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INTEGRAL REPRESENTATIONS FOR SOLUTIONS OF EXPONENTIAL GAUSS-MANIN SYSTEMS

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Abstract. Let $f, g : U \to \mathbb{A}^1$ be two algebraically independent regular functions from the smooth affine complex variety U to the affine line. The associated exponential Gauß-Manin systems on the affine line are defined to be the cohomology sheaves of the direct image of the exponential differential system $\mathcal{O}_U e^g$ with respect to f. We prove that its holomorphic solutions admit representations in terms of period integrals over topological chains with possibly closed support and with rapid decay condition.

1. **Introduction**

Let U be a smooth affine complex variety of dimension n and let $f, g : U \to \mathbb{A}^1$ be two regular functions which we assume to be algebraically independent. We consider the flat algebraic connection $\nabla : \mathcal{O}_U \longrightarrow \Omega^1_U$ on the trivial line bundle $\mathcal{O}_U$ defined as $\nabla u = du + dg \cdot u$. Let us denote the associated holonomic $\mathcal{D}_U$-module by $\mathcal{O}_U e^g$, i.e. the trivial $\mathcal{O}_U$-module on which any vector field ξ on U acts via the associated derivation ∇_ξ induced by the connection. Let $\mathcal{N}$ be the direct image $\mathcal{N} := f_+(\mathcal{O}_U e^g)$ in the theory of $\mathcal{D}$-modules (see e.g. [2]). Then $\mathcal{N}$ is a complex of $\mathcal{D}_{\mathbb{A}^1}$-modules and we consider its k-th cohomology sheaf

$$\mathcal{H}^k f_+(\mathcal{O}_U e^g) \ ,$$

a holonomic $\mathcal{D}_{\mathbb{A}^1}$-module.

There is an alternative point of view of $\mathcal{H}^k\mathcal{N}$ in terms of flat connections: outside a finite subset $\Sigma_1 \subset \mathbb{A}^1$, the $\mathcal{O}_{\mathbb{A}^1}$-module $\mathcal{H}^k f_+(\mathcal{O}_U e^g)$ is locally free, hence a flat connection. This connections coincides with the Gauß-Manin connection on the relative de Rham cohomology

$$\mathcal{H}^k(\mathcal{N})|_{\mathbb{A}^1 \smallsetminus \Sigma_1} \cong \big(\boldsymbol{R}^{k+n-1} f_*(\Omega^{\bullet}_{U|\mathbb{A}^1}, \nabla), \nabla_{GM}\big)|_{\mathbb{A}^1 \smallsetminus \Sigma_1} \ ,$$

i.e. the corresponding higher direct image of the complex of relative differential forms on U with respect to f and the differential induced by the absolute connection, as in [6]. It is a flat connection over $\mathbb{A}^1 \smallsetminus \Sigma_1$.

In the case $g = 0$, which is classically called the Gauß-Manin system of f, it is well-known that the solutions admit integral representations (see [10] and [12]). The aim of the present article is to give such a description of the solutions in the more general case of the Gauß-Manin system of the exponential module $\mathcal{O}_U e^g$, which we will call *exponential Gauß-Manin system*. A major difficulty lies in the definition of the integration involved. In the case of vanishing g, the connection is regular singular at infinity and the topological cycles over which the integrations are performed can be chosen with compact support inside the affine variety U. In the exponential case $g \neq 0$, the connection is irregular singular at infinity and we will have to consider integration paths approaching the irregular locus at infinity.

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A systematic examination of period integrals in more general irregular singular situations over complex surfaces has been carried out in [5] resulting in a perfect dualtiy between the de Rham cohomology and some homology groups, the *rapid decay homology groups*, in terms of period integrals. We will prove in this article that this result generalizes to arbitrary dimension in case of an elementary exponential connection $\mathcal{O}e^g$ as before, see Theorem 2.4.

With this tool at hand, we construct a local system $\widetilde{H_k^{rd}}$ on $\mathbb{A}^1 \smallsetminus \Sigma_2$, the stalk at point t of which is the rapid decay homology $H_k^{rd}(f^{-1}(t), e^{-g_t})$ of the restriction of $\mathcal{O}_U e^{-g}$ to the fibre $f^{-1}(t)$. The set Σ_2 can be described using a compactification $(F,G) : X \to \mathbb{P}^1 \times \mathbb{P}^1$ of the map $(f,g) : U \to \mathbb{A}^1 \times \mathbb{A}^1$ (see Proposition 3.5). Let $\overline{\Delta}$ be the projective variety in $\mathbb{P}^1 \times \mathbb{P}^1$ such that (F,G) is a locally trivial fibration out of $\overline{\Delta}$. The set Σ_2 can be choosen as the set of $t_0 \in \mathbb{A}^1$ such that (t_0, ∞) belongs to the closure of $\overline{\Delta} \setminus (\mathbb{P}^1 \times \{\infty\})$ in $\mathbb{P}^1 \times \mathbb{P}^1$.

Let $\Sigma = \Sigma_1 \cup \Sigma_2$. We prove that given a flat section $c_t \otimes e^{g_t}$ in the local system H_{k+n-1}^{rd} and a relative differential form ω in $\Omega_{rel}^{k+n-1}(f^{-1}(\mathbb{A}^1 \smallsetminus \Sigma))$ (which describes the analytification of $\mathcal{H}^k\mathcal{N}$, see Proposition 3.4), the integral

$$\int_{c_t} \omega|_{f^{-1}(t)} \cdot e^{g_t} \tag{1.1}$$

gives a (multivalued) holomorphic solution of the exponential Gauß-Manin system. As a consequence of the duality proved in Theorem 2.4, we deduce that all (multivalued) holomorphic solutions on $\mathbb{A}^1 \smallsetminus \Sigma$ can be obtained by this construction, i.e. we achieve the following theorem (Theorem 3.7) which is the main result of this article:

Theorem 1.1. *For any simply connected open subset $V \subset \mathbb{C} \smallsetminus \Sigma$, the space of holomorphic solutions of the exponential Gauß-Manin system $\mathcal{H}^k\mathcal{N}$ is isomorphic to the space of flat sections in the local system $\widetilde{H_{k+n-1}^{rd}}$ over V.*

The isomorphism in the theorem is given in terms of the integration (1.1). We want to remark that this theorem generalizes similar results by F. Pham for the Fourier-Laplace transform of the Gauß-Manin systems of a regular function $h : U \to \mathbb{A}^1$ (consider $f : \mathbb{A}^1 \times U \to \mathbb{A}^1$ to be the first canonical projection and $g : \mathbb{A}^1 \times U \to \mathbb{A}^1$ given by $g(x,u) = xh(u)$, cf. [11]). It also includes other well-known examples as the integral representations of the Bessel-functions for instance (cf. the introduction of [1]).

2. The period pairing

Consider the situation described in the introduction, namely $f, g : U \to \mathbb{A}^1$ being two algebraically independent regular functions on the smooth affine complex variety U. The main result of this work is to give a representation of the Gauß-Manin solutions in terms of period integrals. The proof relies on a duality statement between the algebraic de Rham cohomology of the exponential connection associated to g and some Betti homology groups with decay condition. The present section is devoted to the proof of this duality statement, which is a generalization of the analogous result for surfaces in [5] to the case of exponential line bundles in arbitrary dimensions.

2.1. Definition

Let us start with any smooth affine complex variety U_t and a regular function $g_t : U_t \to \mathbb{A}^1$. Note, that the index t is irrelevant in this section but will become meaningful later in the application to the Gauß-Manin system where U_t will denote the smooth fibres $f^{-1}(t)$.

We assume that U_t is embedded into a smooth projective variety X_t with the complement $D_t := X_t \smallsetminus U_t$ being a divisor with normal crossings. Moreover, we require that g_t extends to a meromorphic mapping on X_t. In particular, if $y \in D_t$ is a point at infinity, we can choose coordinates $x_1, \dots, x_n$ of X_t centered at y such that locally $D_t = \{x_1 \cdots x_k = 0\}$ for some $k \in \{1, \dots, n\}$ and such that we have

$$g_t(x) = x_1^{n_1} \cdots x_r^{n_r} \cdot u(x)$$

locally at y, where $r \leq k$ and the exponents $n_i \in \mathbb{Z}$ are either all positive or all negative and u is holomorphic and non-vanishing on D_t (i.e. g_t will not have indeterminancies at D_t).

The function g_t defines a flat meromorphic connection ∇_t of exponential type on U_t, denoted $\mathcal{O}e^{g_t}$ on the trivial line bundle $\mathcal{O}_{U_t}$ with connection defined as

$$\nabla_t 1 = dg_t \in \Omega^1_{U_t|\mathbb{C}} \ .$$

In [5], a duality pairing between the de Rham cohomology of a flat meromorphic connection (admitting a good formal structure) and a certain homology theory, the rapid decay homology, is constructed in the case $\dim(X_t) = 2$, generalizing previous constructions of Bloch and Esnault for curves. For connections of exponential type lying in the main focus of this work, we will now generalized these results to the case of arbitrary dimension. We remark that a crucial point in [5] is to work with a *good* compactification with respect to the given vector bundle and connection which is always fulfilled for the connections of exponential type considered above.

We recall the definition of rapid decay homology as in [5]. Consider the real oriented blow-up $\pi : \widetilde{X_t} \to X_t^{\mathrm{an}}$ of the irreducible components of D_t (cp. [9]). Let $\mathcal{C}^{-p}_{\widetilde{X_t}}$ denote the sheaf associated to the presheaf $V \mapsto S_p(\widetilde{X_t}, \widetilde{X_t} \smallsetminus V)$ of $\mathbb{Q}$-vector spaces, where $S_p(Y)$ denotes the groups of piecewise smooth singular p-chains in Y and $S_p(Y, A)$ the relative chains for a pair of topological spaces $A \subset Y$.

