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Helmut Abels, Gerd Grubb and Ian Geoffrey Wood

Extension Theory and Kreĭn-type Resolvent Formulas for Nonsmooth Boundary Value Problems

Helmut Abels,[*] Gerd Grubb,[†] Ian Geoffrey Wood[‡]

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Abstract

For a strongly elliptic second-order operator A on a bounded domain $\Omega \subset \mathbb{R}^n$ it has been known for many years how to interpret the general closed $L_2(\Omega)$-realizations of A as representing boundary conditions (generally nonlocal), when the domain and coefficients are smooth. The purpose of the present paper is to extend this representation to nonsmooth domains and coefficients, including the case of Hölder $C^{\frac{3}{2}+\varepsilon}$-smoothness, in such a way that pseudodifferential methods are still available for resolvent constructions and ellipticity considerations. We show how it can be done for domains with $B_{2,p}^{\frac{3}{2}}$-smoothness and operators with H_q^1-coefficients, for suitable $p > 2(n-1)$ and $q > n$. In particular, Kreĭn-type resolvent formulas are established in such nonsmooth cases. Some unbounded domains are allowed.

Key words: Elliptic boundary value problems; pseudodifferential boundary operators; extension theory; M-functions; symbol smoothing; nonsmooth domains; nonsmooth coefficients
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1 Introduction

The systematic theory of selfadjoint extensions of a symmetric operator in a Hilbert space H, or more generally, adjoint pairs of extensions of a given dual pair of operators in H, has its origin in fundamental works of Kreĭn [35], Vishik [51] and Birman

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[*]Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany, E-mail `helmut.abels@mathematik.uni-regensburg.de`

[†]Department of Mathematical Sciences, Copenhagen University, Universitetsparken 5, DK-2100 Copenhagen, Denmark. E-mail `grubb@math.ku.dk`

[‡]School of Mathematics, Statistics & Actuarial Science, Cornwallis Building, University of Kent, Canterbury, Kent CT2 7NF, UK. E-mail `i.wood@kent.ac.uk`

[13]. There have been several lines of development since then. For one thing, there are the early works of Grubb [23]–[25] completing and extending the theories and giving an implementation to results for boundary value problems for elliptic PDEs. Another line has been the development by, among others, Kochubei [33], Gorbachuk–Gorbachuk [22], Derkach–Malamud [18], Malamud–Mogilevskii [39], where the tendency has been to incorporate the problems into studies of relations (generalizing operators), with applications to (operator valued) ODEs; keywords in this development are boundary triples, Weyl-Titchmarsh m-functions. More recently this has been applied to PDEs (e.g., Amrein-Pearson [9], Behrndt-Langer [11], Kopachevskiĭ-Kreĭn [34], Ryzhov [45], Brown-Marletta-Naboko-Wood [16]). Further references are given in Brown-Grubb-Wood [15], where a connection between the two lines of development is worked out.

One of the interesting aims is to establish Kreĭn resolvent formulas, linking the resolvent of a general operator with the resolvent of a fixed reference operator by expressing the difference in terms of operators connected to boundary conditions, encoding spectral information.

In the applications to elliptic PDEs, Kreĭn-type resolvent formulas are by now well-established in the case of operators with smooth coefficients on smooth domains, but there remain challenging questions about the validity in nonsmooth cases.

One difficulty in implementing the extension theory in nonsmooth cases lies in the fact that one needs mapping properties of direct and inverse operators not only in the most usual Sobolev spaces, but also in spaces of low order, even of negative order over the boundary. Another nontrivial point is to make it possible to profit from ellipticity considerations for the general boundary conditions that appear, leading to regularity results.

Gesztesy and Mitrea have addressed the questions for the Laplacian on Lipschitz domains, showing Kreĭn-type resolvent formulas in [19] and [20] involving Robin problems under the hypothesis that the boundary is of Hölder class $C^{\frac{3}{2}+\varepsilon}$. More recently, they have described the selfadjoint realizations of the Laplacian in [21] (based on the abstract theory of [23]), under a more general hypothesis of quasi-convexity, which includes convex domains and necessitates nonstandard boundary value spaces. Posilicano and Raimondi gave in [43] an analysis of selfadjoint realizations of second-order problems on $C^{1,1}$-domains. Grubb treated nonselfadjoint realizations on $C^{1,1}$-domains in [28], including Neumann-type boundary conditions

$$\chi u = C\gamma_0 u, \tag{1.1}$$

with C a differential operator of order 1, where the other mentioned works mainly treat cases (1.1) with C of order < 1 or nonlocal. ([28] can be considered as a pilot project for the present paper.)

Our aim in this paper is to set up a construction of general extensions and resolvents that works when the regularity of Ω is in a scale of function spaces larger than $\bigcup_{\varepsilon>0} C^{\frac{3}{2}+\varepsilon}$, the coefficients of the elliptic operator A in another larger scale, yet allowing the use of pseudodifferential calculi that can take ellipticity of boundary

conditions into account. We here choose to work with operators having coefficients in scales of Sobolev spaces and their generalizations to Besov and Bessel-potential spaces, since this allows rather precise multiplication properties, and convenient trace mapping results; then Hölder space properties can be read off using the well-known embedding theorems.

The theory of pseudodifferential boundary value problems (originating in Boutet de Monvel [14] and further developed e.g. in the book of Grubb [27]; introductory material is given in [29]) is well-established for operators with C^∞-coefficients on C^∞ domains. It has recently been extended to nonsmooth cases by Abels [2], along the lines of the extension of pseudodifferential operators on open sets in Kumano-Go and Nagase [37], Marshall [40], Taylor [47], [48]. These results have been applied to studies of the Stokes operator in Abels [3] and Abels and Terasawa [5], which in particular imply optimal regularity results for the instationary Stokes system, cf. Abels [4]. For applications to quasi-linear differential equations and free boundary value problems non-smooth coefficients are essential, cf. e.g. Abels [1] and Abels and Terasawa [6]. The present paper builds on [2] and ideas of [5] and develops additional material.

Our final results will be formulated for operators acting between L_2 Sobolev spaces, but along the way we also need L_p-based variants with $p \neq 2$ for the operator- and domain-coefficients. Here the integral exponent will be called p when we describe the domain Ω and its boundary $\Sigma = \partial\Omega$, and q when we describe the given partial differential operators A and boundary operators and their rules of calculus. There is then an optimal choice of how to link p and q, together with the dimension and the smoothness parameters of the spaces where the operators act; this is expressed in Assumption 2.18.

We originally intended to include $2m$-order operators A with $m > 1$, but the coefficients in Green's formula needed an extra, lengthy development of symbol classes that made us postpone this to a future publication.

Plan of the paper. In Section 2, we recall the facts on Besov and Bessel-potential function spaces that we shall need, define the domains with boundary in these smoothness classes, and establish a useful diffeomorphism property. Nonsmooth pseudodifferential operators are recalled, with mapping- and composition-properties, and Green's formula for second-order nonsmooth elliptic operators A on appropriate nonsmooth domains is established. The Appendix gives further information on pseudodifferential boundary operators (ψdbo's) with nonsmooth coefficients, extending some results of [2] to $S_{1,\delta}^m$ classes. Section 3 recalls the abstract extension theory of [23], [25], [15]. In Section 4 we use the ψdbo calculus to construct the resolvent $(A_\gamma - \lambda)^{-1}$ and Poisson solution operator K_γ^λ for the Dirichlet problem in the nonsmooth situation, by localization and parameter-dependent estimates. The construction shows that the principal part of the resolvent belongs to the class of non-smooth pseudodifferential boundary operators, which is essential for the subsequent analysis. Section 5 gives an extension of Green's formula to low-order spaces, and provides an analysis of K_γ^λ and the associated Dirichlet-to-Neumann operator $P_{\gamma,\chi}^\lambda = \chi K_\gamma^\lambda$, needed for the inter-

pretation of the abstract theory. In particular it is shown that the operators coincide with operators of the pseudodifferential calculi up to lower order operators, which is one of the central results of the paper. Finally, the interpretation is worked out in Section 6, leading to a full validity of the characterization of the closed realizations of A in terms of boundary conditions, and including Kreĭn-type resolvent formulas for all closed realizations $\widetilde{A}$. Section 7 gives a special analysis of the Neumann-type boundary conditions (1.1) entering in the theory, showing in particular that regularity of solutions holds when $C - P^0_{\gamma,\chi}$ is elliptic.

2   Basics on function spaces and operators on non-smooth domains

2.1   Function spaces on nonsmooth domains

For convenience we here recall the definitions and properties of function spaces that will be used throughout this paper. Proofs can be found e.g. in Triebel [49] and Bergh and Löfström [12]. All spaces are Banach spaces, some L_2-based spaces are also Hilbert spaces.

The usual multi-index notation for differential operators with $\partial = \partial_x = (\partial_1, \ldots, \partial_n)$, $\partial_j = \partial_{x_j} = \partial/\partial x_j$, and $D = D_x = (D_1, \ldots, D_n)$, $D_j = D_{x_j} = -i\partial/\partial x_j$, will be employed.

For the spaces defined over $\mathbb{R}^n$, the Fourier transform $\mathcal{F}$ is used to define operators such as $p(D_x)u = \mathcal{F}^{-1}(p(\xi)\mathcal{F}u)$ (also called $\mathrm{Op}(p)u$), for suitable functions $p(\xi)$. In particular, with $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$, $\langle D_x\rangle^s$ stands for $(1 - \Delta)^{s/2}$. $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of smooth, rapidly decreasing functions and $\mathcal{S}'(\mathbb{R}^n)$ its dual space, the space of tempered distributions.

Function spaces. The Bessel potential space in $\mathbb{R}^n$ of order $s \in \mathbb{R}$ is defined for $1 < p < \infty$ by

$$H^s_p(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \langle D_x\rangle^s f \in L_p(\mathbb{R}^n) \right\},$$

normed by $\|f\|_{H^s_p(\mathbb{R}^n)} = \|\langle D_x\rangle^s f\|_{L_p(\mathbb{R}^n)}$. For $s = m$, a non-negative integer, $H^m_p(\mathbb{R}^n)$ equals the space of $L_p(\mathbb{R}^n)$-functions with derivatives up to order m in $L_p(\mathbb{R}^n)$, also denoted $W^m_p(\mathbb{R}^n)$. In the case $p = 2$, we omit the lower index and simply write $H^s(\mathbb{R}^n)$ instead of $H^s_2(\mathbb{R}^n)$. We denote the sesquilinear duality pairing of $u \in H^s(\mathbb{R}^n)$ with $v \in H^{-s}(\mathbb{R}^n)$ by $(u, v)_{s,-s}$ (linear in u, conjugate linear in v).

To describe the regularity, both of domains and of operator-coefficients, we shall also need Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, where $s \in \mathbb{R}, 1 \leq p, q \leq \infty$. These are defined by

$B_{p,q}^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B_{p,q}^s(\mathbb{R}^n)} < \infty \right\}$, where

$$\|f\|_{B_{p,q}^s(\mathbb{R}^n)} = \left(\sum_{j=0}^{\infty} 2^{sjq} \|\varphi_j(D_x)f\|_{L_p(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} \quad \text{if } q < \infty,$$

$$\|f\|_{B_{p,\infty}^s(\mathbb{R}^n)} = \sup_{j \in \mathbb{N}_0} 2^{sj} \|\varphi_j(D_x)f\|_{L_p(\mathbb{R}^n)}.$$

Here, φ_j, $j \in \mathbb{N}_0$, is a partition of unity on $\mathbb{R}^n$ such that $\text{supp}\,\varphi_0 \subseteq \{\xi \in \mathbb{R}^n : |\xi| \le 2\}$ and $\text{supp}\,\varphi_j \subseteq \{\xi \in \mathbb{R}^n : 2^{j-1} \le |\xi| \le 2^{j+1}\}$ if $j \in \mathbb{N}$, chosen such that $\varphi_j(\xi) = \varphi_1(2^{1-j}\xi)$ for all $j \in \mathbb{N}$, $\xi \in \mathbb{R}^n$.

The parameter s indicates the smoothness of the functions. The second parameter p is called the integration exponent. The third parameter q is called the summation exponent; it measures smoothness on a *finer scale* than s, which can be seen by the following simple relations:

$$B_{p,1}^s(\mathbb{R}^n) \hookrightarrow B_{p,q_1}^s(\mathbb{R}^n) \hookrightarrow B_{p,q_2}^s(\mathbb{R}^n) \hookrightarrow B_{p,\infty}^s(\mathbb{R}^n) \quad \text{if } 1 \le q_1 \le q_2 \le \infty, \qquad (2.1)$$
$$B_{p,\infty}^{s+\varepsilon}(\mathbb{R}^n) \hookrightarrow B_{p,1}^s(\mathbb{R}^n), \qquad (2.2)$$

where $s \in \mathbb{R}$, $\varepsilon > 0$, and $1 \le p \le \infty$ are arbitrary. (The sign $\hookrightarrow$ indicates continuous embedding.) The embeddings follow directly from the definition and the fact that $\ell_{q_1}(\mathbb{N}_0) \hookrightarrow \ell_{q_2}(\mathbb{N}_0)$ if $1 \le q_1 \le q_2 \le \infty$. Here, $\ell_q(\mathbb{N}_0)$ is the space of sequences $(a_k)_{k \in \mathbb{N}_0}$ such that $(\sum_{k=0}^{\infty} |a_k|^q)^{\frac{1}{q}} < \infty$ in the case $q < \infty$ and $\sup_{k \in \mathbb{N}_0} |a_k| < \infty$ if $q = \infty$, provided with the hereby defined norm.

We recall that for $p = q$ and $s \in \mathbb{R}_+ \setminus \mathbb{N}$, $B_{p,p}^s(\mathbb{R}^n)$ equals the Sobolev-Slobodetskiĭ space $W_p^s(\mathbb{R}^n)$, whereas for $s \in \mathbb{N}_0$, it is $H_p^s(\mathbb{R}^n)$ that equals $W_p^s(\mathbb{R}^n)$. (In the following, the H- and B-notation will be used for clarity; these scales of spaces have the best interpolation properties.) In the case $p = 2$, all three spaces coincide, for general s:

$$H^s(\mathbb{R}^n) = H_2^s(\mathbb{R}^n) = B_{2,2}^s(\mathbb{R}^n) = W_2^s(\mathbb{R}^n). \qquad (2.3)$$

The spaces $B_{\infty,\infty}^s(\mathbb{R}^n)$, also denoted $\mathcal{C}^s(\mathbb{R}^n)$ when $s > 0$ (Hölder-Zygmund spaces), play a special role. For $s \in \mathbb{R}_+ \setminus \mathbb{N}$, $B_{\infty,\infty}^s(\mathbb{R}^n) = \mathcal{C}^s(\mathbb{R}^n)$ can be identified with the Hölder space $C^{k,\sigma}(\mathbb{R}^n)$, defined for $k + \sigma = s$, $k \in \mathbb{N}_0$ and $\sigma \in (0,1]$, and also denoted $C^s(\mathbb{R}^n)$ when $\sigma \in (0,1)$. For $s = k \in \mathbb{N}$, there are sharp inclusions

$$C_b^k(\mathbb{R}^n) \hookrightarrow C^{k-1,1}(\mathbb{R}^n) \hookrightarrow \mathcal{C}^k(\mathbb{R}^n);$$

here, $C_b^k(\mathbb{R}^n)$ is the usual space of bounded continuous functions with bounded continuous derivatives up to order k.

At this point, let us recall some interpolation results: Denoting the real and complex interpolation functors by $(.,.)_{\theta,q}$ and $(.,.)_{[\theta]}$, respectively, we have that if $s_0, s_1 \in \mathbb{R}$ with $s_0 \ne s_1$, $1 \le p, q_0, q_1, r \le \infty$, and $s = (1-\theta)s_0 + \theta s_1$, $\theta \in (0,1)$, then

$$(B_{p,q_0}^{s_0}(\mathbb{R}^n), B_{p,q_1}^{s_1}(\mathbb{R}^n))_{\theta,r} = B_{p,r}^s(\mathbb{R}^n). \qquad (2.4)$$

If additionally $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ for some $1 \le p_0, p_1 \le \infty$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, then

$$(B_{p_0,q_0}^{s_0}(\mathbb{R}^n), B_{p_1,q_1}^{s_1}(\mathbb{R}^n))_{[\theta]} = B_{p,q}^s(\mathbb{R}^n), \tag{2.5}$$

cf. [12, Theorem 6.4.5] or [49, Section 2.4.1 Theorem]. Using the same notation, we have in particular for the Bessel potential spaces

$$(H_p^{s_0}(\mathbb{R}^n), H_p^{s_1}(\mathbb{R}^n))_{\theta,r} = B_{p,r}^s(\mathbb{R}^n), \tag{2.6}$$
$$(H_{p_0}^{s_0}(\mathbb{R}^n), H_{p_1}^{s_1}(\mathbb{R}^n))_{[\theta]} = H_p^s(\mathbb{R}^n)$$

(cf. [12, Theorem 6.4.5]).

General embedding properties. For any $1 < p < \infty$, we have the following embeddings between Besov spaces and Bessel potential spaces:

$$B_{p,q_1}^{s+\varepsilon}(\mathbb{R}^n) \hookrightarrow H_p^s(\mathbb{R}^n) \hookrightarrow B_{p,q_2}^{s-\varepsilon}(\mathbb{R}^n) \qquad \text{for all } \varepsilon > 0, 1 \le q_1, q_2 \le \infty, s \in \mathbb{R},$$
$$B_{p,\min(2,p)}^s(\mathbb{R}^n) \hookrightarrow H_p^s(\mathbb{R}^n) \hookrightarrow B_{p,\max(2,p)}^s(\mathbb{R}^n) \qquad \text{for all } s \in \mathbb{R}, \tag{2.7}$$

cf. e.g. [12, Theorem 6.4.4].

There are the following Sobolev embeddings for Bessel potential spaces:

$$H_{p_1}^{s_1}(\mathbb{R}^n) \hookrightarrow H_{p_0}^{s_0}(\mathbb{R}^n) \qquad \text{if } s_1 \ge s_0, s_1 - \frac{n}{p_1} \ge s_0 - \frac{n}{p_0}, \tag{2.8}$$
$$H_p^s(\mathbb{R}^n) \hookrightarrow \mathcal{C}^\alpha(\mathbb{R}^n) \qquad \text{if } \alpha = s - \frac{n}{p} > 0, \tag{2.9}$$

provided that $1 < p_1 \le p_0 < \infty$, $1 < p < \infty$. In particular, $H_p^s(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n)$ for $s \ge 0$.

For the Besov spaces a Sobolev-type embedding is given by

$$B_{p_1,q}^{s_1}(\mathbb{R}^n) \hookrightarrow B_{p_0,q}^{s_0}(\mathbb{R}^n) \qquad \text{if } s_1 \ge s_0 \text{ and } s_1 - \frac{n}{p_1} \ge s_0 - \frac{n}{p_0}, \tag{2.10}$$

for any $1 \le q \le \infty$. In particular, combining this with (2.1), we get

$$B_{p_1,q}^{s_1}(\mathbb{R}^n) \hookrightarrow B_{\infty,q}^{s_0}(\mathbb{R}^n) \hookrightarrow B_{\infty,\infty}^{s_0}(\mathbb{R}^n) = \mathcal{C}^{s_0}(\mathbb{R}^n) \tag{2.11}$$

whenever $s_0 = s_1 - \frac{n}{p_1} > 0$. In the opposite direction, we have from (2.1) and (2.2)

$$\mathcal{C}^\alpha(\mathbb{R}^n) = B_{\infty,\infty}^\alpha(\mathbb{R}^n) \hookrightarrow B_{\infty,2}^{\alpha-\varepsilon}(\mathbb{R}^n) \tag{2.12}$$

when $\alpha > 0$, $0 < \varepsilon < \alpha$. We also note that

$$H_{p_1}^{s_1}(\mathbb{R}^n) \cup B_{p_1,p_1}^{s_1}(\mathbb{R}^n) \hookrightarrow H_{p_0}^{s_0}(\mathbb{R}^n) \cap B_{p_0,p_0}^{s_0}(\mathbb{R}^n) \tag{2.13}$$

if $1 < p_1 < p_0 < \infty$ and $s_1 - \frac{n}{p_1} \ge s_0 - \frac{n}{p_0}$; this can be found in [49, Section 2.8.1, equation (17)].

Function spaces over subsets of $\mathbb{R}^n$. The Bessel potential and Besov spaces are defined on a domain $\Omega \subset \mathbb{R}^n$ with $C^{0,1}$-boundary (see Definition 2.4 below) simply by restriction:

$$
\begin{aligned}
H_p^s(\Omega) &= \{f \in \mathcal{D}'(\Omega) : f = f'|_\Omega, f' \in H_p^s(\mathbb{R}^n)\}, & (2.14) \\
B_{p,q}^s(\Omega) &= \{f \in \mathcal{D}'(\Omega) : f = f'|_\Omega, f' \in B_{p,q}^s(\mathbb{R}^n)\},
\end{aligned}
$$

for $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. Here $f'|_\Omega \in \mathcal{D}'(\Omega)$ is defined by $\langle f'|_\Omega, \varphi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \langle f, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)}$ for all $\varphi \in C_0^\infty(\Omega)$, embedded in $C_0^\infty(\mathbb{R}^n)$ by extension by zero. The spaces are equipped with the quotient norms, e.g.,

$$
\|f\|_{B_{p,q}^s(\Omega)} = \inf_{f' \in B_{p,q}^s(\mathbb{R}^n):f'|_\Omega=f} \|f'\|_{B_{p,q}^s(\mathbb{R}^n)}. \tag{2.15}
$$

In particular, $H_p^m(\Omega)$ is for $m \in \mathbb{N}_0$ and $1 < p < \infty$ equal to the usual Sobolev space $W_p^m(\Omega)$ of $L_p(\Omega)$-functions with derivatives up to order m in $L_p(\Omega)$. We recall that there is an extension operator E_Ω which is a bounded linear operator $E_\Omega \colon W_p^m(\Omega) \to W_p^m(\mathbb{R}^n)$, for all $m \in \mathbb{N}_0$, $1 \leq p \leq \infty$, and satisfies $E_\Omega f|_\Omega = f$ for all $f \in W_p^m(\Omega)$. This holds when Ω is merely a Lipschitz domain, cf. e.g. Stein [46, Chapter VI, Section 3.2] and trivially carries over to $H_p^m(\Omega)$ for $1 < p < \infty$. Moreover, in view of the fact that $H_p^s(\Omega)$ is a retract of $H_p^s(\mathbb{R}^n)$, one has that all interpolation and Sobolev embedding results for $H_p^s(\mathbb{R}^n)$ are inherited by the spaces on Ω.

We shall also need the spaces

$$
H_0^s(\overline{\Omega}) = \{u \in H^s(\mathbb{R}^n) : \operatorname{supp} u \subset \overline{\Omega}\}.
$$

Here, $H_0^s(\overline{\Omega})$ identifies in a natural way with the dual space of $H^{-s}(\Omega)$, for all $s \in \mathbb{R}$, cf. [42, Theorem 3.30]. For s integer ≥ 0, $H_0^s(\overline{\Omega})$ equals the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ and is usually denoted $H_0^s(\Omega)$ (see also [42, Theorem 3.33]).

Traces. Next, let us recall the well-known trace theorems: The trace map γ_0 from $\mathbb{R}_+^n$ to $\mathbb{R}^{n-1}$, defined on smooth functions with bounded support, extends by continuity to continuous maps for $s > \frac{1}{p}$, $1 < p < \infty$, $1 \leq q \leq \infty$,

$$
\begin{aligned}
\gamma_0 \colon H_p^s(\mathbb{R}_+^n) &\to B_{p,p}^{s-\frac{1}{p}}(\mathbb{R}^{n-1}), \\
\gamma_0 \colon B_{p,q}^s(\mathbb{R}_+^n) &\to B_{p,q}^{s-\frac{1}{p}}(\mathbb{R}^{n-1}).
\end{aligned}
$$

All of these maps are surjective and have continuous right inverses.

Vector-valued Besov and Bessel potential spaces. In the following let X be a Banach space. Then $L_p(\mathbb{R}^n; X)$, $1 \leq p < \infty$, is defined as the space of strongly measurable functions $f \colon \mathbb{R}^n \to X$ with

$$
\|f\|_{L_p(\mathbb{R}^n;X)} := \left(\int_{\mathbb{R}^n} \|f(x)\|_X^p \, dx \right)^{\frac{1}{p}} < \infty
$$

and $L_\infty(\mathbb{R}^n; X)$ is the space of all strongly measurable and essentially bounded functions. Similarly, $\ell_p(\mathbb{N}_0; X)$, $1 \le p \le \infty$, denotes the X-valued variant of $\ell_p(\mathbb{N}_0)$.

Furthermore, let $\mathcal{S}(\mathbb{R}^n; X)$ be the space of smooth rapidly decreasing functions $f \colon \mathbb{R}^n \to X$ and let $\mathcal{S}'(\mathbb{R}^n; X) := \mathcal{L}(\mathcal{S}(\mathbb{R}^n), X)$ denote the space of tempered X-valued distributions. Then the X-valued variants of the Bessel potential and Besov spaces of order $s \in \mathbb{R}$ are defined as

$$H_p^s(\mathbb{R}^n; X) := \{f \in \mathcal{S}'(\mathbb{R}^n; X) : \langle D_x \rangle^s f \in L_p(\mathbb{R}^n; X)\} \quad \text{if } 1 < p < \infty,$$

$$B_{p,q}^s(\mathbb{R}^n; X) := \{f \in \mathcal{S}'(\mathbb{R}^n; X) : (2^{sj} \varphi_j(D_x) f)_{j \in \mathbb{N}_0} \in \ell_q(\mathbb{N}_0; L_p(\mathbb{R}^n; X))\},$$

where $1 \le p, q \le \infty$. Here, $p(D_x) f \in \mathcal{S}'(\mathbb{R}^n; X)$ is defined by

$$\langle p(D_x) f, \overline{\varphi} \rangle = \langle f, \overline{p(D_x)\varphi} \rangle \qquad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

We will also make use of the Banach space

$$BUC(\mathbb{R}^n; X) = \{f \in L_\infty(\mathbb{R}^n; X) : f \text{ is uniformly continuous}\}$$

with the supremum norm.

The properties of the function spaces discussed above for the scalar case, carry over to the vector-valued case. For details, we refer to e.g. [10, 31] (for the Bochner integral and its properties) and to [8] (for vector-valued function spaces).

In the following we will use some special anisotropic Sobolev spaces.

Definition 2.1 *Let $I = (0, \infty)$ or $\mathbb{R}$, and let $\Omega = \mathbb{R}^{n-1} \times I$, with coordinates (x', x_n). For $k \in \mathbb{N}_0$, $1 \le p < \infty$, set*

$$W_{(2,p)}^k(\Omega) = \left\{ f \in L_2(I; L_p(\mathbb{R}^{n-1})) : \partial_x^\alpha f \in L_2(I; L_p(\mathbb{R}^{n-1})), |\alpha| \le k \right\}.$$

Lemma 2.2 *One has for $k \ge 1$ that*

$$W_{(2,p)}^k(\Omega) \quad \hookrightarrow \quad BUC(I; B_{p,2}^{k-\frac{1}{2}}(\mathbb{R}^{n-1})). \tag{2.16}$$

Here, the trace mapping $u \mapsto u|_{x_n=0}$ is surjective from $W_{(2,p)}^k(\Omega)$ to $B_{p,2}^{k-\frac{1}{2}}(\mathbb{R}^{n-1})$. Namely, when $g(x') \in B_{p,2}^{k-\frac{1}{2}}(\mathbb{R}^{n-1})$, then $G(x', x_n) = (e^{-Ax_n} g)(x')$ is in $W_{(2,p)}^k(\mathbb{R}_+^n)$ with $G(x', 0) = g(x')$ (where e^{-Ax_n} is the semigroup generated by $-A = -\langle D_{x'} \rangle$). $G(x', x_n)$ extends to a function $G \in W_{(2,p)}^k(\mathbb{R}^n)$.

Proof: First of all,

$$W_{(2,p)}^k(\Omega) \quad \hookrightarrow \quad L_2(I; H_p^k(\mathbb{R}^{n-1})) \cap H^k(I; L_p(\mathbb{R}^{n-1})).$$

To obtain

$$L_2(I; H_p^k(\mathbb{R}^{n-1})) \cap H^k(I; L_p(\mathbb{R}^{n-1})) \hookrightarrow BUC(I; B_{p,2}^{k-\frac{1}{2}}(\mathbb{R}^{n-1}))$$

one can apply [12, Corollary 3.12.3] with $\eta_j = \frac{1}{2}, p_j = 2, j = 0, 1$, a result from Lions' trace method of real interpolation, to obtain

$$u|_{x_n=0} \in (L_p(\mathbb{R}^{n-1}), H_p^k(\mathbb{R}^{n-1}))_{1-\frac{1}{2k},2} = B_{p,2}^{k-\frac{1}{2}}(\mathbb{R}^{n-1}), \qquad (2.17)$$

for every $u \in W_{(2,p)}^k(\Omega)$; the identity follows from (2.6). Next, this is combined with the strong continuity of the translations $(\tau_h u)(x) = (x', x_n + h)$, $h \geq 0$, in $L_2(I; H_p^k(\mathbb{R}^{n-1})) \cap H^k(I; L_p(\mathbb{R}^{n-1}))$ as in the proof of [8, Chapter III, Theorem 4.10.2].

