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Abstract

It is proved that under a technical condition the étale cohomology groups $H^1(\mathcal{O}_K[1/S], H^i(\bar{X}, \mathbb{Q}_p(j)),$ where $X \to \operatorname{Spec} \mathcal{O}_K[1/S]$ is a smooth, projective scheme, are generated by twists of norm compatible units in a tower of number fields associated to $H^i(\bar{X}, \mathbb{Z}_p(j))$. This confirms a consequence of the non-abelian Iwasawa main conjecture. Using the "Bloch-Kato-conjecture" a similar result is proven for motivic cohomology with finite coefficients.

Introduction

One of the most astonishing consequences of the equivariant Tamagawa number conjecture is the twist invariance of the zeta elements, which implies that all motivic elements should be twists of norm compatible units in (big) towers of number fields. More precisely one expects that for a $\mathbb{Z}_p$-lattice T in a motive with p-adic realization V the image of the twisting map (see 1.1.2 below)

$$(\varprojlim_n H^1(\mathcal{O}_{K_n}[1/S], \mathbb{Z}_p(1))) \otimes T(j-1) \to H^1(\mathcal{O}_K[1/S], T(j))$$

generates a subgroup of finite index. Here the inverse limit runs over the number fields $K_n := K(T/p^n)$ obtained from K by adjoining the elements T/p^n. Moreover, the image of this map should have a motivic meaning, that is the elements should be in the image of the p-adic regulator from motivic cohomology. (This is explained in [Hu-Ki2] and builds on ideas of Kato [Ka1]).

The philosophy of twisting originates from work of Iwasawa, Tate and Soulé, who considered twisting with the cyclotomic character. This already lead to many interesting results. Here Kato's work [Ka2] on the Birch-Swinnerton-Dyer conjecture is the most spectacular example. Earlier Soulé used this idea in the case of Tate motives in his investigations about the connection of K-theory and étale cohomology for number rings [So1]. He also pointed the way to applications to CM-elliptic curves [So4].

The goal of this paper is to show that the above twisting map has for $j >> 0$ indeed finite cokernel assuming the very reasonable condition that the Iwasawa μ-invariant of the number field K vanishes. In the second part of the paper we consider the statement, that the resulting elements are in the image of the regulator from motivic cohomology. Our results in this direction give a weak hint that the elements obtained as twists of units are motivic. Using the "Bloch-Kato-conjecture" for all fields (as announced by Voevodsky), we prove that there is a twisting map for motivic cohomology compatible with the one for étale cohomology under the cycle class map.

The second author likes to thank J. Coates for useful discussions and for making available the results of [Co-Su] before their publication. The authors are indebted to T. Geisser for insisting to use motivic cohomology with finite coefficients instead of K-theory in the formulation of the results in the second part.

1 Non-commutative twisting in étale cohomology

In this section we describe the étale situation. All cohomology groups in this paper are étale cohomology groups unless explicitly labeled otherwise.

1.1 The twisting map in étale cohomology

Let K be a number field with ring of integers $\mathcal{O}_K$. Fix a prime number $p > 2$ and a finite set of primes S of $\mathcal{O}_K$, which contains the primes dividing p. As usual let $G_S := \mathrm{Gal}(K_S/K)$ be the Galois group of K_S/K, where K_S is the maximal outside of S unramified extension field in a fixed algebraic closure $\bar{K}$ of K. Let T be a finitely generated $\mathbb{Z}_p$-module with a continuous G_S-action

$$\rho : G_S \longrightarrow \mathrm{Aut}_{\mathbb{Z}_p}(T).$$

We will consider T also as étale sheaf on $\mathcal{O}_K[1/S]$ (see e.g. [Fo-Pe, p. 640] using the fixed embedding into $\bar{K}$ as base point) and write

$V := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for the associated G_S-representation and $\mathbb{Q}_p$-sheaf. Let $\mathcal{G} := \operatorname{im} \rho$ be the image of ρ. If we define finite groups

$$G_n := \operatorname{im} \left\{ \rho_n : G_S \longrightarrow \operatorname{Aut}_{\mathbb{Z}/p^n\mathbb{Z}}(T/p^nT) \right\}$$

then we also have $\mathcal{G} \cong \varprojlim_n G_n$. Note that $\mathcal{G}$ is a p-adic Lie group. The *Iwasawa algebra* of $\mathcal{G}$ is by definition the continuous group ring

$$\Lambda(\mathcal{G}) := \varprojlim_n \mathbb{Z}_p[G_n] \cong \varprojlim_n \mathbb{Z}/p^n[G_n].$$

We denote by K_∞ the field fixed by the kernel of ρ and by K_n the field fixed by the kernel of ρ_n, so that $K_\infty = \cup_n K_n$ and $\operatorname{Gal}(K_\infty/K) \cong \mathcal{G}$. Note that $\mathcal{O}_{K_n}[1/S]$ is finite and etale over $\mathcal{O}_K[1/S]$.

Example: To make the above definitions more concrete, consider the following important example. Let E/K be an elliptic curve without complex multiplication and $T_pE := \varprojlim_n E[p^n]$ its Tate-module. We have $K_n := K(E[p^n])$. It is a well-known result of Serre that the image of the Galois group G_S in $\operatorname{Aut}_{\mathbb{Z}_p}(T_pE)$ has finite index and is equal to $\operatorname{Aut}_{\mathbb{Z}_p}(T_pE)$ for almost all p. If we assume the latter case, we have in the above notation $G_n \cong \operatorname{Gl}_2(\mathbb{Z}/p^n)$ and $\mathcal{G} \cong \operatorname{Gl}_2(\mathbb{Z}_p)$.

The following proposition for the étale cohomology should be well-known. For convenience of the reader and to explain the normalizations of the action in detail, we give the proof in an appendix.

Proposition 1.1.1. *(see appendix B) There are canonical isomorphisms of compact finitely generated $\Lambda(\mathcal{G})$-modules*

$$H^i(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})) \otimes_{\mathbb{Z}_p} T \xrightarrow{\cong} H^i(\mathcal{O}_K[1/S], \Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_p} T)$$

and

$$\varprojlim_n H^i(\mathcal{O}_{K_n}[1/S], T/p^n) \cong \varprojlim_n H^i(\mathcal{O}_{K_n}[1/S], T) \cong H^i(\mathcal{O}_K[1/S], \Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_p} T)$$

(limit over the corestriction maps). Here the $\Lambda(\mathcal{G})$-module structure on T is induced by the action of $\mathcal{G}$ and the one on $\Lambda(\mathcal{G})$ is via multiplication with the inverse.

Denote by $\epsilon : \Lambda(\mathcal{G}) \to \mathbb{Z}_p$ the augmentation map.

Definition 1.1.2. *The twisting map*

$$\mathrm{Tw}_T : H^i(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})) \otimes_{\mathbb{Z}_p} T \longrightarrow H^i(\mathcal{O}_K[1/S], T)$$

is the composition of the isomorphism of proposition 1.1.1 with the map

$$\epsilon : H^i(\mathcal{O}_K[1/S], \Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_p} T) \to H^i(\mathcal{O}_K[1/S], T)$$

induced by the augmentation ϵ.

