that $f \circ \psi^{-1}$ is of the form $\sum_I a_I x^I$ with $\|a_I\|_\varepsilon \cdot |I|^t \to 0$ as $|I| \to \infty$ for every $t \in \mathbf{N}_0$. Choose a chart $u' \in V' \xrightarrow{\psi'} B_{\varepsilon'}(0)$ for M' with $\psi'(u') = 0$ such that $\varphi(V') \subset V$. We may assume that the induced map $\tilde\varphi : B_{\varepsilon'}(0) \to B_\varepsilon(0)$ is given by a power series. Since $\tilde\varphi(0) = 0$ it follows that $\tilde\varphi$ has no constant term and therefore that $\|\tilde\varphi\|_{\varepsilon''} \le \frac{\varepsilon''}{\varepsilon'}\|\tilde\varphi\|_{\varepsilon'}$ for every $\varepsilon'' \le \varepsilon$. Thus we may assume that $\|\tilde\varphi\|_{\varepsilon'} \le \varepsilon$. But then $a_I \circ \tilde\varphi$ is a well defined power series in $\mathcal{F}_{\varepsilon'}(K^{r'}, K)$ with $\|a_I \circ \tilde\varphi\|_{\varepsilon'} \le \|a_I\|_\varepsilon$. Now $f \circ \varphi \circ \psi'^{-1} = \sum_I (a_I \circ \tilde\varphi) x^I$ and the claim follows. ☐

We call $f : M \to K_n$ *locally analytic* if it is locally analytic in the above sense under the identification $K_n = K\langle x_1, \dots, x_n\rangle$. One can check that if $g : M \to K\langle x_0, \dots, x_n\rangle$ is locally analytic, so is the induced map $M \to K_n$.

Proposition A.6. *Let $f : M \to K_n$ be locally analytic. Then $M \ni u \mapsto \varphi(u) := \int_{\Delta^n} f(u) dx_1 \dots dx_n \in K$ is locally analytic.*

Proof. Fix a chart $M \supset V \xrightarrow{\psi} B_\varepsilon(0)$ such that $f \circ \psi^{-1} = \sum_I a_I x^I$ with $\|a_I\|_\varepsilon \cdot |I|^s \to 0$ as $|I| \to \infty$ for every $s \in \mathbf{N}_0$ as in the definition. For I fixed the function $B_\varepsilon(0) \ni v \mapsto \int_{\Delta^n} a_I(v) x^I dx_1 \dots dx_n = a_I(v) \int_{\Delta^n} x^I dx_1 \dots dx_n$ is given by the power series $a_I \cdot (\int_{\Delta^n} x^I dx_1 \dots dx_n) \in \mathcal{F}_\varepsilon(K^r, K)$. For $|I| \ge 1$ we have $\|a_I \cdot (\int_{\Delta^n} x^I dx_1 \dots dx_n)\|_\varepsilon \le \|a_I\|_\varepsilon \cdot |(\int_{\Delta^n} x^I dx_1 \dots dx_n)| \le \|a_I\|_\varepsilon \cdot \tilde C |I|^{sn}$ with $\tilde C$ as in the proof of proposition A.4. Since $\|a_I\|_\varepsilon |I|^{sn} \to 0$ as $|I| \to \infty$ it follows that $\sum_I a_I \cdot (\int_{\Delta^n} x^I dx_1 \dots dx_n)$ converges in $\mathcal{F}_\varepsilon(K^r, K)$. The claim follows since obviously $\varphi \circ \psi^{-1} = \sum_I a_I \cdot (\int_{\Delta^n} x^I dx_1 \dots dx_n)$. ☐

We will also write $\int_{\Delta^n} f dx_1 \dots dx_n$ for the function φ in the above proposition.

Proposition A.7. *Assume that $M = B_\varepsilon(0) \subset K^r$ and $f : M \to K_n$ is given by $\sum_I a_I x^I$ with $\|a_I\|_\varepsilon |I|^t \to 0$ as $|I| \to \infty$.*

(i) $\partial_i f := \sum_I (\partial_i a_I) x^I$ is well defined and locally analytic.

(ii) $\int_{\Delta^n} (\partial_i f) dx_1 \dots dx_n = \partial_i \int_{\Delta^n} f dx_1 \dots dx_n$.