For the last assertion, let $A^s = \langle D_{x'} \rangle^s$, $s \in \mathbb{R}$; here $A = A^1$. Then

$$H_p^k(\mathbb{R}^{n-1}) = \{f \in L_p(\mathbb{R}^{n-1}) : \langle D_{x'} \rangle^k f \in L_p(\mathbb{R}^{n-1})\} =: D(A^k)$$

for all $k \in \mathbb{N}_0$, $1 < p < \infty$, as explained in [12, Theorem 6.2.3]. Now when g is given, let $G(x', x_n) = (e^{-Ax_n}g)(x')$ for $x_n \geq 0$. Since

$$W_{(2,p)}^k(\mathbb{R}_+^n) = \bigcap_{0 \leq j \leq k} H^j(0, \infty; H_p^{k-j}(\mathbb{R}^{n-1})),$$

we have by [17, Corollary 3.5.6, Theorem 3.4.2] that $A^k G \in L^2(0, \infty; L_p(\mathbb{R}^{n-1}))$, so $G \in W_{(2,p)}^k(\mathbb{R}_+^n)$. Here, we use that $B_{p,2}^{k-\frac{1}{2}}(\mathbb{R}^{n-1}) = (L_p(\mathbb{R}^{n-1}), H_p^k(\mathbb{R}^{n-1}))_{1-\frac{1}{2k},2}$, as noted above in (2.17). $G(x', x_n)$ is extended to a function in $W_{(2,p)}^k(\mathbb{R}^n)$ by a standard "reflection" in $x_n = 0$, as explained e.g. in [38, Th. I 2.2]. ∎

By use of this lemma we derive the following product estimate, which is essential for the low boundary regularity that we shall allow:

Lemma 2.3 *Let I, Ω and $W_{(2,p)}^k(\Omega)$ be as in Definition 2.1. Then for every $k \in \mathbb{N}$ and $2 \leq p < \infty$ such that $k - \frac{1}{2} - \frac{n-1}{p} > 0$ there is some $C_{k,p} > 0$ such that*

$$\|fg\|_{H^k(\Omega)} \leq C_{k,p} \|f\|_{H^k(\Omega)} \|g\|_{W_{(2,p)}^k(\Omega)},$$

for all $f \in H^k(\Omega)$, $g \in W_{(2,p)}^k(\Omega)$. Moreover, if $k = 1$ and $\tau = \frac{1}{2} - \frac{n-1}{p}$, then

$$\|f \partial_{x_j} g\|_{L_2(\Omega)} \leq C_p \|f\|_{H^{1-\tau}(\Omega)} \|g\|_{W_{(2,p)}^1(\Omega)} \qquad (2.18)$$

uniformly with respect to $f \in H^1(\Omega), g \in W_{(2,p)}^1(\Omega)$.

Proof: First of all

$$W_{(2,p)}^k(\Omega) \hookrightarrow BUC(I; B_{p,2}^{k-\frac{1}{2}}(\mathbb{R}^{n-1})) \hookrightarrow L_\infty(\Omega)$$

by (2.16); the second embedding follows from (2.11) since $k - \frac{1}{2} - \frac{n-1}{p} > 0$. Furthermore,

$$H^k(\Omega) \hookrightarrow BUC(I; H^{k-\frac{1}{2}}(\mathbb{R}^{n-1})) \hookrightarrow BUC(I; L_r(\mathbb{R}^{n-1})), \text{ for } \frac{1}{r} = \frac{1}{2} - \frac{1}{p}.$$

Here we apply (2.16) with $p = 2$ for the first embedding, and for the second embedding we use (2.8), noting that $k - \frac{1}{2} - \frac{n-1}{p} > 0$ is equivalent to $k - \frac{1}{2} - \frac{n-1}{2} > -\frac{n-1}{r}$. Next we observe that for all $|\alpha| \leq k$

$$\partial_x^\alpha(fg) = \sum_{0 \leq \beta \leq \alpha, \beta \neq \alpha} \binom{\alpha}{\beta} \partial_x^{\alpha-\beta} f \partial_x^\beta g + f \partial_x^\alpha g. \tag{2.19}$$

Since $f \in L_\infty(I; L_r)$ and $\partial_x^\alpha g \in L_2(I; L_p)$, where $\frac{1}{r} + \frac{1}{p} = \frac{1}{2}$, we have $f \partial_x^\alpha g \in L_2(\Omega)$. For the other terms where $|\beta| < |\alpha| \leq k$, we note that

$$\partial_x^{\alpha-\beta} f \in L_2(I; H^{k-|\alpha|+|\beta|}(\mathbb{R}^{n-1})), \quad \partial_x^\beta g \in L_\infty(I; B_{p,2}^{k-\frac{1}{2}-|\beta|}(\mathbb{R}^{n-1})). \tag{2.20}$$

One has in general

$$\|uv\|_{L_2(\mathbb{R}^{n-1})} \leq C_{M,M'} \|u\|_{H^M(\mathbb{R}^{n-1})} \|v\|_{B_{p,2}^{M'}(\mathbb{R}^{n-1})}, \tag{2.21}$$

provided that $M, M' \in \mathbb{N}_0$ and $M + M' - \frac{n-1}{p} > 0$. This estimate easily follows from the Sobolev-type embedding theorems: If $M' - \frac{n-1}{p} > \varepsilon > 0$, then from (2.11) we have $B_{p,2}^{M'}(\mathbb{R}^{n-1}) \hookrightarrow \mathcal{C}^\varepsilon(\mathbb{R}^{n-1}) \hookrightarrow L_\infty(\mathbb{R}^{n-1})$ and the statement is trivial. If $M' - \frac{n-1}{p} < 0$, then (2.13) implies $B_{p,2}^{M'}(\mathbb{R}^{n-1}) \hookrightarrow L_r(\mathbb{R}^{n-1})$ with $\frac{1}{r} = \frac{1}{p} - \frac{M'}{n-1}$, and $M + M' - \frac{n-1}{p} > 0$ implies by (2.8) that $H^M(\mathbb{R}^{n-1}) \hookrightarrow L_{\tilde{r}}(\mathbb{R}^{n-1})$ with $\frac{1}{2} = \frac{1}{r} + \frac{1}{\tilde{r}}$. If $M' - \frac{n-1}{p} = 0$, one can choose some $\tilde{p} < p$ such that $M + M' - \frac{n-1}{\tilde{p}} > 0$ and apply the preceding case. The estimate is also a consequence of Hanouzet [30, Théorème 3].

Using (2.21) for products of functions as in (2.20), we obtain altogether that $\partial_x^{\alpha-\beta} f \partial_x^\beta g \in L_2(\mathbb{R}^{n-1})$ for all $0 \leq \beta \leq \alpha, |\alpha| \leq k$.

Finally, if $k = 1$, then we have that

$$H^{1-\tau}(\Omega) \hookrightarrow BUC(I; H^{\frac{1}{2}-\tau}(\mathbb{R}^{n-1})) \hookrightarrow BUC(I; L_r(\mathbb{R}^{n-1})), \ \frac{1}{r} = \frac{1}{2} - \frac{1}{p}.$$

Therefore

$$\|f \partial_{x_j} g\|_{L_2(\Omega)} \leq \|f\|_{L_\infty(I; L_r)} \|\partial_{x_j} g\|_{L_2(I, L_p)} \leq C \|f\|_{H^{1-\tau}(\Omega)} \|g\|_{W^1_{(2,p)}(\Omega)},$$

which proves the last statement.   ∎

Domains with nonsmooth boundary. For the following, let $n \geq 2$, let M be a positive integer, and let $1 \leq p, q \leq \infty$ be such that $M - \frac{3}{2} - \frac{n-1}{p} > 0$.

Definition 2.4 *Let Ω be an open subset of $\mathbb{R}^n$. We say that Ω has a boundary of class $B_{p,q}^{M-\frac{1}{2}}$ in the following three cases:*
 1° $\Omega = \mathbb{R}^n_\gamma$, *where*
$$\mathbb{R}^n_\gamma = \{x \in \mathbb{R}^n : x_n > \gamma(x')\}$$

for a function $\gamma \in B_{p,q}^{M-\frac{1}{2}}(\mathbb{R}^{n-1})$.

$2°$ $\partial\Omega$ *is compact, and each* $x \in \partial\Omega$ *has an open neighborhood* U *satisfying: For a suitable choice of coordinates on* $\mathbb{R}^n$, *there is a function* $\gamma(x') \in B_{p,q}^{M-\frac{1}{2}}(\mathbb{R}^{n-1})$ *such that* $U \cap \Omega = U \cap \mathbb{R}_\gamma^n$ *and* $U \cap \partial\Omega = U \cap \partial\mathbb{R}_\gamma^n$.

$3°$ *For a large ball* $B_R = \{x \in \mathbb{R}^n : |x| < R\}$, $\Omega \setminus B_R$ *equals* $\mathbb{R}_+^n \setminus B_R$. *The points* $x \in B_{R+1} \cap \partial\Omega$ *have the property described in* $2°$.

There are similar definitions with other function spaces.

In the second case, one can cover $\partial\Omega$ by a finite set of such coordinate neighborhoods U. Note that exterior domains are allowed. The third case is included in order to show a simple case with noncompact boundary where a finite system of coordinate neighborhoods suffices (namely finitely many U's covering $\partial\Omega \cap B_{R+1}$ and a trivial one covering $\partial\Omega \setminus B_R$), to describe the smoothness structure. More general such cases can be defined as the "admissible manifolds" in [27].

We shall work under the following general hypothesis:

Assumption 2.5 $n \geq 2$, $M \in \mathbb{N}$, $2 \leq p < \infty$, *with*

$$\tau := M - \tfrac{3}{2} - \tfrac{n-1}{p} > 0. \tag{2.22}$$

Moreover, Ω *is an open subset of* $\mathbb{R}^n$ *with boundary* $\partial\Omega$ *of regularity* $B_{p,2}^{M-\frac{1}{2}}$, *as in Definition 2.4.*

Remark 2.6 Under Assumption 2.5, it follows from (2.11) that $\partial\Omega$ is Hölder continuous with exponent $1+\tau$ (if $\tau \notin \mathbb{N}$). In the converse direction, if $\partial\Omega$ is Hölder continuous with exponent $M - \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$, then in view of (2.12), $\partial\Omega \in B_{\infty,2}^{M-\frac{1}{2}}$. This in turn implies $\partial\Omega \in B_{p,2}^{M-\frac{1}{2}}$ for every $1 \leq p \leq \infty$ if Ω is of type $2°$ or $3°$ in Definition 2.4, since $L^\infty(U) \hookrightarrow L^p(U)$ for every $1 \leq p \leq \infty$ when U is bounded. In other words,

$$\partial\Omega \in C^{M-\frac{1}{2}+\varepsilon} \implies \partial\Omega \in B_{p,2}^{M-\frac{1}{2}} \implies \partial\Omega \in C^{M-\frac{1}{2}-\frac{n-1}{p}} = C^{1+\tau}.$$

if Ω is of type $2°$ or $3°$ and $\tau \notin \mathbb{N}$.

When U and V are subsets of $\mathbb{R}^n$, and $F: V \to U$ is a bijection, we denote the pull-back mapping by F^*:

$$(F^*u)(x) = u(F(x)) \text{ for } x \in V, \quad (F^{-1,*}v)(y) = v(F^{-1}(y)) \text{ for } y \in U,$$

when u is a function on U, v is a function on V. The gradient $\nabla u = (\partial_j u)_{j=1}^n$ is viewed as a column vector.

Proposition 2.7 *Under Assumption 2.5, let* $\gamma \in B_{p,2}^{M-\frac{1}{2}}(\mathbb{R}^{n-1})$, *and let* $\mathbb{R}_\gamma^n = \{x \in \mathbb{R}^n : x_n > \gamma(x')\}$. *Then there is a* C^1-*diffeomorphism* $F_\gamma: \mathbb{R}^n \to \mathbb{R}^n$ *with* $\nabla F_\gamma \in C^{\tau'}(\mathbb{R}^n)^{n^2}$ *for* $\tau' \leq \tau$, $\tau' \in (0, \frac{1}{2})$ *(cf. (2.22)), such that* $F_\gamma(\mathbb{R}_+^n) = \mathbb{R}_\gamma^n$ *and* $F_\gamma^*: H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ *as well as* $F_\gamma^*: H^s(\mathbb{R}_\gamma^n) \to H^s(\mathbb{R}_+^n)$ *for all* $0 \leq s \leq M$.

Proof: We begin by defining $\Gamma(x', x_n)$ as the lifting of $\gamma(x')$ by the construction described in the last statement in Lemma 2.2; then $\Gamma \in W_{(2,p)}^M(\mathbb{R}^n)$. In particular, this implies that

$$\nabla\Gamma \in L_2(\mathbb{R}; H_p^{M-1}(\mathbb{R}^{n-1}))^n \cap H^{M-1}(\mathbb{R}; L_p(\mathbb{R}^{n-1}))^n$$

$$\hookrightarrow BUC(\mathbb{R}; B_{p,2}^{M-\frac{3}{2}}(\mathbb{R}^{n-1}))^n \hookrightarrow BUC(\mathbb{R}; C_b^0(\mathbb{R}^{n-1}))^n$$

in view of (2.16); we here use that $B_{p,2}^{M-\frac{3}{2}}(\mathbb{R}^{n-1}) \hookrightarrow \mathcal{C}^\tau(\mathbb{R}^{n-1}) \hookrightarrow C_b^0(\mathbb{R}^{n-1})$ since $\tau = M - \frac{3}{2} - \frac{n-1}{p} > 0$. Hence $\Gamma \in C_b^1(\mathbb{R}^n)$. Moreover, we have from (2.10)

$$\nabla\Gamma \in H^1(\mathbb{R}; H_p^{M-2}(\mathbb{R}^{n-1}))^n \hookrightarrow C^{\frac{1}{2}}(\mathbb{R}; H_p^{M-2}(\mathbb{R}^{n-1}))^n.$$

Due to (2.8), we have $H_p^{M-2}(\mathbb{R}^{n-1}) \hookrightarrow B_{p,p}^{M-2}(\mathbb{R}^{n-1})$ and using $\nabla\Gamma \in BUC(\mathbb{R}; B_{p,2}^{M-\frac{3}{2}}(\mathbb{R}^{n-1}))^n$, (2.4) yields

$$\nabla\Gamma \in C^{\tau'}(\mathbb{R}; B_{p,1}^{(n-1)/p}(\mathbb{R}^{n-1}))^n \hookrightarrow C^{\tau'}(\mathbb{R}; C_b^0(\mathbb{R}^{n-1}))^n,$$

where $\tau' = \tau$ if $0 < \tau < \frac{1}{2}$ and $\tau' \in (0, \frac{1}{2})$ is arbitrary otherwise. Here one uses the general estimate

$$
\begin{aligned}
\|f(t) - f(s)\|_X &\leq C\|f(t) - f(s)\|_{X_0}^{1-\theta}\|f(t) - f(s)\|_{X_1}^\theta \\
&\leq C\|f\|_{BUC(\mathbb{R};X_0)}^{1-\theta}\|f\|_{C^{\frac{1}{2}}(\mathbb{R};X_1)}^\theta|t - s|^{\theta/2}
\end{aligned}
$$

for all $t, s \in \mathbb{R}$, where $X = (X_0, X_1)_{\theta,1}$. Thus $\nabla\Gamma \in C^{\tau'}(\mathbb{R}^n)^n$. Note that similarly

$$W_{(2,p)}^{M-1}(\mathbb{R}^n) \hookrightarrow C^{\tau'}(\mathbb{R}^n). \tag{2.23}$$

Now we define, for some $\lambda > 0$,

$$F_\gamma(x) = x + \begin{pmatrix} 0 \\ \Gamma(x', \lambda x_n) \end{pmatrix} = \big(F_{\gamma,k}(x)\big)_{k=1}^n.$$

Then $\partial_{x_n} F_{\gamma,n}(x) = 1 + \lambda(\partial_{x_n}\Gamma)(x', \lambda x_n)$. Hence, if λ is sufficiently small, $F_{\gamma,n}(x', \cdot)\colon \mathbb{R} \to \mathbb{R}$ is strictly increasing and surjective for every $x' \in \mathbb{R}^{n-1}$. Therefore $F_\gamma \colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-diffeomorphism with $F_\gamma(\mathbb{R}_+^n) = \mathbb{R}_\gamma^n$, and $\nabla F_\gamma = (\partial_{x_j} F_{\gamma,k})_{j,k=1}^n$ (the transposed functional matrix) is in $C^{\tau'}(\mathbb{R}^n)^{n^2}$.

Next let $u \in H^k(\Omega)$, $0 \leq k \leq M$, $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}_\gamma^n$. We prove $F_\gamma^*(u) \in H^k(\Omega)$ by mathematical induction. If $k = 0$, then the statement is true since F_γ is a C^1-diffeomorphism. For $u \in H^1(\Omega)$,

$$\partial_{x_j}(u(F_\gamma(x))) = (\nabla u)(F_\gamma(x)) \cdot \partial_{x_j} F_\gamma(x) \tag{2.24}$$

where $(\nabla u)(F_\gamma(x)) \in L_2(\Omega)^n$ by the argument for $k = 0$ and and $\partial_{x_j} F_\gamma(x) \in C_b^0(\Omega)^n$. Next we assume that the statement is true for some $1 \leq k < M$. Then for

$u \in H^{k+1}(\Omega)$, we have $(\nabla u)(F_\gamma(x)) \in H^k(\Omega)^n$ by the assumption, and $\partial_{x_j} F_\gamma(x) \in W_{(2,p)}^{M-1}(\Omega)^n \hookrightarrow W_{(2,p_k)}^k(\Omega)^n$ for some $2 \le p_k < \infty$ with $k - \frac{n-1}{p_k} - \frac{1}{2} > 0$. Therefore Lemma 2.3 implies that $\partial_{x_j}(u(F_\gamma(x)) \in H^k(\mathbb{R}_+^n)$ for all $j = 1, \ldots, n$, which implies the statement for $k + 1 \le M$. This proves the statement for integer $0 \le k \le M$. For real $0 \le s \le M$ the statement follows by interpolation. ∎

For the case $M = 2$, we specify the result in the following corollary. We use the notation $C : D = \sum_{j,k=1}^n c_{jk} d_{jk} = \mathrm{tr}(C^T D)$ for the "scalar product" of square matrices $C = (c_{jk})_{j,k=1}^n$ and $D = (d_{jk})_{j,k=1}^n$. $\partial^2 u$ stands for the Hessian $(\partial_{x_j} \partial_{x_k} u)_{j,k=1}^n$, and e_j is the j-th coordinate vector written as a column.

Corollary 2.8 *Let $\gamma \in B_{p,2}^{\frac{3}{2}}(\mathbb{R}^{n-1})$, with $2 \le p < \infty$, $\frac{1}{2} - \frac{n-1}{p} > 0$, and let $\mathbb{R}_\gamma^n$ and F_γ be as in Proposition 2.7. Then for every $j, k = 1, \ldots, n$,*

$$
\begin{aligned}
F_\gamma^* \nabla u &= \Phi(x) \nabla F_\gamma^* u, && (2.25) \\
F_\gamma^* \partial_{x_j} \partial_{x_k} u &= \Phi_{j,k}(x) : \partial^2 F_\gamma^* u + Ru,
\end{aligned}
$$

where $\Phi(x) = (\nabla F_\gamma(x))^{-1} \in W_{(2,p)}^1(\mathbb{R}_+^n)^{n^2}$ and $\Phi_{j,k}(x) = \Phi(x)^T e_j e_k^T \Phi(x)$, and

$$
R \colon H^{2-\tau}(\mathbb{R}_\gamma^n) \to L_2(\mathbb{R}_+^n)
$$

is a bounded operator for $\tau = \frac{1}{2} - \frac{n-1}{p}$.

Proof: The chain rule (2.24) gives that $\nabla(u(F_\gamma(x)) = (\nabla F_\gamma)(x)(\nabla u)(F_\gamma(x))$, which implies the first line in (2.25). In particular,

$$
F_\gamma^* \partial_{x_j} u = e_j^T F_\gamma^*(\nabla u) = e_j^T \Phi F_\gamma^* u, y
$$

where $e_j^T \Phi$ is the j-th row $\begin{pmatrix} \varphi_{j1} & \ldots & \varphi_{jn} \end{pmatrix}$ in Φ. Repeated use gives

$$
\begin{aligned}
F_\gamma^* \partial_j \partial_k u &= e_j^T \Phi(x) \nabla F_\gamma^* \partial_k u = e_j^T \Phi(x) \nabla \left(e_k^T \Phi(x) \nabla F_\gamma^* u \right) \\
&= \begin{pmatrix} \varphi_{j1} & \ldots & \varphi_{jn} \end{pmatrix} \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \left(\begin{pmatrix} \varphi_{k1} & \ldots & \varphi_{kn} \end{pmatrix} \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} F_\gamma^* u \right) \\
&= \sum_{l,m=1}^n \varphi_{jl} \partial_l (\varphi_{km} \partial_m F_\gamma^* u) \\
&= \sum_{l,m=1}^n \varphi_{jl} \varphi_{km} \partial_l \partial_m F_\gamma^* u + \sum_{l,m=1}^n \varphi_{jl}(x)(\partial_l \varphi_{km}(x)) \partial_m F_\gamma^* u
\end{aligned}
$$

for all $u \in H^2(\mathbb{R}_\gamma^n)$. Here $(\varphi_{jl} \varphi_{km})_{l,m=1}^n$ equals the matrix $\Phi^T e_j e_k^T \Phi = \Phi_{j,k}$ for each j, k. This shows the second line in (2.25), where R is estimated by use of (2.18):

$$
\|\varphi_{jl}(x)(\partial_l \varphi_{km}(x)) \partial_m F_\gamma^* u\|_{L_2(\mathbb{R}_+^n)} \le C \|\partial_m F_\gamma^* u\|_{H^{1-\tau}(\mathbb{R}_+^n)} \le C' \|u\|_{H^{2-\tau}(\mathbb{R}_\gamma^n)}.
$$

∎

Remark 2.9 Now we can choose a covering of $\overline{\Omega}$ by a system of open sets $U_0, \ldots, U_J$ with coordinate mappings such that the U_j for $j = 1, \ldots, J$ form a covering of $\partial\Omega$, and $\overline{U}_0 \subset \Omega$. Here in case $2°$ of Definition 2.4, we can assume that for each $1 \leq j \leq J$, U_j is bounded, $U_j \cap \Omega = U_j \cap \mathbb{R}^n_{\gamma_j}$ for some $\gamma_j \in B^{M-\frac{1}{2}}_{p,2}(\mathbb{R}^{n-1})$ (modulo a rotation), and the diffeomorphism $F_j \equiv F_{\gamma_j}$ in $\mathbb{R}^n$ mapping $\mathbb{R}^n_+$ to $\mathbb{R}^n_{\gamma_j}$ defined by Proposition 2.7 is such that F_j^{-1} carries $\Omega \cap U_j$ over to $V_j = \{(y', y_n) : \max_{k<n}|y_k| < a_j, 0 < y_n < a_j\}$ and $\Sigma \cap U_j$ over to $\{(y', y_n) : \max_{k<n}|y_k| < a_j, y_n = 0\}$. In case $3°$ of Definition 2.4, the open sets $U_1, \ldots, U_{J-1}$ are of this kind, and the set U_J equals $\{(x', x_n) : |x| > R, x_n > -1\}$, with F_J being the identity ($\gamma_J = 0$). We shall denote $F_j|_{\partial\mathbb{R}^n_+} = F_{j,0}$, for $j = 1, \ldots, J$.

We also introduce $\eta_j, \psi_j, \varphi_j \in C^\infty(\mathbb{R}^n)$, $j = 0, \ldots, J$, as non-negative functions supported in U_j such that $\varphi_0, \ldots, \varphi_J$ is a partition of unity on $\overline{\Omega}$, and

$$\psi_j = 1 \quad \text{on } \operatorname{supp}\varphi_j, \quad \eta_j = 1 \quad \text{on } \operatorname{supp}\psi_j, \quad \text{for all } j = 0, \ldots, J.$$

It is accounted for in [42] in the case of a Lipschitz boundary (and it also follows in our case by use of the fact that F_γ defined in Proposition 2.7 is a C^1-diffeomorphism), that the surface measure $d\sigma$ on $\partial\mathbb{R}^n_\gamma$ satisfies

$$d\sigma = \kappa(x')\,dx', \text{ where } \kappa(x') = \sqrt{1 + |\nabla_{x'}\gamma(x')|^2};$$

here $\kappa \in B^{M-\frac{3}{2}}_{p,2}(\mathbb{R}^{n-1})$. We define

$$H^s(\partial\mathbb{R}^n_\gamma) = \left\{ u \in L_2(\partial\Omega) : F^*_{\gamma,0}u \in H^s(\mathbb{R}^{n-1}) \right\}, \text{ when } s \geq 0,$$

provided with the inherited Hilbert space norm $\|u\|_{H^s(\partial\mathbb{R}^n_\gamma)} = \|F^*_{\gamma,0}u\|_{H^s(\mathbb{R}^{n-1})}$. Furthermore, we put as in [42], for $s > 0$,

$$\|u\|_{H^{-s}(\partial\mathbb{R}^n_\gamma)} = \left\|\kappa F^*_{\gamma,0}u\right\|_{H^{-s}(\mathbb{R}^{n-1})},$$

for $u \in L_2(\partial\mathbb{R}^n_\gamma)$, and define $H^{-s}(\partial\mathbb{R}^n_\gamma)$ as the completion of $L_2(\partial\mathbb{R}^n_\gamma)$ with respect to this norm. Then

$$\|u\|_{H^{-s}(\partial\mathbb{R}^n_\gamma)} = \sup_{0 \neq v \in H^s(\partial\mathbb{R}^n_\gamma)} \frac{|(u, v)_{L_2(\partial\mathbb{R}^n_\gamma)}|}{\|v\|_{H^s(\partial\mathbb{R}^n_\gamma)}}$$

for all $u \in L_2(\partial\mathbb{R}^n_\gamma)$. Here $H^{-s}(\partial\mathbb{R}^n_\gamma)$ is naturally identified with the dual of $H^s(\partial\mathbb{R}^n_\gamma)$ (more precisely, we hereby mean the space of conjugate linear continuous functionals as in [38], also called the antidual space) in such a way that the sesquilinear duality, denoted $(u, v)_{H^{-s}(\partial\mathbb{R}^n_\gamma), H^s(\partial\mathbb{R}^n_\gamma)}$ or $(u, v)_{-s,s}$ for short, coincides with the L_2-scalar product when $u \in L_2(\partial\mathbb{R}^n_\gamma)$. We also write $\overline{(u, v)}_{-s,s}$ as $(v, u)_{s,-s}$.

Moreover, for $-M + \frac{3}{2} \leq s < 0$ we define $F^{-1,*}_{\gamma,0} : H^s(\mathbb{R}^{n-1}) \to H^s(\partial\mathbb{R}^n_\gamma)$ by

$$(F^{-1,*}_{\gamma,0}v, \varphi)_{H^s(\partial\mathbb{R}^n_\gamma), H^{-s}(\partial\mathbb{R}^n_\gamma)} = (v, \kappa F^*_{\gamma,0}\varphi)_{H^s(\mathbb{R}^{n-1}), H^{-s}(\mathbb{R}^{n-1})} \quad \text{for all } \varphi \in H^{-s}(\partial\mathbb{R}^n_\gamma),$$

consistently with the definition of $F_{\gamma,0}^{-1,*}v$ for $v \in L_2(\mathbb{R}^{n-1})$. Here $F_{\gamma,0}^{-1,*}v \in H^s(\partial\mathbb{R}_\gamma^n)$, the dual space of $H^{-s}(\partial\mathbb{R}_\gamma^n)$, since

$$\left\|\kappa F_{\gamma,0}^*\varphi\right\|_{H^{-s}(\mathbb{R}^{n-1})} \le C\left\|\kappa\right\|_{B_{p,2}^{M-\frac{3}{2}}(\mathbb{R}^{n-1})}\left\|F_{\gamma,0}^*\varphi\right\|_{H^{-s}(\mathbb{R}^{n-1})} \le C'\left\|\varphi\right\|_{H^{-s}(\partial\mathbb{R}_\gamma^n)}$$

for all $\varphi \in H^{-s}(\partial\mathbb{R}_\gamma^n)$, because of $0 < -s \le M - \frac{3}{2}$ and (2.31) below. To incorporate the factor κ, we also introduce the modified pull-back mappings

$$\widetilde{F}_{\gamma,0}^*(u) = \kappa F_{\gamma,0}^*(u) \qquad \text{for all } u \in H^s(\partial\mathbb{R}_\gamma^n), \tag{2.26}$$

$$\widetilde{F}_{\gamma,0}^{-1,*}(v) = F_{\gamma,0}^{-1,*}(\kappa v) \quad \text{for all } v \in H^s(\mathbb{R}^{n-1}), \tag{2.27}$$

for all $0 \le s \le M - \frac{3}{2}$, whereby

$$(\widetilde{F}_{\gamma,0}^*(u), \varphi)_{-s,s} = (u, F_{\gamma,0}^{-1,*}(\varphi))_{-s,s} \qquad \text{for all } u \in H^{-s}(\partial\mathbb{R}_\gamma^n), \varphi \in H^s(\mathbb{R}^{n-1}),$$

$$(\widetilde{F}_{\gamma,0}^{-1,*}(v), \varphi)_{-s,s} = (v, F_{\gamma,0}^*(\varphi))_{-s,s} \qquad \text{for all } v \in H^{-s}(\mathbb{R}^{n-1}), \varphi \in H^s(\partial\mathbb{R}_\gamma^n),$$

$0 \le s \le M - \frac{3}{2}$.