Our goal is to show that the twisting map is surjective in certain cases after tensoring with $\mathbb{Q}_p$. In particular it allows to construct elements in $H^i(\mathcal{O}_K[1/S], T)$ starting from (corestriction or norm compatible) elements in

$$H^i(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})) \cong \varprojlim_n H^i(\mathcal{O}_{K_n}[1/S], \mathbb{Z}_p).$$

We will apply this in the case where $T = H^r(X \times_{\mathcal{O}_K[1/S]} \bar{K}, \mathbb{Z}_p)$.

1.2 The conjectured ranks of the étale cohomology

For the convenience of the reader, we recall the conjecture of Jannsen [Ja] about the ranks of the étale cohomology.

Let X be a smooth, projective scheme over $\mathcal{O}_K[1/S]$ and denote by $\bar{X} := X \times_{\mathcal{O}_K[1/S]} \bar{K}$ the base change to the algebraic closure.

Conjecture 1.2.1. *(Jannsen) For $i + 1 < j$ or $i + 1 > 2j$ one has*

$$H^2(\mathcal{O}_K[1/S], H^i(\bar{X}, \mathbb{Q}_p(j))) = 0.$$

As a consequence one obtains (for $p \neq 2$) the following formula for the dimension of the H^1: for $i + 1 < j$

$$\dim_{\mathbb{Q}_p} H^1(\mathcal{O}_K[1/S], H^i(\bar{X}, \mathbb{Q}_p(j))) = \dim_{\mathbb{R}} H^i(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{R}(j))^+,$$

where "+" denotes the invariants under complex conjugation, which acts on $X \times_{\mathbb{Q}} \mathbb{C}$ and on $\mathbb{R}(j) = (2\pi i)^j \mathbb{R}$.

Moreover in analogy with Beilinson's conjecture that the regulator from K-theory to Beilinson-Deligne cohomology is an isomorphism for $i + 1 < j$, Jannsen also conjectures that the Soulé regulator

$$r_p : H^{i+1}_{mot}(X, \mathbb{Z}(j)) \otimes_{\mathbb{Z}} \mathbb{Q}_p \to H^1(\mathcal{O}_K[1/S], H^i(\bar{X}, \mathbb{Q}_p(j)))$$

is an isomorphism for $i + 1 < j$. It is shown in [So4] that for p-adic K-theory the above regulator is surjective if $j >> 0$. This should be compared with the result in 2.2.4.

1.3 Another description of the twisting map

To make the similarity with the twisting map in motivic cohomology and in p-adic K-theory more apparent, we describe the twisting map at finite level.

Fix an integer $n > 0$. We have by Shapiro's lemma

$$H^i(\mathcal{O}_K[1/S], \mathbb{Z}/p^n\mathbb{Z}[G_n]) \cong H^i(\mathcal{O}_{K_n}[1/S], \mathbb{Z}/p^n\mathbb{Z})$$

using the identification as explained in appendix B. As T/p^nT is a trivial sheaf over $\mathcal{O}_{K_n}[1/S]$, we have $T/p^nT \cong H^0(\mathcal{O}_{K_n}[1/S], T/p^nT)$ and the cup product gives an isomorphism

$$H^i(\mathcal{O}_{K_n}[1/S], \mathbb{Z}/p^n\mathbb{Z}) \otimes T/p^nT \xrightarrow{\cup} H^i(\mathcal{O}_{K_n}[1/S], T/p^nT).$$

Together with the corestriction (=trace map in étale cohomology)

$$H^i(\mathcal{O}_{K_n}[1/S], T/p^nT) \to H^i(\mathcal{O}_K[1/S], T/p^nT)$$

we get a map

$$H^i(\mathcal{O}_K[1/S], \mathbb{Z}/p^n\mathbb{Z}[G_n]) \otimes T/p^nT \to H^i(\mathcal{O}_K[1/S], T/p^nT). \tag{1}$$

Observe that by Mittag-Leffler we have $\varprojlim_n H^i(\mathcal{O}_K[1/S], T/p^nT) \cong H^i(\mathcal{O}_K[1/S], T)$.

Lemma 1.3.1. *The inverse limit with respect to the trace map and reduction on the coefficients of the maps (1) coincides with the twisting map in definition 1.1.2.*

Proof. Straightforward. □

1.4 Tate twist

Let $K^{\mathrm{cyc}} := \cup_n K(\mu_{p^n})$ be the field K with all the p-th power roots of unity μ_{p^∞} adjoined. We will assume that K_∞ contains K^{cyc}. If this is not the case it can be achieved by considering $K_n(\mu_{p^n})$ instead of K_n. Let $\Gamma := \mathrm{Gal}(K^{\mathrm{cyc}}/K)$, then we have a map $\mathcal{G} \to \Gamma$ and we denote its kernel by $\mathcal{H}$. This map induces a surjection

$$\Lambda(\mathcal{G}) \to \Lambda(\Gamma). \tag{2}$$

The cyclotomic character induces an inclusion of Γ in $\mathbb{Z}_p^*$ and the associated free $\mathbb{Z}_p - \Gamma$-module of rank 1 is denoted by $\mathbb{Z}_p(1)$. As usual let $\mathbb{Z}_p(j) := \mathbb{Z}_p(1)^{\otimes j}$ and $T(j) := T \otimes \mathbb{Z}_p(j)$.

We will consider the following important variant of the twisting map, given by combining Definition 1.1.2 and Proposition 1.1.1 for $T = \mathbb{Z}_p(1)$ and $T(j-1)$:

$$(3) \quad H^i(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})(1)) \otimes_{\mathbb{Z}_p} T(j-1) \xrightarrow{\mathrm{Tw}_{T(j-1)}} H^i(\mathcal{O}_K[1/S], T(j)).$$

Note that $H^1(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})(1)) = \varprojlim_n H^1(\mathcal{O}_{K_n}[1/S], \mathbb{Z}_p(1))$ and that by Kummer theory we have an exact sequence

(4)

$$0 \to \mathcal{O}_{K_n}[1/S]^* \otimes \mathbb{Z}/p^n \to H^1(\mathcal{O}_{K_n}[1/S], \mathbb{Z}/p^n(1)) \to \mathrm{Cl}(\mathcal{O}_{K_n}[1/S])[p^n] \to 0.$$

Here $\mathrm{Cl}(\mathcal{O}_{K_n}[1/S])[p^n]$ is the p^n-torsion subgroup of the class group of $\mathcal{O}_{K_n}[1/S]$. Taking the limit over n we get

$$(5) \qquad \varprojlim_n \mathcal{O}_{K_n}[1/S]^* \otimes \mathbb{Z}_p \cong \varprojlim_n H^1(\mathcal{O}_{K_n}[1/S], \mathbb{Z}_p(1)).$$

We have a twisted variant of Lemma 1.3.1. Namely, the map $Tw_{T(j-1)}$ of (3) is again given by taking cup products $H^i(\mathcal{O}_{K_n}[1/S], \mathbb{Z}/p^n(1)) \otimes_{\mathbb{Z}_p} T/p^n(j-1)$ and passing to the limit, using again Proposition 1.1.1.

