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Abstract. The linearized stability of stationary solutions for surface diffusion is studied. We consider three hypersurfaces that lie inside a fixed domain and touch its boundary with a right angle and fulfill a non-flux condition. Additionally they meet at a triple line with prescribed angle conditions and further boundary conditions resulting from the continuity of chemical potentials and a flux balance have to hold at the triple line. We introduce a new specific parametrization with two parameters corresponding to a movement in tangential and normal direction to formulate the geometric evolution law as a system of partial differential equations. For the linearized stability analysis we identify the problem as an H^{-1}-gradient flow, which will be crucial to show self-adjointness of the linearized operator. Finally we study the linearized stability of some examples.

Keywords. surface diffusion, partial differential equations on manifolds, linearized stability, gradient flow, triple lines.

AMS subject classification. 35G30, 35R35, 35B35, 35K55, 53C44.

1 Introduction

We consider three evolving hypersurfaces that meet the boundary of a fixed bounded region Ω at a right angle and also meet each other at a triple line. They evolve due to weighted surface diffusion flow

$$V_i = -m_i\,\gamma_i\,\Delta H_i\,, \tag{1.1}$$

each for $i = 1, 2, 3$. Here V_i is the normal velocity of the evolving hypersurface Γ_i, H_i is the mean curvature and Δ is the Laplace-Beltrami operator. Our sign convention is that H is negative for spheres provided with outer unit normal. Further the constants $\gamma_i, m_i > 0$ are the surface energy density and the mobility of the evolving hypersurface Γ_i. If the three evolving hypersurfaces meet at a triple line $L(t)$, we require that there the following conditions hold.

$$\angle(\Gamma_1(t), \Gamma_2(t)) = \theta_3\,,\ \angle(\Gamma_2(t), \Gamma_3(t)) = \theta_1\,,\ \angle(\Gamma_3(t), \Gamma_1(t)) = \theta_2\,, \tag{1.2}$$

$$\gamma_1\,H_1 + \gamma_2\,H_2 + \gamma_3\,H_3 = 0\,, \tag{1.3}$$

$$m_1\,\gamma_1\,\nabla H_1 \cdot n_{\partial\Gamma_1} = m_2\,\gamma_2\,\nabla H_2 \cdot n_{\partial\Gamma_2} = m_3\,\gamma_3\,\nabla H_3 \cdot n_{\partial\Gamma_3}\,, \tag{1.4}$$

where the quantity $\angle(\Gamma_i(t), \Gamma_j(t))$ denotes the angle between $\Gamma_i(t)$ and $\Gamma_j(t)$ and the angles $\theta_1, \theta_2, \theta_3$ with $0 < \theta_i < \pi$ are related through the identity $\theta_1 + \theta_2 + \theta_3 = 2\pi$ and Young's law, which is

$$\frac{\sin\theta_1}{\gamma_1} = \frac{\sin\theta_2}{\gamma_2} = \frac{\sin\theta_3}{\gamma_3}\,. \tag{1.5}$$

One can show that Young's law (1.5) is equivalent to

$$\gamma_1\,n_{\partial\Gamma_1} + \gamma_2\,n_{\partial\Gamma_2} + \gamma_3\,n_{\partial\Gamma_3} = 0\,, \tag{1.6}$$

which is the force balance at the triple line.

At the fixed outer boundary $\Gamma_i(t) \cap \partial\Omega$ we assume a $90°$ angle condition and a no-flux condition resulting in

$$\Gamma_i(t) \perp \partial\Omega\,, \tag{1.7}$$
$$\nabla H_i \cdot n_{\partial\Gamma_i} = 0\,. \tag{1.8}$$

Here ∇ is the surface gradient and $n_{\partial\Gamma_i}$ is the outer unit conormal of Γ_i at boundary points.

For the derivation of the boundary conditions (1.2)-(1.4) at the triple line and (1.7)-(1.8) as the asymptotic limit of a Cahn-Hilliard system with degenerate mobility, we refer to Garcke and Novick-Cohen [GN00]. The angle conditions (1.2) follow from the balance of forces (1.6) at the triple line, the second condition (1.3) follows from the continuity of chemical potentials and the conditions (1.4) are the flux balance at the triple line $L(t)$.

Smooth solutions Γ_i of (1.1) with boundary conditions (1.2)-(1.4) at the triple line and (1.7)-(1.8) at the outer boundary the properties area-minimizing and volume-preserving in the sense that

$$\frac{d}{dt}A(t) \leq 0 \quad \text{and} \quad \frac{d}{dt}Vol_{ij}(t) = 0\,,$$

where $A(t) = \sum_{i=1}^{3} \gamma_i \int_{\Gamma_i(t)} 1\, \mathrm{d}\mathcal{H}^n$ is the sum of the weighted surface areas and $Vol_{ij}(t)$ denotes the volume of the region enclosed by $\Gamma_i(t)$, $\Gamma_j(t)$ and $\partial\Omega$. The details for this calculation can be found for example in the work of the first author [Dep10].

In the following situations there are some results on stability for surface diffusion. Let three plane curves lie in the fixed region Ω, where $\partial\Omega$ is a rectangle, and evolve due to the weighted surface diffusion flow (1.1) such that the outer boundary conditions (1.7) and (1.8) are fulfilled for each curve. The three plane curves shall also have a triple junction where the conditions (1.2)-(1.4) are fulfilled. In this case Ito and Kohsaka [IK01a] and also Escher, Garcke and Ito [EGI03] showed global existence results when the initial curve is a small perturbation of a certain stationary curve. The same is true if $\partial\Omega$ is a triangle and was shown in [IK01b] from Ito and Kohsaka. In these cases also nonlinear stability of the stationary curve can be shown. The above described planar situation was also considered without a special geometry of Ω in the work of Garcke, Ito and Kohsaka [GIK10], where the authors formulate a linearized stability criterion for stationary curves. Related results for mean curvature flow can be found in the works of Ei, Sato and Yanagida [ESY96] and Garcke, Kohsaka and Ševčovič [GKS09].

This work is the continuation of [Dep11] from Depner, where the case of one hypersurface lying inside a fixed region was considered. We will introduce a linear stability criterion based on the work of Garcke, Ito and Kohsaka [GIK10] for curves in the plane and extend it to the case of hypersurfaces. At the beginning it is very important to come up with a parametrization with good properties to rewrite the geometric evolution laws as partial differential equations for unknown functions. Therefore we use a composition of a curvilinear coordinate system by Vogel [Vog00], that was also used in [Dep11], and a more explicit parametrization near the triple line with two parameters corresponding to a movement in tangential and normal direction.

In this way we consider evolving hypersurfaces given as a graph over some fixed stationary solution. In the next step it is crucial that we can describe the linearized problem as an H^{-1}-gradient flow, because this is the main reason that the linearized operator is self-adjoint. Then we can apply results from spectral theory and relate the asymptotic stability of the zero solution of the linearized problem to the fact that the eigenvalues of the linearized operator are negative. Since we can describe the largest eigenvalue with the help of a bilinear form arising due to the gradient flow structure, we can finally give a criterion for linearized stability of the original geometric problems around stationary states. At the end of the work we discuss some examples.

2 Parametrization

In this section we give our use of parametrization to formulate partial differential equations out of the geometric evolution law (1.1)-(1.8). In detail the problem consists in finding three evolving hypersurfaces $\Gamma_i = \bigcup_{t\in[0,T)}\{t\} \times \Gamma_i(t)$, $i = 1, 2, 3$, with $\Gamma_i(t) \subset \mathbb{R}^{n+1}$ moving due to weighted surface diffusion flow, such that $\Gamma_i(t)$ lies in a fixed bounded region $\Omega \subset \mathbb{R}^{n+1}$ and the following decomposition is fulfilled. The boundary can be seperated disjointly into $\partial\Gamma_i(t) = L_i(t) \cup S_i(t)$, such that $L(t) = L_1(t) = L_2(t) = L_3(t)$ is a triple line and the other parts $S_i(t) = \partial\Gamma_i(t) \cap \partial\Omega$ represent the sections with the outer fixed boundary. Note our implicit assumption that $L(t)$ does not intersect $\partial\Omega$.

In formulas, we have to find hypersurfaces as described above which fulfill the following surface diffusion equation in $\Gamma_i(t)$

$$V_i = -m_i\gamma_i\Delta_{\Gamma_i(t)}H_i\,, \tag{2.1}$$

where the positive constants γ_i and m_i are the surface energy density and the mobility of the interface $\Gamma_i(t)$.

At the outer boundary $S_i(t)$, we require the following right angle and natural boundary conditions.

$$\left\{\begin{array}{rcl} \angle(\Gamma_i(t), \partial\Omega) & = & \frac{\pi}{2}\,, \\ \nabla_{\Gamma_i(t)}H_i \cdot n_{\partial\Gamma_i(t)} & = & 0\,. \end{array}\right. \tag{2.2}$$

At the triple line $L(t)$, we require the following conditions

$$\left\{\begin{array}{l} \angle(\Gamma_1(t), \Gamma_2(t)) = \theta_3\,,\ \ \angle(\Gamma_2(t), \Gamma_3(t)) = \theta_1\,,\ \ \angle(\Gamma_3(t), \Gamma_1(t)) = \theta_2\,, \\ \gamma_1H_1 + \gamma_2H_2 + \gamma_3H_3 = 0\,, \\ m_1\gamma_1\nabla_{\Gamma_1(t)}H_1 \cdot n_{\partial\Gamma_1(t)} = m_2\gamma_2\nabla_{\Gamma_2(t)}H_2 \cdot n_{\partial\Gamma_2(t)} = m_3\gamma_3\nabla_{\Gamma_3(t)}H_3 \cdot n_{\partial\Gamma_3(t)}\,. \end{array}\right. \tag{2.3}$$

With the help of the outer unit conormals $n_{\partial\Gamma_i(t)}$ of $\Gamma_i(t)$ at $\partial\Gamma_i(t)$ we can write the angle conditions at the triple line through the requirement that

$$n_{\partial\Gamma_1(t)} \cdot n_{\partial\Gamma_2(t)} = \cos\theta_3\,, \quad n_{\partial\Gamma_2(t)} \cdot n_{\partial\Gamma_3(t)} = \cos\theta_1\,, \quad n_{\partial\Gamma_3(t)} \cdot n_{\partial\Gamma_1(t)} = \cos\theta_2\,. \tag{2.4}$$

Due to $\theta_1 + \theta_2 + \theta_3 = 2\pi$ two of the above angle conditions already imply the third one.

An important observation is the fact that the three outer unit conormals $n_{\partial\Gamma_1(t)}$, $n_{\partial\Gamma_2(t)}$ and $n_{\partial\Gamma_3(t)}$ all lie in a two-dimensional space, namely the orthogonal complement of the tangent space of the triple line $L(t)$, i.e. $n_{\partial\Gamma_i(t)}(p) \in (T_pL(t))^\perp$. Since $L(t)$ is an $(n-1)$-dimensional submanifold of $\mathbb{R}^{n+1}$, this orthogonal complement is in fact a two-dimensional space.

We choose unit normals $n_j(t)$ of $\Gamma_j(t)$ in an appropriate direction through the requirement that the angle between $n_{\partial\Gamma_i(t)}$ and $n_j(t)$ increases by $\pi/2$ compared to the angle between $n_{\partial\Gamma_i(t)}$ and $n_{\partial\Gamma_j(t)}$, i.e. we have the following formulas

$$n_i(t) \cdot n_j(t) = \cos\theta_k\,, \tag{2.5}$$

$$n_{\partial\Gamma_i(t)} \cdot n_{\partial\Gamma_j(t)} = \cos\theta_k\,, \tag{2.6}$$

$$n_{\partial\Gamma_i(t)} \cdot n_j(t) = \cos(\theta_k + \frac{\pi}{2}) = -\sin\theta_k\,, \tag{2.7}$$

each on $L(t)$ and for $(i, j, k) = (1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$. To be precise we require formula (2.7) at a fixed point of $L(t)$, extend the normals by continuity to all of $\Gamma_j(t)$ and observe the validity of (2.7) on all of $L(t)$ again by continuity. See Figure 1 for a sketch in the two-dimensional situation for curves near the triple line.

With this choice of normals the force balance (1.6) can also be written as

$$\gamma_1n_1(t) + \gamma_2n_2(t) + \gamma_3n_3(t) = 0 \qquad \text{on } \ L(t)\,. \tag{2.8}$$

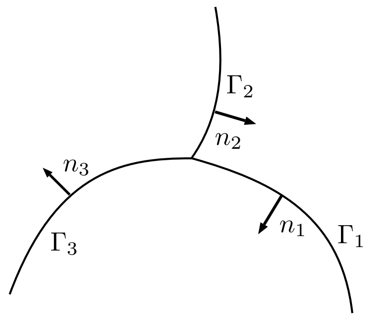

Figure 1: The choice of the normals.

We want to describe the considered hypersurfaces as graphs over some stationary solutions Γ^* of (2.1)-(2.3). This means we consider three hypersurfaces Γ_i^*, which lie in Ω, and the boundary has a decomposition $\partial\Gamma_i^* = L_i^* \cup S_i^*$, such that the three hypersurfaces meet at a triple line $L^* = L_1^* = L_2^* = L_3^*$ and the other parts are intersections with the outer fixed boundary, i.e. $S_i^* = \partial\Gamma_i^* \cap \partial\Omega$. Γ_i^* shall fulfill the surface diffusion equation (2.1) with $V_i = 0$, the conditions (2.2) at S_i^* and (2.3) at the triple line L^*. As above, we choose the normals n_i^* of Γ_i^* so that $\gamma_1 n_1^* + \gamma_2 n_2^* + \gamma_3 n_2^* = 0$. For these stationary solutions the following lemma holds.

Lemma 2.1. *Stationary solutions as described above have constant mean curvature and fulfill the identity*

$$\gamma_1 \kappa_{n_{\partial\Gamma_1^*}} + \gamma_2 \kappa_{n_{\partial\Gamma_2^*}} + \gamma_3 \kappa_{n_{\partial\Gamma_3^*}} = 0 \quad on \quad L^* ,$$

where $\kappa_{n_{\partial\Gamma_i^*}} = \sigma_i^*(n_{\partial\Gamma_i^*}, n_{\partial\Gamma_i^*})$ is the normal curvature of Γ_i^* in direction of $n_{\partial\Gamma_i^*}$ and σ_i^* is our notation for the second fundamental form of Γ_i^* with respect to the unit normal n_i^*.

