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A NEW PROOF OF BRANSON'S CLASSIFICATION OF ELLIPTIC GENERALIZED GRADIENTS

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Abstract. We give a representation theoretical proof of Branson's classification, [4], of minimal elliptic sums of generalized gradients. The original proof uses tools of harmonic analysis, which as powerful as they are, seem to be specific for the structure groups $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$. The different approach we propose is a local one, based on the relationship between ellipticity and optimal Kato constants and on the representation theory of $\mathfrak{so}(n)$. Optimal Kato constants for elliptic operators were computed by Calderbank, Gauduchon and Herzlich, [8]. We extend their method to all generalized gradients (not necessarily elliptic) and recover Branson's result, up to one special case. The interest of this method is that it is better suited to be applied for classifying elliptic sums of generalized gradients of G-structures, for other subgroups G of the special orthogonal group.

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1. Introduction

The classical notion of generalized gradients, also called Stein-Weiss operators, was first introduced by Stein and Weiss, [18], on an oriented Riemannian manifold, as a generalization of the Cauchy-Riemann equations. They are first order differential operators acting on sections of vector bundles associated to irreducible representations of the special orthogonal group (or of the spin group, if the manifold is spin), given by the projections of a metric covariant derivative onto irreducible subbundles. Some of the most important first order differential operators which naturally appear in geometry are generalized gradients, up to normalization. For example, on a Riemannian manifold, the exterior differential, the codifferential and the conformal Killing operator on 1-forms are generalized gradients. On a spin manifold, classical examples are the Dirac operator, the twistor operator and the Rarita-Schwinger operator.

On an oriented Riemannian manifold, generalized gradients naturally give rise, by composition with their formal adjoints, to second order differential operators acting on sections of associated vector bundles. Particularly important are the extreme cases of linear combinations of such second order operators: if the linear combination provides a zero-order operator, then it is a curvature term and one obtains a so-called Weitzenböck formula; if the linear combination is a second order differential operator, then it is interesting to determine when it is elliptic. Whereas Weitzenböck formulas play a key role in relating the local differential geometry to global topological properties by the so-called Bochner method (for recent systematic approaches to the description of all Weitzenböck formulas we refer to [11] and [17]), the importance of elliptic operators is well established, see *e.g.* the seminal paper [1].

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The elliptic second order differential operators constructed this way were completely classified by Branson, [4]. The classical example is the Laplacian acting on differential forms, which is obtained by assembling two generalized gradients, namely the exterior differential and the codifferential. Branson showed that it is enough to take surprisingly few generalized gradients in order to obtain an elliptic operator. Namely, apart from a few known exceptions, each minimal elliptic operator is given by a pair of generalized gradients. The arguments used by Branson are based on techniques of harmonic analysis and explicit computations of the spectra of generalized gradients on the sphere. Partial results were previously obtained by Kalina, Pierzchalski and Walczak, [13], who showed that the only generalized gradient which is strongly elliptic is given by the projection onto the Cartan summand. Furthermore, the projection onto the complement of the Cartan summand is also elliptic, by a result of Stein and Weiss, [18].

In this paper we give a new proof of Branson's classification. The method we use is completely different from the original one in [4], which seems to be specific for the two structure groups $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$. Our approach is based on the one hand on the relationship between ellipticity and Kato constants and on the other hand on the representation theory of $\mathfrak{so}(n)$. The starting point is the remark that these elliptic operators are closely related to the existence of refined Kato inequalities, which was first noticed by Bourguignon, [2]. The explicit computation of the optimal Kato constants for all elliptic differential operators obtained from generalized gradients by the above construction was given by Calderbank, Gauduchon and Herzlich, [8]. In the first part of our proof we extend their computation to all (not necessarily elliptic) sums of generalized gradients and then use it to recover Branson's list of minimal elliptic operators, up to an exceptional case. In the second part of the proof we show that these are *all* minimal elliptic operators. The tool used here is the branching rule for the special orthogonal group.

The construction of the classical generalized gradients can be carried over to G-structures, when there is a reduction of the structure group of the tangent bundle of a Riemannian manifold to a closed subgroup G of $\mathrm{SO}(n)$ (see *e.g.* [15]). The argument of our new approach suggest that they should carry over to other subgroups G of $\mathrm{SO}(n)$, in order to provide the classification of natural elliptic operators constructed from G-generalized gradients.

2. Generalized Gradients

We briefly recall in this section the construction of generalized gradients given by Stein and Weiss, [18], on an oriented Riemannian (spin) manifold.

Recall that the finite-dimensional complex irreducible $\mathfrak{so}(n)$-representations are parametrized by the *dominant weights*, *i.e.* those weights whose coordinates are either all integers or all half-integers, $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}^m \cup (\frac{1}{2} + \mathbb{Z})^m$ and satisfy the inequality:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m-1} \geq |\lambda_m|, \text{ if } n = 2m, \text{ or } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0, \text{ if } n = 2m+1. \quad (2.1)$$

These coordinates are given with respect to the orthonormal basis $\{\varepsilon_i\}_{i=\overline{1,m}}$ dual to the basis $\{e_1 \wedge e_2, \dots, e_{2m-1} \wedge e_{2m}\}$ (where $\{e_1, \dots, e_n\}$ is an oriented orthonormal basis of $\mathbb{R}^n$), which fixes a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}(n)$. With a slight abuse of notation, we use the same symbol for an irreducible representation and its highest weight. For example, the (complex) standard representation, denoted by τ, is given by the weight $(1, 0, \dots, 0)$ and the weight

$(1,\ldots,1,0,\ldots,0)$ (with p ones) corresponds to the p-form representation $\Lambda^p\mathbb{R}^n$. The so-called *classical selection rule* (see [9]) holds:

Lemma 2.1. *An irreducible representation of highest weight μ occurs in the decomposition of $\tau\otimes\lambda$ if and only if the following two conditions are fulfilled:*

(i) *$\mu=\lambda\pm\varepsilon_j$, for some $j=1,\ldots,m$, or $n=2m+1$, $\lambda_m>0$ and $\mu=\lambda$,*
(ii) *μ is a dominant weight,* i.e. *satisfies* (2.1).

We adopt the same terminology as in [17] and call *relevant weights of* λ (and write $\varepsilon\subset\lambda$) the weights ε of τ such that $\lambda+\varepsilon$ occurs in the decomposition of $\tau\otimes\lambda$: $\tau\otimes\lambda=\underset{\varepsilon\subset\lambda}{\oplus}(\lambda+\varepsilon)$. This decomposition is multiplicity-free, *i.e.* the isotypical components are actually irreducible, so that the projections Π_ε onto each irreducible summand $\lambda+\varepsilon$ are well-defined.

