Universität Regensburg
Mathematik

Integrality of Stickelberger elements and the equivariant Tamagawa number conjecture

Andreas Nickel

Preprint Nr. 29/2011

Integrality of Stickelberger elements and the equivariant Tamagawa number conjecture

Andreas Nickel*

Abstract

Let L/K be a finite Galois CM-extension of number fields with Galois group G. In an earlier paper, the author has defined a module $SKu(L/K)$ over the center of the group ring $\mathbb{Z}G$ which coincides with the Sinnott-Kurihara ideal if G is abelian and, in particular, contains many Stickelberger elements. It was shown that a certain conjecture on the integrality of $SKu(L/K)$ implies the minus part of the equivariant Tamagawa number conjecture at an odd prime p for an infinite class of (non-abelian) Galois CM-extensions of number fields which are at most tamely ramified above p, provided that Iwasawa's μ-invariant vanishes. Here, we prove a relevant part of this integrality conjecture which enables us to deduce the equivariant Tamagawa number conjecture from the vanishing of μ for the same class of extensions.

Introduction

Let L/K be a finite Galois extension of number fields with Galois group G. Burns [Bu01] used complexes arising from étale cohomology of the constant sheaf $\mathbb{Z}$ to define a canonical element $T\Omega(L/K)$ of the relative K-group $K_0(\mathbb{Z}G, \mathbb{R})$. This element relates the leading terms at zero of Artin L-functions attached to L/K to natural arithmetic invariants. It was shown that the vanishing of $T\Omega(L/K)$ is equivalent to the equivariant Tamagawa number conjecture (ETNC) for the pair $(h^0(\mathrm{Spec}(L))(0), \mathbb{Z}G)$ (cf. loc.cit., Th. 2.4.1).
The vanishing of $T\Omega(L/K)$ is known to be true if L is absolutely abelian as proved by Burns and Greither [BG03] with the exclusion of the 2-primary part; Flach [Fl04] extended the argument to cover the 2-primary part as well. Slightly weaker results in this cyclotomic case have been settled independently by Ritter and Weiss [RW02, RW03], Huber and Kings [HK03]. Some relatively abelian results are due to Bley [Bl06]; he showed that if L/K is a finite abelian extension, where K is an imaginary quadratic field which has class number one, then the ETNC holds for all intermediate extensions L/E such that $[L : E]$ is odd and divisible only by primes which split completely in $K/\mathbb{Q}$. Finally, if L/K is a CM-extension and p is odd, the ETNC at p naturally decomposes into a plus and a minus part; it was shown by the author [Ni11a] that the minus part of the ETNC at p holds if L/K is abelian and at most tamely ramified above p, and the Iwasawa μ-invariant vanishes if p divides $|G|$ (and some additional technical condition is fulfilled). Note that the vanishing of μ is a

*I acknowledge financial support provided by the DFG
2010 Mathematics Subject Classification 11R42, 11R23
Keywords: Tamagawa number, equivariant L-values, Stickelberger elements

long standing conjecture of Iwasawa theory; the most general result is still due to Ferrero and Washington [FW79] and says that $\mu = 0$ for absolutely abelian extensions.
For non-abelian extensions, the results are even more sparse. Burns and Flach [BF03] have given a proof for an infinite class of quaternion extensions over the rationals and Navilarekallu [Na06] has treated a specific A_4-extension over $\mathbb{Q}$. If L/K is a CM extension, the author [Ni11b] has introduced a module $SKu(L/K)$ over the center of the group ring $\mathbb{Z}G$ which is a non-commutative analogue of the Sinnot-Kurihara ideal (cf. [Si80], p. 193) and was already implicitly used in [Ni] and [BJ11]. An integrality conjecture on $SKu(L/K)$ has been formulated and it was shown that it is implied by the ETNC in many cases and follows from the results in [Ba77], [Ca79], [DR80] if G is abelian. Assuming the validity of this integrality conjecture, the minus part of the ETNC at p was deduced from the conjectural vanishing of μ, provided that the ramification above p is at most tame (and, as in the abelian case, some technical extra assumption holds). Moreover, it follows from the results in [Ni] that for the case at hand the non-abelian analogues of Brumer's conjecture, of the Brumer-Stark conjecture and of the strong Brumer-Stark property (as formulated in loc.cit.) hold, provided that $\mu = 0$ and the integrality conjecture holds.
Most of these results make heavily use of the validity of the equivariant Iwasawa main conjecture (EIMC) attached to the extension L_∞^+/K, where L_∞^+ is the cyclotomic $\mathbb{Z}_p$-extension of L^+ which is the maximal real subfield of L. Note that the EIMC is known for abelian extensions of totally real number fields with Galois group $\mathcal{G}$ such that $\mathcal{G}$ is a p-adic Lie group of dimension 1 (cf. [Wi90, RW02]). More recently, Ritter and Weiss [RWa] have shown that the EIMC (up to its uniqueness statement) holds for arbitrary p-adic Lie groups of dimension 1 provided that μ vanishes. In fact, this can be generalized to higher dimensional p-adic Lie groups as shown by Kakde [Ka] and, independently, by Burns [Bub].
In this paper, we define a variant $SKu'(L/K)$ of the Sinnott-Kurihara module which is contained in $SKu(L/K)$ and in fact equals $SKu(L/K)$ for abelian G. Let $\mathcal{M}(G)$ be a maximal order in $\mathbb{Q}G$ containing $\mathbb{Z}G$; for any ring Λ, we write $\zeta(\Lambda)$ for the subring of all elements which are central in Λ. The first main result is the following theorem which will be proved in §4.