Proof. Standard analysis gives the claim of constant mean curvature and for the details we refer to [Dep10]. Here we just show the remaining identity. For $q \in L^*$, we can decompose the tangent space $T_q\Gamma_i^*$ with the help of the outer unit conormal $n_{\partial\Gamma_i^*}$ of Γ_i^* at L^* into $T_q\Gamma_i^* = T_qL^* \cup \operatorname{span}\{n_{\partial\Gamma_i^*}\}$.
Therefore we can complete $n_{\partial\Gamma_i^*}$ to an orthonormal basis $\{n_{\partial\Gamma_i^*}, t_1, \ldots, t_{n-1}\}$ of $T_q\Gamma_i^*$ with the help of suitable vectors $t_1, \ldots, t_{n-1} \in T_qL^*$. Note that we choose for every $i = 1, 2, 3$ the same set of vectors $t_1, \ldots, t_{n-1}$. Since the mean curvature H_i^* is the trace of the Weingarten map, we obtain the identity

$$\gamma_i H_i^* = \gamma_i \sigma_i^*(n_{\partial\Gamma_i^*}, n_{\partial\Gamma_i^*}) + \gamma_i \sum_{j=1}^{n-1} \sigma_i^*(t_j, t_j) .$$

Now we use the second equation $\gamma_1 H_1^* + \gamma_2 H_2^* + \gamma_3 H_3^* = 0$ on L^* from (2.3) for the stationary hypersurfaces to get

$$0 = \sum_{i=1}^{3} \gamma_i \kappa_{n_{\partial\Gamma_i^*}} + \sum_{i=1}^{3} \gamma_i \sum_{j=1}^{n-1} \sigma_i^*(t_j, t_j) .$$

For the second term we calculate

$$\sum_{i=1}^{3} \gamma_i \sum_{j=1}^{n-1} \sigma_i^*(t_j, t_j) = -\sum_{j=1}^{n-1} \sum_{i=1}^{3} \gamma_i \partial_{t_j} n_i^* \cdot t_j = -\sum_{j=1}^{n-1} \partial_{t_j} \underbrace{\left(\sum_{i=1}^{3} \gamma_i n_i^* \right)}_{=0 \text{ on } L^*} \cdot t_j = 0 ,$$

where the identity holds since t_j is a tangent vector of L^*. $\square$

To describe the considered hypersurfaces $\Gamma_i(t)$, we will use the representation for one hypersurface from Depner [Dep11] resp. Vogel [Vog00] near the fixed boundary $\partial\Omega$, an explicit mapping near the triple line L^* and finally compose them with the help of a cut-off function.

So for $i = 1,2,3$ and small $\varepsilon > 0$ we set up a specific curvilinear coordinate system that takes into account a possible curved boundary $\partial\Omega$ and the fact that the considered hypersurfaces have to stay inside Ω and their boundary has to lie on $\partial\Omega$. Let

$$\Psi_i : \Gamma_i^* \times (-\varepsilon, \varepsilon) \longrightarrow \Omega\,, \qquad (q,w) \mapsto \Psi_i(q,w) \tag{2.9}$$

be a mapping with $\Psi_i(q,0) = q$ for all $q \in \Gamma_i^*$, $\Psi_i(q,w) \in \partial\Omega$ for all $q \in \partial\Gamma_i^* \cap \partial\Omega = S_i^*$ and $\partial_w \Psi_i(q,0) \cdot n_i^*(q) = 1$ for all $q \in \Gamma_i^*$. We also assume that for every (local) parametrization $q : D \to \Gamma^*$ with $D \subset \mathbb{R}^n$ open, the mapping $(y,w) \mapsto \Psi(q(y),w)$ is a locally invertible map from $\mathbb{R}^n$ to $\mathbb{R}^n$. We remind the following lemma, which was shown in [Dep11].

Lemma 2.2. *For $q \in \partial\Gamma^*$, it holds that $\partial_w \Psi(q,0) = n^*(q)$.*

Furthermore, for small $\delta > 0$ let Z_i be a mapping given through

$$\begin{aligned} Z_i : \Gamma_i^* \times (-\varepsilon,\varepsilon) \times (-\delta,\delta) \quad &\longrightarrow \quad \mathbb{R}^{n+1}\,, \\ (q,w,s) \quad &\mapsto \quad Z_i(q,w,s) := q + w\, n_i^*(q) + s\, t_i^*(q)\,, \end{aligned} \tag{2.10}$$

where $i = 1,2,3$ and t_i^* is a tangent vector field on Γ_i^* with support in a neighbourhood of L_i^*, which equals the outer unit conormal $n_{\partial\Gamma_i^*}$ at L_i^*. More precisely we choose an open set $U \subset \mathbb{R}^{n+1}$, such that U is a neighbourhood of the triple line L^* and set $U_i := U \cap \Gamma_i^*$. Then we require for t_i^* that

$$t_i^*(q) \begin{cases} = 0 & \text{for } q \in \Gamma_i^* \backslash \overline{U_i}\,, \\ \in T_q\Gamma_i^* & \text{for } q \in U_i\,, \\ = n_{\partial\Gamma_i^*}(q) & \text{for } q \in L_i^*\,. \end{cases} \tag{2.11}$$

Now we choose a neighbourhood of L^* given by some small tube $B_{2\tau}(L^*)$ around L^*, where $2\tau > 0$ is such that $B_{2\tau}(L^*)$ is compactly included in Ω, i.e. $\overline{B_{2\tau}(L^*)} \subset \Omega$. Since our decomposition of $\partial\Gamma_i^*$ assured that $L^* \subset \Omega$, such a neighbourhood can be found.

An additional assumption is now that the evolution of the triple line shall always stay inside the neighbourhood $B_{2\tau}(L^*)$, in particular the triple line will never touch the outer fixed boundary $\partial\Omega$. To this end, we choose a smooth cut-off function $\eta \in C^\infty(\Omega)$, such that

$$\eta(x) = \begin{cases} 1 & , \quad x \in B_\tau(L^*)\,, \\ 0 & , \quad x \in \Omega \backslash B_{2\tau}(L^*)\,. \end{cases}$$

For $i = 1,2,3$ and functions

$$\rho_i : [0,T) \times \Gamma_i^* \longrightarrow \mathbb{R} \;\text{ and }\; \mu_i : [0,T) \times L^* \longrightarrow \mathbb{R}$$

with $|\rho_i| < \varepsilon$ and $|\mu_i| < \delta$, we define the mappings $\Phi_i = \Phi_i^{\rho_i,\mu_i}$ (we often omit the superscript (ρ_i,μ_i) for shortness) for $i = 1,2,3$ through

$$\begin{aligned} &\Phi_i : [0,T) \times \Gamma_i^* \longrightarrow \Omega\,, \\ &\Phi_i(t,q) := \eta(q)\, Z_i(q, \rho_i(t,q), \mu_i(t, \mathrm{pr}_i(q))) + (1-\eta(q))\, \Psi_i(q, \rho_i(t,q)) \end{aligned} \tag{2.12}$$

Here $\mathrm{pr}_i : \Gamma_i^* \to L_i^*$ a projection on L_i^*, which we define as follows. We let $V \subset \mathbb{R}^{n+1}$ be an open set such that U from the above definition of the tangent vector field t_i^* is compactly embedded in V, i.e. $U \subset\subset V$ and set $V_i := V \cap \Gamma_i^*$. If V is a small enough neighbourhood of L^*, we define the projection pr_i through

$$\mathrm{pr}_i(q) = \begin{cases} u & \text{for } q \in V_i\,, \\ q_0 & \text{for } q \in \Gamma_i^* \backslash \overline{V_i}\,. \end{cases} \tag{2.13}$$

Here q_0 is some fixed point on L_i^* and $u = \mathrm{pr}_i(q)$ is the unique point on L_i^*, that is mapped to q with the geodesic line $\alpha_i(s)$ on Γ_i^* with

$$\alpha_i(0) = u \qquad \text{and} \qquad \alpha_i'(0) = n_{\partial\Gamma_i^*}(q)\,.$$

Note that we need this projection just inside of the small neighbourhood V of L^*, because it is used in the product $\mu_i(t, \mathrm{pr}_i(q))\, t_i^*(q)$, where the second term is 0 outside of the even smaller neighbourhood U of L^*. We set for fixed t the mapping

$$(\Phi_i)_t : \Gamma_i^* \longrightarrow \mathbb{R}^{n+1}\,, \qquad (\Phi_i)_t(q) := \Phi_i(t,q)\,,$$

which is a diffeomorphism onto its image if ε and δ are small enough. Finally we define new hypersurfaces through

$$\Gamma_{\rho_i,\,\mu_i}(t) \quad := \quad \{(\Phi_i)_t(q)\,|\, q \in \Gamma_i^*\}\,. \tag{2.14}$$

Then the resulting hypersurface for $\rho_i \equiv 0$ and $\mu_i \equiv 0$ is simply $\Gamma_{\rho_i\equiv 0,\,\mu_i\equiv 0}(t) = \Gamma_i^*$.

We formulate the condition that the new hypersurfaces meet in one triple line $L(t)$ through

$$\Phi_1(t,q) = \Phi_2(t,q) = \Phi_3(t,q) \qquad \text{for } \; q \in L^* (= L_1^* = L_2^* = L_3^*) \; \text{ and for all } \; t > 0\,. \tag{2.15}$$

For the new hypersurfaces $\Gamma_i(t) := \Gamma_{\rho_i,\mu_i}(t)$ there exists also a decomposition of the boundary $\partial\Gamma_i(t)$ through

$$\partial\Gamma_i(t) = L_i(t) \cup S_i(t)\,,$$

where $S_i(t) = \partial\Gamma_i(t) \cap \partial\Omega$ and from (2.15) we can identify the other parts $L_i(t) = \partial\Gamma_i(t)\backslash S_i(t)$ to one compact $(n-1)$-dimensional submanifold $L(t) = L_1(t) = L_2(t) = L_3(t)$.

Note that (2.15) can be formulated as

$$Z_1(t, \rho_1(t,q), \mu_1(t,q)) = Z_2(t, \rho_2(t,q), \mu_2(t,q)) = Z_3(t, \rho_3(t,q), \mu_3(t,q)) \qquad \text{for } \; q \in L^*\,,$$

since the cut-off function η equals 1 at the triple line L^* and the projections give $\mathrm{pr}_i(q) = q$. The last identity can also be written as

$$\rho_1 n_1^* + \mu_1 n_{\partial\Gamma_1^*} = \rho_2 n_2^* + \mu_2 n_{\partial\Gamma_2^*} = \rho_3 n_3^* + \mu_3 n_{\partial\Gamma_3^*} \qquad \text{on } \; L^*\,. \tag{2.16}$$

Since on L^* the six vectors n_1^*, $n_{\partial\Gamma_1^*}$, n_2^*, $n_{\partial\Gamma_2^*}$, n_3^* and $n_{\partial\Gamma_3^*}$ lie in the two-dimensional space $(T_qL^*)^\perp$, the equations $\Phi_1 = \Phi_2$ and $\Phi_2 = \Phi_3$ on L^* (the third one is then automatically fulfilled) lead to 4 conditions, namely 2 in each case. Therefore it is reasonable to try to find 4 equivalent conditions to (2.15), which is done in the next lemma.

Lemma 2.3. *Equivalent to the equations*

$$\Phi_1 = \Phi_2 \quad \textit{and} \quad \Phi_2 = \Phi_3 \quad \textit{on } \; L^* \tag{2.17}$$

are the following conditions, which describe an identity for the weighted sum of the ρ_i and a linear dependence of μ_i to all of the ρ_i on L^* given through

$$\begin{cases} (i) & \gamma_1\rho_1 + \gamma_2\rho_2 + \gamma_3\rho_3 = 0 \quad \textit{on } \; L^*\,, \\ (ii) & \quad \mu_i = \frac{1}{s_i}\,(c_j\rho_j - c_k\rho_k) \quad \textit{on } \; L^*\,. \end{cases} \tag{2.18}$$

for $(i,j,k) = (1,2,3), (2,3,1)$ and $(3,1,2)$ and where $s_i = \sin\theta_i$ and $c_i = \cos\theta_i$.

