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Naive nonabelianization and resummation of fermion bubble chains

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Abstract

We propose to extend the Brodsky-Lepage-Mackenzie scale-fixing prescription by resumming exactly any number of one-loop vacuum polarization insertions into one-loop diagrams. In this way, one makes maximal use of the information contained in one-loop perturbative corrections combined with the one-loop running of the effective coupling. The scale ambiguity at leading order is converted into an intrinsic uncertainty of perturbative approximations induced by IR renormalons. Practical implementation of this resummation requires only knowledge of one-loop radiative corrections with non-vanishing gluon mass. We find that higher order corrections to the pole mass and the top quark decay width are dominated by renormalons already in low orders and demonstrate the impact of eliminating the pole mass on the convergence of the perturbative series.

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1. With experiments growing ever more accurate, the problem of scheme- and scale-fixing in truncated perturbative expansions for observables

$$R - R_{\rm tree} = \alpha(Q) \sum_{n=0}^{N} r_n \alpha(Q)^n \tag{1}$$

in QCD continues to be of acute practical interest. Inherent to any scale-fixing procedure is a guess of uncalculated higher order corrections and the impossibility of assessing with rigour the quality of such a guess for any particular observable. Yet some prescriptions may be closer to general expectations. Such a prescription has been formulated by Brodsky, Lepage and Mackenzie (BLM) [1]. Motivated by the observation that in QED all effects of the running coupling are associated with the photon vacuum polarization, they suggested to absorb the contribution of one fermion loop insertion into lowest order diagrams in the scale of the lowest-order correction. Thus,

$$\begin{aligned} r_0\alpha(Q) + [r_{10} + r_{11}N_f]\alpha(Q)^2 + \ldots &\equiv r_0\Big\{\alpha(Q) + [\delta_1 + (-\beta_0)\, d_1]\alpha(Q)^2 + \ldots\Big\} \\ &\equiv r_0\Big\{\alpha(Q_1^*) + \delta_1\alpha(Q)^2 + \ldots\Big\} \end{aligned} \tag{2}$$

where $\beta_0 = (-1/(4\pi))(11 - (2N_f)/3)$ is the first coefficient of the QCD beta-function, and the new scale is given by $Q_1^* = Q\, e^{-d_1/2}$. In QCD the main effect of this scale redefinition is not to absorb a relatively small contribution proprotional to N_f from the fermion loop itself, but to absorb contributions of other diagrams as well, via the substituion $N_f \to N_f - 33/2$.

In many cases, such as observables derived from the hadronic vacuum polarization [2] and the pole mass, it turns out that δ_1 is small and $r_0\alpha(Q_1^*)$ is in fact a very good approximation to the exact two-loop result. This suggests that often the bulk contribution to radiative corrections in QCD can be obtained from diagrams with fermion loop insertions, after the full β_0 of QCD is restored, a procedure which we shall refer to as naive nonabelianization[1]. Beyond this empirical fact, additional justification for this extrapolation derives from the observation that in large orders, the series of radiative corrections is expected to grow as $n!\,(-\beta_0)^n\alpha(Q)^n$ from insertion of n fermion loops and dominance of these graphs (plus naive nonabelianization) can persist to relatively low orders. At the same time the fast growth, $\propto n!$, indicates that the effect of including more than one fermion loop insertion into the definition of Q^* can be numerically significant.

In this note, we investigate the effect of multiple one fermion loop insertions and provide a generalization of the leading order BLM scale Q_1^* by summing exactly to all orders all fermion loop insertions in lowest order gluon exchange diagrams. This corresponds to defining $\alpha(Q^*)$ as the average of the one-loop running coupling over the lowest order radiative corrections of a particular process. This proposal can hardly be considered new [3], though it has never been pursued in the literature, apparently because of lack of its computational implementation. We shall show that the resummation of this class of diagrams requires as input only lowest order radiative corrections computed with a finite gluon mass. Resumming fermion loop insertions has at least three merits:

[1] This denomination has been introduced by D. Broadhurst (private communication).