Then we have altogether:

Lemma 2.10 *Under the assumptions above, we have the mapping properties*

$$F_{\gamma,0}^*\colon H^s(\partial\mathbb{R}_\gamma^n) \to H^s(\mathbb{R}^{n-1}), F_{\gamma,0}^{-1,*}\colon H^s(\mathbb{R}^{n-1}) \to H^s(\partial\mathbb{R}_\gamma^n) \text{ if } -M + \tfrac{3}{2} \le s \le M - \tfrac{1}{2},$$

$$\widetilde{F}_{\gamma,0}^*\colon H^s(\partial\mathbb{R}_\gamma^n) \to H^s(\mathbb{R}^{n-1}), \widetilde{F}_{\gamma,0}^{-1,*}\colon H^s(\mathbb{R}^{n-1}) \to H^s(\partial\mathbb{R}_\gamma^n) \text{ if } -M + \tfrac{1}{2} \le s \le M - \tfrac{3}{2},$$

continuously, and $\widetilde{F}_{\gamma,0}^{-1,*}$ is the adjoint of $F_{\gamma,0}^*$, $F_{\gamma,0}^{-1,*}$ is the adjoint of $\widetilde{F}_{\gamma,0}^*$.

Now one can define $H^s(\partial\Omega)$ using suitable partitions of unity, when $\partial\Omega$ is of class $B_{p,2}^{M-\frac{1}{2}}$ as in 2° and 3° of Definition 2.4; cf. Remark 2.9.

Theorem 2.11 *Let $\Omega \subseteq \mathbb{R}^n$ be as in Definition 2.4. For every $s \in (\frac{1}{2}, M]$, there is a continuous linear mapping γ_0 such that $\gamma_0 u = u|_{\partial\Omega}$ for all $u \in H^s(\Omega) \cap C^0(\overline{\Omega})$ and $\gamma_0\colon H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$ is bounded. Moreover, there is a continuous right inverse of $\gamma_0\colon H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$.*

Proof: In the case $\Omega = \mathbb{R}_\gamma^n$, the statement can be reduced to the well-known corresponding fact in the case of $\Omega = \mathbb{R}_+^n$ using F_γ established in Proposition 2.7. With the aid of suitable partitions of unity, the general cases can be reduced to the case $\Omega = \mathbb{R}_\gamma^n$. ∎

We note that Gauß's formula

$$\int_\Omega \operatorname{div} f(x)\,dx = -\int_{\partial\Omega} \vec{n} \cdot f(x)\,d\sigma(x) \tag{2.28}$$

is valid for any $f \in C^1(\overline{\Omega})^n$ with compact support if Ω is a Lipschitz domain. Here $\vec{n} = (n_1, \dots, n_n)$ denotes the interior normal of $\partial\Omega$. A proof can be found e.g. in [42, Theorem 3.34]. Because of [42, Theorems 3.29 and 3.38], (2.28) also holds true for any $f \in H^1(\Omega)^n$.

2.2   Pointwise multiplication and inversion

First of all, we recall the following product estimates: For every $r > 0$, $|s| \leq r$, and $1 \leq p \leq q < \infty$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$ and $r - \frac{n}{q} > 0$ there is some constant $C_{r,s,p,q} > 0$ such that

$$\|fg\|_{H^s_p(\mathbb{R}^n)} \leq C_{r,s,p,q} \|f\|_{H^r_q(\mathbb{R}^n)} \|g\|_{H^s_p(\mathbb{R}^n)}, \quad f \in H^r_q(\mathbb{R}^n), g \in H^s_p(\mathbb{R}^n), \qquad (2.29)$$

cf. e.g. Johnsen [32, Theorems 6.1 and 6.4].

Moreover, due to Hanouzet [30, Théorème 3] we have

$$\|fg\|_{B^s_{p,\max(q_1,q_2)}} \leq C_{r,s,p,q} \|f\|_{B^r_{p_1,q_1}} \|g\|_{B^s_{p,q_2}} \qquad (2.30)$$

for all $f \in B^r_{p_1,q_1}(\mathbb{R}^n), g \in B^s_{p,q_2}(\mathbb{R}^n)$ provided that $1 \leq p \leq p_1 \leq \infty$, $1 \leq q_1, q_2 \leq \infty$, $r > \frac{n}{p_1}$, and

$$-r + n\left(\frac{1}{p_1} + \frac{1}{p} - 1\right)_+ < s \leq r,$$

see also [32, Theorem 6.6]. In particular, this implies

$$\|fg\|_{H^s(\mathbb{R}^n)} \leq C_{s,r,p} \|f\|_{B^r_{p,2}(\mathbb{R}^n)} \|g\|_{H^s(\mathbb{R}^n)} \qquad (2.31)$$

for all $g \in H^s(\mathbb{R}^n), f \in H^s(\mathbb{R}^n)$ provided that $2 \leq p \leq \infty$, $r - \frac{n}{p} > 0$ and $-r < s \leq r$.

Concerning pointwise inversion, let $X = B^s_{p,q}(\mathbb{R}^n)$ with $s > \frac{n}{p}$, $1 \leq p, q \leq \infty$ or $X = H^s_p(\mathbb{R}^n)$ with $s > \frac{n}{p}$, $1 < p < \infty$. Then $G(f) \in X$ for all $G \in C^\infty(\mathbb{R})$ and $f \in X$. This implies that $f^{-1} \in X$ for all $f \in X$ such that $|f| \geq c_0 > 0$. We refer to Runst [44] for an overview, further results, and references.

2.3   Pseudodifferential operators with nonsmooth coefficients

Let X be a Banach space such that $X \hookrightarrow L_\infty(\mathbb{R}^n)$.

Definition 2.12 *For every $m \in \mathbb{R}$ the symbol space $XS^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $n \in \mathbb{N}$, is the set of all $p \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that for every $\alpha \in \mathbb{N}^m_0$ there is some $C_\alpha > 0$ satisfying*

$$\|\partial^\alpha_\xi p(.,\xi)\|_X \leq C_\alpha \langle \xi \rangle^{m-|\alpha|} \qquad \text{for all } \xi \in \mathbb{R}^n.$$

The space $XS^m_{\mathrm{cl}}(\mathbb{R}^n \times \mathbb{R}^n)$ is the set of all $p \in XS^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, which are classical symbols in the sense that there are $p_j \in XS^{m-j}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $j \in \mathbb{N}_0$ that are homogeneous with respect to $|\xi| \geq 1$ and satisfy

$$p(x,\xi) - \sum_{j=0}^{N-1} p_j(x,\xi) \in XS^{m-N}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \qquad \text{for all } N \in \mathbb{N}.$$

In order to define pseudodifferential operators on $\partial\Omega$ with $B^{M-\frac{1}{2}}_{p,2}$-regularity, we recall:

Theorem 2.13 *Let $p \in H_q^r S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n)$ for some $2 \leq q < \infty$ and $r > \frac{n}{q}$. Then*

$$p(x, D_x) \colon H^{s+m}(\mathbb{R}^n) \to H^s(\mathbb{R}^n) \qquad \text{for all} \; -r < s \leq r$$

is a bounded linear operator.

The theorem follows from [40, Theorem 2.2]. We note that, if $p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$ for some $a_\alpha \in H_q^r(\mathbb{R}^n)$, then $p(x, D_x)$ is a differential operator with coefficients in $H_q^r(\mathbb{R}^n)$ and the statement in the theorem easily follows from the product estimate (2.29) provided that $|s| \leq r$, $2 \leq q < \infty$ and $r > \frac{n}{q}$.

Let us recall the so-called symbol-smoothing: For every $p \in C^r S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n)$ and $0 < \delta < 1$ there is a decomposition

$$p = p^\sharp + p^b, \quad \text{where } p^\sharp \in S_{1,\delta}^m(\mathbb{R}^n \times \mathbb{R}^n), p^b \in C^r S_{1,\delta}^{m-r\delta}(\mathbb{R}^n \times \mathbb{R}^n). \tag{2.32}$$

We refer to [47, end of Section 1.3] for a proof. The definition of $C^r S_{1,\delta}^{m-r\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ is given in the appendix.

In order to estimate the remainder term $p^b(x, D_x)$ we will use:

Proposition 2.14 *Let $p \in C^r S_{1,\delta}^m(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, $\delta \in [0, 1]$, $r > 0$. Then*

$$p(x, D_x) \colon H^{s+m}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$$

for all $s \in \mathbb{R}$ with $-r(1 - \delta) < s < r$.

We refer to [41, Theorem 2.1] for a proof.

Now let $\partial\Omega$ be again of class $B_{p,2}^{M-\frac{1}{2}}$, where $M - \frac{3}{2} - \frac{n-1}{p} > 0$. Then we can define a pseudodifferential operator P on $\partial\Omega$ of order $m' \in \mathbb{R}$ and with coefficients in $H_q^r(\mathbb{R}^{n-1})$ for $2 \leq q < \infty$ and $r > \frac{n}{q}$, as

$$Pu = \sum_{j=1}^J \psi_j F_{j,0}^{-1,*} p_j(x', D_{x'}) F_{j,0}^* \varphi_j u, \tag{2.33}$$

where the $p_j \in H_q^r S_{1,0}^{m'}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, and $F_{j,0} \colon V_j \to U_j \subset \partial\Omega$, $j = 1, \ldots, J$, where $V_j \subset \mathbb{R}^{n-1}$, are local charts forming an atlas of $\partial\Omega$. Moreover, φ_j, $j = 1, \ldots, J$ is a partition of unity on $\partial\Omega$ such that $\operatorname{supp}\varphi_j \subset U_j$, and the functions ψ_j satisfy $\psi_j \equiv 1$ on $\operatorname{supp}\varphi_j$ and have $\operatorname{supp}\psi_j \subset U_j$. Here we assume that at least $\varphi_j, \psi_j \in B_{p,2}^{M-\frac{1}{2}}(\partial\Omega)$.

For later purposes we also define the modified pseudodifferential operator

$$\widetilde{P}u = \sum_{j=1}^J \psi_j \widetilde{F}_{j,0}^{-1,*} p_j(x', D_{x'}) F_{j,0}^* \varphi_j u, \tag{2.34}$$

where p_j are as above and $\widetilde{F}_{j,0}^{-1,*} = \widetilde{F}_{\gamma_j,0}^{-1,*}$. For these operators we have the following slightly different continuity results:

Corollary 2.15 *Let $\partial\Omega$ be of class $B_{p,2}^{M-\frac{1}{2}}$, $M - \frac{3}{2} - \frac{n-1}{p} > 0$, $2 \leq p < \infty$, and let $p_j \in H_q^r S_{1,0}^{m'}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ for some $2 \leq q < \infty$ and $r > \frac{n-1}{q}$, $j = 1, \ldots, J$. Then with P as in (2.33) we have for every $s \in \mathbb{R}$ such that $-r < s \leq r$, $s, s + m' \in [-M + \frac{3}{2}, M - \frac{1}{2}]$,*

$$P: H^{s+m'}(\partial\Omega) \to H^s(\partial\Omega)$$

is a well-defined linear and bounded operator. Moreover, for every $s \in \mathbb{R}$ such that $-r < s \leq r$, $s \in [-M + \frac{1}{2}, M - \frac{3}{2}]$, $s + m' \in [-M + \frac{3}{2}, M - \frac{1}{2}]$ and $\widetilde{P}$ as in (2.34) we have:

$$\widetilde{P}: H^{s+m'}(\partial\Omega) \to H^s(\partial\Omega)$$

is a well-defined linear and bounded operator.

Proof: The proof follows immediately from Theorem 2.13 and Lemma 2.10 and local charts. ∎

In particular, we can define differential operators with H_q^r-coefficients in the manner above.

Remark 2.16 We will not address the question of invariance under coordinate transformation of nonsmooth pseudodifferential operators. Therefore we will also not show that the definition (2.33) does not depend in an essential way on the choice of the charts and the cut-off functions φ_j, ψ_j.

We recall from [40, Corollary 3.4]:

Theorem 2.17 *Let $p_j \in H_q^r S_{1,0}^{m_j}(\mathbb{R}^n \times \mathbb{R}^n)$, $m_j \in \mathbb{R}$, $j = 1, 2$, $2 \leq q < \infty$ and $r > \frac{n}{q}$. Then for every $0 < \tau \leq 1$ with $\tau < r - \frac{n}{q}$*

$$p_1(x, D_x)p_2(x, D_x) - (p_1 p_2)(x, D_x): H^{s+m_1+m_2-\tau}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$$

is a bounded operator provided that

$$s, s + m_1 \in (-r + \tau, r]. \tag{2.35}$$

Note here that if the $p_j(x, D_x)$ are differential operators, then Theorem 2.17 can be proved by elementary but lengthy estimates using Sobolev embeddings. As one consequence of the theorem one has that

$$\sum_{|\alpha|,|\beta| \leq m} D_x^\alpha(a_{\alpha,\beta}(x) D_x^\beta u) = \sum_{|\alpha|,|\beta| \leq m} a_{\alpha,\beta}(x) D_x^{\alpha+\beta} u + Ru, \tag{2.36}$$

where

$$R: H^{s+2m-\tau}(\Omega) \to H^s(\Omega) \qquad \text{if } -r + \tau < s \leq r - m \tag{2.37}$$

provided that $a_{\alpha,\beta} \in H_q^r(\Omega)$ and Ω is a Lipschitz domain. In fact, the statement in the case $\Omega = \mathbb{R}^n$ follows from Theorem 2.17, and then for a general Lipschitz domain Ω one can obtain the statement by extension to $\mathbb{R}^n$.

2.4   Green's formula for second order boundary value problems

Since the smoothness properties of the coefficients in Green's formula for general $2m$-order operators are quite complicated to analyse and would take up much space, we shall in the present paper restrict the attention to the second-order case from here on (expecting to take up higher-order problems elsewhere).

Consider a second order strongly elliptic operator A,

$$Au = -\sum_{j,k=1}^{n} \partial_{x_j}(a_{jk}\partial_{x_k}u) + \sum_{j=1}^{n} a_j\partial_{x_j}u + a_0 u, \tag{2.38}$$

with

$$\text{Re} \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geq c_0|\xi|^2, \text{ all } x \in \Omega, \ \xi \in \mathbb{R}^n; \tag{2.39}$$

$c_0 > 0$. We assume that the a_{jk} and a_j are in $H^1_q(\Omega)$ and $a_0 \in L_q(\Omega)$, where $q \geq 2$ and $1 - \frac{n}{q} > 0$, and we apply the expression to $u \in H^2(\Omega)$. In view of (2.29),

$$\|fg\|_{H^s(\Omega)} \leq C_q\|f\|_{H^1_q(\Omega)}\|g\|_{H^s(\Omega)} \qquad \text{for all } |s| \leq 1;$$

hence

$$A\colon H^{2+s}(\Omega) \to H^s(\Omega) \qquad \text{for all } s \in [-2,0]. \tag{2.40}$$

Concerning the domain Ω, we assume that Ω is as in Definition 2.4 $2°$ or $3°$ with $M = 2$ (so $\partial\Omega \in B^{\frac{3}{2}}_{p,2}$ and $\frac{1}{2} - \frac{n-1}{p} > 0$, in particular $p > 2$).

Denote $\partial_{\vec{n}} = \vec{n} \cdot \partial$, the normal derivative, where $\vec{n} = (n_1,\ldots,n_n)$ is the interior unit normal on $\partial\Omega$. We shall denote $\partial_\tau = \text{pr}_\tau\partial$, where $\text{pr}_\tau = I - \vec{n}\otimes\vec{n}$; the "tangential gradient". (Here $\vec{n}\otimes\vec{n}$ is the matrix $(n_jn_k)_{j,k=1,\ldots,n} = \vec{n}\,\vec{n}^T$, $\vec{n}$ used as a column vector.) Setting $\partial_{\tau,j} = e_j \cdot \partial_\tau$, we have (at points of $\partial\Omega$), since

$$\partial_{\tau,j}u = e_j \cdot \partial_\tau u = e_j \cdot \partial u - e_j\,\vec{n}\,\vec{n}^T\partial u = \partial_{x_j}u - n_j\vec{n} \cdot \partial u,$$

that

$$\partial_{x_j}u = n_j\partial_{\vec{n}}u + \partial_{\tau,j}u. \tag{2.41}$$

When $\xi \in \mathbb{C}^n$, we set $\xi_\tau = \text{pr}_\tau\xi = (I - \vec{n} \otimes \vec{n})\xi$. For $j \in \mathbb{N}_0$, we define

$$\gamma_j u := \big((\vec{n} \cdot \partial_x)^j u\big)|_{\partial\Omega} = \partial_{\vec{n}}^j u|_{\partial\Omega}.$$

By our assumptions, $\vec{n} \in B^{\frac{1}{2}}_{p,2}(\partial\Omega)^n \hookrightarrow H^{\frac{1}{2}}_p(\partial\Omega)^n$ (cf. (2.7), recall that $p \geq 2$). The product rule (2.29) applies with $r = s = \frac{1}{2}$, $p = q$, n replaced by $n - 1$, to show that $H^{\frac{1}{2}}_p(\partial\Omega)$ is an algebra with respect to pointwise multiplication. The rule (2.29) also applies with $p = 2$, $r = \frac{1}{2}$, n replaced by $n - 1$ and q replaced by our general p, to show that multiplication by elements of $H^{\frac{1}{2}}_p(\partial\Omega)$ preserves $H^s(\partial\Omega)$ for $|s| \leq \frac{1}{2}$.

Then since $\gamma_0 \partial_{x_j} \colon H^s(\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega)$ continuously for $s \in (\frac{3}{2}, 2]$, $\gamma_1 \colon H^s(\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega)$ continuously for $s \in (\frac{3}{2}, 2]$.

Let

$$a'_{jk}(x) = \overline{a_{kj}(x)}, \quad a'_j(x) = \overline{a_j(x)}, \quad a'_0(x) = \overline{a_0(x)} - \sum_{j=1}^n \partial_{x_j} \overline{a_j(x)}$$

for all $x \in \Omega$, $j, k = 1, \ldots, n$, and let A' denote the operator defined as in (2.38) with a_{jk}, a_j, a_0 replaced by a'_{jk}, a'_j, a'_0; it is the formal adjoint of A.

It will be convenient for the following to have $\frac{1}{2} - \frac{n-1}{p} \le 1 - \frac{n}{q}$; this can be achieved by replacing p by a smaller $p > 2$. Then we can set $\tau = \frac{1}{2} - \frac{n-1}{p}$ as in Assumption 2.5. To sum up, we make the following assumption:

Assumption 2.18 $n \ge 2$, $2 < q \le \infty$ and $2 < p < \infty$, with

$$1 - \tfrac{n}{q} \ge \tfrac{1}{2} - \tfrac{n-1}{p} > 0; \quad \tau := \tfrac{1}{2} - \tfrac{n-1}{p}. \tag{2.42}$$

The domain Ω is as in Definition 2.4 2° or 3°, with boundary $\partial\Omega$ of regularity $B_{p,2}^{\frac{3}{2}}$. In (2.38), the coefficients a_{jk} and a_j are in $H_q^1(\Omega)$ and $a_0 \in L_q(\Omega)$.

The inequalities for q and p mean that $(\frac{n}{q}, \frac{n-1}{p})$ belongs to the polygon $\{(x, y) : 0 \le x < 1, 0 < y < \frac{1}{2}, y \ge x - \frac{1}{2}\}$. All $q > n$ and $p > 2(n-1)$ can occur (but not all at the same time). Under Assumption 2.18,

$$B_{q,q}^{1-\frac{1}{q}}(\partial\Omega) + B_{p,2}^{\frac{1}{2}}(\partial\Omega) \hookrightarrow H_p^{\frac{1}{2}}(\partial\Omega)$$

by (2.13), where we also use that $B_{p,2}^{\frac{1}{2}} \hookrightarrow H_p^{\frac{1}{2}}$. Recall from Remark 2.6 that the boundary regularity $B_{p,2}^{\frac{3}{2}}$ includes $C^{\frac{3}{2}+\varepsilon}$ and is included in $C^{1+\tau}$.

Theorem 2.19 *Under Assumption 2.18, the following Green's formula holds:*

$$(Au, v)_\Omega - (u, A'v)_\Omega = (\chi u, \gamma_0 v)_{\partial\Omega} - (\gamma_0 u, \chi' v)_{\partial\Omega}, \tag{2.43}$$

for all $u, v \in H^2(\Omega)$. Here, setting $B = (a_{jk})_{j,k=1}^n$, $B' = (a'_{jk})_{j,k=1}^n$, and $b'_0 = \sum_{j=1}^n n_j a'_j$, and defining

$$s_0 = \sum_{j,k=1}^n a_{jk}(x) n_j n_k = \vec{n}^T B \vec{n},$$

$$\mathcal{A}_1 = b_1 \cdot \partial_\tau, \text{ with } b_1 = (\vec{n}^T B)_\tau, \quad \mathcal{A}'_1 = b'_1 \cdot \partial_\tau + b'_0, \text{ with } b'_1 = (\vec{n}^T B')_\tau,$$

we have that

$$\chi = s_0 \gamma_1 + \mathcal{A}_1 \gamma_0, \quad \chi' = \bar{s}_0 \gamma_1 + \mathcal{A}'_1 \gamma_0. \tag{2.44}$$

Here $s_0, b'_0 \in H_p^{\frac{1}{2}}(\partial\Omega)$, $b_1, b'_1 \in H_p^{\frac{1}{2}}(\partial\Omega)^n$.

Furthermore, s_0 is invertible with s_0^{-1} likewise in $H_p^{\frac{1}{2}}(\partial\Omega)$.

Proof: It is well-known that when coefficients and boundary are smooth, then the Gauss formula (2.28) implies

$$(Au, v)_\Omega - (u, A'v)_\Omega = (\chi u, \gamma v)_{\partial\Omega} - (\gamma u, \chi' v)_{\partial\Omega},$$

for all $u, v \in H^2(\Omega)$, where

$$\chi u = \sum_{j,k=1}^n n_j \gamma_0 a_{jk} \partial_{x_k} u,$$

$$\chi' u = \sum_{j,k=1}^n n_j \gamma_0 a'_{jk} \partial_{x_k} u + \sum_{j=1}^n n_j \gamma_0 a'_j u.$$

Here we can write, using (2.41),

$$\chi u = \sum_{j,k=1}^n n_j a_{jk} \gamma_0 \partial_{x_k} u = \sum_k (\vec{n}^T B)_k \gamma_0 (\partial_{\tau,k} u + n_k \partial_{\vec{n}} u)$$

$$= \vec{n}^T B \vec{n} \gamma_1 u + \vec{n}^T B \partial_\tau \gamma_0 u = \vec{n}^T B \vec{n} \gamma_1 u + (\vec{n}^T B)_\tau \cdot \partial_\tau \gamma_0 u;$$

similarly,

$$\chi' u = \sum_{j,k=1}^n n_j \gamma_0 a'_{jk} \partial_{x_k} u + \sum_{j=1}^n n_j a'_j \gamma_0 u = \vec{n}^T B' \vec{n} \gamma_1 u + (\vec{n}^T B')_\tau \cdot \partial_\tau \gamma_0 u + \vec{n}^T b'_0 \gamma_0 u.$$

This shows the asserted formulas in the smooth case.

The validity is extended in [42, Theorem 4.4] to the case where $a_{jk}, a_j \in C^{0,1}(\overline{\Omega})$ and $a_0 \in L_\infty(\Omega)$, and Ω is a bounded Lipschitz domain. The unbounded cases in Definition 2.4 are included by adding the appropriate (trivial) coordinate charts.

The case $a_{jk}, a_j \in H_q^1(\Omega)$, $a_0 \in L_q(\Omega)$ can then easily be proved by first replacing a_{jk}, a_j, and a_0 by some smoothed $a_{jk}^\varepsilon, a_j^\varepsilon \in C^{0,1}(\overline{\Omega})$ and $a_0^\varepsilon \in L_\infty(\Omega) \cap L_q(\Omega)$ such that $a_{jk}^\varepsilon \to_{\varepsilon\to 0} a_{jk}$ and $a_j^\varepsilon \to_{\varepsilon\to 0} a_j$ in $H_q^1(\Omega)$ and $a_0^\varepsilon \to a_0$ in $L_q(\Omega)$ and then passing to the limit $\varepsilon \to 0$. For this argument one uses (2.29) with $s = r = 1$ and $p = 2$ to pass to the limit in all terms involving a_{jk}, a'_{jk}, a_j, and a'_j. To pass to the limit in the term involving a_0 one uses $H^{\frac{n}{q}}(\Omega) \hookrightarrow L_r(\Omega)$ with $\frac{1}{r} = \frac{1}{2} - \frac{1}{q}$ since $\frac{1}{q} < \frac{1}{n}$, which implies

$$\|a_0 u\|_{L_2(\Omega)} \le C \|a_0\|_{L_q(\Omega)} \|u\|_{H^{\frac{n}{q}}(\Omega)}. \tag{2.45}$$

Finally, since A is strongly elliptic, $|s_0(x)| \ge C > 0$ for all $x \in \overline{\Omega}$. Then $s_0^{-1} \in H_p^{\frac{1}{2}}(\partial\Omega)$ because of the results at the end of Section 2.2. ∎

Note that since the coefficients in the trace operators χ and χ' are in $H_p^{\frac{1}{2}}(\partial\Omega)$, χ and χ' map $H^s(\Omega)$ continuously into $H^{s-\frac{3}{2}}(\partial\Omega)$ for $s \in (\frac{3}{2}, 2]$.

We shall also need a result on localization of χ and χ', and their surjectivity.

Corollary 2.20 *Let χ and χ' be as in the preceding theorem and let $U \subset \mathbb{R}^n$ be such that $\Omega \cap U$ coincides with $\mathbb{R}^n_\gamma \cap U$ (after a suitable rotation), where $\gamma \in B^{\frac{3}{2}}_{p,2}(\mathbb{R}^{n-1})$. Then there is a trace operator*

$$t(x', D_x) = s_1(x')\gamma_0\partial_{x_n} + \sum_{|\alpha|\leq 1} c_\alpha(x')D^\alpha_{x'}\gamma_0,$$

where $s_1, c_\alpha \in H^{\frac{1}{2}}_p(\mathbb{R}^{n-1})$ for all $|\alpha| \leq 1$, such that

$$\chi(\psi u) = \eta F^{-1,*}_{\gamma,0}t(x', D_x)F^*_\gamma(\psi u) \tag{2.46}$$

for any $u \in H^2(\Omega)$ and $\psi, \eta \in C^\infty_0(U)$ with $\eta \equiv 1$ on $\operatorname{supp}\psi$. Here F_γ is the diffeomorphism from Proposition 2.7 and $F_{\gamma,0} = F_\gamma|_{\partial\mathbb{R}^n_+}$. Moreover, s_1 is invertible.

For every $s \in (\frac{3}{2}, 2]$ there is a continuous right-inverse of

$$\begin{pmatrix} \chi \\ \gamma_0 \end{pmatrix} : H^s(\Omega) \rightarrow \begin{matrix} H^{s-\frac{3}{2}}(\partial\Omega) \\ \times \\ H^{s-\frac{1}{2}}(\partial\Omega) \end{matrix} \; ;$$

this holds in particular with χ replaced by γ_1. The analogous statements hold for χ'.