Proof. (2.18) follows from (2.17) with the help of (2.16) as in [GIK05].
In order to show that (2.18) implies (2.17), some linear algebra is needed. We fix $p \in L^*$ and formulate (2.17) with the help of the matrix

$$A = \begin{pmatrix} n_1^* & -n_2^* & 0 & n_{\partial\Gamma_1^*} & -n_{\partial\Gamma_2^*} & 0 \\ 0 & n_2^* & -n_3^* & 0 & n_{\partial\Gamma_2^*} & -n_{\partial\Gamma_3^*} \end{pmatrix}$$

and the vector $(\rho, \mu) = (\rho_1, \rho_2, \rho_3, \mu_1, \mu_2, \mu_3)$ through

$$(\rho, \mu) \text{ fulfill (2.17)} \Longleftrightarrow A \begin{pmatrix} \rho \\ \mu \end{pmatrix} = 0 \Longleftrightarrow (\rho, \mu) \in \ker A\,.$$

Since $\Phi_1 = \Phi_2$ and $\Phi_2 = \Phi_3$ on L^* are each identities for linear combinations of the vectors n_1^*, n_2^*, n_3^*, $n_{\partial\Gamma_1^*}, n_{\partial\Gamma_2^*}, n_{\partial\Gamma_3^*}$, which lie in a two-dimensional space, the image of A has at most dimension four. From the fact that the first, the third, the fourth and the sixth column in A are linearly independent, we see that in fact $\dim(\mathrm{im}A) = 4$. This leads to $\dim(\ker A) = 6 - 4 = 2$.
Now we observe that (2.18) can be written with the help of the matrix

$$B = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & 0 & 0 & 0 \\ 0 & -\frac{c_2}{s_1} & -\frac{c_3}{s_1} & 1 & 0 & 0 \\ -\frac{c_1}{s_2} & 0 & -\frac{c_3}{s_2} & 0 & 1 & 0 \\ -\frac{c_1}{s_3} & -\frac{c_2}{s_3} & 0 & 0 & 0 & 1 \end{pmatrix}$$

through

$$(\rho, \mu) \text{ fulfill (2.18)} \Longleftrightarrow B \begin{pmatrix} \rho \\ \mu \end{pmatrix} = 0 \Longleftrightarrow (\rho, \mu) \in \ker B\,.$$

Since the third, the fourth, the fifth and the sixth column of B are linearly independent, we see that the rank of B, i.e. the dimension of the image of B, is four. The rank formula leads to $\dim(\ker B) = 6-4 = 2$.
With the above calculations we showed $\ker A \subset \ker B$, and since both kernels have dimension two, we conclude $\ker A = \ker B$, which gives the desired equivalence of the lemma. □

From now on, we always assume condition (2.15) and write the surface diffusion equation (2.1) and the boundary conditions (2.2) and (2.3) over the fixed stationary hypersurfaces Γ_i^* to get partial differential equations for ρ_i and μ_i, $i = 1, 2, 3$. This gives for the surface diffusion equations in Γ_i^*

$$V_i(\Phi_i(t, q)) = -m_i\gamma_i\Delta_{\Gamma_i(t)}H_i(\Phi_i(t, q))\,, \tag{2.19}$$

for the boundary equations on S_i^*

$$\left\{ \begin{aligned} (n_i \cdot \mu)\,(\Phi_i(t, q)) &= 0\,, \\ \nabla_{\Gamma_i(t)}H_i(\Phi_i(t, q)) \cdot n_{\partial\Gamma_i(t)}(\Phi_i(t, q)) &= 0\,, \end{aligned} \right. \tag{2.20}$$

and for the boundary equations at the triple line L^*

$$\left\{ \begin{aligned} & n_1(\Phi_1(t, q)) \cdot n_2(\Phi_2(t, q)) = \cos\theta_3\,, \\ & n_2(\Phi_2(t, q)) \cdot n_3(\Phi_3(t, q)) = \cos\theta_2\,, \\ & \gamma_1 H_1(\Phi_1(t, q)) + \gamma_2 H_2(\Phi_2(t, q)) + \gamma_3 H_3(\Phi_3(t, q)) = 0\,, \\ & m_1\gamma_1\nabla_{\Gamma_1(t)}H_1(\Phi_1(t, q)) \cdot n_{\partial\Gamma_1(t)}(\Phi_1(t, q)) \\ & = m_2\gamma_2\nabla_{\Gamma_2(t)}H_2(\Phi_2(t, q)) \cdot n_{\partial\Gamma_2(t)}(\Phi_2(t, q)) \\ & = m_3\gamma_3\nabla_{\Gamma_3(t)}H_3(\Phi_3(t, q)) \cdot n_{\partial\Gamma_3(t)}(\Phi_3(t, q))\,. \end{aligned} \right. \tag{2.21}$$

3 Linearization

In this section we give the linearization of (2.19)-(2.21) around $(\rho_i, \mu_i) \equiv (0,0)$, which is our interpretation of the linearization of (2.1)-(2.3) around a stationary state Γ_1^*, Γ_2^* and Γ_3^*. To get the linearization, we consider each term separately, write $\varepsilon\rho$ instead of ρ, differentiate with respect to ε and set $\varepsilon = 0$.

We use the results in [Dep11], in particular the linearization of mean curvature and the right angle condition at the fixed boundary, which are summarized in the following lemma whose detailed proof can be found in [Dep10].

Lemma 3.1. *We use the following results.*

$$\left(\frac{d}{d\varepsilon}\Delta_{\Gamma_i(t)} H_i(\Phi_i(t,q))\right|_{\varepsilon=0} = \Delta_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right) \quad in \;\; \Gamma_i^* \,,$$

$$\left(\frac{d}{d\varepsilon}\left(n_i \cdot \mu\right)\left(\Phi_i(t,q)\right)\right|_{\varepsilon=0} = \left(\partial_\mu - S(n_i^*, n_i^*)\right)\rho_i \quad on \;\; S_i^* \,.$$

Herein $\Delta_{\Gamma_i^*}$ is the Laplace-Beltrami operator on Γ_i^* and $|\sigma_i^*|^2$ is the squared norm of the second fundamental form of Γ_i^* with respect to n_i^*, which is given through the sum over the squared principal curvatures. S is the second fundamental form of $\partial\Omega$ with respect to the inwards pointing normal $-\mu$. Note that n_i^* lies in the tangential space of $\partial\Omega$ due to the right angle condition for the stationary state Γ_i^* at S_i^*.

The remaining work is the linearization of the angle conditions $n_i \cdot n_j = \cos\theta_k$ at the triple line L^*. To calculate this linearization at a fixed point $q_0 \in L^* (= L_1^* = L_2^* = L_3^*)$ for $t > 0$, we choose a local parametrization of Γ_i^* around q_0 with nice properties. More precisely, let $U_i \subset \mathbb{R}^{n+1}$ be an open neighbourhood of q_0, $V_i \subset \mathbb{R}^{n+1}$ open and $\varphi_i : U \to V$ a diffeomorphism, such that

$$\varphi_i(U_i \cap \Gamma_i^*) = V_i \cap \left(\mathbb{R}_+^n \times \{0\}\right) \quad \text{with} \quad \left(\varphi_i(q_0)\right)_n = 0 \,.$$

We set $D_i \times \{0\} \coloneqq V_i \cap \left(\mathbb{R}_+^n \times \{0\}\right)$ and let $F_i = \left(\varphi_i^{-1}\right)\big|_{D_i}$, i.e.

$$F_i : D_i \longrightarrow \Gamma_i^* \subset \mathbb{R}^{n+1} \,, \qquad x \mapsto F_i(x) \,. \tag{3.1}$$

This is a local parametrization extended up to the boundary around q_0 with $F(x_0^i) = q_0$ for some $x_0^i \in \partial D_i$. At the fixed point x_0^i, we can demand the following properties.

(A) $\partial_1 F_i(x_0^i), \ldots, \partial_n F_i(x_0^i)$ is an orthonormal basis of $T_{q_0}\Gamma_i^*$,

(B) $\partial_1 F_i(x_0^i) = n_{\partial\Gamma_i^*}(q_0)$, where $n_{\partial\Gamma_i^*}$ is the outer unit conormal of Γ_i^* at $\partial\Gamma_i^*$ and

(C) $\left(\partial_1 F_i \times \ldots \times \partial_n F_i\right)(x_0) = n_i^*(F(x_0^i))$, where we just fix the sign.

The third assumption (C) uses the cross product for n vectors in $\mathbb{R}^{n+1}$, which in this case due to the orthonormality of $\partial_1 F_i(x_0^i), \ldots, \partial_n F_i(x_0^i)$ lies by definition in normal direction and we just want to fix the sign. To calculate the linearization of the angle conditions at the triple line, we need the following properties.

Lemma 3.2. *With the help of the parametrizations F_i it holds for $F_i(x) = q \in \Gamma_i^*$*

(i) $\Psi_i(F_i(x), 0, 0) = F_i(x)$,

(ii) $\partial_j \Psi_i(F_i(x), 0, 0) = \partial_j F_i(x)$, $\partial_w \Psi_i(F_i(x), 0, 0) = n_i^*(F_i(x))$,
$\partial_s \Psi_i(F_i(x), 0, 0) = t_i^*(F_i(x))$.

Additionally, for the fixed point $F_i(x_0^i) = q_0 \in L^*$ it holds

$$(iii)\quad \Big(\partial_1 F_i \times \ldots \times \overbrace{n_i^* \circ F_i}^{l\text{-th pos.}} \times \ldots \times \partial_n F_i\Big)(x_0^i) = (-1)\partial_l F_i(x_0^i),$$

$$(iv)\quad \Big(\partial_1 F_i \times \ldots \times \overbrace{\partial_l\,(n_i^* \circ F_i)}^{l\text{-th pos.}} \times \ldots \times \partial_n F_i\Big)(x_0^i)$$
$$= \Big(\partial_l\,(n_i^* \circ F_i)\cdot \partial_l F_i\Big)(x_0^i)\;\;(n_i^* \circ F_i)\,(x_0^i),$$

$$(v)\quad \Big(\partial_1 F_i \times \ldots \times \overbrace{t_i^* \circ F_i}^{l\text{-th pos.}} \times \ldots \times \partial_n F_i\Big)(x_0^i) = \Big((t_i^* \circ F_i)\cdot \partial_l F_i\Big)(x_0^i)\;\;(n_i^* \circ F_i)\,(x_0^i),$$

$$(vi)\quad \Big(\partial_1 F_i \times \ldots \times \overbrace{\partial_l\,(t_i^* \circ F_i)}^{l\text{-th pos.}} \times \ldots \times \partial_n F_i\Big)(x_0^i)$$
$$= \Big(\partial_l\,(t_i^* \circ F_i)\cdot \partial_l F_i\Big)(x_0^i)\;\;(n_i^* \circ F_i)\,(x_0^i) - \Big(\partial_l\,(t_i^* \circ F_i)\cdot n_i^*\Big)(x_0^i)\;\partial_l F_i(x_0^i).$$

Proof. This is a direct calculation using the properties of the vector product and the parametrizations F_i from (3.1) and will be omitted here for reasons of shortness. □

Now we are in a position to derive the linearization of the angle condition at the triple junction.

Lemma 3.3. *The linearization of*

$$n_i(t, \Phi_i^{\rho_i,\mu_i}(t,q)) \cdot n_j(t, \Phi_j^{\rho_j,\mu_j}(t,q)) = \cos\theta_k \;\; on \;\; L^*$$

around $(\rho,\mu) = (0,0)$*, where* $\rho = (\rho_1,\rho_2,\rho_3)$ *and* $\mu = (\mu_1,\mu_2,\mu_3)$*, is given through*

$$\partial_{n_{\partial\Gamma_i^*}}\rho_i + \kappa_{n_{\partial\Gamma_i^*}}\mu_i = \partial_{n_{\partial\Gamma_j^*}}\rho_j + \kappa_{n_{\partial\Gamma_i^*}}\mu_j \;\; on \;\; L^*\,, \tag{3.2}$$

where $\kappa_{n_{\partial\Gamma_i^*}} = \sigma_i^*(n_{\partial\Gamma_i^*}, n_{\partial\Gamma_i^*})$ *is the normal curvature of* Γ_i^* *in direction* $n_{\partial\Gamma_i^*}$*. Equivalently, we can write this equation as*

$$\partial_{n_{\partial\Gamma_i^*}}\rho_i + \frac{1}{s_i}\Big(c_j\kappa_{n_{\partial\Gamma_j^*}} - c_k\kappa_{n_{\partial\Gamma_k^*}}\Big)\rho_i = \partial_{n_{\partial\Gamma_j^*}}\rho_j + \frac{1}{s_j}\Big(c_k\kappa_{n_{\partial\Gamma_k^*}} - c_i\kappa_{n_{\partial\Gamma_i^*}}\Big)\rho_j \;\; on \;\; L^*\,, \tag{3.3}$$

where $(i,j,k) = (1,2,3)$*,* $(2,3,1)$ *or* $(3,1,2)$*,* $s_i = \sin\theta_i$ *and* $c_i = \cos\theta_i$*.*

Proof. We show the linearization at a fixed point $q_0 \in L^*$ for $t_0 > 0$ and choose parametrizations F_i as in (3.1) with properties (A)-(C) at the fixed point $F_i(x_0^i) = q_0$.
Using the diffeomorphism $(\Phi_i)_t : \Gamma_i^* \to \Gamma_{\rho_i,\mu_i}(t)$ we also get a parametrization of $\Gamma_{\rho_i,\mu_i}(t)$, which we denote by

$$G_i^t : D_i \longrightarrow \Gamma_{\rho_i,\mu_i}(t)\,, \qquad G_i^t(x) \coloneqq \Phi_i(t, F_i(x))\,.$$

Then the normal n_i of $\Gamma_{\rho_i,\mu_i}(t)$ at $p = \Phi_i(t,q) \in \Gamma_{\rho_i,\mu_i}(t)$ for some $q \in \Gamma_i^*$, is given with the help of the cross product of n vectors in $\mathbb{R}^{n+1}$ through

$$n_i(t,p) = n_i(t,\Phi_i(t,q)) = n_i(t, G_i(x)) = \frac{\partial_1 G_i^t(x) \times \ldots \times \partial_n G_i^t(x)}{|\partial_1 G_i^t(x) \times \ldots \times \partial_n G_i^t(x)|}\,. \tag{3.4}$$

A calculation of the partial derivative $\partial_l G_i^t(x)$ gives

$$\partial_l G_i^t = \partial_l F_i + \partial_l \rho_i\, n_i^* + \rho_i\,\partial_l n_i^* + \partial_l \mu_i\, t_i^* + \rho_i\,\partial_l t_i^*\,,$$

where we omitted variables for reasons of shortness. We consider the numerator of n_i from (3.4).

$$\begin{aligned}
\partial_1 G_i^t \times \ldots \times \partial_n G_i^t = & \bigtimes_{l=1}^{n} \left(\partial_l F_i + \partial_l \rho_i\, n_i^* + \rho_i \partial_l n_i^* + \partial_l \mu_i\, t_i^* + \mu_i\, \partial_l t_i^*\right) \\
= & \left(\partial_1 F_i \times \ldots \times \partial_n F_i\right) + \sum_{l=1}^{n} \partial_l \rho_i \left(\partial_1 F_i \times \ldots \times \overbrace{n_i^*}^{\text{l-th pos.}} \times \ldots \times \partial_n F_i\right) \\
& + \sum_{l=1}^{n} \rho_i \left(\partial_1 F_i \times \ldots \times \overbrace{\partial_l n_i^*}^{\text{l-th pos.}} \times \ldots \times \partial_n F_i\right) \\
& + \sum_{l=1}^{n} \partial_l \mu_i \left(\partial_1 F_i \times \ldots \times \overbrace{t_i^*}^{\text{l-th pos.}} \times \ldots \times \partial_n F_i\right) \\
& + \sum_{l=1}^{n} \mu_i \left(\partial_1 F_i \times \ldots \times \overbrace{\partial_l t_i^*}^{\text{l-th pos.}} \times \ldots \times \partial_n F_i\right) + \text{ quadratic terms in } \rho_i \text{ and } \mu_i\,,
\end{aligned}$$

where the quadratic terms are not written down explicitly, because they will not give a contribution to the linearization. Cubic or higher order terms in ρ_i and μ_i do not appear, because the vector product will always vanish for such expressions.
With the help of the results from Lemma 3.2 for the parametrization, we can proceed at the fixed point $q_0 \in L^*$ for $t_0 > 0$ as follows.

