Convergence Acceleration of Logarithmically Convergent Series Avoiding Summation*

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Abstract

Quite often in application, logarithmically convergent series have to be evaluated. There are several convergence acceleration methods that are based on the evaluation of partial sums s_n for relatively large n and thus, normally require the evaluation of all terms a_j with $0 \leq j \leq n$. Here, we show that it is possible to avoid the computation of the partials sums of high order if it is possible to evaluate a few terms a_j for relatively large j. The effectiveness of the approach is demonstrated for the $1/z$ expansion that is a particular difficult example of logarithmic convergence.
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1 Description of the Method

Consider an infinite series $s = \sum_{j=0}^{\infty} a_j$ with partials sums $s_n = \sum_{j=0}^{n} a_j$ and terms a_j. If the partial sums satisfy the equation

$$\lim_{n\to\infty} (s_{n+1} - s)/(s_n - s) = \rho \tag{1}$$

the series is called *linearly convergent* if $0 < |\rho| < 1$, and *logarithmically convergent* for $\rho = 1$. Logarithmically convergent series are rather slowly convergent, and often, one tries to use convergence acceleration methods to speed up the convergence. Some important references on this topic are [1–20]. More general references for extrapolation, convergence acceleration, and summation of divergence are [4, 9, 18, 19].

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As shown by Delahaye and Germain-Bonne [5] there is no single method that is able to provide convergence acceleration for all such series. Thus, there is always a need for new methods to supplement the growing set of methods that are successful for larger subsets of the set of logarithmically convergent sequences [7, 17].

There have been attempts to use only linearly convergent subsequences of the logarithmically convergent sequence [3, 16]. In this way, the usual convergence accelerators for linearly convergent sequences become applicable. In many important cases, linearly convergent subsequences of the form s_{R_ℓ} are obtained for (cp. [16] and references therein)

$$R_0 = 1\,, \quad R_{\ell+1} = \lfloor \sigma R_\ell \rfloor + 1\,, \quad \ell = 0, 1, \ldots\,, \text{ for some } \sigma > 1\,. \tag{2}$$

This implies that the R_ℓ grow exponentially like σ^ℓ [16, Eq. (4.16)]. Thus, also the number of required terms grows like σ^ℓ, i.e., exponentially fast. This drawback may largely be avoided by using a method based of interpolation for the transformation to a linearly convergent sequence as shown recently by the author [21].

Here, we discuss a different approach that is related to the model sequence of the E algorithm or Brezinski–Håvie Protocol (BHP) [19, Chap. 10] after its two main investigators Håvie [22] and Brezinski [23]. The model sequence is

$$s_n = s + \sum_{j=1}^{k-1} c_j g_j(n)\,, \qquad n = 0, 1, \ldots\,, \quad k = 2, 3, \ldots\,, \tag{3}$$

with (anti-)limit s, real or complex coefficients c_j that are considered to be unknown, and known functions $g_j(n)$. The BHP is a relatively complicated recursive scheme that allows the elimination of the coefficients c_j and the exact computation of s. When applied to sequences s_n that are not exactly of the form (3), the scheme produces an approximation to the (anti-)limit s, and hence, the algorithm provides a sequence transformation that is nonlinear if the $g_j(n)$ are chosen to depend on the s_n. Alternatively one may consider Eq. (3) for k different values of n as a linear system of equations for the k unknowns s and c_j, $j = 1, \ldots, k-1$, whence determinantal representations for the E algorithm result, and use one of the usual linear solvers for the actual computation of the approximation to the limit s and also, if desired, of the coefficients c_j. For both alternatives, the input data in general are the s_n for k different values of n and, of course, the functions $g_j(n)$ for $j = 1, \ldots, k-1$ at the same set of n values.

Here, we propose to use Eq. (3) for λ different values of n, and to use the equation

$$a_n = \sum_{j=1}^{k-1} c_j \triangle_n g_j(n-1) \tag{4}$$

for the $\mu = k - \lambda$ values $n_1, \ldots, n_\mu$ of n. Equation (4) is obtained from Eq. (3) by taking differences with respect to n using the forward difference operator $\triangle_n$ acting on n-dependent quantities like $\triangle_n x_n = x_{n+1} - x_n$. In this way, a

system of k linear equations is obtained. The approximation to the limit is obtained by solving the linear system for the unknown s. If Eq. (3) is used for $n = 0, 1, \ldots, \lambda - 1$, then the input data are $s_0, \ldots, s_{\lambda-1}, a_{n_1}, \ldots, a_{n_\mu}$ and thus, the algorithm necessitates only $k = \lambda + \mu$ different *terms* of the series.