Theorem 0.1. *Let L/K be a Galois extension of number fields with Galois group G. If G is nilpotent, then*
$$SKu'(L/K) \subset \zeta(\mathcal{M}(G)).$$

Now let S and T be two finite sets of places of K such that S and T are disjoint and S contains the set S_∞ of all infinite places of K. One can associate to S and T so-called Stickelberger elements θ_S^T which lie in the center of the group ring algebra $\mathbb{Q}G$. These Stickelberger elements are defined via values of Artin L-functions at zero and are closely related to the Sinnott-Kurihara ideal $SKu(L/K)$; more precisely, they lie in $SKu(L/K)$ under suitable hypotheses on S and T. For instance, it suffices that S contains the set S_{ram} of all ramified primes and no non-trivial root of unity in L is congruent to 1 modulo all primes $\mathfrak{P} \in T(L)$; here, for any set T of places of K, we write $T(L)$ for the set of places of L which lie above the places in T.
Now assume that the Galois group G decomposes as $G = H \times C$, where H is nilpotent and C is abelian. As before, let $\mathcal{M}(H)$ be a maximal order in $\mathbb{Q}H$ containing $\mathbb{Z}H$. Then we may view $\mathcal{M}(H)[C]$ as an order in $\mathbb{Q}G$ and we have the following integrality statement for Stickelberger elements.

Theorem 0.2. *Let L/K be a Galois extension of number fields with Galois group $G = H \times C$, where H is nilpotent and C is abelian. Then*

$$\theta_S^T \in \zeta(\mathcal{M}(H)[C]) = \zeta(\mathcal{M}(H))[C]$$

for all suitable finite sets of places S and T.

We will give a more precise statement and its proof in §4. In fact, we will prove a more general result involving also Stickelberger elements which are defined via values of Artin L-functions at negative integers. Now let L/K be a Galois CM-extension of number fields with arbitrary Galois group G. We denote the maximal real subfield of L by L^+. Then Theorem 0.2 is the key in proving our main result.

Theorem 0.3. *Let L/K be a Galois CM-extension of number fields with Galois group G and let p be a non-exceptional prime. If the Iwasawa μ-invariant attached to the cyclotomic $\mathbb{Z}_p$-extension of L^+ vanishes, then the p-minus part of the ETNC for the pair $(h^0(\mathrm{Spec}(L))(0), \mathbb{Z}G)$ is true.*

For a fixed extension L/K there is only a finite number of exceptional primes; for a precise definition see §5, where we will prove Theorem 0.3. Finally, we obtain the following corollaries.

Corollary 0.4. *Let L/K be a Galois CM-extension of number fields with Galois group G and let p be a non-exceptional prime. If the Iwasawa μ-invariant attached to the cyclotomic $\mathbb{Z}_p$-extension of L^+ vanishes, then the p-part of the integrality conjecture holds.*

Corollary 0.5. *Let L/K be a Galois CM-extension of number fields with Galois group G and let p be a non-exceptional prime. If the Iwasawa μ-invariant attached to the cyclotomic $\mathbb{Z}_p$-extension of L^+ vanishes, then the p-parts of the following conjectures hold:*

(i) the non-abelian Brumer conjecture of [Ni], Conj. 2.1

(ii) the non-abelian Brumer-Stark conjecture of [Ni], Conj. 2.6

(iii) the minus part of the central conjecture (Conj. 2.4.1) of [Bua]

(iv) the minus-part of the Lifted Root Number Conjecture of Gruenberg, Ritter and Weiss [GRW99].

