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Marco Hien

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Abstract: We define a period pairing for any flat, irregular singular, rank one connection satisfying a technical condition regarding its stationary set on a complex surface between de Rham cohomology of the connection and a modified singular homology, the rapid decay homology. We prove that this gives a perfect duality.

1 Introduction

Let X be a smooth quasi-projective algebraic variety over the complex numbers and E be a vector bundle on X equipped with an integrable connection

$$\nabla : E \to E \otimes_{\mathscr{O}_X} \Omega^1_{X|\mathbb{C}} \ .$$

Its de Rham cohomology $H^*_{dR}(X;E,\nabla)$ is defined as the hypercohomology of the complex

$$0 \longrightarrow E \xrightarrow{\nabla} E \otimes_{\mathscr{O}_X} \Omega^1_{X|\mathbb{C}} \xrightarrow{\nabla} E \otimes_{\mathscr{O}_X} \Omega^2_{X|\mathbb{C}} \xrightarrow{\nabla} \dots \xrightarrow{\nabla} E \otimes_{\mathscr{O}_X} \Omega^{\dim X}_{X|\mathbb{C}} \longrightarrow 0 \ ,$$

where $\Omega^p_{X|\mathbb{C}}$ denotes the sheaf of Kähler p-forms on X.

In addition to these data, we can consider the analytic manifold X_{an} associated to X as well as the associated analytic vector bundle E_{an} with holomorphic connection ∇_{an}. The hypercohomology of the resulting complex of sheaves of holomorphic forms with values in E gives the analytic de Rham cohomology. If X is projective, it follows from the Poincaré Lemma and Serre's GAGA, that the algebraic and the analytic de Rham cohomology coincide. Equivalently, if $\mathscr{E}^\vee$ denotes the local system of solutions of the dual connection $\nabla^\vee_{an}$ on the dual bundle $E^\vee_{an}$, integration defines a perfect pairing

$$H^*_{dR}(X;E,\nabla) \times H_*(X_{an},\mathscr{E}^\vee) \to \mathbb{C} \tag{1.1}$$

between algebraic de Rham cohomology and singular homology with values in the local system $\mathscr{E}^\vee$.

If we start with a quasi-projective variety U, which we consider to be compactified by a projective variety X, the situation is more complicated. Let $D := X \smallsetminus U$ denote the complement which we assume to be a normal crossing divisor. In [7], P. Deligne introduces the condition for a connection to be regular singular along D generalizing the well-known property for linear differential operators in one variable (Fuchs condition) and proves the comparison isomorphism and hence the perfect duality of (1.1) under this assumption ([7], Théorème II.6.2).

In the irregular singular case, the period pairing as in (1.1) is no longer perfect. The appropriate generalization is known in dimension one only (cp. [3]). On curves, S. Bloch and H. Esnault define a modified homology, the *rapid decay homology groups* $H^{rd}_*(X_{an};E_{an},\nabla_{an})$ and obtain a perfect duality

$$H^*_{dR}(U;E,\nabla) \times H^{rd}_*(X_{an},E^\vee_{an},\nabla^\vee_{an}) \to \mathbb{C} \ ,$$

given by integration. The resulting periods are interesting objects by themselves (the integral representations of the classical Bessel-functions, Gamma-function and confluent hypergeometric functions arise in this way as periods of irregular singular rank one connections on curves) and are mysteriously related to ramification data for certain wildly ramified ℓ-adic sheaves on curves over a finite field (see e.g. [20]).

In the present paper, we want to start the investigation of the higher-dimensional case by studying the period pairing for irregular rank one connections on complex surfaces. We work entirely in the analytic topology, the algebraic aspect we originally have in mind will be mirrored by looking at an integrable connection on the smooth

[1]NWF I – Mathematik, Universität Regensburg, 93040 Regensburg, Germany, email: marco.hien@mathematik.uni-regensburg.de

analytic manifold X which is meromorphic along the normal crossing divisor D at infinity, i .e. the connection is given as

$$\nabla : E(*D) \to E \otimes_{\mathscr{O}_X} \Omega^1_{X|\mathbb{C}}(*D) ,$$

where we use the usual notation $E(*D)$ for the sheaf of sections of E meromorphic along D and we skip the subscript *an* in the following.

Furthermore, we restrict ourselves to the case of line bundles L with irregular singular connections, which we assume to be *good with respect to the divisor D*, which is defined as follows: Consider the formal connection $\widehat{L} := L \otimes \mathscr{O}_{\widehat{X|Y}}(*D)$, where $\mathscr{O}_{\widehat{X|Y}}$ denotes the formal completion of $\mathscr{O}_X$ with respect to the stratum Y of D considered, namely Y being a smooth component of D or a crossing-point. By a standard argument, $\widehat{L}$ is locally isomorphic to the formal completion of a connection of the form $e^\alpha \otimes R$, with $\alpha \in \mathscr{O}_X(*D)$ (cp. [18], Proposition III.2.2.1), where e^α denotes the connection on the trivial bundle $\mathscr{O}_X$ given by $\nabla 1 = d\alpha$ (such that the local solutions are of the form e^α), and R is a regular singular connection.

Definition 1.1 *The connection* ∇ *is* **good with respect to** *D, if its formal completion locally is isomorphic to the formal completion of* $e^\alpha \otimes R$ *with a regular singular R and a local section* $\alpha \in \mathscr{O}_X(*D)$*, such that the divisor* (α) *of* α *is contained in D and negative.*

We thus exclude examples like $\alpha = x_1^{-m_1} - x_2^{-m_2}$. Remark, that the definition means that a good connection ∇ has a local formal presentation as above, locally at the point $x = 0$ with coordinates x_1, x_2 with $D = \{x_1 x_2 = 0\}$ or $D = \{x_1 = 0\}$, such that

$$\alpha = x_1^{-m_1} x_2^{-m_2} \cdot u(x)$$

with $u(0) \neq 0$. We will always assume this. Note that any rank one connection will become good after a finite number of point blow-ups centered at points on D. We will come back to the more general situation (for higher rank connections) in a follwing paper.

We generalize the notion of rapid decay homology groups and prove that the resulting period pairing between the meromorphic de Rham cohomology and rapid decay homology is perfect:

Theorem 1.2 *Let L be a line bundle on a smooth projective complex surface* X*. If* ∇ *is an integrable connection which is meromorphic along the normal crossing divisor* $D \subset X$ *and good with respect to D, the period pairing*

$$H^*_{dR}(X \smallsetminus D; L, \nabla) \times H^{rd}_*(X; L^\vee, \nabla^\vee) \longrightarrow \mathbb{C}$$

is a perfect duality.

In dimension one, the Levelt-Turrittin theorem and the theory of Stokes structures allows to reduce the higher rank case to the case of irregular singular line bundles (cp. [3]). On surfaces, there are analogous partial results for higher rank connections due to C. Sabbah, e.g. an analogue to the Levelt-Turrittin theorem in the case of rank less than or equal to 5. However, there are subtle differences between the one- and the two-dimensional situation, mainly concerning the non-good situation, with interesting consequences regarding the period pairing as well. We will come back to this in a subsequent paper.

Additionally, it turns out to be very difficult to give explicit examples of flat meromorphic connections of higher rank due to the integrability condition imposed. Locally, a rank r connection is given by its connection matrix $A = A_1 dx_1 + A_2 dx_2$ with $r \times r$-matrices A_i having meromorphic functions as entries. The integrability condition reads as

$$\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = [A_1, A_2] ,$$

and it is difficult to find explicit (non-trivial) meromorphic solutions to this equation (cp. the corresponding remark in [18], p. 2). It is however possible to construct higher-rank examples by functoriality (in the category of $\mathscr{D}$-modules), e.g. by pushing forward an irregular singular rank one connection (L, ∇) on X along some map $f : X \to Y$. The resulting Gauß-Manin connection lives on the higher direct image $R^p f_{*,dR}(L, \nabla)$ of the de Rham complex on X. We will give an example, a two-dimensional generalization of the confluent hypergeometric connection, at the end of this introduction, leading to an explicit meromorphic rank 3 connection on $\mathbb{P}^2$.

Examples involving line bundles, however, occur in a natural way in the framework of special functions, more precisely the higher dimensional generalizations of well-studied special functions, such as the generalized hypergeometrics in the sense of Gelfand and Aomoto and their confluent variants (cp. [9], [11]). For example, any closed meromorphic 1-form ω on X with poles along the normal crossing divisor D gives rise to a singular connection on the trivial line bundle given by

$$\nabla = d + \omega \wedge \ .$$

If the 1-form ω depends on additional parameters, say $z \in Z$, and if ω is closed as a 1-form on the product $X \times Z$ (and hence induces an integrable connection on the trivial line bundle $\mathscr{O}_{X\times Z}$), the resulting periods for the relative connection on X satisfy the Gauß-Manin connection on Z. Bessel-functions and more generally confluent hypergeometric functions occur in this way. We want to illustrate this construction with an example.

1.1 An example: two-dimensional confluent hypergeometrics

We fix $a,b,c,\alpha \in \mathbb{C}$, with $a,b,c \notin \mathbb{Z}$ satisfying $a+b+c=-3$, as well as additional parameters x,y. Consider the connection on the trivial line bundle $\mathscr{O}_{\mathbb{P}^2}$ on $\mathbb{P}^2$, which in affine coordinates $[1,u_1,u_2] \in \mathbb{P}^2$ reads as

$$\nabla = d + \big(\frac{a}{u_1} + \frac{c}{1+u_1+u_2} + \alpha x\big)du_1 + \big(\frac{b}{u_2} + \frac{c}{1+u_1+u_2} + \alpha y\big)du_2 \ .$$

Solutions are given by the various branches of

$$U(u_1,u_2,x,y) := u_1^a \cdot u_2^b \cdot (1+u_1+u_2)^c \cdot \exp(\alpha(xu_1+yu_2)) \ .$$

The connection above can be written as $\nabla = d + \mathrm{dlog}_u U$, where dlog_u denotes the logarithmic derivation with respect to the coordinates $u=(u_1,u_2)$, i.e. $\mathrm{dlog}_u U = \mathrm{dlog}_{u_1} U\, du_1 + \mathrm{dlog}_{u_2} U\, du_2$.

If we interpret the parameters x,y as coordinates in the affine space and compactify with a projective plane at infinity, i.e. we read (x,y) as the point $[1,x,y] \in \mathbb{P}^2$, we obtain an integrable meromorphic connection ∇ on the trivial line bundle over $\mathbb{P}^2 \times \mathbb{P}^2$, namely in affine coordinates

$$\nabla_{\mathrm{abs}} = d + \mathrm{dlog}_{(u,x,y)} U = d + \mathrm{dlog}_u U + \alpha u_1 dx + \alpha u_2 dy \ .$$

Let $D := \{u_1u_2 = 0\} \cup \{1+u_1+u_2 = 0\} \cup \{[0,t_1,t_2] \in \mathbb{P}^2\}$ and $X := \mathbb{P}^2 \smallsetminus D$. On one-forms, the connection $\nabla : \Omega^1_X(*D) \to \Omega^2_X(*D)$ reads as $\nabla(f du_1 + g du_2) =$

$$= \big(\frac{\partial g}{\partial u_1} - \frac{\partial f}{\partial u_2} + g\cdot(\frac{a}{u_1} + \frac{c}{1+u_1+u_2} + \alpha x) - f\cdot(\frac{b}{u_2} + \frac{c}{1+u_1+u_2} + \alpha y)\big) du \ ,$$

where we abbreviate $du := du_1 \wedge du_2$. We claim, that $\dim H^2_{dR}(X,\mathscr{O}_X(*D),\nabla) = 3$, a basis is given by the de Rham classes of du, u_1du, u_2du. In order to understand this, we consider the following equalities in $H^2_{dR} := H^2_{dR}(X;\mathscr{O}_X(*D),\nabla)$. First, we have $0 \equiv \nabla\big(u_2(1+u_1+u_2)du_1\big) =$

$$= \big(\alpha x u_1^2 + \alpha x u_1 u_2 + (\alpha x - (1+b))u_1 + (1+a)u_2 + (1+a)\big) du \ , \tag{1.2}$$

as well as $0 \equiv \nabla\big(u_1(1+u_1+u_2)du_2\big) =$

$$= -\big(\alpha y u_2^2 + \alpha y u_1 u_2 + (1+b)u_1 + (\alpha y - (1+a))u_2 + (1+b)\big) du \ . \tag{1.3}$$

Additionally, calculating

$$\nabla\big((\alpha x u_1 + (1+a))u_2(1+u_1+u_2)du_1 + (\alpha y u_2 + (1+b))u_1(1+u_1+u_2)du_2\big) \tag{1.4}$$

induces the following equivalence in H^2_{dR}:

$$\alpha u_1 u_2 du = -\frac{1+b}{y-x} u_1 du + \frac{1+a}{y-x} u_2 du \;. \tag{1.5}$$

Evaluating higher powers $u_1^k u_2^l (1+u_1+u_2) du_i$, $i=1,2$, and similar linear combinations as in (1.4), easily proves the claim about H^2_{dR}.
The period pairing

$$H^2_{dR}(X;\mathscr{O}_X(*D),\nabla) \times H^{rd}_2(\mathbb{P}^2;\mathscr{O}_X(*D),\nabla) \to \mathbb{C} \;,$$

which we are going to define in the next section, produces the following kind of generalized confluent hypergeometric functions on X:

$$F(x,y) = \int_c u_1^a \cdot u_2^b \cdot (1+u_1+u_2)^c \cdot \exp(\alpha(xu_1+yu_2))\omega$$

with ω being either du, u_1du or u_2du. How the topological 2-chain c in $\mathbb{P}^2$ has to be chosen, will be the main point in the definition of the rapid decay homology H^{rd}_*.

These confluent hypergeometric functions F again satisfy another partial differential equation, namely the Gauß-Manin equation on H^2_{dR} derived from ∇. The latter is defined as follows. Let $Z=\mathbb{P}^2$ denote the space for the parameters (x,y) and let $\pi : X\times Z \to Z$ be the projection. We will keep the affine coordinates $u=(u_1,u_2)$ for the points in X and (x,y) in Z. There is a filtration on $\Omega^*_{X\times Z|\mathbb{C}}$ given by

$$\mathrm{Fil}^i\Omega^*_{X\times Z|\mathbb{C}} := \mathrm{im}(\pi^*\Omega^i_{Z|\mathbb{C}} \otimes \Omega^{*-i}_{X\times Z|\mathbb{C}}) \;.$$

Now, the associated graded object fulfills $\mathrm{gr}^i \cong \pi^*\Omega^i_{Z|\mathbb{C}} \otimes \Omega^{*-i}_{X\times Z|Z}$, especially $\mathrm{gr}^0 \cong \Omega^i_{X\times Z|Z}$, the sheaf of relative differential forms, on which the original connection ∇ canonically lives. The short exact sequence $0\to \mathrm{gr}^1 \to \mathrm{Fil}^0/\mathrm{Fil}^2 \to \mathrm{gr}^0 \to 0$ induces the short exact sequence of de Rham complexes

$$0 \to (\Omega^{*-1}_{X\times Z|Z} \otimes \pi^*\Omega^1_{Z|\mathbb{C}}, \nabla_{\mathrm{abs}}\otimes 1) \to \left(\Omega^*_{X\times Z|\mathbb{C}}/\mathrm{Fil}^2, \nabla_{\mathrm{abs}}\right) \to (\Omega^*_{X\times Z|Z},\nabla) \to 0 \;. \tag{1.6}$$

The Gauß-Manin connection is by definition the connecting morphism of the associated long exact sequence of the higher direct images, in our case $\nabla_{\mathrm{GM}} : H^2_{dR} \to H^2_{dR}\otimes \Omega^1_{Z|\mathbb{C}}$ (note that $H^2_{dR} = R^2\pi_*(\Omega^*_{X\times Z|Z},\nabla)$). Chasing the diagram (1.6) gives

$$\nabla_{\mathrm{GM}}(u_1^k u_2^l du) = \alpha dx \otimes u_1^{k+1}u_2^l du + \alpha dy \otimes u_1^k u_2^{l+1} du \;.$$

Applying (1.2), (1.3) and (1.5), we obtain the connection matrix Φ_{GM} with respect to the basis du, u_1du, u_2du of H^2_{dR}:

$$\Phi_{\mathrm{GM}} = \left(\begin{array}{c|c|c} 0 & -\frac{1+a}{x} & 0 \\ \alpha & \frac{(1+b)y}{x(y-x)} - \alpha & -\frac{1+b}{y-x} \\ 0 & -\frac{(1+a)y}{x(y-x)} & \frac{1+a}{y-x} \end{array}\right) dx + \left(\begin{array}{c|c|c} 0 & 0 & -\frac{1+b}{y} \\ 0 & -\frac{1+b}{y-x} & \frac{(1+b)x}{y(y-x)} \\ \alpha & \frac{1+a}{y-x} & -\frac{(1+a)x}{y(y-x)} - \alpha \end{array}\right) dy \;, \tag{1.7}$$

being one of the rather rare explicit examples of an integrable higher rank connection on a surface.