We remark, that an extension of the method is possible, if it is easy to compute (higher) differences of consecutive terms by using further equations obtained by taking differences with respect to n of Eq. (4) in a similar manner. For instance, this situation often applies if the terms of the series are generated from difference equations or recurrence relations. Thus, to obtain the required k linear equations for the unknowns s and $c_1, \ldots, c_{k-1}$, one may use λ equations of the form Eq. (3) as before, supplemented by μ_ℓ equations of the form

$$\triangle_n^{\ell-1} a_n = \sum_{j=1}^{k-1} c_j \triangle_n^\ell g_j(n-1) \qquad (\triangle_n^\ell = \triangle_n \triangle_n^{\ell-1}) \tag{5}$$

for $\ell = 1, ..., L$ such that $\sum_{\ell=1}^{L} \mu_\ell = k - \lambda$ for suitable values of n. This extension, however, is not used in the following for simplicity.

2 A Numerical Example

As a numerical example, this algorithm was applied to the $1/z$ expansion, i.e., the expansion of $1/z$ in terms of modified Bessel functions $K_\nu(z)$ of the second kind as given by (e.g., [24, Eq. (3.2-32)])

$$s_n = \sqrt{2/\pi} \sum_{j=0}^{n} z^{j-1/2} K_{j-1/2}(z)/(2^j\, j!)\,, \quad \lim_{n\to\infty} s_n = 1/z \tag{6}$$

for $z > 0$ that is a particularly difficult logarithmically convergent series [8],[18, p.349]. We have $s_n - 1/z = O(n^{-1/2})$ for large n. Thus, we use $g_j(n) = (n + 1/2)^{1/2-j}$, and $z = 4/5$ (case 1) or $z = 1/2$ (case 2) in order to be able to compare to literature data. All calculations were done using `MAPLE V`$^{\mathrm{TM}}$ Release 3. Using in case 1 `Digits`=32 and $s_0, \ldots, s_8, a_{11}, a_{14}, a_{18}, a_{23}, a_{29}, a_{37}, a_{47}, a_{59}, a_{74}$ as input, according to $n_j = \lfloor 1.25\, n_{j-1} \rfloor + 1$, i.e., using 18 terms, we obtained 15.05 digits (defined as the negative decadic logarithm of the relative error). Reducing the accuracy to `Digits`=16, still 12.21 digits were obtained, while increasing the accuracy to `Digits`=64 also produced 15.05 digits. Thus, the method is relatively stable. In case 2, for `Digits`=32, $\lambda = 7$, $\mu = 3$ and $n_1 = 14$, $n_2 = 31$, and $n_3 = 69$, i.e., using only 10 terms of the series, the absolute error was $1.54 \cdot 10^{-10}$.

These results compare favorably with many known convergence accelerators [8, 18, 25, 26]. In case 1, the E algorithm for $g_j(n) = (n + 1/2)^{1/2-j}$ (computed as solution of the linear system corresponding to $\lambda = k = 18$ using `Digits`=64) produced only 10.21 digits using the partial sums $s_0, \ldots, s_{17}$. From the same input, all the algorithms tested in [18, Tabs. 14-5, 14-6] (θ algorithm, iterated θ_2 algorithm, Weniger's λ algorithm, three versions of the Levin transformation)

produced not more than 12 exact decimal digits in QUADRUPLE PRECISION corresponding to about 32 digits, and not more than 10 digits in DOUBLE PRECISION corresponding to about 16 digits. Using the same data, the iterative ${}_1\mathbf{J}$ tranformation with $\omega_n = (2n-1)!!/(2n)!!$ produced 13 exact decimal digits in QUADRUPLE PRECISION, and 10 digits in DOUBLE PRECISION. [8, Tab. 6] The latter sequence transformation is only slightly inferior to the best currently known algorithms for this example that are due to Bjørstad, Dahlquist, and Grosse (BDG) [1] and Osada [13]. In case 2, for instance, using 10 terms, the absolute errors are $5.58 \cdot 10^{-8}$ for the BDG algorithm and $9.79 \cdot 10^{-8}$ for the Osada method. [26, Tab. 3, $n = 9$] Thus, our method can at least compete with the best currently known methods in this example.

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