Moreover, L/K fulfills the non-abelian strong Brumer-Stark property at p (cf. [Ni], Def. 3.5).

Corollary 0.6. *Let L/K be a tamely ramified Galois CM-extension of number fields with Galois group G and let p be a non-exceptional prime. If the Iwasawa μ-invariant attached to the cyclotomic $\mathbb{Z}_p$-extension of L^+ vanishes, then the minus-p-part of the central conjecture (Conj. 3.3) of [BB07] is valid. If in addition Leopoldt's conjecture is true for all number fields, then the minus-p-part of the ETNC for the pair $(h^0(\mathrm{Spec}(L))(1), \mathbb{Z}G)$ is true. If $K = \mathbb{Q}$, it suffices to assume Leopoldt's conjecture for the number field L.*

1 Preliminaries

1.0.1 K-theory

Let Λ be a left noetherian ring with 1 and $\mathrm{PMod}(\Lambda)$ the category of all finitely generated projective Λ-modules. We write $K_0(\Lambda)$ for the Grothendieck group of $\mathrm{PMod}(\Lambda)$, and $K_1(\Lambda)$ for the Whitehead group of Λ which is the abelianized infinite general linear group. If S is a multiplicatively closed subset of the center of Λ which contains no zero divisors, $1 \in S$, $0 \notin S$, we denote the Grothendieck group of the category of all finitely generated S-torsion Λ-modules of finite projective dimension by $K_0S(\Lambda)$. Writing Λ_S for the ring of quotients of Λ with denominators in S, we have the following Localization Sequence (cf. [CR87], p. 65)

$$K_1(\Lambda) \to K_1(\Lambda_S) \to K_0S(\Lambda) \to K_0(\Lambda) \to K_0(\Lambda_S).$$

In the special case where Λ is an $\mathfrak{o}$-order over a commutative ring $\mathfrak{o}$ and S is the set of all nonzerodivisors of $\mathfrak{o}$, we also write $K_0T(\Lambda)$ instead of $K_0S(\Lambda)$. Moreover, we denote the relative K-group corresponding to a ring homomorphism $\Lambda \to \Lambda'$ by $K_0(\Lambda, \Lambda')$ (cf. [Sw68]). Then we have a Localization Sequence (cf. [CR87], p. 72)

$$K_1(\Lambda) \to K_1(\Lambda') \xrightarrow{\partial_{\Lambda,\Lambda'}} K_0(\Lambda, \Lambda') \to K_0(\Lambda) \to K_0(\Lambda').$$

It is also shown in [Sw68] that there is an isomorphism $K_0(\Lambda, \Lambda_S) \simeq K_0S(\Lambda)$. Let G be a finite group; in the case where Λ' is the group ring $\mathbb{R}G$, the reduced norm map $\mathrm{nr}_{\mathbb{R}G} : K_1(\mathbb{R}G) \to \zeta(\mathbb{R}G)^\times$ is injective, and there exists a canonical map $\hat{\partial}_G : \zeta(\mathbb{R}G)^\times \to K_0(\mathbb{Z}G, \mathbb{R}G)$ such that the restriction of $\hat{\partial}_G$ to the image of the reduced norm equals $\partial_{\mathbb{Z}G,\mathbb{R}G} \circ \mathrm{nr}_{\mathbb{R}G}^{-1}$. This map is called the extended boundary homomorphism and was introduced by Burns and Flach [BF01].

1.0.2 Equivariant L-values

Let us fix a finite Galois extension L/K of number fields with Galois group G. For any prime $\mathfrak{p}$ of K we fix a prime $\mathfrak{P}$ of L above $\mathfrak{p}$ and write $G_\mathfrak{P}$ resp. $I_\mathfrak{P}$ for the decomposition group resp. inertia subgroup of L/K at $\mathfrak{P}$. Moreover, we denote the residual group at $\mathfrak{P}$ by $\overline{G_\mathfrak{P}} = G_\mathfrak{P}/I_\mathfrak{P}$ and choose a lift $\phi_\mathfrak{P} \in G_\mathfrak{P}$ of the Frobenius automorphism at $\mathfrak{P}$.
If S is a finite set of places of K containing the set S_∞ of all infinite places of K, and χ is a (complex) character of G, we denote the S-truncated Artin L-function attached to χ and S by $L_S(s, \chi)$ and define $L_S^*(r, \chi)$ to be the leading coefficient of the Taylor expansion of $L_S(s, \chi)$ at $s = r$, $r \in \mathbb{Z}$. Recall that there is a canonical isomorphism $\zeta(\mathbb{C}G) = \prod_{\chi \in \mathrm{Irr}\,(G)} \mathbb{C}$, where $\mathrm{Irr}\,(G)$ denotes the set of irreducible characters of G. We define the equivariant Artin L-function to be the meromorphic $\zeta(\mathbb{C}G)$-valued function