We remark, that there is a theory of generalized confluent hypergeometrics on the space $Z_{r+1,n+1}$ of complex $(r+1)\times(n+1)$-matrices of full rank for given $1\le r<n$, defined in [11]. The starting point again is a connection of the form $\nabla = d + d\log U$ for a certain class of multi-valued functions $U : \mathbb{P}^r \times Z_{r+1,n+1} \to \mathbb{C}$ with well-defined logarithmic derivative. Actually, one also fixes a composition $\lambda = (1+\lambda_0,\dots,1+\lambda_l)$ of $n+1$, i.e. $\sum(1+\lambda_i) = n+1$ and requires for the solutions to be invariant under the left and right action $\mathrm{GL}_{r+1}(\mathbb{C})\times Z_{r+1,n+1}\times H_\lambda \to Z_{r+1,n+1}$, where $H_\lambda \subset \mathrm{GL}_{n+1}(\mathbb{C})$ denotes the subgroups of all block diagonal matrices with $l+1$ blocks consisting of upper triagonal matrices with the constant entry $h_i^{(k)}$ along the i-th upper diagonal for $i=0,\dots,\lambda_k$ (the entry on the

main diagonal $h_0^{(k)} \neq 0$). Our example above corresponds to the choices $r = 2$, $n = 4$ and $\lambda = (2,1,1,1)$ and the restriction to the subspace of all matrices

$$\left(\begin{array}{cc|c|c|c} 1 & 0 & 0 & 0 & 1 \\ 0 & x & 1 & 0 & 1 \\ 0 & y & 0 & 1 & 1 \end{array}\right) \in Z_{3,5}\ ,$$

which parameterize the *generic stratum* of the double quotient $\mathrm{GL}_3(\mathbb{C})\backslash Z_{3,5}/H_\lambda$ (we refer to [11] and [9] for further details).

The paper is organized as follows. In section 2, we define the rapid decay homology groups for line bundles on X and its pairing with the meromorphic de Rham cohomology. Afterwards, we reduce the problem of perfectness of the pairing to local questions according to the canonical stratification of the normal crossing divisor D into crossing points and smooth components. Perfectness of the resulting local pairings is proved in several steps in section 4, completing the proof of the main theorem.

2 Rapid decay homology and the pairing with de Rham cohomology

2.1 Rapid decay homology

Let X be a n-dimensional smooth projective complex manifold and $D \subset X$ a divisor with normal crossings (i.e. in suitable local coordinates $z_1,\dots,z_n$ it is of the form $D = \{z_1 \cdots z_k = 0\}$ for some $1 \leq k \leq n$ — such coordinates will be called *good* w.r.t D). We further consider a line bundle L over X together with an integrable meromorphic connection on $X \smallsetminus D$ with possibly irregular singularities at D. In the usual notation $L(*D)$ for the sheaf of local sections in L meromorphic along D, the connection then reads as $\nabla : L(*D) \longrightarrow L \otimes \Omega^1(*D)$. The rank one local system of horizontal sections in L on the complement $U := X \smallsetminus D$ will be denoted by

$$\mathscr{L} := \ker(L|_U \xrightarrow{\nabla} L|_U \otimes \Omega^1_U) \subset L|_U\ .$$

The dual connection $\nabla^\vee$ on $L^\vee$ is characterized by $d < e,\varphi > = < \nabla e, \varphi > + < e, \nabla^\vee \varphi >$ for local sections e of L and φ of $L^\vee$. Let $\mathscr{L}^\vee$ denote the corresponding local system.

We assume that the complement $U := X \smallsetminus D$ is Stein, which we can always obtain by joining additional hypersurfaces, where the connection is not singular at all, to D. These do not affect our procedure. Then the de Rham cohomology is given by the cohomology of the global sections in U, i.e.

$$H^p_{dR}(U; L(*D), \nabla) = H^p(\ldots \to \Gamma(L \otimes \Omega^q(*D)) \xrightarrow{\nabla} \Gamma(L \otimes \Omega^{q+1}(*D)) \to \ldots)\ .$$

The homology we are going to define will be a generalization of the usual notion of singular homology with coefficients in a local system $\mathscr{V}$, where one considers chain complexes built from pairs of a topological chain together with an element of the stalk of $\mathscr{V}$ at the barycentre of the chain. In our situation, we will have to allow the topological chains to be able to meet the divisor D, where the local system $\mathscr{L} := (L|_U)^\nabla$ is not defined. To this end, we will make the following definition:

Definition 2.1 *For any $x \in X$, we define the* **stalk** *$\mathscr{L}_x$ of $\mathscr{L}$ to be the usual stalk of $\mathscr{L}$ if $x \in U$ and to be the coinvariants*

$$\mathscr{L}_x := (\mathscr{L}_y)_{\pi_1(U,y)} := \mathscr{L}_y \big/ \{v - \sigma v \mid v \in \mathscr{L}_y, \sigma \in \pi_1(U,y)\} \qquad \textit{for } x \in D\ ,$$

where $y \in U$ is any point near $x \in D$ and we are taking coinvariants w.r.t. the local monodromy action $\pi_1(U,y) \to \mathrm{Aut}(\mathscr{L}_y)$.

Now, we can define the notion of rapidly decaying topological chains in analogy to the definition by S. Bloch and H. Esnault in [3] on curves. In the following, we denote by Δ^p the standard p-simplex with barycentre $b \in \Delta^p$ and we call a function $f : Y \to \mathbb{C}$ from any subset $Y \subset U$ of the open complex manifold U analytic, if it is the restriction of an analytic function on an open neighborhood of Y.

Definition 2.2 *A* **rapid decay** p**-chain** *is a pair* (c,ε) *consisting of a continuous map* $c : \Delta^p \to X$*, such that the pre-image* $c^{-1}(D)$ *is a union of complete subsimplices of* Δ^p*, and an element* $\varepsilon \in \mathscr{L}_{c(b)}$*. If* $c(\Delta^p) \not\subset D$*, we require that* $\varepsilon \in \mathscr{L}_{c(b)} \subset L_{c(b)}$ *is* rapidly decaying *in the following sense:*

For any $y \in c(\Delta^p) \cap D$*, let* e *denote a local trivialization of* $L(*D)$ *and* $z_1,\ldots,z_n$ *local coordinates at* y *such that locally* $D = \{z_1 \cdots z_k = 0\}$ *and that the coordinates of* y *fulfill* $y_1 = \ldots = y_k = 0$*. With respect to the trivialization* e *of* L *restricted to* U*,* ε *becomes an analytic function*

$$f := e^* \varepsilon : c(\Delta^p) \smallsetminus D \to \mathbb{C}\ ,\ (z_1,\ldots,z_n) \mapsto f(z)\ .$$

We require that this function has rapid decay at y*, i.e. that for all* $N \in \mathbb{N}^k$ *there is a* $C_N > 0$ *such that*

$$|f(z)| \leq C_N \cdot |z_1|^{N_1} \cdots |z_k|^{N_k}$$

for all $z \in c(\Delta^p) \smallsetminus D$ *with small* $|z_1|,\ldots,|z_k|$*.*

We stress that we do not impose any decay condition on pairs (c,ε) with $c(\Delta^p) \subset D$; nevertheless we call those pairs rapidly decaying as well.

Now, let $S_p^D X$ be the free vector-space over all singular chains $\Delta^p \to X$ meeting D only in full subsimplices and let $\mathscr{K}_p^{rd}(X;L)$ be the $\mathbb{C}$-vector space of all maps

$$\psi : S_p^D X \to \bigsqcup_{x \in X} \mathscr{L}_x,$$

such that $\psi(c) = 0$ for all but finitely many c and that $(c,\psi(c))$ has rapid decay. We will write $c \otimes \varepsilon \in \mathscr{K}_p^{rd}(X;L)$ for the element ψ which takes the value ε at c and zero otherwise. We remark that any element of $\mathscr{K}_p^{rd}(X;L)$ can be written as a finite sum $\sum_{i=1}^r c_i \otimes \varepsilon_i$. The notation $\otimes$ is justified by the fact, that it is linear in each of the entries. There is a natural boundary map $\partial : \mathscr{K}_p^{rd}(X;L) \to \mathscr{K}_{p-1}^{rd}(X;L)$, $c \otimes \varepsilon \mapsto \sum_j (-1)^j c_j \otimes \varepsilon_j$, where the sum runs over the faces c_j of c and the elements ε_j in the stalk of $\mathscr{L}$ at $c(b_j)$, with b_j being the barycentre of the j^{th} face of Δ^p, are given as follows:

i) if $c(b_j) \notin D$, there is a unique homotopy class of paths from $c(b)$ to $c(b_j)$ (e.g. induced by the line $[b,b_j]$ in Δ^p) and ε_j is defined as the analytic continuation along a representative.

ii) if $c(b_j) \in D$, we just define ε_j to be the element represented by ε in the corresponding stalk of coinvariants under monodromy.

It can be easily seen, that $\partial \circ \partial = 0$ and that ∂ respects the support of the chains, so that we can define the rapid decay homology as follows:

Definition 2.3 *For* $\emptyset \subseteq Z \subseteq Y \subseteq X$*, put*

$$\mathscr{C}_p^{rd}(Y,Z;L) := \mathscr{K}_p^{rd}(Y;L) \big/ \mathscr{K}_p^{rd}(Z;L) + \mathscr{K}_p^{rd}(D \cap Y;L) \quad \textit{(relative version)}$$

and $\mathscr{C}_p^{rd}(Y;L) := \mathscr{C}_p^{rd}(Y,\emptyset;L)$ *for the absolute version. The* **rapid decay homology** *is defined as the homology of the corresponding complexes and denoted by* $H_p^{rd}(Y,Z;L)$ *and* $H_p^{rd}(Y;L)$ *respectively.*

Note that we have moduled out the chains that are mapped completely into D as they will not play any role in the pairing with meromorphic differential forms as described below. Nevertheless, one has to include them a priori into the definition of rapid decay chains in order to be able to define the boundary map ∂. We also include L into the notation to remind that the rd-homology does not depend on the local system alone, but on the connection.

2.2 The pairing and statement of the main result

Now, if we have a rapidly decaying chain $c \otimes \varepsilon \in \mathscr{C}_p^{rd}(X;L^\vee)$ in the dual bundle (with the dual connection) and a meromorphic p-form $\omega \in L\otimes\Omega^p(*D)$, then the integral $\int_c <\varepsilon,\omega>$ converges because the rapid decay of ε along c annihilates the moderate growth of the meromorphic ω. Let c_t denote the topological chain one gets by cutting off a small tubular neighborhood with radius t around the boundary $\partial\Delta^p$ from the given topological chain c. Then, for $c\otimes\varepsilon \in \mathscr{C}_p^{rd}(X;L^\vee)$ and $\eta \in L\otimes\Omega^{p-1}(*D)$ a meromorphic $p-1$-form, we have the **'limit Stokes formula'**

$$\int_c <\varepsilon,\nabla\eta> = \lim_{t\to 0}\int_{c_t} <\varepsilon,\nabla\eta> = \lim_{t\to 0}\int_{\partial c_t} <\varepsilon,\eta> = \int_{\partial c - D} <\varepsilon,\eta>$$

where in the last step we used that by the given growth/decay conditions the integral over the faces of ∂c_t 'converging' against the faces of ∂c contained in D vanishes.

The limit Stokes formula easily shows in the standard way that integrating a closed differential form over a given rd-cycle (i.e. with vanishing boundary value) only depends on the de Rham class of the differential form and the rd-homology class of the cycle. Thus, we have:

Proposition 2.4 *Integration induces a well defined bilinear pairing*

$$H^p_{dR}(U;L(*D),\nabla)\times H_p^{rd}(X;L^\vee) \longrightarrow \mathbb{C}\ ,\ ([\omega],[c\otimes\varepsilon]) \mapsto \int_c <\omega,\varepsilon>\ , \tag{2.1}$$

which we call the **period pairing** *of* (L,∇).

Our main result is the following

Theorem 2.5 *On a complex surface X, the period pairing is perfect for any irregular singular line bundle which is good with respect to D.*

Remark, that for $\dim(X) = 1$ (and arbitrary rank) the perfectness was proved by S. Bloch and H. Esnault [3] (in the one-dimensional case every line bundle is good). The case of arbitrary dimension $\dim(X) \geq 2$ is not known in general.

2.3 The irregularity pairing

Let $j : U := X \smallsetminus D \hookrightarrow X$ denote the inclusion. In addition to the de Rham cohomology $H^*_{dR}(U;L(*D))$ of meromorphic sections, we will also have to consider the de Rham cohomology over U allowing essential singularities as well. We will denote the corresponding sheaves by

$$L_{\text{mero}} := L(*D) \quad \text{and} \quad L_{\text{ess}} := j_*(L|_U)\ .$$

Note, that the de Rham cohomology of the connection induced on L_{ess} pairs classically with the singular homology $H^*(U;\mathscr{L}^\vee)$ of U with coefficients in the restricted local system $\mathscr{L}^\vee|_U$. Let $C_*(U;\mathscr{L}^\vee)$ denote the corresponding singular chain complex. Consider the short exact sequence of de Rham complexes on X:

$$0 \longrightarrow \mathrm{DR}(L_{\text{mero}}) \longrightarrow \mathrm{DR}(L_{\text{ess}}) \longrightarrow \mathrm{DR}(L_{\text{ess}}/L_{\text{mero}}) \longrightarrow 0$$

as well as the following sequence of complexes of abelian groups

$$0 \longrightarrow C_*(U;\mathscr{L}^\vee) \longrightarrow \mathscr{C}_*^{rd}(X;L^\vee) \longrightarrow \mathscr{C}_*^{rd}(X,U;L^\vee) \longrightarrow 0\ ,$$

whose exactness is obvious.

Now, for $\omega \in L^p_{\text{ess}}$ with $\nabla\omega \in L^{p+1}_{\text{mero}}$ and $c \otimes \varepsilon \in \mathscr{C}^{rd}_{p+1}(X)$ with $\partial c \in C_p(X \smallsetminus D) + C_p(D)$, we define

$$< [\omega], [c \otimes \varepsilon] > := \int_c < \varepsilon, \nabla\omega > - \int_{\partial c - D} < \varepsilon, \omega > \ .$$

This gives a well-defined pairing $H^p_{dR}(L_{\text{ess}}/L_{\text{mero}}) \times H^{rd}_{p+1}(X,U) \to \mathbb{C}$, which fits into the long exact sequences induced:

$$\begin{array}{ccccccccc}
\ldots \leftarrow & H^{p+1}_{dR}(L_{\text{mero}}) & \leftarrow & H^p_{dR}(L_{\text{ess}}/L_{\text{mero}}) & \leftarrow & H^p_{dR}(L_{\text{ess}}) & \leftarrow & H^p_{dR}(L_{\text{mero}}) & \leftarrow \ldots \\
\ldots \rightarrow & H^{rd}_{p+1}(X;L^\vee) & \rightarrow & H^{rd}_{p+1}(X,U;L^\vee) & \rightarrow & H_p(U;\mathscr{L}^\vee) & \rightarrow & H^{rd}_p(X;L^\vee) & \rightarrow \ldots
\end{array} .$$

We want to fix the given connection $(L(*D),\nabla)$ and drop $L^\vee$ and $\mathscr{L}^\vee$ in the notation of the rd-homology groups from now on. We will refer to the pairing above as the **irregularity pairing**. Note, that the de Rham complex of $L_{\text{ess}}/L_{\text{mero}}$ coincides up to a shift of degrees with the irregularity complex introduced by Z. Mebkhout (cp. [17]).

3 Localization according to stratification of the divisor

From now on, we concentrate on the case $\dim(X) = 2$. In this section, we want to reduce the question of perfectness of the irregularity pairing in two steps. The first one, being rather standard, reduces to the local situation according to an open covering of the divisor. This leaves us with the task to study two different local situations, namely with D being of the form $D = \{x_1 = 0\}$ or $D = \{x_1x_2 = 0\}$ in suitable local coordinates. In order to understand the situation at a crossing point, we will further split the pairing into one concentrating at the crossing point and one determining the contribution of the connection along the two local components meeting at 0.

