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Farid Madani and Mounir Nisse

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ON THE VOLUME OF COMPLEX AMOEBAS

FARID MADANI AND MOUNIR NISSE

Abstract. The paper deals with amoebas of k-dimensional algebraic varieties in the algebraic complex torus of dimension $n \geq 2k$. First, we show that the area of complex algebraic curve amoebas is finite. Moreover, we give an estimate of this area in the rational curve case in terms of the degree of the rational parametrization coordinates. We also show that the volume of the amoeba of k-dimensional algebraic variety in $(\mathbb{C}^*)^n$, with $n \geq 2k$, is finite.

1. Introduction

Amoebas have proven to be a very useful tool in several areas of mathematics, and they have many applications in real algebraic geometry, complex analysis, mirror symmetry, algebraic statistics and in several other areas (see [M1-02], [M2-04], [M3-00], [FPT-00], [PR1-04], [PS-04] and [R-01]). They degenerate to a piecewise-linear object called *tropical varieties* (see [M1-02], [M2-04], and [PR1-04]). Moreover, we can use amoebas as an intermediate link between the classical and the tropical geometry.

The amoeba $\mathscr{A}$ of an algebraic variety $V \subset (\mathbb{C}^*)^n$ is a closed subsets of $\mathbb{R}^n$, and its (Lebesgue) volume is well-defined. Passare and Rullgård [PR1-04], proved that the area of complex plane curve amoebas is finite and the bound is given in terms of the Newton polygon. In this paper, we prove that the amoeba area of any algebraic curve in $(\mathbb{C}^*)^n$ is finite (the area here is with respect to the induced Euclidean metric of $\mathbb{R}^n$). Moreover, we generalize our result, for any algebraic variety V of dimension k in the algebraic complex torus $(\mathbb{C}^*)^{2k+m}$ with $m \geq 0$. Our main result is the following theorem:

Theorem 1.1. *Let $V \subset (\mathbb{C}^*)^n$ be an algebraic variety of dimension k in the complex torus $(\mathbb{C}^*)^{2k+m}$ with $m \geq 0$, such that no irreducible component of V is contained in a sub-torus of dimension less than $2k$. Then, the volume of its amoeba is finite.*

Note that if V contains an irreducible ï¿½component in an algebraic sub-torus of dimension strictly less than $2k$, then the volume of its amoeba is infinite.

The remainder of this paper is organized as follows. In Section 2, we review some properties of the amoebas, and also the theorem structure of the logarithmic limit set defined by Bergman [B-71], which is used as tools in the proof of our results. In Section 3, we prove our main result for complex algebraic curves in $(\mathbb{C}^*)^n$ for any $n \geq 2$. In section 4, we give an estimate of the bound for algebraic rational complex curves, and some examples. In Section 5, we prove the main theorem of this paper.

2. Preliminaries

Let V be an algebraic variety in $(\mathbb{C}^*)^n$. The *amoeba* $\mathscr{A}$ of V is by definition (see M. Gelfand, M.M. Kapranov and A.V. Zelevinsky [GKZ-94]) the image of V under

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the map :

$$\begin{array}{rccl} \mathrm{Log} & : & (\mathbb{C}^*)^n & \longrightarrow & \mathbb{R}^n \\ & & (z_1,\dots,z_n) & \longmapsto & (\log|z_1|,\dots,\log|z_n|). \end{array}$$

Passare and Rullgård prove the following (see [PR1-04]):

Theorem 2.1. *Let f be a Laurent polynomial in two variables. Then the area of the amoeba of the curve with defining polynomial f is not greater than π^2 times the area of the Newton polytope of f.*

In [MR-00], Mikhalkin and Rullgård show that up to multiplication by a constant in $(\mathbb{C}^*)^2$, the algebraic plane curves with Newton polygon Δ with maximal amoeba area are defined over $\mathbb{R}$. Furthermore, their real loci are isotopic to so-called Harnack curves (possibly singular with ordinary real isolated double points).

Recall that the Hausdorff distance between two closed subsets A, B of a metric space (E, d) is defined by:

$$d_{\mathcal{H}}(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},$$

where we take $E = \mathbb{R}^n \times (S^1)^n$, with the distance defined as the product of the Euclidean metric on $\mathbb{R}^n$ and the flat metric on $(S^1)^n$.

The *logarithmic limit set* of a complex algebraic variety V, denoted by $\mathscr{L}^\infty(V)$, is the set of limit points of $\mathscr{A}$ in the sphere $S^{n-1} = (\mathbb{R}^n)^*/\mathbb{R}_+$. In other word, if S^{n-1} denotes the boundary of the unit ball B^n and r the map defined by:

$$\begin{array}{rccl} r & : & \mathbb{R}^n & \longrightarrow & B^n \\ & & x & \longmapsto & r(x) = \frac{x}{1+|x|}, \end{array}$$

then $\mathscr{L}^\infty(V) = \overline{r(\mathscr{A})} \cap S^{n-1}$. Bergman [B-71] proved that if $V \subset (\mathbb{C}^*)^n$ is an algebraic variety of dimension k, then the cone over $\mathscr{L}^\infty(V)$ is contained in a finite union of k-dimensional subspaces of $\mathbb{R}^n$ defined over $\mathbb{Q}$. On the other hand, Bieri and Groves [BG-84] proved that this cone is a finite union of rational polyhedral convex cones of dimension at most k, and the maximal dimension in this union is achieved by at least one polyhedral P in this union. Moreover, one has $\dim_{\mathbb{R}} \mathscr{L}^\infty(V) = (\dim_{\mathbb{C}} V) - 1$. More precisely, we have the following theorem structure:

Theorem 2.2 (Bergman, Bieri-Groves). *The logarithmic limit set $\mathscr{L}^\infty(V)$ of an algebraic variety V in $(\mathbb{C}^*)^n$ is a finite union of rational convex polyhedrals. The maximal dimension of a polyhedral in this union is achieved at least by one polyhedral P in this union, and we have $\dim_{\mathbb{R}} \mathscr{L}^\infty(V) = \dim_{\mathbb{R}} P = (\dim_{\mathbb{C}} V) - 1$.*

3. Area of complex algebraic curve amoeba

The main result of this section, is the following theorem:

Theorem 3.1. *Let $\mathcal{C} \subset (\mathbb{C}^*)^n$ be an algebraic curve with $n \geq 2$. Assume that no irreducible component of $\mathcal{C}$ is contained in a sub-torus of dimension less than 2. Then, the area of its amoeba is finite.*

We start by proving Theorem 3.1 in the rational curve case (see Theorem 3.2). Recall that a complex algebraic curve is contained in a sub-torus of dimension one, means that the curve is the sub-torus of dimension one itself (sometimes called a holomorphic annulus). Moreover, its amoeba is a straight line in $\mathbb{R}^n$, and this case is not interesting for us because it's not generic.
Let n and k be two positive integers such that $2k \leq n$. Let $f : \mathbb{C}^k \longrightarrow (\mathbb{C}^*)^n$ be a rational map, and V be the variety in $(\mathbb{C}^*)^n$ defined by the image of f. We denote by $\{z_j\}_{1\leq j\leq k}$ the complex coordinates of $\mathbb{C}^k$, and by $\{f_j\}_{1\leq j\leq n}$ the coordinates of

f in $\mathbb{C}^n$. For simplicity, we denote by $\mathrm{Log}\, f$ the composition $\mathrm{Log} \circ f$.

Let $\mathscr{A}_f$ be the amoeba of V (i.e. $\mathscr{A}_f = \mathrm{Log}\,(V)$). Let S be the set of points in $\mathbb{C}^k$ defined by

$$S = \{z \in \mathbb{C}^k \mid \text{rank } \mathrm{d}_z \mathrm{Log}\, f < 2k\}$$

and $\mathscr{S}_f = \mathrm{Log}\, f(S)$ is the set of critical values of $\mathrm{Log}\, f$.

By construction, $\mathrm{Log}\, f$ is an immersion from $\mathbb{C}^k \setminus S$ to $\mathbb{R}^n$. Hence, the set $\mathscr{A}_f \setminus \mathscr{S}_f = \mathrm{Log}\, f(\mathbb{C}^k \setminus S)$ is a $2k-$real dimensional immersed submanifold in $\mathbb{R}^n$. We endow $\mathscr{A}_f \setminus \mathscr{S}_f$ with the induced Riemannian metric $\imath^* \mathcal{E}_n$, where $\mathcal{E}_n$ is the Euclidean metric of $\mathbb{R}^n$ and $\imath : \mathscr{A}_f \setminus \mathscr{S}_f \hookrightarrow \mathbb{R}^n$ is the inclusion map. Let $U_f \subset \mathbb{C}^k \setminus S$ be an open set such that $\mathrm{Log}\, f|_{U_f}$ is an injective immersion and $\mathrm{Log}\, f(U_f) = \mathscr{A}_f \setminus \mathscr{S}_f$. We claim that

$$vol(\mathscr{A}_f \setminus \mathscr{S}_f, \imath^* \mathcal{E}_n) = vol(U_f, (\mathrm{Log}\, f)^* \mathcal{E}_n) \tag{1}$$

where $vol(\mathscr{A}_f \setminus \mathscr{S}_f, \imath^* \mathcal{E}_n)$ is the volume of $\mathscr{A}_f \setminus \mathscr{S}_f$ with respect to the metric $\imath^* \mathcal{E}_n$. Let ψ_{2k} be a reel $2k-$vector field in $\Lambda^{2k}\mathbb{C}^k$ which doesn't vanish on $\mathbb{C}^k$ and $\mathrm{d}v_{(\mathrm{Log}\, f)^* \mathcal{E}_n}$, $\mathrm{d}v_{\mathcal{E}_{2k}}$ be the volume forms defined over U_f associated to the metrics $(\mathrm{Log}\, f)^* \mathcal{E}_n$ and $\mathcal{E}_{2k}$ respectively. These two forms are related by the following formula:

$$|\psi_{2k}|_{\mathcal{E}_{2k}} \mathrm{d}v_{(\mathrm{Log}\, f)^* \mathcal{E}_n} = |\psi_{2k}|_{(\mathrm{Log}\, f)^* \mathcal{E}_n} \mathrm{d}v_{\mathcal{E}_{2k}} \tag{2}$$