1.5 The cokernel of the twisting map

To study the cokernel of the twisting map, we factor the augmentation into $\Lambda(\mathcal{G}) \to \Lambda(\Gamma) \to \mathbb{Z}_p$ using (2) and get:

$$(6) \quad H^i(\mathcal{O}_K[1/S], \Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_p} T(j)) \to H^i(\mathcal{O}_K[1/S], \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T(j)).$$

The analysis of the cokernel of the twisting map $Tw_{T(j-1)}$ will proceed in two steps. The first is to investigate the cokernel of (6). The second step treats then the cokernel of the map induced by the augmentation $\Lambda(\Gamma) \to \mathbb{Z}_p$:

$$(7) \qquad H^i(\mathcal{O}_K[1/S], \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T(j)) \to H^i(\mathcal{O}_K[1/S], T(j)).$$

Lemma 1.5.1. *There is a spectral sequence*

$$E_2^{r,s} = \mathrm{Tor}_r^{\Lambda(\mathcal{G})}\left(H^s(\mathcal{O}_K[1/S], \Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_p} T(j)), \Lambda(\Gamma)\right) \Rightarrow H^{s-r}(\mathcal{O}_K[1/S], \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T(j)).$$

Proof. The projection formula (see e.g. [We, Exercise 10.8.3]) in the derived category gives

$$R\Gamma(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})\otimes_{\mathbb{Z}_p} T(j))\otimes^{\mathbb{L}}_{\Lambda(\mathcal{G})}\Lambda(\Gamma) \cong R\Gamma(\mathcal{O}_K[1/S], \Lambda(\Gamma)\otimes_{\mathbb{Z}_p} T(j)).$$

Taking cohomology gives the desired spectral sequence. □

Corollary 1.5.2. *There is an exact sequence*

$$\begin{aligned} H^1(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})(1))\otimes_{\mathbb{Z}_p} T(j-1) \xrightarrow{Tw_{T(j-1)}} H^1(\mathcal{O}_K[1/S], \Lambda(\Gamma)\otimes_{\mathbb{Z}_p} T(j)) \\ \to \mathrm{Tor}_1^{\Lambda(\mathcal{G})}\left(H^2(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})\otimes_{\mathbb{Z}_p} T(j)), \Lambda(\Gamma)\right). \end{aligned}$$

Proof. As $p > 2$ the cohomological p-dimension of $\mathcal{O}_K[1/S]$ is 2 and the result follows from the spectral sequence and the fact that the twisting map factors through

$$H^1(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})\otimes_{\mathbb{Z}_p} T(j)) \to H^1(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})\otimes_{\mathbb{Z}_p} T(j))\otimes_{\Lambda(\mathcal{G})}\Lambda(\Gamma).$$

□

Lemma 1.5.3. *The canonical isomorphism* $\Lambda(\mathcal{G})\otimes_{\Lambda(\mathcal{H})}\mathbb{Z}_p\cong\Lambda(\Gamma)$ *induces isomorphisms for all* r*:*

$$\begin{aligned} \mathrm{Tor}_r^{\Lambda(\mathcal{G})}\left(H^2(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})\otimes_{\mathbb{Z}_p} T(j)), \Lambda(\Gamma)\right) \cong \\ \mathrm{Tor}_r^{\Lambda(\mathcal{H})}\left(H^2(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})\otimes_{\mathbb{Z}_p} T(j)), \mathbb{Z}_p\right). \end{aligned}$$

In particular, one gets from corollary 1.5.2 an exact sequence

$$\begin{aligned} H^1(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})(1))\otimes_{\mathbb{Z}_p} T(j-1) \to H^1(\mathcal{O}_K[1/S], \Lambda(\Gamma)\otimes_{\mathbb{Z}_p} T(j)) \to \\ \to \mathrm{Tor}_1^{\Lambda(\mathcal{H})}\left(H^2(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})\otimes_{\mathbb{Z}_p} T(j)), \mathbb{Z}_p\right). \end{aligned}$$

Proof. The isomorphism $\Lambda(\mathcal{G})\otimes_{\Lambda(\mathcal{H})}\mathbb{Z}_p\cong\Lambda(\Gamma)$ can be checked at finite level as $\mathbb{Z}_p$ is a finitely generated $\Lambda(\mathcal{H})$-module. Then $\mathbb{Z}_p[G_n]\otimes_{\mathbb{Z}_p[H_n]}\mathbb{Z}_p \cong \mathbb{Z}_p[G_n/H_n]$ and the claim is obvious. In particular, for finitely generated $\Lambda(\mathcal{G})$-modules M is the functor $M \mapsto M\otimes_{\Lambda(\mathcal{G})}\Lambda(\Gamma)$ isomorphic to $M \mapsto M\otimes_{\Lambda(\mathcal{H})}\mathbb{Z}_p$. □

From this lemma and the factorization of the twisting map it is clear that the cokernel of the twisting map is controlled by

$$\mathrm{Tor}_1^{\Lambda(\mathcal{H})}\left(H^2(\mathcal{O}_K[1/S], \Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_p} T(j)), \mathbb{Z}_p\right)$$

and by $\mathrm{Tor}_1^{\Lambda(\Gamma)}\left(H^2(\mathcal{O}_K[1/S], \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T(j)), \mathbb{Z}_p\right)$. To say something about these groups we need some results of Coates and Sujatha on the finiteness of the H^2's involved.

1.6 Finiteness conditions for H^2

This section contains only slight modifications of results of Coates and Sujatha [Co-Su]. We thank them very much for making these results available to us before their publication. One should also compare this section with the appendix B in Perrin-Riou [Pe] prop. B.2.

Let L^{cyc} (resp. L_∞) be the maximal unramified abelian p-extension of K^{cyc} (resp. K_∞), in which every prime above p splits completely.

Proposition 1.6.1. *(Coates-Sujatha [Co-Su]) Assume that* $\mathcal{G} = \mathrm{Gal}(K_\infty/K)$ *is a pro-p-group, then the following conditions are equivalent:*

i) $\mathrm{Gal}(L^{\mathrm{cyc}}/K^{\mathrm{cyc}})$ *is a finitely generated* $\mathbb{Z}_p$*-module*

ii) $\mathrm{Gal}(L_\infty/K_\infty)$ *is a finitely generated* $\Lambda(\mathcal{H})$*-module*

iii) $H^2(\mathcal{O}_K[1/S], \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T)$ *is a finitely generated* $\mathbb{Z}_p$*-module*

iv) $H^2(\mathcal{O}_K[1/S], \Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_p} T)$ *is a finitely generated* $\Lambda(\mathcal{H})$*-module.*

In particular, if these equivalent conditions are satisfied, the $\Lambda(\Gamma)$*-module* $H^2(\mathcal{O}_K[1/S], \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T)$ *is torsion, i.e., the weak Leopold conjecture is true.*

Remarks: The first condition is equivalent to the famous $\mu = 0$ conjecture of Iwasawa (see [NSW] Ch. XI thm. 11.3.18), which is known to be true for $K/\mathbb{Q}$ abelian.