$$L_S(s) := (L_S(s, \chi))_{\chi \in \mathrm{Irr}\,(G)}.$$

We put $L_S^*(r) = (L_S^*(r, \chi))_{\chi \in \mathrm{Irr}\,(G)}$ and abbreviate $L_{S_\infty}(s)$ by $L(s)$. If T is a second finite set of places of K such that $S \cap T = \emptyset$, we define $\delta_T(s) := (\delta_T(s, \chi))_{\chi \in \mathrm{Irr}\,(G)}$, where $\delta_T(s, \chi) = \prod_{\mathfrak{p} \in T} \det(1 - N(\mathfrak{p})^{1-s} \phi_\mathfrak{P}^{-1} | V_\chi^{I_\mathfrak{P}})$ and V_χ is a G-module with character χ. We put

$$\Theta_{S,T}(s) := \delta_T(s) \cdot L_S(s)^\sharp,$$

where we denote by $\sharp : \mathbb{C}G \to \mathbb{C}G$ the involution induced by $g \mapsto g^{-1}$. These functions are the so-called (S,T)-modified G-equivariant L-functions and we define Stickelberger elements

$$\theta_S^T(r) := \Theta_{S,T}(r) \in \zeta(\mathbb{C}G).$$

For convenience, we also put $L_S^T(s,\chi) := \delta_T(s,\check{\chi}) \cdot L_S(s,\chi)$, where we write $\check{\chi}$ for the character contragredient to χ. Thus

$$\theta_S^T(r)^\sharp = (L_S^T(r,\chi))_{\chi\in\mathrm{Irr}\,(G)}.$$

We will also write $L_S^T(L/K,s,\chi)$ for $L_S^T(s,\chi)$ if the extension L/K is not clear from the context, and similarly for $\theta_S^T(r)$. If T is empty, we abbreviate $\theta_S^T(r)$ by $\theta_S(r)$. Now a result of Siegel [Si70] implies that

$$\theta_S^T(r) \in \zeta(\mathbb{Q}G) \tag{1}$$

for all $r \le 0$. Let us fix an embedding $\iota : \mathbb{C} \rightarrowtail \mathbb{C}_p$; then the image of θ_S^T in $\zeta(\mathbb{Q}_pG)$ via the canonical embedding

$$\zeta(\mathbb{Q}G) \rightarrowtail \zeta(\mathbb{Q}_pG) = \bigoplus_{\chi\in\mathrm{Irr_p}\,(G)/\sim} \mathbb{Q}_p(\chi),$$

is given by $\sum_\chi L_S^T(r,\check{\chi}^{\iota^{-1}})^\iota$. Here, the sum runs over all $\mathbb{C}_p$-valued irreducible characters of G modulo Galois action. Note that we will frequently drop ι and ι^{-1} from the notation. Finally, for an irreducible character χ with values in $\mathbb{C}$ (resp. $\mathbb{C}_p$) we put $e_\chi = \frac{\chi(1)}{|G|}\sum_{g\in G}\chi(g^{-1})g$ which is a central idempotent in $\mathbb{C}G$ (resp. $\mathbb{C}_pG$).

1.0.3 Ray class groups

Let T and S be as above. We write cl_L^T for the ray class group of L to the ray $\mathfrak{M}_T := \prod_{\mathfrak{P}\in T(L)}\mathfrak{P}$ and $\mathfrak{o}_S$ for the ring of $S(L)$-integers of L. We denote the $S(L)$-units of L by E_S and define $E_S^T := \{x \in E_S : x \equiv 1 \bmod \mathfrak{M}_T\}$. If $S = S_\infty$, we also write E_L^T for $E_{S_\infty}^T$. All these modules are equipped with a natural G-action. Now let L/K be a Galois CM-extension, i.e. L is a CM-field, K is totally real and complex conjugation induces a unique automorphism j of L which lies in the center of G. If R is a subring of either $\mathbb{C}$ or $\mathbb{C}_p$ for a prime p such that 2 is invertible over R, we put $RG_- := RG/(1+j)$ which is a ring, since the idempotent $\frac{1-j}{2}$ lies in RG. For any RG-module M we define $M^- = RG_- \otimes_{RG} M$ which is an exact functor since $2 \in R^\times$. We define

$$A_L^T := (\mathbb{Z}[\tfrac{1}{2}] \otimes_\mathbb{Z} \mathrm{cl}_L^T)^-.$$

If M is a finitely generated $\mathbb{Z}$-module and p is a prime, we put $M(p) := \mathbb{Z}_p \otimes_\mathbb{Z} M$. For odd primes p, we will in particular consider $A_L^T(p)$, the minus p-part of the ray class group cl_L^T.