We adopt the standard sign convention for the resulting complex $\mathcal{C}^{-\bullet}_{\widetilde{X_t}}$. i.e. the differential of $\mathcal{C}^{-\bullet}_{\widetilde{X_t}}$ will be given by $(-1)^r$ times the topological boundary operator ∂ on $\mathcal{C}^{-r}_{\widetilde{X_t}}$. If $d := \dim_{\mathbb{C}}(X_t)$, the complex $\mathcal{C}^{-\bullet}_{\widetilde{X_t}}$ is a resolution of the sheaf $\widetilde{\jmath}_! \mathbb{C}_{U_t^{\mathrm{an}}}[2d]$ which in turn is the dualizing sheaf on the compact real manifold $\widetilde{X_t}$ with boundary $\widetilde{D_t} = \widetilde{X_t} \smallsetminus U_t^{\mathrm{an}}$ (cp. [19]).

Now, let $\mathcal{C}^{-\bullet}_{\widetilde{X_t}, \widetilde{D_t}}$ be the complex of chains relative to the boundary $\widetilde{D_t}$, i.e. the sheaf associated to

$$V \mapsto S_\bullet(\widetilde{X_t}, (\widetilde{X_t} \smallsetminus V) \cup \widetilde{D_t}) \ .$$

Let $c \in \Gamma(V, \mathcal{C}^{-\bullet}_{\widetilde{X_t}, \widetilde{D_t}})$ be a local section over some open $V \subset \widetilde{X_t}$, represented by a piecewise smooth map from the standard p-simplex Δ^p to $\widetilde{X_t}$. We consider c always together with the solution e^{g_t} of the dual of the given connection of exponential type and will therefore write $c \otimes e^{g_t}$ for c in the following definition (in the terminology of [5], we consider $c \otimes e^{g_t}$ as a section in $\mathcal{C}^{-\bullet}_{\widetilde{X_t}, \widetilde{D_t}} \otimes_{\mathbb{Q}} \widetilde{\jmath}_* \mathcal{E}$ where $\mathcal{E}$ denoted the local system associated to the dual bundle $\mathcal{O}_{U_t} e^{-g_t}$, in our case the trivial system with basis e^{g_t}).

Now, consider a point $y \in c(\Delta^p) \cap \widetilde{D_t} \cap V$. Choose local coordinates $x_1, \dots, x_d$ of X_t around $y = 0$ such that $D_t = \{x_1 \cdots x_k = 0\}$.

Definition 2.1. *We call $c \otimes e^{g_t}$ a* rapid decay chain*, if for all y as above, the function $e^{g_t(x)}$ has rapid decay for the argument approaching $\widetilde{D_t}$, i.e. if for all*

$N \in \mathbb{N}^k$ there is a $C_N > 0$ such that

$$|e^{g_t(x)}| \le C_N \cdot |x_1|^{N_1} \cdots |x_k|^{N_k}$$

for all $x \in (c(\Delta^p) \smallsetminus \widetilde{D_t}) \cap V$ with small $|x_1|, \ldots, |x_k|$.

In other terms, the requirement is that $\arg(g_t(x)) \in (\frac{\pi}{2}, \frac{3\pi}{2})$ for all x on c with small $|x|$.

In case $V \cap \widetilde{D_t} = \emptyset$, we do not impose any condition on $c \otimes e^{g_t}$. Let $\mathcal{C}^{\mathrm{rd},-p}_{\widetilde{X_t}}(e^{-g_t})$ denote the subsheaf of $\mathcal{C}^{-p}_{\widetilde{X_t},\widetilde{D_t}} \otimes e^{g_t}$ generated by all rapid decay p-chains and all chains inside U_t. As for the sign notation, note that the dual exponential module is denoted by $\mathcal{O}e^{-g_t}$ and admits e^{g_t} as holomorphic solution. Together with the usual boundary operator of chains, these give the **complex of rapid decay chains** $\mathcal{C}^{\mathrm{rd}}_{\widetilde{X_t}}(e^{-g_t}) := (\mathcal{C}^{\mathrm{rd},-\bullet}_{\widetilde{X_t}}(e^{-g_t}), \partial)$.

Definition 2.2. *The* **rapid decay homology** *of $\mathcal{O}e^{-g_t}$ is the hypercohomology*

$$H_k^{rd}(U_t, e^{-g_t}) := \mathbb{H}^{-k}(\widetilde{X_t}, \mathcal{C}^{\mathrm{rd}}_{\widetilde{X_t}}(e^{-g_t})) \ ,$$

where $\pi : \widetilde{X_t} \to X_t^{\mathrm{an}}$ denotes the oriented real blow-up of the normal crossing divisor $D_t := X_t \smallsetminus U_t$.

In the same way as in [5], Prop. 3.10, one proves that this definition does not depend on the choice of the compactification X_t. We will comment on this later.

The usual barycentric subdivision operator on $\mathcal{C}_{\widetilde{X_t},\widetilde{D_t}}$ induces a subdivision operator on the rapid decay complex. The latter is therefore easily seen to be a homotopically fine complex of sheaves (cp. [18], p. 87). Hence, the rapid decay homology can be computed as the cohomology of the cooresponding complex of global sections:

$$H_k^{rd}(U_t, e^{-g_t}) = H^{-k}\left(\Gamma(\widetilde{X_t}, \mathcal{C}^{\mathrm{rd}}_{\widetilde{X_t}}(e^{-g_t}))\right) \ .$$

In other words, we can regard the elements of this group as relative topological cycles c in $H^k(X_t, D_t, \mathbb{C})$ which satisfy the rapid decay condition for g_t.

Since U_t is assumed to be affine, the de Rham cohomology of $\mathcal{O}_{U_t}e^{g_t}$ can also be computed on the level of global sections

$$H^p_{dR}(U_t, e^{g_t}) := \mathbb{H}^p(U_t, (\Omega^\bullet_{U_t|k}, \nabla_t)) \cong H^p(\ldots \to \Gamma_{U_t}(\Omega^q_{U_t|k}) \xrightarrow{\nabla_t} \Gamma_{U_t}(\Omega^{q+1}_{U_t|k}) \to \ldots) \ ,$$

with $\nabla_t(\omega) = d\omega + dg_t \wedge \omega$ for a local section ω of $\Omega^q_{U_t|k}$.

Now, if we have a global rapid decay chain $c \otimes e^{g_t} \in \Gamma(\widetilde{X_t}, \mathcal{C}^{\mathrm{rd}}_p(e^{-g_t}))$ with respect to the dual bundle $\mathcal{O}_{U_t}e^{-g_t}$ and a meromorphic p-form ω, then the integral $\int_c \omega e^{g_t}$ converges because the rapid decay of e^{g_t} along c annihilates the moderate growth of the meromorphic ω. Let c_τ denote the topological chain one gets by cutting off a small tubular neighborhood with radius τ around the boundary $\partial\Delta^p$ from the given topological chain c. Then, for $c \otimes e^{g_t} \in \mathcal{C}^{\mathrm{rd}}_p(e^{-g_t})(\widetilde{X_t})$ and $\eta \in \Omega^{p-1}(*D_t)$ a meromorphic $(p-1)$-form, we have the **'limit Stokes formula'**

$$\int_c (\nabla_t \eta) e^{g_t} = \lim_{\tau \to 0} \int_{c_\tau} (\nabla_t \eta) e^{g_t} = \lim_{\tau \to 0} \int_{\partial c_\tau} \eta \, e^{g_t} = \int_{\partial c - D_t} \eta \, e^{g_t}$$

where in the last step we used that by the given growth/decay conditions the integral over the faces of ∂c_τ 'converging' against the faces of ∂c contained in D_t vanishes.

The limit Stokes formula easily shows in the standard way that integrating a closed differential form over a given rd-cycle (i.e. with vanishing boundary value) only depends on the de Rham class of the differential form and the rd-homology class of the cycle. Thus, we have:

Proposition 2.3. *Integration induces a well defined bilinear pairing*

$$H^p_{dR}(U_t, e^{g_t}) \times H^{rd}_p(U_t, e^{-g_t}) \longrightarrow \mathbb{C}\ ,\ ([\omega], [c \otimes e^{g_t}]) \mapsto \int_c \omega\, e^{g_t}\ , \tag{2.1}$$

which we call the **period pairing** *of* $\mathcal{O}_{U_t} e^{g_t}$.

2.2. **Duality**

We will now prove the following duality theorem for the connection $\mathcal{O}_{U_t} e^{g_t}$ in arbitrary dimension $\dim U_t$.

Theorem 2.4. *The period pairing* (2.1) *is a perfect pairing of finite dimensional vector spaces.*

Note that one has the analogous result for any connection (E, ∇) with good formal structure in dimension $\dim U_t \le 2$ ([5], or [1] for $\dim U_t = 1$ resp.). The proof given in the following is a generalization of the one in [5] for the special case of a connection $\mathcal{O}_{U_t} e^{g_t}$ to arbitrary dimension $\dim(X_t) \in \mathbb{N}$. For the readers's convenience, we give the outline of the proof and also comment on the necessary changes for higher dimensions.

Proof. The proof will rely on a local duality on $\widetilde{X_t}$. To prepare for it, we will describe the period pairing as a local pairing of complexes of sheaves on $\widetilde{X_t}$. We recall the corresponding notions of [5].

Let $\mathcal{A}^{\mathrm{mod} D_t}_{\widetilde{X_t}}$ denote the sheaf of functions on $\widetilde{X_t}$ which are holomorphic on $U_t^{\mathrm{an}} \subset \widetilde{X_t}$ and of moderate growth along $\pi^{-1}(D_t)$ and let $\mathcal{A}^{<D_t}_{\widetilde{X_t}}$ denote the subsheaf of such functions that are infinitely flat along $\widetilde{D_t}$, i.e. all of whose partial derivatives of arbitrary order vanish on $\widetilde{D_t}$.