Let now (M,g) be an oriented Riemannian manifold, SO_gM denotes the principal $\mathrm{SO}(n)$-bundle of oriented orthonormal frames and ∇ any metric connection. If M has, in addition, a spin structure, then we consider the corresponding principal $\mathrm{Spin}(n)$-bundle, Spin_gM, and the induced metric connection ∇. We consider vector bundles $V_\lambda M$, associated to SO_gM (or Spin_gM) and irreducible $\mathrm{SO}(n)$ (or $\mathrm{Spin}(n)$)-representations of highest weight λ, with the induced connection ∇. The above decomposition carries over to the associated vector bundles:

$$\mathrm{T^*M}\otimes V_\lambda M\cong\mathrm{TM}\otimes V_\lambda M\cong\underset{\varepsilon\subset\lambda}{\oplus}V_{\lambda+\varepsilon}M \tag{2.2}$$

and the corresponding projections are also denoted by Π_ε.

Definition 2.2. For each relevant weight ε of λ, *i.e.* for each irreducible component in the decomposition of $\mathrm{T^*M}\otimes V_\lambda M$, there is a *generalized gradient* P_ε defined by the composition:

$$\Gamma(V_\lambda M)\xrightarrow{\nabla}\Gamma(\mathrm{T^*M}\otimes V_\lambda M)\xrightarrow{\Pi_\varepsilon}\Gamma(V_{\lambda+\varepsilon}M). \tag{2.3}$$

Generalized gradients may be thus defined by any metric connection. Those defined by the Levi-Civita connection play an important role since they are conformal invariant ([16]). The following examples are of this type.

Example 2.3 (Generalized Gradients on Differential Forms)**.** We consider the bundle of p-forms, Λ^pM, on a Riemannian manifold (M^n,g) and assume for simplicity that $n=2m+1$ and $p\le m-1$. The highest weight of the representation is $\lambda_p=(1,\ldots,1,0,\ldots,0)$, so that by Lemma 2.1, there are three relevant weights for λ_p, namely $-\varepsilon_p$, ε_{p+1} and ε_1. Then we have $\mathrm{TM}\otimes\Lambda^pM\cong\Lambda^{p-1}M\oplus\Lambda^{p+1}M\oplus\Lambda^{p,1}M$, where the last irreducible component is the Cartan summand. The generalized gradients are, up to a constant factor, the following: the codifferential, δ, the exterior derivative, d, and respectively the so-called *twistor operator*, T.

Example 2.4 (Dirac and Twistor Operator)**.** The spinor representation $\rho_n:\mathrm{Spin}(n)\to\mathrm{Aut}(\Sigma_n)$, with n odd, is irreducible of highest weight $(\frac{1}{2},\ldots,\frac{1}{2})$, so that, on a spin manifold, we have $\mathrm{TM}\otimes\Sigma\mathrm{M}\cong\Sigma\mathrm{M}\oplus\ker(c)$, where $c:\mathrm{TM}\times\Sigma\mathrm{M}\to\Sigma\mathrm{M}$ denotes the *Clifford multiplication* of a vector field with a spinor. There are thus two generalized gradients: the *Dirac operator D* and the *twistor (Penrose) operator T*: $T_X\varphi=\nabla_X\varphi+\frac{1}{n}X\cdot D\varphi$. For n even, the spinor representation splits into the so-called *positive*, respectively *negative half-spinors*, $\Sigma_n=\Sigma_n^+\oplus\Sigma_n^-$, and for each bundle we similarly obtain two generalized gradients, also called the Dirac and the twistor operator.

Essentially the same construction as above may be used to define generalized gradients associated to a G-structure. For a study of these G-generalized gradients, where G is one of the subgroups of $\mathrm{SO}(n)$ from Berger's list of holonomy groups, we refer the reader *e.g.* to [15].

3. Branson's Classification of Elliptic Generalized Gradients

We recall that if E and F are smooth vector bundles over the manifold M and $P:\Gamma(E)\to\Gamma(F)$ is a linear differential operator, P is *elliptic* if its principal symbol $\sigma_\xi(P;x)$ is an isomorphism for every real section $\xi \in \mathrm{T}^*_x\mathrm{M} \setminus \{0\}$ at all points $x \in M$. Obviously, a necessary condition for the existence of an elliptic operator between two vector bundles is that they have the same rank, so that, one may more generally consider the following weaker notion of ellipticity:

Definition 3.1. A linear differential operator $P : \Gamma(E) \to \Gamma(F)$ is *underdetermined elliptic at a point* $x \in M$ if its symbol $\sigma_\xi(P;x)$ is surjective for every real section $\xi \in \mathrm{T}^*_x\mathrm{M} \setminus \{0\}$. P is *overdetermined elliptic at a point* $x \in M$ if $\sigma_\xi(P;x)$ is injective for every real section $\xi \in \mathrm{T}^*_x\mathrm{M}\setminus\{0\}$. P is called *(injectively) strongly elliptic* if $\sigma_\xi(P;x)$ is injective for every complex cotangent vector $\xi \in (\mathrm{T}^*_x\mathrm{M})^{\mathbb{C}} \setminus \{0\}$.

Remark 3.2. Since the principal symbol of a generalized gradient P_ε is given by the projection Π_ε defining it, the above notion of ellipticity may be rephrased as follows: P_ε is underdetermined (respectively overdetermined) elliptic if and only if the map $\Pi_\varepsilon \circ (\xi \otimes \cdot) : V_\lambda \to V_{\lambda+\varepsilon}$ is surjective (respectively injective), for each nonzero section $\xi \in \Gamma(\mathrm{T}^*_x\mathrm{M})$. Thus, the generalized gradient P_ε is (strongly) injectively elliptic if and only if Π_ε is non-vanishing on each decomposable element. A strongly elliptic operator is obviously elliptic. The converse is not true and a counterexample is provided by the Dirac operator D on a spin manifold, whose principal symbol is given by the Clifford multiplication: $\sigma_\xi(D)(\varphi) = \xi \cdot \varphi$.

Let now (M, g) be a Riemannian (spin) manifold, λ a dominant weight of $\mathfrak{so}(n)$ and $V_\lambda M$ the associated vector bundle. For any subset I of the set of relevant weights of λ, we consider the following second order differential operator: $\sum_{\varepsilon\in I} P^*_\varepsilon P_\varepsilon$, where $P_\varepsilon := \Pi_\varepsilon \circ \nabla$ is the generalized gradient. It is then natural to ask for a given λ, for which subsets I is this operator elliptic? The complete answer to this question was given by Branson, [4].

The problem may be reduced to first order differential operators. More precisely, if we denote by P_I the following first order operator:

$$P_I := \sum_{\varepsilon\in I} P_\varepsilon, \tag{3.1}$$

then $\sum_{\varepsilon\in I} P^*_\varepsilon P_\varepsilon$ is elliptic if and only if P_I is injectively elliptic (in the sequel we shall shortly say that P_I is elliptic), *i.e.* the projection $\Pi_I := \sum_{\varepsilon\in I} \Pi_\varepsilon : T \otimes V_\lambda \to \bigoplus_{\varepsilon\in I} V_{\lambda+\varepsilon}$ is injective when restricted to the set of decomposable elements in $T \otimes V_\lambda$. Thus, the study of ellipticity is reduced to a question on the representation theory of $\mathfrak{so}(n)$, without reference to any particular manifold.