Proof: To prove the first statement, let $\Omega \cap U$ coincide with $\mathbb{R}^n_\gamma \cap U$ after a suitable rotation of Ω and let $\psi, \eta \in C^\infty_0(U)$ with $\eta \equiv 1$ on $\operatorname{supp}\psi$. Then with B as in Corollary 2.8,

$$\begin{aligned} s_0\gamma_1(\psi u) &= \eta s_0\gamma_0(\vec{n} \cdot \nabla(\psi u)) = \eta s_0\gamma_0(\vec{n} \cdot F^{-1,*}_\gamma B\nabla F^*_\gamma(\psi u)) \\ &= \eta s_0 F^{-1,*}_{\gamma,0}a_0\gamma_0(\partial_{x_n}F^*_\gamma(\psi u)) + \eta s_0 F^{-1,*}_{\gamma,0}B'\nabla_{x'}\gamma_0 F^*_\gamma(\psi u) \end{aligned}$$

where $a_0 = (F^*_{\gamma,0}\vec{n}) \cdot B(x', 0)e_n \in H^{\frac{1}{2}}_p(\mathbb{R}^{n-1})$ and $B' = (F^*_{\gamma,0}\vec{n}) \cdot B(x', 0)(I - e_n \otimes e_n) \in H^{\frac{1}{2}}_p(\mathbb{R}^{n-1})$. Moreover,

$$\begin{aligned} (F^*_{\gamma,0}\vec{n}) \cdot (B(x',0)e_n) &= \frac{1}{\sqrt{1+|\nabla_{x'}\gamma|^2}}\begin{pmatrix} -\nabla_{x'}\gamma(x') \\ 1 \end{pmatrix} \cdot \frac{1}{b(x')}\begin{pmatrix} -\nabla_{x'}\gamma(x') \\ 1+b(x') \end{pmatrix} \\ &= \frac{\sqrt{1+|\nabla_{x'}\gamma|^2}}{b(x')} + \frac{1}{\sqrt{1+|\nabla_{x'}\gamma|^2}} \geq c > 0 \end{aligned}$$

where $b(x') = 1 + \lambda\partial_{x_n}\Gamma(x', 0) \in [\frac{1}{2}, \frac{3}{2}]$ as in the proof of Proposition 2.7. Since A is elliptic, $|s_0(x)| \geq c_0 > 0$ for all $x \in \partial\Omega$, too. Therefore $s_1 = s_0 F^{-1,*}_{\gamma,0}a_0 \in H^{\frac{1}{2}}_p(\partial\Omega)$ is invertible. It is easy to observe that $\mathcal{A}_1\gamma_0 u = \eta F^{-1,*}_{\gamma,0}\sum_{|\alpha'|\leq 1} c'_\alpha D_{x'}\gamma_0\psi u$ for some $c'_\alpha \in H^{\frac{1}{2}}_p(\mathbb{R}^{n-1})$. This proves the first statement.

To prove the last statement, we first note that there is a linear extension operator K such that $K : H^{s-\frac{3}{2}}(\mathbb{R}^{n-1}) \rightarrow H^s(\mathbb{R}^n_+)$ for all $s \in (\frac{3}{2}, 2]$ and $\gamma_1 Kv = v$ and $\gamma_0 Kv = 0$

for all $v \in H^{s-\frac{3}{2}}(\mathbb{R}^{n-1})$. Let $F_j, F_{j,0}, \varphi_j, \psi_j, \eta_j$ be as in Remark 2.9 for $M = 2$. Using K and the coefficients $s_{1,j}$ in (2.46) with respect to $\mathbb{R}^n_{\gamma_j}$, we define

$$K_1 v = \sum_{j=1}^{N} \psi_j F_j^{-1,*} K s_{1,j}^{-1} F_{j,0}^* \varphi_j v.$$

Then $\gamma_0 K_1 v = 0$ and therefore

$$\chi K_1 v = \sum_{j=1}^{N} \psi_j F_j^{-1,*} s_{1,j} \gamma_1 K s_{1,j}^{-1} F_{j,0}^* \varphi_j v = \sum_{j=1}^{N} \psi_j F_j^{-1,*} s_{1,j} s_{1,j}^{-1} F_{j,0}^* \varphi_j v = v,$$

where we have applied (2.46) with respect to $\mathbb{R}^n_{\gamma_j}$.

Now we define

$$\mathcal{K} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = K_0 v_2 + K_1 (v_1 - \chi K_0 v_2)$$

for all $v_1 \in H^{s-\frac{3}{2}}(\partial\Omega)$, $v_2 \in H^{s-\frac{1}{2}}(\partial\Omega)$, where $K_0 \colon H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega)$ is a right inverse of γ_0, which exists in view of Theorem 2.11. Then $\mathcal{K}$ is a right-inverse of $\begin{pmatrix} \chi \\ \gamma_0 \end{pmatrix}$. In the special case $A = -\Delta$, $\chi = \gamma_1$.

■

3 Extension theory

In this section we briefly recall some elements of the theory of extensions of dual pairs established in Grubb [23] (building on works of Krein [35], Vishik [51] and Birman [13]) and its relation to M-functions shown in Brown-Grubb-Wood [15].

We start with a pair of closed, densely defined linear operators $A_{\min}$, $A'_{\min}$ in a Hilbert space H satisfying:

$$A_{\min} \subset (A'_{\min})^* = A_{\max}, \quad A'_{\min} \subset (A_{\min})^* = A'_{\max};$$

a so-called dual pair. By $\mathcal{M}$ we denote the set of linear operators lying between the minimal and maximal operator:

$$\mathcal{M} = \{\widetilde{A} \mid A_{\min} \subset \widetilde{A} \subset A_{\max}\}, \quad \mathcal{M}' = \{\widetilde{A}' \mid A'_{\min} \subset \widetilde{A}' \subset A'_{\max}\}.$$

Here we write $\widetilde{A}u$ as Au for any $\widetilde{A}$, and $\widetilde{A}'u$ as $A'u$ for any $\widetilde{A}'$. We assume that there exists an $A_\gamma \in \mathcal{M}$ with $0 \in \varrho(A_\gamma)$; then $A_\gamma^* \in \mathcal{M}'$ with $0 \in \varrho(A_\gamma^*)$.

Denote

$$Z = \ker A_{\max}, \quad Z' = \ker A'_{\max},$$

and define the basic non-orthogonal decompositions

$$D(A_{\max}) = D(A_\gamma) \dot{+} Z, \text{ denoted } u = u_\gamma + u_\zeta = \mathrm{pr}_\gamma\, u + \mathrm{pr}_\zeta\, u,$$
$$D(A'_{\max}) = D(A^*_\gamma) \dot{+} Z', \text{ denoted } v = v_{\gamma'} + v_{\zeta'} = \mathrm{pr}_{\gamma'}\, v + \mathrm{pr}_{\zeta'}\, v;$$

here $\mathrm{pr}_\gamma = A_\gamma^{-1} A_{\max}$, $\mathrm{pr}_\zeta = I - \mathrm{pr}_\gamma$, and $\mathrm{pr}_{\gamma'} = (A^*_\gamma)^{-1} A'_{\max}$, $\mathrm{pr}_{\zeta'} = I - \mathrm{pr}_{\gamma'}$. By $\mathrm{pr}_V\, u = u_V$ we denote the *orthogonal projection* of u onto a subspace V.

The following "abstract Green's formula" holds for $u \in D(A_{\max})$, $v \in D(A'_{\max})$:

$$(Au, v) - (u, A'v) = ((Au)_{Z'}, v_{\zeta'}) - (u_\zeta, (A'v)_Z). \tag{3.1}$$

It can be used to show that when $\widetilde{A} \in \mathcal{M}$ and we set $W = \overline{\mathrm{pr}_{\zeta'} D(\widetilde{A^*})}$, then

$$\{\{u_\zeta, (Au)_W\} \mid u \in D(\widetilde{A})\} \text{ is a graph.}$$

Denoting the operator with this graph by T, we have:

Theorem 3.1 [23] *For the closed* $\widetilde{A} \in \mathcal{M}$, *there is a 1–1 correspondence*

$$\widetilde{A} \text{ closed } \longleftrightarrow \begin{cases} T : V \to W, \text{ closed, densely defined} \\ \text{with } V \subset Z, \ W \subset Z', \text{ closed subspaces.} \end{cases}$$

Here $D(T) = \mathrm{pr}_\zeta D(\widetilde{A})$, $V = \overline{D(T)}$, $W = \overline{\mathrm{pr}_{\zeta'} D(\widetilde{A^*})}$, *and*

$$Tu_\zeta = (Au)_W \text{ for all } u \in D(\widetilde{A}), \text{ (the } \textbf{defining equation}\text{)}. \tag{3.2}$$

In this correspondence,
   (i) $\widetilde{A}^*$ *corresponds similarly to* $T^* : W \to V$.
   (ii) $\ker \widetilde{A} = \ker T;$ $\mathrm{ran}\, \widetilde{A} = \mathrm{ran}\, T + (H \ominus W)$.
   (iii) When $\widetilde{A}$ *is invertible,*

$$\widetilde{A}^{-1} = A_\gamma^{-1} + \mathrm{i}_V\, T^{-1}\, \mathrm{pr}_W.$$

Here i_V indicates the injection of V into H (it is often left out).
Now provide the operators with a spectral parameter λ, then this implies, with

$$Z_\lambda = \ker(A_{\max} - \lambda), \quad Z'_{\bar{\lambda}} = \ker(A'_{\max} - \bar{\lambda}),$$
$$D(A_{\max}) = D(A_\gamma) \dot{+} Z_\lambda, \quad u = u_\gamma^\lambda + u_\zeta^\lambda = \mathrm{pr}_\gamma^\lambda\, u + \mathrm{pr}_\zeta^\lambda\, u, \text{ etc.:}$$

Corollary 3.2 *Let* $\lambda \in \varrho(A_\gamma)$. *For the closed* $\widetilde{A} \in \mathcal{M}$, *there is a 1–1 correspondence*

$$\widetilde{A} - \lambda \longleftrightarrow \begin{cases} T^\lambda : V_\lambda \to W_{\bar{\lambda}}, \text{ closed, densely defined} \\ \text{with } V_\lambda \subset Z_\lambda, \ W_{\bar{\lambda}} \subset Z'_{\bar{\lambda}}, \text{ closed subspaces.} \end{cases}$$

Here $D(T^\lambda) = \mathrm{pr}_\zeta^\lambda\, D(\widetilde{A})$, $V_\lambda = \overline{D(T^\lambda)}$, $W_{\bar\lambda} = \overline{\mathrm{pr}_{\zeta'}^{\bar\lambda}\, D(\widetilde{A}^*)}$, and

$$T^\lambda u_\zeta^\lambda = ((A - \lambda)u)_{W_{\bar\lambda}} \ \text{for all } u \in D(\widetilde{A}).$$

Moreover,
(i) $\ker(\widetilde{A} - \lambda) = \ker T^\lambda;\quad \mathrm{ran}(\widetilde{A} - \lambda) = \mathrm{ran}\, T^\lambda + (H \ominus W_{\bar\lambda})$.
(ii) When $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$,

$$(\widetilde{A} - \lambda)^{-1} = (A_\gamma - \lambda)^{-1} + \mathrm{i}_{V_\lambda}(T^\lambda)^{-1}\,\mathrm{pr}_{W_{\bar\lambda}}. \tag{3.3}$$

This gives a Kreĭn-type resolvent formula for any closed $\widetilde{A} \in \mathcal{M}$.

The operators T and T^λ are related in the following way: Define

$$E^\lambda = I + \lambda(A_\gamma - \lambda)^{-1}, \quad F^\lambda = I - \lambda A_\gamma^{-1},$$
$$E'^{\bar\lambda} = I + \bar\lambda(A_\gamma^* - \bar\lambda)^{-1}, \quad F'^{\bar\lambda} = I - \bar\lambda(A_\gamma^*)^{-1},$$

then $E^\lambda F^\lambda = F^\lambda E^\lambda = I$, $E'^{\bar\lambda} F'^{\bar\lambda} = F'^{\bar\lambda} E'^{\bar\lambda} = I$ on H. Moreover, E^λ and $E'^{\bar\lambda}$ restrict to homeomorphisms

$$E_V^\lambda : V \xrightarrow{\sim} V_\lambda, \quad E_W'^{\bar\lambda} : W \xrightarrow{\sim} W_{\bar\lambda},$$

with inverses denoted F_V^λ resp. $F_W'^{\bar\lambda}$. In particular, $D(T^\lambda) = E_V^\lambda D(T)$.

Theorem 3.3 *Let $G_{V,W}^\lambda = -\mathrm{pr}_W\, \lambda E^\lambda\, \mathrm{i}_V$; then*

$$(E_W'^{\bar\lambda})^* T^\lambda E_V^\lambda = T + G_{V,W}^\lambda. \tag{3.4}$$

In other words, T and T^λ are related by the commutative diagram

$$
\begin{array}{ccc}
V_\lambda & \xleftarrow[E_V^\lambda]{\sim} & V \\[4pt]
T^\lambda \downarrow & \quad \downarrow T + G_{V,W}^\lambda & \qquad D(T^\lambda) = E_V^\lambda D(T). \\[4pt]
W_{\bar\lambda} & \xrightarrow[(E_W'^{\bar\lambda})^*]{\sim} & W
\end{array}
\tag{3.5}
$$

This is a straightforward elaboration of [25, Prop. 2.6].

It was shown in [15] how this relates to formulations in terms of M-functions. First there is the following result in the case where $V = Z$, $W = Z'$, i.e., $\mathrm{pr}_\zeta\, D(\widetilde{A})$ is dense in Z and $\mathrm{pr}_{\zeta'}\, D(\widetilde{A}^*)$ is dense in Z':

Theorem 3.4 *Let $\widetilde{A}$ correspond to $T : Z \to Z'$ by Theorem 3.1. There is a holomorphic operator family $M_{\widetilde{A}}(\lambda) \in \mathcal{L}(Z', Z)$ defined for $\lambda \in \varrho(\widetilde{A})$ by*

$$M_{\widetilde{A}}(\lambda) = \mathrm{pr}_\zeta(I - (\widetilde{A} - \lambda)^{-1}(A_{\max} - \lambda))A_\gamma^{-1}\, \mathrm{i}_{Z'},$$

Here $M_{\widetilde{A}}(\lambda)$ relates to T and T^λ by:

$$M_{\widetilde{A}}(\lambda) = -(T + G_{Z,Z'}^\lambda)^{-1} = -F_Z^\lambda(T^\lambda)^{-1}(F_{Z'}'^{\bar\lambda})^*, \ \text{for } \lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma). \tag{3.6}$$

This is directly related to M-functions (Weyl-Titchmarsh functions) introduced by other authors, see details and references in [15]. Moreover, the construction extends in a natural way to all the closed $\widetilde{A} \in \mathcal{M}$, giving the following result:

Theorem 3.5 *Let $\widetilde{A}$ correspond to $T : V \to W$ by Theorem 3.1. For any $\lambda \in \varrho(\widetilde{A})$, there is a well-defined $M_{\widetilde{A}}(\lambda) \in \mathcal{L}(W, V)$, holomorphic in λ and satisfying*

(i) $M_{\widetilde{A}}(\lambda) = \mathrm{pr}_\zeta (I - (\widetilde{A} - \lambda)^{-1}(A_{\max} - \lambda)) A_\gamma^{-1} \mathrm{i}_W$.

(ii) *When $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$,*

$$M_{\widetilde{A}}(\lambda) = -(T + G^\lambda_{V,W})^{-1}.$$

(iii) *For $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$, it enters in a Kreĭn-type resolvent formula*

$$(\widetilde{A} - \lambda)^{-1} = (A_\gamma - \lambda)^{-1} - \mathrm{i}_{V_\lambda} E^\lambda_V M_{\widetilde{A}}(\lambda)(E'^{\bar\lambda}_W)^* \mathrm{pr}_{W_{\bar\lambda}} . \qquad (3.7)$$

Other Kreĭn-type resolvent formulas in a general framework of *relations* can be found e.g. in Malamud and Mogilevskiĭ [39, Section 5.2].

4   The resolvent construction

4.1   Realizations

The abstract extension theory in the preceding section was implemented for boundary value problems for elliptic operators A with smooth coefficients on smooth domains Ω in [23]–[25], with further results worked out in [15] on Kreĭn resolvent formulas and M-functions. Our aim in the present paper is to extend the validity to the nonsmooth situation introduced in Section 2.4. An important ingredient in this is to show that the Dirichlet problem for A has a resolvent and a Poisson solution operator with appropriate mapping properties.

As $A_{\min}$, $A'_{\min}$, $A_{\max}$ and $A'_{\max}$ we take the operators in $L_2(\Omega)$ defined by

$$A_{\min} \text{ resp. } A'_{\min} = \text{ the closure of } A|_{C_0^\infty(\Omega)} \text{ resp. } A'|_{C_0^\infty(\Omega)}. \qquad (4.1)$$

$$A_{\max} = (A'_{\min})^*, \quad A'_{\max} = (A_{\min})^*.$$

Then $A_{\max}$ acts like A with domain consisting of the functions $u \in L_2(\Omega)$ such that Au, defined weakly, is in $L_2(\Omega)$. $A'_{\max}$ is defined similarly from A'.

By extension of the coefficients a_{jk}, a_j, to all of $\mathbb{R}^n$ (preserving the degree of smoothness) we can extend A to a uniformly strongly elliptic operator A_e on $\mathbb{R}^n$; by addition of a constant, if necessary, we can assume that it has a positive lower bound. By a variant of the resolvent construction described below (easier here, since there is no boundary) we get unique solvability of the equation $A_e u = f$ on $\mathbb{R}^n$ with $f \in L_2(\mathbb{R}^n)$, with a solution $u \in H^2(\mathbb{R}^n)$. Then the graph-norm $(\|Au\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\Omega)})^{\frac{1}{2}}$ and the H^2-norm are equivalent on $H_0^2(\Omega)$, so

$$D(A_{\min}) = H_0^2(\Omega). \qquad (4.2)$$

As A_γ we take the Dirichlet realization of A; it is the restriction of $A_{\max}$ with domain
$$D(A_\gamma) = D(A_{\max}) \cap H_0^1(\Omega).$$
and equals the operator defined by variational theory (Lions' version of the Lax-Milgram lemma, the notation used here is as in [29], Ch. 12), applied to the sesquilinear form
$$a(u,v) = \Sigma_{j,k=1}^n (a_{jk} D_{x_k} u, D_{x_j} v) + (\Sigma_{j=1}^n a_j i D_{x_j} u + a_0 u, v), \tag{4.3}$$
with domain $H_0^1(\Omega) \subset L_2(\Omega)$. A_γ also has positive lower bound. The analogous operator for A' is its Dirichlet realization A'_γ; it equals the adjoint of A_γ. The inequality (2.39) implies that the principal symbol takes its values in a sector $\{\lambda \in \mathbb{C} : |\arg \lambda| \leq \pi/2 - \delta\}$ with $\delta > 0$. The resolvent $(A_\gamma - \lambda)^{-1}$ is well-defined and $O(\langle\lambda\rangle^{-1})$ for large $|\lambda|$ on the rays $\{re^{i\eta}\}$ with $\eta \in (\pi/2 - \delta, 3\pi/2 + \delta)$.

The linear operators $\widetilde{A}$ with $A_{\min} \subset \widetilde{A} \subset A_{\max}$ are the realizations of A.

In the detailed study of the Dirichlet problem that now follows, we first treat a half-space case by pseudodifferential methods, and then use this to treat the general case by localization.

4.2 The halfspace case

In this subsection, we consider the case of a uniformly strongly elliptic second order operator $a(x, D_x)$ on $\mathbb{R}_+^n$ in x-form (i.e., defined from $\sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k$ by the formula (A.2), not in divergence form). More precisely, we assume that $a(x, D_x)u := \sum_{j,k=1}^n a_{jk}(x) D_{x_j} D_{x_k} u$, where $a_{jk} \in C^\tau(\overline{\mathbb{R}}_+^n)$ for some $0 < \tau \leq 1$. The case of a general domain will be treated by the help of this situation, using that $H_p^{\frac{1}{2}}(\mathbb{R}^{n-1}) \hookrightarrow C^\tau(\mathbb{R}^{n-1})$ and $W_{(2,p)}^1(\mathbb{R}_+^n) \hookrightarrow C^\tau(\overline{\mathbb{R}}_+^n)$, where $\tau = \frac{1}{2} - \frac{n-1}{p}$, cf. (2.23). For the construction of a parametrix on $\mathbb{R}_+^n$ it will be enough to use the C^τ-regularity of the coefficients.

We define
$$\mathcal{A}^\times := \begin{pmatrix} a(x, D_x) \\ \gamma_0 \end{pmatrix} : H^{s+2}(\mathbb{R}_+^n) \to \begin{matrix} H^s(\mathbb{R}_+^n) \\ \times \\ H^{s+\frac{3}{2}}(\mathbb{R}^{n-1}) \end{matrix}, \tag{4.4}$$
which maps continuously for all $|s| < \tau$, since $a(x, D_x) : H^{s+2}(\mathbb{R}_+^n) \to H^s(\mathbb{R}_+^n)$ for all $|s| < \tau$ by Proposition 2.14.

To prepare for an application of Theorem A.8, we apply order-reducing operators (cf. Remark A.9) to reduce to H^s-preserving operators, introducing
$$\mathcal{A}_1 = \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^{\frac{3}{2}} \end{pmatrix} \mathcal{A}^\times \Lambda_{-,+}^{-2} = \begin{pmatrix} a(x, D_x)\Lambda_{-,+}^{-2} \\ \Lambda_0^{\frac{3}{2}} \gamma_0 \Lambda_{-,+}^{-2} \end{pmatrix} : H^s(\mathbb{R}_+^n) \to \begin{matrix} H^s(\mathbb{R}_+^n) \\ \times \\ H^s(\mathbb{R}^{n-1}) \end{matrix}, \tag{4.5}$$
continuous for $|s| < \tau$; it is again in x-form with C^τ-smoothness in x. Since γ_0 is of class 1, $\Lambda_0^{\frac{3}{2}} \gamma_0 \Lambda_{-,+}^{-2}$ is of class -1 (and order $-\frac{1}{2}$); $a(x, D_x)\Lambda_{-,+}^{-2}$ is in fact of class -2.

(The notion of *class* is recalled at the end of Section A.1 and extended to negative values in Remark A.9.)

By Theorem A.8 2°, $\mathcal{A}_1$ has a parametrix $\mathcal{B}_1^0$ in x-form, of class -1, defined from the inverse symbol;

$$
\mathcal{B}_1^0 = \begin{pmatrix} R_1^0 & K_1^0 \end{pmatrix} : \begin{array}{c} H^s(\mathbb{R}_+^n) \\ \times \\ H^s(\mathbb{R}^{n-1}) \end{array} \to H^s(\mathbb{R}_+^n), \tag{4.6}
$$

continuous for $|s| < \tau$. (We omit the class related condition $s > -\frac{3}{2}$, since $\tau \le 1$.) In particular, R_1^0 is of order 0, and K_1^0 is a Poisson operator of order $\frac{1}{2}$, having symbol-kernel in $C^\tau S_{1,0}^{-\frac{1}{2}}(\mathbb{R}^N \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$. The remainder $\mathcal{R}_1 = \mathcal{A}_1 \mathcal{B}_1^0 - I$ satisfies

$$
\mathcal{R}_1 : \begin{array}{c} H^{s-\theta}(\mathbb{R}_+^n) \\ \times \\ H^{s-\theta}(\mathbb{R}^{n-1}) \end{array} \to \begin{array}{c} H^s(\mathbb{R}_+^n) \\ \times \\ H^s(\mathbb{R}^{n-1}) \end{array}, \tag{4.7}
$$

when $0 < \theta < \tau$,

$$
-\tau + \theta < s < \tau, \quad s > -\tfrac{3}{2} + \theta. \tag{4.8}
$$

Then the equation $\mathcal{A}_1 \mathcal{B}_1^0 = I + \mathcal{R}_1$, also written

$$
\begin{pmatrix} I & 0 \\ 0 & \Lambda_0^{\frac{3}{2}} \end{pmatrix} \mathcal{A}^\times \Lambda_{-,+}^{-2} \mathcal{B}_1^0 = I + \mathcal{R}_1,
$$

implies by composition to the left with $\begin{pmatrix} I & 0 \\ 0 & \Lambda_0^{-\frac{3}{2}} \end{pmatrix}$ and to the right with $\begin{pmatrix} I & 0 \\ 0 & \Lambda_0^{\frac{3}{2}} \end{pmatrix}$:

$$
\mathcal{A}^\times \Lambda_{-,+}^{-2} \mathcal{B}_1^0 \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^{\frac{3}{2}} \end{pmatrix} = I + \mathcal{R}', \text{ with } \mathcal{R}' = \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^{-\frac{3}{2}} \end{pmatrix} \mathcal{R}_1 \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^{\frac{3}{2}} \end{pmatrix}.
$$

Hence

$$
\mathcal{B}^0 = \Lambda_{-,+}^{-2} \mathcal{B}_1^0 \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^{\frac{3}{2}} \end{pmatrix} = \begin{pmatrix} \Lambda_{-,+}^{-2} R_1^0 & \Lambda_{-,+}^{-2} K_1^0 \Lambda_0^{\frac{3}{2}} \end{pmatrix} = \begin{pmatrix} R^0 & K^0 \end{pmatrix} \tag{4.9}
$$

is a parametrix of the x-form operator $\mathcal{A}^\times$, with

$$
\mathcal{A}^\times \mathcal{B}^0 = I + \mathcal{R}', \tag{4.10}
$$

$$
\mathcal{B}^0 : \begin{array}{c} H^s(\mathbb{R}_+^n) \\ \times \\ H^{s+\frac{3}{2}}(\mathbb{R}^{n-1}) \end{array} \to H^{s+2}(\mathbb{R}_+^n), \quad \mathcal{R}' : \begin{array}{c} H^{s-\theta}(\mathbb{R}_+^n) \\ \times \\ H^{s+\frac{3}{2}-\theta}(\mathbb{R}^{n-1}) \end{array} \to \begin{array}{c} H^s(\mathbb{R}_+^n) \\ \times \\ H^{s+\frac{3}{2}}(\mathbb{R}^{n-1}) \end{array}, \tag{4.11}
$$

for s and θ as in (4.8).

4.3   General domains

Now we consider the situation where the domain Ω and the differential operator A are as in Assumption 2.18. From now on, we use the notation $\partial\Omega = \Sigma$. We recall that the assumption implies that Σ is $B^{\frac{3}{2}}_{p,2}$, $\frac{1}{2} - \frac{n-1}{p} > 0$, the principal part of A is in divergence form with $H^1_q(\Omega)$-coefficients, and $\tau = \frac{1}{2} - \frac{n-1}{p} \leq 1 - \frac{n}{q}$. We have the direct operator with A as in (2.38)

$$\mathcal{A} = \begin{pmatrix} A \\ \gamma_0 \end{pmatrix} : H^{s+2}(\Omega) \to \begin{matrix} H^s(\Omega) \\ \times \\ H^{s+\frac{3}{2}}(\Sigma) \end{matrix} ; \tag{4.12}$$

it is continuous for $-\frac{3}{2} < s \leq 0$.

First we replace the differential operator A by its principal part in x-form, namely

$$a(x, D_x)u := \sum_{j,k=1}^n a_{jk}(x) D_{x_j} D_{x_k} u.$$

Then $a(x, D_x)$ has C^τ-coefficients since $H^1_q(\Omega) \hookrightarrow C^\tau(\overline{\Omega})$, and we have:

$$A - a(x, D_x) : H^{s+2-\theta}(\Omega) \to H^s(\Omega) \tag{4.13}$$

for all $-1 < s \leq 0$ and $0 < \theta < \min(\tau, s+1)$. This statement follows from Theorem 2.17 applied to the principal part (see also (2.36), (2.37)) and from (2.45) since $\frac{n}{q} \leq 1 - \tau$.

Let $\mathcal{A}^\times$ be the operator obtained from $\mathcal{A}$ by replacing A by $a(x, D_x)$. Then $\mathcal{A}^\times$ has the mapping properties

$$\mathcal{A}^\times = \begin{pmatrix} a(x, D_x) \\ \gamma_0 \end{pmatrix} : H^{s+2}(\Omega) \to \begin{matrix} H^s(\Omega) \\ \times \\ H^{s+\frac{3}{2}}(\Sigma) \end{matrix} \tag{4.14}$$

continuously for $|s| \leq 1$, since $a(x, D_x) : H^{s+2}(\Omega) \to H^s(\Omega)$ for all $|s| \leq 1$ in view of (2.29) and the fact that $a_{jk} \in H^1_q(\Omega)$.

We shall use a system of local coordinates and cutoff functions as introduced in Remark 2.9, with $M = 2$.

When the differential operator A is transformed to local coordinates, the principal part of the resulting operator $\underline{A}$ is an x-form operator with C^τ-coefficients since $H^1_q(\mathbb{R}^n_+) \hookrightarrow C^\tau(\overline{\mathbb{R}}^n_+)$. More precisely, because of Corollary 2.8,

$$F_j^* a(x, D_x) F_j^{-1,*} = \underline{a}_j(x, D_x) + \mathcal{R},$$

where $\mathcal{R} : H^{2-\tau}(\mathbb{R}^n_+) \to L^2(\mathbb{R}^n_+)$ and

$$\underline{a}_j(x, \xi) = \sum_{k,l=1}^n a_{kl}(F_j(x))(\Phi_j(x)\xi)^{\alpha+\beta},$$

with $\Phi_j(x) = (\nabla F_j(x))^{-1} \in W^1_{(2,p)}(\mathbb{R}^n_+)^{n^2} \hookrightarrow C^\tau(\overline{\mathbb{R}}^n_+)^{n^2}$. Hence $\underline{a}_j(x, D_x)$ has coefficients in $C^\tau(\overline{\mathbb{R}}^n_+)$.