$$\begin{aligned}
& \partial_1 G_i \times \ldots \times \partial_n G_i - \text{ quadratic terms from above} \\
= & n_i^* - \sum_{l=1}^{n} \partial_l \rho_i\, \partial_l F_i + \sum_{l=1}^{n} \rho_i \left(\partial_l n_i^* \cdot \partial_l F_i\right)\, n_i^* + \sum_{l=1}^{n} \partial_l \mu_i \left(t_i^* \cdot \partial_l F_i\right)\, n_i^* \\
& - \sum_{l=1}^{n} \mu_i \left(\partial_l t_i^* \cdot n_i^*\right)\, \partial_l F_i \\
= & \left(1 + \sum_{l=1}^{n} \rho_i \left(\partial_l n_i^* \cdot \partial_l F_i\right) + \sum_{l=1}^{n} \partial_l \mu_i \left(t_i^* \cdot \partial_l F_i\right) + \sum_{l=1}^{n} \mu_i \left(\partial_l t_i^* \cdot \partial_l F_i\right)\right) n_i^* \\
& - \sum_{l=1}^{n} \partial_l \rho_i\, \partial_l F_i - \sum_{l=1}^{n} \mu_i \left(\partial_l t_i^* \cdot n_i^*\right)\, \partial_l F_i \\
=: & R_i(\rho_i, \mu_i)\,,
\end{aligned}$$

where we use the abbreviation R_i. We want to linearize the relation

$$\frac{R_i(\rho_i, \mu_i)}{|R_i(\rho_i, \mu_i)|} \cdot \frac{R_j(\rho_j, \mu_j)}{|R_j(\rho_j, \mu_j)|} \quad = \quad \cos\theta_k \tag{3.5}$$

around $(\rho_i, \mu_i) \equiv (0, 0)$. Replacing ρ_i and μ_i by $\varepsilon\rho_i$ and $\varepsilon\mu_i$ and setting $Q_i(\varepsilon) := R_i(\varepsilon\rho_i, \varepsilon\mu_i)$ we have to compute the term

$$\frac{d}{d\varepsilon}\left(\frac{Q_i(\varepsilon)}{|Q_i(\varepsilon)|} \cdot \frac{Q_j(\varepsilon)}{|Q_j(\varepsilon)|}\right)\Big|_{\varepsilon=0}\,.$$

We see the identity $Q_i(0) = R_i(0\rho_i, 0\mu_i) = R_i(0, 0) = n_i^*$ and can therefore calculate abstractly

$$\begin{aligned}
\frac{d}{d\varepsilon}\left(\frac{Q_i(\varepsilon)}{|Q_i(\varepsilon)|}\right)\Big|_{\varepsilon=0} &= \frac{|Q_i(0)|\, Q_i'(0) - Q_i(0)\, \frac{d}{d\varepsilon}\left(|Q_i(\varepsilon)|\right)\big|_{\varepsilon=0}}{|Q_i(0)|^2} = Q_i'(0) - Q_i(0)\, \frac{Q_i(0) \cdot Q_i'(0)}{|Q_i(0)|} \\
&= Q_i'(0) - n_i^*\, \left(Q_i'(0) \cdot n_i^*\right) = \left(Q_i'(0)\right)^T\,,
\end{aligned}$$

where we used the projection on the tangent space of Γ_i^* given by $(y)^T = y - (y \cdot n_i^*)\, n_i^*$.
With the relation $Q_i'(0) = \frac{d}{d\varepsilon} R_i(\varepsilon\rho_i, \varepsilon\mu_i)\big|_{\varepsilon=0}$ and with the definition of R we see

$$(Q_i'(0))^T = \left(\frac{d}{d\varepsilon} R_i(\varepsilon\rho_i, \varepsilon\mu_i)\Big|_{\varepsilon=0} \right)^T \quad = \quad -\sum_{l=1}^{n} \partial_l\rho_i\, \partial_l F_i - \mu_i \sum_{l=1}^{n} (\partial_l t_i^* \cdot n_i^*)\, \partial_l F_i \,.$$

Therefore, we get

$$\begin{aligned}
\frac{d}{d\varepsilon}\left(\frac{Q_i(\varepsilon)}{|Q_i(\varepsilon)|} \cdot \frac{Q_j(\varepsilon)}{|Q_j(\varepsilon)|} \right)\Bigg|_{\varepsilon=0} &= (Q_i'(0))^T \cdot \frac{Q_j(0)}{|Q_j(0)|} + \frac{Q_i(0)}{|Q_i(0)|} \cdot \left(Q_j'(0)\right)^T \\
&= \left(-\sum_{l=1}^{n} \partial_l\rho_i\, \partial_l F_i - \mu_i \sum_{l=1}^{n} (\partial_l t_i^* \cdot n_i^*)\, \partial_l F_i \right) \cdot n_j^* \\
&\quad + n_i^* \cdot \left(-\sum_{l=1}^{n} \partial_l\rho_j\, \partial_l F_j - \mu_j \sum_{l=1}^{n} \left(\partial_l t_j^* \cdot n_j^*\right) \partial_l F_j \right) .
\end{aligned}$$

Here we use that $\partial_1 F_i$ equals the outer unit conormal $n_{\partial\Gamma_i^*}$ at the fixed point x_0^i, compare (B). Because of the orthogonality of $\partial_1 F_i, \ldots, \partial_n F_i$, we can conclude that the tangent vectors $\partial_2 F_i, \ldots, \partial_n F_i$ are all perpendicular to $n_{\partial\Gamma_i^*}$. Of course, they are also perpendicular to the normal n_i^*, everything at the fixed point $q_0 = F(x_0^i) \in L^*$. Furthermore we observed at the beginning that the vectors n_1^*, $n_{\partial\Gamma_1^*}$, n_2^*, $n_{\partial\Gamma_2^*}$, n_3^* and $n_{\partial\Gamma_3^*}$, all lie in a two-dimensional space, namely the space which is orthogonal to the tangent space of L^*. So we can write n_j^* as a linear combination of n_i^* and $n_{\partial\Gamma_i^*}$. Therefore in the above linearization of the angle conditions the scalar products involving $\partial_2 F_i, \ldots, \partial_n F_i$ and also $\partial_2 F_j, \ldots, \partial_n F_j$ all cancel out and the following terms remain

$$\begin{aligned}
&-\frac{d}{d\varepsilon}\left(\frac{Q_i(\varepsilon)}{|Q_i(\varepsilon)|} \cdot \frac{Q_j(\varepsilon)}{|Q_j(\varepsilon)|} \right)\Bigg|_{\varepsilon=0} \\
&= \Big(\partial_1\rho_i\, \partial_1 F_i + \mu_i\, (\partial_1 t_i^* \cdot n_i^*)\, \partial_1 F_i \Big) \cdot n_j^* + n_i^* \cdot \Big(\partial_1\rho_j\, \partial_1 F_j + \mu_j \left(\partial_1 t_j^* \cdot n_j^*\right) \partial_1 F_j \Big) \\
&= \Big(\partial_1\rho_i\, n_{\partial\Gamma_i^*} + \mu_i\, (\partial_1 t_i^* \cdot n_i^*)\, n_{\partial\Gamma_i^*} \Big) \cdot n_j^* + n_i^* \cdot \Big(\partial_1\rho_j\, n_{\partial\Gamma_j^*} + \mu_j \left(\partial_1 t_j^* \cdot n_j^*\right) n_{\partial\Gamma_i^*} \Big) \\
&= (\partial_1\rho_i + \mu_i\, (\partial_1 t_i^* \cdot n_i^*)) \left(n_{\partial\Gamma_i^*} \cdot n_j^*\right) + \left(\partial_1\rho_j + \mu_j \left(\partial_1 t_j^* \cdot n_j^*\right)\right) \Big(n_{\partial\Gamma_j^*} \cdot n_i^* \Big) .
\end{aligned}$$

Due to the angle conditions for the stationary reference hypersurfaces Γ_i^*, it holds that one of the terms $\Big(n_{\partial\Gamma_i^*} \cdot n_j^*\Big)$ and $\Big(n_{\partial\Gamma_j^*} \cdot n_i^*\Big)$ is $\sin\theta_k$ and the other one is $-\sin\theta_k$. Since $\sin\theta_k \neq 0$, we obtain the linearization of the angle condition as follows

$$\partial_1\rho_i + \mu_i\, (\partial_1 t_i^* \cdot n_i^*) \quad = \quad \partial_1\rho_j + \mu_j \left(\partial_1 t_j^* \cdot n_j^*\right)$$

for $(i, j) = (1, 2)$ and $(2, 3)$.
In geometric terms, the derivative ∂_1 here is a directional derivative in direction of the conormal, which follows from (B), so we get

$$\begin{aligned}
&\partial_1\rho_i = \partial_{n_{\partial\Gamma_i^*}}\rho_i = \nabla_{\Gamma_i^*}\rho_i \cdot n_{\partial\Gamma_i^*} \quad \text{and} \\
&(\partial_1 t_i^* \cdot n_i^*) = \Big(\partial_{n_{\partial\Gamma_i^*}} n_{\partial\Gamma_i^*} \cdot n_i^* \Big) = -n_{\partial\Gamma_i^*} \cdot \partial_{n_{\partial\Gamma_i^*}} n_i^* = \sigma_i^*(n_{\partial\Gamma_i^*}, n_{\partial\Gamma_i^*}) = \kappa_{n_{\partial\Gamma_i^*}} ,
\end{aligned}$$

where σ_i^* is the second fundamental form of Γ_i^* with respect to n_i^* and $\kappa_{n_{\partial\Gamma_i^*}}$ is the normal curvature of Γ_i^* in direction of the conormal $n_{\partial\Gamma_i^*}$.
The linearization of the angle condition then reads as follows

$$\partial_{n_{\partial\Gamma_i^*}}\rho_i + \kappa_{n_{\partial\Gamma_i^*}}\mu_i \quad = \quad \partial_{n_{\partial\Gamma_j^*}}\rho_j + \kappa_{n_{\partial\Gamma_j^*}}\mu_j \,,$$

for $(i,j) = (1,2)$ and $(2,3)$.

To derive (3.3) from this identity, we use the fact $\sum_{i=1}^{3} \gamma_i \kappa_{\partial\Gamma_i^*} = 0$ from Lemma 2.1 and analogue calculations as in [GIK10]. The details can be found in [Dep10]. □

To proceed, we abbreviate for reasons of shortness the following terms on L^*.

$$a_1 := \frac{1}{s_1}\left(c_2\,\kappa_{n_{\partial\Gamma_2^*}} - c_3\,\kappa_{n_{\partial\Gamma_3^*}}\right)\,, \tag{3.6}$$

$$a_2 := \frac{1}{s_2}\left(c_3\,\kappa_{n_{\partial\Gamma_3^*}} - c_1\,\kappa_{n_{\partial\Gamma_1^*}}\right) \quad \text{and} \tag{3.7}$$

$$a_3 := \frac{1}{s_3}\left(c_1\,\kappa_{n_{\partial\Gamma_1^*}} - c_1\,\kappa_{n_{\partial\Gamma_1^*}}\right)\,. \tag{3.8}$$

Altogether we obtain the linearized problem for $i = 1,2,3$ and $t > 0$

$$\partial_t \rho_i = -m_i\gamma_i \Delta_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right) \qquad \text{in } \Gamma_i^* \tag{3.9}$$

with the boundary conditions on S_i^*

$$\left\{\begin{array}{rcl} \Big(\partial_\mu - S(n_i^*, n_i^*)\Big)\,\rho_i &=& 0\,, \\ \partial_\mu\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right) &=& 0\,, \end{array}\right. \tag{3.10}$$

and the boundary conditions on the triple line L^*

$$\left\{\begin{array}{l} \gamma_1\rho_1 + \gamma_2\rho_2 + \gamma_3\rho_3 = 0\,, \\ \partial_{n_{\partial\Gamma_1^*}}\rho_1 + a_1\rho_1 = \partial_{n_{\partial\Gamma_2^*}}\rho_2 + a_2\,\rho_2 = \partial_{n_{\partial\Gamma_3^*}}\rho_3 + a_3\,\rho_3\,, \\ \gamma_1\left(\Delta_{\Gamma_1^*}\rho_1 + |\sigma_1^*|^2\rho_1\right) + \gamma_2\left(\Delta_{\Gamma_2^*}\rho_2 + |\sigma_2^*|^2\rho_2\right) + \gamma_3\left(\Delta_{\Gamma_3^*}\rho_3 + |\sigma_3^*|^2\rho_3\right) = 0\,, \\ m_1\gamma_1\partial_{n_{\partial\Gamma_1^*}}\left(\Delta_{\Gamma_1^*}\rho_1 + |\sigma_1^*|^2\rho_1\right) = m_2\gamma_2\partial_{n_{\partial\Gamma_2^*}}\left(\Delta_{\Gamma_2^*}\rho_2 + |\sigma_2^*|^2\rho_2\right) \\ \quad = m_3\gamma_3\partial_{n_{\partial\Gamma_3^*}}\left(\Delta_{\Gamma_3^*}\rho_3 + |\sigma_3^*|^2\rho_3\right)\,. \end{array}\right. \tag{3.11}$$

4 Stability analysis

In this section we derive conditions for the asymptotic stability of the zero solution of the linearized problem (3.9)-(3.11). We first show that (3.9)-(3.11) can be interpreted as a gradient flow with respect to an energy E given by a bilinear form I. Then we can show that the solution operator $\mathcal{A}$ of (3.9)-(3.11) is self-adjoint and we will study its spectrum. Finally, we describe asymptotic stability through the condition that I is positive.