- It makes maximal use of the information contained in the one-loop radiative corrections combined with one-loop running of the coupling and is thus the logical realization of the BLM prescription at one loop.
- Using naive nonabelianization it absorbs a class of supposedly large corrections into the normalization of the lowest order coupling. Numerically, this turns out to be most relevant in heavy quark physics.
- It affords a semiquantitative estimate of the ultimate accuracy of perturbative approximations due to renormalon divergence.

We would like to note that this resummation of a subclass of diagrams which is not based on any systematic parameter is delicate in the sense that it can only be justified a posteriori, akin to, say, tadpole improvement in lattice perturbation theory [3] or resummation of "Sudakov-π^2's". It is for this reason that we prefer the discussion in the context of scale-fixing methods, though this is irrelevant for practical purposes.

In the following section, we introduce our notation and formulate the resummation prescription in precise terms. In section 3, we collect the techniques necessary to calculate any number of fermion loop insertions and their sum. The final section presents an application to the pole mass and the top quark width. This example demonstrates the numerical importance of higher order loop insertions and the impact of eliminating the pole mass from heavy quark decay widths on the convergence of perturbation series even in low orders. A more detailed account of techniques as well as their application to the phenomenologically most relevant case of beauty and charm decays will be presented in a separate publication [4].

2. Our focus is on renormalization group invariant quantities, which are dominated by a large scale Q, such as in Eq. (1). We shall also assume that lowest order radiative corrections r_0 are given by one gluon exchange and do not involve the gluon self-coupling. We fix a renormalization scheme that does not introduce artificial N_f-dependence, for instance $\overline{\mathrm{MS}}$. Then $(n+1)$-th order corrections can be written as

$$r_n = r_{n0} + r_{n1} N_f + \ldots + r_{nn} N_f^n, \tag{3}$$

where r_{nn} originates from n fermion-loop insertions into the lowest order radiative corrections. We rewrite r_n as

$$r_0 \left[\delta_n + (-\beta_0)^n d_n\right], \tag{4}$$

where d_n is determined by the requirement that it absorbs the largest power of N_f. In particular, $d_n = (-6\pi)^n r_{nn}$. Again we note that through β_0, the second term in brackets absorbs the contributions from diagrams with less than n fermion loops associated with the gluonic part of the one-loop running of the coupling. It will be absorbed into the scale of the lowest order correction. To this end, for the perturbative expansion, truncated at order $N+1$, we introduce

$$M_N(-\beta_0\alpha(Q)) = 1 + \sum_{n=1}^{N} d_n(-\beta_0\alpha(Q))^n \tag{5}$$

as a measure of how much the lowest order correction is modified by summing N one-loop vacuum polarization insertions and define

$$\alpha(Q_N^*) = \alpha(Q)\, M_N(a) \qquad a \equiv -\beta_0\alpha(Q). \tag{6}$$

With these definitions

$$R - R_{\rm tree} = r_0 \left(\alpha(Q_N^*) + \sum_{n=1}^{N} \delta_n \alpha(Q)^{n+1} \right), \tag{7}$$

where the sum contains the "genuine" higher order corrections, not related to the scale dependence of the coupling[2].

We remark that for $N > 1$, the solution Q_N^* to Eq. (6) depends on the value of $\alpha(Q)$. The coupling dependence of any generalization of the BLM prescription beyond leading order has previously been noted in [5, 6]. For $N = \infty$, $\alpha(Q^*)$ ($Q^* \equiv Q_\infty^*$) is transparently interpreted as the running coupling averaged over the lowest order corrections. Isolating the integration over gluon virtuality in the lowest order correction (ignoring renormalization for now), we may write

$$r_0\alpha(Q) = \alpha(Q) \int {\rm d}^4k\, F(k,Q)\, \frac{1}{k^2}. \tag{8}$$

Then

$$r_0\alpha(Q^*) = \frac{r_0\alpha(Q)}{1 - \beta_0\alpha(Q)\ln(Q^{*2}/Q^2)} = \int {\rm d}^4k\, F(k,Q)\, \frac{\alpha\left(k\exp[C/2]\right)}{k^2}, \tag{9}$$

where the scheme-dependent constant C is the constant part of the fermion loop, see below. The integral in Eq. (9) is not well defined due to the Landau pole of the one-loop running coupling. Equivalently $\lim_{N\to\infty} M_N$ does not exist, because the d_n exhibit the familiar renormalon divergence, $\propto n!$. However, assuming perturbative series are asymptotic, Q^* can still be defined – see below – up to terms suppressed by a power of the large scale Q. After resummation $\alpha(Q^*)$ is formally independent of the finite renormalization C for the fermion loop. All scheme-dependence introduced by one-loop counterterms is eliminated and the accuracy of perturbative predictions is limited only by the divergence of the series and scheme-dependence in unknown genuine higher order corrections.