The fact that each projection Π_I is onto a different direct summand has the following straightforward, but important consequence: if instead of the operators P_I given by (3.1), we consider, more generally, operators of the form $\sum_{\varepsilon\in I} a_\varepsilon P_\varepsilon$ with nonzero coefficients, then such an operator is elliptic if and only if P_I is. Thus, ellipticity only depends on the subset I, unlike for Weitzenböck formulas, where these coefficients play a very important role. Moreover, if

$I_1 \subset I_2$ and P_{I_1} is elliptic, then also P_{I_2} is elliptic. Hence, the interesting operators are the *minimal elliptic* operators P_I (*i.e.* such that no proper subset of I defines an elliptic operator).

Example 3.3. In Example 2.3 the complement of the Cartan projection defines the operator $P = d + \delta$, which is (injectively) elliptic and, by the above construction, just gives rise to the *Laplacian* acting on p-forms: $\Delta = d\delta + \delta d = (d+\delta)^*(d+\delta)$.

Branson's classification essentially says that the Laplacian is not a special case, but the generalized gradients usually break up into pairs or singletons which are elliptic.

Theorem 3.4 (Branson, [4])**.** *Let (M, g) be an n-dimensional Riemannian (spin) manifold and $V_\lambda M$ the associated vector bundle to an irreducible $\mathrm{SO}(n)$- (or $\mathrm{Spin}(n)$)-representation of highest weight λ. For any subset I of the set of relevant weights of λ, the corresponding operator $P_I = \sum_{\varepsilon\in I} \Pi_\varepsilon \circ \nabla$ is a minimal elliptic operator if and only if I is one of the following sets, depending on the parity of n:*

(a) if n is odd, $n = 2m+1$:
- *(1) $\{\varepsilon_1\}$ (strongly elliptic),*
- *(2) $\{0\}$, if λ is properly half-integral,*
- *(3) $\{-\varepsilon_i, \varepsilon_{i+1}\}$, for $i = 1, \dots, m-1$,*
- *(4) $\{-\varepsilon_m, 0\}$, if λ is integral.*

(b) if n is even, $n = 2m$:
- *(1) $\{\varepsilon_1\}$ (strongly elliptic),*
- *(2) $\{-\varepsilon_m\}$, if $\lambda_m > 0$,*
- *(3) $\{\varepsilon_m\}$, if $\lambda_m < 0$,*
- *(4) $\{-\varepsilon_i, \varepsilon_{i+1}\}$, for $i = 1, \dots, m-2$,*
- *(5) $\{-\varepsilon_{m-1}, \varepsilon_m\}$, if $\lambda_m \geq 0$,*
- *(6) $\{-\varepsilon_{m-1}, -\varepsilon_m\}$, if $\lambda_m \leq 0$.*

Note that in the list of minimal elliptic operators no operator $P^*_\varepsilon P_\varepsilon$ appears twice, except for $P^*_{-\varepsilon_{m-1}} P_{-\varepsilon_{m-1}}$ in the case when n is even and $\lambda_m = 0 \neq \lambda_{m-1}$. The list is also exhaustive, except for n odd and λ properly half-integral, when $P^*_{-\varepsilon_m} P_{-\varepsilon_m}$ does not occur in the list. Thus, apart from these exceptions, the subsets I defining the minimal elliptic operators form a partition of the set of weights of the standard representation τ.

Remark 3.5. A priori it is not clear that the ellipticity of $P_I : \Gamma(V_\lambda M) \to \Gamma(\bigoplus_{\varepsilon\in I} V_{\lambda+\varepsilon} M)$, defined by a certain subset I, is independent of the given highest weight λ (of course here are considered only those highest weights for which all the elements in I are relevant weights). This follows from Theorem 3.4 and no other direct way of proving it is known.

4. A New Proof of the Classification

The aim of this section is to give a local proof of Branson's classification of minimal elliptic (sums of) generalized gradients, [4], stated here in our notation in Theorem 3.4. In a first step we extend to all (not necessarily elliptic) generalized gradients the computation of the Kato constant provided by Calderbank, Gauduchon and Herzlich, [8]. The main idea is to reverse, in a certain sense, the argument: while in [8] the purpose is to establish for each natural elliptic operator an explicit formula of its optimal Kato constant, assuming known the list of minimal elliptic operators, our goal is to analyze to which extend the computations of the Kato constants rely on this assumption of ellipticity and how Branson's list could be recovered.

The new proof of Branson's classification will follow from Propositions 4.7 and 4.11 and Remark 4.2.

4.1. **Elliptic Operators and Refined Kato Inequalities.** We first briefly recall how refined Kato inequalities are related to the ellipticity of differential operators. The principle underlying the existence of refined Kato inequalities was first remarked by J.-P. Bourguignon, [2]. Calderbank, Gauduchon and Herzlich, [8], proved that for each injectively elliptic operator P_I, there exists an optimal constant $k_I < 1$ such that the refined Kato inequality holds:

$$|d|\varphi|| \leq k_I |\nabla\varphi|, \qquad \text{for all } \varphi \in \ker(P_I), \tag{4.1}$$

and gave an explicit formula for k_I, in terms of the translated conformal weights (see Theorem 4.3).

In the sequel we show how this computation can be extended to all generalized gradients and in order give our argument we first need to briefly review the main steps in [8] (see also [7], [10]). We recall that the conformal weights are the eigenvalues of the so-called conformal weight operator defined as follows:

Definition 4.1. The *conformal weight operator* of an $\mathrm{SO}(n)$-representation $\lambda : \mathrm{SO}(n) \to \mathrm{Aut}(V)$, is the symmetric endomorphism defined as follows:

$$B : (\mathbb{R}^n)^* \otimes V \to (\mathbb{R}^n)^* \otimes V, \quad B(\alpha \otimes v) = \sum_{i=1}^{n} e_i^* \otimes d\lambda(e_i \wedge \alpha)v, \tag{4.2}$$

where $\{e_i\}_{\overline{1,n}}$ is an orthonormal basis of $\mathbb{R}^n$ and $\{e_i^*\}_{\overline{1,n}}$ its dual basis. We also denote by B the induced endomorphism on the associated bundle $\mathrm{T}^*M \otimes V_\lambda M$.

As pointed out in [8], the computations are simplified if one considers the *translated conformal weight operator*: $\widetilde{B} := B + \frac{n-1}{2}\mathrm{Id}$, whose eigenvalues, *translated conformal weights*, are explicitly known (see *e.g.* [9]):

$$\widetilde{w}_0(\lambda) = 0, \ \widetilde{w}_{i,+}(\lambda) = \lambda_i - i + \frac{n+1}{2}, \widetilde{w}_{i,-}(\lambda) = -\lambda_i + i - \frac{n-1}{2}, \text{ for } i = 1, \dots, m. \tag{4.3}$$

The key property used in the sequel is that the (translated) conformal weights are strictly ordered, with the exception of the case when n is even, $n = 2m$, $\lambda_m = 0$ and $\tilde{w}_{m,+} = \tilde{w}_{m,-}$, which is due to the fact that the two corresponding $\mathrm{SO}(n)$-irreducible representations are exchanged by a change of orientation, while their sum is an irreducible $\mathrm{O}(n)$-representation. In this exceptional case these two representations are considered as one summand, so that the conformal weights of distinct projections are always different from each other. The strict ordering of the translated conformal weights allows us to rename them (and the corresponding summands in the decomposition of the tensor product $(\mathbb{R}^n)^* \otimes V_\lambda$) and to index them in a decreasing ordering as follows: $(\mathbb{R}^n)^* \otimes V_\lambda = \overset{N}{\underset{i=1}{\oplus}} V_i$, with $\widetilde{w}_1(\lambda) > \widetilde{w}_2(\lambda) > \cdots > \widetilde{w}_N(\lambda)$, where N is the number of summands in the decomposition, *i.e.* the number of relevant weights for λ. This reordering of the indices carries over to the corresponding weights of the standard representation and thus, the subsets I defining the operators P_I are subsets of $\{1, \dots, N\}$.