Note also that the statements i) and ii) in the proposition are independent of the Galois representation T.

Proof. The proof of the proposition can be found in Coates and Sujatha [Co-Su] in the case of the Tate module for an elliptic curve. The case for an arbitrary Galois representation T is the same. More precisely, the equivalence $i) \iff ii)$ is lemma 3.7., $i) \iff iii)$ is thm. 3.4. in loc. cit. To prove $iii) \iff iv)$, we have from the spectral sequence in lemma 1.5.1 and the vanishing of étale cohomology for $s > 2$ that

$$H^2(\mathcal{O}_K[1/S], \Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_p} T) \otimes_{\Lambda(\mathcal{H})} \mathbb{Z}_p \cong H^2(\mathcal{O}_K[1/S], \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T)$$

and the claim follows from Nakayama's lemma. □

Example: Let $E/\mathbb{Q}$ be an elliptic curve over $\mathbb{Q}$ and $F/\mathbb{Q}$ be an abelian extension such that $E_{p^\infty}(F) \neq 0$. Then it is easy to see (cf. [Co-Su] cor. 3.6.) that $F(E_p^\infty)/F(\mu_p)$ is a pro-p extension. Thus these elliptic curves provide examples where the above proposition 1.6.1 applies. More specific examples are $E : y^2 + xy = x^3 - x - 1$ and $F = \mathbb{Q}(\mu_7)$ or $E : y^2 + xy = x^3 - 3x - 3$ and $F = \mathbb{Q}(\mu_5)$ (see loc. cit. 4.7. and 4.8.).

1.7 Étale cohomology classes as twists of units

Recall that L^{cyc} is the maximal unramified abelian p-extension of K^{cyc}, in which every prime above p splits completely, and that we have an isomorphism $H^1(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})(1)) \cong \varprojlim_n \mathcal{O}_{K_n}[1/S]^* \otimes \mathbb{Z}_p$ by Proposition 1.1.1 and (5).

Theorem 1.7.1. *Suppose that* $\mathrm{Gal}(L^{\mathrm{cyc}}/K^{\mathrm{cyc}})$ *is a finitely generated* $\mathbb{Z}_p$*-module, then there exists a* $J \in \mathbb{N}$ *such that for all* $j \geq J$ *the twisting map*

$$H^1(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})(1)) \otimes_{\mathbb{Z}_p} T(j-1) \xrightarrow{\mathrm{Tw}_{T(j-1)}} H^1(\mathcal{O}_K[1/S], T(j))$$

has finite cokernel. In particular, for $j \geq J$ *all elements in* $H^1(\mathcal{O}_K[1/S], V(j))$ *(where* $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *as before) are "twists" of norm compatible units in*

$$H^1(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})(1)) = \varprojlim_n H^1(\mathcal{O}_{K_n}[1/S], \mathbb{Z}_p(1))$$

with a basis in the lattice $T(j-1)$.

Remark: The choice of the twist 1 in $H^1(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})(1))$ and hence of the group of norm compatible units instead of any other twist is just for esthetic reasons. For the application to Euler systems and the construction of p-adic L-functions the units are certainly the most interesting case. In particular, we see this theorem as a strong confirmation of the philosophy explained in [Hu-Ki2], that all p-adic properties of motives in connection with L-values should be encoded in the associated tower of number fields.

It is an interesting question to investigate $H^2(\mathcal{O}_K[1/S], T(j))$ with the above methods and to compare this with the results by McCallum and Sharifi [Mc-Sh].

Proof. It follows from proposition 1.6.1 that under our conditions

$$\mathrm{Tor}_1^{\Lambda(\mathcal{H})}\left(H^2(\mathcal{O}_K[1/S], \Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_p} T(j)), \mathbb{Z}_p\right)$$

is a finitely generated $\mathbb{Z}_p$-module. Indeed $H^2(\mathcal{O}_K[1/S], \Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_p} T(j))$ is a finitely generated $\Lambda(\mathcal{H})$-module and thus the groups

$$\mathrm{Tor}_r^{\Lambda(\mathcal{H})}\left(H^2(\mathcal{O}_K[1/S], \Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_p} T(j)), \mathbb{Z}_p\right)$$

are also finitely generated $\mathbb{Z}_p$-modules by standard homological algebra. The exact sequence 1.5.2 implies that the cokernel of

$$H^1(\mathcal{O}_K[1/S], \Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_p} T(j)) \otimes_{\Lambda(\mathcal{G})} \Lambda(\Gamma) \to H^1(\mathcal{O}_K[1/S], \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T(j))$$

is a $\Lambda(\Gamma)$-module, say $M(j)$, which is finitely generated as a $\mathbb{Z}_p$-module, hence torsion as $\Lambda(\Gamma)$-module. By the classification of torsion $\Lambda(\Gamma)$-modules, the coinvariants of $M(j) \otimes_{\Lambda(\Gamma)} \mathbb{Z}_p$ are finite for sufficiently big j. We get an exact sequence

$$\begin{aligned} &H^1(\mathcal{O}_K[1/S], \Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_p} T(j)) \otimes_{\Lambda(\mathcal{G})} \mathbb{Z}_p \to \\ &\qquad H^1(\mathcal{O}_K[1/S], \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T(j)) \otimes_{\Lambda(\Gamma)} \mathbb{Z}_p \to M(j) \otimes_{\Lambda(\Gamma)} \mathbb{Z}_p \to 0. \end{aligned}$$

To get the twisting map we have to compose with the first map in the following exact sequence (which is similar to Corollary 1.5.2)

$$\begin{aligned} &H^1(\mathcal{O}_K[1/S], \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T(j)) \otimes_{\Lambda(\Gamma)} \mathbb{Z}_p \to \\ &\quad H^1(\mathcal{O}_K[1/S], T(j)) \to \mathrm{Tor}_1^{\Lambda(\Gamma)}\left(H^2(\mathcal{O}_K[1/S], \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T(j)), \mathbb{Z}_p\right). \end{aligned}$$

By our condition and proposition 1.6.1 $H^2(\mathcal{O}_K[1/S], \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T(j))$ is also a finitely generated $\mathbb{Z}_p$-module and thus $\Lambda(\Gamma)$-torsion. As Γ is cyclic ($\mathcal{G}$ and hence Γ is pro-p), the $\mathrm{Tor}_1^{\Lambda(\Gamma)}$ term identifies with the Γ-invariants of $H^2(\mathcal{O}_K[1/S], \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T(j))$. Again for j big enough these are finite. □

Remark: In fact, if M is the p-adic realization of a motive (say $M = H^n(X \times_K \bar{K}, \mathbb{Q}_p)$ for X/K smooth, projective), one should expect that the $\Lambda(\Gamma)$-module

$$\mathrm{Tor}_1^{\Lambda(\mathcal{G})}\left(H^2(\mathcal{O}_K[1/S], \Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_p} T(j)), \Lambda(\Gamma)\right) \otimes_{\Lambda(\Gamma)} \mathbb{Z}_p$$

is finite for all $j \geq n+1$. Compare this with Jannsen's conjecture 1.2.1 about the vanishing of $H^2(\mathcal{O}_K[1/S], H^n(X \times_K \bar{K}, \mathbb{Q}_p(j)))$ for $j \geq n+1$.