If $\mathscr{U}$ denotes an open covering of X, we have two cohomological spectral sequences, the first one considered being

$$E_2^{p,q} = H^p(\mathscr{U}, \mathscr{H}^q_{dR}(L_{\text{ess}}/L_{\text{mero}})) \Longrightarrow H^{p+q}_{dR}(X; L_{\text{ess}}/L_{\text{mero}}) ,$$

where $\mathscr{H}^q_{dR}(L_{\text{ess}}/L_{\text{mero}})$ denotes the presheaf $U' \mapsto H^p_{dR}(U', L_{\text{ess}}/L_{\text{mero}})$. In the same way, one has a homological spectral sequence involving the sheafified rapid decay complex $\mathscr{C}^{rd}_*$. To define the latter, we contruct the sheaf $\mathscr{C}^*_{rd}$ of rapid decay cochains as the sheaf associated to the presheaf $U' \mapsto \mathrm{Hom}(\mathscr{C}^{rd}_*(U', U' \smallsetminus D; D), \mathbb{C})$ and let

$$\mathscr{C}^{rd}_* := \mathscr{H}om_{\mathbb{C}}(\mathscr{C}^*_{rd}, \mathbb{C})$$

be its dual sheaf. We consider the dual (cohomological) spectral sequence

$$\widetilde{E}_2^{p,q} = H_p(\mathscr{U}, \mathscr{C}^{rd}_q)^\vee \Longrightarrow H^{rd}_{p+q}(X, X \smallsetminus D; D)^\vee \ .$$

Obviously, the pairing above induces a morphism between these two spectral sequences. In order to prove perfectness of (2.1), we therefore can do so for the local situation. i.e. a suitable open covering. For small enough open U', we thus have to consider two cases, the one at a crossing point and the one at a smooth point of D. Thus, we can assume, that X is a small bi-disc $X = \Delta \cong D^2 \times D^2 \subset \mathbb{C}^2$ and D reads as either the smooth divisor $D = \{x_1 = 0\}$ or the union of two coordinate planes $D = \{x_1x_2 = 0\}$ with crossing point $(0,0)$.

3.1 Distinguishing the contribution of the local strata

3.1.1 The local (co-)homology groups at a crossing-point

We consider the second case from above, i.e. $X = \Delta$ a small bi-disc around the crossing-point $0 \in D = \{x_1x_2 = 0\}$ in suitable local coordinates. We write $D = D_1 \cup D_2$ for the two local components, $D_\nu = \{x_\nu = 0\}$. Let $j_\nu : \Delta \smallsetminus D_\nu \to \Delta$ denote the inclusion for $\nu = 1,2$. We will write L^p_ν for the subsheaf of $L^p_{\text{ess}} := L_{\text{ess}} \otimes \Omega^p$ defined as

$$L^p_\nu := j_{\nu *} j^*_\nu (L(*D) \otimes \Omega^p) \ ,$$

so that a local section u of L_ν^p is an analytic p-form with values in L defined on the complement of D, which is meromorphic along $D_{\neq\nu} \smallsetminus 0$ and arbitrary along D_ν. One might think of the function $1/z_2 \cdot \exp(1/z_1)$ as a typical example of an element in L_1^0. The given connection ∇, being meromorphic, obviously carries L_ν^p to L_ν^{p+1} and therefore gives the following complex of sheaves on Δ:

$$\mathrm{DR}(L_\nu): \quad 0 \longrightarrow L_\nu^0 \xrightarrow{\nabla} \ldots \longrightarrow L_\nu^p \xrightarrow{\nabla} L_\nu^{p+1} \longrightarrow \ldots .$$

In the same way, we write $L_{\mathrm{mero}}^p := L_{\mathrm{mero}} \otimes \Omega^p$. Now $L_1 \cap L_2 = L_{\mathrm{mero}}$ and we have the following short exact sequence

$$0 \leftarrow \mathrm{DR}(L_{\mathrm{ess}}/L_1 + L_2) \leftarrow \mathrm{DR}(L_{\mathrm{ess}}/L_{\mathrm{mero}}) \leftarrow \mathrm{DR}(L_1/L_{\mathrm{mero}} \oplus L_2/L_{\mathrm{mero}}) \leftarrow 0$$

of complexes of sheaves supported on D, where the map at the right is given by $[\eta_1] + [\eta_2] \mapsto [\eta_1 + \eta_2]$. It gives rise to the long exact cohomology sequence

$$\ldots \leftarrow H_{dR}^{p-1}(L_{\mathrm{ess}}/L_1 + L_2) \leftarrow H_{dR}^{p-1}(L_{\mathrm{ess}}/L_{\mathrm{mero}}) \leftarrow \bigoplus_{i=1,2} H_{dR}^{p-1}(L_i/L_{\mathrm{mero}}) \leftarrow \ldots \tag{3.1}$$

which is the long exact sequence we are going to pair with the corresponding sequence of rapid decay homology.

As for the rapid decay homology groups, an easy argument using the subdivision morphism shows that $\mathscr{C}_*^{rd}(\Delta \smallsetminus D_1) + \mathscr{C}_*^{rd}(\Delta \smallsetminus D_2) = \mathscr{C}_*^{rd}(\Delta \smallsetminus 0)$ and $\mathscr{C}_*^{rd}(\Delta \smallsetminus D_1) \cap \mathscr{C}_*^{rd}(\Delta \smallsetminus D_2) = \mathscr{C}_*^{rd}(V \smallsetminus D)$. It follows that the following sequence is exact:

$$0 \to \mathscr{C}_*^{rd}(\Delta, \Delta \smallsetminus D; D) \to \bigoplus_{i=1,2} \mathscr{C}_*^{rd}(\Delta, \Delta \smallsetminus D_i; D) \to \mathscr{C}_*^{rd}(\Delta, \Delta \smallsetminus 0; D) \to 0 .$$

Here the first map is given by $[\alpha] \mapsto [\alpha] + [\alpha]$ and the second one by $[\alpha] + [\beta] \mapsto [\alpha - \beta]$. The corresponding long exact homology sequence reads as

$$\ldots \to H_p^{rd}(\Delta, \Delta \smallsetminus D; D) \to \bigoplus_{i=1,2} H_p^{rd}(\Delta, \Delta \smallsetminus D_i; D) \to H_p^{rd}(\Delta, \Delta \smallsetminus 0; D) \to \ldots \tag{3.2}$$

The members of the latter may be rewritten in the following form, where we retract $\Delta \smallsetminus D_\nu$ to the boundary ∂V_ν of a small tubular neighborhood V_ν of D_ν, i.e. if Δ reads as the product of two disc $\Delta = D^2 \times D^2$ in local coordinates, ∂V_1 will be $\partial V_1 = S^1 \times D^2$.

Lemma 3.1 *There are natural isomorphisms* $H_p^{rd}(\Delta, \Delta \smallsetminus D_\nu; D) \cong H_p^{rd}(V_\nu, \partial V_\nu; D)$ *for* $\nu = 1, 2$ *and* $H_p^{rd}(\Delta, \Delta \smallsetminus 0; D) \cong H_p^{rd}(\Delta, \partial\Delta; D)$.

Proof: This follows easily by excision with the help of the subdivision morphism, *subd* $\simeq$ *id*, on the complex of rapidly decaying cycles, as well as the observation that one can chose retracts of e.g. $\Delta \smallsetminus D_\nu$ to the boundary ∂V_ν respecting the rapid decay condition on the cycles $c \otimes \varepsilon$.

□

With these isomorphisms, the exact sequence (3.2) reads as

$$\ldots \to H_p^{rd}(\Delta, \Delta \smallsetminus D; D) \to \bigoplus_{i=1,2} H_p^{rd}(V_1, \partial V_i; D) \to H_p^{rd}(\Delta, \partial\Delta; D) \to \ldots \tag{3.3}$$

Our aim in this section is to define a pairing between the long exact sequences (3.1) and (3.3), that is a bilinear pairing of their members, such that the obvious diagrams all commute.

3.1.2 Definition of the pairings

In perfect analogy to the case of the complex $\mathrm{DR}(L_{\mathrm{ess}})$ itself, one defines for given

i) $\omega \in L_\nu^p$ with $\nabla\omega \in L_{\mathrm{mero}}^{p+1}$ and ii) $c\otimes\varepsilon \in \mathscr{C}^{rd}(V_1)$ with $\partial c \in \mathscr{C}^{rd}(\partial V_1) + \mathscr{C}^{rd}(D)$:

$$< \omega, c\otimes\varepsilon > := \int\limits_c < \varepsilon, \nabla\omega > - \int\limits_{\partial c - D_1} < \varepsilon, \omega > \ . \tag{3.4}$$

We remark, that the first integral exists, as $\nabla\omega \in L_{\mathrm{mero}}^{p+1}$ is meromorphic along D and therefore pairs with the rapidly decaying $c\otimes\varepsilon$. In the same way the second integral converges, because $\omega \in L_1^p$ is meromorphic along $D_2 - 0$ and ε decreases rapidly as $\partial c - D$ possibly approaches D_2.

Again, by using the limit-Stokes formula, one easily observes that (3.4) induces a well-defined pairing $H_{dR}^p(L_\nu / L_{\mathrm{mero}}) \times H_{p+1}^{rd}(V_\nu, \partial V_\nu; D) \longrightarrow \mathbb{C}$, which fits into the long exact sequence:

$$\begin{array}{ccccccccc}
\ldots \leftarrow H_{dR}^{p+1}(L_{\mathrm{mero}}) & \leftarrow & H_{dR}^p(L_\nu/L_{\mathrm{mero}}) & \leftarrow & H_{dR}^p(L_\nu) & \leftarrow & H_{dR}^p(L_{\mathrm{mero}}) & \leftarrow \ldots \\
\ldots \rightarrow H_{p+1}^{rd}(\Delta; D) & \rightarrow & H_{p+1}^{rd}(V_\nu, \partial V_\nu; D) & \rightarrow & H_p^{rd}(\Delta \smallsetminus D_\nu; D_{\neq\nu}) & \rightarrow & H_p^{rd}(\Delta; D) & \rightarrow \ldots
\end{array}$$

Returning to the long exact sequences (3.1) and (3.3), which we want to relate to each other, we now prove compatibility of the pairings defined so far:

Lemma 3.2 *The maps*

$$\begin{array}{ccc}
H_{dR}^p(L_{\mathrm{ess}}/L_{\mathrm{mero}}) & \longleftarrow & H_{dR}^p(L_1/L_{\mathrm{mero}}) \oplus H_{dR}^p(L_2/L_{\mathrm{mero}}) \\
H_{p+1}^{rd}(\Delta, \Delta\smallsetminus D; D) & \longrightarrow & H_{p+1}^{rd}(V_1, \partial V_1; D_1) \oplus H_{p+1}^{rd}(V_2, \partial V_2; D_2)
\end{array}$$

are compatible with the pairings between the groups in the columns of the above diagram.

Proof: Suppose we have $\eta_\nu \in L_\nu^p$ with $\nabla\eta \in L_{\mathrm{mero}}^{p+1}$ and $c\otimes\varepsilon \in \mathscr{C}_{p+1}^{rd}(\Delta)$ with $\partial(c\otimes\varepsilon) \in \mathscr{C}_p^{rd}(\Delta\smallsetminus D) + \mathscr{C}_p^{rd}(D)$. Now, we can assume (by subdivision) that the topological chain c decomposes as a sum $c = c_1 + c_2$, such that $c_\nu \otimes \varepsilon \in \mathscr{C}_{p+1}^{rd}(V_\nu, \partial V_\nu; D)$ with vanishing $\partial(c_\nu\otimes\varepsilon)$ (mod $\mathscr{C}_p^{rd}(\partial V_\nu) + \mathscr{C}_p^{rd}(D)$). Restricting the section ε of $\mathscr{L}^\vee$ to c_ν does not affect its rapid decay properties.

The diagram above reads as $[\eta_1+\eta_2] \leftarrow [\eta_1] + [\eta_2]$ in the top row and $[c\otimes\varepsilon] \mapsto [c_1\otimes\varepsilon] + [c_2\otimes\varepsilon]$ in the bottom row and we have to prove that $< [\eta_1+\eta_2], [c\otimes\varepsilon] > = < [\eta_1], [c_1\otimes\varepsilon] > + < [\eta_2], [c_2\otimes\varepsilon] >$. Now, we consider the decomposition $c = c_1 + c_2$ and observe, that $c_2 \cap D_1 = \emptyset$, the section ε is rapidly decreasing as c_2 possibly approaches D_2 and that $\eta_1 \in L_1^p$ is meromorphic along $D_2 - 0$, so that we can apply the 'limit Stokes formula' and obtain

$$\int\limits_c < \varepsilon, \nabla\eta_1 > - \int\limits_{\partial c - D} < \varepsilon, \nabla\eta_1 > = \int\limits_{c_1} < \varepsilon, \nabla\eta_1 > - \int\limits_{\partial c_1 - D} < \varepsilon, \nabla\eta_1 > \ .$$

The same argument applies to c_2, η_2 instead of c_1, η_1 proving the assertion.

□

In the next step, we define a pairing $H_{dR}^{p-1}(L_{\mathrm{ess}}/L_1 + L_2) \times H_{p+1}^{rd}(\Delta, \partial\Delta; D) \to \mathbb{C}$. To this end, let there be given

i) $\omega \in L_{\mathrm{ess}}^{p-1}$ with $\nabla\omega =: \eta_1 + \eta_2 \in L_1^p + L_2^p$ and

ii) $c\otimes\varepsilon \in \mathscr{C}_{p+1}^{rd}(\Delta)$ with $\partial(c\otimes\varepsilon) \in \mathscr{C}_p^{rd}(\partial\Delta) + \mathscr{C}_p^{rd}(D)$.

We have $\partial\Delta = \partial V_1 \cup \partial V_2$. By subdivision we can decompose c as the sum $c = c_1 + c_2$ with $\partial c_\nu \cap \partial\Delta \subset \partial V_\nu$. With these choices, we define $\alpha(\eta_1, \eta_2, c_1\otimes\varepsilon, c_2\otimes\varepsilon) :=$

$$:= \sum_{i=1,2} (-1)^{i+1} \Big(\int\limits_{c_i} < \varepsilon, \nabla\eta_i > - \int\limits_{\partial c_i \cap \partial\Delta - D} < \varepsilon, \eta_i > \Big) + \int\limits_{\partial(\partial c_1 \cap \partial c_2) - D} < \varepsilon, \omega > \ . \tag{3.5}$$

A few words should be said about this definition:

i) Integrability of ∇ ensures that $0 = \nabla\nabla\omega = \nabla\eta_1 + \nabla\eta_2$ and therefore $\nabla\eta_1 = -\nabla\eta_2 \in L_1^{p-1} \cap L_2^{p-1} = L_{\mathrm{mero}}^{p-1}$. This shows convergence of the first and third integral in (3.5), as ε is rapidly decaying.

ii) $\eta_\nu \in L_\nu^p$ is meromorphic along $D_{\neq\nu} \setminus 0$ and therefore $< \varepsilon, \eta_\nu >$ can be integrated over the chain ∂c_ν not meeting D_ν, hence the second and fourth integral in (3.5) converge.

iii) $\partial c_1 \cap \partial c_2$ consist of the subsimplices that arose in the chosen decomposition of $c = c_1 + c_2$, which we equip with the orientation induced from c_1. These simplices are either fully contained in $\Delta \setminus D$ or in D. In the last integral in (3.5) we integrate over the first type completely contained in $\Delta \setminus D$, hence the integral is well-defined.

Lemma 3.3 *Mapping* $([\omega], [c \otimes \varepsilon]) \mapsto \alpha(\eta_1, \eta_2, c_1 \otimes \varepsilon, c_2 \otimes \varepsilon)$ *gives a well-defined pairing*

$$H_{dR}^{p-1}(L_{\mathrm{ess}} / L_1 + L_2) \times H_{p+1}^{rd}(\Delta, \partial\Delta; D) \to \mathbb{C} .$$

Proof: Let $\alpha := \alpha(\eta_1, \eta_2, c_1 \otimes \varepsilon, c_2 \otimes \varepsilon)$ and let c_1^t, c_2^t denote the simplices if we cut off a small tube of radius t from D, so that c_i^t converges to c_i for $t \to 0$. Then α is the limit $t \to 0$ of the sum of the integrals as in (3.5) the cut-off c_i^t instead of c_i. We decompose $\partial c_i^t = \gamma_i^t + \nu_i^t + \zeta^t$, where $\gamma_i^t \subset \partial\Delta$, ζ^t denotes the simplices of $\partial c_1^t \cap \partial c_2^t$ not running completely into D and ν_i^t the simplices of ∂c_i^t running completely into D for $t \to 0$, all of them taken with their natural orientations induced from c (especially ζ^t carries different orientations viewed as part of c_1^t or c_2^t). The simplices of $\partial\zeta^t =$ again decompose into those contained in $\Delta \setminus D$ for all t and those running into D. The first type will be denoted by γ^t, the second type by ν^t. Applying the usual Stokes formula to the integrals over c_i^t gives

$$\int_{c_1^t} < \nabla\eta_i, \varepsilon > - \int_{\partial c_i^t - D} < \eta_1, \varepsilon > = \int_{\nu_i^t} < \eta_i, \varepsilon > + (-1)^i \int_{\zeta^t} < \eta_i, \varepsilon > ,$$

and $\alpha = \lim_{t\to 0}(\int_{\nu_1^t} <\eta_1, \varepsilon> - \int_{\nu_2^t} <\eta_2, \varepsilon> + \int_{\zeta^t} <\eta_1 + \eta_2, \varepsilon> + \int_{\gamma^t} <\omega, \varepsilon>)$. Now, $\eta_1 + \eta_2 = \nabla\omega$ and the Stokes formula gives

$$\alpha = \lim_{t\to 0}(\int_{\nu_1^t} < \eta_1, \varepsilon > - \int_{\nu_2^t} < \eta_2, \varepsilon > + \int_{\nu^t} < \omega, \varepsilon >) . \tag{3.6}$$

With this presentation of α independence of the choices made is easily shown. If e.g. $\nabla\omega = \eta_1 + \eta_2 = \eta_1' + \eta_2'$ are two different choices, it follows that $\eta_1 - \eta_1' = \eta_2' - \eta_2 \in L_1^p \cap L_2^p = L_{\mathrm{mero}}^p$, and hence the rapid decay of ε implies that $\int_{\nu_i^t} < \eta_i - \eta_i', \varepsilon > = 0$.