The following abbreviations for function spaces resp. dual spaces will be useful. For $k \in \mathbb{N}$, we set

$$\begin{array}{rcl} \mathcal{H}^k &:=& H^k(\Gamma_1^*) \times H^k(\Gamma_2^*) \times H^k(\Gamma_3^*)\,, \\ \left(\mathcal{H}^k\right)' &:=& \left(H^k(\Gamma_1^*)\right)' \times \left(H^k(\Gamma_2^*)\right)' \times \left(H^k(\Gamma_3^*)\right)'\,, \\ \mathcal{Y} &:=& \left\{(\xi_1,\xi_2,\xi_3) \in \mathcal{H}^1 \mid \xi_1+\xi_2+\xi_3 = 0 \text{ on } L^* \text{ and } \displaystyle\int_{\Gamma_1^*}\xi_1 = \int_{\Gamma_2^*}\xi_2 = \int_{\Gamma_3^*}\xi_3\right\}\,, \\ \widetilde{\mathcal{Y}} &:=& \left\{(\xi_1,\xi_2,\xi_3) \in \mathcal{H}^1 \mid \xi_1+\xi_2+\xi_3 = 0 \text{ on } L^*\right\}\,, \\ \mathcal{E} &:=& \left\{(v_1,v_2,v_3) \in \mathcal{H}^1 \mid \gamma_1 v_1+\gamma_2 v_2+\gamma_3 v_3 = 0 \text{ on } L^* \text{ and } \displaystyle\int_{\Gamma_1^*}v_1 = \int_{\Gamma_2^*}v_2 = \int_{\Gamma_3^*}v_3\right\}\,, \\ \mathcal{H}^{-1} &:=& \left\{(w_1,w_2,w_3) \in \left(\mathcal{H}^1\right)' \mid \langle w_1,1\rangle = \langle w_2,1\rangle = \langle w_3,1\rangle\right\}\,. \end{array}$$

Here $\langle .\,,.\rangle$ is the duality pairing between the dual space $\left(H^1(\Gamma_i^*)\right)'$ and the Sobolev space $H^1(\Gamma_i^*)$. We will also denote the duality pairing between $w = (w_1, w_2, w_3) \in \mathcal{H}^{-1}$ and $u = (u_1, u_2, u_3) \in \mathcal{H}^1$ with the same symbol, i.e.

$$\langle w, u\rangle = \langle w_1, u_2\rangle + \langle w_2, u_2\rangle + \langle w_3, u_3\rangle \ .$$

We will show that the linearized problem (3.9) - (3.11) is a gradient flow with respect to the $\mathcal{H}^{-1}$ inner product.

Definition 4.1. *We say that $u^w = (u_1^w, u_2^w, u_3^w) \in \mathcal{Y}$ for a given $w = (w_1, w_2, w_3) \in \mathcal{H}^{-1}$ is a weak solution of*

$$\begin{cases} -m_i \Delta_{\Gamma_i^*} u_i^w = w_i & \text{in } \Gamma_i^* \quad (i = 1, 2, 3)\,, \\ u_1^w + u_2^w + u_3^w = 0 & \text{on } L^*\,, \\ m_1 \nabla_{\Gamma_1^*} u_1^w \cdot n_{\partial\Gamma_1^*} = m_2 \nabla_{\Gamma_2^*} u_2^w \cdot n_{\partial\Gamma_2^*} = m_3 \nabla_{\Gamma_3^*} u_3^w \cdot n_{\partial\Gamma_3^*} & \text{on } L^*\,, \\ \nabla_{\Gamma_i^*} u_i^w \cdot n_{\partial\Gamma_i^*} = 0 & \text{on } S_i^* \quad (i = 1, 2, 3)\,, \end{cases} \tag{4.1}$$

if and only if $u^w \in \mathcal{Y}$ satisfies

$$\langle w, \xi\rangle = \sum_{i=1}^{3} m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} u_i^w \cdot \nabla_{\Gamma_i^*} \xi_i \tag{4.2}$$

for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathcal{Y}$.

For later use we show in the next lemma that the above weak formulation (4.2) can also be written with the help of test functions from the larger space $\widetilde{\mathcal{Y}}$ instead of $\mathcal{Y}$.

Lemma 4.2. *Equation (4.2) can be written equivalently with test functions $\xi \in \widetilde{\mathcal{Y}}$ instead of $\mathcal{Y}$. In detail this means for $w \in \mathcal{H}^{-1}$ and $u^w \in \mathcal{Y}$ the equivalence between the following two equations*

(i) $\langle w, \xi\rangle = \sum_{i=1}^{3} m_i \nabla_{\Gamma_i^*} u_i^w \cdot \nabla_{\Gamma_i^*} \xi_i$ *for all* $\xi \in \mathcal{Y}$ *and*

(ii) $\left\langle w, \widetilde{\xi}\right\rangle = \sum_{i=1}^{3} m_i \nabla_{\Gamma_i^*} u_i^w \cdot \nabla_{\Gamma_i^*} \widetilde{\xi}_i$ *for all* $\widetilde{\xi} \in \widetilde{\mathcal{Y}}$.

Proof. The inclusion $\mathcal{Y} \subset \widetilde{\mathcal{Y}}$ leads to the implication $(ii) \Rightarrow (i)$.
For the other implication let $\widetilde{\xi} = (\widetilde{\xi}_1, \widetilde{\xi}_2, \widetilde{\xi}_3) \in \widetilde{\mathcal{Y}}$ be given, i.e. $\widetilde{\xi}_i \in H^1(\Gamma_i^*)$ and $\widetilde{\xi}_1 + \widetilde{\xi}_2 + \widetilde{\xi}_3 = 0$ on L^*. We want to find constants (c_1, c_2, c_3), such that

$$\xi := (\widetilde{\xi} - c) := (\widetilde{\xi}_1 - c_1, \widetilde{\xi}_2 - c_2, \widetilde{\xi}_3 - c_3) \quad \in \quad \mathcal{Y}\,.$$

This means, we have to find constants $c = (c_1, c_2, c_3)$ such that

$$c_1 + c_2 + c_3 = 0 \quad \text{ and } \quad \int_{\Gamma_1^*} \left(\widetilde{\xi}_1 - c_1\right) = \int_{\Gamma_2^*} \left(\widetilde{\xi}_2 - c_2\right) = \int_{\Gamma_3^*} \left(\widetilde{\xi}_3 - c_3\right)\,.$$

We formulate these conditions as a linear system of three equations for the unknowns (c_1, c_2, c_3) and observe that the corresponding matrix

$$M := \begin{pmatrix} 1 & 1 & 1 \\ -|\Gamma_1^*| & |\Gamma_2^*| & 0 \\ 0 & -|\Gamma_2^*| & |\Gamma_3^*| \end{pmatrix}$$

is invertible due to $\det M = |\Gamma_2^*| \cdot |\Gamma_3^*| + |\Gamma_1^*| \cdot |\Gamma_2^*| + |\Gamma_1^*| \cdot |\Gamma_3^*| > 0$. Therefore we can find c with the above properties and $\xi = \widetilde{\xi} - c$ fulfills $\xi \in \mathcal{Y}$ and can be used as a test function in (i) to get

$$\left\langle w, \widetilde{\xi} - c \right\rangle = \sum_{i=1}^{3} m_i \nabla_{\Gamma_i^*} u_i^w \cdot \nabla_{\Gamma_i^*} \widetilde{\xi}_i \,,$$

where the constant on the right side has vanished. Due to $\langle w_1, 1\rangle = \langle w_2, 1\rangle = \langle w_3, 1\rangle$ the left side can be written as

$$\left\langle w, \widetilde{\xi} - c \right\rangle = \left\langle w, \widetilde{\xi} \right\rangle - \sum_{i=1}^{3} \langle w_i, c_i\rangle = \left\langle w, \widetilde{\xi} \right\rangle - \langle w_1, 1\rangle \underbrace{\sum_{i=1}^{3} c_i}_{=0} = \left\langle w, \widetilde{\xi} \right\rangle$$

and we proved (ii). □

Since the problem (4.1) is a bit unusual due to the different domains of definition Γ_i^*, we want to show equivalence of strong and weak solutions in the smooth case.

Lemma 4.3. *Let $w \in \mathcal{H}^{-1}$ be smooth, so that we can assume $\langle w, \xi\rangle = \sum_{i=1}^{3} \int_{\Gamma_i^*} w_i \, \xi_i$ for the duality pairing. Then $u^w \in \mathcal{Y}$ is a smooth solution of (4.1) if and only if $u^w \in \mathcal{Y}$ is smooth and fulfills (4.2).*

Proof. Let $u^w \in \mathcal{Y}$ be a smooth solution of (4.1). By testing with $\xi \in \mathcal{Y}$, we get with the help of integration by parts

$$\begin{aligned}
\langle w, \xi\rangle &= \sum_{i=1}^{3} \int_{\Gamma_i^*} w_i \, \xi_i = \sum_{i=1}^{3} \int_{\Gamma_i^*} (-m_i \Delta_{\Gamma_i^*} u_i^w) \, \xi_i \\
&= \sum_{i=1}^{3} m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} u_i^w \cdot \nabla_{\Gamma_i^*} \xi_i - \sum_{i=1}^{3} m_i \int_{S_i^*} \underbrace{\left(\nabla_{\Gamma_i^*} u_i^w \cdot n_{\partial\Gamma_i^*}\right)}_{=0} \xi_i \\
&\quad - \sum_{i=1}^{3} m_i \int_{L^*} \left(\nabla_{\Gamma_i^*} u_i^w \cdot n_{\partial\Gamma_i^*}\right) \xi_i \\
&= \sum_{i=1}^{3} m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} u_i^w \cdot \nabla_{\Gamma_i^*} \xi_i - \int_{L^*} m_1 \left(\nabla_{\Gamma_1^*} u_1^w \cdot n_{\partial\Gamma_1^*}\right) \underbrace{\sum_{i=1}^{3} \xi_i}_{=0} \\
&= \sum_{i=1}^{3} m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} u_i^w \cdot \nabla_{\Gamma_i^*} \xi_i \,.
\end{aligned}$$

Conversely, let $u^w \in \mathcal{Y}$ be smooth and fulfill (4.2) for test functions $\xi \in \widetilde{\mathcal{Y}}$, which is possible due to Lemma 4.2. Integration by parts then gives

$$\begin{aligned}
\sum_{i=1}^{3} \int_{\Gamma_i^*} w_i \, \xi_i &= \sum_{i=1}^{3} m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} u_i^w \cdot \nabla_{\Gamma_i^*} \xi_i \\
&= -\sum_{i=1}^{3} m_i \int_{\Gamma_i^*} \Delta_{\Gamma_i^*} u_i^w \, \xi_i + \sum_{i=1}^{3} m_i \int_{L^*} \left(\nabla_{\Gamma_i^*} u_i^w \cdot n_{\partial\Gamma_i^*}\right) \xi_i + \sum_{i=1}^{3} m_i \int_{S_i^*} \left(\nabla_{\Gamma_i^*} u_i^w \cdot n_{\partial\Gamma_i^*}\right) \xi_i \,.
\end{aligned}$$

Therefore it holds

$$
\begin{aligned}
0 = &\sum_{i=1}^{3} \int_{\Gamma_i^*} \left(w_i + m_i \Delta_{\Gamma_i^*} u_i^w\right) \xi_i \\
&+ \sum_{i=1}^{3} \int_{L^*} m_i \left(\nabla_{\Gamma_i^*} u_i^w \cdot n_{\partial \Gamma_i^*}\right) \xi_i + \sum_{i=1}^{3} \int_{S_i^*} m_i \left(\nabla_{\Gamma_i^*} u_i^w \cdot n_{\partial \Gamma_i^*}\right) \xi_i
\end{aligned}
$$

for all $\xi_i \in H^1(\Gamma_i^*)$ with $\xi_1 + \xi_2 + \xi_3 = 0$ on L^*.
By setting two of the ξ_i to zero and using zero boundary conditions for the remaining one, we get with the help of the fundamental lemma $w_i = -m_i \Delta_{\Gamma_i^*} u_i^w$ on Γ_i^*. Since ξ_i is arbitrary at S_i^*, we also get the boundary condition $\nabla_{\Gamma_i^*} u_i^w \cdot n_{\partial \Gamma_i^*} = 0$ at S_i^*. It remains the identity

$$
0 = \sum_{i=1}^{3} \int_{L^*} m_i \left(\nabla_{\Gamma_i^*} u_i^w \cdot n_{\partial \Gamma_i^*}\right) \xi_i \,.
$$

Here we use $\xi_1 + \xi_2 + \xi_3 = 0$ at L^* to get

$$
m_1 \nabla_{\Gamma_1^*} u_1^w \cdot n_{\partial \Gamma_1^*} = m_2 \nabla_{\Gamma_2^*} u_2^w \cdot n_{\partial \Gamma_2^*} = m_3 \nabla_{\Gamma_3^*} u_3^w \cdot n_{\partial \Gamma_3^*} \text{ at } L^* \,.
$$

Altogether we showed that u^w is a strong solution of (4.1). □

The next step is to show a Poincaré-type inequality for functions in $\mathcal{E}$ resp. in $\mathcal{Y}$. Therefore we use the notation for $\rho = (\rho_1, \rho_2, \rho_3)$

$$
\|\rho\| := \left(\sum_{i=1}^{3} \|\rho_i\|^2_{L^2(\Gamma_i^*)} \right)^{1/2} \quad \text{and} \quad \|\nabla_{\Gamma^*} \rho\| := \left(\sum_{i=1}^{3} \|\nabla_{\Gamma_i^*} \rho_i\|^2_{L^2(\Gamma_i^*)} \right)^{1/2} . \tag{4.3}
$$

Lemma 4.4. *There exists a constant $C > 0$, such that*

$$
\|\rho\| \leq C \|\nabla_{\Gamma^*} \rho\|
$$

holds for all $\rho = (\rho_1, \rho_2, \rho_3) \in \mathcal{E}$. The statement is also true for functions $\rho = (\rho_1, \rho_2, \rho_3) \in \mathcal{Y}$.