In each of these charts one constructs a parametrix $\underline{\mathcal{B}}^0_j = \begin{pmatrix} R^0_j & K^0_j \end{pmatrix}$ for $\begin{pmatrix} \underline{a}_j(x, D_x) \\ \gamma_0 \end{pmatrix}$ as in Section 4.2 (the coefficients of $\underline{A}$ can be assumed to be extended to $\overline{\mathbb{R}}^n_+$ preserving ellipticity); for U_0 which is disjoint from the boundary one takes a parametrix R^0_0 of $a(x, D_x)$. Then one defines $\mathcal{B}^0 = \begin{pmatrix} R^0 & K^0 \end{pmatrix}$ by

$$R^0 f \;=\; \sum_{j=1}^J \psi_j F_j^{-1,*} R^0_j F_j^* \varphi_j f + \psi_0 R^0_0 \varphi_0 f \tag{4.15}$$

$$K^0 g \;=\; \sum_{j=1}^J \psi_j F_j^{-1,*} K^0_j F_{j,0}^* \varphi_j g \tag{4.16}$$

for all $f \in H^s(\Omega)$, $g \in H^{s+\frac{3}{2}}(\Sigma)$, where $-\tau + \theta < s \leq 0$ and φ_j, ψ_j as in Remark 2.9. Here we recall from (4.9) that $K^0_j = \Lambda^{-2}_{-,+} \tilde{k}^0_j(x', D_x) \Lambda^{\frac{3}{2}}_0$ with $\tilde{k}^0_j \in C^\tau S^{-\frac{1}{2}}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$. Then it follows directly from the results so far that

$$\mathcal{A}\mathcal{B}^0 \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} + \mathcal{R}_1 \begin{pmatrix} f \\ g \end{pmatrix},$$

where

$$\mathcal{R}_1 : \begin{array}{ccc} H^{s-\theta}(\Omega) & & H^s(\Omega) \\ \times & \rightarrow & \times \\ H^{s+\frac{3}{2}-\theta}(\Sigma) & & H^{s+\frac{3}{2}}(\Sigma) \end{array}$$

is a bounded operator for all θ and $s \in \mathbb{R}$ such that

$$0 < \theta < \tau, \quad -\tau + \theta < s \leq 0. \tag{4.17}$$

In the present construction, we shall actually carry a spectral parameter along, which will be useful for discussions of invertibility and resolvents. So we now replace the originally given A by $A - \lambda$, to be studied for large λ in a sector around $\mathbb{R}_-$. The parameter is taken into the order-reducing operators as well, by replacing $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$ by $(1 + |\lambda| + |\xi|^2)^{\frac{1}{2}}$.

The parametrix will be of the form

$$\mathcal{B}^0(\lambda) = \begin{pmatrix} R^0(\lambda) & K^0(\lambda) \end{pmatrix} : \begin{array}{c} H^s(\Omega) \\ \times \\ H^{s+\frac{3}{2}}(\Sigma) \end{array} \rightarrow H^{s+2}(\Omega); \tag{4.18}$$

with H^1_q-smoothness in x, where $-\tau < s \leq 0$. The remainder maps as follows:

$$\mathcal{R}(\lambda) = \mathcal{A}(\lambda)\mathcal{B}^0(\lambda) - I : \begin{array}{ccc} H^{s-\theta}(\Omega) & & H^s(\Omega) \\ \times & \rightarrow & \times \\ H^{s+\frac{3}{2}-\theta}(\Sigma) & & H^{s+\frac{3}{2}}(\Sigma) \end{array} \tag{4.19}$$

for s and θ as in (4.17).

In order to get hold of exact inverses, we shall use a variant of an old trick of Agmon [7], which implies a useful λ-dependent estimate of the remainder. (The technique was developed further and applied to ψdbo's in [27], which could also be invoked here; but in the present simple case of differential operators the trick can be used more directly.)

Consider λ on a ray outside the sector where the principal symbol $\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k$ takes its values, i.e., we set $\lambda = e^{i\eta}\mu^2$ ($\mu \geq 0$) with $\eta \in (\pi/2 - \delta, 3\pi/2 + \delta)$. For the study of $A - \lambda$, introduce an extra variable $t \in S^1$, and replace μ by $D_t = -i\partial_t$, letting

$$\widehat{A} = A - e^{i\eta}D_t^2 \text{ on } \Omega \times S^1. \tag{4.20}$$

Then $\widehat{A}$ is elliptic on $\Omega \times S^1$ and its Dirichlet problem is elliptic, and by the preceding construction (carried out with local coordinates respecting the product structure),

$$\widehat{\mathcal{A}} = \begin{pmatrix} \widehat{A} \\ \gamma_0 \end{pmatrix} \text{ has a parametrix } \widehat{\mathcal{B}}^0,$$

with mapping properties of $\widehat{\mathcal{B}}^0$ and the remainder $\widehat{\mathcal{R}} = \widehat{\mathcal{A}}\widehat{\mathcal{B}}^0 - I$ as in (4.18) and (4.19) with Ω, Σ replaced by $\widehat{\Omega} = \Omega \times S^1$, $\widehat{\Sigma} = \Sigma \times S^1$.

For functions w of the form $w(x,t) = u(x)e^{i\mu t}$,

$$\widehat{\mathcal{A}}w = \begin{pmatrix} (A - e^{i\eta}\mu^2)w \\ \gamma_0 w \end{pmatrix} = \begin{pmatrix} (A - \lambda)w \\ \gamma_0 w \end{pmatrix},$$

and similarly, the parametrix $\widehat{\mathcal{B}}^0$ and the remainder $\widehat{\mathcal{R}}$ act on such functions like $\mathcal{B}^0(\lambda)$ and $\mathcal{R}(\lambda)$ applied in the x-coordinate.

Moreover, for $w(x,t) = u(x)e^{i\mu t}$, $u \in \mathcal{S}(\mathbb{R}^n)$, $\mu \in 2\pi\mathbb{Z}$,

$$\|w\|_{H^s(\mathbb{R}^n \times S^1)} \simeq \|(1 - \Delta + \mu^2)^{s/2}u(x)\|_{L_2(\mathbb{R}^n)} \simeq \|(1 + |\xi|^2 + \mu^2)^{s/2}\hat{u}(\xi)\|_{L_2},$$

with similar relations for Sobolev spaces over other sets. Norms as in the right-hand side are called $H^{s,\mu}$-norms; they were extensively used in [27], see the Appendix there for the definition on subsets. For the parametrix $\mathcal{B}^0(\lambda)$ this implies

$$\|\mathcal{B}^0(\lambda)\{f,g\}\|_{H^{s+2,\mu}(\Omega)} \leq c_s\|\{f,g\}\|_{H^{s,\mu}(\Omega) \times H^{s+\frac{3}{2},\mu}(\Sigma)}. \tag{4.21}$$

The important observation is now that when $s' < s$ and $w(x,t) = u(x)e^{i\mu t}$, then

$$\begin{aligned}
\|w\|_{H^{s'}(\mathbb{R}^n \times S^1)} &\simeq \|(1 + |\xi|^2 + \mu^2)^{s'/2}\hat{u}(\xi)\|_{L_2} \\
&\leq \langle\mu\rangle^{s'-s}\|(1 + |\xi|^2 + \mu^2)^{s/2}\hat{u}(\xi)\|_{L_2} \simeq \langle\mu\rangle^{s-s'}\|w\|_{H^s(\mathbb{R}^n \times S^1)},
\end{aligned}$$

with constants independent of u and μ. Analogous estimates hold with $\mathbb{R}^n$ replaced by Ω or Σ.

Applying this principle to the estimates of the remainder $\widehat{\mathcal{R}}$, we find that

$$\|\mathcal{R}(\lambda)\{f,g\}\|_{H^{s,\mu}(\Omega)\times H^{s+\frac{3}{2},\mu}(\Sigma)} \leq c_s\|\{f,g\}\|_{H^{s-\theta,\mu}(\Omega)\times H^{s+\frac{3}{2}-\theta,\mu}(\Sigma)}$$
$$\leq c_s'\langle\mu\rangle^{-\theta}\|\{f,g\}\|_{H^{s,\mu}(\Omega)\times H^{s+\frac{3}{2},\mu}(\Sigma)} \qquad (4.22)$$

for s as in (4.17), $\lambda = e^{i\eta}\mu^2$ with $\mu \in 2\pi\mathbb{N}_0$. One way to extend the observation to arbitrary λ on the ray, is to write $\lambda = e^{i\eta}\mu^2 = e^{i\eta}(\mu_0 + \mu')^2$ with $\mu_0 \in 2\pi\mathbb{N}_0$, $\mu' \in [0, 2\pi)$, and set $\lambda_0 = e^{i\eta}\mu_0^2$. Using (4.21) and observing that $(1 + |\xi|^2 + \mu_0^2)^{t/2} \simeq (1 + |\xi|^2 + (\mu_0 + \mu')^2)^{t/2}$ uniformly in ξ, μ_0, μ', by elementary inequalities, we find for

$$\mathcal{R}(\lambda) = \mathcal{A}(\lambda)\mathcal{B}^0(\lambda_0) - I = \mathcal{A}(\lambda_0)\mathcal{B}^0(\lambda_0) - I + \begin{pmatrix} \lambda_0 - \lambda \\ 0 \end{pmatrix}\mathcal{B}^0(\lambda_0) \qquad (4.23)$$

that

$$\|\mathcal{R}(\lambda)\{f,g\}\|_{H^{s,\mu}(\Omega)\times H^{s+\frac{3}{2},\mu}(\Sigma)} \leq c_s''\|\mathcal{R}(\lambda)\{f,g\}\|_{H^{s,\mu_0}(\Omega)\times H^{s+\frac{3}{2},\mu_0}(\Sigma)}$$
$$\leq c_s'''\|\{f,g\}\|_{H^{s-\theta,\mu_0}(\Omega)\times H^{s+\frac{3}{2}-\theta,\mu_0}(\Sigma)} \leq c_s''''\|\{f,g\}\|_{H^{s-\theta,\mu}(\Omega)\times H^{s+\frac{3}{2}-\theta,\mu}(\Sigma)}.$$

So (4.22) also holds for general λ, when we define $\mathcal{B}^0(\lambda) = \mathcal{B}^0(\lambda_0)$.

For each s, consider $\lambda = e^{i\eta}\mu^2$ with $\mu \geq \mu_1$, where μ_1 is taken so large that $c_s'\langle\mu\rangle^{-\theta} \leq \frac{1}{2}$ for $\mu \geq \mu_1$. Then $I + \mathcal{R}(\lambda)$ has the inverse $I + \mathcal{R}'(\lambda) = I + \sum_{k\geq 1}(-\mathcal{R}(\lambda))^k$ (converging in the operator norm for operators on $H^{s,\mu}(\Omega) \times H^{s+\frac{3}{2},\mu}(\Sigma)$), and, by definition of $\mathcal{B}^0(\lambda)$,
$$\mathcal{A}(\lambda)\mathcal{B}^0(\lambda)(I + \mathcal{R}'(\lambda)) = I.$$

This gives a right inverse

$$\mathcal{B}(\lambda) = \mathcal{B}^0(\lambda) + \mathcal{B}^0(\lambda)\mathcal{R}'(\lambda) = \begin{pmatrix} R(\lambda) & K(\lambda) \end{pmatrix},$$

with the same Sobolev space continuity (4.21) as $\mathcal{B}^0(\lambda)$, and $\mathcal{B}^0(\lambda)\mathcal{R}'(\lambda)$ of lower order:

$$\|\mathcal{B}^0(\lambda)\mathcal{R}'(\lambda)\{f,g\}\|_{H^{s+2,\mu}(\Omega)} \leq c_s\|\{f,g\}\|_{H^{s-\theta,\mu}(\Omega)\times H^{s-\theta+\frac{3}{2},\mu}(\Sigma)}$$
$$\leq c_s'\langle\mu\rangle^{-\theta}\|\{f,g\}\|_{H^{s,\mu}(\Omega)\times H^{s+\frac{3}{2},\mu}(\Sigma)}. \qquad (4.24)$$

Since
$$\mathcal{A}(\lambda)\mathcal{B}(\lambda) = \begin{pmatrix} (A-\lambda)R(\lambda) & (A-\lambda)K(\lambda) \\ \gamma_0 R(\lambda) & \gamma_0 K(\lambda) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad (4.25)$$

$R(\lambda)$ solves
$$(A-\lambda)u = f, \quad \gamma_0 u = 0, \qquad (4.26)$$

and $K(\lambda)$ solves
$$(A-\lambda)u = 0, \quad \gamma_0 u = \varphi. \qquad (4.27)$$

Since $R(\lambda)$ maps $L_2(\Omega)$ into $H^2(\Omega) \cap H_0^1(\Omega) \subset D(A_\gamma)$, it must coincide with the resolvent $(A_\gamma - \lambda)^{-1}$ of A_γ defined in Section 4.1 by variational theory. The operator $K(\lambda)$ is the Poisson-type solution operator of the Dirichlet problem with zero interior data; it is often denoted by K_γ^λ and we shall also use this notation here. The operators have the mapping properties, for each $\lambda = e^{i\eta}\mu^2$, $\mu \geq \mu_1$,

$$(A_\gamma - \lambda)^{-1}: H^s(\Omega) \to H^{s+2}(\Omega), \quad K_\gamma^\lambda: H^{s+\frac{3}{2}}(\Sigma) \to H^{s+2}(\Omega), \tag{4.28}$$

for s satisfying (4.17).

Moreover, the mapping properties extend to all the λ for which the resolvents and Poisson operators exist as solution operators to (4.26), (4.27), in particular to $\lambda = 0$. For A_γ^{-1}, this goes as follows: When $u \in H^1(\Omega)$ and $f \in H^s(\Omega)$ with $s < 1$, $f + \lambda u$ is likewise in $H^s(\Omega)$. Then $A_\gamma u = f + \lambda u$ allows the conclusion $u \in H^{s+2}(\Omega)$. The argument works for all $|s| < \tau$. Moreover, since $A_\gamma^{-1} - (A_\gamma - \lambda)^{-1} = -\lambda A_\gamma^{-1}(A_\gamma - \lambda)^{-1}$ is of lower order than A_γ^{-1}, A_γ^{-1} coincides with $R^0(0)$ plus a lower order remainder. The Poisson operator solving (4.27) can be further described as follows: There is a right inverse $\mathcal{K}: H^{s+\frac{3}{2}}(\Sigma) \to H^{s+2}(\Omega)$ of γ_0 for $-\frac{3}{2} < s \leq 0$ (cf. Theorem 2.11). When we set $v = u - \mathcal{K}\varphi$, we find that v should solve

$$(A - \lambda)v = -(A - \lambda)\mathcal{K}\varphi, \quad \gamma_0 v = 0,$$

to which we apply the preceding results; then when $\lambda \in \varrho(A_\gamma)$,

$$K_\gamma^\lambda = \mathcal{K} - (A_\gamma - \lambda)^{-1}(A - \lambda)\mathcal{K}; \tag{4.29}$$

solves (4.27) uniquely. Thus K_γ^λ exists for all $\lambda \in \varrho(A_\gamma)$.

Since the formal adjoint A' of A is similar to A (with regards to strong ellipticity and smoothness properties of the coefficients in its divergence form), the same construction works for the adjoint Dirichlet problem, so also here we get the mapping properties

$$(A_\gamma' - \bar{\lambda})^{-1}: H^s(\Omega) \to H^{s+2}(\Omega), \quad K_\gamma'^{\bar{\lambda}}: H^{s+\frac{3}{2}}(\Sigma) \to H^{s+2}(\Omega), \tag{4.30}$$

for $-\tau < s \leq 0$.

The above analysis shows moreover that

$$R(\lambda) = R^0(\lambda) + S(\lambda), \qquad K(\lambda) = K^0(\lambda) + S'(\lambda), \tag{4.31}$$

where

$$\|R^0(\lambda)\|_{\mathcal{L}(H^{s,\mu}(\Omega),H^{s+2,\mu}(\Omega))}, \ \|K^0(\lambda)\|_{\mathcal{L}(H^{s+\frac{3}{2},\mu}(\Sigma),H^{s+2,\mu}(\Omega))} \text{ are } O(1),$$

$$\|S(\lambda)\|_{\mathcal{L}(H^{s-\theta,\mu}(\Omega),H^{s+2,\mu}(\Omega))}, \ \|S'(\lambda)\|_{\mathcal{L}(H^{s+\frac{3}{2}-\theta,\mu}(\Sigma),H^{s+2,\mu}(\Omega))} \text{ are } O(1), \tag{4.32}$$

$$\|S(\lambda)\|_{\mathcal{L}(H^{s,\mu}(\Omega),H^{s+2,\mu}(\Omega))}, \ \|S'(\lambda)\|_{\mathcal{L}(H^{s+\frac{3}{2},\mu}(\Sigma),H^{s+2,\mu}(\Omega))} \text{ are } O(\langle\lambda\rangle^{-\theta/2}),$$

for λ going to infinity on rays $\lambda = e^{i\eta}\mu^2$, $\eta \in (\pi/2 - \delta, 3\pi/2 + \delta)$, when s, θ are as in (4.17). Here $R^0(\lambda)$, $K^0(\lambda)$ are explicit parametrices as in (4.15)–(4.16) (modified to depend on λ). For "stationary" norms, one has in particular

$$\|R(\lambda)f\|_{s+2} + \langle\lambda\rangle^{1+s/2}\|R(\lambda)f\|_s \le C_s \min\{\|f\|_s, \langle\lambda\rangle^{s/2}\|f\|_0\}, \tag{4.33}$$

$$\|K(\lambda)g\|_{s+2} + \langle\lambda\rangle^{1+s/2}\|K(\lambda)g\|_s \le C_s'\big(\|g\|_{s+\frac{3}{2}} + \langle\lambda\rangle^{3/4+s/2}\|g\|_0\big). \tag{4.34}$$

Note that $\|R(\lambda)\|_{\mathcal{L}(L_2(\Omega))}$ is $O(\langle\lambda\rangle^{-1})$ on the ray.

Summing up, we have proved:

Theorem 4.1 *Let Ω, τ, and A be as in Assumption 2.18 and let $-\tau < s \le 0$. Then for $\lambda \in \varrho(A_\gamma)$, the operator*

$$\begin{pmatrix} A - \lambda \\ \gamma_0 \end{pmatrix} : H^{s+2}(\Omega) \to \begin{matrix} H^s(\Omega) \\ \times \\ H^{s+\frac{3}{2}}(\Sigma) \end{matrix}; \tag{4.35}$$

has an inverse

$$\begin{pmatrix} R(\lambda) & K(\lambda) \end{pmatrix} = \begin{pmatrix} (A_\gamma - \lambda)^{-1} & K_\gamma^\lambda \end{pmatrix} : \begin{matrix} H^s(\Omega) \\ \times \\ H^{s+\frac{3}{2}}(\Sigma) \end{matrix} \to H^{s+2}(\Omega). \tag{4.36}$$

On the rays $\lambda = e^{i\eta}\mu^2$ with $\eta \in (\pi/2 - \delta, 3\pi/2 + \delta)$ (outside the range of the principal symbol), the inverse exists for $|\lambda|$ sufficiently large. $R(\lambda)$ and $K(\lambda)$ have the structure in (4.31) and satisfy estimates (4.32)–(4.34).

Similar statements hold for A'.

There is a class related condition $s > -\frac{3}{2}$, cf. Theorem A.8 and the beginning of Section 4, that prevents the above construction (even if τ were > 2) from defining the Poisson operator to start in the space $H^{-\frac{1}{2}}(\Sigma)$, but that will be needed for an analysis as in Section 3. Fortunately, it is possible to get supplementing information in other ways, as we shall see below.

5   Dirichlet-to-Neumann operators

5.1   An extension of Green's formula

For a general treatment of realizations of A, we need to extend the trace and Poisson operators to low-order Sobolev spaces. We begin by establishing an extension of Green's formula.

For $\lambda \in \varrho(A_\gamma)$, $s \in [0, 2]$, let

$$Z_\lambda^s(A) = \{u \in H^s(\Omega) \mid (A - \lambda)u = 0\}; \tag{5.1}$$

it is a closed subspace of $H^s(\Omega)$. It follows from Theorem 2.11 that the trace operator γ_0 is continuous:

$$\gamma_0 \colon Z_\lambda^s(A) \to H^{s-\frac{1}{2}}(\Sigma), \tag{5.2}$$

for $s \in (\frac{1}{2}, 2]$. Moreover, in view of the solvability properties shown in Section 4, it defines a homeomorphism

$$\gamma_0 \colon Z_\lambda^s(A) \xrightarrow{\sim} H^{s-\frac{1}{2}}(\Sigma), \tag{5.3}$$

for $s \in (2 - \tau, 2]$, with inverse $K_\gamma^\lambda = K(\lambda)$.

As shown in Section 2.4, the trace operators γ_1, χ and χ' define continuous maps

$$\gamma_1, \chi, \chi' \colon Z_\lambda^s(A) \to H^{s-\frac{3}{2}}(\Sigma) \tag{5.4}$$

for all $s \in (\frac{3}{2}, 2]$.

We need an extension of these mapping properties to all $s \in [0, 2]$, along with an extension of Green's formula to $u \in D(A_{\max})$, $v \in H^2(\Omega)$. This is shown by the method of Lions and Magenes [38]. Here we use the restriction operator r_Ω (restricting distributions from $\mathbb{R}^n$ to Ω) and the extension-by-zero operator e_Ω (extending functions on Ω by zero on $\mathbb{R}^n \setminus \Omega$).

An important ingredient is the following denseness result:

Proposition 5.1 *The space $C_{(0)}^\infty(\overline{\Omega}) = r_\Omega C_0^\infty(\mathbb{R}^n)$ is dense in $D(A_{\max})$ (provided with the graph-norm).*

Proof: This follows if we show that when ℓ is a continuous antilinear (conjugate linear) functional on $D(A_{\max})$ which vanishes on $C_{(0)}^\infty(\overline{\Omega})$, then $\ell = 0$. So let ℓ be such a functional; it can be written as

$$\ell(u) = (f, u)_{L_2(\Omega)} + (g, Au)_{L_2(\Omega)} \tag{5.5}$$

for some $f, g \in L_2(\Omega)$. We know that $\ell(\varphi) = 0$ for $\varphi \in C_{(0)}^\infty(\overline{\Omega})$. Any such φ is the restriction to Ω of a function $\Phi \in C_0^\infty(\mathbb{R}^n)$, and in terms of such functions we have

$$\ell(r_\Omega \Phi) = (e_\Omega f, \Phi)_{L_2(\mathbb{R}^n)} + (e_\Omega g, A_e \Phi)_{L_2(\mathbb{R}^n)} = 0, \text{ all } \Phi \in C_0^\infty(\mathbb{R}^n). \tag{5.6}$$

The equations to the right in (5.6) imply, in terms of the formal adjoint A_e' on $\mathbb{R}^n$,

$$\langle e_\Omega f + A_e' e_\Omega g, \overline{\Phi} \rangle = 0, \text{ all } \Phi \in C_0^\infty(\mathbb{R}^n),$$

i.e.,

$$e_\Omega f + A_e' e_\Omega g = 0, \text{ or } A_e' e_\Omega g = -e_\Omega f, \tag{5.7}$$

as distributions on $\mathbb{R}^n$. Here we know that $e_\Omega g$ and $e_\Omega f$ are in $L_2(\mathbb{R}^n)$, and the solvability properties of A_e' then imply that $e_\Omega g \in H^2(\mathbb{R}^n)$. Since it has support in $\overline{\Omega}$, it identifies with a function in $H_0^2(\Omega)$, i.e., $g \in H_0^2(\Omega)$. Then by (4.2), g is in $D(A_{\min}')$. And (5.7) implies that $A'g = -f$. But then, for any $u \in D(A_{\max})$,

$$\ell(u) = (f, u)_{L_2(\Omega)} + (g, Au)_{L_2(\Omega)} = -(A'g, u)_{L_2(\Omega)} + (g, Au)_{L_2(\Omega)} = 0,$$

since $A_{\max}$ and $A'_{\min}$ are adjoints.   ∎

We shall show:

Theorem 5.2 *The collected trace operator $\{\gamma_0, \chi\}$, defined on $C^\infty_{(0)}(\overline{\Omega})$, extends by continuity to a continuous mapping from $D(A_{\max})$ to $H^{-\frac{1}{2}}(\Sigma) \times H^{-\frac{3}{2}}(\Sigma)$. Here Green's formula (2.43) extends to the formula*

$$(Au, v)_{L_2(\Omega)} - (u, A'v)_{L_2(\Omega)} = (\chi u, \gamma_0 v)_{-\frac{3}{2},\frac{3}{2}} - (\gamma_0 u, \chi' v)_{-\frac{1}{2},\frac{1}{2}}, \qquad (5.8)$$

for $u \in D(A_{\max})$, $v \in H^2(\Omega)$.

Proof: Let $u \in D(A_{\max})$. We want to define $\{\gamma_0 u, \chi u\}$ as a continuous antilinear functional on $H^{\frac{1}{2}}(\Sigma) \times H^{\frac{3}{2}}(\Sigma)$, depending continuously (and of course linearly) on $u \in D(A_{\max})$. For this we use that

$$\begin{pmatrix} \gamma_0 \\ \\ \chi' \end{pmatrix} : H^2(\Omega) \to \begin{matrix} H^{\frac{3}{2}}(\Sigma) \\ \times \\ H^{\frac{1}{2}}(\Sigma) \end{matrix} \quad \text{has a continuous right inverse } \mathcal{K}' = \begin{pmatrix} \mathcal{K}'_0 & \mathcal{K}'_1 \end{pmatrix},$$

a lifting operator, cf. Corollary 2.20. For a given $\varphi = \{\varphi_0, \varphi_1\} \in H^{\frac{1}{2}}(\Sigma) \times H^{\frac{3}{2}}(\Sigma)$, we set

$$w_\varphi = \mathcal{K}'_0 \varphi_1 - \mathcal{K}'_1 \varphi_0; \text{ then } \gamma_0 w_\varphi = \varphi_1, \; \chi' w_\varphi = -\varphi_0.$$

Now we define

$$\begin{aligned} \ell_u(\varphi) &= (Au, w_\varphi) - (u, A' w_\varphi), \text{ noting that} \\ |\ell_u(\varphi)| &\le C \|u\|_{D(A_{\max})} \|w_\varphi\|_{H^2(\Omega)} \le C' \|u\|_{D(A_{\max})} \|\varphi\|_{H^{\frac{1}{2}}(\Sigma) \times H^{\frac{3}{2}}(\Sigma)}. \end{aligned} \qquad (5.9)$$

So, ℓ_u is a continuous antilinear functional on $\varphi \in H^{\frac{1}{2}}(\Sigma) \times H^{\frac{3}{2}}(\Sigma)$, hence defines an element $\psi = \{\psi_0, \psi_1\} \in H^{-\frac{1}{2}}(\Sigma) \times H^{-\frac{3}{2}}(\Sigma)$ such that

$$\ell_u(\varphi) = (\psi_0, \varphi_0)_{-\frac{1}{2},\frac{1}{2}} + (\psi_1, \varphi_1)_{-\frac{3}{2},\frac{3}{2}}. \qquad (5.10)$$

Moreover, it depends continuously on $u \in D(A_{\max})$, in view of the estimates in (5.9). If u is in $C^\infty_{(0)}(\overline{\Omega})$, the defining formula in (5.9) can be rewritten using Green's formula (2.43), which leads to

$$\ell_u(\varphi) = (Au, w_\varphi) - (u, A' w_\varphi) = (\chi u, \gamma_0 w_\varphi) - (\gamma_0 u, \chi' w_\varphi) = (\chi u, \varphi_1) + (\gamma_0 u, \varphi_0)$$

for such u. Since φ_0 and φ_1 run through full Sobolev spaces, it follows by comparison with (5.10) that $\psi_0 = \gamma_0 u$, $\psi_1 = \chi u$, when $u \in C^\infty_{(0)}(\overline{\Omega})$, so the functional ℓ_u is consistent with $\{\gamma_0 u, \chi u\}$ then. Since $C^\infty_{(0)}(\overline{\Omega})$ is dense in $D(A_{\max})$, we have found the unique continuous extension.

Identity (5.8) is now obtained in general by extending (2.43) by continuity from $u \in C^\infty_{(0)}(\overline{\Omega})$, $v \in H^2(\Omega)$.   ∎

In particular, the validity of the mapping properties of γ_0 and χ in (5.2) and (5.4) extend to $s = 0$.

5.2 Poisson operators

The next step is to extend the action of the Poisson operators to low-order spaces.