It is worth noting that the resummation of vacuum polarization insertions could be extended to two (and consecutively higher) loop insertions by including the effect of

[2]In the present form, the δ_n still contain two and higher loop vacuum polarization insertions, which upon incorporation of higher loop running of the coupling would also be absorbed into Q_N^*, see ref. [6] for the case $N = 2$.

two-loop evolution of the coupling into Q^* and introducing a second scale Q^{**} for the genuine two-loop correction plus an infinite number of one-fermion-loop insertions into these, see ref. [6] for one insertion. The introduction of new BLM scales at each order closely parallels the $1/N_f$-expansion, employed in the analysis of renormalon singularities. Since it is known that the nature as well as normalization of renormalon singularities is not correctly provided by the leading term in the $1/N_f$-expansion, one might therefore question the usefulness of the restriction to Q^* alone. However, contributions of higher order in $1/N_f$ to the normalization of renormalons and therefore the estimate of ambiguity of perturbative approximations, start at increasingly larger number of loops. Thus, in all phenomenologically interesting cases, where the divergence of the series starts at comparatively low orders in perturbation theory, the incalculability of the normalization is practically irrelevant and the ambiguity from resumming multiple one-fermion loop insertions should provide a good guide to the limits of perturbation theory.

3. To compute the coefficient r_n with n fermion loop insertions ("n bubbles") into lowest order radiative corrections, it is often useful to calculate the Borel transform

$$B[R](u) = \sum_n \frac{r_n}{n!}(-\beta_0)^{-n} u^n \tag{10}$$

directly and use it as a generating function for the coefficients r_n [7]:

$$r_n = (-\beta_0)^n \frac{d^n}{du^n} B[R](u)_{|u=0}. \tag{11}$$

In many cases, in particular observables involving more than one mass scale, the exact Borel transform is difficult to obtain or even if it is obtainable, taking derivatives is not always a simple task. We can avoid this complication by exploiting that the information about n bubble coefficients is contained in the lowest order radiative corrections, calculated with non-vanishing gluon mass λ (in Landau gauge):

$$r_0(\lambda^2) = \int d^4k\, F(k,Q) \frac{1}{k^2 - \lambda^2}\,. \tag{12}$$

Indeed it has been emphasized [8] that the residues of IR renormalon poles are given by nonanalytic terms in λ^2 in a small-λ^2 expansion, and in ref. [9] the insertion of one bubble has been expressed as an integral over λ^2. We fill the gap between one bubble and the asymptotic behaviour and provide a representation for r_n and the sum of contributions with any number of one-fermion loop insertions ("bubble sum") in terms of the lowest order radiative correction with finite gluon mass as only input. We restrict attention to observables in Euclidean space or such that can be obtained upon analytic continuation from Euclidean space. We shall also assume that no explicit renormalization is needed except for the fermion loop insertions. This assumption can easily be relaxed [4]. It is convenient to use the Landau gauge, the final result being of course gauge-independent.

We start with the "bubble sum". The gluon propagator with summation of an arbitrary number of bubbles can be written as (in Landau gauge)

$$D^{AB}(k) = i\delta^{AB}\frac{k_\mu k_\nu - k^2 g_{\mu\nu}}{k^4}\frac{1}{1+\Pi(k^2)} \tag{13}$$

where

$$\Pi(k^2) = a\ln\left(\frac{-k^2}{Q^2}e^C\right) \tag{14}$$

For a generic contribution to a certain physical amplidude with Euclidian external momenta, one can separate the integration over the gluon momenta to write the contribution of the bubble sum as (cf. Eq. (9))

$$r_0\alpha(Q)M(a) = \int d^4k\, F(k,Q)\frac{1}{k^2}\frac{\alpha(Q)}{1+\Pi(k^2)}. \tag{15}$$