Proof. We argument by contradiction and assume that we can find a sequence $(\tilde{\rho}^n)_{n \in \mathbb{N}} \in \mathcal{E}$, such that

$$
\|\tilde{\rho}^n\| > n \|\nabla_{\Gamma^*} \tilde{\rho}^n\| \,.
$$

In particular, this gives $\|\tilde{\rho}^n\| > 0$ and normalizing $\rho^n := \frac{\tilde{\rho}^n}{\|\tilde{\rho}^n\|}$ leads to a sequence $\rho^n \in \mathcal{E}$ with $\|\rho^n\| = 1$ and $1 > n \|\nabla_{\Gamma^*} \rho^n\|$. For the components, we get the bound

$$
\|\rho_i^n\|_{L^2(\Gamma_i^*)} \leq \sum_{j=1}^{3} \|\rho_j^n\|_{L^2(\Gamma_j^*)} \leq \sqrt{3} \left(\sum_{j=1}^{3} \|\rho_j^n\|^2_{L^2(\Gamma_j^*)} \right)^{\frac{1}{2}} = \sqrt{3} \|\rho^n\| = \sqrt{3} \,.
$$

For the surface gradient of the components, we observe the convergence

$$
\|\nabla_{\Gamma_i^*} \rho_i^n\|_{L^2(\Gamma_i^*)} \leq \sqrt{3} \|\nabla_{\Gamma^*} \rho^n\| \leq \frac{\sqrt{3}}{n} \longrightarrow 0 \quad \text{for } n \to \infty \,.
$$

Therefore, we can deduce the weak convergence $\rho_i^n \rightharpoonup C_i$ in $H^1(\Gamma_i^*)$ for constants $C_i \in \mathbb{R}$. The Rellich embedding theorem gives $\rho_i^n \longrightarrow C_i$ in $L^2(\Gamma_i^*)$ for $n \to \infty$. Furthermore, the integral condition

$\int_{\Gamma_1^*} \rho_1 = \int_{\Gamma_2^*} \rho_2 = \int_{\Gamma_3^*} \rho_3$ leads to $|\Gamma_1^*| \cdot C_1 = |\Gamma_2^*| \cdot C_2 = |\Gamma_3^*| \cdot C_3$, so that we can conclude that the constants C_i all have the same sign.
Finally, the boundary condition $\gamma_1\rho_1^n + \gamma_2\rho_2^n + \gamma_3\rho_3^n = 0$ on L^* gives $\gamma_1 C_1 + \gamma_2 C_2 + \gamma_3 C_3 = 0$ and therefore $C_1 = C_2 = C_3 = 0$. More precisely, we have to use the compact embedding $H^1(\Gamma_i^*) \hookrightarrow L^2(\partial\Gamma_i^*)$ here.
But this is a contradiction to $\|\rho^n\| = 1$ for all $n \in \mathbb{N}$. $\square$

With the above Poincaré-type inequality one can show unique existence of a weak solution from problem (4.1) by means of the Lax-Milgram theorem. Now we are able to define the $\mathcal{H}^{-1}$-inner product, a symmetric bilinear form and an energy on $\mathcal{H}^1$.

Definition 4.5. *For $v, w \in \mathcal{H}^{-1}$ we define the inner product*

$$(v, w)_{-1} := \sum_{i=1}^{3} m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} u_i^v \cdot \nabla_{\Gamma_i^*} u_i^w \,,$$

where $u^v = (u_1^v, u_2^v, u_3^v)$, $u^w = (u_1^w, u_2^w, u_3^w) \in \mathcal{Y}$ are the weak solutions of (4.1) for given $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3) \in \mathcal{H}^{-1}$. We remark that the identity $(v, w)_{-1} = \langle v, u^w \rangle$ holds for all $u, w \in \mathcal{H}^{-1}$.

Definition 4.6. *For $\rho = (\rho_1, \rho_2, \rho_3)$ and $\eta = (\eta_1, \eta_2, \eta_3)$ in $\mathcal{H}^1$ we define*

$$I(\rho, \eta) := \sum_{i=1}^{3} \gamma_i \left(\int_{\Gamma_i^*} \left(\nabla_{\Gamma_i^*} \rho_i \, \nabla_{\Gamma_i^*} \eta_i - |\sigma_i^*|^2 \rho_i \, \eta_i \right) \mathrm{d}\mathcal{H}^n - \int_{S_i^*} S(n_i^*, n_i^*) \rho_i \, \eta_i \, \mathrm{d}\mathcal{H}^{n-1} + \int_{L^*} a_i \rho_i \, \eta_i \, \mathrm{d}\mathcal{H}^{n-1} \right)$$

and the associated energy for $\rho \in \mathcal{H}^1$ by $E(\rho) := \frac{1}{2} I(\rho, \rho)$. We remind that a_i are the abbreviations from (3.6)-(3.8).

Now we want to show that the linearized problem (3.9) - (3.11) is the gradient flow of E with respect to the $\mathcal{H}^{-1}$ inner product $(.,.)_{-1}$. Therefore we introduce the following time independent problem.

Definition 4.7. *For a given $w = (w_1, w_2, w_3) \in \mathcal{H}^{-1}$ we say that $\rho = (\rho_1, \rho_2, \rho_3) \in \mathcal{H}^3$ with $\int_{\Gamma_1^*} \rho_1 = \int_{\Gamma_2^*} \rho_2 = \int_{\Gamma_3^*} \rho_3$ is a weak solution of the boundary value problem*

$$w_i = -m_i \gamma_i \Delta_{\Gamma_i^*} \left(\Delta_{\Gamma_i^*} \rho_i + |\sigma_i^*|^2 \rho_i \right) \quad \text{in } \Gamma_i^* \,, \tag{4.4}$$

with the boundary conditions (3.10) on S_i^* and the boundary conditions (3.11) on the triple line L^*, if and only if ρ satisfies

$$\langle w, \xi \rangle = \sum_{i=1}^{3} m_i \gamma_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} \left(\Delta_{\Gamma_i^*} \rho_i + |\sigma_i^*|^2 \rho_i \right) \cdot \nabla_{\Gamma_i^*} \xi_i \tag{4.5}$$

for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathcal{Y}$ and fulfills the boundary conditions

$$\left(\partial_\mu - S(n_i^*, n_i^*) \right) \rho_i = 0 \tag{4.6}$$

on S_i^* and

$$\begin{cases} \gamma_1 \rho_1 + \gamma_2 \rho_2 + \gamma_3 \rho_3 = 0 \,, \\ \partial_{n_{\partial\Gamma_1^*}} \rho_1 + a_1 \rho_1 = \partial_{n_{\partial\Gamma_2^*}} \rho_2 + a_2 \, \rho_2 = \partial_{n_{\partial\Gamma_3^*}} \rho_3 + a_3 \, \rho_3 \,, \\ \gamma_1 \left(\Delta_{\Gamma_1^*} \rho_1 + |\sigma_1^*|^2 \rho_1 \right) + \gamma_2 \left(\Delta_{\Gamma_2^*} \rho_2 + |\sigma_2^*|^2 \rho_2 \right) + \gamma_3 \left(\Delta_{\Gamma_3^*} \rho_3 + |\sigma_3^*|^2 \rho_3 \right) = 0 \end{cases} \tag{4.7}$$

on the triple line L^*.

Lemma 4.8. *Let $w = (w_1, w_2, w_3) \in \mathcal{H}^{-1}$ and $\rho = (\rho_1, \rho_2, \rho_3) \in \mathcal{E}$ be given. Then ρ is a weak solution of (4.4) if and only if*

$$(w, \xi)_{-1} = -I(\rho, \xi)$$

for all $\xi \in \mathcal{E}$.

Proof. Let $\rho \in \mathcal{E}$ be a weak solution of (4.4). Due to $\xi \in \mathcal{E} \subset \mathcal{H}^{-1}$ through $\langle \xi, u \rangle = \sum_{i=1}^3 \int_{\Gamma_i^*} \xi_i \, u_i$ for $u \in \mathcal{H}^1$ we get from Definition 4.5 $(w, \xi)_{-1} = \langle w, u^\xi \rangle$.
Using $u^\xi \in \mathcal{Y}$ as a test function in the weak formulation of (4.4), we observe

$$\langle w, u^\xi \rangle = \sum_{i=1}^3 m_i \gamma_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} \left(\Delta_{\Gamma_i^*} \rho_i + |\sigma_i^*|^2 \rho_i \right) \cdot \nabla_{\Gamma_i^*} u_i^\xi = \sum_{i=1}^3 m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} \Theta_i \cdot \nabla_{\Gamma_i^*} u_i^\xi \,,$$

where we defined for shortness $\Theta_i = \gamma_i \left(\Delta_{\Gamma_i^*} \rho_i + |\sigma_i^*|^2 \rho_i \right)$. The third boundary condition on L^* from problem (4.4) yields $\Theta_1 + \Theta_2 + \Theta_3 = 0$ on L^*. Due to Lemma 4.2 we can use $\Theta = (\Theta_1, \Theta_2, \Theta_3)$ as a test function in (4.2) to get

$$\sum_{i=1}^3 \int_{\Gamma_i^*} \xi_i \cdot \Theta_i = \sum_{i=1}^3 m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} \Theta_i \cdot \nabla_{\Gamma_i^*} u_i^\xi \,.$$

Here we used the inclusion $\xi \in \mathcal{E} \subset \mathcal{H}^{-1}$ through $\langle \xi, \Theta \rangle = \sum_{i=1}^3 \int_{\Gamma_i^*} \xi_i \, \Theta_i$.
Thus we can conclude with integration by parts

$$\begin{aligned}
(w, \xi)_{-1} &= \sum_{i=1}^3 \int_{\Gamma_i^*} \xi_i \, \gamma_i \left(\Delta_{\Gamma_i^*} \rho_i + |\sigma_i^*|^2 \rho_i \right) \\
&= -\sum_{i=1}^3 \gamma_i \int_{\Gamma_i^*} \left(\nabla_{\Gamma_i^*} \xi_i \cdot \nabla_{\Gamma_i^*} \rho_i - |\sigma_i^*|^2 \xi_i \, \rho_i \right) + \sum_{i=1}^3 \gamma_i \int_{\partial \Gamma_i^*} \xi_i \, \left(\nabla_{\Gamma_i^*} \rho_i \cdot n_{\partial \Gamma_i^*} \right) \\
&= -\sum_{i=1}^3 \gamma_i \int_{\Gamma_i^*} \left(\nabla_{\Gamma_i^*} \xi_i \cdot \nabla_{\Gamma_i^*} \rho_i - |\sigma_i^*|^2 \xi_i \, \rho_i \right) + \sum_{i=1}^3 \gamma_i \int_{L^*} \xi_i \, \partial_{n_{\partial \Gamma_i^*}} \rho_i + \sum_{i=1}^3 \gamma_i \int_{S_i^*} \xi_i \, \partial_\mu \rho_i \,.
\end{aligned}$$

Using $\gamma_1 \xi_1 + \gamma_2 \xi_2 + \gamma_3 \xi_3 = 0$ at L^* for $\xi \in \mathcal{E}$ and the third boundary condition on L^* for the weak solution ρ of (4.4), we get

$$\begin{aligned}
\sum_{i=1}^3 \gamma_i \int_{L^*} \xi_i \, \partial_{n_{\partial \Gamma_i^*}} \rho_i &= \sum_{i=1}^3 \gamma_i \int_{L^*} \xi_i \, \left(\partial_{n_{\partial \Gamma_i^*}} \rho_i + a_i \rho_i \right) - \sum_{i=1}^3 \gamma_i \int_{L^*} a_i \, \xi_i \, \rho_i \\
&= \int_{L^*} \left(\partial_{n_{\partial \Gamma_1^*}} \rho_1 + a_1 \rho_1 \right) \underbrace{\sum_{i=1}^3 \gamma_i \xi_i}_{=0} - \sum_{i=1}^3 \gamma_i \int_{L^*} a_i \, \xi_i \, \rho_i \,.
\end{aligned}$$

From the first boundary condition on S_i^* for the weak solution ρ of (4.4) we get

$$\sum_{i=1}^3 \gamma_i \int_{S_i^*} \xi_i \cdot \partial_\mu \rho_i \quad = \quad \sum_{i=1}^3 \gamma_i \int_{S_i^*} \xi_i \cdot S(n_i^*, n_i^*) \rho_i \,.$$

Altogether, we arrive at $(w, \xi)_{-1} = -I(\rho, \xi)$ for all $\xi \in \mathcal{E}$.

Conversely, assume that $\rho \in \mathcal{E}$ satisfies $(w,\xi)_{-1} = -I(\rho,\xi)$ for all $\xi \in \mathcal{E}$. Now let $\zeta \in \mathcal{H}^3 \cap \mathcal{Y}$ be a given function with

$$m_1 \left(\nabla_{\Gamma_1^*}\zeta_1 \cdot n_{\partial\Gamma_1^*}\right) = m_2 \left(\nabla_{\Gamma_2^*}\zeta_2 \cdot n_{\partial\Gamma_2^*}\right) = m_3 \left(\nabla_{\Gamma_3^*}\zeta_3 \cdot n_{\partial\Gamma_3^*}\right) \quad \text{on } L^*, \tag{4.8}$$

$$\left(\nabla_{\Gamma_i^*}\zeta_i \cdot n_{\partial\Gamma_i^*}\right) = 0 \quad \text{on } S_i^* \text{ and} \tag{4.9}$$

$$\gamma_1 m_1 \Delta_{\Gamma_1^*}\zeta_1 + \gamma_2 m_2 \Delta_{\Gamma_2^*}\zeta_2 + \gamma_3 m_3 \Delta_{\Gamma_3^*}\zeta_3 = 0 \quad \text{on } L^*. \tag{4.10}$$

With the help the abbreviation $m\Delta_{\Gamma^*}\zeta = \left(m_1\Delta_{\Gamma_1^*}\zeta_1, m_2\Delta_{\Gamma_2^*}\zeta_2, m_3\Delta_{\Gamma_3^*}\zeta_3\right)$ we set $\xi := m\Delta_{\Gamma^*}\zeta$. One can directly verify the property $\xi \in \mathcal{E}$ for ξ, so that we can plug it into the assumption in this part of the proof. Since ζ is a solution of problem (4.1) for the right side ξ, we see with our above notation that $\zeta = u^\xi$ and from Definition 4.5 we get $-I(\rho,\xi) = (w,\xi)_{-1} = \langle w,\zeta\rangle$. This leads to the following equation

$$\begin{aligned}
\langle w,\zeta\rangle &= -I(\rho,\xi) = I(\rho, m\Delta_{\Gamma^*}\zeta) \\
&= \sum_{i=1}^3 m_i\gamma_i \int_{\Gamma_i^*} \left(\nabla_{\Gamma_i^*}\rho_i \cdot \nabla_{\Gamma_i^*}\Delta_{\Gamma_i^*}\zeta_i - |\sigma_i^*|^2\rho_i\,\Delta_{\Gamma_i^*}\zeta_i\right) \\
&\quad - \sum_{i=1}^3 m_i\gamma_i \int_{S_i^*} S(n_i^*,n_i^*)\,\rho_i\,\Delta_{\Gamma_i^*}\zeta_i + \sum_{i=1}^3 m_i\gamma_i \int_{L^*} a_i\,\rho_i\,\Delta_{\Gamma_i^*}\zeta_i\,.
\end{aligned}$$