Lemma 5.3 *The composed operator $\chi'(A'_\gamma - \bar{\lambda})^{-1} \colon L_2(\Omega) \to H^{\frac{1}{2}}(\Sigma)$ has as adjoint an operator $(\chi'(A'_\gamma - \bar{\lambda})^{-1})^* \colon H^{-\frac{1}{2}}(\Sigma) \to L_2(\Omega)$ extending K^λ_γ (originally known to map $H^{s-\frac{1}{2}}(\Sigma)$ to $Z^s_\lambda(A)$ for $s \in (2-\tau, 2]$). Moreover, $(\chi'(A'_\gamma - \bar{\lambda})^{-1})^*$ ranges in $Z^0_\lambda(A)$.*

Proof: Let $\varphi \in H^{s-\frac{1}{2}}(\Sigma)$ for some $s \in (2-\tau, 2]$, let $u = K^\lambda_\gamma \varphi$. For any $f \in L_2(\Omega)$, let $v = (A'_\gamma - \bar{\lambda})^{-1} f$. Note that $(A - \lambda)u = 0$ and $\gamma_0 v = 0$. Then by (5.8),

$$
\begin{aligned}
-(K^\lambda_\gamma \varphi, f) &= ((A - \lambda)u, v) - (u, (A' - \bar{\lambda})v) \\
&= -(\gamma_0 u, \chi' v)_{-\frac{1}{2}, \frac{1}{2}} = -(\varphi, \chi'(A'_\gamma - \bar{\lambda})^{-1} f)_{-\frac{1}{2}, \frac{1}{2}}.
\end{aligned}
$$

This shows that the adjoint of $\chi'(A'_\gamma - \bar{\lambda})^{-1}$ acts like K^λ_γ on functions $\varphi \in H^{s-\frac{1}{2}}(\Sigma)$, $s \in (2-\tau, 2]$.

To see that $(\chi'(A'_\gamma - \bar{\lambda})^{-1})^*$ maps into the nullspace of $A - \lambda$, let $\varphi \in H^{-\frac{1}{2}}(\Sigma)$ and let $v \in C^\infty_0(\Omega)$. Then, using the definition of A in the weak sense,

$$
\begin{aligned}
\langle (A - \lambda)(\chi'(A'_\gamma - \bar{\lambda})^{-1})^* \varphi, \bar{v} \rangle_\Omega &= \langle (\chi'(A'_\gamma - \bar{\lambda})^{-1})^* \varphi, \overline{(A' - \bar{\lambda})v} \rangle_\Omega \\
&= ((\chi'(A'_\gamma - \bar{\lambda})^{-1})^* \varphi, (A' - \bar{\lambda})v)_{L_2(\Omega)} \\
&= (\varphi, \chi'(A'_\gamma - \bar{\lambda})^{-1}(A' - \bar{\lambda})v)_{-\frac{1}{2}, \frac{1}{2}} = 0,
\end{aligned}
$$

since $v \in C^\infty_0(\Omega)$ implies $(A'_\gamma - \bar{\lambda})^{-1}(A' - \bar{\lambda})v = v$ (since $\gamma_0 v = 0$), and $\chi' v = 0$. Thus $(A - \lambda)(\chi'(A'_\gamma - \bar{\lambda})^{-1})^* \varphi = 0$ in the weak sense, so since $(\chi'(A'_\gamma - \bar{\lambda})^{-1})^* \varphi \in L_2(\Omega)$, it lies in $Z^0_\lambda(A)$. ∎

Since $(\chi'(A'_\gamma - \bar{\lambda})^{-1})^*$ extends K^λ_γ and maps into $Z^0_\lambda(A)$, we *define* this to be the operator K^λ_γ for $s = 0$:

$$
K^\lambda_\gamma = (\chi'(A'_\gamma - \bar{\lambda})^{-1})^* \colon H^{-\frac{1}{2}}(\Sigma) \to L_2(\Omega). \tag{5.11}
$$

Theorem 5.4 *Let Ω and A satisfy Assumption 2.18. The operator K^λ_γ defined in (5.11) maps $H^{s-\frac{1}{2}}(\Sigma)$ to $H^s(\Omega)$ continuously for $0 \le s \le 2$. Moreover, γ_0 defined in Theorem 5.2 is a homeomorphism (5.3) for all $s \in [0, 2]$, with K^λ_γ acting as its inverse.*
There is a similar result for $K'^{\bar{\lambda}}_\gamma$.

Proof: Since the composed operator is continuous:

$$
\chi'(A'_\gamma - \lambda)^{-1} \colon H^s(\Omega) \to H^{s+\frac{1}{2}}(\Sigma)
$$

for $-\tau < s \le 0$, it follows by duality that

$$
K^\lambda_\gamma \colon H^{s'-\frac{1}{2}}(\Sigma) \to H^{s'}(\Omega), \tag{5.12}
$$

when $0 \leq s' < \tau$ (recall that $\tau < \frac{1}{2}$, cf. (2.42)). Taking this together with the larger values that were covered by (4.28), we find that (5.12) holds for

$$0 \leq s' \leq 2; \tag{5.13}$$

the intermediate values are included by interpolation. We can replace this s' by s.

The identities

$$\gamma_0 K_\gamma^\lambda \varphi = \varphi \text{ for } \varphi \in H^{s-\frac{1}{2}}(\Sigma), \quad K_\gamma^\lambda \gamma_0 z = z \text{ for } z \in Z_\lambda^s(A),$$

were shown in Section 4 to hold for $s \in (2 - \tau, 2]$. The first identity now extends by continuity to $H^{-\frac{1}{2}}(\Sigma)$, since $H^{\frac{3}{2}}(\Sigma)$ is dense in this space. The second identity will extend by continuity to $Z_\lambda^0(A)$, if we can prove that $Z_\lambda^2(A)$ is dense in $Z_\lambda^0(A)$. Indeed, this follows from Proposition 5.1:

Let $z \in Z_\lambda^0(A)$. By Proposition 5.1 applied to $A - \lambda$, there is a sequence $u_k \in C_{(0)}^\infty(\overline{\Omega})$ such that $u_k \to z$ and $(A - \lambda)u_k \to 0$ in $L_2(\Omega)$. Then $v_k = (A_\gamma - \lambda)^{-1}(A - \lambda)u_k \to 0$ in $H^2(\Omega)$. Let $z_k = u_k - v_k$; then $z_k \in H^2(\Omega)$, $(A - \lambda)z_k = 0$, and $z_k \to z$ in $L_2(\Omega)$. Hence, z_k is a sequence of elements of $Z_\lambda^2(A)$ that converges to z in $Z_\lambda^0(A)$, showing the desired denseness.

Thus the identities are valid for $s = 0$, and hence for all $s \in [0, 2]$. In particular, K_γ^λ maps $H^{s-\frac{1}{2}}(\Sigma)$ bijectively onto $Z_\lambda^s(A)$, and γ_0 maps $Z_\lambda^s(A)$ bijectively onto $H^{s-\frac{1}{2}}(\Sigma)$, for $s \in [0, 2]$, as inverses of one another.

The proof in the primed situation is analogous.   ∎

The adjoints also extend, e.g.

$$(K_\gamma'^{\bar\lambda})^* \colon H_0^s(\overline{\Omega}) \to H^{s+\frac{1}{2}}(\Sigma), \text{ for } -2 \leq s \leq 0; \tag{5.14}$$

recall that $H_0^s(\overline{\Omega}) = H^s(\Omega)$ when $|s| < \frac{1}{2}$.

From (4.32) we conclude moreover that when $0 \leq s < \tau$,

$$\|\chi'(A_\gamma' - \bar\lambda)^{-1}\|_{\mathcal{L}(H^{-s,\mu}(\Omega),H^{-s+\frac{1}{2},\mu}(\Sigma))} \text{ and } \|K_\gamma^\lambda\|_{\mathcal{L}(H^{s-\frac{1}{2},\mu}(\Sigma),H^{s,\mu}(\Omega))} \text{ are } O(1), \quad (5.15)$$

for λ going to infinity on rays $\lambda = e^{i\eta}\mu^2$, $\eta \in (\pi/2 - \delta, 3\pi/2 + \delta)$. In particular,

$$\|K_\gamma^\lambda \varphi\|_0 \leq C \min\{\|\varphi\|_{-\frac{1}{2}}, \langle\lambda\rangle^{-1/4}\|\varphi\|_0\}. \tag{5.16}$$

We shall now analyze the structure somewhat further. It should be noted that a Poisson operator maps a Sobolev space over Σ to a Sobolev space over Ω; the co-restriction of K_γ^λ mapping into $Z_\lambda^0(A)$ will be denoted $\gamma_{Z_\lambda}^{-1}$ further below, cf. (6.8).

Theorem 5.5 *Let* $0 < \delta < 1$, *and let* Ω *and* A *satisfy Assumption* 2.18.

$1°$ K_γ^λ *is the sum of a Poisson operator of the form*

$$K_\gamma^{0,\lambda} v = \sum_{j=1}^J \psi_j F_j^{-1,*} \Lambda_{-,+}^{-2}(\lambda) k_{j,\lambda}(x', D_x) F_{j,0}^* \varphi_j v, \tag{5.17}$$

where $k_{j,\lambda}$ has symbol-kernel $\tilde{k}_{j,\lambda} \in C^\tau S_{1,0}^1(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, and a remainder $\mathcal{S}(\lambda)$ that for $s \in (2 - \tau, 2]$ maps $H^{s-\frac{1}{2}-\theta}(\mathbb{R}^{n-1}) \to H^s(\mathbb{R}_+^n)$, when $0 < \theta < s - 2 + \tau$.

2° K_γ^λ is a generalized Poisson operator in the sense that it is the sum of a Poisson operator of the form

$$K_\gamma^{\sharp\lambda} v = \sum_{j=1}^J \psi_j F_j^{-1,*} \Lambda_{-,+}^{-2}(\lambda) k_{j,\lambda}^\sharp(x', D_x) F_{j,0}^* \varphi_j v, \qquad (5.18)$$

with $\tilde{k}_{j,\lambda}^\sharp \in S_{1,\delta}^1(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, and a remainder $\mathcal{R}(\lambda)$ that for $s \in (0, 2]$ maps $H^{s-\frac{1}{2}-\varepsilon}(\mathbb{R}^{n-1}) \to H^s(\mathbb{R}_+^n)$, for some $\varepsilon = \varepsilon(s) > 0$.

Here ψ_j, φ_j, F_j, and $F_{j,0}$ are as in Remark 2.9.

There are similar statements for the primed version $K_\gamma'^{\bar{\lambda}}$.

Proof: We give the proof of the statements for K_γ^λ; the proofs for $K_\gamma'^{\bar{\lambda}}$ are analogous.

The first statement follows from the construction in Section 4.3, applied to $A - \lambda$ and with λ-dependent order-reducing operators (where ξ is replaced by (ξ, μ), $\mu = |\lambda|^{\frac{1}{2}} \in \overline{\mathbb{R}}_+$). The composition $k_{j,\lambda}$ of the λ-dependent variant of k_j^0 and $\Lambda_0^{\frac{3}{2}}(\lambda)$ is of order 2 and has symbol-kernel $\tilde{k}_{j,\lambda}$ in $C^\tau S_{1,0}^1(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$ for each λ. The mapping properties of $\mathcal{S}(\lambda)$ follow from (4.32).

For the second statement, observe that we have from 1° that

$$K_\gamma^\lambda = K_\gamma^{0,\lambda} + S(\lambda), \quad \text{where } S(\lambda) \in \mathcal{L}(H^{\frac{3}{2}-\theta}(\Sigma), H^2(\Omega)), \qquad (5.19)$$

for every $0 < \theta < \tau$. Now, applying Lemma A.7 for $\delta \in (0, 1)$, we obtain that $K_\gamma^{0,\lambda} = K_\gamma^{\sharp\lambda} + S'(\lambda)$, where $K_\gamma^{\sharp\lambda}$ is as described in (5.18) and

$$S'(\lambda): H^{\frac{3}{2}-\tau\delta}(\Sigma) \to H^2(\Omega).$$

Since also $K_\gamma^\lambda, K_\gamma^{\sharp\lambda} \in \mathcal{L}(H^{s-\frac{1}{2}}(\Sigma), H^s(\Omega))$ for all $s \in [0, 2]$, interpolation yields that for every $s \in (0, 2]$ there is some $\varepsilon = \varepsilon(s) > 0$ such that

$$K_\gamma^\lambda - K_\gamma^{\sharp\lambda} \in \mathcal{L}(H^{s-\frac{1}{2}-\varepsilon}(\Sigma), H^s(\Omega)).$$

This proves the theorem. ∎

Remark 5.6 As a technical observation we note that the above approximate Poisson solution operators $K_\gamma^{0,\lambda}$ and $K_\gamma^{\sharp\lambda}$ are constructed in such a way that their symbol-kernels are smooth in $(\xi', \mu) \in \overline{\mathbb{R}}_+^n$ (outside a neighborhood of zero); this is the case of symbols "of regularity $+\infty$" in the sense of [27].

5.3   Dirichlet-to-Neumann operators

Finally, we shall study the composed operators $P_{\gamma,\chi}^{\lambda} = \chi K_{\gamma}^{\lambda}$ and $P_{\gamma,\chi'}^{\prime\bar{\lambda}} = \chi' K_{\gamma}^{\prime\bar{\lambda}}$; often called Dirichlet-to-Neumann operators.

It follows immediately from Theorem 5.4 that they are continuous for $s \in [0, 2]$,

$$P_{\gamma,\chi}^{\lambda}, P_{\gamma,\chi'}^{\prime\bar{\lambda}} \colon H^{s-\frac{1}{2}}(\Sigma) \to H^{s-\frac{3}{2}}(\Sigma). \tag{5.20}$$

Applying Green's formula (5.8) to functions u, v with $Au = 0$, $A'v = 0$, we see that

$$(P_{\gamma,\chi}^{\lambda}\varphi, \psi)_{-\frac{3}{2},\frac{3}{2}} = (\varphi, P_{\gamma,\chi'}^{\prime\bar{\lambda}}\psi)_{-\frac{1}{2},\frac{1}{2}}$$

for all $\varphi \in H^{-\frac{1}{2}}(\Sigma), \psi \in H^{\frac{3}{2}}(\Sigma)$, so $P_{\gamma,\chi}^{\lambda}$ and $P_{\gamma,\chi'}^{\prime\bar{\lambda}}$ are consistent with each other's adjoints.

Theorem 5.7 *Assumptions as in Theorem 5.5. $P_{\gamma,\chi}^{\lambda}$ maps $H^{s-\frac{1}{2}}(\Sigma)$ continuously to $H^{s-\frac{3}{2}}(\Sigma)$ for $s \in [0, 2]$, and satisfies:*

1° $P_{\gamma,\chi}^{\lambda}$ is the sum of a first-order ψdo of the form

$$S^{\lambda}v = \sum_{j=1}^{J} \eta_j \widetilde{F}_{j,0}^{-1,*} \Lambda_0^{-\frac{1}{2}} s_{j,\lambda}(x', D_{x'}) F_{j,0}^* \varphi_j v,$$

where $s_{j,\lambda} \in C^{\tau} S_{1,0}^{\frac{3}{2}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, and a remainder, such that for every $s \in (2-\tau, 2]$, the remainder maps $H^{s-\frac{1}{2}-\varepsilon}(\Sigma)$ continuously to $H^{s-\frac{3}{2}}(\Sigma)$ for some $\varepsilon = \varepsilon(s) > 0$.

2° There is a pseudodifferential operator $P_{\gamma,\chi}^{\sharp\lambda}$ on Σ with symbol in $S_{1,\delta}^{1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ (cf. (5.21)), such that for every $s \in (0, 2]$, $P_{\gamma,\chi}^{\lambda}$ is a generalized ψdo of order 1 in the sense that it is the sum of $P_{\gamma,\chi}^{\sharp\lambda}$ and a remainder mapping $H^{s-\frac{1}{2}-\varepsilon}(\Sigma)$ continuously to $H^{s-\frac{3}{2}}(\Sigma)$ for some $\varepsilon = \varepsilon(s) > 0$. Here $\varepsilon > 0$ can be chosen uniformly with respect to $s \in [s', 2]$ for every $s' \in (0, 2]$.

3° There is a pseudodifferential operator $P_{\gamma,\chi}^{\sharp\lambda 1}$ on Σ with symbol in $S_{1,\delta}^{1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ (cf. (5.23)), such that for $s = 0$, $P_{\gamma,\chi}^{\lambda}$ is the sum of $P_{\gamma,\chi}^{\sharp\lambda 1}$ and a remainder mapping $H^{-\frac{1}{2}}(\Sigma)$ continuously to $H^{-\frac{3}{2}+\varepsilon}(\Sigma)$ for some $\varepsilon > 0$.

There are similar statements for $P_{\gamma,\chi'}^{\prime\bar{\lambda}}$; it acts like an adjoint of $P_{\gamma,\chi}^{\lambda}$.

Proof: The first and last statements were shown above. We shall prove statements 1°–3° for $P_{\gamma,\chi}^{\lambda}$; the treatment of $P_{\gamma,\chi'}^{\prime\bar{\lambda}}$ is analogous.

For 1°, we note that since K_{γ}^{λ} coincides in highest order with the approximation (5.17), $P_{\gamma,\chi}^{\lambda}$ coincides in the highest order with

$$\chi K_{\gamma}^{0,\lambda}v = \sum_{j=1}^{J} \chi \psi_j(x) F_j^{-1,*} \Lambda_{-,+}^{-2} k_{j,\lambda}(x', D_x) F_{j,0}^* \varphi_j v;$$

here $v \in H^{s-\frac{1}{2}}(\Sigma)$, $s \in (2-\tau, 2]$, and $\tilde{k}_{j,\lambda} \in C^{\tau} S_{1,0}^{1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$.

Moreover, because of Corollary 2.20, for every $\eta_j \in C_{(0)}^\infty(\overline{\Omega})$ with $\eta_j \equiv 1$ on supp ψ_j and supp $\eta_j \subset U_j$ we have the representation

$$\chi(\psi_j F_j^{-1,*} v_j) \quad = \quad \eta_j s_0 \gamma_1(\psi_j F_j^{-1,*} v_j) + \eta_j \sum_{|\alpha| \leq 1} c_{\alpha,j} D_{x'}^\alpha \psi_j \gamma_0 F_j^{-1,*} v_j$$

$$= \quad \eta_j \widetilde{F}_{j,0}^{-1,*} t_j(x', D_x) v_j,$$

where the operators $t_j(x', D_x)$ are differential trace operators of order 1 and class 2 on $\mathbb{R}_+^n$ (in x-form) with coefficients in $H_p^{\frac{1}{2}}(\mathbb{R}^{n-1})$. — Note that the factor κ in the definition of $\widetilde{F}_{j,0}^{-1,*}$, cf. (2.27), can be absorbed into the coefficients of $t_j(x', D_x)$. — Now if we set $t_{j,\lambda}'(x', \xi', D_n) = \langle (\xi', \mu) \rangle^{\frac{1}{2}} t_j(x', \xi', D_n)$ (where as usual $\mu = |\lambda|^{\frac{1}{2}}$ and $\langle (\xi', \mu) \rangle^r$ is the symbol of $\Lambda_0^r(\lambda)$), we have from Theorem 2.17 that

$$\Lambda_0^{\frac{1}{2}} t_j(x', D_x) - t_{j,\lambda}'(x', D_x) \colon H^{s-\varepsilon}(\mathbb{R}_+^n) \to H^{s-2}(\mathbb{R}^{n-1}),$$

for all s, ε such that $\frac{3}{2} + \varepsilon < s \leq 2$ and $0 < \varepsilon < \tau = \frac{1}{2} - \frac{n-1}{p}$. In particular, we can choose $s = 2$. Moreover, since $t_j(x', D_x)$ is a differential trace operator, we have that

$$t_{j,\lambda}'(x', D_x)\Lambda_{-,+}^{-2} \quad = \quad b_{j,1,\lambda}(x', D_{x'})\gamma_1 \Lambda_{-,+}^{-2} + b_{j,0,\lambda}(x', D_{x'})\gamma_0 \Lambda_{-,+}^{-2}$$

for some $b_{j,k,\lambda} \in H_p^{\frac{1}{2}} S_{1,0}^{\frac{3}{2}-k}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, $k = 0, 1$, which implies that $t_{j,\lambda}'(x', D_x)\Lambda_{-,+}^{-2} = t_{j,\lambda}''(x', D_x)$ with $\tilde{t}_{j,\lambda}'' \in C^\tau S_{1,0}^{-\frac{1}{2}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\mathbb{R}^{n-1}))$, since $H_p^{\frac{1}{2}}(\mathbb{R}^{n-1}) \hookrightarrow C^\tau(\mathbb{R}^{n-1})$. (More precisely, $b_{j,1,\lambda}(x', \xi') = s_0(x')\langle (\xi', \mu) \rangle^{\frac{1}{2}}$ and $b_{j,0,\lambda}(x', \xi') = \sum_{|\alpha| \leq 1} c_{\alpha,j}(x')(\xi')^\alpha \langle (\xi', \mu) \rangle^{\frac{1}{2}}$.) Set

$$s_{j,\lambda}(x', \xi') = t_{j,\lambda}''(x', \xi', D_n) k_{j,\lambda}(x', \xi', D_n);$$

it is in $C^\tau S_{1,0}^{\frac{3}{2}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$. Then we can apply the composition rules for Green operators with C^τ-coefficients, cf. [2, Theorem 4.13.3], to conclude that

$$t_{j,\lambda}'(x', D_x)\Lambda_{-,+}^{-2} k_{j,\lambda}(x', D_x) - s_{j,\lambda}(x', D_{x'}) \colon H^{\frac{3}{2}-\varepsilon}(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^{n-1}),$$

for some $\varepsilon > 0$.

Summing up, we have that

$$P_{\gamma,\chi}^\lambda v = \chi K_\gamma^\lambda v \quad = \quad \sum_{j=1}^J \eta_j \widetilde{F}_{j,0}^{-1,*} t_j(x', D_x)\Lambda_{-,+}^{-2} k_j^\lambda(x', D_x) F_{j,0}^* \varphi_j v + \mathcal{R}(\lambda)v$$

$$= \quad \sum_{j=1}^J \eta_j \widetilde{F}_{j,0}^{-1,*} \Lambda_0^{-\frac{1}{2}} s_{j,\lambda}(x', D_{x'}) F_{j,0}^* \varphi_j v + \mathcal{R}'(\lambda)v,$$

where $\mathcal{R}(\lambda)$ and $\mathcal{R}'(\lambda)$ are continuous from $H^{\frac{3}{2}-\varepsilon}(\Sigma)$ to $H^{\frac{1}{2}}(\Sigma)$ for some $\varepsilon > 0$.

Hence, if we define

$$S^\lambda v = \sum_{j=1}^{J} \eta_j \widetilde{F}_{j,0}^{-1,*} \Lambda_0^{-\frac{1}{2}} s_{j,\lambda}(x', D_{x'}) F_{j,0}^* \varphi_j v,$$

then $P_{\gamma,\chi}^\lambda - S^\lambda \colon H^{\frac{3}{2}-\varepsilon}(\Sigma) \to H^{\frac{1}{2}}(\Sigma)$ is a bounded operator for some $\varepsilon > 0$. Moreover, because of Proposition 2.14, $S^\lambda \in \mathcal{L}(H^{s-\frac{1}{2}}(\Sigma), H^{s-\frac{3}{2}}(\Sigma))$ for every $s \in (2-\tau, 2]$. Finally, interpolation yields that for every $s \in (2-\tau, 2]$ there is some $\varepsilon > 0$ such that

$$P_{\gamma,\chi}^\lambda - S^\lambda \colon H^{s-\frac{1}{2}-\varepsilon}(\Sigma) \to H^{s-\frac{3}{2}}(\Sigma)$$

is bounded. This proves the first statement.

To prove the second statement we apply symbol smoothing, cf. (A.1), to $s_{j,\lambda}$. Then we obtain for any $0 < \delta < 1$ that

$$s_{j,\lambda}(x', D_{x'}) = s_{j,\lambda}^\sharp(x', D_{x'}) + s_{j,\lambda}^b(x', D_{x'})$$

where $s_{j,\lambda}^\sharp \in S_{1,\delta}^{\frac{3}{2}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ and $s_{j,\lambda}^b \in C^\tau S_{1,\delta}^{\frac{3}{2}-\tau\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$. Hence,

$$\Lambda_0^{-\frac{1}{2}} s_{j,\lambda}(x', D_{x'}) - \Lambda_0^{-\frac{1}{2}} s_{j,\lambda}^\sharp(x', D_{x'}) \colon H^{\frac{3}{2}-\tau\delta}(\mathbb{R}^{n-1}) \to H^{\frac{1}{2}}(\mathbb{R}^{n-1}),$$

since

$$s_{j,\lambda}^b(x', D_{x'}) \colon H^{\frac{3}{2}-\tau\delta}(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^{n-1}),$$

by Proposition 2.14. Moreover, by the composition rules of the (smooth) $S_{1,\delta}^m$-calculus, cf. e.g. [36, Theorem 1.7, Chapter II],

$$\Lambda_0^{-\frac{1}{2}} s_{j,\lambda}^\sharp(x', D_{x'}) = p_{j,\lambda}^\sharp(x', D_{x'})$$

for some $p_{j,\lambda}^\sharp \in S_{1,\delta}^1(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$. Altogether, if we define

$$P_{\gamma,\chi}^{\sharp\lambda} v = \sum_{j=1}^{J} \eta_j \widetilde{F}_{j,0}^{-1,*} p_{j,\lambda}^\sharp(x', D_{x'}) F_{j,0}^* \varphi_j v, \tag{5.21}$$

then

$$P_{\gamma,\chi}^\lambda - P_{\gamma,\chi}^{\sharp\lambda} \colon H^{\frac{3}{2}-\varepsilon_0}(\Sigma) \to H^{\frac{1}{2}}(\Sigma) \tag{5.22}$$

is a bounded operator for some $\varepsilon_0 > 0$. Since the $p_{j,\lambda}^\sharp$ are smooth symbols, we have that

$$P_{\gamma,\chi}^\lambda, P_{\gamma,\chi}^{\sharp\lambda} \in \mathcal{L}(H^{s-\frac{1}{2}}(\Sigma), H^{s-\frac{3}{2}}(\Sigma))$$

for every $s \in [0, 2]$, by Proposition 2.14. Hence, interpolation implies that for every $s \in (0, 2]$ there is some $\varepsilon = \varepsilon(s) > 0$ such that

$$P_{\gamma,\chi}^\lambda - P_{\gamma,\chi}^{\sharp\lambda} \colon H^{s-\frac{1}{2}-\varepsilon}(\Sigma) \to H^{s-\frac{3}{2}}(\Sigma)$$

is a bounded operator. Here $\varepsilon = \varepsilon_0 \frac{s}{2}$, which shows that ε can be chosen uniformly with respect to $s \in [s', 2]$ for every $s' \in (0, 2]$.

Finally, the case $s = 0$ will be treated differently. To this end let $P'^{\sharp\bar{\lambda}}_{\gamma,\chi'}$ denote the approximation to $P'^{\bar{\lambda}}_{\gamma,\chi'}$ defined analogously to (5.21). Then

$$P^\lambda_{\gamma,\chi} - (P'^{\sharp\bar{\lambda}}_{\gamma,\chi'})^* = (P'^{\bar{\lambda}}_{\gamma,\chi'})^* - (P'^{\sharp\bar{\lambda}}_{\gamma,\chi'})^* \colon H^{-\frac{1}{2}}(\Sigma) \to H^{-\frac{3}{2}+\varepsilon}(\Sigma),$$

by (5.22) for the primed operators. Moreover, by the standard calculus for pseudo-differential operators with $S^m_{1,\delta}$-symbols,

$$(P'^{\sharp\bar{\lambda}}_{\gamma,\chi'})^* v = \sum_{j=1}^J \varphi_j \widetilde{F}^{-1,*}_{j,0} p'^{\sharp*}_{j,\lambda}(x', D_{x'}) F^*_{j,0} \eta_j v + \mathcal{R}(\lambda) v$$

$$= \sum_{j=1}^J \eta_j \widetilde{F}^{-1,*}_{j,0} p'^{\sharp*}_{j,\lambda}(x', D_{x'}) F^*_{j,0} \varphi_j v + \mathcal{R}'(\lambda) v = P^{\sharp\lambda 1}_{\gamma,\chi} v + \mathcal{R}'(\lambda) v \qquad (5.23)$$

for all $v \in H^{-\frac{1}{2}}(\Sigma)$, where $p'^{\sharp*}_{j,\lambda} \in S^1_{1,\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, and $\mathcal{R}(\lambda)$ and $\mathcal{R}'(\lambda)$ are bounded from $H^{-\frac{1}{2}}(\Sigma)$ to $H^{-\frac{1}{2}}(\Sigma)$. $\blacksquare$

Operators $P^\lambda_{\gamma,\beta}$ can be defined for any trace operator β, as $P^\lambda_{\gamma,\beta} = \beta K^\lambda_\gamma$. For $\beta = \gamma_1$, one has that $P^\lambda_{\gamma,\gamma_1} = \gamma_1 K^\lambda_\gamma$ is *elliptic* (the principal symbol is invertible), cf. [7], [24] (it is also documented in Ch. 11 of [29], Exercise 11.7ff.). Note that

$$P^\lambda_{\gamma,\chi} = s_0 P^\lambda_{\gamma,\gamma_1} + \mathcal{A}_1,$$

where $\mathcal{A}_1$ is the first-order differential operator on Σ explained in Theorem 2.19. The ellipticity of $P^\lambda_{\gamma,\chi}$ depends on $\mathcal{A}_1$, which is defined from the coefficients in the divergence form (2.38). It should be recalled that the choice of coefficients in (2.38) is not unique for a given operator A, see the discussion in [24]; it is shown there in the smooth case that the choice of coefficients in (2.38) can be adapted to give *any desired first-order differential operator on Σ* in the place of $\mathcal{A}_1$. Ellipticity of $P^\lambda_{\gamma,\chi}$ holds if and only if the system $\{A, \chi\}$ is elliptic.