Now we choose ψ_{2k} such that $\mathrm{d}v_{\mathcal{E}_{2k}}(\psi_{2k}) = |\psi_{2k}|_{\mathcal{E}_{2k}} = 1$. From (1) and (2) we deduce that

$$vol(\mathscr{A}_f \setminus \mathscr{S}_f) = \int_{U_f} |\psi_{2k}|_{(\mathrm{Log}\, f)^* \mathcal{E}_n} \mathrm{d}v_{\mathcal{E}_{2k}}, \tag{3}$$

and $area := vol$ if $k = 1$. This definition of the volume doesn't depend on the choice of coordinates. It is more convenient to use the following integral vol_{2k} defined as follows:

$$vol_{2k}(\mathscr{A}_f \setminus \mathscr{S}_f) = \int_{\mathbb{C}^k - S} |\psi_{2k}|_{(\mathrm{Log}\, f)^* \mathcal{E}_n} \mathrm{d}v_{\mathcal{E}_{2k}}. \tag{4}$$

Remark 3.1. *$vol_{2k}(\mathscr{A}_f \setminus \mathscr{S}_f)$ can be viewed as the weighted volume of $\mathscr{A}_f \setminus \mathscr{S}_f$. In fact, we know that there exist a positive integer m and a family of open connected components on $\mathscr{A}_f \setminus \mathscr{S}_f$, denoted by $\{R_\alpha\}$, such that*

$$\mathscr{A}_f \setminus \mathscr{S}_f = \bigsqcup_{\alpha=1}^{m} R_\alpha, \quad (\mathrm{Log}\, f)^{-1} R_\alpha = \bigsqcup_{\beta=1}^{p_\alpha} U_{\alpha\beta} \quad \textit{and} \quad \bigsqcup_{\alpha=1}^{m} \bigsqcup_{\beta=1}^{p_\alpha} U_{\alpha\beta} = \mathbb{C}^k \setminus S,$$

for all $\alpha \leq m$ and $1 \leq \beta \leq p_\alpha$ the map

$$\mathrm{Log}\, f : U_{\alpha\beta} \longrightarrow R_\alpha$$

is a diffeomorphism.

For a chosen β, we set $U_f = \bigsqcup_{\alpha=1}^m U_{\alpha\beta}$. It yields

$$vol_{2k}(\mathscr{A}_f \setminus \mathscr{S}_f) = \sum_{\alpha=1}^{m} p_\alpha \int_{U_{\alpha\beta}} \left|\psi_{2k}\right|_{(\mathrm{Log}\, f)^*\mathcal{E}_n} dv_{\mathcal{E}_{2k}}, \tag{5}$$

$$vol(\mathscr{A}_f \setminus \mathscr{S}_f) = \sum_{\alpha=1}^{m} \int_{U_{\alpha\beta}} \left|\psi_{2k}\right|_{(\mathrm{Log}\, f)^*\mathcal{E}_n} dv_{\mathcal{E}_{2k}}. \tag{6}$$

If we define $p = \min_{1\leq\alpha\leq m} p_\alpha$ and $P = \max_{1\leq\alpha\leq m} p_\alpha$, then

$$\frac{vol_{2k}(\mathscr{A}_f \setminus \mathscr{S}_f)}{P} \leq vol(\mathscr{A}_f \setminus \mathscr{S}_f) \leq \frac{vol_{2k}(\mathscr{A}_f \setminus \mathscr{S}_f)}{p}. \tag{7}$$

On $\mathbb{C}^k$, $\mathrm{d}v_{\mathcal{E}_{2k}}$ and ψ_{2k} are given by

$$\mathrm{d}v_{\mathcal{E}_{2k}} = i^k \mathrm{d}z \wedge \mathrm{d}\bar{z}, \qquad \psi_{2k} = (-i)^k \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \bar{z}}$$

where $\mathrm{d}z = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_k$ and $\frac{\partial}{\partial z} = \frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_k}$
Let us compute $\left|\psi_{2k}\right|_{(\mathrm{Log}\, f)^*\mathcal{E}_n}$:
We have

$$\frac{\partial \mathrm{Log}\, f}{\partial z} \wedge \frac{\partial \mathrm{Log}\, f}{\partial \bar{z}} = \sum_{I=\{i_1<\cdots<i_{2k}\}} \det(\partial_j \mathrm{Log}\, f_I)_{1\leq j\leq 2k} e_I.$$

Where for all $I = \{i_1 < \cdots < i_{2k}\} \subset \{1, \cdots, n\}$, $f_I = (f_{i_1}, \cdots, f_{i_{2k}})$ and $e_I := e_{i_1} \wedge \cdots \wedge e_{i_{2k}}$. We denote by $\partial_j = \frac{\partial}{\partial z_j}$ if $j \leq k$ and by $\partial_j = \frac{\partial}{\partial \bar{z}_{j-k}}$ if $j \geq k+1$. $\{e_j\}_{1\leq j\leq n}$ is an orthonormal basis of $T^*\mathbb{R}^n$. It implies that $\{e_I\}_{I\subset\{1,\cdots,n\},|I|=2k}$ is an orthonormal basis of $\wedge^{2k} T^*\mathbb{R}^n$ with respect to the Euclidean metric.