Since $w \in \mathcal{H}^{-1}$, we obtain from regularity theory that $\rho \in \mathcal{H}^3$. Then we can integrate by parts to see

$$\begin{aligned}
\langle w,\zeta\rangle &= \sum_{i=1}^3 m_i\gamma_i \Bigg(\int_{\Gamma_i^*} -\left(\Delta_{\Gamma_i^*}\rho_i\,\Delta_{\Gamma_i^*}\zeta_i - \nabla_{\Gamma_i^*}\left(|\sigma_i^*|^2\rho_i\right)\cdot\nabla_{\Gamma_i^*}\zeta_i\right) \\
&\quad + \int_{\partial\Gamma_i^*} \left(\left(\nabla_{\Gamma_i^*}\rho_i \cdot n_{\partial\Gamma_i^*}\right)\,\Delta_{\Gamma_i^*}\zeta_i - |\sigma_i^*|^2\rho_i\,\left(\nabla_{\Gamma_i^*}\zeta_i \cdot n_{\partial\Gamma_i^*}\right)\right) \\
&\quad - \int_{S_i^*} S(n_i^*,n_i^*)\,\rho_i\,\Delta_{\Gamma_i^*}\zeta_i + \int_{L^*} a_i\,\rho_i\,\Delta_{\Gamma_i^*}\zeta_i \Bigg) \\
&= \sum_{i=1}^3 m_i\gamma_i \Bigg(\int_{\Gamma_i^*} -\left(\Delta_{\Gamma_i^*}\rho_i\,\Delta_{\Gamma_i^*}\zeta_i - \nabla_{\Gamma_i^*}\left(|\sigma_i^*|^2\rho_i\right)\cdot\nabla_{\Gamma_i^*}\zeta_i\right) + \int_{S_i^*} \left(\partial_\mu\rho_i - S(n_i^*,n_i^*)\,\rho_i\right)\,\Delta_{\Gamma_i^*}\zeta_i \\
&\quad + \gamma_i \int_{L^*} \left(\partial_{n_{\partial\Gamma_i^*}}\rho_i + a_i\,\rho_i\right)\,\Delta_{\Gamma_i^*}\zeta_i - \int_{\partial\Gamma_i^*} |\sigma_i^*|^2\rho_i\,\underbrace{\left(\nabla_{\Gamma_i^*}\zeta_i \cdot n_{\partial\Gamma_i^*}\right)}_{=0 \text{ on } S_i^*} \Bigg) \\
&= \sum_{i=1}^3 m_i\gamma_i \Bigg(\int_{\Gamma_i^*} \nabla_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right)\cdot\nabla_{\Gamma_i^*}\zeta_i + \int_{S_i^*} \left(\partial_\mu\rho_i - S(n_i^*,n_i^*)\,\rho_i\right)\,\Delta_{\Gamma_i^*}\zeta_i \\
&\quad + \int_{L^*} \left(\partial_{n_{\partial\Gamma_i^*}}\rho_i + a_i\,\rho_i\right)\,\Delta_{\Gamma_i^*}\zeta_i - \int_{L_i^*} \left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right)\,\left(\nabla_{\Gamma_i^*}\zeta_i \cdot n_{\partial\Gamma_i^*}\right) \Bigg).
\end{aligned}$$

Since the term on the left side and the surface integrals over Γ_i^* form a bounded linear functional on $\mathcal{H}^1$, we can use a similar argument as in [Dep10] to conclude that the above equality also holds without them, as we shall demonstrate in the following. To this end, let $h \in C^\infty(L^*)$ and $g_i^n \in C^\infty(\Gamma_i^*)$ with given boundary data $g_i^n|_{L^*} = g_i$ with $\gamma_1 g_1 + \gamma_2 g_2 + \gamma_3 g_3 = 0$ on L^*, which fulfill $\|g_i^n\|_{L^2(\Gamma_i^*)} \to 0$ for $n \to \infty$. Then we solve the problem

$$\begin{cases}
\Delta_{\Gamma_i^*}\eta_i^n = g_i^n & \text{in } \Gamma_i^*, \\
\nabla_{\Gamma_i^*}\eta_i^n \cdot n_{\partial\Gamma_i^*} = 0 & \text{on } S_i^*, \\
m_i\nabla_{\Gamma_i^*}\eta_i^n \cdot n_{\partial\Gamma_i^*} = h & \text{on } L^*
\end{cases}$$

with additional condition $\eta^n = (\eta_1^n, \eta_2^n, \eta_3^n) \in \mathcal{Y}$. A solution fulfills $\|\eta_i^n\|_{H^1} \to 0$ for $n \to \infty$, which leads to the following boundary integrals

$$\begin{aligned}0 =& \sum_{i=1}^{3} m_i \gamma_i \int_{S_i^*} \left(\partial_\mu \rho_i - S(n_i^*, n_i^*)\, \rho_i\right)\, g_i + \sum_{i=1}^{3} m_i \gamma_i \int_{L^*} \left(\partial_{n_{\partial\Gamma_i^*}} \rho_i + a_i\, \rho_i\right)\, g_i \\ &- \sum_{i=1}^{3} m_i \gamma_i \int_{L_i^*} \left(\Delta_{\Gamma_i^*} \rho_i + |\sigma_i^*|^2 \rho_i\right)\, h\end{aligned}$$

for arbitrary $h \in C^\infty(L^*)$, $g_i \in C^\infty(\partial\Gamma_i^*)$ with $\gamma_1 g_1 + \gamma_2 g_2 + \gamma_3 g_3 = 0$ on L^* and arbitrary on S_i^*. This yields the boundary conditions

$$\begin{aligned}\partial_\mu \rho_i - S(n_i^*, n_i^*)\rho_i = 0 \qquad & \text{on } S_i^*\,,\\ \gamma_1\left(\Delta_{\Gamma_1^*}\rho_1 + |\sigma_1^*|^2\rho_1\right) + \gamma_2\left(\Delta_{\Gamma_2^*}\rho_2 + |\sigma_2^*|^2\rho_2\right) + \gamma_3\left(\Delta_{\Gamma_3^*}\rho_3 + |\sigma_3^*|^2\rho_3\right) = 0 \qquad & \text{on } L^*\,,\\ \partial_{n_{\partial\Gamma_1^*}}\rho_1 + a_1\rho_1 = \partial_{n_{\partial\Gamma_2^*}}\rho_2 + a_2\,\rho_2 = \partial_{n_{\partial\Gamma_3^*}}\rho_3 + a_3\,\rho_3 \qquad & \text{on } L^*\,.\end{aligned}$$

Using the derived boundary equations, it remains the equality

$$\langle w, \zeta\rangle = \sum_{i=1}^{3} m_i \gamma_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right) \cdot \nabla_{\Gamma_i^*}\zeta_i\,.$$

for all $\zeta \in \mathcal{H}^3 \cap \mathcal{Y}$ satisfying (4.8)-(4.9). With a similar argumentation as in [Dep10] we can use such functions with prescribed Neumann-boundary to approximate arbitrary functions $\xi \in \mathcal{Y}$ in the H^1-norm.
Altogether we showed that $\rho \in \mathcal{E}$ is a weak solution of problem (4.4). □

We define the linearized operator corresponding to the linearized problem (3.9) - (3.11) through

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{H}^{-1}$$

with

$$\mathcal{D}(\mathcal{A}) = \left\{\rho = (\rho_1, \rho_2, \rho_3) \in \mathcal{H}^3 \mid \rho \text{ satisfies (4.6) on } S_i^* \text{ and (4.7) on } L^*\,, \text{and } \int_{\Gamma_1^*}\rho_1 = \int_{\Gamma_2^*}\rho_2 = \int_{\Gamma_3^*}\rho_3\right\} \tag{4.11}$$

by

$$\langle \mathcal{A}\rho, \xi\rangle = \sum_{i=1}^{3} m_i \gamma_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right) \cdot \nabla_{\Gamma_i^*}\xi_i \tag{4.12}$$

for all $\rho \in \mathcal{D}(\mathcal{A})$ and $\xi \in \mathcal{H}^1$.
The boundary value problem (4.4) is then related to the problem in finding a $\rho \in \mathcal{D}(\mathcal{A})$ with $\mathcal{A}\rho = w$. By Lemma 4.8, we observe for all $\xi \in \mathcal{E}$ the identity $(\mathcal{A}\rho, \xi)_{-1} = -I(\rho, \xi)$. With this property we can show symmetry of $\mathcal{A}$.

Lemma 4.9. *The operator $\mathcal{A}$ is symmetric with respect to the inner product $(\,.\,,.)_{-1}$.*

Proof. The proof follows along the lines of a similar proof in Depner [Dep10]. □

To study the spectrum of $\mathcal{A}$ as in the previous chapter, we need the following inequalities to get as a corollary an upper bound for the eigenvalues of $\mathcal{A}$.

Lemma 4.10. *For all $\delta > 0$ there exists a $C_\delta > 0$, such that for all $\rho = (\rho_1, \rho_2, \rho_3) \in \mathcal{E}$ and each $i = 1, 2, 3$ the inequality*

$$\|\rho_i\|^2_{L^2(\partial\Gamma_i^*)} \leq \delta \|\nabla_{\Gamma^*}\rho\|^2 + C_\delta \|\rho\|^2_{-1} \,,$$

holds, where we used the $\|.\|_{-1}$-norm on $\mathcal{H}^{-1}$ from Definition 4.6 and the Definition of $\|\nabla_{\Gamma^*}\rho\|$ from (4.3).

Proof. With the help of the Poincaré-type inequality from Lemma 4.4 we can apply a similar argument as in Depner [Dep10] for the case of one hypersurface without a triple line. Thus we omit it. □

Lemma 4.11. *There exist positive constants C_1 and C_2, such that*

$$\|\nabla_{\Gamma^*}\rho\|^2 \leq C_1 \, \|\rho\|^2_{-1} + C_2 \, I(\rho, \rho)$$

for all $\rho \in \mathcal{E}$.

Proof. Using the previous Lemma 4.10 and the Poincaré-type inequality from Lemma 4.4 we again just refer to a similar argument in [Dep10]. □

Lemma 4.12. *The largest eigenvalue of $\mathcal{A}$ is bounded from above by $\frac{C_1}{C_2}$, where C_1 and C_2 are the positive constants from Lemma 4.11.*

Proof. See [Dep10]. □

Lemma 4.13. *The operator $\mathcal{A}$ is self-adjoint with respect to the $(.\,,.)_{-1}$ inner product.*

Proof. We use the following theorem of operator theory from the book of Weidmann [Weid76]. If there exists an $\omega \in \mathbb{R}$, such that

$$\mathrm{im}\,(\omega Id - \mathcal{A}) = \mathcal{H}^{-1} \,,$$

then the properties symmetry and self-adjointness of $\mathcal{A}$ are equivalent.
So we have to show that there exists an $\omega \in \mathbb{R}$ such that for a given $f \in \mathcal{H}^{-1}$ there exists a $\rho \in \mathcal{D}(\mathcal{A})$ with $\omega\rho - \mathcal{A}\rho = f$. This means that ρ is a weak solution of the boundary value problem

$$\begin{cases} \Delta_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right) + \omega\rho_i = f & \text{in } \Gamma_i^* \,, \\ \rho \text{ satisfies (3.10)} & \text{on } S_i^* \,, \\ \rho \text{ satisfies (3.11)} & \text{on } L^* \,. \end{cases}$$

In detail the weak solution consists in finding a $\rho \in \mathcal{H}^3$ with the boundary condition (4.6) on S_i^* and (4.7) on the triple line L^* such that

$$-\sum_{i=1}^{3}\left(m_i\gamma_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right) \cdot \nabla_{\Gamma_i^*}\xi_i - \omega \int_{\Gamma_i^*} \rho_i \, \xi_i \right) = \langle f, \xi \rangle$$

holds for all $\xi \in \mathcal{Y}$. One can verify that such a weak solution fulfills $\int_{\Gamma_1^*}\rho_1 = \int_{\Gamma_2^*}\rho_2 = \int_{\Gamma_3^*}\rho_3$, so that $\rho \in \mathcal{D}(\mathcal{A})$. To get a solution, we use the minimizing problem

$$F(\rho) := \frac{1}{2}\left(I(\rho, \rho) + \omega\|\rho\|^2_{-1}\right) - \sum_{i=1}^{3} \int_{\Gamma_i^*} u_i^f \rho_i \; \longrightarrow \min$$

for all $\rho \in \mathcal{E}$, where $u^f \in \mathcal{Y}$ is the weak solution of (4.1) with respect to $f \in \mathcal{H}^{-1}$. With the help of Lemma 4.11 we can show that F is coercive on $\mathcal{E}$ for large ω, so that the minimizing problem has a solution $\overline{\rho} = (\overline{\rho}_1, \overline{\rho}_2, \overline{\rho}_3) \in \mathcal{E}$, when ω is large enough. Taking the first variation of F we get

$$I(\overline{\rho}, v) + \omega\,(\overline{\rho}, v)_{-1} = \sum_{i=1}^{3} \int_{\Gamma_i^*} u_i^f\, v_i$$

for all $v \in \mathcal{E}$. By the Definition of $u^{\overline{\rho}} \in \mathcal{Y}$ as weak solution of (4.1) with respect to $\overline{\rho} \in \mathcal{E} \subset \mathcal{H}^{-1}$ and Definition 4.5 we observe that

$$\omega\,(\overline{\rho}, v)_{-1} = \omega\,\langle v, u^{\overline{\rho}}\rangle = \sum_{i=1}^{3} \int_{\Gamma_i^*} u_i^{\overline{\rho}}\, v_i$$

for all $v \in \mathcal{E}$. So the above first variation is the weak version of the boundary value problem

$$\begin{cases} -\gamma_i\left(\Delta_{\Gamma_i^*}\rho_i + |\sigma_i^*|^2\rho_i\right) + \omega u_i^{\overline{\rho}} + c_i = u_i^f & \text{in } \Gamma_i^*\,, \\ \rho \text{ satisfies the first condition in (3.10)} & \text{on } S_i^*\,, \\ \rho \text{ satisfies the first and second condition in (3.11)} & \text{on } L^*\,. \end{cases} \tag{4.13}$$