Remark 4.2. Notice that, in the above notation, the list of minimal elliptic operators of the form P_I established by Branson (see Theorem 3.4) is the following:

(1) $P_{\{1\}}$;
(2) $P_{\{\ell+1\}}$ if $N = 2\ell$ and $\lambda_m \neq 0$;
(3) $P_{\{\ell\}}$ if $N = 2\ell - 1$ and λ is properly half-integral;

(4) $P_{\{i,N+2-i\}}$ for $i = 2, \dots, \ell - 1$;
(5) $P_{\{\ell,\ell+2\}}$ if $N = 2\ell$;
(6) $P_{\{\ell,\ell+1\}}$ if $N = 2\ell - 1$ and λ is integral.

In particular, the list of the minimal elliptic operators depends only on the ordering of the conformal weights.

Let $\widehat{I}$ denote the complement of I. The following formula for the optimal Kato constant in (4.1) reduces the problem to an algebraic one (*cf.* [8]):

$$k_I := \sup_{|\alpha|=|v|=1} |\Pi_{\widehat{I}}(\alpha \otimes v)| = \sqrt{1 - \inf_{|\alpha|=|v|=1} |\Pi_I(\alpha \otimes v)|^2}, \tag{4.4}$$

where $\alpha \in (\mathbb{R}^n)^*$ and $v \in V_\lambda$. Furthermore, equality holds at a point if and only if $\nabla\varphi = \Pi_{\widehat{I}}(\alpha \otimes \varphi)$ for a 1-form α at that point, such that: $|\Pi_{\widehat{I}}(\alpha \otimes \varphi)| = k_I |\alpha \otimes \varphi|$.

The norm of each projection Π_j, $j = 1, \dots, N$, is then expressed as an affine function as follows[1] for $N = 2\ell - 1$:

$$|\Pi_j(\alpha \otimes v)|^2 = \frac{\widetilde{w}_j^{2(\ell-1)} - \sum_{k=2}^{\ell} (-1)^k \widetilde{w}_j^{2(\ell-k)} Q_k}{\prod_{k \neq j} (\widetilde{w}_j - \widetilde{w}_k)} =: \pi_j(Q_2, \dots, Q_\ell), \tag{4.5}$$

with the variables Q_k given by $Q_k := (-1)^{k-1} \langle A_{2k-2}(\alpha \otimes v), \alpha \otimes v \rangle$, $k = 2, \dots, \ell$, where $A_k := \sum_{\ell=0}^{k} (-1)^\ell \sigma_\ell(\widetilde{w}) \widetilde{B}^{k-\ell}$ and $\sigma_i(\widetilde{w})$ is the i-th elementary symmetric function in the translated conformal weights $\widetilde{w}_1, \dots, \widetilde{w}_N$.

Hence, the problem of estimating $\inf_{|\alpha|=|v|=1} |\Pi_I(\alpha \otimes v)|^2$ (for a subset I corresponding to an elliptic operator) is reduced to minimizing this affine function over the admissible region in the $(\ell-1)$-dimensional affine space. The admissible region consists of the points Q of coordinates $\{Q_k\}_{k=\overline{2,\ell}}$, such that there exist unitary vectors $\alpha \in (\mathbb{R}^n)^*$ and $v \in V_\lambda$ with the property that for each $k = 2, \dots, \ell$ the following relation holds: $Q_k = (-1)^{k-1} \langle A_{2k-2}(\alpha \otimes v), \alpha \otimes v \rangle$. Thus, the search for Kato constants mainly reduces to linear programming.

The admissible region is contained in a convex in the Q-space, since $|\Pi_j(\alpha \otimes v)|^2 = \pi_j(Q)$ and each norm is non-negative and smaller than 1, if Q is an admissible point. More precisely, from (4.5) it follows that the point $Q = (Q_2, \dots, Q_\ell)$ is in the convex region $\mathcal{P}$ in $\mathbb{R}^{\ell-1}$ defined by the following system of linear inequalities:

$$\sum_{k=2}^{\ell} (-1)^{j+k} \widetilde{w}_j^{2(\ell-k)} Q_k \geq (-1)^j \widetilde{w}_j^{2(l-1)}, \quad j = 1, \dots, 2\ell - 1, \tag{4.6}$$

with equality if and only if $|\Pi_j(\alpha \otimes v)|^2 = \pi_j(Q) = 0$. The convex region $\mathcal{P}$ defined by (4.6) is proven in [8] to be compact, hence polyhedral. Since the norms are affine in the Q_k's, it then suffices to minimize over the set of vertices.

[1]In the sequel we recall the computation only for N odd, since for N even the argument is similar and the details can be found in [8].

For a subset $J \subset \{1, \dots, N\}$ with $\ell - 1$ elements, the intersection of the corresponding hyperplanes is the point denoted by Q^J: $\{Q^J\} := \bigcap_{j\in J} \{\pi_j(Q_2, \dots Q_\ell) = 0\}$, whose coordinates are given by the elementary symmetric functions in the squares of the translated conformal weights: $Q^J_k = \sigma_{k-1}\big((\widetilde{w}_j^2)_{j\in J}\big)$. At the point Q^J, the affine functions π_j, defined by (4.5), take the values

$$\pi_j(Q^J) = \frac{\prod_{k\in J}(\widetilde{w}_j^2 - \widetilde{w}_k^2)}{\prod_{k\neq j}(\widetilde{w}_j - \widetilde{w}_k)} = \frac{\prod_{k\in J, k\neq j}(\widetilde{w}_j + \widetilde{w}_k)}{\prod_{k\in \widehat{J}, k\neq j}(\widetilde{w}_j - \widetilde{w}_k)}\, \varepsilon_j(J), \tag{4.7}$$

where $\varepsilon_j(J) = 0$ if $j \in J$ and 1 otherwise.