where the transverse projector that appears in the gluon propagator in Eq. (13) is assumed to be included in the function $F(k,Q)$. The next step is to substitute $(1+\Pi(k^2))^{-1}$ by the dispersion relation

$$\frac{1}{1+\Pi(k^2)} = \frac{1}{\pi}\int_0^\infty d\lambda^2\,\frac{1}{k^2-\lambda^2}\frac{\mathrm{Im}\Pi(\lambda^2)}{|1+\Pi(\lambda^2)|^2} + \int_{-\infty}^\infty d\lambda^2\,\frac{1}{k^2-\lambda^2}\frac{\lambda_L^2}{a}\delta(\lambda^2-\lambda_L^2) \tag{16}$$

where

$$\lambda_L^2 = -Q^2\exp[-1/a - C] \tag{17}$$

is the position of the Landau pole. Further writing

$$\frac{1}{k^2-\lambda^2} = \frac{1}{\lambda^2}\left(\frac{k^2}{k^2-\lambda^2} - 1\right), \tag{18}$$

interchanging the order of integrations in k and λ^2, we arrive at

$$r_0M(a) = -\int_{-\infty}^\infty \frac{d\lambda^2}{\lambda^2}\left\{\frac{a\,\theta(\lambda^2)}{|1+\Pi(\lambda^2)|^2} - \frac{\lambda_L^2}{a}\delta(\lambda^2-\lambda_L^2)\right\}\left[r_0(\lambda^2) - r_0(0)\right]. \tag{19}$$

Since by assumption the integral over gluon momentum in Eq. (12) is ultraviolet finite, $r_0(\lambda^2)$ decreases to zero at $\lambda\to\infty$. Thus, integrating by parts, we finally get

$$r_0M(a) = \int_0^\infty d\lambda^2\,\Phi(\lambda)\,r_0'(\lambda^2) + \frac{1}{a}[r_0(\lambda_L^2) - r_0(0)] \tag{20}$$

where $r_0'(\lambda^2) \equiv (d/d\lambda^2)r_0(\lambda^2)$ and

$$\Phi(\lambda) = -\frac{1}{a\pi}\arctan\left[\frac{a\pi}{1+a\ln(\lambda^2/Q^2e^C)}\right] - \frac{1}{a}\,\theta(-\lambda_L^2-\lambda^2). \tag{21}$$

Note that the term with the θ-function exactly cancels the jump of the arctan at $\lambda^2 = -\lambda_L^2$.

Eq. (20) presents the desired answer for the sum of diagrams with any number of fermion bubbles in terms of an integral over gluon mass. This result has a very transparent structure: the quantity $r_0'(\lambda^2)$ (with certain reservations) can be considered as the contribution to the integral from gluons of virtuality of order λ^2, and the function $\Phi(\lambda)$ can be understood as an effective charge. At large scales $\Phi(\lambda)$ essentially coincides with α_v, the QCD coupling in so-called V-scheme [1], but in difference to it remains finite at small λ. The absence of a Landau pole in this effective coupling exhibits another welcome feature of Eq. (20). The integral is a well-defined number and the fact that we have started with an ill-defined expression in Eq. (15) due to the Landau pole (equivalently, attempted to sum a non-Borel summable series) is isolated in the Landau pole contribution $r_0(\lambda_L^2)$. Whenever IR renormalons are present, $r_0(\lambda^2)$ develops a cut at negative λ^2. One can show that the real part of the above prescription for the bubble sum coincides with the principal value of the Borel integral [4] and, in particular, coincides with the Borel sum, when it exists. The imaginary part provided by $r_0(\lambda_L^2)$ coincides with the the imaginary part of the Borel integral, when the contour is deformed above (or below) the positive real axis. We therefore adopt the real part of Eq. (20) as a defnition of the bubble sum contribution and include the imaginary part (divided by π) as an estimate of intrinsic uncertainty from resumming a (non-Borel summable) divergent series without any additional nonperturbative input. Note that this imaginary part is proportional to a power of $\lambda_L^2/Q^2 \sim \exp(-1/a(Q))$ and therefore suppressed by powers of $\Lambda_{\rm QCD}/Q$.