Here c_i are constants as in the proof of Lemma 4.2 that appear due to the condition $\int_{\Gamma_1^*} v_1 = \int_{\Gamma_2^*} v_2 = \int_{\Gamma_2^*} v_2$ for the test functions.
Since $u^{\overline{\rho}}$ and u^f lie in $\mathcal{H}^1$, regularity theory gives us $\overline{\rho} \in \mathcal{H}^3$ and the fact that the identities in (4.13) hold pointwise. Summing the first line in (4.13) leads to the third condition in (3.11), since $\sum_{i=1}^3 c_i = 0$, $\sum_{i=1}^3 u_i^{\overline{\rho}} = 0$ and $\sum_{i=1}^3 u_i^f = 0$, where the last two identities hold on L^* due to $u^{\overline{\rho}}$, $u^f \in \mathcal{Y}$. We arrive at

$$-\sum_{i=1}^{3} m_i\gamma_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\overline{\rho}_i + |\sigma_i^*|^2\overline{\rho}_i\right)\cdot\nabla_{\Gamma_i^*}\xi_i + \sum_{i=1}^{3} \omega\, m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} u_i^{\overline{\rho}} \cdot \nabla_{\Gamma_i^*}\xi_i = \sum_{i=1}^{3} m_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*} u_i^f \cdot \nabla_{\Gamma_i^*}\xi_i\,,$$

where we differentiated the first line in (4.13) and tested with $m_i\nabla_{\Gamma_i^*}\xi_i$ for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathcal{Y}$. Using (4.2) in the definition of the weak solutions $u^{\overline{\rho}}$ and u^f we can rewrite the last equation to

$$-\sum_{i=1}^{3} m_i\gamma_i \int_{\Gamma_i^*} \nabla_{\Gamma_i^*}\left(\Delta_{\Gamma_i^*}\overline{\rho}_i + |\sigma_i^*|^2\overline{\rho}_i\right)\cdot\nabla_{\Gamma_i^*}\xi_i + \sum_{i=1}^{3} \omega \int_{\Gamma_i^*} \overline{\rho}_i\,\xi_i \quad = \quad \underbrace{\sum_{i=1}^{3} \int_{\Gamma_i^*} \langle f_i, \xi_i\rangle}_{=\langle f,\xi\rangle}$$

for all $\xi \in \mathcal{Y}$. So we found a $\overline{\rho} \in \mathcal{D}(\mathcal{A})$ with $\omega\overline{\rho} - \mathcal{A}\overline{\rho} = f$ for ω large enough, which was remaining to get the assertion. □

With the help of the previous results we are able to apply standard theory of self-adjoint operators and the theory of semigroups to get the following theorem.

Theorem 4.14.

(i) The spectrum of $\mathcal{A}$ consists of countable many real eigenvalues.

(ii) The initial value problem (3.9) - (3.11) is solvable for given initial data in $\mathcal{H}^{-1}$.

(iii) The zero solution of the linearized problem (3.9) - (3.11) is asymptotically stable if and only if the largest eigenvalue of $\mathcal{A}$ is negative.

Proof. With the same abstract arguments as in [Dep10] and [GIK10] we can show the assertions with the help of Lemma 4.12 and Lemma 4.13. □.

The next lemma relates eigenvalues of $\mathcal{A}$ to the bilinear form I, so that we can formulate our linearized stability criterion.

Lemma 4.15. *Let*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$$

be the eigenvalues of $\mathcal{A}$ (taken into account the multiplicity).

(*i*) *For all $n \in \mathbb{N}$, the following description of the eigenvalues of $\mathcal{A}$ holds.*

$$\lambda_n = \inf_{W \in \Sigma_{n-1}} \sup_{\rho \in W \setminus \{0\}} -\frac{I(\rho,\rho)}{(\rho,\rho)_{-1}},$$
$$-\lambda_n = \sup_{W \in \Sigma_{n-1}} \inf_{\rho \in W^{\perp} \setminus \{0\}} \frac{I(\rho,\rho)}{(\rho,\rho)_{-1}},$$

where Σ_n is the collection of n-dimensional subspaces of $\mathcal{E}$ and $W^{\perp}$ is the orthogonal complement with respect to the $(.,.)_{-1}$- inner product.

(*ii*) *The eigenvalues λ_n depend continuously on $S(n_i^*, n_i^*)$, $\kappa_{n_{\partial\Gamma_i^*}}$ and $|\sigma_i^*|$ in the L^∞-norm.*

Proof. As in [Dep10] and [GIK10], for the first part we just refer to the classical work of Courant and Hilbert [CH68]. The second part follows directly from the structure of I. □

Lemma 4.16. *For the largest eigenvalue λ_1 of $\mathcal{A}$ we have the description*

$$-\lambda_1 = \min_{\rho \in \mathcal{E} \setminus \{0\}} \frac{I(\rho,\rho)}{(\rho,\rho)_{-1}}, \tag{4.14}$$

which can be seen directly from the second description of λ_1 in Lemma 4.15 through $-\lambda_1 = \sup_{W \in \Sigma_0} \inf_{\rho \in W^{\perp} \setminus \{0\}} \frac{I(\rho,\rho)}{(\rho,\rho)_{-1}}$ and $\Sigma_0 = \emptyset$ and therefore $W^{\perp} = \mathcal{E}$. The fact that the minimum is attained also follows from the classical work of Courant and Hilbert [CH68].

From Theorem 4.14 we have asymptotic stability of the linearized problem (3.9) - (3.11) if and only if $\lambda_1 < 0$. This leads to the following main conclusion of the final section.

Theorem 4.17. *The linearized problem (3.9) - (3.11) is asymptotically stable if and only if*

$$I(\rho,\rho) \quad > \quad 0$$

for all $\rho \in \mathcal{E} \setminus \{0\}$, where

$$\begin{aligned} I(\rho,\rho) &:= \sum_{i=1}^{3} \gamma_i \int_{\Gamma_i^*} \left(|\nabla_{\Gamma_i^*} \rho_i|^2 - |\sigma_i^*|^2 \rho_i^2 \right) \mathrm{d}\mathcal{H}^n - \sum_{i=1}^{3} \gamma_i \int_{S_i^*} S(n_i^*, n_i^*) \rho_i^2 \, \mathrm{d}\mathcal{H}^{n-1} \\ &+ \int_{L^*} \frac{\gamma_1}{s_1} \left(c_2 \kappa_{n_{\partial\Gamma_2^*}} - c_3 \kappa_{n_{\partial\Gamma_3^*}} \right) \rho_1^2 \, \mathrm{d}\mathcal{H}^{n-1} + \int_{L^*} \frac{\gamma_2}{s_2} \left(c_3 \kappa_{n_{\partial\Gamma_3^*}} - c_1 \kappa_{n_{\partial\Gamma_1^*}} \right) \rho_2^2 \, \mathrm{d}\mathcal{H}^{n-1} \\ &+ \int_{L^*} \frac{\gamma_3}{s_3} \left(c_1 \kappa_{n_{\partial\Gamma_1^*}} - c_2 \kappa_{n_{\partial\Gamma_2^*}} \right) \rho_3^2 \, \mathrm{d}\mathcal{H}^{n-1}. \end{aligned}$$

For this time we wrote out the corresponding terms for the abbreviations a_i.

5 Examples

Without the outer fixed boundary, the bilinear form from Theorem 4.17 is the same as in the proof of the double bubble conjecture by Hutchings et al. [HMRR02] for surfaces in $\mathbb{R}^3$ meeting at a triple line with an angle of 120 degree. We remind that in this paper it is shown that the so called non-standard double bubble is not stable, a fact which is also derived numerically in the work of Barrett, Garcke and Nürnberg [BGN09]. Stability holds for the so called standard double bubble, which is the main conclusion of [HMRR02].

Now we want to discuss an example and we will specify a region Ω together with three hypersurfaces Γ_i^*, which are a stationary solution of problem (2.1)-(2.3). The hypersurfaces will have mean curvature zero, so that we can determine a characteristic behaviour concerning the linearized stability for a related geometric problem, the so called mean curvature flow with triple lines and outer boundary contact. In the work of the first author [Dep10] also this problem was considered and an analogue conclusion as Theorem 4.17 was derived. This states that the same bilinear form is positive but now for functions which just fulfill the identity $\rho_1 + \rho_2 + \rho_3 = 0$ at the triple line L^* without the integral constraints.

For $l > 0$ and $\overline{u} = \ln\sqrt{3}$ let Ω be the cylinder $\Omega = B_l(0) \times (-\overline{u}, \overline{u}) \subset \mathbb{R}^3$. As stationary states of the problem (2.1)-(2.3) we consider three hypersurfaces Γ_1^*, Γ_2^* and Γ_3^* lying inside Ω which touch the boundary $\partial\Omega$ at a right angle and meet each other at a triple line with angles of $\theta_i = \frac{2}{3}\pi$. Γ_1^* and Γ_2^* are parts of catenoids and Γ_3^* is a circular ring with width $b = l - \cosh\overline{u}$ given through

$$\begin{aligned}
\Gamma_1^* &= \{(\cosh \cos v, \cosh u \sin v, \overline{u} - u) \mid u \in (0, \overline{u}),\ v \in (0, 2\pi]\}\,,\\
\Gamma_2^* &= \{(\cosh \cos v, \cosh u \sin v, u - \overline{u}) \mid u \in (0, \overline{u}),\ v \in (0, 2\pi]\} \quad \text{and}\\
\Gamma_3^* &= \{(u \cos v, u \sin v, 0) \mid u \in (\cosh\overline{u}, l),\ v \in (0, 2\pi]\}\,.
\end{aligned}$$

The triple line is then given through

$$L^* = \partial\Gamma_1^* \cap \partial\Gamma_2^* \cap \partial\Gamma_3^* = \{(\cosh\overline{u}\cos v, \cosh\overline{u}\sin v, 0) \mid v \in (0, 2\pi]\}\,,$$

which is illustrated in Figure 2.

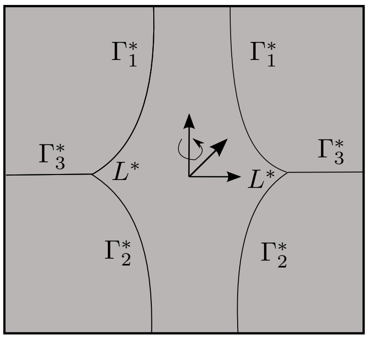

Figure 2: Example with specific geometry

To determine a characteristic behaviour concerning the linearized stability, we have to consider for $\rho = (\rho_1, \rho_2, \rho_3)$ the quadratic form from Theorem 4.17

$$I(\rho,\rho) = \sum_{i=1}^{3} \int_{\Gamma_i^*} \left(|\nabla_{\Gamma_i^*}\rho_i|^2 - |\sigma_i^*|^2\rho_i^2\right) - \sum_{i=1}^{3} \int_{S_i^*} S(n_i^*, n_i^*)\rho_i^2 + \frac{1}{\sqrt{3}} \int_{L^*} \left(\kappa_{n_{\partial\Gamma_k^*}} - \kappa_{n_{\partial\Gamma_j^*}}\right)\rho_i^2\,,$$

where $(i,j,k) = (1,2,3)$, $(2,3,1)$, $(3,1,2)$ and we already calculated the angle functions. With the indicated parametrizations F_i from the introduction of the surfaces Γ_i^* we let $\tilde{\rho}_i = \rho_i \circ F_i$ and observe for

the gradients $|\nabla_{\Gamma_i^*}\rho_i|^2 = \frac{1}{\cosh^2 u}|\nabla\tilde{\rho}_i|^2$ for $i = 1, 2$ and $|\nabla_{\Gamma_3^*}\rho_i|^2 = (\partial_u\tilde{\rho}_3)^2 + \frac{1}{u^2}(\partial_v\tilde{\rho}_3)^2$. Straightforward calculations give then $|\sigma_i^*|^2 = \frac{2}{\cosh^4 u}$ for $i = 1, 2$, $|\sigma_3^*|^2 = 0$, $S(n_i^*, n_i^*) = 0$ and for the normal curvatures of Γ_i^* in direction of $n_{\partial\Gamma_i^*}$ at the triple line $\kappa_{n_{\partial\Gamma_1^*}} = \frac{1}{\cosh^2 \overline{u}} = -\kappa_{n_{\partial\Gamma_2^*}}$ and $\kappa_{n_{\partial\Gamma_3^*}} = 0$.
So we have to consider the quadratic form

$$I(\rho,\rho) = \sum_{i=1}^{2}\int_0^{\overline{u}}\int_0^{2\pi}\left(|\nabla\tilde{\rho}_i|^2 - \frac{2}{\cosh^2 u}\tilde{\rho}_i^2\right) du dv + \int_{\cosh\overline{u}}^{l}\int_0^{2\pi}\left((\partial_u\tilde{\rho}_3)^2 + \frac{1}{u^2}(\partial_v\tilde{\rho}_3)^2\right) u\, du dv \tag{5.1}$$
$$+ \frac{1}{\sqrt{3}\cosh\overline{u}}\int_0^{2\pi}\left(\tilde{\rho}_1^2 + \tilde{\rho}_2^2 - 2(\tilde{\rho}_1 + \tilde{\rho}_2)^2\right)\big|_{u=\overline{u}}\, dv$$

for $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_3)$ with $\rho_i \in H^1(I_i \times (0, 2\pi))$ and $\tilde{\rho}_1 + \tilde{\rho}_2 + \tilde{\rho}_3 = 0$ for $u = \cosh\overline{u}$ to determine the linearized stability of the mean curvature flow problem as described above and in the thesis [Dep10].

With the specific functions $\tilde{\rho}_1 = \tilde{\rho}_2 \equiv C > 0$ and $\tilde{\rho}_3 \equiv -2C$ we observe that $I(\rho,\rho) < 0$, which means that the above geometry is not stable under mean curvature flow.

For related stability results with special outer geometry without triple lines we refer to the works about drops between parallel planes from Vogel [Vog87, Vog89] and Athanassenas [Ath87].

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