If $\omega \equiv 0 \bmod L_1 + L_2$, say $\omega = \omega_1 + \omega_2 \in L_1^{p-1} + L_2^{p-1}$, one can take $\eta_i := \nabla\omega_i$ and obtains that $\alpha = 0$, since $\partial\nu_i^t$ consists of ν^t and some simplices running into $D \setminus 0$, let us denote them by μ^t. Then

$$\lim_{t\to 0} \int_{\nu_1^t} < \nabla\omega_1, \varepsilon > = \lim_{t\to 0}(\int_{\nu^t} < \omega_1, \varepsilon > + \int_{\mu^t} < \omega_1, \varepsilon >) = \lim_{t\to 0} \int_{\nu^t} < \omega_1, \varepsilon > ,$$

the last equality following from the rapid decay of ε and the at worst meromorphic behavior of ω_1 in the limit process involved. Taking care of the orientations (that are all induced by the one of c) gives that $\alpha = 0$ in this case. The cases $\omega = \nabla\Omega$ or the corresponding cases to show independence of the choices of the rapid decay chains are similar and omitted here.

□

3.1.3 Compatibility with the long exact sequences

We now want to show that the pairings defined in the last section commute with the mappings of the long exact sequences (3.1) and (3.3). The first step toward this has already been done in Lemma 3.2.

First, we take a more precise look at the rd-homology sequence (3.3). In it we find the following mapping, the first row of the diagram:

$$\begin{array}{ccccc} H^{rd}_{p+1}(V_1,\partial V_1;D) & \oplus & H^{rd}_{p+1}(V_2,\partial V_2;D) & \longrightarrow & H^{rd}_{p+1}(\Delta,\partial\Delta;D) \\ \uparrow\cong & & \uparrow\cong & & \uparrow\cong \\ H^{rd}_{p+1}(\Delta,\Delta\smallsetminus D_1;D_1) & \oplus & H^{rd}_{p+1}(\Delta,\Delta\smallsetminus D_2;D_2) & \longrightarrow & H^{rd}_{p+1}(\Delta,\Delta\smallsetminus 0;D) \\ [c_1\otimes\varepsilon_1] & + & [c_2\otimes\varepsilon_2] & \mapsto & [c_1\otimes\varepsilon_1]-[c_2\otimes\varepsilon_2] \end{array} \quad . \tag{3.7}$$

The vertical mapping at the right hand side is defined by 'capping off at $\partial\Delta$'. More precisely, let $c\otimes\varepsilon\in\mathscr{C}^{rd}_{p+1}(\Delta)$. By subdivision, we can decompose the topological chain c into a sum $c=\widetilde{c}+\gamma$ such that $\widetilde{c}=c\cap\Delta$ and we have $H^{rd}_{p+1}(\Delta,\Delta\smallsetminus 0;D)\xrightarrow{\cong}H^{rd}_{p+1}(\Delta,\partial\Delta;D)$, $[c\otimes\varepsilon]\mapsto[\widetilde{c}\otimes\varepsilon]$. Now suppose we have a $\omega\in L^{p-1}_{\text{ess}}$ with $\nabla\omega=\eta_1+\eta_2\in L^p_1+L^p_2$ (and therefore also $\nabla\eta_1=-\nabla\eta_2\in L^p_{\text{mero}}$) and $c_\nu\otimes\varepsilon_\nu\in\mathscr{C}^{rd}_{p+1}(V_\nu)$ with $\partial(c_\nu\otimes\varepsilon_\nu)\in\mathscr{C}^{rd}_p(\partial V_\nu)+\mathscr{C}^{rd}_p(D)$.

We decompose $c_\nu=\widetilde{c}_\nu+\gamma_\nu$ as before. Then $\gamma_\nu\cap D_{\neq\nu}=\emptyset$ and the 'limit Stokes formula'

$$\int\limits_{\gamma_1}<\varepsilon_1,\nabla\eta_1>=-\int\limits_{\gamma_1}<\varepsilon_1,\nabla\eta_2>=-\int\limits_{\partial\gamma_1-D_1}<\varepsilon_1,\eta_2>$$

tells us that

$$\int\limits_{c_\nu}<\varepsilon_\nu,\nabla\eta_\nu>-\int\limits_{\partial c_\nu-D_\nu}<\varepsilon_\nu,\eta_\nu>=\int\limits_{\widetilde{c}_\nu}<\varepsilon_\nu,\nabla\eta_\nu>-\int\limits_{\partial\widetilde{c}_\nu-D_\nu}<\varepsilon_\nu,\eta_\nu>\ . \tag{3.8}$$

Lemma 3.4 *The maps in the diagram*

$$\begin{array}{ccc} H^p_{dR}(L_1/L_{\text{mero}})\oplus H^p_{dR}(L_2/L_{\text{mero}}) & \longleftarrow & H^{p-1}_{dR}(L_{\text{ess}}/L_1+L_2) \\ H^{rd}_{p+1}(V_1,\partial V_1;D_1)\oplus H^{rd}_{p+1}(V_2,\partial V_2;D_2) & \longrightarrow & H^{rd}_{p+1}(\Delta,\partial\Delta;D) \end{array}$$

are compatible with the given pairings.

Proof: With the notation introduced right above the lemma, the diagram maps the given elements as $[\eta_1]+[\eta_2]\leftarrow[\omega]$ and $[c_1\otimes\varepsilon_1]+[c_2\otimes\varepsilon_2]\mapsto[\widetilde{c}_1\otimes\varepsilon_1]-[\widetilde{c}_2\otimes\varepsilon_2]$. The desired equation $<[\omega],[\widetilde{c}_1\otimes\varepsilon_1]>-<[\omega],[\widetilde{c}_2\otimes\varepsilon_2]>=<[\eta_1],[c_1\otimes\varepsilon_1]>+<[\eta_2],[c_2\otimes\varepsilon_2]>$ follows directly from the definitions using (3.8).

□

It remains to prove

Lemma 3.5 *The maps in the diagram*

$$\begin{array}{ccc} H^{p-1}_{dR}(L_{\text{ess}}/L_1+L_2) & \longleftarrow & H^{p-1}_{dR}(L_{\text{ess}}/L_{\text{mero}}) \\ H^{rd}_{p+1}(\Delta,\partial\Delta;D) & \longrightarrow & H^{rd}_p(\Delta,\Delta\smallsetminus D;D) \end{array}$$

are compatible with the given pairings.

Proof: We compute the connecting morphism ψ in the long exact sequence induced by

$$0\to\mathscr{C}^{rd}_*(\Delta,\Delta\smallsetminus D;D)\to\bigoplus_{i=1,2}\mathscr{C}^{rd}_*(\Delta,\Delta\smallsetminus D_i;D)\to\mathscr{C}^{rd}_*(\Delta,\Delta\smallsetminus 0;D)\to 0\ .$$

If we write $(\mathscr{C}_\nu)_p:=\mathscr{C}^{rd}_p(\Delta\smallsetminus D_\nu)+\mathscr{C}^{rd}_p(D)$ and $\mathscr{C}_p:=\mathscr{C}^{rd}_p(\Delta)$, we have $\mathscr{C}^{rd}_p(\Delta\smallsetminus 0)+\mathscr{C}^{rd}_p(D)=(\mathscr{C}_1)_p+(\mathscr{C}_2)_p$ and $\mathscr{C}^{rd}_p(\Delta\smallsetminus D)+\mathscr{C}^{rd}_p(D)=(\mathscr{C}_1)_p\cap(\mathscr{C}_2)_p$. Therefore the homological yoga to determine $\psi([c\otimes\varepsilon])$ for a given chain $c\otimes\varepsilon\in\mathscr{C}_p$ with $\partial(c\otimes\varepsilon)\in(\mathscr{C}_1+\mathscr{C}_2)_{p-1}$ reads:

$$\begin{array}{cccccccc} & & & (\mathscr{C}/\mathscr{C}_1)_p\oplus(\mathscr{C}/\mathscr{C}_2)_p & \to & (\mathscr{C}/\mathscr{C}_1+\mathscr{C}_2)_p & \to 0 \\ & & & \downarrow & & \downarrow & \\ 0\to & (\mathscr{C}/\mathscr{C}_1\cap\mathscr{C}_2)_{p-1} & \to & (\mathscr{C}/\mathscr{C}_1)_{p-1}\oplus(\mathscr{C}/\mathscr{C}_2)_{p-1} & \to & (\mathscr{C}/\mathscr{C}_1+\mathscr{C}_2)_{p-1} & \to 0 \end{array}$$

where $c\otimes\varepsilon$ makes its way through this diagram as follows (to be read from the upper right to the lower left corner):

$$\begin{array}{ccccccc} & & & & 0+[-c\otimes\varepsilon] & \mapsto & [c\otimes\varepsilon] \\ & & & & \downarrow & & \downarrow \\ 0\mapsto & [-\partial(c_1\otimes\varepsilon)] & \mapsto & & [-\partial(c_1\otimes\varepsilon)]+[-\partial(c_1\otimes\varepsilon)] & \mapsto & 0\,. \end{array} \tag{3.9}$$

Here we decomposed the topological chain $c\in C_p(\Delta)$ into the sum $c=c_1+c_2$ with $\partial c_\nu\subset\Delta\setminus D_\nu$. It follows, that $0+[-\partial(c\otimes\varepsilon)]=[-\partial(c_1\otimes\varepsilon)]+[-\partial(c_1\otimes\varepsilon)]\in(\mathscr{C}/\mathscr{C}_1)_{p-1}\oplus(\mathscr{C}/\mathscr{C}_2)_{p-1}$.

Now, let $\omega\in L^{p-1}_{\text{ess}}$ be given such that $\nabla\omega\in L^p_{\text{mero}}$. To keep notation, we let $\eta_1:=\nabla\omega\in L^p_{\text{mero}}\subset L^p_1$. Suppose we started in (3.9) above with $c\otimes\varepsilon\in\mathscr{C}^{rd}_p(\Delta)$ such that $\partial c\subset\mathscr{C}^{rd}_{p-1}(\partial\Delta)+\mathscr{C}^{rd}_{p-1}(D)$. Then we have to consider the diagram

$$\begin{array}{ccccccc} H^{p-1}_{dR}(L_{\text{ess}}/L_1+L_2) & \leftarrow & H^{p-1}_{dR}(L_{\text{ess}}/L_{\text{mero}}) & & [\omega] & \leftarrow & [\omega] \\ \times & & \times & & & & \\ H^{rd}_{p+1}(\Delta,\partial\Delta;D) & \rightarrow & H^{rd}_p(\Delta,\Delta\setminus D;D) & & [c\otimes\varepsilon] & \mapsto & [-\partial(c_1\otimes\varepsilon)]\,. \end{array}$$

Starting with the left hand side we obtain

$$<[\omega],[c\otimes\varepsilon]>\overset{\text{def}}{=}\alpha(\eta_1,0,c_1\otimes\varepsilon,c_2\otimes\varepsilon)=-\int\limits_{\partial c_1\cap\partial\Delta-D}<\varepsilon,\eta_1>+\int\limits_{\partial(\partial c_1\cap\partial c_2)-D}<\varepsilon,\omega>=<[\omega],[-\partial(c_1\otimes\varepsilon)]>\,,$$

which is the right hand side.

$\square$

3.2 The Localization Lemma

In the section above, we have constructed a pairing between the long exact sequences (3.1) and (3.3). It follows, that in order to prove perfectness of the irregularity pairing, it suffices to do so for $H^p_{dR}(L_{\text{ess}}/L_1+L_2)\times H^{rd}_{p+1}(\Delta,\partial\Delta;D)\to\mathbb{C}$ and $H^{p-1}_{dR}(L_\nu/L_{\text{mero}})\times H^{rd}_{p+1}(V_\nu,\partial V_\nu;D_\nu)\to\mathbb{C}$.
Now, consider the short exact sequence of de Rham complexes

$$0\leftarrow\mathrm{DR}(L_{\text{ess}}/L_1+L_2)\leftarrow\mathrm{DR}(L_{\text{ess}}/L_2)\leftarrow\mathrm{DR}(L_1/L_{\text{mero}})\leftarrow 0 \tag{3.10}$$

where the mapping is defined via the isomorphism $L^p_1/L^p_{\text{mero}}\xrightarrow{\cong}L^p_1+L^p_2/L^p_2$, which follows from $L^p_1\cap L^p_2=L^p_{\text{mero}}$.
In the same manner we have a short exact sequence of complexes of rd-chains:

$$0\to\mathscr{C}^{rd}_*(\Delta\setminus D_2,\Delta\setminus D;D)\to\mathscr{C}^{rd}_*(\Delta,\Delta\setminus D_1;D)\to\mathscr{C}^{rd}_*(\Delta,\Delta\setminus 0;D)\to 0\,. \tag{3.11}$$

Excision and retraction to the boundary again induces a natural isomorphism $H^{rd}_p(\Delta\setminus D_2,\Delta\setminus D;D)\cong H^{rd}_p(V_1\setminus D_2,\partial V_1;D)$. Again, for given

i) $\omega\in L^p_{\text{ess}}$ with $\nabla\omega\in L^p_2$ and

ii) $c\otimes\varepsilon\in\mathscr{C}^{rd}_p(V_1\setminus D_2)$ with $\partial c\in C_{p-1}(\partial V_1\setminus D_2)+C_{p-1}(D)$.

we define

$$<[\omega],[c\otimes\varepsilon]>:=\int\limits_c<\varepsilon,\nabla\omega>-\int\limits_{\partial c_1-D_1}<\varepsilon,\omega>\,,$$

where the first integral exists, because c does not intersect D_2 and $\nabla\omega$ is meromorphic along D_1, and the second one does, as $\partial c-D_1$ lies in $X\setminus D$. With the same arguments already used several times above one shows that this gives indeed a well-defined pairing of the long exact sequences corresponding to (3.10) and (3.11).

We have thus reduced the problem of perfectness of the period pairing at a crossing-point to the perfectness of a pairing involving (co-)homology groups supported at the crossing-point 0 and a pairing involving groups supported on the complement $D\smallsetminus\{0\}$. Together with the considerations at the beginning of this section, we have now achieved the Localization Lemma, which we will state using the following notion:

Definition 3.6 *We say that the rank* 1 *connection* (L,∇) *on* X*, singular along* D*, satisfies* **local perfectness***, if there is an open covering* $\mathscr{U}$ *of the tubular neighborhood* V *of* D *consisting of small enough bi-discs* Δ*, such that the following holds:*

i) *for each local situation* $D=D_1=\{x_1=0\}\subset\Delta$*, the pairing*

$$H^{p-1}_{dR}(\Delta;L_{\mathrm{ess}}/L_{\mathrm{mero}})\times H^{rd}_{p}(\Delta,\Delta\smallsetminus D_1;D_1)\longrightarrow\mathbb{C}\quad\textit{is perfect, and}$$

ii) *for each local situation* $D=\{x_1x_2=0\}\subset\Delta$*, the pairings*

$$H^{p-1}_{dR}(\Delta;L_{\mathrm{ess}}/L_{\nu})\times H^{rd}_{p}(\Delta\smallsetminus D_{\neq\nu},\Delta\smallsetminus D;D)\to\mathbb{C}\ ,$$

$(\nu=1,2)$*, and*

$$H^{p-1}_{dR}(\Delta;L_{\mathrm{ess}}/L_1+L_2)\times H^{rd}_{p+1}(\Delta,\Delta\smallsetminus D;D)\to\mathbb{C}\ ,$$

are perfect.