We remark that Eq. (20) applies without modification to quantities like inclusive decay rates, which can be obtained starting from a suitable amplitude in Euclidian space and taking the total imaginary part upon analytic continuation to Minkowski space. The structure of the λ^2-integral remains unaffected, and it is only the quantity $r_0'(\lambda^2)$ which should be substituted by the corresponding decay rate calculated with finite gluon mass (in addition, no explicit renormalization is needed, when the decay rates are expressed in terms of pole masses).

To obtain the coefficient r_n with n bubble insertions it suffices to find a representation for the Borel transform, see Eq. (11). We relax the requirement of no explicit renormalization, but assume at most a logarithmic ultraviolet divergence. Regularizing in $d = 4-2\epsilon$ dimensions, the *bare* lowest order correction calculated with the finite gluon mass has the following asymptotic form:

$$r_0^{bare}(\lambda^2,\epsilon) \stackrel{\lambda\to\infty}{=} -\frac{r_\infty(\epsilon)}{\epsilon}\left(\frac{Q^2}{\lambda^2}\right)^\epsilon . \tag{22}$$

Inverting

$$r_0(\lambda^2) = \frac{1}{2\pi i}\int_C du\,\Gamma(-u)\Gamma(1+u)\left(\frac{\lambda^2}{Q^2}e^C\right)^u B[R](u) \tag{23}$$

from ref. [8] and using the expressions for renormalization of the Borel transform in App. A of ref. [10], we find [4]

$$B[R](u) = -\frac{\sin(\pi u)}{\pi u}\int_0^\infty d\lambda^2 \left(\frac{\lambda^2}{Q^2}e^C\right)^{-u}\left[r_0'(\lambda^2) - \frac{r_\infty}{\lambda^2}\theta(\lambda^2 - Q^2)\right] + R(u) \tag{24}$$

where

$$R(u) = \frac{1}{u}\left(\tilde{G}_0(u) - \frac{\sin(\pi u)}{\pi u} r_\infty e^{-uC}\right), \tag{25}$$

$$G_0(u) \equiv \sum_{n=0}^\infty g_n u^n = \frac{1}{(4\pi)^{-u}}\frac{\sin(\pi u)}{\pi u}\frac{\Gamma(4+2u)}{6\Gamma(1-u)\Gamma(2+u)^2}\, r_\infty(u)\,, \tag{26}$$

$$\tilde{G}_0(u) = \sum_{n=0}\frac{g_n}{n!}u^n\,, \tag{27}$$

and $r_\infty \equiv r_\infty(0)$. Thus r_n can basically be expressed in terms of the integrals

$$\int_0^\infty d\lambda^2\, \ln^k(\lambda^2/Q^2)\, r_0'(\lambda^2) \qquad k \le n\,. \tag{28}$$

4. As a first application, we consider the relation between the pole and the $\overline{\rm MS}$ mass of a heavy quark, defined as

$$m_{\rm pole} = m_{\overline{\rm MS}}(m)\left[1 + \frac{C_F\alpha}{4\pi}\sum_{n=0}^\infty r_n(-\beta_0\alpha(m))^n\right] \tag{29}$$

The one-loop mass shift with finite gluon mass, $r_0(\lambda^2)$, equals ($x \equiv \lambda^2/m^2$):

$$r_0(\lambda^2) = 4 + x - \frac{x^2}{2}\ln x - \frac{\sqrt{x}(8+2x-x^2)}{\sqrt{4-x}}\left\{\arctan\left[\frac{2-x}{\sqrt{x(4-x)}}\right] + \arctan\left[\frac{\sqrt{x}}{\sqrt{4-x}}\right]\right\} \tag{30}$$

Using the formulae collected in sect. 3, we easily compute the contributions of finite number of bubbles, collected in the second column[3] of Table 1. The third column contains coefficients for n fermion loop insertions into one-loop radiative corrections to the top decay width (in the limit $m_t \gg m_W$),

$$\Gamma = \frac{G_F}{\sqrt{2}}\frac{m_t^3}{8\pi}\left[1 + 2\frac{C_F\alpha}{4\pi}\sum_{n=0}^\infty g_n(-\beta_0\alpha(m_t))^n\right] \tag{31}$$

The one-loop correction with finite gluon mass $g_0(\lambda^2)$ has been taken from ref. [9] and m_t is the pole mass of the top. The coupling α is always taken in the $\overline{\rm MS}$ scheme ($C_{\overline{\rm MS}} = -5/3$). For both, the pole mass and the top decay width, the coefficients grow very rapidly, and roughly in the same proportion:

$$r_n/r_{n-1} \sim g_n/g_{n-1} \sim 2n \tag{32}$$

[3]In this case the exact Borel transform is simple [10]. We have checked that the coefficients obtained from differentiating Eq. (24) agree with the derivatives of the exact expression of ref. [10].