2 Twisting for motivic cohomology with p-adic coefficients

In this section X will always be a smooth and projective scheme over $D = \mathcal{O}_K[1/S]$.

The goal in this section is to study the twisting map in motivic cohomology with finite coefficients. The general assumption is that the "Bloch-Kato-conjecture" for motivic cohomology holds as announced by Voevodsky in [Vo2] (do not confuse this with the Tamagawa number conjecture). This implies, using the Beilinson-Lichtenbaum conjecture, that we have to deal with étale cohomology of X.

2.1 Review of motivic cohomology with finite coefficients over Dedekind domains

For a variety X smooth over a Dedekind ring D, we define motivic cohomology groups as the hypercohomology of Bloch's cycle complex $\mathbb{Z}(j)$. As usual, $\Delta_D^s := \mathrm{Spec}(D[t_0, ..., t_s]/\sum_i t_i - 1)$ denotes the standard algebraic s-simplex.

For a variety X smooth over a Dedekind ring D, let $z^j(X, i)$ be the free abelian group on closed integral subschemes of codimension j on $X \times_D \Delta_D^i$ which intersect all faces properly. The associated complex of presheaves (with $z^j(X, 2j - i)$ in degree i) is denoted $\mathbb{Z}(j)$, and $\mathbb{Z}/n(j) := \mathbb{Z}(j) \otimes^{\mathbb{L}} \mathbb{Z}/n$. The complex $\mathbb{Z}(j)$ (and thus also $\mathbb{Z}/n(j)$) is a complex of sheaves for the étale topology [Ge, Lemma 3.1], and we write $\mathbb{Z}/n(j)_{et}$ resp. $\mathbb{Z}/n(j)_{Zar}$ when considering it as a complex of étale resp. Zariski sheaves.

Definition 2.1.1. *(compare [Ge, p. 5]) The* motivic cohomology *of X is the hypercohomology*

$$H^i_{mot}(X, \mathbb{Z}/n(j)) := H^i(X, \mathbb{Z}/n(j)_{Zar}). \tag{8}$$

Calling this motivic cohomology is justified by Voevodsky's [Vo1] theorem $Hom_{DM^{eff,-}(K)}(M(X), \mathbb{Z}(j)[i]) =: H^i_{mot}(X, \mathbb{Z}(j)) \cong CH^j(X, 2j-i)$ if $D = K$ is a field. In this case, higher Chow groups are defined by taking just cohomology and not hypercohomology. By [Ge, Theorem 3.2] both definitions coincide not only over a field but still if the base D is a discrete valuation ring.

Observe [Ge, section 3] that H^i_{mot} is covariant for proper maps (with degree shift) and contravariant for flat maps. The latter applies in particular to the structural morphisms $p_n : X_n \to D_n$.

The étale cycle class cl factors through the étale sheafification $\mathbb{Z}/n(j)_{et}$ via the map $\mathbb{Z}/n(j)_{Zar} \to \mathbf{R}\pi_*\mathbb{Z}/n(j)_{et}$ induced by the morphism of sites $\pi : (Sm/D)_{et} \to (Sm/D)_{Zar}$.

For us the most important consequence of the Bloch-Kato conjecture is the truth of the Beilinson-Lichtenbaum conjecture:

Theorem 2.1.2. *(Geisser [Ge, Theorem 1.2 (2)(4)]) Assume that X is a smooth scheme over a Dedekind domain D with $n \in D^\times$ and that the "Bloch-Kato-conjecture" holds.*

1) For all i and j there is an isomorphism

$$H^i(X, \mathbb{Z}/n(j)_{et}) \cong H^i(X, \mathbb{Z}/n(j))$$

of the étale hypercohomology of $\mathbb{Z}/n(j)_{et}$ with the étale cohomology.

2) The étale cycle class map induces isomorphisms for $0 \leq i \leq j$

$$H^i_{mot}(X, \mathbb{Z}/n(j)) \cong H^i(X, \mathbb{Z}/n(j))$$

of motivic with étale cohomology.

2.2 The geometric twisting map

We are going to define a geometric twisting map, which will allow to relate our results for étale cohomology with motivic cohomology. The main difficulty is that the cup-product is not compatible with corestriction maps. We use the compatibility of the Hochschild-Serre spectral sequence with cup-product to overcome this and to reduce to an observation due to Soulé.

In this section we consider $X \to \operatorname{Spec} \mathcal{O}_K[1/S]$ smooth and proper. We denote by $\bar{X} := X \times \bar{K}$ and let $T := H^i(\bar{X}, \mathbb{Z}_p)$. This is a Galois-module and finitely generated $\mathbb{Z}_p$-module. As in section 1.1 this defines a tower of number fields K_n and a p-adic Lie group $\mathcal{G} := \operatorname{Gal}(K_\infty/K)$. Let

$$X_n := X \times_{\mathcal{O}_K[1/S]} \mathcal{O}_{K_n}[1/S].$$

To construct a twisting map for motivic cohomology we use the pull-back

$$H^1(\mathcal{O}_{K_n}[1/S], \mathbb{Z}/p^n(1)) \to H^1(X_n, \mathbb{Z}/p^n(1))$$

and the cup-product with $H^i(X_n, \mathbb{Z}/p^n(j-1))$. This produces elements in $H^{i+1}(X_n, \mathbb{Z}/p^n(j))$. The Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(\mathcal{O}_{K_n}[1/S], H^q(\bar{X}, \mathbb{Z}/p^n(j))) \Rightarrow H^{p+q}(X_n, \mathbb{Z}/p^n(j))$$

allows to relate these elements with $H^1(\mathcal{O}_{K_n}[1/S], H^i(\bar{X}, \mathbb{Z}/p^n(j)))$ as follows: Let

$$H^{i+1}(X_n, \mathbb{Z}/p^n(j))^0 := \ker\left(H^{i+1}(X_n, \mathbb{Z}/p^n(j)) \xrightarrow{\gamma} H^0(\mathcal{O}_{K_n}[1/S], H^{i+1}(\bar{X}, \mathbb{Z}/p^n(j)))\right)$$

be the kernel of the edge morphism γ. As the Hochschild-Serre spectral sequence is compatible with cup-products, we get a map

(9)
$$H^1(\mathcal{O}_{K_n}[1/S], \mathbb{Z}/p^n(1)) \times H^i(X_n, \mathbb{Z}/p^n(j-1)) \to H^{i+1}(X_n, \mathbb{Z}/p^n(j))^0.$$

As $E_2^{1,i} = E_\infty^{1,i}$ we have a surjection,

(10)
$$H^{i+1}(X_n, \mathbb{Z}/p^n(j))^0 \to H^1(\mathcal{O}_{K_n}[1/S], H^i(\bar{X}, \mathbb{Z}/p^n(j))),$$

which we compose with the corestriction map

(11)
$$H^1(\mathcal{O}_{K_n}[1/S], H^i(\bar{X}, \mathbb{Z}/p^n(j))) \xrightarrow{\text{cores}} H^1(\mathcal{O}_K[1/S], H^i(\bar{X}, \mathbb{Z}/p^n(j))).$$