$$\left|\psi_{2k}\right|^2_{(\mathrm{Log}\, f)^*\mathcal{E}_n} = \sum_{I=\{i_1<\cdots<i_{2k}\}} \left|\det(\partial_j \mathrm{Log}\, f_I)_{1\leq j\leq 2k}\right|^2. \tag{8}$$

Hence, we deduce the following inequality:

$$\left|\psi_{2k}\right|_{(\mathrm{Log}\, f)^*\mathcal{E}_n} \leq \sum_{I=\{i_1<\cdots<i_{2k}\}} \left|\det(\partial_j \mathrm{Log}\, f_I)_{1\leq j\leq 2k}\right|. \tag{9}$$

We have the following result:

Theorem 3.2. *If $k = 1$, the area of $\mathscr{A}_f$ with respect to the Euclidean metric of $\mathbb{R}^n$ is finite.*

Lemma 3.1. *Let $f = (f_1, f_2)$ be a rational map from $\mathbb{C}$ to $\mathbb{C}^2$. The function $i \det(\partial_z \mathrm{Log}\, f, \partial_{\bar{z}} \mathrm{Log}\, f)$ is a real valued rational function. Moreover,*

- (i) *It has simple poles.*
- (ii) *There exist $P, Q \in \mathbb{R}[X, Y]$ such that $i \det(\partial_z \mathrm{Log}\, f, \partial_{\bar{z}} \mathrm{Log}\, f) = \frac{P}{Q}$ with* $\deg Q \geq \deg P + 3$.

Proof. We have

$$i \det(\partial_z \mathrm{Log}\, f, \partial_{\bar{z}} \mathrm{Log}\, f) = \frac{i}{4}\left(\frac{f_1'}{f_1}\frac{\bar{f}_2'}{\bar{f}_2} - \frac{\bar{f}_1'}{\bar{f}_1}\frac{f_2'}{f_2}\right).$$

It is trivial that this function, which is the Jacobian of $\mathrm{Log}\, f$ times i, is a real rational function. Its poles are zeros and poles of f and their order is equal to one (even if there is a common pole or zero between f_1 and f_2, one can check that this pole is also simple). On the other hand, if $i \det(\partial_z \mathrm{Log}\, f, \partial_{\bar{z}} \mathrm{Log}\, f) = \frac{P}{Q}$ then deg$Q \geq$ deg$P + 2$. We can improve it. In fact, by elementary computations on the deg$P$ and deg$Q$, we show that $\deg Q \geq \deg P + 3$.

□

Proof of Theorem 3.2. First of all, we don't have to worry about the area of $\mathscr{S}_f$. Indeed, by Sard's theorem, we know that this area is equal to zero. If $\mathscr{A}_f \setminus \mathscr{S}_f$ is empty, then the area of $\mathscr{A}_f$ is zero. From now, we assume that $\mathscr{A}_f \setminus \mathscr{S}_f$ is not empty. Hence, it is a surface defined on $\mathbb{R}^n$, and $\mathscr{A}_f \setminus \mathscr{S}_f$, $\mathscr{A}_f$ have the same area. The area of $\mathscr{A}_f \setminus \mathscr{S}_f$ is given by (3). Hence, it is sufficient to prove that $|\psi_2|$ is integrable over $\mathbb{C}$. Inequality (9) implies

$$|\psi_2|_{(\mathrm{Log}\, f)^*\mathcal{E}_n} \leq \sum_{1\leq j<k\leq n} |\det(\partial_z \mathrm{Log}\, f_j, \partial_{\bar{z}} \mathrm{Log}\, f_k)|. \tag{10}$$

We claim that all the functions in the right hand side of (10) are integrable. Indeed, using Lemma 3.1, (i), we have the integrability in a neighborhood of any pole. By (ii), we get the integrability at infinity. □

Proof of Theorem 3.1. Let $\mathcal{C}$ be an algebraic curve in $(\mathbb{C}^*)^n$, then its closure $\overline{\mathcal{C}}$ in $\mathbb{CP}^n$ is an algebraic curve. Hence, any end of $\mathcal{C}$ corresponds to a local branch of $\overline{\mathcal{C}}$ at some point $p \in \partial\overline{\mathcal{C}} \setminus \mathcal{C}$. After a monomial map of $(\mathbb{C}^*)^n$ if necessary, we can assume that p corresponding to an end of $\mathcal{C}$ is the origin of $\mathbb{C}^n$ in $\mathbb{CP}^n$.
A local parametrization ρ_p of a branch of $\overline{\mathcal{C}}$ at p can be written in terms of vectorial Puiseux series in t near zero as follows:

$$\begin{array}{rccl} \rho_p & : & \mathbb{C}^* & \longrightarrow & (\mathbb{C}^*)^n \\ & & t & \longmapsto & (b_1 t^{u_1}, \ldots, b_n t^{u_n}), \end{array}$$