As there exists a set of minimal elliptic operators, there also exists a set of *maximal non-elliptic* operators. Let $\mathcal{NE}$ denote the set of subsets of $\{1, \dots, N\}$ whose elements are obtained by choosing exactly one index in each of the sets $\{j, N+2-j\}$ for $2 \leq j \leq \ell$, if $N = 2\ell - 1$ or $N = 2\ell$, giving $2^{\ell-1}$ elements:

$$\mathcal{NE} = \{J \subset \{1, \dots, N\} \,|\, |J \cap \{i, N+2-i\}| = 1, \text{for } 2 \leq i \leq \ell\}. \tag{4.8}$$

Notice that each subset in $\mathcal{NE}$ has exactly $\ell - 1$ elements, where ℓ gives the parity of N, *i.e.* $N = 2\ell - 1$ or $N = 2\ell$. In the sequel we call $\mathcal{NE}$ the set of *virtually maximal non-elliptic operators*, since by Theorem 3.4, the elements of $\mathcal{NE}$ are precisely the subsets of $\{1, \dots, N\}$ corresponding to the maximal non-elliptic operators, unless n is odd, $N = 2\ell - 1$ and λ is properly half-integral, in which case the subsets containing ℓ (which corresponds to the zero weight) are elliptic. This is called the *exceptional case* and is the only one when the Kato constant provided by Theorem 4.3 might not be optimal. More precisely, in [8] it is proven on the one hand, that the vertices of the admissible polyhedron $\mathcal{P}$ are contained in $\mathcal{NE}$ and, on the other hand, that the points of $\mathcal{NE}$ corresponding to maximal non-elliptic operators are vertices of $\mathcal{P}$. Thus, minimizing in the expression (4.4) of k_I the affine functions given by (4.5) over the set $\mathcal{NE}$ yields an optimal value for k_I if $\mathcal{NE}$ is equal to the set of maximal non-elliptic operators and a possibly non-optimal one if $\mathcal{NE}$ is larger. The result is as follows:

Theorem 4.3 (Calderbank, Gauduchon and Herzlich, [8]). *Let I be a subset of $\{1, \dots, N\}$ corresponding to an injectively elliptic operator $P_I = \sum_{i\in I} \Pi_i \circ \nabla$ acting on sections of $V_\lambda M$. Then a refined Kato inequality holds: $|d|\varphi|| \leq k_I |\nabla\varphi|$, for any section $\varphi \in \ker(P_I)$, outside the zero set of φ.*
If N is odd, the Kato constant k_I is given by the following expressions:

$$k_I^2 = \max_{J\in\mathcal{NE}} \left(\sum_{i\in \widehat{I}\cap\widehat{J}} \frac{\prod_{j\in J}(\widetilde{w}_i + \widetilde{w}_j)}{\prod_{j\in\widehat{J}\setminus\{i\}}(\widetilde{w}_i - \widetilde{w}_j)} \right) = 1 - \min_{J\in\mathcal{NE}} \left(\sum_{i\in I\cap\widehat{J}} \frac{\prod_{j\in J}(\widetilde{w}_i + \widetilde{w}_j)}{\prod_{j\in\widehat{J}\setminus\{i\}}(\widetilde{w}_i - \widetilde{w}_j)} \right). \tag{4.9}$$

If N is even, the Kato constant k_I is similarly given by:

$$k_I^2 = \max_{J\in\mathcal{NE}} \left(\sum_{i\in \widehat{I}\cap\widehat{J}} \frac{(\widetilde{w}_i - \frac{1}{2})\prod_{j\in J}(\widetilde{w}_i + \widetilde{w}_j)}{\prod_{j\in\widehat{J}\setminus\{i\}}(\widetilde{w}_i - \widetilde{w}_j)} \right) = 1 - \min_{J\in\mathcal{NE}} \left(\sum_{i\in I\cap\widehat{J}} \frac{(\widetilde{w}_i - \frac{1}{2})\prod_{j\in J}(\widetilde{w}_i + \widetilde{w}_j)}{\prod_{j\in\widehat{J}\setminus\{i\}}(\widetilde{w}_i - \widetilde{w}_j)} \right). \tag{4.10}$$

These Kato constants are optimal, unless in the exceptional case when n and N are odd, $N = 2\ell + 1$, λ is properly half-integral and the set J achieving the extremum contains $\ell + 1$.

The starting point in our new proof is the following straightforward observation:

Lemma 4.4. *Let k_I be the optimal Kato constant for the operator P_I, given by (4.4): $k_I = \sup_{|\alpha|=|v|=1} |\Pi_{\widehat{I}}(\alpha \otimes v)|$. Then P_I is an elliptic operator if and only if $k_I < 1$.*

Proof: If $|\alpha| = |v| = 1$, then $1 = |\alpha \otimes \varphi|^2 = |\Pi_I(\alpha \otimes \varphi)|^2 + |\Pi_{\widehat{I}}(\alpha \otimes \varphi)|^2$, so that k_I is always smaller or equal to 1. Then, by negation, the equivalence in the statement is the same as the following equivalence: P_I is not elliptic if and only if $k_I = 1$, which in turn is a consequence of the definitions: $k_I = 1$ if and only if there exist α and v of norm 1 such that $|\Pi_{\widehat{I}}(\alpha \otimes v)| = 1$, which is then the same as $|\Pi_I(\alpha \otimes \varphi)| = 0$, or, equivalently, $\alpha \otimes \varphi \in \ker(P_I)$, meaning that P_I is not elliptic. □

Lemma 4.4 implies that the ellipticity of a natural first order differential operator P_I follows from the computation of its optimal Kato constant k_I. Thus, as soon as we are able to compute explicitly k_I (without using the ellipticity assumption) or to show that k_I is strictly less than 1, it follows that the operator P_I is elliptic. In the sequel we show that k_I is strictly bounded from above by 1 for the operators in Branson's list (in the notation given by the decreasing ordering of the translated conformal weights, for all operators enumerated in Remark 4.2), except for one case, which corresponds to the zero weight.

We use the same notation as above and notice that for the construction of the convex region $\mathcal{P}$, as well as for establishing its compactness, the only ingredient needed is the ordering of the translated conformal weights, which is provided by the explicit formulas (4.3).

The key observation is that the only step in the proof of Theorem 4.3 in [8] where the ellipticity of the operators is used, is in the identification of the vertices of the polyhedral region. If we now consider the set $\mathcal{NE}$ introduced in (4.8), then one inclusion still holds, without any ellipticity assumption on the operators. More precisely, we obtain:

Lemma 4.5. *The vertices of the polyhedron $\mathcal{P}$ are given by a subset of $\mathcal{NE}$.*

Proof: Let us denote by $\mathcal{V}$ the set of vertices of the polyhedron $\mathcal{P}$ in $\mathbb{R}^{\ell-1}$, which are characterized as follows:

$$\mathcal{V} = \{Q^J \,|\, |J| = \ell - 1, \Pi_j(Q^J) = 0, \text{for all } j \in J; \Pi_j(Q^J) > 0, \text{for all } j \in \widehat{J}\}.$$

Then we have to show the following inclusion: $\mathcal{V} \subset \{Q^J | \, J \in \mathcal{NE}\}$. Or, equivalently, we prove that $J \notin \mathcal{NE}$ implies $Q^J \notin \mathcal{V}$ (where J is a subset of $\{1, \ldots, N\}$ with $\ell-1$ elements, for $N = 2\ell$ or $N = 2\ell - 1$).

Let $J \notin \mathcal{NE}$. In order to show that Q^J is not a vertex of the polyhedron $\mathcal{P}$ it is enough to find an element $i \in \{1, \ldots, N\}$ such that $\pi_i(Q^J) < 0$.