In the papers [15], [28], χ is replaced by $\nu_1 = s_0 \gamma_1$ in order to have an elliptic Dirichlet-to-Neumann operator (then Green's formula looks slightly different). However, the modified trace operator Γ^λ introduced further below is actually independent of the choice of $\mathcal{A}_1$ (cf. Remark 6.2).

6 Boundary value problems

We shall now apply the abstract results from Section 3 to boundary value problems. The realizations $A_{\max}$, $A_{\min}$ and A_γ of A in $H = L_2(\Omega)$ introduced in the beginning of Section 4 have the properties described in Section 3, with the realizations $A'_{\min}$, $A'_{\max}$

and A'_γ of A' representing the adjoints. Then the general constructions of Section 3 can be applied. The operators $\widetilde{A} \in \mathcal{M}$ are the general realizations of A. Note that

$$Z_0^0(A) = Z, \quad Z_0^0(A') = Z', \quad Z_\lambda^0(A) = Z_\lambda, \quad Z_{\bar{\lambda}}^0(A') = Z'_{\bar{\lambda}}. \tag{6.1}$$

For an interpretation of the correspondence between $\widetilde{A}$ and $T\colon V \to W$, we need a modified version of Green's formula (as introduced originally in [23]). Here we use the trace operators (in the sense of the ψdbo calculus) defined by:

$$\Gamma^\lambda = \chi - P_{\gamma,\chi}^\lambda \gamma_0, \quad \Gamma'^{\bar{\lambda}} = \chi' - P_{\gamma,\chi'}^{\bar{\lambda}} \gamma_0 \tag{6.2}$$

for $\lambda \in \varrho(A_\gamma)$; we call them the *reduced Neumann trace operators*.

Theorem 6.1 *The trace operators Γ^λ and $\Gamma'^{\bar{\lambda}}$ map $D(A_{\max})$ resp. $D(A'_{\max})$ continuously onto $H^{\frac{1}{2}}(\Sigma)$, and satisfy*

$$\Gamma^0 = \chi A_\gamma^{-1} A_{\max}, \quad \Gamma'^0 = \chi'(A_\gamma^*)^{-1} A'_{\max}, \tag{6.3}$$

$$\Gamma^\lambda = \chi(A_\gamma - \lambda)^{-1}(A_{\max} - \lambda), \quad \Gamma'^{\bar{\lambda}} = \chi'(A_\gamma^* - \bar{\lambda})^{-1}(A'_{\max} - \bar{\lambda}). \tag{6.4}$$

In particular, Γ^0 vanishes on Z, etc.

With these trace operators there is a modified Green's formula valid **for all** $u \in D(A_{\max})$, $v \in D(A'_{\max})$:

$$(Au, v)_\Omega - (u, A'v)_\Omega = (\Gamma^0 u, \gamma_0 v)_{\frac{1}{2}, -\frac{1}{2}} - (\gamma_0 u, \Gamma'^0 v)_{-\frac{1}{2}, \frac{1}{2}}; \tag{6.5}$$

in particular,

$$(Au, w)_\Omega = (\Gamma^0 u, \gamma_0 w)_{\frac{1}{2}, -\frac{1}{2}}, \text{ when } w \in Z'. \tag{6.6}$$

Similarly, for all $u \in D(A_{\max})$, $v \in D(A'_{\max})$,

$$((A - \lambda)u, v)_\Omega - (u, (A' - \bar{\lambda})v)_\Omega = (\Gamma^\lambda u, \gamma_0 v)_{\frac{1}{2}, -\frac{1}{2}} - (\gamma_0 u, \Gamma'^{\bar{\lambda}} v)_{-\frac{1}{2}, \frac{1}{2}}, \tag{6.7}$$

which is also equal to $(Au, v)_\Omega - (u, A'v)_\Omega$.

Proof: With the preparations we have made, this goes exactly as in the smooth case [23]; we give some details for the convenience of the reader. Take $\lambda = 0$. Writing $u = u_\gamma + u_\zeta$, where $u_\gamma = A_\gamma^{-1} A_{\max} u$ and $u_\zeta \in Z$, we have that $\gamma_0 u = \gamma_0 u_\zeta$, and $u_\zeta = K_\gamma^0 \gamma_0 u$. Then

$$\Gamma^0 u = \chi u - P_{\gamma,\chi}^0 \gamma_0 u = \chi u - P_{\gamma,\chi}^0 \gamma_0 u_\zeta = \chi u - \chi u_\zeta = \chi u_\gamma = \chi A_\gamma^{-1} A_{\max} u.$$

This shows (6.3), and since $A_\gamma^{-1} A_{\max}$ is surjective from $D(A_{\max})$ to $D(A_\gamma)$, and χ is surjective from $D(A_\gamma) = H^2(\Omega) \cap H_0^1(\Omega)$ to $H^{\frac{1}{2}}(\Sigma)$, we get the surjectiveness from $D(A_{\max})$ to $H^{\frac{1}{2}}(\Sigma)$. (The continuity is with respect to the graph-norm on $D(A_{\max})$.)

Let $u \in D(A_{\max})$, $v \in D(A'_{\max})$. Then, using the fact that $(Au_\gamma, v_{\gamma'}) - (u_\gamma, A'v_{\gamma'}) = 0$, and using the extended Green's formula (5.8), we find

$$
\begin{aligned}
(Au, v) - (u, A'v) &= (Au_\gamma, v_{\gamma'} + v_{\zeta'}) - (u_\gamma + u_\zeta, A'v_{\gamma'}) \\
&= (Au_\gamma, v_{\zeta'}) - (u_\zeta, A'v_{\gamma'}) \\
&= (Au_\gamma, v_{\zeta'}) - (u_\gamma, A'v_{\zeta'}) + (Au_\zeta, v_{\gamma'}) - (u_\zeta, A'v_{\gamma'}) \\
&= (\chi u_\gamma, \gamma_0 v_{\zeta'})_{\frac{1}{2}, -\frac{1}{2}} - (\gamma_0 u_\gamma, \chi' v_{\zeta'})_{\frac{3}{2}, -\frac{3}{2}} + (\chi u_\zeta, \gamma_0 v_{\gamma'})_{-\frac{3}{2}, \frac{3}{2}} - (\gamma_0 u_\zeta, \chi' v_{\gamma'})_{-\frac{1}{2}, \frac{1}{2}} \\
&= (\chi u_\gamma, \gamma_0 v)_{\frac{1}{2}, -\frac{1}{2}} - (\gamma_0 u, \chi' v_{\gamma'})_{-\frac{1}{2}, \frac{1}{2}} = (\Gamma^0 u, \gamma_0 v)_{\frac{1}{2}, -\frac{1}{2}} - (\gamma_0 u, \Gamma'^0 v)_{-\frac{1}{2}, \frac{1}{2}}.
\end{aligned}
$$

This shows (6.5); (6.6) is a special case. The proof for general λ is similar. ∎

Remark 6.2 Note that when $\chi = s_0 \gamma_1 + \mathcal{A}_1 \gamma_0$, then

$$
\Gamma^\lambda = \chi - P^\lambda_{\gamma, \chi} \gamma_0 = s_0 \gamma_1 + \mathcal{A}_1 \gamma_0 - (P^\lambda_{\gamma, s_0 \gamma_1} - \mathcal{A}_1) \gamma_0 = s_0 \gamma_1 - P^\lambda_{\gamma, s_0 \gamma_1};
$$

so in fact Γ^λ is independent of the choice of coefficient of γ_0 in χ (cf. (2.44)).

By composition with suitable isometries (order-reducing operators), (6.5) can be turned into a formula with L_2-scalar products over the boundary, but since this leads to more overloaded formulas, we shall not pursue that line of thought here.

We denote by γ_{Z_λ} the restriction of γ_0 to a mapping from Z_λ (closed subspace of $L_2(\Omega)$) to $H^{-\frac{1}{2}}(\Sigma)$; its adjoint $\gamma^*_{Z_\lambda}$ goes from $H^{\frac{1}{2}}(\Sigma)$ to Z_λ:

$$
\gamma_{Z_\lambda} \colon Z_\lambda \overset{\sim}{\to} H^{-\frac{1}{2}}(\Sigma), \text{ with adjoint } \gamma^*_{Z_\lambda} \colon H^{\frac{1}{2}}(\Sigma) \overset{\sim}{\to} Z_\lambda. \tag{6.8}
$$

The inverse $\gamma^{-1}_{Z_\lambda}$ gives by composition with i_{Z_λ} the Poisson operator K^λ_γ:

$$
K^\lambda_\gamma = \mathrm{i}_{Z_\lambda} \gamma^{-1}_{Z_\lambda} \colon H^{-\frac{1}{2}}(\Sigma) \to L_2(\Omega).
$$

There is a similar notation for the primed operators. When $\lambda = 0$, this index can be left out.

For the study of general realizations $\widetilde{A}$ of A, the homeomorphisms (6.8) make it possible to translate the characterization in terms of operators $T \colon V \to W$ in Section 3 into a characterization in terms of operators L over the boundary.

First let $\lambda = 0$. For $V \subset Z$, $W \subset Z'$, let $X = \gamma_0 V$, $Y = \gamma_0 W$, with the notation for the restrictions of γ_0 to homeomorphisms:

$$
\gamma_V \colon V \overset{\sim}{\to} X, \quad \gamma_W \colon W \overset{\sim}{\to} Y. \tag{6.9}
$$

The map $\gamma_V \colon V \overset{\sim}{\to} X$ has the adjoint $\gamma^*_V \colon X^* \overset{\sim}{\to} V$. Here X^* denotes the antidual space of X, with a duality coinciding with the scalar product in $L_2(\Sigma)$ when applied to elements that come from $L_2(\Sigma)$. The duality is written $(\psi, \varphi)_{X^*, X}$. We also write $\overline{(\psi, \varphi)_{X^*, X}} = (\varphi, \psi)_{X, X^*}$. Similar conventions are applied to Y.

When A is replaced by $A - \lambda$ for $\lambda \in \varrho(A_\gamma)$, we use a similar notation where Z, Z', V and W are replaced by Z_λ, $Z'_{\bar\lambda}$, V_λ, $W_{\bar\lambda}$. Since $\gamma_0 E^\lambda z = \gamma_0 z$ (cf. Section 3), the mapping defined by γ_0 on V_λ has *the same range space X as when $\lambda = 0$*. Similarly, the mapping defined by γ_0 on $W_{\bar\lambda}$ has the range space Y for all λ. So γ_0 defines homeomorphisms

$$\gamma_{V_\lambda} \colon V_\lambda \xrightarrow{\sim} X, \quad \gamma_{W_{\bar\lambda}} \colon W_{\bar\lambda} \xrightarrow{\sim} Y, \tag{6.10}$$

For $\lambda \in \varrho(A_\gamma)$, we denote

$$K_{\gamma,X}^\lambda = \mathrm{i}_{V_\lambda}\, \gamma_{V_\lambda}^{-1} \colon X \to V_\lambda \hookrightarrow H, \quad K_{\gamma,Y}'^{\bar\lambda} = \mathrm{i}_{W_{\bar\lambda}}\, \gamma_{W_{\bar\lambda}}^{-1} \colon Y \to W_{\bar\lambda} \hookrightarrow H. \tag{6.11}$$

Now a given $T \colon V \to W$ is carried over to a closed, densely defined operator $L \colon X \to Y^*$ by the definition

$$L = (\gamma_W^{-1})^* T \gamma_V^{-1}, \quad D(L) = \gamma_V D(T); \tag{6.12}$$

it is expressed in the diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\;\;\sim\;\;}_{\gamma_V} & X \\[2pt]
{\scriptstyle T}\Big\downarrow & & \Big\downarrow{\scriptstyle L} \\[2pt]
W & \xrightarrow[{(\gamma_W^{-1})^*}]{\;\;\sim\;\;} & Y^*
\end{array}
\tag{6.13}
$$

When $\widetilde{A}$ corresponds to $T \colon V \to W$ and $L \colon X \to Y^*$, we can write

$$(Tu_\zeta, w) = (T\gamma_V^{-1}\gamma_0 u, \gamma_W^{-1}\gamma_0 w) = (L\gamma_0 u, \gamma_0 w)_{Y^*,Y}, \text{ all } u \in D(\widetilde{A}), w \in W. \tag{6.14}$$

The formula $(Au)_W = Tu_\zeta$ in (3.2) is then in view of (6.6) turned into

$$(\Gamma^0 u, \gamma_0 w)_{\frac{1}{2},-\frac{1}{2}} = (L\gamma_0 u, \gamma_0 w)_{Y^*,Y}, \text{ all } w \in W,$$

or, since γ_0 maps W bijectively onto Y,

$$(\Gamma^0 u, \varphi)_{\frac{1}{2},-\frac{1}{2}} = (L\gamma_0 u, \varphi)_{Y^*,Y} \text{ for all } \varphi \in Y. \tag{6.15}$$

To simplify the notation, note that the injection $\mathrm{i}_Y \colon Y \to H^{-\frac{1}{2}}(\Sigma)$ has as adjoint the mapping $\mathrm{i}_Y^* \colon H^{\frac{1}{2}}(\Sigma) \to Y^*$ that sends a functional ψ on $H^{-\frac{1}{2}}(\Sigma)$ over into a functional $\mathrm{i}_Y^* \psi$ on Y by:

$$(\mathrm{i}_Y^* \psi, \varphi)_{Y^*,Y} = (\psi, \varphi)_{\frac{1}{2},-\frac{1}{2}} \text{ for all } \varphi \in Y. \tag{6.16}$$

With this notation (also indicated in [25] after (5.23)), (6.15) may be rewritten as

$$\mathrm{i}_Y^* \,\Gamma^0 u = L\gamma_0 u,$$

or, when we use that $\Gamma^0 = \chi - P_{\gamma,\chi}^0 \gamma_0$,

$$\mathrm{i}_Y^* \chi u = (L + \mathrm{i}_Y^* P_{\gamma,\chi}^0)\gamma_0 u. \tag{6.17}$$

We have then obtained:

Theorem 6.3 *For a closed operator $\widetilde{A} \in \mathcal{M}$, the following statements* (i) *and* (ii) *are equivalent:*

(i) $\widetilde{A}$ *corresponds to* $T \colon V \to W$ *as in Section* 3.

(ii) $D(\widetilde{A})$ *consists of the functions* $u \in D(A_{\max})$ *that satisfy the boundary condition*

$$\gamma_0 u \in D(L), \quad \mathrm{i}_Y^* \chi u = (L + \mathrm{i}_Y^* P^0_{\gamma,\chi})\gamma_0 u. \tag{6.18}$$

Here $T \colon V \to W$ *and* $L \colon X \to Y^*$ *are defined from one another as described in* (6.9)–(6.13).

Note that when Y is the full space $H^{-\frac{1}{2}}(\Sigma)$, i_{Y^*} is superfluous, and (6.18) takes the form

$$\gamma_0 u \in D(L), \quad \chi u = (L + P^0_{\gamma,\chi})\gamma_0 u. \tag{6.19}$$

The whole construction can be carried out with A replaced by $A - \lambda$, when $\lambda \in \varrho(A_\gamma)$. We define L^λ from T^λ as in (6.12)–(6.13) with $T \colon V \to W$ replaced by $T^\lambda \colon V_\lambda \to W_{\bar\lambda}$ and use of (6.10); here L^λ maps from X to Y^*;

$$L^\lambda = (\gamma_{W_{\bar\lambda}}^{-1})^* T^\lambda \gamma_{V_\lambda}^{-1}, \quad D(L^\lambda) = \gamma_{V_\lambda} D(T) = D(L). \tag{6.20}$$

This can be expressed in the following diagram, where we also take (3.5) into account:

$$
\begin{array}{ccccc}
V & \xrightarrow[E_V^\lambda]{\sim} & V_\lambda & \xrightarrow[\gamma_{V_\lambda}]{\sim} & X \\
{\scriptstyle T+G_{V,W}^\lambda}\downarrow & & {\scriptstyle T^\lambda}\downarrow & & \downarrow{\scriptstyle L^\lambda} \\
W & \xrightarrow[(F_W'^{\bar\lambda})^*]{\sim} & W_{\bar\lambda} & \xrightarrow[(\gamma_{W_{\bar\lambda}}^*)^{-1}]{\sim} & Y^*
\end{array}
$$

Here the horizontal homeomorphisms compose as

$$\gamma_{V_\lambda} E_V^\lambda = \gamma_V, \quad (\gamma_{W_{\bar\lambda}}^*)^{-1}(F_W'^{\bar\lambda})^* = (\gamma_W^*)^{-1}, \tag{6.21}$$

so

$$L^\lambda = (\gamma_W^*)^{-1}(T + G_{V,W}^\lambda)\gamma_V^{-1}. \tag{6.22}$$

In this λ-dependent situation, Theorem 6.3 takes the form:

Theorem 6.4 *Let* $\lambda \in \varrho(A_\gamma)$. *For a closed operator* $\widetilde{A} \in \mathcal{M}$, *the following statements* (i) *and* (ii) *are equivalent:*

(i) $\widetilde{A} - \lambda$ *corresponds to* $T^\lambda \colon V_\lambda \to W_{\bar\lambda}$ *as in Section* 3.

(ii) $D(\widetilde{A})$ *consists of the functions* $u \in D(A_{\max})$ *such that*

$$\gamma_0 u \in D(L), \quad \mathrm{i}_Y^* \chi u = (L^\lambda + \mathrm{i}_Y^* P^\lambda_{\gamma,\chi})\gamma_0 u. \tag{6.23}$$

Observe that since the boundary conditions (6.18) and (6.23) define the same realization, we obtain moreover the information that

$$(L^\lambda + \mathrm{i}_Y^* P_{\gamma,\chi}^\lambda)\gamma_0 u = (L + \mathrm{i}_Y^* P_{\gamma,\chi}^0)\gamma_0 u, \text{ for } \gamma_0 u \in D(L) = D(L^\lambda),$$

i.e.,

$$L^\lambda = L + \mathrm{i}_Y^*(P_{\gamma,\chi}^0 - P_{\gamma,\chi}^\lambda) \text{ on } D(L) = D(L^\lambda). \tag{6.24}$$

Note also that in view of (6.12) and (6.22),

$$L^\lambda = L + (\gamma_W^*)^{-1} G_{V,W}^\lambda \gamma_V^{-1} \text{ on } D(L).$$

This has the particular consequence:

$$\mathrm{i}_Y^*(P_{\gamma,\chi}^0 - P_{\gamma,\chi}^\lambda) = (\gamma_W^*)^{-1} G_{V,W}^\lambda \gamma_V^{-1} \text{ on } D(L). \tag{6.25}$$

Since the last statement will hold for fixed choices of V, W, X, Y, regardless of how the operator L is chosen (it can e.g. be taken as the zero operator), we conclude that

$$\mathrm{i}_Y^*(P_{\gamma,\chi}^0 - P_{\gamma,\chi}^\lambda)\, \mathrm{i}_X = (\gamma_W^*)^{-1} G_{V,W}^\lambda \gamma_V^{-1}, \tag{6.26}$$

as bounded operators from X to Y^*. In particular, in the case $X = Y = H^{-\frac{1}{2}}(\Sigma)$:

$$P_{\gamma,\chi}^0 - P_{\gamma,\chi}^\lambda = (\gamma_{Z'}^*)^{-1} G_{Z,Z'}^\lambda \gamma_Z^{-1}; \tag{6.27}$$

bounded operators from $H^{-\frac{1}{2}}(\Sigma)$ to $H^{\frac{1}{2}}(\Sigma)$.

We can now connect the description with M-functions and establish Kreĭn-type resolvent formulas. (The following formulation differs slightly from that in [15] using order-reducing operators carrying $H^{\pm\frac{1}{2}}(\Sigma)$ over to $L_2(\Sigma)$ and orthogonal projections.)

Theorem 6.5 *Let $\widetilde{A}$ correspond to $T\colon V \to W$, carried over to $L\colon X \to Y^*$, whereby $\widetilde{A}$ represents the boundary condition (6.18), as well as (6.23) when $\lambda \in \varrho(A_\gamma)$.*

(i) For $\lambda \in \varrho(A_\gamma)$, L and L^λ satisfy (6.24), where $P_{\gamma,\chi}^0 - P_{\gamma,\chi}^\lambda \in \mathcal{L}(H^{-\frac{1}{2}}(\Sigma), H^{\frac{1}{2}}(\Sigma))$. The relations to $G_{V,W}^\lambda$ are as described in (6.26), (6.27).

(ii) For $\lambda \in \varrho(\widetilde{A})$, there is a related M-function $\in \mathcal{L}(Y^*, X)$:

$$M_L(\lambda) = \gamma_0\big(I - (\widetilde{A} - \lambda)^{-1}(A_{\max} - \lambda)\big)A_\gamma^{-1}\, \mathrm{i}_W\, \gamma_W^*.$$

(iii) For $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$,

$$M_L(\lambda) = -(L + \mathrm{i}_Y^*(P_{\gamma,\chi}^0 - P_{\gamma,\chi}^\lambda)\, \mathrm{i}_X)^{-1} = -(L^\lambda)^{-1}.$$

(iv) For $\lambda \in \varrho(A_\gamma)$ (recall (6.11)),

$$\ker(\widetilde{A} - \lambda) = K_{\gamma,X}^\lambda \ker L^\lambda,$$
$$\mathrm{ran}(\widetilde{A} - \lambda) = \gamma_{W_{\bar\lambda}}^* \mathrm{ran}\, L^\lambda + H \ominus W_{\bar\lambda}. \tag{6.28}$$

(v) *For $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$ there is a Kreĭn-type resolvent formula:*

$$(\widetilde{A} - \lambda)^{-1} = (A_\gamma - \lambda)^{-1} - \mathrm{i}_{V_\lambda}\, \gamma_{V_\lambda}^{-1} M_L(\lambda)(\gamma_{W_{\bar\lambda}}^*)^{-1}\, \mathrm{pr}_{W_{\bar\lambda}}$$
$$= (A_\gamma - \lambda)^{-1} - K_{\gamma,X}^\lambda M_L(\lambda)(K_{\gamma,Y}^{\prime\bar\lambda})^* \qquad (6.29)$$
$$= (A_\gamma - \lambda)^{-1} + K_{\gamma,X}^\lambda (L^\lambda)^{-1}(K_{\gamma,Y}^{\prime\bar\lambda})^*.$$

Proof: Statement (i) was accounted for before the theorem.

In (ii), the M-function $M_L(\lambda)$ is obtained from $M_{\widetilde{A}}(\lambda)$ in Theorem 3.5 (i) by composition to the right with γ_W^* and to the left with γ_V:

$$M_L(\lambda) = \gamma_V M_{\widetilde{A}}(\lambda)\gamma_W^* = \gamma_V \,\mathrm{pr}_\zeta (I - (\widetilde{A} - \lambda)^{-1}(A_{\max} - \lambda))A_\gamma^{-1}\, \mathrm{i}_W\, \gamma_W^* \qquad (6.30)$$
$$= \gamma_0(I - (\widetilde{A} - \lambda)^{-1}(A_{\max} - \lambda))A_\gamma^{-1}\, \mathrm{i}_W\, \gamma_W^*.$$

Statement (iii) follows from Theorem 3.5 (ii) in a similar way.

For (iv), we use the homeomorphism properties of γ_{V_λ} and $\gamma_{W_{\bar\lambda}}$ and their adjoints.

For (v), we calculate the last term in (3.7), using (6.30):

$$-\mathrm{i}_{V_\lambda} E_V^\lambda M_{\widetilde{A}}(\lambda)(E_W^{\prime\bar\lambda})^*\, \mathrm{pr}_{W_{\bar\lambda}} = -\mathrm{i}_{V_\lambda} E_V^\lambda \gamma_V^{-1} M_L(\lambda)(\gamma_W^*)^{-1}(E_W^{\prime\bar\lambda})^*\, \mathrm{pr}_{W_{\bar\lambda}}$$
$$= -\mathrm{i}_{V_\lambda}\, \gamma_{V_\lambda}^{-1} M_L(\lambda)(\gamma_{W_{\bar\lambda}}^{-1})^*\, \mathrm{pr}_{W_{\bar\lambda}}$$
$$= -K_{\gamma,X}^\lambda M_L(\lambda)(K_{\gamma,Y}^{\prime\bar\lambda})^*;$$

cf. (6.21) and (6.11). ∎

Hereby, Kreĭn-type resolvent formulas are established for all closed realizations of A in the present nonsmooth case.

In the cases where X and Y differ from $H^{-\frac{1}{2}}(\Sigma)$, the formulas are quite different from those established in [21] for selfadjoint realizations of the Laplacian, where an M-function acting between full boundary Sobolev spaces is used (if the domain is $C^{\frac{3}{2}+\varepsilon}$).

7 Neumann-type conditions

The case where $X = Y = H^{-\frac{1}{2}}(\Sigma)$, i.e., $V = Z$, $W = Z'$, is particularly interesting for applications of the theory. Here the boundary condition has the form in (6.19) and we say that $\widetilde{A}$ represents a *Neumann-type condition* (this includes the information that $D(L)$ is a dense subset of $H^{-\frac{1}{2}}(\Sigma)$). It may be written

$$\chi u = C\gamma_0 u, \text{ with } C = L + P_{\gamma,\chi}^0, \quad D(C) = D(L). \qquad (7.1)$$

An interesting case is where C acts like a differential operator (or pseudodifferential operator) of order 1. In the differential operator case we can assume, to match the smoothness properties of s_0 and $\mathcal{A}_1$ in Green's formula, that

$$C = c \cdot D_\tau + c_0, \text{ where } c = (c_1, \ldots, c_n), \ c_j \in H_p^{\frac{1}{2}}(\Sigma) \text{ for } j = 0, 1, \ldots, n. \qquad (7.2)$$

As a ψdo C we can take an operator constructed from local first-order pieces as in (2.34) with $H_q^{\frac{1}{2}} S_{1,0}^1$- symbols.

Then L acts as a pseudodifferential operator of order 1,

$$L = C - P_{\gamma,\chi}^0;$$

cf. Theorem 5.7 for the properties of $P_{\gamma,\chi}^0$.

It should be noted that L is determined in a precise way from $\widetilde{A}$ as an operator from $D(L) \subset H^{-\frac{1}{2}}(\Sigma)$ to $H^{\frac{1}{2}}(\Sigma)$; it is generally *unbounded* from $H^{-\frac{1}{2}}(\Sigma)$ to $H^{\frac{1}{2}}(\Sigma)$ since it is of order 1.

In the study of boundary conditions, the situation is sometimes set up in a slightly different way:

It is C that is given as a first-order operator, and we define $\widetilde{A}$ as the restriction of $A_{\max}$ with domain

$$D(\widetilde{A}) = \{u \in D(A_{\max}) \mid \chi u = C\gamma_0 u\}. \tag{7.3}$$

Note that for the $H^2(\Omega)$-functions satisfying $\chi u = C\gamma_0 u$, the Dirichlet data $\gamma_0 u$ fill out the space $H^{\frac{3}{2}}(\Sigma)$, since $\{\gamma_0, \chi\}$ is surjective from $H^2(\Omega)$ to $H^{\frac{3}{2}}(\Sigma) \times H^{\frac{1}{2}}(\Sigma)$. When $u \in D(\widetilde{A})$,

$$L\gamma_0 u = \Gamma^0 u = \chi u - P_{\gamma,\chi}^0 \gamma_0 u = (C - P_{\gamma,\chi}^0)\gamma_0 u,$$

so necessarily $\gamma_0 u$ belongs to the subset of $H^{-\frac{1}{2}}(\Sigma)$ that is mapped by $C - P_{\gamma,\chi}^0$ into $H^{\frac{1}{2}}(\Sigma)$. When u lies there, it moreover has to satisfy $\Gamma^0 u = (C - P_{\gamma,\chi}^0)\gamma_0 u$, in order to belong to $D(\widetilde{A})$. This shows:

Lemma 7.1 *Let C be a first-order operator on Σ as described above and define the realisation $\widetilde{A}$ by (7.3). Then L is the operator acting like $C - P_{\gamma,\chi}^0$ with domain*

$$D(L) = \{\varphi \in H^{-\frac{1}{2}}(\Sigma) \mid (C - P_{\gamma,\chi}^0)\varphi \in H^{\frac{1}{2}}(\Sigma)\}.$$

$D(L)$ contains $H^{\frac{3}{2}}(\Sigma)$, hence is dense in $H^{-\frac{1}{2}}(\Sigma)$.