These sheaves are both flat over $\pi^{-1}(\mathcal{O}_{X_t})$ and we call

$$\mathrm{DR}^{\mathrm{mod} D_t}_{\widetilde{X_t}}(e^{g_t}) := \mathcal{A}^{\mathrm{mod} D_t}_{\widetilde{X_t}} \otimes_{\pi^{-1}(\mathcal{O}_{X_t})} \pi^{-1}(\mathrm{DR}_{X_t^{\mathrm{an}}}(\mathcal{O}_{X_t}[*D_t] e^{g_t}))$$

the *moderate de Rham complex* of $\mathcal{O}_{U_t} e^{g_t}$ and

$$\mathrm{DR}^{<D_t}_{\widetilde{X_t}}(e^{g_t}) := \mathcal{A}^{<D_t}_{\widetilde{X_t}} \otimes_{\pi^{-1}(\mathcal{O}_{X_t})} \pi^{-1}(\mathrm{DR}_{X_t^{\mathrm{an}}}(\mathcal{O}_{X_t}[*D_t] e^{g_t}))$$

the *asymptotically flat de Rham complex.* Note that the first one computes the meromorphic de Rham cohomology of the meromorphic connection $\mathcal{O}_{X_t}[*D_t] e^{g_t}$ (cp. [15], Corollaire 1.1.8).

Now, the wedge product obviously defines a pairing of complexes of sheaves on $\widetilde{X_t}$:

$$\mathrm{DR}^{\mathrm{mod} D_t}_{\widetilde{X_t}}(e^{g_t}) \otimes_{\mathbb{C}} \mathrm{DR}^{<D_t}_{\widetilde{X_t}}(e^{-g_t}) \to \mathrm{DR}^{<D_t}_{\widetilde{X_t}}(\mathcal{O}_{X_t}, d)\ , \tag{2.2}$$

where the right hand side denotes the asymptotically flat de Rham complex of the trivial connection. We will compare this pairing to the periods pairing from above.

Before that, let us introduce the sheaf $\mathfrak{Db}^{\mathrm{rd}, -s}_{\widetilde{X_t}}$ of rapid decay distributions on $\widetilde{X_t}$, the local section of which are distributions

$$\varphi \in \mathfrak{Db}^{-s}_{\widetilde{X_t}}(V) := \mathfrak{Db}^{-s}_{\widetilde{X_t}}(V) := \mathrm{Hom}_{\mathrm{cont}}(\Gamma_c(V, \Omega^{\infty, s}_{\widetilde{X_t}}), \mathbb{C})$$

on the space $\Omega^{\infty, s}_{\widetilde{X_t}}$ of C^∞ differential forms on $\widetilde{X_t}$ of degree s with compact support in $\widetilde{X_t}$ satisfying the following condition: we choose coordinates $x_1, \ldots, x_n$ on X_t such that locally on V one has $D_t = \{x_1 \cdots x_k = 0\}$. Then we require that for any compact $K \subset V$ and any element $N \in \mathbb{N}^k$ there are $m \in \mathbb{N}$ and $C_{K,N} > 0$ such that for any test form η with compact support in K the estimate

$$|\varphi(\eta)| \le C_{K,N} \sum_i \sup_{|\alpha|\le m} \sup_K \{|x|^N |\partial^\alpha f_i|\} \tag{2.3}$$

holds, where α runs over all multi-indices of degree less than or equal to m and ∂^α denotes the α-fold partial derivative of the coefficient functions f_i of η in the chosen coordinates.

Integration along chains induces a pairing of complexes of sheaves

$$\mathrm{DR}^{\mathrm{mod} D_t, s}_{\widetilde{X_t}}(e^{g_t}) \otimes \mathcal{C}^{\mathrm{rd},-r}_{\widetilde{X_t}}(e^{-g_t}) \to \mathfrak{Db}^{\mathrm{rd},s-r}_{\widetilde{X_t}} \tag{2.4}$$

to be defined as follows: for any local s-form ω of $\mathrm{DR}^{\mathrm{mod} D_t}_{\widetilde{X_t}}(e^{g_t})$, any rapid decay chain $c \otimes e^{g_t} \in \Gamma(V, \mathcal{C}^{\mathrm{rd},-r}_{\widetilde{X_t}}(e^{-g_t}))$ and any test form $\eta \in \Gamma_c(V, \Omega^{\infty,p}_{\widetilde{X_t}})$ with $p = r - s$, the decay and growth assumptions ensure that the integral

$$\int_c \eta \wedge (e^{g_t} \cdot \omega) \tag{2.5}$$

converges and moreover, that the distribution that maps η to (2.5) satisfies the condition (2.3) from above.

Note that $\mathfrak{Db}^{\mathrm{rd},-\bullet}_{\widetilde{X_t}}$ is a fine resolution of $\widetilde{\jmath}_! \mathbb{C}_{U_t^{\mathrm{an}}}[2d]$ where $\widetilde{\jmath} : U_t^{\mathrm{an}} \hookrightarrow \widetilde{X_t}$ denotes the inclusion and $d := \dim_{\mathbb{C}}(X_t)$. It follows that $\mathbb{H}^0(\widetilde{X_t}, \mathfrak{Db}^{\mathrm{rd},\bullet}_{\widetilde{X_t}}) \cong \mathbb{C}$ in a standard way. The resulting pairing $H^p_{dR}(U_t, e^{g_t}) \times H^{rd}_p(U_t, e^{-g_t}) \to \mathbb{C}$ induced by (2.4) on cohomology in degree zero coincides with the period pairing (Proposition 2.3).

The proof of Theorem 2.4 now splits into the following intermediate steps (cp. with [5]):

Claim 1. *The complexes* $\mathrm{DR}^{\mathrm{mod} D_t}_{\widetilde{X_t}}(e^{g_t})$ *and* $\mathrm{DR}^{<D_t}_{\widetilde{X_t}}(e^{-g_t})$ *have cohomology in degree zero only.*

Proof. — In both cases the argument relies on an existence result of a certain linear partial differential system with moderate or rapidly decaying coefficients. To be more precise, consider the local situation at some point $x_0 \in D_t = \{x_1 \cdots x_k = 0\}$ and let $\vartheta \in \pi^{-1}(x_0) \simeq (S^1)^k$ be a direction in $\widetilde{D_t}$ over x_0. Then the complex of stalks at ϑ which we have to consider is given as

$$\ldots \longrightarrow \Big(\mathcal{A}^{?D_t}_{\widetilde{X_t}} \otimes_{\pi^{-1}\mathcal{O}_{X_t}} \pi^{-1}\Omega^p_{X_t}\Big)_\vartheta \xrightarrow{\nabla_t} \Big(\mathcal{A}^{?D_t}_{\widetilde{X_t}} \otimes_{\pi^{-1}\mathcal{O}_{X_t}} \pi^{-1}\Omega^{p+1}_{X_t}\Big)_\vartheta \longrightarrow \ldots\ , \tag{2.6}$$

where ? stands for either $<$ or mod. If we describe $\Omega^p_{X_t}$ in terms of the local basis dx_I for all $I = \{1 \le i_1 < \ldots < i_p \le d\}$, the covariant derivation ∇_t in degree p reads as

$$\sum_{\#I=p} w_I\, dx_I \mapsto \sum_{\#J=p+1} \Big(\sum_{j\in J} sgn_J(j)(Q_j w_{J\smallsetminus\{j\}})\Big) dx_J$$

where

$$Q_j u := \frac{\partial}{\partial x_j} u \pm \frac{\partial g_t}{\partial x_j} \cdot u\ ,$$

with the negative sign in case of e^{g_t} and the positive in case of e^{-g_t}. Also, we let $sgn_J(j) = (-1)^\nu$ for $J = \{j_1 < \ldots < j_{p+1}\}$ and $j_\nu = j$.

Now, let ω be a germ of a section of $\mathcal{A}^{?D_t}_{\widetilde{X_t}} \otimes \pi^{-1}\Omega^{p+1}_{X_t}$ written in the chosen coordinates as

$$\omega = \sum_{\#J=p+1} w_J\, dx_J\ ,$$

such that $\nabla_t \omega = 0$. We have to find a p-form η with appropriate growth condition such that $\nabla_t \eta = \omega$.

To this end, let $r \in \mathbb{N}$ be an integer such that $w_J = 0$ for all J with $J \cap \{1, \dots, r-1\} \neq \emptyset$ (which is an empty condition for $r = 1$). We prove that we can find a p-form η with coefficients in $\mathcal{A}^{?D_t}_{\widetilde{X}_t}$ such that

$$(\omega - \nabla_t \eta) \in \sum_{J \cap \{1,\dots,r\} = \emptyset} \mathcal{A}^{?D_t}_{\widetilde{X}_t, \vartheta}\, dx_J \ . \tag{2.7}$$

Successive application of this argument will prove the claim.

By assumption

$$0 = \nabla_t \omega =: \sum_{\# K = p+2} \Big(\sum_{k \in K} sgn_K(k) (Q_k w_{K \smallsetminus \{k\}}) \Big) dx_K \ . \tag{2.8}$$

Taking $k < r$ and $r \in J$ and examining the summand of (2.8) corresponding to the set $K := \{s\} \cup J$ we see that

$$Q_s w_J = 0 \quad \text{for all such } s = 1, \dots, r-1 \ . \tag{2.9}$$

Now, consider the system (Σ_J) of partial differential equations for the unknown function u_J, where J is a fixed subset $J \subset \{r, \dots, d\}$ of cardinality $p+1$ with $r \in J$:

$$(\Sigma_J) : \begin{cases} Q_s u_J = 0 & \text{for all } s = 1, \dots, r-1 \\ Q_r u_J = w_J \ , & \end{cases}$$

together with the integability assumption (2.9). Systems of this type had been studied by Majima ([9]) before. See for example [16], Appendix A for a presentation from which follows that this system always has a solution in $\mathcal{A}^{<D_t}_{\widetilde{X}_t, \vartheta}$ if the coefficients of the system belong to the same space of functions. In the case of moderate growth coefficients/solutions, the assertion follows from [5], Theorem A.1 (which is formulated for dimension 2 only, but generalizes one-to-one to the case of arbitrary dimension).