For N odd, equation (4.7) implies that for each $i \notin J$, $\Pi_i(Q^J)$ is nonzero and its sign is:

$$\operatorname{sgn}(\pi_i(Q^J)) = (-1)^{i-1} \operatorname{sgn}\Big(\prod_{j \in J} (\widetilde{w}_i^2 - \widetilde{w}_j^2)\Big).$$

There are exactly $\ell - 1$ couples of the type $(s, N + 2 - s)$ and, since $J \notin \mathcal{NE}$ and has $\ell - 1$ elements, there exists at least one such couple not contained in J.
The ordering of the squares of the translated conformal weights, that can be directly checked by the formulas (4.3), is the following ($N = 2\ell - 1$):

$$\widetilde{w}_1^2 > \widetilde{w}_{N+1}^2 > \widetilde{w}_2^2 > \widetilde{w}_N^2 > \cdots > \widetilde{w}_i^2 > \widetilde{w}_{N+2-i}^2 > \cdots > \widetilde{w}_\ell^2 > \widetilde{w}_{N+2-\ell}^2.$$

It then follows that for a couple $(s, N+2-s)$, $\widetilde{w}_s^2$ and $\widetilde{w}_{N+2-s}^2$ are adjacent in this ordering, so that the following signs are the same:

$$\operatorname{sgn}(\prod_{j\in J}(\widetilde{w}_s^2-\widetilde{w}_j^2))=\operatorname{sgn}(\prod_{j\in J}(\widetilde{w}_{N+2-s}^2-\widetilde{w}_j^2)).$$

Since N is odd, s and $N+2-s$ have different parity, showing that $\pi_s(Q^J)$ and $\pi_{N+2-s}(Q^J)$ have opposite signs. For N even a similar argument holds. □

From the inclusion $\mathcal{V}\subset\mathcal{NE}$ given by Lemma 4.5, the formula (4.4) for the Kato constant k_I and the expression (4.5) for the norms of the projections, we obtain the following upper bound:

Proposition 4.6. *Let I be a subset of $\{1,\dots,N\}$ and the operator $P_I=\sum_{i\in I}\Pi_i\circ\nabla$ acting on sections of $V_\lambda M$. Then, the corresponding Kato constant k_I satisfies the following inequality:*

$$k_I^2=\max_{Q\in\mathcal{P}}\left(\sum_{j\in\widehat{I}}\pi_j(Q)\right)=\max_{Q\in\mathcal{V}}\left(\sum_{j\in\widehat{I}}\pi_j(Q)\right)\le\max_{J\in\mathcal{NE}}\left(\sum_{j\in\widehat{I}}\pi_j(Q^J)\right)=:c_I. \tag{4.11}$$

Thus, if $c_I<1$ for a subset $I\subset\{1,\dots,N\}$, it follows by Lemma 4.4 that the corresponding operator P_I is elliptic.

We notice that the formulas for the optimal Kato constant in Theorem 4.3 actually compute the values of the upper bound c_I, if we do not assume the ellipticity of any operator involved. This straightforward, but important remark provides the main argument in our proof of Branson's classification.

From Theorem 4.3 applied to the special case when the set I has only one element or two elements of the form $\{i, N+2-i\}$, we recover the list of minimal elliptic operators as follows.

Proposition 4.7. *The upper bound c_I is strictly smaller than 1 for any of the following subsets I:*

(1) $I=\{1\}$;
(2) $I=\{\ell+1\}$ if $N=2\ell$ and $\lambda_m\neq 0$;
(3) $I=\{i,N+2-i\}$ for $i=2,\dots,\ell$.

From the above discussion it follows that the corresponding operators P_I are elliptic.

Proof: By Theorem 4.3, the upper bound c_I is given by the following formula, if $N=2\ell-1$:

$$c_I=\max_{J\in\mathcal{NE}}\left(\sum_{j\in\widehat{I}}\pi_j(Q^J)\right)=1-\min_{J\in\mathcal{NE}}\left(\sum_{i\in I\cap\widehat{J}}\frac{\prod_{j\in J}(\widetilde{w}_i+\widetilde{w}_j)}{\prod_{j\in\widehat{J}\setminus\{i\}}(\widetilde{w}_i-\widetilde{w}_j)}\right), \tag{4.12}$$

and if $N=2\ell$:

$$c_I=\max_{J\in\mathcal{NE}}\left(\sum_{j\in\widehat{I}}\pi_j(Q^J)\right)=1-\min_{J\in\mathcal{NE}}\left(\sum_{i\in I\cap\widehat{J}}\frac{(\widetilde{w}_i-\frac{1}{2})\prod_{j\in J}(\widetilde{w}_i+\widetilde{w}_j)}{\prod_{j\in\widehat{J}\setminus\{i\}}(\widetilde{w}_i-\widetilde{w}_j)}\right). \tag{4.13}$$

These expressions are particularly simple if the set I has just a few elements, as in our case.

(1) Substituting $I=\{1\}$ in (4.12) and (4.13), the sums reduce to one element, since $I\cap\widehat{J}=\{1\}$ for any $J\in\mathcal{NE}$, and we get:

$$c_{\{1\}}=1-\min_{J\in\mathcal{NE}}\left(\frac{\prod_{j\in J}(\widetilde{w}_1+\widetilde{w}_j)}{\prod_{j\in\widehat{J}\setminus\{i\}}(\widetilde{w}_1-\widetilde{w}_j)}\right),\quad \text{if } N=2\ell-1, \tag{4.14}$$

$$c_{\{1\}}=1-\min_{J\in\mathcal{NE}}\left(\left(\widetilde{w}_1-\frac{1}{2}\right)\frac{\prod_{j\in J}(\widetilde{w}_1+\widetilde{w}_j)}{\prod_{j\in\widehat{J}\setminus\{1\}}(\widetilde{w}_1-\widetilde{w}_j)}\right),\quad \text{if } N=2\ell, \tag{4.15}$$

which implies that $c_{\{1\}}<1$, because $\widetilde{w}_1$ is the biggest translated conformal weight: $\widetilde{w}_1^2>\widetilde{w}_j^2$, for any $2\leq j\leq N$ and $\widetilde{w}_1=\lambda_1+\frac{n-1}{2}>\frac{1}{2}$ (we assume always $n\geq 2$ and $\lambda_1\neq 0$, otherwise λ is just the trivial representation).