In the λ-dependent setting,

$$L^\lambda \text{ acts like } C - P_{\gamma,\chi}^\lambda \text{ with } D(L^\lambda) = D(L).$$

Further information can be obtained in the *elliptic* case. This is the case where the model operator, defined from the principal symbols at each (x', ξ') in local coordinates, is invertible:

$$\begin{pmatrix} \underline{a}^0(x', 0, \xi', D_n) \\ \\ \underline{\chi}^0(x', \xi', D_n) - \underline{c}^0(x', \xi')\gamma_0 \end{pmatrix} : H^2(\mathbb{R}_+) \overset{\sim}{\to} \begin{matrix} L_2(\mathbb{R}_+) \\ \times \\ \mathbb{C} \end{matrix},$$

for all x', all $|\xi'| \geq 1$. Using the various reductions introduced above on this one-dimensional level, we find that L has the principal symbol

$$\underline{l}^0(x',\xi') = \underline{c}^0(x',\xi') - \underline{p}^0(x',\xi').$$

where $\underline{c}^0(x',\xi')$ and $\underline{p}^0(x',\xi')$ are the principal symbols of C and $P_{\gamma,\chi}^0$; moreover, ellipticity holds if and only if $\underline{l}^0(x',\xi') \neq 0$ for $|\xi'| \geq 1$. These considerations take place pointwise in x' regardless of smoothness with respect to x'.

Theorem 7.2 *For a given first-order pseudodifferential operator C as described above, let $\widetilde{A}$ be defined by (7.3). Assume that $C - P_{\gamma,\chi}^0$ is elliptic. Then $D(L) = H^{\frac{3}{2}}(\Sigma)$, and $D(\widetilde{A}) \subset H^2(\Omega)$.*

Proof: Let $\varphi \in D(L)$; then we know to begin with that $\varphi \in H^{-\frac{1}{2}}(\Sigma)$ and $(C - P_{\gamma,\chi}^0)\varphi \in H^{\frac{1}{2}}(\Sigma)$. It follows from Theorem 5.7 3° that $(C - P_{\gamma,\chi}^0)\varphi = (C - P_{\gamma,\chi}^{\sharp 01})\varphi + \psi$, where $\psi \in H^{-\frac{3}{2}+\varepsilon}(\Sigma)$. This together with $(C - P_{\gamma,\chi}^0)\varphi \in H^{\frac{1}{2}}(\Sigma)$ implies $(C - P_{\gamma,\chi}^{\sharp 01})\varphi \in H^{-\frac{3}{2}+\varepsilon}(\Sigma)$. Here $C - P_{\gamma,\chi}^{\sharp 01}$ is defined as in (5.21), constructed from localized pieces with elliptic smooth ψdo symbols, and it follows by use of a parametrix in each localization that $\varphi \in H^{-\frac{1}{2}+\varepsilon}(\Sigma)$. (Details on cutoffs and partitions of unity in parametrix constructions can e.g. be found in [29], Sect. 8.2.)

Next, let $s' = -\frac{3}{2} + \varepsilon$. By Theorem 5.7 2° there is an $\varepsilon' > 0$ such that $P_{\gamma,\chi}^0 - P_{\gamma,\chi}^{\sharp 0}$ is continuous from $H^{s-\frac{1}{2}-\varepsilon'}(\Sigma)$ to $H^{s-\frac{3}{2}}(\Sigma)$ for all $s \in [s', 2]$. In a similar way as in the preceding construction, one finds by use of a parametrix of $C - P_{\gamma,\chi}^{\sharp 0}$ that $(C - P_{\gamma,\chi}^0)\varphi \in H^{\frac{1}{2}}(\Sigma)$ together with $\varphi \in H^{s-\frac{1}{2}-\varepsilon'}(\Sigma)$ imply $\varphi \in H^{s-\frac{1}{2}}(\Sigma)$, when $s \in [s', 2]$. Starting from $s = s' - \varepsilon'$ and applying the argument successively with $s = s' + k\varepsilon'$, $k = -1, 0, 1, 2, \ldots$, we reach the conclusion $\varphi \in H^{\frac{3}{2}}(\Sigma)$ in finitely many steps.

Since we know from Lemma 7.1 that $D(L) \supset H^{\frac{3}{2}}(\Sigma)$, this shows that $D(L) = H^{\frac{3}{2}}(\Sigma)$. Now any $u \in D(\widetilde{A})$ satisfies $\gamma_0 u \in H^{\frac{3}{2}}(\Sigma)$, so it follows from our knowledge of the Dirichlet problem that $u \in H^2(\Omega)$. ∎

We also have:

Theorem 7.3 *For a given first-order differential or pseudodifferential operator C as described around (7.2), let $\widetilde{A}$ be defined by (7.3). If there is a $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$ such that $(\widetilde{A} - \lambda)^{-1}$ is continuous from $L_2(\Omega)$ to $H^2(\Omega)$, then $D(L) = H^{\frac{3}{2}}(\Sigma)$.*

In this case, there is a Kreĭn resolvent formula for $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$:

$$(\widetilde{A} - \lambda)^{-1} = (A_\gamma - \lambda)^{-1} + K_\gamma^\lambda (L^\lambda)^{-1} (K_\gamma'^{\bar\lambda})^*, \tag{7.4}$$

where the operators L^λ and K_γ^λ are a ψdo and a Poisson operator, respectively, belonging to the nonsmooth calculus, acting on $H^{\frac{3}{2}}(\Sigma)$ (the case $s = 2$ in Theorems 5.5 and 5.7).

Proof: According to Theorem 6.5 (ii), $M_L(\lambda)$ has the form

$$M_L(\lambda) = \gamma_0\big(I - (\widetilde{A} - \lambda)^{-1}(A_{\max} - \lambda)\big)A_\gamma^{-1}\,\mathrm{i}_{Z'}\,\gamma_{Z'}^*.$$

Here the mapping property of $(\widetilde{A}-\lambda)^{-1}$ assures that $I-(\widetilde{A}-\lambda)^{-1}(A_{\max}-\lambda)$ preserves $H^2(\Omega)$, which implies that $M_L(\lambda)$ maps $H^{\frac{1}{2}}(\Sigma)$ continuously into $H^{\frac{3}{2}}(\Sigma)$. Moreover, by (iii), $-M_L(\lambda)$ is the inverse of L^λ, whose domain contains $H^{\frac{3}{2}}(\Sigma)$ by Lemma 7.1. Then the domain must equal $H^{\frac{3}{2}}(\Sigma)$. Since $D(L) = D(L^\lambda)$, it follows that also $D(L) = H^{\frac{3}{2}}(\Sigma)$.

The next statement follows from the definition of L, and the last statement is a consequence of the fact that $D(L) = H^{\frac{3}{2}}(\Sigma)$. ∎

This theorem includes general Neumann-type boundary conditions with C of order 1 in the discussion, where earlier treatments such as [43] and [19] had conditions of compactness relative to order 1 or lower order than 1 in the picture (Robin conditions). [20] has a somewhat more general class of nonlocal operators C, also used in [21], for selfadjoint realizations of $A = -\Delta$.

As a sufficient condition for the validity of the assumptions in Theorem 7.3 we can mention parameter-ellipticity of the system $\{A - \lambda, \chi - C\gamma_0\}$ on a ray in $\mathbb{C}$ in the sense of [27], when C is a differential operator (7.2). Here one can construct the resolvent in an exact way for large λ on the ray, using Agmon's trick in this situation in the same way as in the resolvent construction for A_γ we described above; this is also accounted for at the end of [28]. (It is used that $C - P_{\gamma,\chi}^\lambda$ has "regularity $\nu = +\infty$", cf. Remark 5.6.) We can denote $\widetilde{A} = A_{\chi-C\gamma_0}$ in these cases. The case $C = 0$ (the oblique Neumann problem) satisfies the hypothesis for rays $re^{i\eta}$ with $\eta \in (\pi/2 - \delta', 3\pi/2 + \delta')$, some $\delta' > 0$, when the sesquilinear form $a(u, v)$ satisfies

$$\mathrm{Re}\,a(u, u) \geq c_1\|u\|_1^2 - k\|u\|_0^2, \quad u \in H^1(\Omega), \tag{7.5}$$

with $c_1 > 0$. It defines the realization A_χ.

If A is symmetric, A_γ is selfadjoint (and then $\widetilde{A}$ will be selfadjoint if and only if $X = Y$ and $L\colon X \to X^*$ is selfadjoint, cf. [23]). If, moreover, $a(u, u)$ is real for $u \in H^1(\Omega)$ and satisfies (7.5), A_χ will be selfadjoint with domain in $H^2(\Omega)$.

The preceding choices of L are the most natural ones in connection with applications to physical problems. One can of course more generally choose L abstractly to be of a convenient form and derive C from it as in (7.1).

Besides the Kreĭn-type resolvent formulas, there are many other applications of the characterization of realizations in terms of the operators L. Let us mention numerical range estimates and lower bounds, and coerciveness estimates, as e.g. in [24], and spectral asymptotics estimates as e.g. in [25], for the smooth case. Both papers are followed up in the recent literature with further developments, and there remain unsolved questions, particularly for nonsmooth domains.

A Pseudodifferential boundary value problems with nonsmooth coefficients

A.1 Definitions, symbol smoothing

Definition A.1 *Let X be a Banach space and let $X^\tau = C^\tau$ or $X^\tau = \mathcal{C}^\tau$. The symbol space $X^\tau S_{1,\delta}^m(\mathbb{R}^n \times \mathbb{R}^n; X)$, $\tau > 0$, $\delta \in [0,1]$, $m \in \mathbb{R}$, is the set of all functions $p \colon \mathbb{R}^n \times \mathbb{R}^n \to X$ that are smooth with respect to ξ and are in X^τ with respect to x satisfying the estimates*

$$\|D_\xi^\alpha D_x^\beta p(.,\xi)\|_{L^\infty(\mathbb{R}^n;X)} \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\alpha|+\delta|\beta|},$$
$$\|D_\xi^\alpha p(.,\xi)\|_{X^\tau(\mathbb{R}^n;X)} \leq C_\alpha \langle \xi \rangle^{m-|\alpha|+\delta\tau},$$

and if $X = C^\tau$ additionally

$$\|D_\xi^\alpha p(.,\xi)\|_{C^j(\mathbb{R}^n;X)} \leq C_\alpha \langle \xi \rangle^{m-|\alpha|+\delta j} \quad \text{for all } j \in \mathbb{N}_0, \ j \leq [\tau],$$

for all $\alpha \in \mathbb{N}_0^n$ and $|\beta| \leq [\tau]$.

Obviously, $\bigcap_{\tau > 0} C^\tau S_{1,\delta}^m(\mathbb{R}^n \times \mathbb{R}^n; X)$ coincides with the usual Hörmander class $S_{1,\delta}^m(\mathbb{R}^n \times \mathbb{R}^n; X)$ in the vector-valued variant.

In particular, we are interested in the case $\delta = 0$, where we simply say that the symbols (and operators) have X^τ-smoothness in x. But we need the classes $C^\tau S_{1,\delta}^m$ with $\delta > 0$ when working with the technique called *symbol smoothing*: If $p \in C^\tau S_{1,\delta}^m(\mathbb{R}^n \times \mathbb{R}^n; X)$, $\delta \in [0,1)$, then for every $\gamma \in (\delta, 1)$ there is a decomposition $p = p^\# + p^b$ with

$$p^\# \in S_{1,\gamma}^m(\mathbb{R}^n \times \mathbb{R}^n; X), \qquad p^b \in C^\tau S_{1,\gamma}^{m-(\gamma-\delta)\tau}(\mathbb{R}^n \times \mathbb{R}^n; X), \tag{A.1}$$

cf. [47, Equation (1.3.21)]. We note that the proofs in [47] are formulated for scalar symbols only; but they still hold in the X-valued setting since they are based on direct estimates.

In the case where $X = \mathcal{L}(X_0, X_1)$ is the space of all bounded linear operators $A \colon X_0 \to X_1$ for some Banach spaces X_0 and X_1, we define the pseudodifferential operator with symbol $p \in C^\tau S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(X_0, X_1))$ by

$$p(x, D_x)u = \mathrm{OP}(p)u = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\,d\xi \quad \text{for } u \in \mathcal{S}(\mathbb{R}^n; X_0), \tag{A.2}$$

where $d\xi := (2\pi)^{-n} d\xi$. — An operator defined from a symbol $p(x,\xi)$ by formula (A.2) is said to be "in x-form" in contrast to more general formulas, e.g. where $\hat{u}(\xi) = \int e^{-iy\cdot\xi} u(y)\,dy$ is inserted, and p is allowed to depend also on y. Compositions often lead to more general formulas.

Proposition A.2 *Let $1 < q < \infty$ and let $p \in C^r S_{1,\delta}^m(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(H_0, H_1))$, $m \in \mathbb{R}$, $\delta \in [0, 1]$, $r > 0$, where H_0 and H_1 are Hilbert spaces. Then $p(x, D_x)$ is continuous*

$$p(x, D_x) \colon H_q^{s+m}(\mathbb{R}^n; H_0) \to H_q^s(\mathbb{R}^n; H_1)$$

for all $s \in \mathbb{R}$ with $-r(1 - \delta) < s < r$.

Proof: The proposition is an operator-valued variant of [47, Proposition 2.1.D]. As indicated in [3, Appendix] the proof given in [47] directly carries over to the present setting by using the Mihlin multiplier theorem in the $\mathcal{L}(H_0, H_1)$-valued version, where it is essential that H_0 and H_1 are Hilbert spaces. ∎

We denote by $\mathcal{S}(\overline{\mathbb{R}}_+)$ the space of restrictions to $\overline{\mathbb{R}}_+$ of functions in $\mathcal{S}(\mathbb{R})$.

Definition A.3 *The space $C^\tau S_{1,\delta}^d(\mathbb{R}^N \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, $d \in \mathbb{R}$, $n, N \in \mathbb{N}$, consists of all functions $\tilde{f}(x, \xi', y_n)$, which are smooth in $(\xi', y_n) \in \mathbb{R}^{n-1} \times \overline{\mathbb{R}}_+$, are in $C^\tau(\mathbb{R}^N)$ with respect to x, and satisfy*

$$\sup_{x' \in \mathbb{R}^N} \|y_n^l \partial_{y_n}^{l'} D_{\xi'}^\alpha \tilde{f}(x', \xi', .)\|_{L_{y_n}^2(\mathbb{R}_+)} \quad \leq \quad C_{\alpha, l, l'} \langle \xi' \rangle^{d + \frac{1}{2} - l + l' - |\alpha|} \tag{A.3}$$

$$\|y_n^l \partial_{y_n}^{l'} D_{\xi'}^\alpha \tilde{f}(., \xi', .)\|_{C^\tau(\mathbb{R}^N; L_{y_n}^2(\mathbb{R}_+))} \quad \leq \quad C_{\alpha, l, l'} \langle \xi' \rangle^{d + \frac{1}{2} - l + l' - |\alpha| + |\delta|\tau} \tag{A.4}$$

for all $\alpha \in \mathbb{N}_0^{n-1}$, $l, l' \in \mathbb{N}_0$. Moreover, we define $S_{1,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}(\overline{\mathbb{R}}_+)) := \cap_{k \in \mathbb{N}} C^k S_{1,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}(\overline{\mathbb{R}}_+))$.

Now we define an $S_{1,\delta}^m$-variant of the Poisson operators with nonsmooth coefficients as studied in [2].

Definition A.4 *Let $\tilde{k} = \tilde{k}(x', \xi', y_n) \in C^\tau S_{1,\delta}^{d-1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, $d \in \mathbb{R}$, $0 \leq \delta < 1$, $\tau > 0$. Then we define the* Poisson operator *of order d by*

$$k(x', D_x)v = \mathcal{F}_{\xi' \mapsto x'}^{-1} \left[\tilde{k}(x', \xi', x_n) \hat{v}(\xi') \right], \qquad v \in \mathcal{S}(\mathbb{R}^{n-1}),$$

where the Fourier transform is applied in the x'-variables. $\tilde{k}$ is called a Poisson symbol-kernel of order d. The associated boundary symbol operator $k(x', \xi', D_n) \in \mathcal{L}(\mathbb{C}, \mathcal{S}(\overline{\mathbb{R}}_+))$ is defined by

$$(k(x', \xi', D_n)v)(x_n) = \tilde{k}(x', \xi', x_n)v \qquad \text{for all } x_n \geq 0, v \in \mathbb{C}.$$

Theorem A.5 *Let $\tilde{k} = \tilde{k}(x', \xi', y_n) \in C^\tau S_{1,\delta}^{d-1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, $d \in \mathbb{R}$, $0 \leq \delta < 1$. Then $k(x', D_x)$ extends to a bounded operator*

$$k(x', D_x) \colon H^{s+d-\frac{1}{2}}(\mathbb{R}^{n-1}) \to H^s(\mathbb{R}_+^n) \tag{A.5}$$

for every $-(1 - \delta)\tau < s < \tau$. In particular, (A.5) holds for every $s \in \mathbb{R}$ if $\tilde{k} \in S_{1,\delta}^{d-1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$.

Proof: From the symbol estimates one easily derives that

$$k(x', \xi', D_n) \in C^\tau S_{1,\delta}^{d-\frac{1}{2}+s}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(\mathbb{C}, H^s(\mathbb{R}_+^n)))$$

for every $s \geq 0$, cf. [2, Proof of Lemma 4.5] for details. Hence Proposition A.2 implies that

$$k(x', D_x) \colon H^{s+d-\frac{1}{2}}(\mathbb{R}^{n-1}) \to H^s(\mathbb{R}^{n-1}; L^2(\mathbb{R}_+))$$

is a bounded operator for every $-(1-\delta)\tau < s < \tau$. If $s \leq 0$, then $H^s(\mathbb{R}^{n-1}; L^2(\mathbb{R}_+)) \hookrightarrow H^s(\mathbb{R}_+^n)$ and the theorem is proved. In the case $s \geq 0$, Proposition A.2 additionally implies that

$$k(x', D_x) \colon H^{s+d-\frac{1}{2}}(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^{n-1}; H^s(\mathbb{R}_+))$$

is a bounded operator for every $s \geq 0$. Since

$$H^s(\mathbb{R}_+^n) = H^s(\mathbb{R}^{n-1}; L^2(\mathbb{R}_+)) \cap L^2(\mathbb{R}^{n-1}; H^s(\mathbb{R}_+)) \quad \text{if } s \geq 0,$$

the theorem is also obtained in this case. ∎

Remark A.6 One can easily modify the arguments in the proof of [2, Th.~4.8] to even get that

$$k(x', D_x) \colon B_{p,p}^{s+d-\frac{1}{2}}(\mathbb{R}^{n-1}) \to H_p^s(\mathbb{R}_+^n)$$

is bounded for every $-(1-\delta)\tau < s < \tau$ and $1 < p < \infty$.

We note that $\tilde{f} \in C^\tau S_{1,\delta}^d(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}(\overline{\mathbb{R}}_+))$ if and only if

$$x_n^l \partial_{x_n}^{l'} f(x', \xi', D_n) \in C^\tau S_{1,\delta}^{d+\frac{1}{2}-l+l'}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{L}(L^2(\mathbb{R}_+))) \quad \text{for all } l, l' \in \mathbb{N}_0,$$

where $(x_n^l \partial_{x_n}^{l'} f(x', \xi', D_n)v)(x_n) = x_n^l \partial_{x_n}^{l'} \tilde{f}(x', \xi', x_n)v$ for all $v \in \mathbb{C}$, $x_n \geq 0$. Hence, applying symbol smoothing with respect to (x', ξ'), we obtain that $\tilde{f} = \tilde{f}^\sharp + \tilde{f}^b$, where

$$\tilde{f}^\sharp \in S_{1,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}(\overline{\mathbb{R}}_+)), \quad \tilde{f}^b \in C^\tau S_{1,\delta}^{m-\tau\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}(\overline{\mathbb{R}}_+)),$$

which can be proved the same way as e.g. [47, Prop. 1.3.E]. As a consequence we derive directly from Theorem A.5:

Lemma A.7 *Let* $\tilde{k} = \tilde{k}(x', \xi', y_n) \in C^\tau S_{1,0}^{d-1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, $d \in \mathbb{R}$, $0 < \delta < 1$. *Then* $k(x', D_x) = k^\sharp(x', D_x) + k^b(x', D_x)$, *where*

$$\tilde{k}^\sharp \in S_{1,\delta}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}(\overline{\mathbb{R}}_+)), \quad \tilde{k}^b \in C^\tau S_{1,\delta}^{m-\tau\delta}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}(\overline{\mathbb{R}}_+)).$$

In particular, we have that

$$k(x', D_x) - k^\sharp(x', D_x) \colon H^{s+d-\delta\tau-\frac{1}{2}}(\mathbb{R}^{n-1}) \to H^s(\mathbb{R}_+^n)$$

for every $-(1-\delta)\tau < s < \tau$, *and* $k^\sharp(x', D_x) \in \mathcal{L}(H^{s+d-\frac{1}{2}}(\mathbb{R}^{n-1}), H^s(\mathbb{R}_+^n))$ *for every* $s \in \mathbb{R}$.

Let us recall the definition of trace operators from [2]: A trace operator of order $m \in \mathbb{R}$ and class $r \in \mathbb{N}_0$ with C^τ-coefficients (in x-form) is defined as

$$
\begin{aligned}
t(x', D_x)f &= \sum_{j=0}^{r-1} s_j(x', D_{x'})\gamma_j f + t_0(x', D_x)f \\
t_0(x', D_x)f &= \mathcal{F}_{\xi' \mapsto x'}^{-1} \left[\int_0^\infty \tilde{t}_0(x', \xi', y_n)\acute{f}(\xi', y_n)dy_n \right],
\end{aligned}
$$

where $\tilde{t}_0 \in C^\tau S_{1,0}^m(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, $s_j \in C^\tau S_{1,0}^{m-j}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, $j = 0, \ldots, r-1$, $\acute{f}(\xi', x_n) = \mathcal{F}_{x' \mapsto \xi'}[f(., x_n)]$ (a partial Fourier transform) and $\gamma_j f = \partial_n^j f|_{x_n=0}$. The associated *boundary symbol operator* $t(x', \xi', D_n)$ is defined by applying the above definition to $f \in \mathcal{S}(\overline{\mathbb{R}}_+)$ for every fixed $x', \xi' \in \mathbb{R}^{n-1}$. Since γ_j is well-defined on $H^k(\mathbb{R}_+)$ ($k \in \mathbb{N}_0$) if and only if $k > j$, the boundary symbol operators of class r are those that are well-defined on $H^r(\mathbb{R}_+)$ for all (x', ξ').

In particular, $t(x', D_x)$ is called a *differential trace operator* of order m and class r with C^τ-coefficients if $t_0 \equiv 0$ and the $s_j(x', \xi')$ are polynomials in ξ'.

For the precise definitions of singular Green and Green operators of order m and class r with C^τ-coefficients (in x-form) as well as the definition of the (global) transmission condition for $p \in C^\tau S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{Z}$, we refer to [2]. The precise definitions are not important for our considerations in the present paper.

A.2   A parametrix result

Theorem A.8 $1°$ *Let*

$$
\mathcal{A} = \begin{pmatrix} P_+ + G \\ T \end{pmatrix},
$$

where P_+ is the truncation to $\mathbb{R}_+^n$ of a zero-order ψdo satisfying the transmission condition at $x_n = 0$, G is a zero-order singular Green operator, such that $P_+ + G$ is of class $r \in \mathbb{Z}$, and T is a trace operator of order $-\frac{1}{2}$ and class r, all in x-form with C^τ-smoothness in x. Then $\mathcal{A}$ maps continuously

$$
\mathcal{A} = \begin{pmatrix} P_+ + G \\ T \end{pmatrix} : H^s(\mathbb{R}_+^n) \to \begin{matrix} H^s(\mathbb{R}_+^n) \\ \times \\ H^s(\mathbb{R}^{n-1}) \end{matrix}, \tag{A.6}
$$

when

   (i) $|s| < \tau$,
   (ii) $s > r - \frac{1}{2}$ *(class restriction).*

$2°$ *Let $\mathcal{A}$ be as in $1°$, and polyhomogeneous and uniformly elliptic with principal symbol a^0. Then the operator $\mathcal{B}^0$ with symbol $(a^0)^{-1}$,*

$$
\mathcal{B}^0 = \begin{pmatrix} R^0 & K^0 \end{pmatrix},
$$

with R^0 of order 0 and class r (being the sum of a truncated ψdo and a singular Green operator), K^0 a Poisson operator of order $\frac{1}{2}$, all in x-form with C^τ-smoothness in x, satisfies that $\mathcal{B}^0$ is continuous in the opposite direction of $\mathcal{A}$, and $\mathcal{R} = \mathcal{A}\mathcal{B}^0 - I$ is continuous:

$$\mathcal{R}\colon \begin{matrix} H^{s-\theta}(\mathbb{R}^n_+) \\ \times \\ H^{s-\theta}(\mathbb{R}^{n-1}) \end{matrix} \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^s(\mathbb{R}^{n-1}) \end{matrix}, \tag{A.7}$$

when

  (i) *$-\tau + \theta < s < \tau$;*
  (ii) *$s - \theta > r - \frac{1}{2}$ (class restriction).*

Proof: The first part of the theorem follows from [2, Th. 1.1 and 1.2]. The second part of the theorem essentially follows from [2, Th. 6.4] with the only difference that there is an additional restriction $|s - \frac{1}{2}| < \tau$. This comes from the fact that for the parametrix construction there, the trace operator is reduced to order $m = 0$ (the same order as the order of P_+ and G). But when we take the trace operator to be of order $-\frac{1}{2}$, the proof of [2, Th. 6.4] applies to the present situation and the restriction $|s - \frac{1}{2}| < \tau$ is not needed. ∎

$\mathcal{B}^0$ is called a parametrix of $\mathcal{A}$.

A more general version than the above is quoted in [28, Th. 6.3].

Remark A.9 To make the above theorem useful for systems where the elements have other orders we need the so-called "order-reducing operators". There are two types, one acting over the domain and one acting over the boundary:

$$\Lambda^r_{-,+} = \mathrm{OP}(\lambda^r_-(\xi))_+\colon H^t(\mathbb{R}^n_+) \overset{\sim}{\to} H^{t-r}(\mathbb{R}^n_+), \tag{A.8}$$
$$\Lambda^s_0 = \mathrm{OP}'(\langle \xi' \rangle^s)\colon H^t(\mathbb{R}^{n-1}) \overset{\sim}{\to} H^{t-s}(\mathbb{R}^{n-1}), \text{ all } t \in \mathbb{R},$$

$r \in \mathbb{Z}$ and $s \in \mathbb{R}$, with inverses $\Lambda^{-r}_{-,+}$ resp. Λ^{-s}_0. Here λ^r_- is the "minus-symbol" defined in [26, Prop. 4.2] as a refinement of $(\langle \xi' \rangle - i\xi_n)^r$. Composition of an operator in x-form with an order-reducing operator to the right gives an operator in x-form (since the order-reducing operator acts on the symbol level essentially as a multiplication by an x-independent symbol). Composition with the order-reducing operator to the left gives a more complicated expression when applied to an x-form operator.

It should be noted that when e.g. $S = s(x', D_{x'})$ is a ψdo of order m on $\mathbb{R}^{n-1}$ with C^τ-smoothness, then it maps $H^{s+m}(\mathbb{R}^{n-1}) \to H^s(\mathbb{R}^{n-1})$ for $|s| < \tau$, so by (A.8),

$$\Lambda^r_0 S\colon H^{s+m}(\mathbb{R}^{n-1}) \to H^{s-r}(\mathbb{R}^{n-1}) \text{ for } -\tau < s < \tau. \tag{A.9}$$

The composition rule Theorem 2.17 (in a version for C^τ-smooth symbols) shows that $\Lambda^r_0 S$ can be written as the sum of an operator in the calculus $\mathrm{OP}'(\langle \xi' \rangle^r s(x', \xi'))$ in x-form and a remainder, such that the sum maps $H^{s'+m+r}(\mathbb{R}^n_+) \to H^{s'}(\mathbb{R}^n_+)$ for $-\tau < s' < \tau$; this gives a mapping property like in (A.9) but with a shifted interval

$-\tau + r < s < \tau + r$. This extends the applicability, but one has to keep in mind that the new decomposition produces different operators; $\Lambda_0^r S$ is not in x-form but is an operator in x-form composed to the left with Λ_0^r, not equal to $\mathrm{OP}'(\langle \xi' \rangle^r s(x', \xi'))$.

The operators $\Lambda_{-,+}^r$ allow an extension of the class concept for trace operators to negative values: When $T = T_0 \Lambda_{-,+}^{-k}$, where T_0 is of class 0 and $k \in \mathbb{N}_0$, T is said to be of class $-k$. Then the boundary symbol operator is well-defined on $H^{-k}(\mathbb{R}_+)$. There is a similar concept for operators $P_+ + G$. More details on operators of negative class can be found in [2, Section 5.4], [26], or [27]. With this extension, Theorem A.8 is valid for $r \in \mathbb{Z}$.

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