If we compose the cup-product in (9) with this composition and using again the compatibility of the spectral sequence with products we see that the cup-product has to factor through the edge morphism

(12)
$$H^i(X_n, \mathbb{Z}/p^n(j-1)) \xrightarrow{\gamma} H^0(\mathcal{O}_{K_n}[1/S], H^i(\bar{X}, \mathbb{Z}/p^n(j-1))).$$

Thus we get

$$(13)\quad H^1(\mathcal{O}_{K_n}[1/S], \mathbb{Z}/p^n(1)) \times H^0(\mathcal{O}_{K_n}[1/S], H^i(\bar{X}, \mathbb{Z}/p^n(j-1))) \to \\ \to H^1(\mathcal{O}_K[1/S], H^i(\bar{X}, \mathbb{Z}/p^n(j))).$$

It is an important observation by Soulé that, although the cup-product in general is not compatible with corestriction, the map in (13) is compatible. To formulate the observation of Soulé properly we need:

Definition 2.2.1. *A sequence of elements $\alpha_n \in H^r(X_n, \mathbb{Z}/p^n)$ is* norm compatible *if the restriction of the coefficients modulo $\mathbb{Z}/p^{n-1}$ of $cores(\alpha_n)$ is α_{n-1} for all $n \geq 2$. The sequence $\{\alpha_n\}$ is* reduction compatible *if the reduction modulo p^{n-1} of α_n is the pull-back of α_{n-1} for all $n \geq 2$.*

Note that the elements $\{\alpha_n\}$ of any $\mathbb{Z}_p$-lattice $T \subset H^i(\bar{X}, \mathbb{Q}_p(j)))$ are reduction compatible. Soulé proves the following:

Lemma 2.2.2. *If the sequence $\{\alpha_n\}$ is norm compatible and $\{\beta_n\}$ is reduction compatible, then $\{\alpha_n \cup \beta_n\}$ is norm compatible.*

Proof. This is just the projection formula, see [So4, Lemma 1.4]. □

Taking the projective limit over n in (13) gives:

Corollary 2.2.3. *Cup-product gives a map*

$$H^1(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})(1)) \otimes H^i(\bar{X}, \mathbb{Z}_p(j-1)) \to H^1(\mathcal{O}_K[1/S], H^i(\bar{X}, \mathbb{Z}_p(j))).$$

Combining this with theorem 1.7.1 and with the isomorphism of motivic and étale cohomology from theorem 2.1.2 we obtain:

Corollary 2.2.4. *Under the condition of theorem 1.7.1 there is an integer m such that for all $n \geq m$ the cokernel of the cup-product map in (9) composed with the maps in (10) and (11)*

$$H^1_{mot}(\mathcal{O}_{K_n}[1/S], \mathbb{Z}/p^n(1)) \times H^i_{mot}(X_n, \mathbb{Z}/p^n(j-1)) \to \\ \to H^1(\mathcal{O}_K[1/S], H^i(\bar{X}, \mathbb{Z}/p^n(j)))$$

is annihilated by p^m for all $j \geq J$.

Remarks: a) More generally, the above construction is possible for any theory A^*, which is covariant for proper maps, contravariant for flat maps and satisfying the projection formula $f_*(a \cup f^*(b)) = f_*(a) \cup b$

for flat proper (or at least finite étale) maps f. We explain the case of K-theory with finite coefficients in appendix A.

b) Soulé applies the above construction to get non-torsion elements in the K-groups of rings of integers or elliptic curves with complex multiplication. In these cases the schemes X_n are base changes of X to the ring $\mathcal{O}_{K_n}[1/S]$, where K_n is defined by adjoining p^n-th roots of unity or p^n-th division points of the elliptic curve. The towers of fields are in these cases abelian. It is shown in the cyclotomic case in [Hu-Wi] and [Hu-Ki1] (with another method) and in the case of CM-elliptic curves in [Ki] that these twisted elements are in fact motivic, i.e., are in the image of motivic cohomology.

2.3 Compatibility of cup products in motivic and étale cohomology

The aim of this technical section is to establish the compatibility of cup products in étale and motivic cohomology. The problem is that the cup product for motivic cohomology over Dedekind rings is only defined if one factor consists of equi-dimensional cycles (see definition 2.3.2 below). We will show that we have a commutative diagram

(14)

$$\begin{tikzcd}
(D_n^\times \otimes \mathbb{Z}/p^n) \times H^i_{mot}(X_n, \mathbb{Z}/p^n(j-1)) \arrow[r, "\cup_{mot}\circ\phi\times \mathrm{id}"] \arrow[d, "\phi\times cl"] & H^{i+1}_{mot}(X_n, \mathbb{Z}/p^n(j)) \arrow[d, "cl"] \\
H^1(D_n, \mathbb{Z}/p^n(1)) \times H^i(X_n, \mathbb{Z}/p^n(j-1)) \arrow[r, "\cup"] \arrow[d, "1\times\gamma"] & H^{i+1}(X_n, \mathbb{Z}/p^n(j)) \arrow[d, "\tilde{\gamma}"] \\
H^1(D_n, \mathbb{Z}/p^n(1)) \times H^i((\bar{X}, \mathbb{Z}/p^n(j-1)) \arrow[r, "Tw"] & H^1(D_n, H^i(\bar{X}, \mathbb{Z}/p^n(j))
\end{tikzcd}$$

where $D_n := Spec(\mathcal{O}_{K_n}[1/S])$, γ and $\tilde{\gamma}$ are the edge maps as before, the vertical arrows cl are étale cycle class maps and the cup product $\cup_{mot}$ as well as the map ϕ are defined below. The commutativity of the lower square follows from lemma 1.3.1 and the compatiblity of the Hochschild-Serre spectral sequence with cup products. The commutativity of the upper square of (14) is discussed below; this generalizes the classical result for the usual cycle class map (see e. g. [Mi, Proposition VI.9.5]). Recall that by [Su, Corollary 4.3] we have an isomorphism $H^i_{mot}(\bar{X}, \mathbb{Z}/p^n(j)) \cong H^i(\bar{X}, \mathbb{Z}/p^n(j))$.

Recall from [Le, section 1.7] that an irreducible scheme $Z \to D$ is *equi-dimensional* if it is dominant over D. The relative dimension $\dim_D Z$ is then defined to be the dimension of the generic fibre. Now we define the relative higher Chow group complex for our smooth $X \to D$ as follows: $z^j(X/D,p)$ to be the free abelian group generated by irreducible closed subsets $Z \subset X \times_D \Delta_D^p$, such that for each face F of Δ_D^p the irreducible components Z' of $Z \cap (X \times F)$ are equi-dimensional over D and $\dim_D Z' = \dim_D F + d - j$. Note that we have an inclusion of complexes $z^j(X/D,*) \subset z^j(X,*)$. We define equi-dimensional motivic cohomology $H^i_{mot}(X/D,\mathbb{Z}(j))$ to be the Zariski hypercohomology of the complex which has in degree i the Zariski sheafification of the presheaf $U \mapsto z^j(U/D, 2j-i)$. To define $H^i_{mot}(X/D,\mathbb{Z}/p^n(j))$ we use the same complex tensored with $\otimes^{\mathbb{L}}\mathbb{Z}/p^n$.