(2) If the dimension n is odd, $n=2m+1$, the case $N=2\ell$ can only occur if $\lambda_m=\frac{1}{2}$, as it can be easily seen by the selection rule in Lemma 2.1, since in all the other cases the weights come in pairs. In this case, the index $\ell+1$, given by the decreasing ordering of the translated conformal weights, stays for the weight 0. If $n=2m$ and $N=2\ell$, then it follows that the index $\ell+1$ stays either for the weight $-\varepsilon_m$, if $\lambda_m>0$, or for the weight ε_m, if $\lambda_m<0$ (since again the indices are given by the decreasing ordering of the translated conformal weights and $\widetilde{w}_{m,+}-\widetilde{w}_{m,-}=2\lambda_m$). Substituting $I=\{\ell+1\}$ in (4.13) reduces again the sum to one element and yields the following expression:

$$c_{\{\ell+1\}}=1-\min_{J\in\mathcal{NE}}\left(\frac{\widetilde{w}_{\ell+1}-\frac{1}{2}}{\widetilde{w}_{\ell+1}-\widetilde{w}_1}\cdot\frac{\prod_{j\in J}(\widetilde{w}_{\ell+1}+\widetilde{w}_j)}{\prod_{j\in\widehat{J}\setminus\{1,\ell+1\}}(\widetilde{w}_{\ell+1}-\widetilde{w}_j)}\right). \tag{4.16}$$

From the explicit values of the translated conformal weights given by (4.3), namely: $\widetilde{w}_{m,-}=-\lambda_m+m-\frac{n-1}{2}$ and $\widetilde{w}_{m,+}=\lambda_m-m+\frac{n+1}{2}$, it follows that for $n=2m+1$, as well as for $n=2m$, the term $\left(\widetilde{w}_{\ell+1}-\frac{1}{2}\right)$ is strictly negative, and thus $\frac{\widetilde{w}_{\ell+1}-\frac{1}{2}}{\widetilde{w}_{\ell+1}-\widetilde{w}_1}$ is strictly positive. From the way the sets $J\in\mathcal{NE}$ are defined, by choosing exactly one element from each pair $\{i,2\ell+2-i\}$ for $2\leq i\leq\ell$, it follows that in the product in (4.16), there occur only factors of one of the following two types: $\frac{\widetilde{w}_{\ell+1}+\widetilde{w}_i}{\widetilde{w}_{\ell+1}-\widetilde{w}_{2\ell+2-i}}$ or $\frac{\widetilde{w}_{\ell+1}+\widetilde{w}_{2\ell+2-i}}{\widetilde{w}_{\ell+1}-\widetilde{w}_i}$ for some $2\leq i\leq\ell$. From the ordering of the translated conformal weights it turns out that each such factor is strictly positive, showing thus that $c_{\{\ell+1\}}<1$.

(3) The ordering of the translated conformal weights implies the following inequalities, for any $i\in\{1,\dots,N\}$, $j\in\{1,\dots,\ell\}$ and $j\neq i,N+2-i$: $\frac{\widetilde{w}_i+\widetilde{w}_j}{\widetilde{w}_i-\widetilde{w}_{N+2-j}}>\frac{\widetilde{w}_i+\widetilde{w}_{N+2-j}}{\widetilde{w}_i-\widetilde{w}_j}>0$, if $i<j$ or $N+2-j<i$, and $\frac{\widetilde{w}_i+\widetilde{w}_{N+2-j}}{\widetilde{w}_i-\widetilde{w}_j}>\frac{\widetilde{w}_i+\widetilde{w}_j}{\widetilde{w}_i-\widetilde{w}_{N+2-j}}>0$, if $j<i<N+2-j$.

If $N=2\ell-1$, then substituting I in (4.12) with a set formed by a pair of type $I=\{i,N+2-i\}$, with $i\in\{2,\dots,\ell\}$, and using the above relations yields the following expression for the upper bound of the Kato constant:

$$c_I=1-\min\left(\frac{\widetilde{w}_i+\widetilde{w}_{2\ell+1-i}}{\widetilde{w}_i-\widetilde{w}_1},\frac{\widetilde{w}_i+\widetilde{w}_{2\ell+1-i}}{\widetilde{w}_{2\ell+1-i}-\widetilde{w}_1}\right).$$

Similarly, if $N=2\ell$, then substituting $I=\{i,N+2-i\}$ in (4.13) yields:

$$c_I=1-\min\left(\frac{(\widetilde{w}_i+\widetilde{w}_{2\ell+2-i})(\widetilde{w}_i-\frac{1}{2})}{(\widetilde{w}_i-\widetilde{w}_{\ell+1})(\widetilde{w}_i-\widetilde{w}_1)},\frac{(\widetilde{w}_i+\widetilde{w}_{2\ell+2-i})(\widetilde{w}_{2\ell+2-i}-\frac{1}{2})}{(\widetilde{w}_{2\ell+2-i}-\widetilde{w}_{\ell+1})(\widetilde{w}_{2\ell+2-i}-\widetilde{w}_1)}\right).$$

The same argument as in the case 2. shows that $c_I < 1$. □

Proposition 4.7 proves that all the operators that come up in Branson's classification (listed in Remark 4.2 in our notation) are elliptic, except for one special case explained in Remark 4.9. However, our aim is to determine *all* minimal elliptic operators, so that we still have to eliminate the other possibilities. Namely, on the one hand, we have to show that the generalized gradients corresponding to an element in one of the sets obtained in the case (3) of Proposition 4.7 are not elliptic, and on the other hand, that there are no other combinations which provide elliptic operators. Thus, we have to find the maximal non-elliptic operators, in order to conclude that the elliptic operators found in Proposition 4.7 are *all* the minimal elliptic operators.

4.2. **Non-elliptic generalized gradients and branching rules.** The main tool we need here is the branching rule of the special orthogonal group and the following necessary condition for ellipticity (see also [8]):

Lemma 4.8. *Let $P_I : \Gamma(V_\lambda) \to \Gamma(\bigoplus_{i\in I} V_i)$ be the operator corresponding to a subset I of $\{1,\dots,N\}$. If there exists an irreducible $\mathrm{SO}(n-1)$-subrepresentation of V_λ that does not occur as $\mathrm{SO}(n-1)$-subrepresentation of V_i for any $i \in I$, then P_I is not elliptic.*

Proof: By Definition 3.1, P_I is elliptic if its principal symbol, $\Pi_I : (\mathbb{R}^n)^* \otimes V_\lambda \to \bigoplus_{i\in I} V_i$, is injective when restricted to the set of decomposable elements, *i.e.* if for any vector $\alpha \in (\mathbb{R}^n)^*$, $\alpha \neq 0$, the linear map $V_\lambda \to \bigoplus_{i\in I} V_i, v \mapsto \Pi_I(\alpha\otimes v)$, is injective. Since $\mathrm{SO}(n)$ acts transitively on the unit sphere in $(\mathbb{R}^n)^*$, one may, without loss of generality, take α to be a unit vector. Then, the above map is $\mathrm{SO}(n-1)$-equivariant, where $\mathrm{SO}(n-1)$ is the stabilizer group of α under the $\mathrm{SO}(n)$-action on the sphere. The existence of an injective and $\mathrm{SO}(n-1)$-equivariant map between V_λ and $\bigoplus_{i\in I} V_i$ shows that any $\mathrm{SO}(n-1)$-subrepresentation of V_λ occurs in some V_i.□

Remark 4.9. There is one exceptional case where we cannot apply Lemma 4.8. Namely, when n is odd, $N = 2\ell - 1$ and $\lambda_m > 0$, then the zero weight is relevant. If λ is moreover properly half-integral, then the corresponding operator $P_\ell : V_\lambda M \to V_\lambda M$ is elliptic (by Branson's result), while if λ is integral, P_ℓ is not elliptic. Unfortunately, this special case cannot be recovered by our approach, since in this case the source and the target representation are isomorphic. In general, our argument only involves the translated conformal weights, which are associated to the Lie algebra $\mathfrak{so}(n)$, so that it does not distinguish between the groups $\mathrm{Spin}(n)$ and $\mathrm{SO}(n)$.