where $(b_1, \ldots, b_n) \in (\mathbb{C}^*)^n$, and $(u_1, \ldots, u_n) \in \mathbb{Q}^n_{\geq 0}$. Indeed, the Bergman logarithmic limit set of a curve is a finite number of points $\{v_i\}$ in the sphere S^{n-1} (see [B-71]). By Bieri and Groves (see [BG-84]), if O denotes the origin of $\mathbb{R}^n$, then the slope $\vec{u}$ of the real line (Ov_i) in $\mathbb{R}^n$ is rational (the slope here means the direction vector of the line, and rational means that its coordinates are rational). Hence, there exist real lines $L_{\vec{u},j}$ in $\mathbb{R}^n$ parametrized by:

$$x \longmapsto (a_1^j + xu_1, \ldots, a_n^j + xu_n)$$

with $x \in \mathbb{R}$ and $(a_1^j, \ldots, a_n^j) \in \mathbb{R}^n$, such that the amoebas $\mathscr{A}_h$ of h in the defining ideal of the curve $\mathcal{C}$ reaches all these lines at the infinity in the direction $\vec{u}$. So, for each line $L_{\vec{u},j}$ there exists $(b_1, \ldots, b_n) \in (\mathbb{C}^*)^n$ such that the Hausdorff distance between the complex line $\mathcal{L}_{\vec{u},j}$ parametrized by:

$$t \longmapsto (b_1 t^{u_1}, \ldots, b_n t^{u_n}),$$

and $V_h \cap \mathrm{Log}^{-1}(L_{\vec{u},j} \setminus B(O,R))^+$ tends to zero when R goes to infinity, where $(L_{\vec{u},j} \setminus B(O,R))^+$ denotes the component which is in the direction of $\vec{u}$. Hence, the Hausdorff distance between the intersection of the curve with $\mathrm{Log}^{-1}(L_{\vec{u},j} \setminus B(O,R))^+$ and $\mathcal{L}_{\vec{u},j}$ tends to zero when R is sufficiently large. Now, using Theorem 3.2 and the fact that the number of ends of an algebraic curve is finite, we obtain the result. □

4. An estimate for the area of rational curve amoebas

In this section, we assume that $k = 1$. Recall that f is the rational map defined in Section 3 with $k = 1$. Hence, for any integer $j \in [1, n]$:

$$f_j(z) = c \prod_{\ell=1}^{d_j} (z - a_{j\ell})^{m_{j\ell}}$$

where $a_{j\ell}$ are distinct poles or zeros of f_j, $m_{j\ell}$ are their multiplicities ($m_{j\ell}$ are negative in the case of poles) and d_j is the number of distinct zeros and poles of f_j. We define the positive integers $n_j := \sum_{\ell=1}^{d_j} |m_{j\ell}|$. These integers represent the number of poles and zeros of f_j counted with their multiplicities.

Theorem 4.1. *Let p be the positive integer defined by*

$$p = \min_{x \in \mathscr{A}_f \setminus \mathscr{S}_f} \#(\operatorname{Log} f)^{-1} x.$$

The following inequalities always hold

$$p \cdot area(\mathscr{A}_f) \leq vol_2(\mathscr{A}_f) \leq \pi^2 \sum_{1 \leq j_1 < j_2 \leq n} n_{j_1} n_{j_2} \tag{11}$$

Moreover, $p \cdot area(\mathscr{A}_f) = vol_2(\mathscr{A}_f)$ if and only if $\operatorname{Log} f : \mathbb{C}^k \setminus S \longrightarrow \mathscr{A}_f \setminus \mathscr{S}_f$ is a covering with exactly p-sheets.

Proof. Recall that vol_2 is defined by (4).
If $n = 2$, using (4), (8), we obtain

$$vol_2(\mathscr{A}_f) \leq \sum_{\ell=1, \ell'=1}^{d_1, d_2} |m_{1\ell} m_{2\ell'}| vol_2(\mathscr{A}_{f_{a_{1\ell}, a_{2\ell'}}}),$$

with $f_{a_{1\ell}, a_{2\ell'}}(z) = (z - a_{1\ell}, z - a_{2\ell})$. We know that $vol_2(\mathscr{A}_{f_{a_{1\ell}, a_{2\ell'}}}) = \pi^2$ (we can prove it using the substitution $z = (a_{1\ell} - a_{2\ell})t + a_{1\ell}$ and Example 1. below for $m = 1$ which is a plane line). Hence,

$$vol_2(\mathscr{A}_f) \leq \pi^2 \sum_{\ell=1, \ell'=1}^{d_1, d_2} |m_{1\ell} m_{2\ell'}| \tag{12}$$

If $n \geq 2$, using (9) and integrating, we obtain

$$vol_2(\mathscr{A}_f) \leq \sum_{1 \leq j_1 < j_2 \leq n} vol_2(\mathscr{A}_{(f_{j_1}, f_{j_2})}) \tag{13}$$

A combination of (12) and (13) yields

$$vol_2(\mathscr{A}_f) \leq \pi^2 \sum_{1 \leq j_1 < j_2 \leq n} \sum_{\ell=1, \ell'=1}^{d_{j_1}, d_{j_2}} |m_{j_1 \ell} m_{j_2 \ell'}| \tag{14}$$