In each case, we can always find a solution $u_J \in \mathcal{A}^{?D_t}_{\widetilde{X}_t, \vartheta}$ for any such $J \subset \{r, \dots, d\}$ of cardinality $p+1$ and $r \in J$ and if we let

$$\eta := \sum_{J \text{ as above}} u_J \, dx_J \ ,$$

we easily see that (2.7) is satisfied.

Claim 1 □

Claim 2. *The local pairing* (2.2) *is perfect in the derived sense (i.e. the induced morphism*

$$\mathrm{DR}^{\mathrm{mod} D_t}_{\widetilde{X}_t}(e^{g_t}) \to \boldsymbol{R}\mathrm{Hom}_{\widetilde{X}_t}(\mathrm{DR}^{<D_t}_{\widetilde{X}_t}(e^{-g_t}), \widetilde{\jmath}_! \mathbb{C})$$

and the analogous one with $\mathrm{DR}^{<D_t}_{\widetilde{X}_t}$ *and* $\mathrm{DR}^{\mathrm{mod} D_t}_{\widetilde{X}_t}$ *exchanged are isomorphisms).*

Proof. — According to Claim 1 both complexes have cohomology in degree 0 only, i.e. we are reduced to look at the pairing of sheaves

$$\mathcal{S}^{\mathrm{mod} D_t} \otimes {}^{\vee}\mathcal{S}^{<D_t} \to \widetilde{\jmath}_! \mathbb{C}_{U_t}$$

with $\mathcal{S}^{\mathrm{mod} D_t} := \mathcal{H}^0(\mathrm{DR}^{\mathrm{mod} D_t}_{\widetilde{X}_t}(e^{g_t}))$ and ${}^{\vee}\mathcal{S}^{<D_t} := \mathcal{H}^0(\mathrm{DR}^{<D_t}_{\widetilde{X}_t}(e^{-g_t}))$. Again, we consider the local situation $D_t = \{x_1 \cdots x_k = 0\}$ and $g_t(x) = x_1^{-m_1} \cdots x_k^{-m_k} u(x)$, which we restrict to a samll enough open polysector $V \subset \widetilde{X}_t \xrightarrow{\pi} X_t$. We define the Stokes multi-directions of g_t along D_t inside V to be

$$\Sigma^{D_t}_{g_t} := St^{-1}\big((\frac{\pi}{2}, \frac{3\pi}{2}) \big) \ ,$$

where $St : \pi^{-1}(D_t) \cap V \to \mathbb{R}/2\pi\mathbb{Z}$ denotes the map

$$St(r_i, \vartheta_i) := -\sum_{i=1}^{k} m_i \vartheta_i + \arg(u \circ \pi(r_i, \vartheta_i)) \ .$$

Let $V_{g_t} := (V \smallsetminus \pi^{-1}(D_t)) \cup \Sigma_{g_t}^{D_t}$, i.e. $V_{g_t} \cap \pi^{-1}(D_t)$ are the directions in which $e^{g_t(x)}$ has rapid decay for x radially approaching D_t. If $\widetilde{\jmath}_{g_t} : V_{g_t} \hookrightarrow V$ denotes the inclusion, one obviously has (possibly after shrinking V):

$$\mathcal{S}^{\mathrm{mod} D_t}|_V = \widetilde{\jmath}_{-g_t}(e^{-g_t(x)} \cdot \mathbb{C}_{U_t})|_V \quad \text{and} \quad {}^{\vee}\mathcal{S}^{<D_t}|_V = \widetilde{\jmath}_{g_t}(e^{g_t(x)} \cdot \mathbb{C}_{U_t})|_V \ . \tag{2.10}$$

Consequently we have the following commutative diagramm:

$$\begin{tikzcd}
\boldsymbol{R}\mathcal{H}om({}^{\vee}\mathcal{S}^{<D_t}, \widetilde{\jmath}_! \mathbb{C}_{X_t \smallsetminus D_t})|_V \arrow[r] \arrow[d, "\cong"'] & \mathcal{S}^{\mathrm{mod} D_t}|_V \arrow[d, "\cong"] \\
\boldsymbol{R}\mathcal{H}om\big((\widetilde{\jmath}_{-g_t})_! \mathbb{C}_{V \smallsetminus D_t} \, , \, \widetilde{\jmath}_! \mathbb{C}_{V \smallsetminus D_t}\big)|_V \arrow[r] & (\widetilde{\jmath}_{g_t})_! \, \mathbb{C}_{V_{g_t}}
\end{tikzcd} \tag{2.11}$$

By the factorization $\widetilde{\jmath} = \widetilde{\jmath}_{-g_t} \circ \widetilde{\iota}_{-g_t}$ with $\widetilde{\iota}_{-g_t} : V \smallsetminus \pi^{-1}(D_t) \hookrightarrow V_{-g_t}$, we see that

$$\begin{aligned}
\boldsymbol{R}\mathcal{H}om\big((\widetilde{\jmath}_{-g_t})_! \mathbb{C}_{V_{-g_t}} \, , \, \widetilde{\jmath}_! \mathbb{C}_{X_t \smallsetminus D_t}\big) &\cong (\widetilde{\jmath}_{-g_t})_* \boldsymbol{R}\mathcal{H}om\big(\mathbb{C}_{V_{-g_t}}, (\widetilde{\iota}_{-g_t})_! \mathbb{C}_{V \smallsetminus \pi^{-1}(D_t)}\big) \\
&\cong (\widetilde{\jmath}_{-g_t})_* \mathcal{H}om\big(\mathbb{C}_{V_{-g_t}}, (\widetilde{\iota}_{-g_t})_! \mathbb{C}_{V \smallsetminus \pi^{-1}(D_t)}\big) = (\widetilde{\jmath}_{-g_t})_! \, \mathbb{C}_{V_{g_t}} \ ,
\end{aligned}$$

since $(V \smallsetminus V_{g_t}) \cap \pi^{-1}(D_t)$ coincides with the closure of $V_{-g_t} \cap \pi^{-1}(D_t)$ inside $\pi^{-1}(D_t)$. Hence, the bottom line of (2.11) is an isomorphism and thus

$$\mathcal{S}^{\mathrm{mod} D_t} \cong \boldsymbol{R}\mathcal{H}om_{\widetilde{X_t}}\big({}^{\vee}\mathcal{S}^{<D_t}, \widetilde{\jmath}_! \mathbb{C}_{X_t \smallsetminus D_t}\big)$$

locally on $\widetilde{X_t}$ over an arbitrary point of D_t. Interchanging ${}^{\vee}\mathcal{S}^{<D_t}$ and $\mathcal{S}^{\mathrm{mod} D_t}$ gives the analogous isomorphism.

Claim 2 $\square$

Claim 3. *There is a canonical isomorphism*

$$\mathcal{C}^{\mathrm{rd}}_{\widetilde{X_t}}(e^{-g_t}) \cong \mathrm{DR}^{<D_t}_{\widetilde{X_t}}(e^{-g_t})[2d]$$

in the derived category $D^b(\mathbb{C}_{\widetilde{X_t}})$, where $d = \dim_{\mathbb{C}}(X_t)$.

Proof. — Here, the arguments of [5], Theorem 3.6, apply directly, we therefore only briefly give the main line of the proof: By Claim 1 we have

$$\mathrm{DR}^{<D_t}_{\widetilde{X_t}}(e^{-g_t}) \xleftarrow{\sim} \mathcal{H}^0(\mathrm{DR}^{<D_t}_{\widetilde{X_t}}(e^{-g_t})) =: {}^{\vee}\mathcal{S}^{<D_t} \ .$$

Additionally, there is a canonical morphism

$$\mathcal{C}^{-\bullet}_{\widetilde{X_t}, \widetilde{D_t}} \otimes {}^{\vee}\mathcal{S}^{<D_t} \longrightarrow \mathcal{C}^{\mathrm{rd}}_{\widetilde{X_t}}(e^{-g_t}) \ ,$$

since for any open $V \subset \widetilde{X_t}$ and an asymptotically flat section $\sigma \in \Gamma_V({}^{\vee}\mathcal{S}^{<D_t})$, the section $\sigma \in {}^{\vee}\mathcal{S}^{<D_t}(V) \subset \widetilde{\jmath}_*(e^{g_t}\mathbb{C}_{U_t})$ is rapidly decaying along any chain in $c \in \mathcal{C}^{-\bullet}_{\widetilde{X_t}, \widetilde{D_t}}(V)$, hence $c \otimes \sigma \in \Gamma(V, \mathcal{C}^{\mathrm{rd}}_{\widetilde{X_t}}(e^{-g_t}))$. We prove that this morphism is a quasi-isomorphism. Restricted to $U_t \subset \widetilde{X_t}$, this is obvious.

Next, consider a small open polysector V around some $\vartheta \in \pi^{-1}(x_0)$ with $x_0 \in D_t$. Let $\Sigma^{D_t}_{\pm g_t} \subset \pi^{-1}(x_0)$ denote the set of Stokes-directions of $e^{\pm g_t}$ as in the proof of Claim 1.

If $\vartheta \in \Sigma^{D_t}_{-g_t}$ then for V being a small enough polysector, we have ${}^{\vee}\mathcal{S}^{<D_t}|_V = \widetilde{\jmath}_!\big(e^{g_t}\mathbb{C}_{U_t}\big)|_V$. For a smooth topological chain c in $\widetilde{X_t}$, the local section e^{g_t} will not have rapid decay along c in V as required by the definition unless the chain does not meet $\widetilde{D_t} \cap V$. Hence

$$\mathcal{C}^{\mathrm{rd}}_{\widetilde{X_t}}(e^{-g_t})|_V = \mathcal{C}^{-\bullet}_{\widetilde{X_t}, \widetilde{D_t}} \otimes {}^{\vee}\mathcal{S}^{<D_t}|_V \ .$$

If $\vartheta \in \Sigma_{g_t}^{D_t}$, we can assume that V is an open polysector such that all the arguments of points in V are contained in $\Sigma_{g_t}^{D_t}$. Then ${}^\vee\mathcal{S}^{<D_t}|_V \cong \widetilde{\jmath}_*(e^{g_t}\mathbb{C}_V)$. Similarly, all twisted chains $c \otimes e^{g_t}$ will have rapid decay inside V and again both complexes considered are equal to $\mathcal{C}^{-\bullet}_{\widetilde{X}_t,\widetilde{D}_t} \otimes \widetilde{\jmath}_*(e^{g_t}\mathbb{C}_{U_t})$.