In order to use Lemma 4.8 we have to apply the branching rule for the restriction of an $\mathrm{SO}(n)$-representation to $\mathrm{SO}(n-1)$, which we recall in the sequel (see, *e.g.* Theorem 9.16, [14]). We consider, as usual, the parametrization of irreducible $\mathrm{SO}(n)$-representations by dominant weights, *i.e.* the weights satisfying the inequalities (2.1).

Proposition 4.10 (Branching Rule for $\mathrm{SO}(n)$)**.**

- **(a):** *For the group $\mathrm{SO}(2m+1)$, the irreducible representation with highest weight $\lambda = (\lambda_1,\dots,\lambda_m)$ decomposes with multiplicity 1 under $\mathrm{SO}(2m)$, and the representations of $\mathrm{SO}(2m)$ that appear are exactly those with highest weights $\gamma = (\gamma_1,\dots,\gamma_m)$ such that*

$$\lambda_1 \geq \gamma_1 \geq \lambda_2 \geq \gamma_2 \geq \cdots \geq \lambda_{m-1} \geq \gamma_{m-1} \geq \lambda_m \geq |\gamma_m|. \tag{4.17}$$

(b): *For the group* SO$(2m)$, *the irreducible representation with highest weight* $\lambda = (\lambda_1, \dots, \lambda_m)$ *decomposes with multiplicity* 1 *under* SO$(2m-1)$, *and the representations of* SO$(2m-1)$ *that appear are exactly those with highest weights* $\gamma = (\gamma_1, \dots, \gamma_{m-1})$ *such that*

$$\lambda_1 \geq \gamma_1 \geq \lambda_2 \geq \gamma_2 \geq \cdots \geq \lambda_{m-1} \geq \gamma_{m-1} \geq |\lambda_m|. \tag{4.18}$$

From Lemma 4.8 and Proposition 4.10 we obtain:

Proposition 4.11. *The maximal non-elliptic operators P_J are given exactly by the sets J in $\mathcal{NE}$, apart from the special case when n is odd, $N = 2\ell - 1$ and $\lambda_m \geq 1$. In this case the sets J of $\mathcal{NE}$ that do not contain ℓ (which corresponds to the weight* 0*) are maximal non-elliptic.*

Proof: We recall that the coordinates of a dominant weight λ are given with respect to the basis $\{\varepsilon_i\}_{i=\overline{1,m}}$ introduced in § 2. Here it is more convenient to consider the elements of a set J as weights of the standard representation, instead of the notation with indices corresponding to the ordering of the translated conformal weights.

Let J be a subset in $\mathcal{NE}$, *i.e.* J has cardinality $\ell - 1$, where $N = 2\ell$ or $N = 2\ell - 1$. If $n = 2m$, then J is obtained by choosing exactly one weight from each pair of relevant weights of type $\{-\varepsilon_i, \varepsilon_{i+1}\}$, for $1 \leq i \leq m-2$ and one weight from $\{-\varepsilon_{m-1}, \varepsilon_m\}$, if $\lambda_m > 0$, or one weight from $\{-\varepsilon_{m-1}, -\varepsilon_m\}$, if $\lambda_m < 0$. If $n = 2m+1$, then we consider the sets $J \in \mathcal{NE}$ obtained by choosing exactly one weight from each pair of relevant weights of type $\{-\varepsilon_i, \varepsilon_{i+1}\}$, for $1 \leq i \leq m-1$ and the weight $-\varepsilon_m$, if it is relevant.

For each such set J, it is enough to find an SO$(n-1)$-subrepresentation of V_λ that does not occur in $\underset{\varepsilon \in J}{\oplus} V_{\lambda+\varepsilon}$. By Lemma 4.8 it will then follow that the corresponding operator P_J is not elliptic. When enlarging the set J to some set J' by adding any other relevant weight, there is at least one subset I of J' which is equal to one of those listed in Proposition 4.7, showing that J' is elliptic. This means that J is maximal non-elliptic.

For $n = 2m$ we choose the irreducible SO$(2m-1)$-subrepresentation of λ with highest weight $\gamma = (\gamma_1, \dots, \gamma_{m-1})$, where the coordinates are defined by the following rule, for each $1 \leq i \leq m-2$:

$$\gamma_i = \begin{cases} \lambda_i, & \text{if } \lambda_i = \lambda_{i+1} \text{ or } -\varepsilon_i \in J \\ \lambda_{i+1}, & \text{if } \varepsilon_{i+1} \in J, \end{cases} \tag{4.19}$$

and

$$\gamma_{m-1} = \begin{cases} \lambda_{m-1}, & \text{if } \lambda_{m-1} = \lambda_m = 0 \text{ or } -\varepsilon_{m-1} \in J \\ \lambda_m, & \text{if } \varepsilon_m \in J \text{ and } \lambda_m > 0 \\ -\lambda_m, & \text{if } -\varepsilon_m \in J \text{ and } \lambda_m < 0. \end{cases} \tag{4.20}$$

We recall that the condition $\lambda_i = \lambda_{i+1}$, for $1 \leq i \leq m-2$, is equivalent to the fact that the weights $\{-\varepsilon_i, \varepsilon_{i+1}\}$ are not relevant for λ and $\lambda_{m-1} = \lambda_m = 0$ is the only case when $-\varepsilon_{m-1}$ is not relevant. The coordinates of γ fulfill the inequalities (4.18) for the representation λ, showing that γ is an irreducible SO$(2m-1)$-subrepresentation of λ. On the other hand, it can be directly checked that the inequalities (4.18) are not satisfied anymore for any of the SO$(2m)$-representations of highest weight $\lambda + \varepsilon$ with $\varepsilon \in J$, showing that γ does not occur as SO$(2m-1)$-subrepresentation in $\oplus_{\varepsilon \in J} V_{\lambda+\varepsilon}$.

For $n = 2m+1$ we similarly choose an irreducible SO($2m$)-subrepresentation of λ with highest weight $\gamma = (\gamma_1, \dots, \gamma_m)$, whose coordinates are defined by the following rule, for each $1 \leq i \leq m-1$:

$$\gamma_i = \begin{cases} \lambda_i, & \text{if } \lambda_i = \lambda_{i+1} \text{ or } -\varepsilon_i \in J \\ \lambda_{i+1}, & \text{if } \varepsilon_{i+1} \in J, \end{cases} \tag{4.21}$$

and $\gamma_m = \lambda_m$. It follows also in this case that the inequalities (4.17) are fulfilled for λ, but fail for any $\lambda + \varepsilon$ with $\varepsilon \in J$. The branching rule then implies that γ is an irreducible SO($2m$)-subrepresentation of V_λ which does not occur as subrepresentation in $\oplus_{\varepsilon \in J} V_{\lambda+\varepsilon}$. □

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