Hence

$$vol_2(\mathscr{A}_f) \leq \pi^2 \sum_{1 \leq j_1 < j_2 \leq n} n_{j_1} n_{j_2} \tag{15}$$

which gives the second inequality of Theorem 4.1. The first one is a consequence of (7).
Assume that $p \cdot area(\mathscr{A}_f) = vol_2(\mathscr{A}_f)$. Using (5), (6), we obtain that for all $\alpha \leq m$, $p_\alpha = p$. This means that the number of connected components in $(\operatorname{Log} f)^{-1} R_\alpha$ doesn't depend on α and $\operatorname{Log} f$ is $p-$sheet covering map.
If we suppose that $\operatorname{Log} f : \mathbb{C}^k \setminus S \longrightarrow \mathscr{A}_f \setminus \mathscr{S}_f$ is a covering with exactly p-sheets then (7) becomes equality. $\square$

Examples.

(1) Let us compute the area of the amoeba $\mathscr{A}_f$ with $f(z) = (z, z^m - 1)$ and $m \in \mathbb{Z}^*$. The set of singular points S is the union of the half lines given by:

$$S = \bigcup_{j=1}^{2|m|} \{z \in \mathbb{C} \mid \arg z = \frac{j\pi}{|m|}\}$$

The set of critical values is $\mathscr{S}_f = \operatorname{Log} f(S)$ which is a curve in $\mathbb{R}^2$ and bounds $\mathscr{A}_f$. The map $\operatorname{Log} f$ is a covering map with exactly $2m$ sheets and for any integer $j \in [1, 2|m|]$ we have:

$$\operatorname{Log} f : U_j = \{z \in \mathbb{C}^* \mid \frac{(j-1)\pi}{|m|} < \arg z < \frac{j\pi}{|m|}\} \longrightarrow \mathscr{A}_f - \mathscr{S}_f$$

is a diffeomorphism. Hence, $\operatorname{Log} f$ is a $2|m|-$sheets covering map. By (3) and (8) we obtain

$$area(\mathscr{A}_f) = \int_{U_1} \left| \det(\frac{\partial \operatorname{Log} f}{\partial z}, \frac{\partial \operatorname{Log} f}{\partial \bar{z}}) \right| i \mathrm{d}z \wedge \mathrm{d}\bar{z} = \int_{U_1} \frac{|m||z^m - \bar{z}^m|}{4|z|^2|z^m - 1|^2} i \mathrm{d}z \wedge \mathrm{d}\bar{z} = \frac{\pi^2}{2|m|},$$

with $area = vol$. We deduce that $vol_2(\mathscr{A}_f) = 2|m| area(\mathscr{A}_f) = \pi^2$.

(2) Now, we consider the real line in $(\mathbb{C}^*)^3$ parametrized by $g(z) = (z, z + \frac{1}{2}, z - \frac{3}{2})$. The amoeba $\mathscr{A}_g$ is a surface in $\mathbb{R}^3$ with boundary as we can see in Figure 1 (this fact is proven in [JNP-10]). The set of singular points is the line of real points, and $\operatorname{Log} g$ is a 2-sheets covering map. It is complicated to compute the area of $\mathscr{A}_g$. However, using the estimate given in Theorem 4.1, we deduce that $2\, area(\mathscr{A}_g) = vol_2(\mathscr{A}_g) \leq 3\pi^2$.

(3) Let $h(z) = (z, z + 1, z - 2i)$ be the parametrization of a complex line in $(\mathbb{C}^*)^3$. The amoeba $\mathscr{A}_h$ is a surface in $\mathbb{R}^3$ without boundary as we can see in the Figure 2, and topologically it is a Riemann sphere with four marked points. Notice that this line is not real and the set of critical values of Log restricted to this line is empty. The map $\operatorname{Log} h : \mathbb{C} - \{-1, 0, 2i\} \longrightarrow \mathscr{A}_h$ is a diffeomorphism. By Theorem 4.1, $area(\mathscr{A}_h) = vol_2(\mathscr{A}_h) \leq 3\pi^2$.

Passare and Rullgård give an estimate for amoeba areas of complex algebraic plane curves (see Theorem 2.1). Our estimate works only for rational curves immersed in $(\mathbb{C}^*)^n$. However, in this case we have a fine estimate (see Theorem 4.1). Indeed, if we consider Example 1, $area(\mathscr{A}_f) = \frac{\pi^2}{2m}$, $n_1 = 1$, $n_2 = m$, $p = 2m$ and the area of Newton's polygon is $\frac{m}{2}$. Inequality (11) gives $area(\mathscr{A}_f) \leq \frac{\pi^2}{2}$, and Passare-Rullgård estimate gives $area(\mathscr{A}_f) \leq \frac{m\pi^2}{2}$.