Lemma 3.7 *Let* X *be of dimension* $\dim(X)=2$ *and* (L,∇) *be an integrable meromorphic rank* 1 *connection on* X *with singularities along the normal crossing divisor* D*. Then the period pairing* $H^{p}_{dR}(U;L(*D),\nabla)\times H^{rd}_{p}(X;L^{\vee})\to\mathbb{C}$ *is perfect if and only if* (L,∇) *satisfies local perfectness.*

4 Local perfectness

In any of the local situations above, consider the formal connection $\widehat{L}:=L\otimes\mathscr{O}_{\widehat{X|Y}}(*D)$, where $\mathscr{O}_{\widehat{X|Y}}$ denotes the formal completion of $\mathscr{O}_X$ with respect to the stratum Y of D considered, namely Y being a smooth component of D or a crossing-point. By assumption (Definition 1.1), $\widehat{L}|_\Delta$ is isomorphic to the formal completion of a connection of the form $e^{\alpha}\otimes R$, with $\alpha(x):=x_1^{-m_1}x_2^{-m_2}u(x)$ such that $u(0)\neq0$ and R a regular singular connection. It follows that $(L,\nabla)|_\Delta$ itself is isomorphic to $e^{\alpha}\otimes R$ (since $L\otimes e^{-\alpha}$ is necessarily regular singular).

4.1 Local pairing supported on a crossing-point

We consider the local situation ii) of Definition 3.6, i.e. Δ is a small bi-disc around the crossing-point $0\in D$.

4.1.1 Local rd-homology

We introduce the following notions (after chosing fixed local coordinates) in the situation of a connection of the form $e^{\alpha}\otimes R$ with $\alpha(x):=x_1^{-m_1}x_2^{-m_2}u(x)$ with $u(0)\neq0$ and R a regular singular connection.

Definition 4.1 *The* **Stokes directions** *of* $e^{\alpha}\otimes R$ *at* 0 *are the elements of*

$$\Sigma_0:=\{(\vartheta_1,\vartheta_2)\mid -m_1\vartheta_1-m_2\vartheta_2+\arg(u(0))\in(\frac{\pi}{2},\frac{3\pi}{2})\}\subset\pi^{-1}(0)\cong S^1\times S^1\ .$$

The **Stokes bisectors** *are the bisectors* $\mathfrak{S}_0:=\bigcup_{(\vartheta_1,\vartheta_2)\in\Sigma_0}\mathbb{R}^+e^{i\vartheta_1}\times\mathbb{R}^+e^{i\vartheta_2}\subset\mathbb{C}^2$.

We now calculate the local rapid decay homology groups at a crossing point. In the following, we will abbreviate $H^{rd}_*(Y,\partial):=H^{rd}_*(Y,\partial Y;L)$ and in the same way for the rapid decay chains $\mathscr{C}^{rd}_*(Y,\partial)$ for any $Y\subset\Delta$.

Proposition 4.2 *In the situation above, we have*

$$H_*^{rd}(\Delta,\partial) \cong H_*(\Delta\cap\mathfrak{S}_0, \partial\Delta\cup D)\ ,$$

where the right hand side denotes the usual singular homology group.

Proof: We first observe, that we have a natural homomorphism

$$H_*(Y\cap\mathfrak{S}_0,\partial Y\cup D) \longrightarrow H_*^{rd}(Y,\partial) \tag{4.1}$$

for any $Y\subset\Delta$, since e^{α} is rapidly decaying along any simplex contained in the Stokes region. We claim, that this is an isomorphism for $Y=\Delta$.

Now, both sides of the morphism (4.1) can be embedded in Mayer-Vietoris sequences by decomposing Δ into bisectors in as follows. Let $\Delta_1=\bigcup_{i=1}^n S(\nu_i)$ and $\Delta_2=\bigcup_{j=1}^m S(\mu_j)$ be a decomposition of the disc into closed sectors $S(\nu_i)$ and $S(\mu_j)$, where the ν_i and μ_j are intervals in S^1, cyclically ordered, such that $\bigcup_{i=1}^n \nu_i = S^1 = \bigcup_{j=1}^m \mu_j$. We denote by

$$[0,p_{i+1}] := \lambda_{i+1} := S(\nu_i)\cap S(\nu_{i+1}) \quad \text{and} \quad [0,q_{j+1}] := \kappa_{j+1} := S(\mu_j)\cap S(\mu_{j+1})$$

the common lines of two consecutive sectors. We assume that the intersection $(\nu_i\times\mu_j)\cap\mathfrak{S}_0$ is either empty or connected. The decomposition $\Delta=\bigcup_{j=1}^m \Delta_1\times S(\mu_j) =: \bigcup_{j=1}^m Z_j$ gives rise to the following Mayer-Vietoris sequence with $Z_{ij} := Z_i\cap Z_j$:

$$0\leftarrow \mathscr{C}_*^{rd}(\Delta,\partial)\leftarrow \bigoplus_{j=1}^m \mathscr{C}_*^{rd}(Z_i,\partial)\leftarrow \bigoplus_{1\le i<j\le m}\mathscr{C}_*^{rd}(Z_{ij},\partial)\leftarrow \dots\ .$$

Because of $Z_{ijk}=\Delta_1\times(S(\mu_i)\cap S(\mu_j)\cap S(\mu_k))=\Delta_1\times 0\subset D$, this sequences reduces to a short exact sequence (recall that in the definition of the rapid decaying chains, we have moduled out all chains contained in D) and we obtain the following long exact sequence of rapid decay homology groups:

$$\dots\to H_{n+1}^{rd}(\Delta,\partial)\to \bigoplus_{1\le i<j\le m} H_n^{rd}(Z_{ij},\partial)\to \bigoplus_{i=1}^m H_n^{rd}(Z_i,\partial)\to H_n^{rd}(\Delta,\partial)\to\dots \tag{4.2}$$

The same holds for the left hand side of (4.1) and thus (4.1) induces a natural morphism between these long exact sequences.

In a similar way, we fix j and consider the decomposition of $\Delta_1\times S(\mu) := Z := Z_j$ according to the decomposition of Δ_1 from above:

$$Z=\Delta_1\times S(\mu)=\bigcup_{i=1}^n S(\nu_i)\times S(\mu)\ .$$

Let $B_i := S(\nu_i)\times S(\mu)$. The analogous Mayer-Vietoris sequence of the $\mathscr{C}^{rd}$-groups is again short exact (as $B_{ijk} := B_i\cap B_j\cap B_k\subset D$) and therefore gives rise to the following exact Mayer-Vietoris sequence:

$$\dots\to H_{n+1}^{rd}(Z,\partial)\to \bigoplus_{1\le i<k\le n} H_n^{rd}(B_{ik},\partial)\to \bigoplus_{i=1}^m H_n^{rd}(B_i,\partial)\to H_n^{rd}(Z,\partial)\to\dots\ , \tag{4.3}$$

together with a map form the corresponding MV-sequence for $H_*(Z\cap\mathfrak{S}_0,\partial Z\cup D)$. Observe that

$$B_{ik}=\begin{cases} 0\times S(\mu)\subset D & \text{for } |i-k|\ge 2\\ \lambda_{i+1}\times S(\mu) & \text{for } k=i+1\end{cases} \quad\text{and } Z_{ij}=\begin{cases}\Delta_1\times 0\subset D & \text{for } |i-j|\ge 2\\ \Delta_1\times\kappa_i & \text{for } j=i+1\end{cases}\ .$$

We keep j fixed and decompose $\Delta_1\times\kappa=\bigcup_{i=1}^n S(\nu_i)\times\kappa$ for $\kappa=\kappa_j$. This leads to the Mayer-Vietoris sequence

$$\dots\to H_{n+1}^{rd}(Z_{i,i+1},\partial)\to \bigoplus_{1\le k<l\le n} H_n^{rd}(C_{kl},\partial,D)\to \bigoplus_{k=1}^m H_n^{rd}(C_k,\partial)\to\dots \tag{4.4}$$

where $C_k := S(\nu_k) \times \kappa$ and thus $C_{kl} := C_k \cap C_l = \begin{cases} 0 \times \kappa \subset D & \text{for } |k-l| \geq 2 \\ \lambda_k \times \kappa & \text{for } l = k+1 \end{cases}$. The similar assertion holds for the left hand side of (4.1). The proposition now follows from the following lemma:

Lemma 4.3 *There are natural isomorphisms*

$$\begin{array}{llll} \text{i)} & H_*^{rd}(\lambda \times \kappa, \partial) & \rightarrow & H_*((\lambda \times \kappa) \cap \mathfrak{S}_0, \partial(\lambda \times \kappa) \cup D) \quad \textit{and} \\ \text{ii)} & H_*^{rd}(S(\nu) \times \kappa, \partial) & \rightarrow & H_*((S(\nu) \times \kappa) \cap \mathfrak{S}_0, \partial(S(\nu) \times \kappa) \cup D) \ . \end{array}$$

Proof: Recall that $(L, \nabla) \cong e^\alpha \otimes R$ for some $\alpha = x_1^{-m_1} x_2^{-m_2} u(x)$ with $u(0,0) \neq 0$. We start with the proof for i). We claim, that if $\lambda \times \kappa \not\subset \mathfrak{S}_0$, then there is no rapidly decaying chain $c \otimes \varepsilon$ approaching 0 inside $\lambda \times \kappa$, which is not entirely contained in D. For, let $l = [0, e^{i\vartheta_1}]$ and $\kappa = [0, e^{i\vartheta_2}]$ and suppose that $\varepsilon = e^{\alpha(x)}$ is rapidly decaying as x varies in c, it follows that

$$\exp(|x_1|^{-m_1}|x_2|^{-m_2} \cdot |u(x)| \cdot \cos(-m_1\vartheta_1 - m_2\vartheta_2 + \arg(u(x)))) \leq C_{NM}|x_1|^N|x_2|^M$$

for $x \in c$. But then $-m_1\vartheta_1 - m_2\vartheta_2 + \arg(u(x)) \in (\pi/2, 3\pi/2)$ for x small enough and therefore $(\vartheta_1, \vartheta_2) \in \overline{\Sigma}_0$ and, because of the properties of the chosen decomposition with respect to the Stokes bisectors, finally $(\vartheta_1, \vartheta_2) \in \Sigma_0$, thus $\lambda \times \kappa \subset \mathfrak{S}_0$. For $\lambda \times \kappa \subset \mathfrak{S}_0$, however, the assertion is clear.

In order to prove part ii), consider $(S(\nu) \times \kappa) \cap \mathfrak{S}_0$, which is either empty or connected by assumption. Let $\nu =: [\xi, \xi'] \subset S^1$ and let ζ be the direction of the line κ, i.e. $\kappa = [0, e^{i\zeta}] \subset \mathbb{C}$. Then $(S(\nu) \times \kappa) \cap \mathfrak{S}_0$ is the union of the radii with directions contained in $(\nu \times \{\zeta\}) \cap \Sigma_0 \subset S^1 \times S^1$, with Σ_0 being the Stokes directions. Let ρ be the interval $\rho := \nu \cap \Sigma_0$. If ρ is empty or $\rho = \nu$ the assertion is clear. Otherwise, assume that $\rho = [\xi, \eta] \subsetneq [\xi, \xi'] = \nu$. Let λ be the radius with direction ξ, so that we now have $\lambda \times \kappa \subset \mathfrak{S}_0$. If $h : [0,1] \times (\nu \times \{\zeta\}) \to (\nu \times \{\zeta\})$ denotes the linear retraction of ν to $\{\zeta\}$, i.e. $h(t, (x, \zeta)) = ((1-t)x + t\xi, \zeta)$, then

$$H : [0,1] \times (S(\nu) \times \kappa) \to (S(\nu) \times \kappa) \ , \ (t, (r_1 e^{ix}, r_2 e^{i\zeta})) \mapsto (r_1 e^{ih(t,x,\zeta)}, r_2 e^{i\zeta})$$

retracts $S(\nu) \times \kappa$ to $\lambda \times \kappa \subset \mathfrak{S}_0$. We claim, that H preserves the rapid decay condition. To prove this, consider a curve $\gamma : [0,1] \to S(\nu) \times \kappa$ with $\gamma^{-1}(D) = 0$, such that $\varepsilon = e^{\alpha(x)}$ is rapidly decaying along γ. We have to show that ε then is rapidly decaying along all the curves $H(t,.) \circ \gamma$, $t \in [0,1]$. Let $\gamma(s) = (r_1(s)e^{i\theta(s)}, r_2(s)e^{i\zeta})$. From the rapid decay of ε along γ, it is clear that the direction of γ at the point $\gamma(0) \in D$ is in the closure of the Stokes-directions, i.e. $(\theta(0), \zeta) \in \overline{\Sigma_0}$. But then, the direction of the transposed curve $H(t,.) \circ \gamma$ at the point $H(t, \gamma(0))$ by construction lies in Σ_0, whence ε is rapidly decaying along this curve. Thus, H induces isomorphisms

$$H_*^{rd}(S(\nu) \times \kappa, \partial) \cong H_*^{rd}(\lambda \times \kappa, \partial) \cong H_*((\lambda \times \kappa) \cap \mathfrak{S}_0, \partial) \cong H_*((S(\nu) \times \kappa) \cap \mathfrak{S}_0, \partial) \ .$$

□

The proposition now follows immediately using the 5-lemma applied to the morphism (4.1) induces on the various Mayer-Vietoris sequences above ((4.2) – (4.4)).

□

Computing the singular homology groups appearing in the proposition, we obtain the following

Theorem 4.4 *For a rank* 1 *connection L with formal model* $e^\alpha \otimes R$*, we have*

$$\dim H_*^{rd}(\Delta, \partial) = \begin{cases} 0 & for \quad * = 1 \text{ or } * \geq 4 \\ & \\ (m_1, m_2) & for \quad * = 2 \text{ or } 3, \end{cases}$$

where $\alpha = x_1^{-m_1} x_2^{-m_2} \cdot u_\alpha(x)$ *with* $u_\alpha(0) \neq 0$ *and* (m, n) *denotes the greatest common divisor of two non-negative integers* m, n*. Note, that the homology groups in degree* 0 *will play no role here.*

Proof: The subspace of the Stokes directions $\Sigma_0 \subset S^1 \times S^1$ of α are homotopy equivalent to a torus knot of type (m_1, m_2), which we denote by $\mathscr{K}$. Let $\mathscr{R} \subset \Delta$ be the union of the radial sheets with directions in $\mathscr{K}$, a radial sheet being the product of the two radii in each direction, then the homology group to be computed is isomorphic to $H_*(\mathscr{R}, \partial\Delta \cup D)$. Now, consider a decomposition $\Delta = \bigcup S(\nu_i) \times S(\mu_j)$ in bisectors as above, where we assume that the intersection of $\mathfrak{S}_0$ with any bisector has at most one connected component. Now observe that for two radii λ and κ and a sector $S(\nu)$ one has homotopy equivalences

$$(\lambda \times \kappa)\big/_{\partial\Delta \cup D} \simeq S^2 \quad \text{and} \quad (l \times S(\nu))\big/_{\partial\Delta \cup D} \simeq S^2 \,. \tag{4.5}$$

Starting with this observation, we can make our way through the various Mayer-Vietoris sequences induced from the decomposition of Δ in direct analogy to the sequences (4.2) – (4.4). One easily deduces that the homology groups to be computed vanish in degree 1 and greater than or equal to 4. As for the remaining degrees $2,3$, one sees that the one-dimensional contributions coming from $H_2(S^2) = \mathbb{C}$ via (4.5) distinguish each other as long as they come from points on the torus knot $\mathscr{K}$ (i.e. the endpoint of $\lambda \times \kappa$ or any direction of some $S(\nu) \times \kappa$ respectively) that belong to the same connected component of the torus knot. A careful book-keeping thus gives the desired result for H_2 and H_3, the integers (m_1, m_2) being the number of connected components of the torus knot. (Another way to look at it, is to cut the torus along the knot $\mathscr{K}$ and to use a Mayer-Vietoris argument to see that the dimension of the homology groups in question are given as the dimension of the analogous homology groups in the special case $m_1 = 1 = m_2$, which is easily seen to be 1, times the number of connected components of the torus knot $\mathscr{K}$, namely (m_1, m_2)).

□

4.1.2 Local de Rham cohomology

We are now going to compute the local de Rham cohomology $H^*_{dR}(L_{\mathrm{ess}}/L_1 + L_2)$ at the crossing point, where the rank one connection L has the formal model $e^\alpha \otimes R$ with $\alpha = x_1^{-m_1} x_2^{-m_2} \cdot u_\alpha(x_1, x_2)$ with $u_\alpha(0,0) \neq 0$.