This result can be expected because the asymptotic behaviour of the perturbation series in high orders in both cases is governed by an infrared renormalon $r_n \sim n!(1/u)^n \alpha(m)^n$ with $u = 1/2$ [10, 11]. It is remarkable that the numerical values are close to their asymptotic ones already in low orders.

In particular, the growth of coefficients for the top decay width is completely due to the parametriziation in terms of the pole mass. It has already been conjectured on the evidence of cancellation of the leading IR renormalon singularities [8, 11] that radiative corrections to heavy particle decays are strongly reduced in high orders if the pole mass is eliminated in favour of a mass parameter defined at short distances, e.g. the $\overline{\rm MS}$ mass. That this phenomenon is already relevant to low orders is clearly displayed by the fourth column of Table 1, where the coefficients are given, when Eq. (31) is expressed in terms of the $\overline{\rm MS}$ mass: the size of coefficients $g_n^{\overline{\rm MS}} = g_n + (3/2) r_n$ is drastically reduced. In very large orders, the coefficients are now sign-alternating and are governed by an ultraviolet renormalon at $u = -1$. The odd pattern of size of coefficients comes from interplay of an infrared renormalon at $u = 3/2$ and the ultraviolet renormalon at $u = -1$, which finally takes over at $n \approx 7$. Note, that due to the factor $\exp[-uC_{\overline{MS}}] = \exp(5/3u)$ in the normalization of renormalons, the $\overline{\rm MS}$ scheme strongly favours IR renormalon dominance. For this reason, quantities which have a leading infrared renormalon (at $u = 1/2$ above) approach the asymptotic regime at comparatively low orders, wheras onset of the asymptotic regime is delayed, when UV renormalons are leading [12].

In Table 2 we examine the modification of the lowest order radiative correction through summation of N fermion loops. We show the factors $M_N(a)$ defined in Eq. (5) and the values of the BLM scales Q_1^* and Q_∞^* (see Eq. (6))

$$Q_1^* = \exp\left[-\frac{1}{2a}(M_1(a) - 1)\right] \qquad Q_\infty^* = \exp\left[-\frac{1}{2a}\left(1 - \frac{1}{M(a)}\right)\right], \tag{33}$$

taking representative values $\alpha(m_c) = 0.35$, $\alpha(m_b) = 0.2$, $\alpha(m_t) = 0.1$. The first two columns display results for the c-quark and b-quark pole masses, respectively. We conclude that at the charm scale, the relation between the pole and $\overline{\rm MS}$ cannot be improved beyond a two-loop calculation. In the third column we give the results for the top decay width. In the last two columns we extrapolate the result for the top width to the scale $m_b \approx 4.8\,\text{GeV}$ and show partial sums before and after elimination of the pole mass. This corresponds to b-quark semileptonic decay width with zero invariant mass transferred to leptons and gives an anticipation of the size of radiative corrections to be expected for semileptoinic decays [4]. In both cases – with and without pole mass – the BLM scale turns out to be rather low. The important point is that a low *value* of the BLM scale alone does not indicate a breakdown of perturbation theory. The failure of perturbation theory due to its ultimate divergence is indicated by the *uncertainty* in Q^* (or, equivalently, $M(a)$, see Table 2), estimated by the imaginary part of the Landau pole contribution to Eq. (20):

$$\delta M(a) = \frac{1}{\pi a r_0} \mathrm{Im}\, r_0(\lambda_L^2) \tag{34}$$

Note that in fact there is no convincing argument to include the factor $1/\pi$ in this estimate, except that upon inspection of Table 2 we find this estimate closer to the estimate of uncertainty from the minimal term in the perturbative series. This ambiguity, and also the fact that the nature of the renormalon singularity is not determined correctly in the bubble sum approximation, indicates that the given error is only a semiquantitative estimate.