5. Volume of a generic complex algebraic variety amoebas

In this section, we assume that $n = 2k + m$ is an integer with $k \geq 1$, and $m \geq 0$. Let $V \subset (\mathbb{C}^*)^n$ be an algebraic variety of dimension k with defining ideal $\mathcal{I}(V)$, and $\mathscr{L}^\infty(V)$ be its logarithmic limit set. We denote by $\mathrm{V}ert(\mathscr{L}^\infty(V))$ the set of vertices of $\mathscr{L}^\infty(V)$. Let $V \subset (\mathbb{C}^*)^n$ be a generic algebraic variety of dimension k. Let $v \in \mathrm{V}ert(\mathscr{L}^\infty(V))$, and let us denote by $\mathscr{H}_{R,v}$ the hyperplane in $\mathbb{R}^n$ with normal the vector $\vec{Ov}$ such that $d(O, \mathscr{H}_{R,v}) = R$, where O is the origin of $\mathbb{R}^n$ and $R \in \mathbb{R}_+$ is sufficiently large. We denote by $\mathscr{H}_{R,v}^-$ the half space with boundary $\mathscr{H}_{R,v}$ containing the origin.
In this section, we prove Theorem 1.1, using the following proposition:

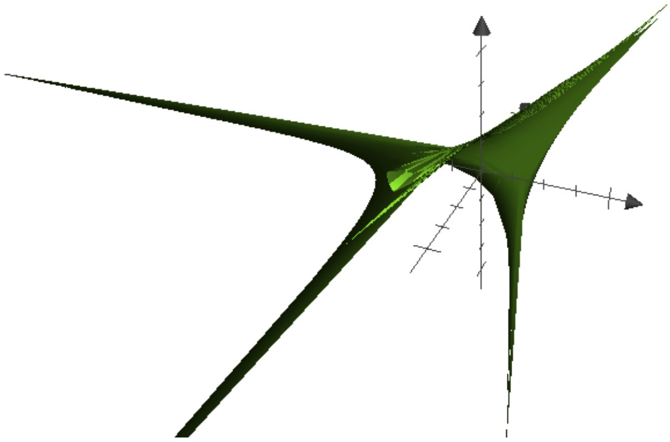

FIGURE 1. The amoeba of the real line in $(\mathbb{C}^*)^3$ given by the parametrization $g(z) = (z, z + \frac{1}{2}, z - \frac{3}{2})$.

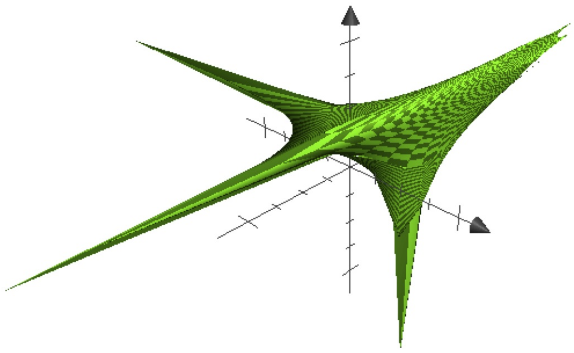

FIGURE 2. The complex line in $(\mathbb{C}^*)^3$ given by the parametrization $h(z) = (z, z + 1, z - 2i)$.

Proposition 5.1. *With the above notations $V \setminus \mathrm{Log}^{-1}(\mathscr{H}_{R,v}^-)$ is a fibration over an algebraic variety V_v of dimension $k-1$ contained in $(\mathbb{C}^*)^{n-1}$, and its fibers are the ends of algebraic curves in $(\mathbb{C}^*)^n$. Moreover, these ends have a rational parametrization with the same slope (i.e., their image under* Log *are lines with the same slope).*

We start by proving the following lemma:

Lemma 5.1. *Let $V \subset (\mathbb{C}^*)^n$ be an algebraic variety of dimension k. Then, for each vertex v of its logarithmic limit set $\mathscr{L}^\infty(V)$ we have the following: there exists a complex algebraic variety $V_v \subset (\mathbb{C}^*)^{n-1}$ of dimension $k-1$ such that the boundary of the Zariski closure $\overline{V}$ of V in $(\mathbb{C}^*)^{n-1} \times \mathbb{C}$ is equal to V_v (i.e., $\partial\overline{V} = \overline{V} \setminus V = V_v$, where $(\mathbb{C}^*)^{n-1} = (\mathbb{C}^*)^{n-1} \times \{0\} \subset (\mathbb{C}^*)^{n-1} \times \mathbb{C}$).*

Proof. If v belongs to $\mathrm{V}ert(\mathscr{L}^\infty(V))$, then after a monomial map defined by a matrix $A_v \in GL_n(\mathbb{Z})$ if necessary, we can assume that $v = (0, \dots, 0, -1) \in \mathbb{S}^{n-1}$. Let $\mathbb{C}[z_1^{\pm 1}, \dots, z_{n-1}^{\pm 1}, z_n] \subset \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ be the inclusion of rings, and $\phi : \mathbb{C}[z_1^{\pm 1}, \dots, z_{n-1}^{\pm 1}, z_n] \to \mathbb{C}[z_1^{\pm 1}, \dots, z_{n-1}^{\pm 1}]$ be the homomorphism which sends z_n to zero. Let $\mathscr{J} = \mathcal{I}(V) \cap \mathbb{C}[z_1^{\pm 1}, \dots, z_{n-1}^{\pm 1}, z_n]$, and $\mathcal{I}_v$ be its image in $\mathbb{C}[z_1^{\pm 1}, \dots, z_{n-1}^{\pm 1}]$. We denote by $V_{(n-1)}$ the subvariety of $(\mathbb{C}^*)^{n-1} \times \mathbb{C}$ defined by $\mathscr{J}$, and $V_v = V_{(n-1)} \cap (\mathbb{C}^*)^{n-1} \times \{0\}$ be the subvariety defined by $\mathcal{I}_v$. We check that $V_{(n-1)} = \overline{V}$ where $\overline{V}$ denotes the Zariski closure of V in $(\mathbb{C}^*)^{n-1} \times \mathbb{C}$, and V_v is the boundary of $\overline{V}$ i.e., $\partial\overline{V} = \overline{V} \setminus V = V_v$. □