Since the sheaf $L_{\mathrm{ess}}/L_1 + L_2$ has support at the origin, it is sufficient to consider the stalk at 0. We will keep the same notation L_{ess}, L_1 and L_2 but in the following think of these as the stalks. The main result of this section is the following

Theorem 4.5 *In the situation above, we have*

$$\dim H^*_{dR}(L_{\mathrm{ess}}\big/L_1 + L_2) = \begin{cases} 0 & \text{for} \quad * \neq 0,1 \\ (m_1, m_2) & \text{for} \quad * = 0,1 \end{cases}$$

Proof: The regular singular part does not give any contribution to $L_{\mathrm{ess}}\big/L_1 + L_2$, so that we can omit it. Since $u_\alpha(0) \neq 0$, we can locally transform the connection by multiplication with u_α^{-1} and obtain a new connection L' whose induced complex $L'_{\mathrm{ess}}\big/L'_1 + L'_2$ is quasi-isomorphic to the analogous complex for e^α. Therefore we can also omit the factor $u_\alpha(x)$ and take $\alpha = x_1^{-m_1} x_2^{-m_2}$. We then have to consider the following complex

$$L_{\mathrm{ess}}\big/L_1 + L_2 \xrightarrow{\varphi} L_{\mathrm{ess}}\big/L_1 + L_2 \oplus L_{\mathrm{ess}}\big/L_1 + L_2 \xrightarrow{\psi} L_{\mathrm{ess}}\big/L_1 + L_2$$

with

$$\varphi(u) := \Big(\frac{\partial u}{\partial x_1} - m_1 x_1^{-m_1-1} x_2^{-m_2} u\,,\ \frac{\partial u}{\partial x_2} - m_2 x_1^{-m_1} x_2^{-m_2-1} u\Big) \quad \text{and}$$
$$\psi(\omega_1, \omega_2) := \frac{\partial \omega_1}{\partial x_1} - \frac{\partial \omega_2}{\partial x_2} + m_2 x_1^{-m_1} x_2^{-m_2-1} \omega_1 - m_1 x_1^{-m_1-1} x_2^{-m_2} \omega_2 \,.$$

1. Step: If we let

$$K_{MN} := \{ \sum_{k,l \in \mathbb{Z}} f_{kl} x_1^k x_2^l \in L_1 + L_2 \mid f_{kl} = 0 \text{ for } k < M \text{ and } l < N \} \subset L_1 + L_2 \,,$$

it follows that $K_{1,1} \xrightarrow{\varphi} K_{-m_1,-m_2+1} \oplus K_{-m_1+1,-m_2} \xrightarrow{\psi} K_{-2m_1,-2m_2}$. Consider the following diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L_1+L_2/K_{1,1} & \to & L_{\mathrm{ess}}/K_{1,1} & \to & L_{\mathrm{ess}}/L_1+L_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \begin{matrix} L_1+L_2/K_{-m_1,-m_2+1} \\ \oplus \\ L_1+L_2/K_{-m_1+1,-m_2} \end{matrix} & \to & \begin{matrix} L_{\mathrm{ess}}/K_{-m_1,-m_2+1} \\ \oplus \\ L_{\mathrm{ess}}/K_{-m_1+1,-m_2} \end{matrix} & \to & \begin{matrix} L_{\mathrm{ess}}/L_1+L_2 \\ \oplus \\ L_{\mathrm{ess}}/L_1+L_2 \end{matrix} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L_1+L_2/K_{-2m_1,-2m_2} & \to & L_{\mathrm{ess}}/K_{-2m_1,-2m_2} & \to & L_{\mathrm{ess}}/L_1+L_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{4.6}$$

where the vertical arrows are induced by the connection ∇. We denote the columns of this diagram as $\mathscr{K}'$, $\mathscr{K}$ and $\overline{\mathscr{K}}$, so that the diagram reads as the short exact sequence $0 \to \mathscr{K}' \to \mathscr{K} \to \overline{\mathscr{K}} \to 0$.

Lemma 4.6 *The first column $\mathscr{K}'$ of* (4.6) *is acyclic.*

Proof: We denote the maps by $\overline{\varphi}$ and $\overline{\psi}$ respectively. To show that $\ker(\overline{\varphi}) = 0$, consider an element $u = \sum f_{kl} x_1^k x_2^l$ in the kernel. This is equivalent to

$$k f_{k,l} - m_1 f_{k+m_1,l+m_2} = 0 = l f_{k,l} - m_2 f_{k+m_1,l+m_2} \quad \text{for } k \leq -m_1 \text{ and } l \leq -m_2. \tag{4.7}$$

Now, $u \in L_1 + L_2$, so that $f_{kl} = 0$ if both k and l are sufficiently negative and successive application of (4.7) gives $u \in K_{1,1}$.

Next, consider $(\overline{g}, \overline{h}) \in \ker(\overline{\psi})$, i.e. $\frac{\partial h}{\partial x_1} - \frac{\partial g}{\partial x_2} + m_2 x_2^{-1} \alpha g - m_1 x_1^{-1} \alpha h \in K_{-2m_1,-2m_2}$. By definition, there are numbers N, M with $g, h \in K_{-M,-N}$. We want to solve $\overline{\varphi}(\overline{f}) = (\overline{g}, \overline{h})$. To this end, let $f_{kl} := 0$ for all k, l with $k < -M + m_1 + 1$ and $l < -N + m_2 + 1$, but $(k,l) \neq (-M+m_1, -N+m_2)$. The desired equation for f induces the necessary equality of

$$f_{-M+m_1,-N+m_2} = -\frac{1}{m_1} g_{-M-1,-N} = -\frac{1}{m_2} h_{-M,-N-1}\ .$$

Now, $\overline{\psi}(\overline{g}, \overline{h}) = 0$ reads as

$$(k+1) h_{k+1,l} - (l+1) g_{k,l+1} + m_2 g_{k+m_1,l+m_2+1} - m_1 h_{k+m_1+1,l+m_2} = 0 \tag{4.8}$$

for $k < -2m_1$ and $l < -2m_2$. We can choose representatives with $g_{kl} = 0$ for $k \geq -m_1$ or $l \geq -m_2 + 1$, and $h_{kl} = 0$ for $k \geq -m_1 + 1$ or $l \geq -m_2$. We can assume that $M > m_1 + 1$ and $N > m_2 + 1$. Using (4.8) for $k = -M - m_1$, $l = -N - m_2$ gives the desired equality $m_2 g_{-M-1,-N} = m_1 h_{-M,-N-1}$ and therefore the well-definedness of $f_{-M+m_1,-N+m_2}$.

In the same manner, we can solve the equation $k f_{kl} - m_1 f_{k+m_1,l+m_2} = g_{k-1,l}$ successively and get the coefficients for the remaining indices in $A := \{(k,l) | k \leq 1, l \leq 1\} \setminus \{(k,l) | k \leq -M + m_1, l \leq -N + m_2\}$. We obtain

$$f_{kl} = -\frac{1}{m_1} g_{k-m_1-1,l-m_2} - \frac{k-m_1}{m_1^2} g_{k-2m_1-1,l-m_2} - \ldots - \frac{(k-m_1)\cdots(k-rm_1)}{m_1^{r+1}} g_{k-rm_1,l-rm_2}$$

with $r := \max\{\frac{M}{m_1}, \frac{N}{m_2}\}$. From this it follows that $f := \sum_{(k,l)\in A} f_{kl} x_1^k x_2^l \in L_1 + L_2$ solves $\mathrm{pr}_1(\overline{\varphi}(f)) = \overline{g}$ where pr_1 is the projection to the first direct summand. The same computations give a solution $\widetilde{f} \in L_1 + L_2$ for the second direct summand, i.e. $\mathrm{pr}_2(\overline{\varphi}(\widetilde{f})) = \overline{h}$. Using $\overline{\psi}(\overline{g}, \overline{h}) = 0$ as above, we easily see that the above calculations give the same coefficients $f_{kl} = \widetilde{f}_{kl}$, so that we have found an $f \in L_1 + L_2$ with $\widetilde{\varphi}(f) = (\overline{g}, \overline{h})$.

To see the surjectivity of $\overline{\psi}$, let $u \in {L_1+L_2}/{K_{-2m_1,-2m_2}}$ be given, represented by $u \in K_{-M,-N}$ for suitable M,N. Let $h := 0$. We then have to find g whose coefficients satisfy

$$-lg_{kl} + m_2 g_{k+m_1,l+m_2} = u_{kl} \text{ for } k < -2m_1 \text{ and } l < -2m_2+1 \ . \tag{4.9}$$

Especially, $lg_{kl} = m_2 g_{k+m_1,l+m_2}$ for $k < -M$ and $l < -N-1$, so that we can take $g_{kl} := 0$ for $k < -M+m_1$ and $l < -N+m_2+1$. Successively solving (4.9), it follows that

$$g_{kl} = \frac{1}{m_2} u_{k-m_1,l-m_2+1} + \frac{l-m_2}{m_2^2} u_{k-2m_2,l-2m_2+1} + \ldots + \frac{(l-m_2)\cdots(l-am_2)}{m_2^{a+1}} u_{k-am_1,l-am_2+1}$$

for $a \geq \max\{\frac{M}{m_1}, \frac{N}{m_2}\}$. One obtains a solution $g \in {K_{-M+m_1,-N+m_2+1}}/{K_{-m_1,-m_2+1}}$ and hence the desired surjectivity of $\overline{\psi}$.

□

2. Step: In order to calculate the cohomology of ${L_{\mathrm{ess}}}/{L_1+L_2}$, we thus have to do so for the middle column $\mathscr{K}$ of (4.6). After transformation $x_i \mapsto x_i^{-1}$ for $i = 1,2$, we have to consider the following complex

$$0 \to \mathscr{H} \xrightarrow{D} x_1^{m_1+1} x_2^{m_2} \mathscr{H} \oplus x_1^{m_1} x_2^{m_2+1} \mathscr{H} \xrightarrow{E} x_1^{2m_1+1} x_2^{2m_2+1} \mathscr{H} \to 0 \ , \tag{4.10}$$

where $\mathscr{H}$ denotes the ring of power-series in two variables which converge in the entire complex plane $\mathbb{C}^2$. The maps are given as follows. Define $P_{MN} := \{\sum_{k,l\geq 0} f_{kl} x_1^k x_2^l \in \mathscr{H} \mid f_{kl} = 0 \text{ if } k > M \text{ and } l > N\}$. Then $x_1^{M+1} x_2^{N+1} \mathscr{H} \cong {\mathscr{H}}/{P_{MN}}$ and we put

$$\begin{aligned} D(u) &:= -(x_1^2 \tfrac{\partial u}{\partial x_1} + m_1 x_1^{m_1+1} x_2^{m_2} u, x_2^2 \tfrac{\partial u}{\partial x_2} + m_2 x_1^{m_1} x_2^{m_2+1} u) \bmod P_{m_1,m_2-1} \oplus P_{m_1-1,m_2} \\ E(\omega_1,\omega_2) &:= -x_1^2 \tfrac{\partial \omega_2}{\partial x_1} + x_2^2 \tfrac{\partial \omega_1}{\partial x_2} + m_2 x_2 \alpha^{-1} \omega_1 - m_1 x_1 \alpha^{-1} \omega_2 \bmod P_{2m_1,2m_2} \ . \end{aligned}$$

Claim 1: $\dim \ker D = (m_1, m_2)$.
Consider an element $u = \sum_{k,l\geq 0} u_{kl} x_1^k x_2^l \in \mathscr{H}$. Then $u \in \ker D$ translates into the following condition on the power series coefficients:

$$k u_{kl} + m_1 u_{k-m_1,l-m_2} = 0 \qquad \text{and} \qquad l u_{kl} + m_2 u_{k-m_1,l-m_2} = 0 \tag{4.11}$$

for all $k \geq m_1$ and $l \geq m_2$. It follows that for u to be in the kernel of D, it is necessary that its non-vanishing coefficients u_{kl} lie on the line $\mathscr{L} := \{(k,l) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid l m_1 = k m_2\}$. Moreover, choosing values for the coefficients u_{kl} on this line in the region $0 \leq k < m_1, 0 \leq l < m_2$ gives an element $u \in \ker D$ by means of (4.11) (the convergence of the solution so obtained is easily seen). Claim 1 follows as the line $\mathscr{L}$ intersects the integer lattice $\mathbb{Z} \times \mathbb{Z}$ in exactly (m_1, m_2) points in this region.
Claim 2: $\dim \operatorname{coker} E = 0$.
Let $\eta \in x_1 x_2 \alpha^{-2} \mathscr{H}$ be given. We have to find $(\omega_1, \omega_2) \in x_1 \alpha^{-1} \mathscr{H} \oplus x_2 \alpha^{-1} \mathscr{H}$ such that

$$-x_1^2 \frac{\partial \omega_2}{\partial x_1} + x_2^2 \frac{\partial \omega_1}{\partial x_2} + m_2 x_2 \alpha^{-1} \omega_1 - m_1 x_1 \alpha^{-1} \omega_2 = \eta \ . \tag{4.12}$$

We let ω_1 be arbitrary and write $\omega_2 = \exp(-x_1^{m_1} x_2^{m_2}) \cdot \rho$ with an element $\rho \in \mathscr{H}$ yet to be determined. Then (ω_1, ω_2) solves (4.12) if and only if

$$\frac{\partial \rho}{\partial x_1} = \exp(x_1^{m_1} x_2^{m_2}) \cdot x_1^{-2} \left(x_2^2 \frac{\partial \omega_1}{\partial x_2} + m_2 x_2 \alpha^{-1} \omega_1 - \eta \right) \ .$$

The right hand side defines an element in $\mathscr{H}$ which can be integrated in x_1 direction, so that such a ρ exists, proving Claim 2.

3. Step: We prove that the complex given in (4.10) has vanishing Euler characteristic. To this end, consider the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & 0 & \to & \mathscr{H} & \xrightarrow{id} & \mathscr{H} & \to & 0 \\ & & \downarrow & & \downarrow D & & \downarrow D & & \\ 0 & \to & P_{m_1,m_2-1} \oplus P_{m_1-1,m_2} & \to & \mathscr{H} \oplus \mathscr{H} & \to & x_1\alpha^{-1}\mathscr{H} \oplus x_2\alpha^{-1}\mathscr{H} & \to & 0 \\ & & \downarrow & & \downarrow E & & \downarrow E & & \\ 0 & \to & P_{2m_1,2m_2} & \to & \mathscr{H} & \to & x_1x_2\alpha^{-1}\mathscr{H} & \to & 0 \end{array} . \tag{4.13}$$

To complete this step, we prove the following

Lemma 4.7 *The operator*

$$E : P_{m_1,m_2-1} \oplus P_{m_1-1,m_2} \longrightarrow P_{2m_1,2m_2}$$

is Fredholm with index -1*. More precisely, one has* $\dim\ker(E) = d+2$ *and* $\dim\operatorname{coker}(E) = d+3$*, where* $d := (m_1, m_2)$ *denotes the greatest common divisor of* m_1 *and* m_2*.*

Proof: We first calculate the dimension of the cokernel. Let $u := \sum_{k,l} u_{kl} x_1^k x_2^l \in P_{2m_1,2m_2}$. We have to solve $E(g,h) = u$. In terms of the coefficients in Laurent series expansions this reads as

$$\psi_{kl} := -(k-1)h_{k-1,l} + (l-1)g_{k,l-1} + m_2 g_{k-m_1,l-m_2-1} - m_1 h_{k-m_1-1,l-m_2} = 0 \tag{4.14}$$

for all $(k,l) \in A$, where $A := \{(k,l) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid k \le 2m_1 \text{ or } l \le 2m_2\}$. Observe that in most cases either the first two or the last two summands vanish, as $g_{kl} = h_{kl} = 0$ for indices $k,l \notin A$. There are several different cases of pairs $(k,l) \in A$ to be considered:

i) $(k < m_1$ or $l < m_2)$ and $(k > 0$ and $l > 0)$: then (4.14) for the pair (k,l) and the pair $(k+m_1, l+m_2)$ gives

$$(k-1)h_{k-1,l} + (l-1)g_{k,l-1} = u_{kl} \text{ and } -m_1 h_{k-1,l} + m_2 g_{k,l-1} = u_{k+m_1,l+m_2} .$$

ii) $k = 0$ and $l > 0$: Then (4.14) gives $\psi_{0,l} = (l-1)g_{0,l-1} = u_{0,l}$, $\psi_{m_1,l+m_2} =$

$$= -(m_1-1)h_{m_1-1,m_2+l} + (m_2+l-1)g_{m_1,m_2+l-1} + m_2 g_{0,l-1} = u_{m_1,m_2+l}$$

and $\psi_{2m_1,2m_2+l} = m_2 g_{m_1,m_2+l-1} - m_1 h_{m_1-1,m_2+l} = u_{2m_1,2m_2+l}$.

iii) $k > 0$ and $l = 0$: in analogy to the case ii), this gives $\psi_{k,0} = -(k-1)h_{k-1,0} = u_{k,0}$, $\psi_{m_1+k,m_2} =$

$$= -(m_1+k-1)h_{m_1+k-1,m_2} + (m_2-1)g_{m_1+k,m_2-1} - m_1 h_{k-1,0} = u_{m_1+k,m_2}$$

and $\psi_{2m_1+k,2m_2} = m_2 g_{m_1+k,m_2-1} - m_1 h_{m_1+k-1,m_2} = u_{2m_1+k,2m_2+l}$.

iv) $(k,l) = (0,0)$: Gives the equation $u_{0,0} = 0$.

v) $(k,l) \in \{(m_1,m_2),(2m_1,2m_2)\}$: Then (4.14) gives

$$\begin{aligned} \psi_{m_1,m_2} &= -(m_1-1)h_{m_1-1,m_2} + (m_2-1)g_{m_1,m_2-1} = u_{m_1,m_2} \text{ and} \\ \psi_{2m_1,2m_2} &= -m_1 h_{m_1-1,m_2} + m_2 g_{m_1,m_2-1} = u_{2m_1,2m_2} . \end{aligned}$$

By some simple matrix calculations one sees that case i) gives a d-dimensional contribution to $\operatorname{coker}(E)$ if $m_1 \neq m_2$ and a $(d-1)$-dimensional contribution if $m_1 = m_2$; cases ii) and iii) each give a one-dimensional contribution. Case iv) contributes with a one-dimensional subspace, whereas case v) adds another dimension if $m_1 \neq m_2$ and no contribution for $m_1 = m_2$. Summing everything up, gives the result

$$\dim\operatorname{coker}(E) = d+3$$

as claimed above. We remark that the convergence of the solutions we obtain by the combinatorics of the Laurent coefficients above is an easy exercise.