Finally, if ϑ separates the Stokes regions of e^{g_t} and e^{-g_t}, we have with the notation of (2.10):

$$ {}^\vee\mathcal{S}^{<D_t}|_V \cong (\widetilde{\jmath}_{g_t})_!(e^{g_t}\mathbb{C}_{U_t})|_V \ . $$

The subspace V_{g_t} is characterized by the property that $V_{g_t} \cap \widetilde{D_t}$ consists of those directions along which $e^{g_t(x)}$ has rapid decay for x approaching $\widetilde{D_t}$. In particular, $c \otimes e^{g_t}$ is a rapid decay chain on V if and only if the topological chain c in $\widetilde{X_t}$ approaches $\widetilde{D_t} \cap V$ in V_{g_t} at most. Hence

$$ \mathcal{C}^{\mathrm{rd}}_{\widetilde{X}_t}(e^{-g_t})|_V = \mathcal{C}^{-\bullet}_{\widetilde{X}_t,\widetilde{D}_t} \otimes (\widetilde{\jmath}_{g_t})_!(e^{g_t}\mathbb{C}_{U_t}) = \mathcal{C}^{-\bullet}_{\widetilde{X}_t,\widetilde{D}_t} \otimes {}^\vee\mathcal{S}^{<D_t}|_V \ . $$

In summary, we have the following composition of quasi-isomorphisms

$$ {}^\vee\mathcal{S}^{<D_t}[2d] \xrightarrow{\simeq} \mathbb{C}_{\widetilde{X}_t}[2d] \otimes {}^\vee\mathcal{S}^{<D_t} \xrightarrow{\simeq} \mathcal{C}^{-\bullet}_{\widetilde{X}_t,\widetilde{D}_t} \otimes {}^\vee\mathcal{S}^{<D_t} \xrightarrow{\simeq} \mathcal{C}^{\mathrm{rd}}_{\widetilde{X}_t}(e^{-g_t}) \ . $$

Claim 3 □

In order to prove Theorem 2.4, consider the resolution

$$ \mathcal{A}^{<D_t}_{\widetilde{X}_t} \otimes_{\pi^{-1}\mathcal{O}_{X_t}} \pi^{-1}\Omega^r_{X_t} \hookrightarrow \big(\mathcal{P}^{<D_t}_{\widetilde{X}_t} \otimes_{\pi^{-1}C^\infty_{X_t}} \pi^{-1}\Omega^{\infty,(r,\bullet)}_{X_t}, \overline{\partial}\big) \ , $$

where $\mathcal{P}^{<D_t}_{\widetilde{X}_t}$ denotes the sheaf of C^∞-functions flat at $\pi^{-1}(D_t)$ and $\Omega^{\infty,(r,s)}_{X_t}$ denotes the sheaf of C^∞ forms on X_t of degree (r,s). We then have the bicomplex

$$ \mathcal{R}d^{\bullet,\bullet} := \big(\mathcal{P}^{<D_t}_{\widetilde{X}_t} \otimes_{\pi^{-1}C^\infty_{X_t}} \pi^{-1}\Omega^{\infty,(\bullet,\bullet)}_{X_t}, \partial, \overline{\partial}\big) \ , $$

whose total complex $\mathcal{R}d^\bullet$ computes is a fine resolution of $\widetilde{\jmath}_!\mathbb{C}_{U_t}$.

According to Claim 1, the local duality pairing (2.2) reads as ${}^\vee\mathcal{S}^{<D_t} \otimes \mathcal{S}^{\mathrm{mod}D_t} \to \widetilde{\jmath}_!\mathbb{C}_{U_t^{\mathrm{an}}}$. We now have a canonical quasi-isomorphism

$$ \beta : \mathcal{C}^{-\bullet}_{\widetilde{X}_t,\widetilde{D}_t} \otimes \mathcal{R}d^\bullet \xrightarrow{\simeq} \mathfrak{Db}^{\mathrm{rd},-\bullet}_{\widetilde{X}_t} $$

of complexes mapping an element $c \otimes \rho \in \mathcal{C}^{-r}_{\widetilde{X}_t,\widetilde{D}_t} \otimes \mathcal{R}d^s(V)$ of the left hand side over some open $V \subset \widetilde{X_t}$ to the distribution given by $\eta \mapsto \int_c \eta \wedge \rho$ for a test form η with compact support in V. The latter represents a local section of $\mathfrak{Db}^{\mathrm{rd},s-r}_{\widetilde{X}_t}$ since ρ is rapidly decaying. Additionally, we have to consider the morphism

$$ \gamma : \mathcal{C}^{\mathrm{rd}}_{\widetilde{X}_t}(e^{-g_t}) \otimes \mathcal{S}^{\mathrm{mod}D_t} \to \mathfrak{Db}^{\mathrm{rd},-\bullet}_{\widetilde{X}_t} \ , \ (c \otimes e^{g_t}) \otimes \sigma \mapsto (\eta \mapsto \int_c e^{g_t} \cdot \sigma \cdot \eta) \ , $$

which induces the period pairing after taking cohomology.

These morphisms fit into the following commutative diagramm

$$ \begin{array}{ccc} {}^\vee\mathcal{S}^{<D_t}[2d] \otimes \mathcal{S}^{\mathrm{mod}D_t} & \longrightarrow & \widetilde{\jmath}_!\mathbb{C}_{U_t^{\mathrm{an}}}[2d] \\ \simeq\big\downarrow & & \big\downarrow\simeq \\ \mathcal{C}^{-\bullet}_{\widetilde{X}_t,\widetilde{D}_t} \otimes {}^\vee\mathcal{S}^{<D_t} \otimes \mathcal{S}^{\mathrm{mod}D_t} & \longrightarrow & \mathcal{C}^{-\bullet}_{\widetilde{X}_t,\widetilde{D}_t} \otimes \mathcal{R}d^\bullet \\ \simeq\big\downarrow \text{Claim 3} & & \simeq\big\downarrow \beta \\ \mathcal{C}^{\mathrm{rd}}_{\widetilde{X}_t}(e^{-g_t}) \otimes \mathcal{S}^{\mathrm{mod}D_t} & \xrightarrow{\gamma} & \mathfrak{Db}^{\mathrm{rd},-\bullet}_{\widetilde{X}_t} \ . \end{array} \tag{2.12} $$

The top row is a perfect duality in the derived sense by Claim 2. Applying Poincaré-Verdier duality thus induces an isomorphism

$$(2.13)\quad \boldsymbol{R}\Gamma_{\widetilde{X}_t}\mathrm{DR}^{<D_t}_{\widetilde{X}_t}(e^{-g_t})[2d] \cong \boldsymbol{R}\Gamma_{\widetilde{X}_t}\boldsymbol{R}\mathcal{H}om_{\widetilde{X}_t}(\mathrm{DR}^{\mathrm{mod}D_t}_{\widetilde{X}_t}(e^{g_t}), \widetilde{\jmath}_!\mathbb{C}_{U_t}[2d]) \cong$$
$$\cong \mathrm{Hom}^{\bullet}_{\mathbb{C}}(\boldsymbol{R}\Gamma_{\widetilde{X}_t}\mathrm{DR}^{\mathrm{mod}D_t}_{\widetilde{X}_t}(e^{g_t}), \mathbb{C})\ .$$

The commutativity of (2.12) shows that the morphism induced by the period pairing i.e. by γ, namely

$$\boldsymbol{R}\Gamma_{\widetilde{X}_t}\mathcal{C}^{\mathrm{rd}}_{\widetilde{X}_t}(e^{-g_t}) \longrightarrow \mathrm{Hom}^{\bullet}_{\mathbb{C}}(\boldsymbol{R}\Gamma_{\widetilde{X}_t}\mathrm{DR}^{\mathrm{mod}D_t}_{\widetilde{X}_t}(e^{g_t}), \mathbb{C})\ ,$$

is an isomorphism. The same holds after exchanging $\mathcal{C}^{\mathrm{rd}}_{\widetilde{X}_t}$ and $\mathrm{DR}^{\mathrm{mod}D_t}_{\widetilde{X}_t}$. Taking p-th cohomology therefore yields the perfect period pairing

$$H^p_{dR}(U_t, e^{g_t}) \otimes_{\mathbb{C}} H^{rd}_p(U_t, e^{-g_t}) \to \mathbb{C}$$

completing the proof of Theorem 2.4.

□

We will now study the exponential Gauß-Manin systems and apply these considerations to describe the holomorphic solutions of the latter.

3. Holomorphic solutions of exponential Gauß-Manin systems

Consider the situation described in the introduction. Let $f, g : U \to \mathbb{A}^1$ be two algebraically independant regular functions on the smooth affine variety U. We denote by X a smooth projective compactification of U, on which $f, g : U \to \mathbb{A}^1$ extend to functions $F, G : X \to \mathbb{P}^1$. Let $D := X \setminus U$ and assume that it is a normal crossing divisor.