The units $D_n^\times$ we use for twisting are all equi-dimensional:

Lemma 2.3.1. *The map*

$$\phi : D_n^\times \to H^1_{mot}(D_n,\mathbb{Z}(1))$$

induced by sending $u \neq 1 \in D_n^\times$ to the graph of the rational map

$$\left(\frac{1}{1-u}, \frac{u}{u-1}\right) : \operatorname{Spec} D_n \to \Delta_D^1$$

(i.e. to a cycle in $D_n \times_D \Delta_D^1$) factors through $H^1_{mot}(D_n/D,\mathbb{Z}(1))$. The induced map

$$D_n^\times \otimes \mathbb{Z}/p^n \to H^1_{mot}(D_n/D,\mathbb{Z}/p^n(1))$$

is injective.

Proof. In [Le, Lemma 11.2] Levine constructs a map $D_n^\times \to CH^1(D_n,1)$ using the graph of $\left(\frac{1}{1-u}, \frac{u}{u-1}\right)$. Together with the natural map $CH^1(D_n,1) \to H^1_{mot}(D_n,\mathbb{Z}(1))$ this defines ϕ and hence a map

$$D_n^\times \otimes \mathbb{Z}/p^n \to H^1_{mot}(D_n,\mathbb{Z}/p^n(1)).$$

If we compose this with the isomorphism in 2.1.2, we get a map

$$D_n^\times \otimes \mathbb{Z}/p^n \to H^1_{et}(D_n,\mathbb{Z}/p^n(1)),$$

which is obviously (reduce to the case of a field) the map induced by the Kummer sequence, hence injective. It remains to show that the map factors through $H^1_{mot}(D_n/D, \mathbb{Z}/p^n(1))$. As the graph of $\left(\frac{1}{1-u}, \frac{u}{u-1}\right)$ is an equi-dimensional cycle, this follows from the definition. $\square$

Now we define the upper horizontal map $\cup_{mot}$ of (14).

Definition 2.3.2. *For $f : X_n \to Spec(D_n)$ smooth, we define*

$$\cup_{mot} : H^1_{mot}(D_n/D, \mathbb{Z}/p^n(1)) \times H^i_{mot}(X_n, \mathbb{Z}/p^n(j-1)) \to H^{i+1}_{mot}(X_n, \mathbb{Z}/p^n(j))$$

as the composition

$$H^1_{mot}(D_n/D, \mathbb{Z}/p^n(1)) \times H^i_{mot}(X_n, \mathbb{Z}/p^n(j-1)) \xrightarrow{(f\times \mathrm{id})^* \circ \cup^{r,s}_{D_n/D,X_n}}$$
$$\to H^{i+1}_{mot}(X_n \times_D D_n, \mathbb{Z}/p^n(j)) \xrightarrow{(id,p_n)^*} H^{i+1}_{mot}(X_n, \mathbb{Z}/p^n(j)).$$

Here $\cup^{r,s}_{D_n/D,X} : z^s(D_n/D, *) \otimes z^r(X) \to z^{r+s}(X \times_D D_n)$ is the exterior product with integral coefficients defined by Levine [Le, section 8]. The product of the complexes of presheaves induces a product of complexes of sheaves and (using Godement resolutions as in [Ge-Le]) on the hypercohomology groups.

Now we return to the commutativity of (14). By definition of the twisting map at finite level in 1.3, it is enough to consider the diagram

$$\begin{array}{ccc}
H^1_{mot}(D_n/D, \mathbb{Z}/p^n(1)) \times H^i_{mot}(X_n, \mathbb{Z}/p^n(j-1)) & \xrightarrow{\cup_{mot}} & H^{i+1}_{mot}(X_n \times_D D_n, \mathbb{Z}/p^n(j)) \\
\downarrow{\scriptstyle cl} & & \downarrow{\scriptstyle cl} \\
H^1(D_n, \mathbb{Z}/p^n(1)) \times H^i(X_n, \mathbb{Z}/p^n(j-1)) & \xrightarrow{\cup} & H^{i+1}(X_n \times_D D_n, \mathbb{Z}/p^n(j)).
\end{array}$$

As pointed out in [Ge, p. 13], the proof of [Ge-Le, Proposition 4.7] for the commutativity of the corresponding diagram of varieties over fields carries over to Dedekind domains. The argument in the proof of [Ge-Le, Proposition 4.7] that $\cup$ equals the product $\cup'$ of *loc. cit.* constructed in a way compatible with $\cup_{D_n/D,X}$ is still valid over Dedekind domains. Hence the commutativity of (14).

A Twisting in p-adic K-theory

In this appendix, we will reinterpret our results in terms of p-adic K-theory.

As usual, we can define K-theory with coefficients of the exact category $Vect(X)$ of vector bundles on X using Quillen's Q-construction and homotopy groups with finite coefficients:

Definition A.0.3. *Let*

$$K_m(X, \mathbb{Z}/q) := \pi_m(\Omega BQVect(X), \mathbb{Z}/q)$$

for $r > 0$ *and* $K_0(X, \mathbb{Z}/q) := K_0(X)/q$. *Moreover, we set*

$$K_r(X, \mathbb{Z}_p) := \varprojlim_n K_r(X, \mathbb{Z}/p^n).$$

and define $K_r(X, \mathbb{Q}_p) := K_r(X, \mathbb{Z}_p) \otimes_{Z_p} \mathbb{Q}_p$.

Here we use that we have maps $\epsilon_n : K_r(X, \mathbb{Z}/p^n) \to K_r(X, \mathbb{Z}/p^{n-1})$ given by reduction of coefficients. Applying $\varprojlim$ to the short exact sequence

$$0 \to K_r(X)/p^n \to K_r(X, \mathbb{Z}/p^n) \to_{p^n} K_{r-1}(X) \to 0$$

shows that $rk_{\mathbb{Z}} K_r(X) = rk_{\mathbb{Z}_p} K_r(X, \mathbb{Z}_p)$, provided the groups $K_r(X)$ and $K_{r-1}(X)$ are finitely generated as generally conjectured ("Bass conjecture") and proved if $X = \mathrm{Spec}(\mathcal{O}_K)$ by Quillen [Qu].

We assume as before that X is smooth over $\mathcal{O}_K[1/S]$, of relative dimension d. Adams operations carry over to finite coefficients and their eigenspaces are denoted by $K(X, \mathbb{Z}/p^n)^{(j)}$ as usual. By [So3, Proposition 6] the transfer maps $(f_n)_*$ respect these eigenspace decomposition (the hypothesis of *loc. cit.* is satisfied as the field extension K_n/K_{n-1} is finite).