Note that Q^*_∞ can be *larger* than Q^*_1 because the one-loop correction $M(a) = 1 + d_1 a + \ldots$ is always large, and keeping the first term only in the expansion of $M(a)$ in the denominator of the second expression in Eq. (33) — which is the only consistent way to compute Q^*_1 — is in fact a bad approximation. Also note that the BLM scale Q^*_1 after elimination of the pole mass in favour of the $\overline{\mathrm{MS}}$ mass in the decay width is much smaller than the BLM scale computed with the pole mass, although the individual coefficients in the perturbative expansion are reduced, see Table 1. This highlights once more than the low value of the BLM scale by itself has no meaning with respect to failure of the perturbation theory.

5. To conclude, we have proposed to extend the BLM scale-fixing prescription by resumming exactly all one-loop vacuum polarization insertions in lowest order radiative corrections, and have worked out a technical framework for the implementation of this program. This generalization is natural, in the spirit of BLM, building a bridge between the problem of setting the scale in low orders and divergences of perturbative expansions in large orders, but delicate in a sense that resummation of a particular subclass of diagrams combined with naive nonabelianization is not based on any systematic parameter. We find that renormalon ambiguities in summation of the perturbative series are translated to uncertainties of the BLM scales. In estimating the overall uncertainty of the calculation, these must be combined with the uncertainty from unknown genuine corrections in higher orders. If naive nonabelianization works — which can only be justified *a posteriori* — the latter may even be smaller than the uncertainty due to infrared renormalons. The renormalon uncertainties in BLM scales, rather than small values of the scales themselves, indicate the ultimate accuracy of perturbation theory.

The few examples considered above indicate that this procedure can be most fruitful for heavy quark physics, where we observe that the behaviour of perturbative series is consistent with the asymptotic expansion already in low orders. Thus, matching explicit calculations of lower order corrections with asymptotic formulae one can expect to get a significant benefit. Immediate phenomenological applications include inclusive B-decays, in which case the one-loop fermion loop insertion has been calculated in [13], and will be considered elsewhere [4].

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n	r_n	g_n	$g_n^{\overline{\rm MS}}$
0	4	$(-2\pi^2)/3+5/2$	$(-2\pi^2)/3+17/2$
1	$71/8+\pi^2$	-17.15684	10.96007
2	70.490601	-94.02309	11.7128
3	439.43538	-626.0250	33.128
4	3495.6957	-5208.637	34.907
5	35358.744	-52654.86	383.3
6	423257.13	-634891.3	-5
7	5939873.7	-8898590	1.05×10^4
8	94962946	-1.42473×10^7	-2.83×10^4

Table 1: Coefficients for n fermion loop insertions into one-loop radiative corrections for the pole mass and the top decay width normalized to the pole and $\overline{\rm MS}$ mass.

N	c-quark mass	b-quark mass	I	II	III
0	1	1	1	1	1
1	2.176	1.623	1.252	1.559	1.759
2	3.286	1.935	1.335	1.967	1.867
3	5.024	2.193	1.368	2.328	1.908
4	8.492	2.467	1.385	2.727	1.913
5	17.30	2.835	1.395	3.265	1.922
6	43.76	3.420	1.402	4.126	1.922
7	137.0	4.514	1.408	5.732	1.926
8	511.0	6.838	1.414	9.151	1.924
∞	1.712 ± 0.608	2.041 ± 0.201	1.408 ± 0.068	2.094 ± 0.295	1.928 ± 0.001
Q_1^*	$0.096\, m_c$	$0.096\, m_b$	$0.122\, m_t$	$0.122\, m_b$	$0.058\, m_b$
Q_∞^*	$0.437\, m_c$	$0.147\, m_b$	$0.089\, m_t$	$0.140\, m_b$	$0.164\, m_b$

Table 2: Modification $M_N(a)$ of the lowest order correction through summation of N fermion loops. For charm, we have taken $a = 0.251$, for beauty $a = 0.133$ and for top $a = 0.06$. Column I is for top quark width with pole mass at $a = 0.06$, columns II and III extrapolate to $a = 0.133$ before (II) and after elimination of the pole mass (III).