3.1. Definition

We define an exponential Gauß-Manin systems as being the cohomology sheaves of the direct image by f of the module of exponential type $\mathcal{O}_U e^g$:

$$\mathcal{G}^k := \mathcal{H}^k f_+(\mathcal{O}_U e^g).$$

Let Ω^k_U be the sheaf of algebraic differential forms of degree k on U. Using the canonical factorisation of f into a closed embedding and a projection, we prove as in [4] that:

Proposition 3.1. *$f_+(\mathcal{O}_U e^g)$ is isomorphic to $f_*(\Omega^{\bullet+n}_U \otimes_{\mathbb{C}} \mathbb{C}[\partial_t])$, where the differential ∇ on $\Omega^{\bullet+n}_U \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$ is given by*

$$\nabla(w \otimes \partial_t^i) = dw \otimes \partial_t^i + dg \wedge w \otimes \partial_t^i - df \wedge w \otimes \partial_t^{i+1}.$$

Its cohomology modules $\mathcal{G}^k$ are equipped with a structure of $\mathcal{D}_{\mathbb{A}^1}$-module given by:

$$t[w\partial_t^i] = [fw\partial_t^i - iw\partial_t^{i-1}],$$
$$\partial_t[w\partial_t^i] = [w\partial_t^{i+1}].$$

They are holonomic $\mathcal{D}_{\mathbb{A}^1}$-modules with irregular singularities on the affine line $\mathbb{A}^1$ and at infinity.

Remark 3.2. *The interpretation of the regular singularities of $\mathcal{G}^k$ in terms of invariants associated to f and g is still unknown. Nevertheless, in* [13] *and* [14]*, a complete description of the irregular singularities is given. Let $\Delta_k \subset \mathbb{A}^1 \times \mathbb{A}^1$ be the singular support of the holonomic $\mathcal{D}_{\mathbb{A}^1\times\mathbb{A}^1}$-module $\mathcal{H}^k(f, g)_+(\mathcal{O}_U)$. Its closure $\overline{\Delta_k}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is included in the projective variety $\overline{\Delta}$, such that $(F, G) : (F, G)^{-1}((\mathbb{P}^1 \times \mathbb{P}^1) \setminus \overline{\Delta}) \to (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \overline{\Delta}$ is a locally trivial fibration. $t_0 \in \mathbb{A}^1 \cup \{\infty\}$ is an irregular* *singularity of $\mathcal{G}^k$ if and only if the germ of $\overline{\Delta_k}$ at (t_0,∞) is not empty and not included in $\{t_0\}\times\mathbb{P}^1$.*

3.2. **Results on the exponential Gauß-Manin systems**

3.2.1. **Exponential Gauß-Manin connection and its sheaf of flat sections**

Let Σ_1 be a finite subset of $\mathbb{A}^1$ such that:

- $f : f^{-1}(\mathbb{A}^1\setminus\Sigma_1)\to\mathbb{A}^1\setminus\Sigma_1$ is a locally trivial fibration. For any $t\in\mathbb{A}^1\setminus\Sigma_1$, $f^{-1}(t)$ is a smooth affine variety. We will write g_t instead of $g_{|f^{-1}(t)}$.
- $\mathcal{G}^k_{|\mathbb{A}^1\setminus\Sigma_1}$ is locally free, hence a flat connection, for any $k\in\mathbb{Z}$ ($\mathcal{G}^k$ is holonomic).

Let $(\Gamma_{f^{-1}(\mathbb{A}^1\setminus\Sigma_1)}(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1}),\nabla)$ be the complex of relative algebraic differential forms on $f^{-1}(\mathbb{A}^1\setminus\Sigma_1)$ with respect to f equipped with the differential defined by $\nabla(w)=dw+dg\wedge w$.

Proposition 3.3. i) *$\Gamma(\mathbb{A}^1\setminus\Sigma_1,\mathcal{G}^k)$ is a $\mathcal{D}(\mathbb{A}^1\setminus\Sigma_1)$-module isomorphic to $H^k(\Gamma_{f^{-1}(\mathbb{A}^1\setminus\Sigma_1)}(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1}),\nabla)$. The action of $\mathcal{D}(\mathbb{A}^1\setminus\Sigma_1)$ on a form $[w]\in H^k(\Gamma_{f^{-1}(\mathbb{A}^1\setminus\Sigma_1)}(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1}),\nabla)$ is given by:*

$$\begin{aligned} t.[w] &= [fw],\\ \partial_t.[w] &= [\eta],\end{aligned}$$

where $[\eta]\in H^k(\Gamma_{f^{-1}(\mathbb{A}^1\setminus\Sigma_1)}(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1}),\nabla)$ satisfies $dw+dg\wedge w=df\wedge\eta$ (which comes from the condition $\nabla(w)=0$).

ii) *The sheaf of flat sections of $\mathcal{G}^k$ is a local system on $\mathbb{A}^1\setminus\Sigma_1$ and the stalk at the point t is $H^{k+n-1}_{dR}(f^{-1}(t),e^{g_t})$.*

This local system on $\mathbb{A}^1\setminus\Sigma_1$ will be denoted by $\widetilde{H^{k+n-1}_{dR}}$ and we will consider (H^{k+n-1}_{dR},∇) its associated flat vector bundle.

The action of t,∂_t does not depend on the class of w in $H^k(\Gamma_{f^{-1}(\mathbb{A}^1\setminus\Sigma_1)}(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1}),\nabla)$. Indeed, if $[w]=0$, we have $w=df\wedge\alpha+d\beta+dg\wedge\beta$, where α and β are some forms. Then,

- $t.[w]=[fw]=[df\wedge(f\alpha-\beta)+d(f\beta)+dg\wedge f\beta]=0$.
- As $dw+dg\wedge w=-df\wedge(d\alpha+dg\wedge\alpha)$, we have $\partial_t.[w]=-[d\alpha+dg\wedge\alpha]=0$.

Proof.

i) As $f_{|f^{-1}(\mathbb{A}^1\setminus\Sigma_1)}$ is smooth, $f_+(\mathcal{O}_Ue^g)_{|\mathbb{A}^1\setminus\Sigma_1}=\boldsymbol{R}f_*(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1})_{|\mathbb{A}^1\setminus\Sigma_1}$ (cf. Proposition 1.4 in [3]). Then, $\Gamma(\mathbb{A}^1\setminus\Sigma_1,\mathcal{G}^k)=\Gamma(\mathbb{A}^1\setminus\Sigma_1,\mathcal{H}^k\boldsymbol{R}f_*(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1}))$. But we have the spectral sequence:

$$H^i(\mathbb{A}^1\setminus\Sigma_1,\mathcal{H}^{k-i}\boldsymbol{R}f_*(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1}))\Longrightarrow\mathbb{H}^k(\mathbb{A}^1\setminus\Sigma_1,\boldsymbol{R}f_*(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1})).$$

As $\mathbb{A}^1\setminus\Sigma_1$ is affine and $\mathcal{H}^{k-i}\boldsymbol{R}f_*(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1})$ is coherent, this spectral sequence degenerates and we have:

$$\Gamma(\mathbb{A}^1\setminus\Sigma_1,\mathcal{H}^k\boldsymbol{R}f_*(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1}))=\mathbb{H}^k(\mathbb{A}^1\setminus\Sigma_1,\boldsymbol{R}f_*(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1})).$$

Then,

$$\begin{aligned}\Gamma(\mathbb{A}^1\setminus\Sigma_1,\mathcal{G}^k) &= \mathbb{H}^k(\mathbb{A}^1\setminus\Sigma_1,\boldsymbol{R}f_*(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1})),\\ &= \mathbb{H}^k(f^{-1}(\mathbb{A}^1\setminus\Sigma_1),\Omega^{\bullet+n-1}_{U|\mathbb{A}^1}),\\ &= H^k(\Gamma_{f^{-1}(\mathbb{A}^1\setminus\Sigma_1)}(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1}),\nabla).\end{aligned}$$

The last isomorphism comes from the coherence of $\Omega^{\bullet+n-1}_{U|\mathbb{A}^1\ |f^{-1}(\mathbb{A}^1\setminus\Sigma_1)}$ and the fact that $f^{-1}(\mathbb{A}^1\setminus\Sigma_1)$ is affine.

Concerning the action of t and ∂_t on $H^k(\Gamma_{f^{-1}(\mathbb{A}^1\setminus\Sigma_1)}(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1}),\nabla)$, we have to explicite the previous isomorphism. In fact, we can prove that it is given by:

$$\begin{array}{rlcl}\Psi: & H^k(\Gamma_{f^{-1}(\mathbb{A}^1\setminus\Sigma_1)}(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1}),\nabla) & \to & \Gamma(\mathbb{A}^1\setminus\Sigma_1,\mathcal{G}^k)) \\ & [w] & \mapsto & [-df\wedge w\otimes 1].\end{array}$$

The injectivity and surjectivity of Ψ is proved with similar calculations as in [10] (p.160) and is left to the reader. It is a morphism of $\mathcal{D}(\mathbb{A}^1\setminus\Sigma_1)$-modules because:

- $\psi(t[w])=\psi([fw])=[-fdf\wedge w\otimes 1]=t.[-df\wedge w\otimes 1]=t.\psi([w])$.
- $\psi(\partial_t[w])=\psi([\eta])=[-df\wedge\eta\otimes 1]=[-(dw+dg\wedge w)\otimes 1]=[-df\wedge w\otimes \partial_t]=\partial_t.[-df\wedge w\otimes 1]=\partial_t.\psi([w])$.

ii) We recall the proof given in [8]. As $\mathcal{G}^k$ is a flat connection on $\mathbb{A}^1\setminus\Sigma_1$, the sheaf of flat sections of $\mathcal{G}^k$ is a local system on $\mathbb{A}^1\setminus\Sigma_1$ and the stalk at the point t is $i_t^+\mathcal{G}^k=\mathcal{H}^k i_t^+ f_+(\mathcal{O}_U e^g)$. Now, according to the base change formula (cf. Theorem 8.4 p. 267 in [2]) and as $f_{|f^{-1}(\mathbb{A}^1\setminus\Sigma_1)}$ is smooth,

$$\begin{array}{rcl} i_t^+\mathcal{G}^k & = & \mathcal{H}^k f_{t+}(\mathcal{O}_{f^{-1}(t)}e^{g_t}),\ \text{with } f_t:=f_{|f^{-1}(t)}:f^{-1}(t)\to\{t\},\\ & = & H^{k+n-1}_{dR}(f^{-1}(t),e^{g_t}).\end{array}$$

□

3.2.2. Analytification of $\mathcal{G}^k$

We will differentiate the analytic and the algebraic settings by writing an exponent ${}^{\mathrm{an}}$ in the analytic situation.