Thomason constructs an algebraic Bott element $\beta \in K_2(X, \mathbb{Z}/p^n)$ and proves that there is an isomorphism $K_*(X, \mathbb{Z}/p^n)[\beta^{-1}] \xrightarrow{\cong} K_*^{et}(X, \mathbb{Z}/p^n)$ [Th1, Theorem 4.11], that $\phi_j : K_j(X, \mathbb{Z}/p^n) \to K_j^{et}(X, \mathbb{Z}/p^n)$ is an epimorphism if $j \geq N$ and β^N annihilates $ker(\phi_j)$ for all $j \geq 0$, where $N = 2/3(d+2)(d+3)(d+4)$ [Th2, Corollary 3.6]. Multyplying the short exact (for $j \geq 2N$) sequence $\ker(\phi_j) \to K_j(X, \mathbb{Z}/p^n) \to K_j^{et}(X, \mathbb{Z}/p^n)$ with β^N and applying the snake lemma, we get a splitting $K_{j+2N}(X, \mathbb{Z}/p^n) \to ker(\phi_{j+2N})$ and thus étale K-theory is a natural direct summand of K-theory in these degrees. So if $2j - i - 2 \geq (8/3)(d+2)(d+3)(d+4)$, we obtain a pairing

$$K_1^{et}(\mathcal{O}_{K_n}[1/S], \mathbb{Z}/p^n) \times K_{2j-i-2}^{et}(X_n, \mathbb{Z}/p^n) \to K_{2j-i-1}^{et}(X_n, \mathbb{Z}/p^n)$$

which is a direct summand of the corresponding pairing in algebraic K-theory with finite coefficients. Concerning the first factor, we even have an isomorphism between K_1 and K_1^{et} by [Dw-Fr, Proposition 8.2].

Remark: For $p = 2$, the bounds for j such that $K_j(X, \mathbb{Z}/p^n) \xrightarrow{\cong} K_j^{et}(X, \mathbb{Z}/p^n)$ have been improved by Kahn [Kah, Theoprem 2] provided X is "non-exceptional". He shows that it is an isomorphism if $j \geq cd_2X - 1$ As he points out [Kah, p. 104], these improved bounds will carry over to odd p (without the non-exceptional restriction) assuming the Bloch-Kato conjecture for K holds.

The next step is to observe that the étale Atiyah-Hirzebruch spectral sequence degenerates ($E_2 = E_\infty$) provided $p > (cd_pX/2) + 1$ where cd_pX is the p-cohomological dimension of X, which is at most $2d + 3$ (see [SGA4, Exposé X]). Moreover, the Adams filtration on K-theory and the weight filtration on étale cohomology coincide in a certain range [So2, Theorem 2], so that the left hand side of the above pairing for K^{et} is isomorphic to

$$H^1(\mathcal{O}_{K_n}[1/S], \mathbb{Z}/p^n(1)) \otimes H^i(\bar{X}, \mathbb{Z}/p^n(j-1))$$

provided $p \geq (j + cd_pX + 3)/2$. As $H^i(\bar{X}, \mathbb{Z}/p^n(j-1))$ is a trivial $\mathcal{O}_{K_n}[1/S]$-sheaf, the twist of Definition 1.1.2 yields an isomorphism

$$H^1(\mathcal{O}_{K_n}[1/S], \mathbb{Z}/p^n(1)) \otimes H^i(\bar{X}, \mathbb{Z}/p^m(j-1)) \cong H^1(\mathcal{O}_{K_n}[1/S], H^i(\bar{X}, \mathbb{Z}/p^n(j))).$$

It is now possible to construct elements having property R and N for algebraic K-theory, and to proceed as in the previous section. The p-adic cycle class map has to be replaced by the p-adic regulator (take the inverse limit of [Gi, Definition 2.22, Example 1.4.(iii)])

$$r_p : K_{2j-i-1}(X, \mathbb{Z}_p)^{(j)} \to H^{i+1}(X, \mathbb{Q}_p(j))).$$

B Calculation of the inverse limit of Galois cohomology

Here we give the proof of proposition 1.1.1. Let $G := G_S$ and $H' \subset H \subset G$ subgroups defining K_n and K_m, so that $G/H' \cong G_m$ and $G/H \cong G_n$ (hence H/H' is finite).

By Shapiro's lemma we have

$$H^i(\mathcal{O}_{K_n}[1/S], T) \cong H^i(\mathcal{O}_K[1/S], \mathrm{Hom}_H(G, T)),$$

where $\mathrm{Hom}_H(G,T)$ denotes the continuous maps $f : G \to T$ such that $f(hg) = hf(g)$. The group G acts on this via $(gf)(x) := f(xg)$. The corestriction on the left hand side

$$\mathrm{cores} : H^i(\mathcal{O}_{K_m}[1/S], T) \to H^i(\mathcal{O}_{K_n}[1/S], T)$$

is induced on the right hand side by the map

$$\begin{aligned} \mathrm{tr} : \mathrm{Hom}_{H'}(G,T) &\to \mathrm{Hom}_H(G,T) \\ f &\mapsto \{g \mapsto \sum_{h \in H/H'} hf(h^{-1}g)\} \end{aligned}$$

A straightforward calculation shows that this is well-defined. Consider now T/p^n so that the G action factors through G_n. Define ${}^g f(x) := gf(g^{-1}x)$. Then because $H \subset G$ is a normal subgroup this defines another G action on $\mathrm{Hom}_H(G, T/p^n)$ where H acts trivially. Consider the G-isomorphism

$$\begin{aligned} \phi : \mathrm{Hom}_H(G, T/p^n) &\xrightarrow{\cong} \mathbb{Z}_p[G_n] \otimes_{\mathbb{Z}_p} T/p^n \\ f &\mapsto \sum_{x \in G_n} (x) \otimes f(x^{-1}). \end{aligned}$$

Then we have $gf \mapsto \sum_{x \in G_n} (gx) \otimes f(x^{-1})$ and ${}^g f \mapsto \sum_{x \in G_n} (g^{-1}x) \otimes gf(x^{-1})$. If we put all this together we obtain that the corestriction is induced by

$$\pi \otimes \mathrm{pr} : \mathbb{Z}_p[G_m] \otimes_{\mathbb{Z}_p} T/p^m \to \mathbb{Z}_p[G_n] \otimes_{\mathbb{Z}_p} T/p^n$$

where $\pi : \mathbb{Z}_p[G_m] \to \mathbb{Z}_p[G_n]$ is the canonical surjection (integration over the fibers) and $\mathrm{pr} : T/p^m \to T/p^n$ the canonical projection. This proves that

$$\varprojlim_n H^i(\mathcal{O}_{K_n}[1/S], T) \cong H^i(\mathcal{O}_K[1/S], \Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_p} T).$$

The G action on $\Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_p} T$ is only via the first factor so that

$$H^i(\mathcal{O}_K[1/S], \Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_p} T) \cong H^i(\mathcal{O}_K[1/S], \Lambda(\mathcal{G})) \otimes_{\mathbb{Z}_p} T.$$

Note that the $\Lambda(\mathcal{G})$ action is induced by the G action on T and is by multiplication with the inverse on $\Lambda(\mathcal{G})$. This proves the proposition.

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