To compute the dimension of $\ker(E)$ we proceed in a similar manner, looking at the cases i) – v) above with $u=0$. Again, one easily sees that case i) give a d-dimensional subspace if $m_1 \neq m_2$ and a $(d-1)$-dimensional one for $m_1 = m_2$, cases ii) and iii) each add one dimension, case iv) gives no contribution to $\ker(E)$ and case v) contributes with a one-dimensional subspace for $m_1 = m_2$ and gives no solution for $m_1 \neq m_2$. Hence, we have

$$\dim \ker(E) = d+2 \ ,$$

from which the lemma follows.

□

We are left with the task to calculate the index of the complex given by the middle column of (4.13). To this end, again consider the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{H} & \xrightarrow{id} & \mathscr{H} & \longrightarrow & 0 & \to & 0 \\ & & \downarrow D' & & \downarrow D & & \downarrow & & \\ 0 & \to & \mathscr{H}\oplus\mathscr{H} & \xrightarrow{-x_1^2,-x_2^2} & \mathscr{H}\oplus\mathscr{H} & \longrightarrow & \mathscr{H}/x_1^2\mathscr{H} \oplus \mathscr{H}/x_2^2\mathscr{H} & \to & 0 \\ & & \downarrow E' & & \downarrow E & & \downarrow A & & \\ 0 & \to & \mathscr{H} & \xrightarrow{x_1^2x_2^2} & \mathscr{H} & \longrightarrow & \mathscr{H}/x_1^2x_2^2\mathscr{H} & \to & 0 \end{array} \tag{4.15}$$

where $A(\omega_1,\omega_2) := -x_1^2\frac{\partial\omega_2}{\partial x_1} + x_2^2\frac{\partial\omega_1}{\partial x_2} \bmod x_1^1x_2^2\mathscr{H}$. Obviously, $\ker(A)$ is the $\mathbb{C}$-span of $(0,1),(0,x_2),(1,0),(x_1,0)$ and $\operatorname{coker}(A) = \operatorname{span}_{\mathbb{C}}\{1,x_1,x_2,x_1x_2\}$. It follows that $\dim\ker(A) = \dim\operatorname{coker}(A) = 4$ and the Euler characteristic of the first and the second column of (4.15) coincide. Thus vanishing of the Euler-characteristic of (4.10) follows from

Lemma 4.8 *The Euler-characteristic of* $0 \to \mathscr{H} \xrightarrow{D'} \mathscr{H}\oplus\mathscr{H} \xrightarrow{E'} \mathscr{H} \to 0$ *equals* 1.

Proof: Decompose the operators as follows:

$$D'u = \big((\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2}) + (m_1x_1^{m_1-1}x_2^{m_2}, m_2x_1^{m_1}x_2^{m_2-1})\big)u \qquad \text{and} \tag{4.16}$$

$$E'(\omega_1,\omega_2) = (\frac{\partial\omega_2}{\partial x_1} - \frac{\partial\omega_1}{\partial x_2}) + (m_1x_1^{m_1-1}x_2^{m_2}\omega_2 - m_2x_1^{m_1}x_2^{m_2-1}\omega_1)\ . \tag{4.17}$$

For any $R>0$, consider the space $B_R^{(k_1,k_2)}$ of all holomorphic functions on $D_R\times D_R$ that are of type C^{k_i} with respect to the variable x_i on $\overline{D}_R\times\overline{D}_R$, where $D_R\subset\mathbb{C}$ denotes the open disc around 0 with radius R and $\overline{D}_R$ its closure. The complex to be considered induces a complex $B_R^{(1,1)} \xrightarrow{D'} B_R^{(0,1)}\oplus B_R^{(1,0)} \xrightarrow{E'} B_R^{(0,0)}$. The decompositions (4.16) and (4.17) are decompositions of complexes in the sense that taking either the first terms in (4.16) and (4.17) or the second terms again give a complex. The second terms in (4.16) and (4.17) consist of compact operators and so the Euler-characteristic remains unchanged if one omits these terms by Vasilescu's generalization of the well-known compact perturbation theorem for Fredholm operators (cp. [21] and [22]). The Euler-characteristic of the unperturbed complex is easily seen to be 1, the only contribution coming from the constant functions being the kernel of $(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2})$ and the lemma follows by taking the limit $R\to\infty$. Compare with [16], Theorem 1.4 for the analogous arguments in the one-dimensional case.

□

This lemma completes the proof of Theorem 4.5.

□

4.1.3 Non-degeneracy from the left

Finally, we will prove non-degeneracy of the local pairing at a crossing-point from the left. Note that perfectness from the left is better accessible than perfectness from the right, the reason lying in the difficulty of constructing sufficiently good 'test forms' $\omega \in L_{\text{ess}}$ with $\nabla\omega \in L_1 + L_2$, whereas 'test cycles' $c \otimes \varepsilon$ are easier to handle. The arguments used in the proof are similar to the one-dimensional case of [3].

Theorem 4.9 *The pairing*

$$H^p_{dR}(L_{\text{ess}}/L_1 + L_2) \times H^{rd}_{p+2}(\Delta, \partial\Delta; D) \to \mathbb{C} \ , \quad p = 0, 1 \ ,$$

is non-degenerate from the left, i.e. if $< [\omega], [c \otimes \varepsilon] >= 0$ *for all* $[c \otimes \varepsilon]$*, then* $[\omega] = 0$.

Proof: We start with the case $p = 0$. Let $[\omega] \in H^0_{dR}(L_{\text{ess}}/L_1 + L_2)$ be given. Let ε be a basis of $\mathscr{L}$ and $\varepsilon^\vee$ denote the dual basis of $\mathscr{L}^\vee$. Then ω can be written as $\omega = a \cdot \varepsilon$ with an analytic function a and

$$\nabla\omega = \varepsilon \otimes da = e \otimes \eta_1 + e \otimes \eta_2 \in L^1_1 + L^1_2 \ ,$$

where e denotes a meromorphic local basis of L. We have to show that $a\varepsilon \in L^0_1 + L^0_2$. Let c be the radial sheet $c = [0, p_1] \times [0, p_2] \subset \Delta \cong D^2 \times D^2$, with $p = (p_1, p_2) \in \partial\Delta$.

First, consider the case that ε and $\varepsilon^\vee$ are sections of $L^0_1 + L^0_2$ in the notation used before. Since

$$da =< \nabla\omega, \varepsilon^\vee >=< e, \varepsilon^\vee > \cdot \eta \ ,$$

with $\eta = \eta_1 + \eta_2$, it follows that $da \in \mathscr{O}^0_1 + \mathscr{O}^0_2$, hence also a and therefore $a\varepsilon \in L^0_1 + L^0_2$, where as before $\mathscr{O}^p_\nu$ denotes the p-forms meromorphic along $D_{\neq\nu}$ and arbitrary along D_ν.

Next, assume that $\varepsilon^\vee$ is rapidly decaying along c. Then by assumption

$$a(p)\varepsilon^\vee(p) = (< \omega, \varepsilon^\vee > \cdot \varepsilon^\vee)(p) = \Big(\int_c < \nabla\eta_2, \varepsilon^\vee > + \int_{\partial c_1 - D} < \eta_1, \varepsilon^\vee > - \int_{\partial c_2 - D} < \eta_2, \varepsilon^\vee > \Big) \cdot \varepsilon^\vee(p) \ , \quad (4.18)$$

with the decompositions $c = c_1 + c_2$ and $\nabla\omega = \eta_1 + \eta_2 \in L^1_1 + L^1_2$ as in the definition of the pairing, see Lemma 3.3. Now, $\nabla\eta_2 = -\nabla\eta_1 = e \otimes \rho$ with a meromorphic two-form ρ. With these notations, one has

$$\int_c < \nabla\eta_2, \varepsilon^\vee >= \int_c < e, \varepsilon^\vee > \rho \ .$$

In local coordinates, the section $\varepsilon^\vee$ is asymptotically equal to $\exp(-kx_1^{-m_1} x_2^{-m_2})$ times a meromorphic section of $\mathscr{L}^\vee$. Therefore, in order to understand the first term in (4.18) we have to study the behavior of

$$e^{kp_1^{-m_1} p_2^{-m_2}} \cdot \int_{[0,p_1]\times[0,p_2]} e^{-kx_1^{-m_1} x_2^{-m_2}} \cdot x_1^{-r_1} x_2^{-r_2} \, dx_1 \, dx_2$$

for $(p_1, p_2) \to (0, 0)$ and some $r_1, r_2 \in \mathbb{Z}$. Similar to [3] in one variable, substituting variables $y_i = x_i^{-1}$, $q_i = p_i^{-1}$ and $u_i = y_i - q_i$, the latter integral reads as

$$\int_0^\infty \int_0^\infty e^{k(-u_1^{m_1} u_2^{m_2} - f(u,q))} \cdot (u_1 + q_1)^{r_1 - 2} (u_2 + q_2)^{r_2 - 2} \, du_1 \, du_2$$

with a polynomial f with positive coefficients. This integral has at worst moderate growth as $(q_1, q_2) \to (\infty, \infty)$, so that its contribution vanishes modulo $L_1 + L_2$.

In a similar manner, the second summand in (4.18)

$$\varepsilon^\vee(p)\cdot\int_{\partial c_1 - D} <\eta_1,\varepsilon^\vee> = \Big(\int_{\{p_1\}\times[0,p_2]} <e,\varepsilon^\vee>\eta_1\Big)\cdot\varepsilon^\vee(p)$$

leads us to study the integral

$$e^{kp_1^{-m_1}p_2^{-m_2}}\cdot\int_0^{p_2} e^{-kp_1^{-m_1}p_2^{-m_2}}\cdot x_2^{-r_2}\cdot\widetilde{\eta}(p_1)\,dx_2$$

for $p_1\to 0$ with arbitrary $\widetilde{\eta}$. As before, this has at most moderate growth for fixed p_2 and $p_1\to 0$ and thus lies in L_2. The same argument shows that the third term in (4.18) vanishes modulo L_1. It follows, that $a\varepsilon$ lies in $L_1^0+L_2^0$ provided that $\varepsilon^\vee$ is rapidly decaying along c.

It remains to consider the case, where $\varepsilon^\vee$ is rapidly increasing along c, i.e. the radial sheet c does not lie in any Stokes bisector belonging to $\varepsilon^\vee$. Then ε is rapidly decaying along c. Again, by

$$\nabla\omega=\varepsilon\otimes da\in L_1^1+L_2^1$$

it follows that $\varepsilon\otimes da\in L_1^1+L_2^1$. If we write $\varepsilon=\psi\cdot e$ with the analytic function ψ, this reads as $\psi da\in\mathscr{O}_1^1+\mathscr{O}_2^1$, i.e.

$$\psi\frac{\partial a}{\partial x_1}\in\mathscr{O}_1^0+\mathscr{O}_2^0 \quad\text{and}\quad \psi\frac{\partial a}{\partial x_2}\in\mathscr{O}_1^0+\mathscr{O}_2^0\ . \tag{4.19}$$

Now, ψ has rapid decay along c by assumption. In order to prove that (4.19) induces $\psi\cdot a\in\mathscr{O}_1^0+\mathscr{O}_2^0$, we have to show that for any two functions g,a in the variables (x_1,x_2) such that g has rapid decay and

$$g\frac{\partial a}{\partial x_i}\in\mathscr{O}_1^0+\mathscr{O}_2^0 \text{ for } i=1,2\ ,$$

it follows that $ga\in\mathscr{O}_1^0+\mathscr{O}_2^0$. We apply the mean value theorem at a position (x_1,x_2) after choosing a fixed point $q=(q_1,q_2)$ with $0<|x_i|<|q_i|$ in order to find a point $r=(r_1,r_2)$ in between such that

$$g(x)a(x)=g(x)\cdot\Big(a(q)+\big(\frac{\partial a}{\partial x_1}(r),\frac{\partial a}{\partial x_2}(r)\big)\cdot\begin{pmatrix}h_1\\h_2\end{pmatrix}\Big)\ ,$$

where $h=(h_1,h_2):=q-x$. By (4.19), we can find functions $\varphi_\nu\in\mathscr{O}_\nu^0$ such that $g(x)a(x)=g(x)a(q)+\frac{g(x)}{g(r)}\cdot(\varphi_1(r)+\varphi_2(r))$. Now, $g(x)$ is rapidly decaying as $(x_1,x_2)\to(0,0)$ along c, so that there is no growth contribution coming from the first term. As for the second term, the function φ_1 has moderate growth in x_2-direction and thus for fixed x_1 and r_1,

$$\Big|\frac{g(x)}{g(r)}\varphi_1(r)\Big|\leq C_1\cdot|r_2|^{-N}\leq C_1\cdot|x_2|^{-N}$$

proving moderate growth of the second term in x_2-direction also. The same argument applies for the third term, proving $ga\in\mathscr{O}_1^0+\mathscr{O}_2^0$.

Next, we consider the pairing $H^1_{dR}(L_{\mathrm{ess}}/L_1+L_2)\times H_2^{rd}(\Delta,\partial\Delta;D)\to\mathbb{C}$, i.e. the case $p=1$. We have the following Mayer-Vietoris sequence for the de Rham cohomology of $\mathscr{M}:=L_{\mathrm{ess}}/L_1+L_2$ as well as the dual of the corresponding sequence in rapid decay homology (see (4.2))

$$\begin{array}{ccccccccc}
0\leftarrow & H^1_{dR}(\pi^{-1}(0);\mathscr{M}) & \leftarrow & \bigoplus_{j=1}^m H^0_{dR}(T_{j,j+1};\mathscr{M}) & \leftarrow & \bigoplus_{j=1}^m H^0_{dR}(T_j;\mathscr{M}) & \leftarrow & H^0_{dR}(\pi^{-1}(0);\mathscr{M}) & \leftarrow 0\\
 & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \downarrow & \\
0\leftarrow & H_3^{rd}(\Delta,\partial)^\vee & \leftarrow & \bigoplus_{j=1}^m H_2^{rd}(Z_{j,j+1},\partial)^\vee & \leftarrow & \bigoplus_{j=1}^m H_2^{rd}(Z_j,\partial)^\vee & \leftarrow & H_2^{rd}(\Delta,\partial)^\vee & \leftarrow 0
\end{array}$$

according to the decomposition of the torus $\pi^{-1}(0) = \bigcup_{j=1}^m T_j$ with $T_j := S^1 \times \mu_j$ where S^1 decomposes into small enough intervals as $S^1 = \bigcup_{j=1}^m \mu_j$. The radial sheets with directions in T_j are denoted by Z_j as before, i.e.

$$Z_j := \bigcup_{(\vartheta_1,\vartheta_2)\in T_j} \mathbb{R}_0^+ e^{i\vartheta_1} \times \mathbb{R}_0^+ e^{i\vartheta_2} \cap \Delta\ .$$

Again, $T_{ij} := T_i \cap T_j$ and $Z_{ij} := Z_i \cap Z_j$. The vertical arrows are induced from the local pairing. The arguments above show that β and γ are injective. Additionally, our knowledge about the dimensions of the (co-)homology groups involved (Theorem 4.4 and Theorem 4.5) then induce that γ is surjective, hence α is injective by the 5-lemma. This proves perfectness from the left for $p = 1$.