Let $(\mathcal{G}^k)^{\mathrm{an}}:=\mathcal{O}^{\mathrm{an}}_{\mathbb{C}}\otimes_{\mathcal{O}_{\mathbb{A}^1}}\mathcal{G}^k$ be the analytification of $\mathcal{G}^k$.

¿From Proposition 3.3 we easily deduce:

Proposition 3.4. *Let V be an open subset of $\mathbb{C}\setminus\Sigma_1$.*

i) *$(\mathcal{G}^k)^{\mathrm{an}}(V)$ is isomorphic to the space of holomorphic sections of (H^{k+n-1}_{dR},∇) on V.*

ii) *$(\mathcal{G}^k)^{\mathrm{an}}(V)$ is isomorphic to $\mathcal{O}^{\mathrm{an}}(V)\otimes_{\mathbb{C}[t]}H^k(\Gamma_{f^{-1}(\mathbb{A}^1\setminus\Sigma_1)}(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1}),\nabla)$; the action of $\mathcal{D}^{\mathrm{an}}(V)$ on $a\otimes[w]\in\mathcal{O}^{\mathrm{an}}(V)\otimes_{\mathbb{C}[t]}H^k(\Gamma_{f^{-1}(\mathbb{A}^1\setminus\Sigma_1)}(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1}),\nabla)$ is given by:*

$$\begin{array}{rcl} h.(a\otimes[w]) & = & (ha)\otimes[w],\ \textit{where } h\in\mathcal{O}^{\mathrm{an}}(V),\\ \partial_t.(a\otimes[w]) & = & \frac{\partial a}{\partial t}\otimes[w]+a\otimes[\eta],\end{array}$$

where $[\eta]\in H^k(\Gamma_{f^{-1}(\mathbb{A}^1\setminus\Sigma_1)}(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1}),\nabla)$ satisfies $dw+dg\wedge w=df\wedge\eta$.

3.3. Family of rapid decay cycles

Proposition 3.5. *There exists a finite subset Σ_2 of $\mathbb{C}$ such that*

$$\bigcup_{t\in\mathbb{C}\setminus\Sigma_2} H_k^{rd}(f^{-1}(t),e^{-g_t})\to\mathbb{C}\setminus\Sigma_2$$

is a flat vector bundle. We will denote $\widetilde{H_k^{rd}}$ its associated local system, it being the sheaf of family of cycles in $H_k^{rd}(f^{-1}(t),e^{-g_t})$ which depend continuously on t.

Let $\mathcal{W}$ be a Whitney stratification of X such that $G^{-1}(\infty)$, D and U are union of strata (cf. Theorem (2.2) in [20]). The central argument of this proof is the following lemma.

Lemma 3.6. *There exists a finite subset Σ_2 of $\mathbb{C}$ such that $F : F^{-1}(\mathbb{C}\setminus\Sigma_2) \to \mathbb{C}\setminus\Sigma_2$ is a locally trivial fibration which respect the stratification on $F^{-1}(\mathbb{C} \setminus \Sigma_2)$ induced by $\mathcal{W}$.*

i) *This means that for any $t_0 \in \mathbb{C} \setminus \Sigma_2$, there exists a small disc $S \subset \mathbb{C} \setminus \Sigma_2$ centered at t_0 and a homeomorphism $\phi : F^{-1}(t_0) \times S \to F^{-1}(S)$ which respect the stratification induced by $\mathcal{W}$ and such that*
$$\forall (x,t) \in F^{-1}(t_0) \times S,\ F \circ \phi(x,t) = t.$$
ii) *Moreover, we can assume that there exists a real positive number R big enough such that*
$$\forall (x,t) \in \big(f^{-1}(t_0) \cap g^{-1}(\{|\rho| > R\})\big) \times S,\ g \circ \phi(x,t) = g(x).$$

Proof of Lemma 3.6. — The first point of this Lemma is the first isotopy Lemma. Here, we have to adapt its proof in the way of checking that there exists a local trivialisation such that the second point is true. If we follow the proof of the first isotopy Lemma given in [20] (cf. Theorem (4.14)), this can be done by constructing some well-chosen rugous tangent vector fields on $(F^{-1}(\mathbb{C} \setminus \Sigma_2), \mathcal{W})$. We just give a sketch of the proof.

According to the Sard's Theorem (cf. Theorem (3.3) in [20]), and the fact that f and g are algebraically independant, we know that

i) there exists a finite subset Σ_0 of $\mathbb{C}$ such that $F : F^{-1}(\mathbb{C} \setminus \Sigma_0) \to \mathbb{C} \setminus \Sigma_0$ is a morphism which is transverse to the stratification on $F^{-1}(\mathbb{C} \setminus \Sigma_0)$ induced by $\mathcal{W}$;
ii) there exists a projective variety $\overline{\Delta}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ of dimension at most one such that $(F,G) : (F,G)^{-1}(\mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}) \to \mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$ is a morphism which is transverse to the stratification on $(F,G)^{-1}(\mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta})$ induced by $\mathcal{W}$.

We define Σ_2 as the union of Σ_0 and the set of $t_0 \in \mathbb{C}$ such that (t_0, ∞) belongs to the closure of $\overline{\Delta} \setminus (\mathbb{P}^1 \times \{\infty\})$ in $\mathbb{P}^1 \times \mathbb{P}^1$.

Let $t_0 \in \mathbb{C} \setminus \Sigma_2$. Then, $\mathbb{C} \setminus \Sigma_2$ is isomorphic to $I_1 \times I_2$ in a neighbourhood of $t_0 = (t_0^1, t_0^2)$, where I_1 and I_2 are open intervals in $\mathbb{R}$. Let $R > 0$ such that $\big((I_1 \times I_2) \times \{|\rho| > R\}\big) \cap \overline{\Delta} = \emptyset$.

If t_i is a coordinate on I_i, $i = 1, 2$, we consider η^i (resp. $\widetilde{\eta}^i$) the vector field $\frac{\partial}{\partial t_i}$ on $I_1 \times I_2$ (resp. $(I_1 \times I_2) \times \{|\rho| > R\} \subset \overline{\Delta}$). According to Proposition (4.6) in [20], there exists a rugous tangent vector field ξ^i on $(F^{-1}(I_1 \times I_2), \mathcal{W})$ such that

i) for all $x \in F^{-1}(I_1 \times I_2)$, $TF(\xi^i_x) = \eta^i_{F(x)}$;
ii) for all $x \in (f,g)^{-1}((I_1 \times I_2) \times \{|\rho| > R\})$, $T(f,g)(\xi^i_x) = \widetilde{\eta}^i_{(f,g)(x)}$.

Let $\phi^i : U^i \subset F^{-1}(I_1 \times I_2) \times \mathbb{R} \to F^{-1}(I_1 \times I_2)$ be the flow of ξ^i, where U^i is an open neighbourhood in $F^{-1}(I_1 \times I_2) \times \mathbb{R}$ of $F^{-1}(I_1 \times I_2) \times \{t_0^i\}$ (cf. Proposition (4.8) in [20]).

As F is a proper map, there exist $\widetilde{I_1} \subset I_1$ and $\widetilde{I_2} \subset I_2$ two open intervals such that
$$\begin{array}{ccc} F^{-1}(t_0) \times (\widetilde{I_1} \times \widetilde{I_2}) & \to & F^{-1}(\widetilde{I_1} \times \widetilde{I_2}) \\ (x, (t_1, t_2)) & \mapsto & \phi^2(\phi^1(x, t_1), t_2) \end{array}$$
is a homeomorphism which satisfies properties i) and ii) of Lemma 3.6.

$\square$

Proof of Proposition 3.5. — We recall that a cycle in $H_k^{rd}(f^{-1}(t), e^{-g_t})$ is a relative cycle in $H_k(F^{-1}(t), F^{-1}(t) \cap D, \mathbb{C})$ with a rapid decay condition on e^{g_t}. Lemma 3.6 i) implies that
$$\pi : \bigcup_{t \in \mathbb{C} \setminus \Sigma_2} H_k(F^{-1}(t), F^{-1}(t) \cap D, \mathbb{C}) \to \mathbb{C} \setminus \Sigma_2$$

is a flat vector bundle. If $t_0 \in \mathbb{C} \setminus \Sigma_2$ and (S, ϕ) is as in Lemma 3.6, a trivialisation

$$T : \bigcup_{t\in S} H_k(F^{-1}(t), F^{-1}(t) \cap D, \mathbb{C}) \to H_k(F^{-1}(t_0), F^{-1}(t_0) \cap D, \mathbb{C}) \times S$$

is given by $T(c_t) = (p_1 \circ \phi^{-1} \circ c_t, t)$ for any $c_t \in H_k(F^{-1}(t), F^{-1}(t) \cap D, \mathbb{C})$, where $p_1 : F^{-1}(t_0) \times S \to F^{-1}(t_0)$ is the canonical projection. Moreover, for any $t \in S$, the restriction $T_{tt_0} : H_k(F^{-1}(t), F^{-1}(t) \cap D, \mathbb{C}) \to H_k(F^{-1}(t_0), F^{-1}(t_0) \cap D, \mathbb{C})$ of T is a linear isomorphism. Indeed, the inclusion of the pair $(F^{-1}(t), F^{-1}(t) \cap D)$ in $(F^{-1}(S), F^{-1}(S) \cap D)$ induces an isomorphism $i_t : H_k(F^{-1}(t), F^{-1}(t) \cap D, \mathbb{C}) \to H_k(F^{-1}(S), F^{-1}(S) \cap D, \mathbb{C})$ and i_t^{-1} is homotope to $p_1 \circ \phi^{-1}$. Then, we have the following diagram:

$$\begin{tikzcd}
& H_k(F^{-1}(t_0), F^{-1}(t_0) \cap D, \mathbb{C}) \\
H_k(F^{-1}(S), F^{-1}(S) \cap D, \mathbb{C}) \arrow[ur, "p_1\circ\phi^{-1}", "\simeq"'] & \\
& H_k(F^{-1}(t), F^{-1}(t) \cap D, \mathbb{C}), \arrow[ul, "\simeq", "i_t"'] \arrow[uu, "\simeq", "T_{tt_0}"']
\end{tikzcd}$$

which proves that T_{tt_0} is an isomorphism. This gives the structure of vector bundle. Now, to show that it is flat, we just have to remark that the changes of trivialisations correspond to changes of basis in $H_k(F^{-1}(S), F^{-1}(S) \cap D, \mathbb{C})$, for some open set S and thus, are constant.