Proof of Proposition 5.1. For each point x in V_v there exists an algebraic curve $\mathcal{C}_x$ in V such that its closure in $(\mathbb{C}^*)^{n-1} \times \mathbb{C}$ contains x and its logarithmic limit set contains the ponit v. Indeed, we have the following commutative diagram:

$$\begin{tikzcd}
(\mathbb{C}^*)^{n-1} \times \mathbb{C} \arrow[r, "\pi^{\mathbb{C}}_{(n-1)}"] \arrow[d, "\mathrm{Log}_{|(\mathbb{C}^*)^n}"'] & (\mathbb{C}^*)^{n-1} \arrow[d, "\mathrm{Log}"] \\
\mathbb{R}^n \arrow[r, "\pi^{\mathbb{R}}_{(n-1)}"] & \mathbb{R}^{n-1},
\end{tikzcd}$$

where $\pi^{\mathbb{C}}_{(n-1)}$ and $\pi^{\mathbb{R}}_{(n-1)}$ are the projections on $(\mathbb{C}^*)^{n-1}$ and $\mathbb{R}^{n-1}$ respectively. The limit of $\pi^{\mathbb{C}}_{(n-1)}(V \setminus \mathscr{H}^-_{v,R})$ when R goes to the infinity is equal to V_v(with respect to the Hausdorff metric on compact sets). Furthermore, the limit of $\pi^{\mathbb{R}}_{(n-1)}(\mathrm{Log}\,(V \setminus \mathscr{H}^-_{v,R}))$ when R goes to the infinity is equal to the amoeba of V_v. Hence, the limit of $\pi^{\mathbb{C}}_{(n-1)}(\mathcal{C}_x \setminus \mathscr{H}^-_{v,R})$ when R goes to the infinity contains the point x. The end of $\mathcal{C}_x$ corresponding to v and containing x is parametrized as follows:

$$\begin{array}{rccl}
\rho_v & : & \mathbb{C}^* & \longrightarrow & (\mathbb{C}^*)^n \\
 & & t & \longmapsto & (b_{x,1}t^{u_1}, \dots, b_{x,n}t^{u_n}),
\end{array}$$

where the coefficients $b_{x,j}$ depend on the holomorphic branch of $\overline{\mathcal{C}_x}$ at x, and the powers u_j depend only on v. Moreover, for any $x_1 \neq x_2$ in V_v the end of the curve $\mathcal{C}_{x_1}$ corresponding to x_1, and the end of the curve $\mathcal{C}_{x_2}$ corresponding to x_2 have an empty intersection. May be the curves $\mathcal{C}_{x_1}$ and $\mathcal{C}_{x_2}$ are the same. In fact, in this case this means that the curve has more than one end corresponding to v. In other words, if $\overline{\mathcal{C}}_{x_1}$ is the Zariski closure of $\mathcal{C}_{x_1}$ in $\mathbb{CP}^n$, then it has more than one holomorphic branch at v. Indeed, if the intersection of these ends is not empty, then the fact that they are holomorphic and with the same slope, they should be equal; it is a contradiction with the assumption on x_1 and x_2. Hence, for R sufficiently large $V \setminus \mathscr{H}^-_{v,R}$ is a fibration overs V_v. □

Proof of Theorem 1.1. Recall that the volume is always computed with the respect to the induced measure of the ambient space. Using induction on the dimension of the variety k, Proposition 5.1, and Theorem 4.1, there exists a rational number q_v depending only on v and the variety V_v such that the following inequality:

$$vol(\mathrm{Log}\,(V \setminus \mathscr{H}^-_{v,R})) \leq \pi^2 q_v vol(\mathscr{A}(V_v))$$

holds. There are a finite number of vertices of the logarithmic limit set, so there exists a positive real number K such that we have the following inequality:

$$vol(\mathscr{A}(V)) \leq K + \sum_{v\in Vert(\mathscr{L}^\infty(V))} \pi^2 q_v vol(\mathscr{A}(V_v))$$

$\square$

Remark 5.1. *In the forthcoming paper, we give an explicit bound of the volume of the amoebas of some special class of algebraic varieties of dimension k in the complex torus $(\mathbb{C}^*)^{2k}$, and the relation of its sharpness with real algebraic varieties. Moreover, we study the coamoebas, the image of the varieties under the argument map, of such class and their volume counted with multiplicity.*

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NWF I-Mathematik, Universität Regensburg, 93040 Regensburg, Germany.
E-mail address: `Farid.Madani@mathematik.uni-regensburg.de`

Department of Mathematics, Texas A & M University, College Station, TX 77843-3368, USA.
E-mail address: `nisse@math.tamu.edu`