□

4.2 Local pairing at a crossing-point involving one direction

To complete the investigation of the situation at a crossing point of D, we have to consider the pairing involving the contributions from the one-dimensional local strata of D. Let Δ denote a small bi-disc around the crossing point $(0,0) \in D$, where we chose local coordinates such that $D = \{x_1x_2 = 0\} = D_1 \cup D_2$. We prove the following

Theorem 4.10 *The local pairing*

$$H^{p-1}_{dR}(\Delta; L_{\mathrm{ess}}/L_2) \times H_p^{rd}(\Delta \smallsetminus D_2, \Delta \smallsetminus D; D) \to \mathbb{C} \tag{4.20}$$

for the local contribution of the connection along D_1 at the crossing point is perfect.

Proof: We omit Δ from the notation of de Rham cohomology. Recall that L has the elementary model $e^\alpha \otimes R$, where $\alpha(x) = x_1^{-m_\alpha} \cdot u(x)$ with $u(0) \neq 0$.

We now prove the analogue of Theorem 4.4, namely

Proposition 4.11 *In the situation above, we have*

$$\dim H_*^{rd}(\Delta \smallsetminus D_2, \Delta \smallsetminus D; D) = \begin{cases} 0 & \text{for } * \geq 3 \\ m & \text{for } * = 1,2\ . \end{cases}$$

Proof: We have the **Stokes directions** in $\pi^{-1}(D_1^\times) \cong S^1 \times D_1^\times$:

$$\Sigma_1 := \{(\vartheta_1, x_2) \in \pi^{-1}(0,x_2) \mid -m_1\vartheta_1 + \arg(u(0,x_2)) \in (\frac{\pi}{2}, \frac{\pi}{2})\}\ ,$$

where $D_1^\times := D_1 \smallsetminus 0$, and the fibration by **Stokes sectors**, which reads in local coordinates as

$$\mathfrak{S}_1 := \bigcup_{(\vartheta_1,x_2)\in\Sigma_1} \mathbb{R}^+ e^{i\vartheta_1} \times \{x_2\}\ .$$

Note, that one may regard $\mathfrak{S}_1$ as a subset of the disc bundle associated to the normal bundle $DN(D_1^\times)$ of $D_1^\times$ in X, which we identify with $\Delta \smallsetminus D_2$. In each fiber over $(0,x_2) \in D_1^\times$ it is determined by the Stokes sectors of e^α, i.e. the sectors, where e^α has rapid decay for $x_1 \to 0$.

Decomposing S^1 into small enough intervals and thereby $\Delta \smallsetminus D_2$ into the corresponding fibration by sectors, we can apply Mayer-Vietoris sequences in the same way as we did in the proof of Proposition 4.2. With the same arguments as before, the statement analogous to Proposition 4.2 follows, namely a decomposition as ordinary singular homology groups:

$$H_*^{rd}(\Delta \smallsetminus D_2, \Delta \smallsetminus D; D) \cong H_*(V_1 \cap \mathfrak{S}_1, \partial V_1 \cup D)\ ,$$

where $\mathfrak{S}_1$ denotes the fibration by Stokes sectors corresponding to e^α and V_1 denotes a small tubular neighborhood of $D_1^\times := D_1 \smallsetminus 0$.

It remains to show, that

$$\dim H_*(V_1\cap\Sigma_1,\partial V_1\cup D;D)=\begin{cases}0 & \text{for } *\geq 3\\ m_\alpha & \text{for } *=1,2\end{cases} \tag{4.21}$$

The topological situation looks as follows. Consider the fiber $DN(D_1^\times)_x$ of the normal disc bundle of $D_1^\times$ at some point $x:=(0,x_2)\in D_1^\times$. We denote by $\sigma^\alpha\subset DN(D_1^\times)_x$ the Stokes sectors inside this disc, i.e. the intersection $\sigma^\alpha=\Sigma_1^\alpha\cap DN(D_1^\times)_x$. Now, we can retract the tubular neighborhood V_1 of $D_1^\times$ to the trivial bundle with base space being a circle S^1 around D_2 and with fiber $DN(D_1^\times)_x$, as well as the Stokes sectors σ_α in this fiber to the union of m_α radii, one for each Stokes sector. Now, we have to identify all points in the boundary ∂V_1 as well as those in D. In the fiber $DN(D_1^\times)_x$, this result in a wedge of m circles, and we finally are left with a trivial bundle over the circle S^1 with fiber the wedge $\bigvee_{m_\alpha}S^1$, where we still have to identify the 'zero'-section $S^1\times\{pt\}\subset D_1^\times$, $\{pt\}$ being the base point of the wedge of the m_α circles. Thus,

$$H_*(V_1^\times\cap\Sigma_1^\alpha,\partial V_1\cup D)\cong H_*\big((S^1\times\bigvee_{m_\alpha}S^1)\big/(S^1\times\{pt\})\big)\ .$$

Using the Künneth isomorphism and the effect of collapsing the one-cell $S^1\times\{pt\}$, one easily sees that the latter singular homology spaces obviously have the desired dimensions as in (4.21).

□

Next, we calculate the dimension of the corresponding de Rham cohomology groups from the pairing (4.20):

Proposition 4.12 *In the situation from above, we have*

$$\dim H^*_{dR}(L_{\text{ess}}/L_2)=\begin{cases}0 & \textit{for } *\neq 0,1\\ m & \textit{for } *=0,1\ .\end{cases}$$

Proof: As before, the regular singular part R gives no contribution to the dimension, so that we can assume it is trivial. Again, we write $\alpha=x_1^{-m}\cdot u_\alpha(x)$ with $u_\alpha(0)\neq 0$. After a local transformation by u_α^{-1}, we can assume $u_\alpha\equiv 1$. We thus have to consider the complex

$$L_{\text{ess}}/L_2\xrightarrow{\varphi}L_{\text{ess}}/L_2\oplus L_{\text{ess}}/L_2\xrightarrow{\psi}L_{\text{ess}}/L_2$$

with

$$\varphi(u):=\Big(\frac{\partial u}{\partial x_1}-mx_1^{-m-1}u,\frac{\partial u}{\partial x_2}\Big)\text{ and }\psi(\omega_1,\omega_2):=\frac{\partial\omega_2}{\partial x_1}-mx_1^{-m-1}\omega_2-\frac{\partial\omega_1}{\partial\omega_2}\ . \tag{4.22}$$

Now, let $\mathscr{O}$ denote the ring of holomorphic germs in one variable z at 0 and $\mathfrak{M}$ denote the meromorphic germs. Let $\rho:\mathscr{O}/\mathfrak{M}\to\mathscr{O}/\mathfrak{M}$ be the map induced form the one-variable connection $\nabla 1:=z^{-m-1}dz$, i.e. $\rho(f)=f'(z)-mz^{-m-1}f(z)$. Consider the diagram

$$\begin{array}{ccccccc} f & \in & \mathscr{O}/\mathfrak{M} & \xrightarrow{\rho} & \mathscr{O}/\mathfrak{M} & \ni & g\\ \downarrow & & \downarrow\beta & & \downarrow\gamma & & \downarrow\\ f(x_1) & \in & L_{\text{ess}}/L_2 & \xrightarrow{\varphi} & L_{\text{ess}}/L_2\oplus L_{\text{ess}}/L_2 & \ni & (g(x_1),0)\ .\end{array}$$

We claim, that β induces an isomorphism of the kernels of the horizontal arrows. To this end, consider $u=\sum_{i,j}u_{ij}x_1^ix_2^j$. Then $u\bmod L_2\in\ker\varphi$ if and only if

$$\text{i)}\quad\frac{\partial u}{\partial x_2}=:\eta\in L_2\quad\text{and}\quad\text{ii)}\quad\frac{\partial u}{\partial x_1}-mx_1^{-m-1}u\in L_2$$

are both meromorphic in x_1-direction. Assume that $\eta=\sum_{i\geq -N}\sum_j\eta_{ij}x_1^ix_2^j$, then by i) it follows that

$$u_{i,j+1}=\frac{1}{j+1}\cdot\eta_{i,j}\quad\text{for all } j\neq -1\ ,$$

especially $u_{i,j} = 0$ for all $j \neq 0$ and all $i < -N$. Let $\widetilde{u} := \sum_{i \geq -N} \sum_j u_{ij} x_1^i x_2^j \in L_2$, then

$$u(x_1,x_2) - \widetilde{u}(x_1,x_2) = \sum_{i<-N} u_{i,0} x_1^i =: w(x_1) \in \mathrm{im}(\beta) .$$

Now, $[u] = [\beta(w)] \in L_{\mathrm{ess}}/L_2$ and obviously $[u] \in \ker\varphi \Leftrightarrow [w] \in \ker\rho$. From the computation of the one-variable case ([16]), we deduce $\dim\ker\varphi = \dim\ker\rho = m$.

Next, we want to compute $H^2(L_{\mathrm{ess}}/L_2) = 0$, i.e. for given $\eta \in L_{\mathrm{ess}}$, we have to find $(\omega_1, \omega_2) \in L_{\mathrm{ess}} \oplus L_{\mathrm{ess}}$ and $\nu \in L_2$, such that

$$\frac{\partial \omega_2}{\partial x_1} - \frac{\partial \omega_1}{\partial x_2} - m x_1^{-m-1} \omega_2 = \eta + \nu .$$

We can do so by setting $\omega_1 := 0$. Writing $\omega_2 = e^{-x_1^{-m}} \cdot f$, we obtain a solution if and only if $\frac{\partial f}{\partial x_1} = e^{x_1^{-m}}(\eta + \nu)$, which is a question of vanishing of the residue in x_1-direction (i.e. the residue of the function in the variable x_1 for fixed x_2). A simple calculation shows that this can be achieved by

$$\nu := -\sum_j \eta_{-mk-1,j} \cdot x_2^j x_1^{-mk-1} \in L_2 .$$

It remains to prove that $\dim H^1_{dR}(L_{\mathrm{ess}}/L_2) = m$. Consider an element $[(\omega_1, \omega_2)] \in \ker(\psi)/\mathrm{im}(\varphi)$. We can write $\omega_2 = \sum_j a_j(x_1) \cdot x_2^j$ and then find some $u \in L_{\mathrm{ess}}$ such that $\frac{\partial u}{\partial x_2} = \sum_{j \neq -1} a_j(x_1) x_2^j$. Modulo $\mathrm{im}(\varphi)$ we may therefore assume that $\omega_2 = a(x_1) \cdot x_2^{-1}$. We write ω_1 in the form $\omega_1 = \sum_j b_j(x_1) x_2^j$. Since $[\omega_1, \omega_2] \in \ker(\psi)$, there exists an $\eta \in L_2$, written as $\eta = \sum_j \eta_j(x_1) \cdot x_2^j$, such that

$$\frac{\partial \omega_2}{\partial x_1} - m x_1^{-m-1} \omega_2 = \eta + \frac{\partial \omega_1}{\partial x_2} ,$$

which reads as $(a'(x_1) - m x_1^{-m-1} a(x_1)) x_2^{-1} = \sum_j (\eta_j(x_1) + (j+1) b_j(x_1)) x_2^j$. It follows that $b_j(x_1) = \frac{-1}{j+1} \eta_j(x_1) \in \mathfrak{M}$ for any $j \neq -1$, so that modulo L_2, we may again assume that $\omega_1 = b(x_1) x_2^{-1}$ with $b(x_1) := b_{-1}(x_1)$. Furthermore the function $a(x_1)$ must satisfy

$$a'(x_1) - m x_1^{-m-1} a(x_1) = \eta_{-1}(x_1) \in \mathfrak{M} .$$

It follows that $\frac{\partial \omega_2}{\partial x_1} - m x_1^{-m-1} \omega_2 \in L_2$ and thus

$$-b(x_1) x_2^{-2} = \frac{\partial \omega_1}{\partial x_2} = \frac{\partial \omega_2}{\partial \omega_1} - m x_1^{-m-1} \omega_2 - \eta \in L_2 .$$

But then $\omega_1 \in L_2$ and we have obtained

$$H^1_{dR}(L_{\mathrm{ess}}/L_2) = \{[(0, a(x_1) x_2^{-1})] \mid a'(x_1) - m x_1^{-m-1} a(x_1) \in \mathfrak{M}\} .$$

From the theory in the case of one variable, we know that its dimension is m as we wanted to prove.

□

Thus, we have proved that the (co-)homology spaces in (4.20) have the same dimension. It remains to prove perfectness from one side:

Proposition 4.13 *The local pairing* (4.20) *is non-degenerate from the left, i.e. assuming* $< [\omega], [c \otimes \varepsilon] >= 0$ *for all* $[c \otimes \varepsilon]$, *then* $[\omega] = 0$.

Proof: The proof uses literally the same arguments as the one for the corresponding statement for the pairing at a crossing-point including both directions D_1 and D_2 (Theorem 4.9) and is therefore omitted here.

□

The proposition completes the proof of Theorem 4.10.

□

4.3 Local pairing at a smooth point

It remains to study the local pairing at a smooth point of D, i.e. we may now assume that in local coordinates $D := D_1 = \{x_1 = 0\}$ and we have to consider a small bi-disc Δ around the smooth point $(0,0) \in D$. Our task is to prove perfectness of the local pairing

$$H^p_{dR}(\Delta, \Delta \smallsetminus D_1; D_1) \times H^{rd}_{p+1}(\Delta; {}^{L_{\mathrm{ess}}}\!/\!_{L_{\mathrm{mero}}}) \longrightarrow \mathbb{C}\ . \tag{4.23}$$

Theorem 4.14 *The pairing* (4.23) *is perfect.*

Proof: The proves of the following propositions is very similar to the proves of Proposition 4.11, 4.12 and 4.13, we will only briefly mention on the necessary changes. Let $e^{\alpha} \otimes R$ be the elementary model with $\alpha(x) = x_1^{-m_\alpha} u(x)$ as before.

Proposition 4.15 *One has* $\dim H^{rd}_*(\Delta, \Delta \smallsetminus D_1; D_1) = \begin{cases} 0 & \text{for } * \geq 2 \\ m_\alpha & \text{for } * = 1 \end{cases}$.

Proof: As in Proposition 4.12, the rapid decay homology groups decomposes as the ordinary singular homology groups of the Stokes regions in the tubular neighborhood of D_1, i.e. the normal disc bundle of D_1 modulo boundary and basis D_1. Here, the topological situation is even simpler as before, as we can retract the base of the bundle, namely D_1, to a point and end up with the Stokes radii in the fiber over a chosen point $(0, x_2) \in D_1$. Modulo boundary and modulo base point x, this is just the wedge of m_α circles. Thus

$$\dim H^{rd}_*(\Delta, \Delta \smallsetminus D_1; D_1) = \dim H_*(\bigvee_{m_\alpha} S^1)\ .$$

□

Proposition 4.16 *One has* $\dim H^*_{dR}(\Delta; {}^{L_{\mathrm{ess}}}\!/\!_{L_{\mathrm{mero}}}) = \begin{cases} 0 & \text{for } * \neq 0 \\ m_\alpha & \text{for } * = 0 \end{cases}$.

Proof: The proof for degree 0 and 2 is nearly literally the same as in Proposition 4.11: We have to consider

$${}^{L_{\mathrm{ess}}}\!/\!_{L_{\mathrm{mero}}} \xrightarrow{\varphi} {}^{L_{\mathrm{ess}}}\!/\!_{L_{\mathrm{mero}}} \oplus {}^{L_{\mathrm{ess}}}\!/\!_{L_{\mathrm{mero}}} \xrightarrow{\psi} {}^{L_{\mathrm{ess}}}\!/\!_{L_{\mathrm{mero}}}$$

with φ and ψ defined as in (4.22). Now, we proceed in the same way considering the Laurent expansions. In the case here, these expansions have no polar part in x_2-direction, but the arguments used for degree 0 and 2 still remain valid and give the desired result.

The non-existence of polar parts in x_2-direction, however, induces vanishing of $H^1_{dR}({}^{L_{\mathrm{ess}}}\!/\!_{L_{\mathrm{mero}}})$: Let $[(\omega_1, \omega_2)] \in \ker(\psi)$. Now, ω_2 has no polar part in x_2-direction, hence no residue in x_2 (with fixed x_1), so that

$$\frac{\partial u}{\partial x_2} = \omega_2$$

has a solution in L_{ess}. Therefore, modulo the image of φ, we can assume that $\omega_2 = 0$. But then ω_1 has to fulfill the equation

$$\frac{\partial \omega_1}{\partial x_2} = \eta = \sum_{j \geq 0} \eta_j(x_1) x_2^j$$

for some $\eta \in L_{\mathrm{mero}}$, i.e. with meromorphic functions η_j. But this forces ω_1 to be meromorphic also, hence $[(\omega_1, \omega_2)] = 0 \in H^1_{dR}({}^{L_{\mathrm{ess}}}\!/\!_{L_{\mathrm{mero}}})$.

□

Proposition 4.17 *The pairing* (4.23) *is non-degenerate from the left.*

Proof: The proof uses the same arguments as in Theorem 4.9 and is therefore omitted here.

□

We thus have proved local perfectness in all three situations to be considered (cp. Definition 3.6). This finally completes the proof of the main result (Theorem 2.5).

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