Then, we have to verify that if we require the rapid decay condition, we obtain a subbundle of this bundle. It is sufficient to prove that the rapid decay condition is stable with respect to the isomorphism T_{tt_0}. In the following, we will denote by $\psi_{tt_0} : F^{-1}(t) \to F^{-1}(t_0)$, the homeomorphism $p_1 \circ \phi^{-1}$.

Let $c_t \otimes e^g$ be a rapid decay chain in $F^{-1}(t)$. We want to prove that $T_{tt_0}(c_t) \otimes e^g$ is a rapid decay chain in $F^{-1}(t_0)$. Let $y \in T_{tt_0}(c_t)(\Delta^p) \cap D$. Then there exists $\widetilde{y} \in c_t(\Delta^p) \cap D$ such that $\psi_{tt_0}(\widetilde{y}) = y$. As e^g has rapid decay on c_t, $G(\widetilde{y}) = \infty$ and as ψ_{tt_0} respects $G^{-1}(\infty)$, we have also $G(y) = \infty$.

But according to Lemma 3.6 ii), there exists a neighbourhood V of y such that for all $z \in f^{-1}(t_0) \cap V$, $g \circ \psi_{tt_0}^{-1}(z) = g(z)$. Then, if e^g has rapid decay on c_t it has rapid decay on $T_{tt_0}(c_t)$. The converse can be proved similary.

□

3.4. **Holomorphic solutions of the exponential Gauss-Manin systems**

We want to describe the holomorphic solutions of $\mathcal{G}^k$, i.e. the morphisms of $\mathcal{D}_{\mathbb{C}}^{\mathrm{an}}$-modules $(\mathcal{G}^k)^{\mathrm{an}} \to \mathcal{O}_{\mathbb{C}}^{\mathrm{an}}$.

Let $\Sigma = \Sigma_1 \cup \Sigma_2$. We recall that $(\mathcal{G}^k)^{\mathrm{an}}(V)$ is isomorphic to $\mathcal{O}^{\mathrm{an}}(V) \otimes_{\mathbb{C}[t]} H^k(\Gamma_{f^{-1}(\mathbb{C}\setminus\Sigma)}(\Omega^{\bullet+n-1}_{U|\mathbb{A}^1}), \nabla)$ for any open subset V in $\mathbb{C} \setminus \Sigma$ (cf. Proposition 3.4) and $\widetilde{H^{rd}}_{k+n-1}(V)$ is the space of families on V of cycles in $H^{rd}_{k+n-1}(f^{-1}(t), e^{-g_t})$ which depend continuously on t.

Theorem 3.7. *Let V be a simply connected open subset of $\mathbb{C} \setminus \Sigma$. The morphism*

$$\begin{array}{rccl} \Psi : & \widetilde{H^{rd}}_{k+n-1}(V) & \to & \mathrm{Hom}_{\mathcal{D}^{\mathrm{an}}(V)}((\mathcal{G}^k)^{\mathrm{an}}(V), \mathcal{O}^{\mathrm{an}}(V)) \\ & (c_t \otimes e^{g_t})_{t\in V} & \mapsto & \left(a \otimes [w] \mapsto (I : t \mapsto \int_{c_t} a(t) w_{|f^{-1}(t)}\ e^{g_t})\right) \end{array}$$

is well-defined and is an isomorphism.

In other words, the space of holomorphic solutions of $\mathcal{G}^k$ on V is isomorphic to $\widetilde{H^{rd}}_{k+n-1}(V)$.

Proof.

i) As $\nabla_t(a(t)w_{|f^{-1}(t)}) = a(t)\nabla(w)$ and as $\int_{\cdot} \cdot\, e^{g_t}$ defined a pairing between $H^{rd}_{k+n-1}(f^{-1}(t), e^{-g_t})$ and $H^{k+n-1}_{dR}(f^{-1}(t), e^{g_t})$ (cf. Proposition 2.3 for $U_t = f^{-1}(t)$), $I(t)$ is well-defined for any $t \in V$.

ii) Now we have to prove that I is holomorphic on V and that we have constructed a morphism of $\mathcal{D}^{\mathrm{an}}(V)$-modules.

Let $t_0 \in V$. We consider a small disc $S \subset \mathbb{C} \setminus \Sigma$ centered at t_0 such that i) and ii) of Lemma 3.6 are fulfilled. We denote by $\widetilde{\phi}$ the restriction of ϕ to $f^{-1}(t_0) \times S$. It is a homeomorphism from $f^{-1}(t_0) \times S$ to $f^{-1}(S)$.

Let $t \in S$. Then,

$$\begin{aligned} I(t) - I(t_0) &= \int_{c_t} a(t)w_{|f^{-1}(t)}e^{g_t} - \int_{c_{t_0}} a(t_0)w_{|f^{-1}(t_0)}e^{g_{t_0}}, \\ &= \int_{c_t - c_{t_0}} (a \circ f)we^g. \end{aligned}$$

Let $C = \bigcup\limits_{s \in [t_0,t]} c_s$. Then $\partial C = c_t - c_{t_0}$ and according to the Stokes' Formula, $I(t) - I(t_0) = \int_C d((a \circ f)we^g)$.

But as $\nabla([w]) = 0$, there exists $\eta \in \Gamma_{f^{-1}(\mathbb{C}\setminus\Sigma)}(\Omega^{k+n-1}_U)$ such that $dw + dg \wedge w = df \wedge \eta$. Then,

$$\begin{aligned} d((a \circ f)we^g) &= \big(df \wedge ((\tfrac{\partial a}{\partial t} \circ f)w) + (a \circ f)(dw + dg \wedge w)\big)e^g, \\ &= df \wedge \big((\tfrac{\partial a}{\partial t} \circ f)w + (a \circ f)\eta\big)e^g, \end{aligned}$$

and by Fubini's Theorem,

$$I(t) - I(t_0) = \int_{t_0}^{t} \left(\int_{c_s} ((\frac{\partial a}{\partial t} \circ f)w + (a \circ f)\eta)e^g \right) ds.$$

Set $\alpha = (\frac{\partial a}{\partial t} \circ f)w + (a \circ f)\eta$. We have $\int_{c_s} \alpha e^g = \int_{c_{t_0}} \widetilde{\phi}^*(\alpha e^{g_t})$.

According to Lemma 3.6 ii), there exists a compact K in U such that $\widetilde{\phi}^*(\alpha e^{g_t}) = \widetilde{\phi}^*(\alpha)e^{g_{t_0}}$ on $c_{t_0} \setminus K$.

- As $\widetilde{\phi}^*(\alpha e^{g_t})$ is continuous in s and K is compact, $\int_{c_{t_0} \cap K} \widetilde{\phi}^*(\alpha e^{g_t})$ is continuous in s.
- As $e^{g_{t_0}}$ does not depend on s and has rapid decay on $c_{t_0} \setminus K$, the integral $\int_{c_{t_0} \setminus K} \widetilde{\phi}^*(\alpha e^{g_t}) = \int_{c_{t_0} \setminus K} \widetilde{\phi}^*(\alpha)e^{g_{t_0}}$ is continuous on s.

We conclude that $\int_{c_{t_0}} \widetilde{\phi}^*(\alpha e^{g_t})$ is continuous on s and

$$\lim_{t \to t_0} \frac{I(t) - I(t_0)}{t - t_0} = \int_{c_{t_0}} \alpha e^g.$$

Then, I is holomorphic on V.

Moreover, as $\partial_t(a \otimes [w]) = \frac{\partial a}{\partial t} \otimes [w] + a \otimes [\eta]$, the morphism $\big(a \otimes [w] \mapsto (I : t \mapsto \int_{c_t} a(t)w_{|f^{-1}(t)}e^{g_t})\big)$ is a morphism of $\mathcal{D}^{\mathrm{an}}(V)$-modules.

iii) At last, we have to prove that Ψ is an isomorphism. According to Theorem 2.4, we have an isomorphism:

$$\begin{array}{lcl} H^{rd}_{k+n-1}(f^{-1}(t), e^{-g_t}) & \to & \mathrm{Hom}_{\mathbb{C}}(H^{k+n-1}_{dR}(f^{-1}(t), e^{g_t}), \mathbb{C}) \\ c_t \otimes e^{g_t} & \mapsto & (w_t \mapsto \int_{c_t} w_t e^{g_t}). \end{array}$$

Then, $\widetilde{H^{rd}}_{k+n-1}(V) \simeq \mathrm{Hom}_{\mathbb{C}}(\widetilde{H^{k+n-1}_{dR}}(V), \mathbb{C})$.

As $\mathrm{Hom}_{\mathbb{C}}(\widetilde{H^{k+n-1}_{dR}}(V), \mathbb{C}) \simeq \mathrm{Hom}_{\mathcal{D}^{\mathrm{an}}(V)}((\mathcal{G}^k)^{\mathrm{an}}(V), \mathcal{O}^{\mathrm{an}}(V))$ (cf. Corollary 7.1.1 p. 71 of [10]), we conclude that Ψ is an isomorphism.

□

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