(iii) If $g : M \to K_n$ is of the same type then $\partial_i(fg) = (\partial_i f)g + f(\partial_i g)$.

Proof. One easily sees that $\partial_i : \mathcal{F}_\varepsilon(K^r, K) \to \mathcal{F}_\varepsilon(K^r, K)$ is well defined and continuous with $\|\partial_i a\|_\varepsilon \le \varepsilon^{-1}\|a\|_\varepsilon$. Thus (i) follows. Then $\int_{\Delta^n} (\partial_i f) dx_1 \dots dx_n = \sum_I (\partial_i a_I) \int_{\Delta^n} x^I dx_1 \dots dx_n = \partial_i \left(\sum_I a_I \int_{\Delta^n} x^I dx_1 \dots dx_n\right) = \partial_i \int_{\Delta^n} f dx_1 \dots dx_n$ by definition of the integral and the continuouity of ∂_i. The last assertion is clear. ☐

More generally, if P is a free K_n-module of finite rank we say that a function $f : M \to P$ is locallay analytic if all comoponent functions with repsect to a given basis of P are locally analytic. Then analogues of the above propositions hold. In particular we are interested in the case where $P = \mathrm{Mat}_N(K_n)$ or $P = \mathrm{Mat}_N(K_n) \otimes_{K_n} \Omega^r(\Delta^n)$.

Now we can prove that the cocycle f in theorem 2.4 is in fact locally analytic.

Proposition A.8. *The function* $U_N(R)^{\times(2n-1)} \to K$,

$$(g_1, \dots, g_{2n-1}) \mapsto \mathrm{Tr} \int_{\Delta^{2n-1}} (d\nu \cdot \nu^{-1})^{2n-1}$$

where $\nu = \nu(g_1, \dots, g_{2n-1}) = \sum_{i=0}^{2n-1} x_i g_{i+1} \cdots g_{2n-1} \in \mathrm{GL}_N(R_{2n-1})$ *is locally analytic.*

Proof. It suffices to show that $\nu^{-1} : (g_1, \dots, g_{2n-1}) \mapsto \nu(g_1, \dots, g_{2n-1})^{-1}$ and $d\nu : (g_1, \dots, g_{2n-1}) \mapsto d(\nu(g_1, \dots, g_{2n-1}))$ are locally analytic, where the above functions are considered as functions on $U_N(R)^{\times(2n-1)}$ with values in $N \times N$-matrices with coefficients in $\Omega^0(\Delta^{2n-1}) = K_{2n-1}$ resp. $\Omega^1(\Delta^{2n-1})$.

This is clear for $d\nu$. Set $\varepsilon := |\pi|$ and consider the global chart $\psi : U_N(R)^{\times(2n-1)} \to \pi\mathrm{Mat}_N(R) = B_\varepsilon(0) \subset K^{N\times N}$ whose inverse is given by $(M_1, \dots, M_{2n-1}) \mapsto (1 + M_1, \dots, 1 + M_{2n-1})$. Then $\nu^{-1} \circ \psi^{-1}$ is given by $\sum_{k=0}^{\infty} (\sum_{i=0}^{2n-1} x_i h_i)^k$ where $h_i : \pi\mathrm{Mat}_N(R) \to \mathrm{Mat}_N(K)$ is the function $(M_1, \dots, M_{2n-1}) \mapsto 1 - (1 + M_{i+1}) \cdots (1 + M_{2n-1})$ (cf. the proof of lemma 2.3). Since h_i has no constant term and only integral coefficients we have $\|h_i\|_\varepsilon \leq \varepsilon$. The coefficient of x^I in the above expansion of $\nu^{-1} \circ \psi^{-1}$ is of the form $h^I +$ permutations and thus $\|(\text{coefficient of } x^I)\|_\varepsilon \leq \|h_0\|_\varepsilon^{i_0} \cdots \|h_{2n-1}\|_\varepsilon^{i_{2n-1}} \leq \varepsilon^{|I|}$. Since $\varepsilon < 1$ it follows that $\|(\text{coefficient of } x^I)\|_\varepsilon \cdot |I|^t$ tends to zero as $|I|$ tends to infinity for every $t \in \mathbf{N}_0$ and thus that ν^{-1} is locally analytic. ☐

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