2 The integrality conjectures

Let L/K be a Galois extension with Galois group G. Let S and T be two finite sets of places of K such that

(i) S contains all the infinite places of K and all the places which ramify in L/K, i.e. $S \supset S_{\mathrm{ram}} \cup S_\infty$.

(ii) $S \cap T = \emptyset$.

(iii) E_S^T is torsionfree.

We refer to the above hypotheses as $Hyp(S,T)$. For a fixed set S we define $\mathfrak{A}_S$ to be the $\zeta(\mathbb{Z}G)$-submodule of $\zeta(\mathbb{Q}G)$ generated by the elements $\delta_T(0)$, where T runs through the finite sets of places of K such that $Hyp(S,T)$ is satisfied. Note that $\mathfrak{A}_S$ equals the $\mathbb{Z}G$-annihilator of the roots of unity of L if G is abelian by [Ta84], Lemma 1.1, p. 82.
For any finite group H we put $N_H := \sum_{h \in H} h$. For a finite prime $\mathfrak{p}$ of K, we define a $\mathbb{Z}G_{\mathfrak{P}}$-module $U_{\mathfrak{p}}$ by

$$U_{\mathfrak{p}} := \langle N_{I_{\mathfrak{P}}}, 1 - \varepsilon_{\mathfrak{p}} \phi_{\mathfrak{P}}^{-1} \rangle_{\mathbb{Z}G_{\mathfrak{P}}} \subset \mathbb{Q}G_{\mathfrak{P}},$$

where $\varepsilon_{\mathfrak{p}} = |I_{\mathfrak{P}}|^{-1} N_{I_{\mathfrak{P}}}$. Note that $U_{\mathfrak{p}} = \mathbb{Z}G_{\mathfrak{P}}$ if $\mathfrak{p}$ is unramified in L/K such that the definition of the following $\zeta(\mathbb{Z}G)$-module is indeed independent of the set S as long as S contains the ramified primes:

$$U := \langle \prod_{\mathfrak{p} \in S \setminus S_\infty} \mathrm{nr}(u_{\mathfrak{p}}) | u_{\mathfrak{p}} \in U_{\mathfrak{p}} \rangle_{\zeta(\mathbb{Z}G)} \subset \zeta(\mathbb{Q}G).$$

Definition 2.1. *Let S be a finite set of primes which contains $S_{\mathrm{ram}} \cup S_\infty$. We define a $\zeta(\mathbb{Z}G)$-module by*

$$SKu(L/K,S) := \mathfrak{A}_S \cdot U \cdot L(0)^\sharp \subset \zeta(\mathbb{Q}G).$$

We call $SKu(L/K) := SKu(L/K, S_{\mathrm{ram}} \cup S_\infty)$ the (fractional) Sinnott-Kurihara ideal.

For abelian G, this definition coincides with the Sinnott-Kurihara ideal $SKu(L/K)$ in [Gr07] (see also [Si80], p. 193).
Let $\mathcal{I}(G)$ be the $\zeta(\mathbb{Z}G)$-module generated by the elements $\mathrm{nr}(H)$, $H \in M_{n \times n}(\mathbb{Z}G)$, $n \in \mathbb{N}$. Actually, $\mathcal{I}(G)$ is a commutative ring and we have inclusions

$$\zeta(\mathbb{Z}G) \subset \mathcal{I}(G) \subset \zeta(\mathfrak{M}(G)),$$

where $\mathfrak{M}(G)$ is a maximal order in $\mathbb{Q}G$ containing $\mathbb{Z}G$. The integrality conjecture as formulated in [Ni11b] (where L/K is assumed to be a CM-extension; but we will not assume this here) now asserts the following:

Conjecture 2.2. *The Sinnott-Kurihara ideal $SKu(L/K)$ is contained in $\mathcal{I}(G)$.*

Remark 2.3. *(i) Since clearly $SKu(L/K,S) \subset SKu(L/K,S')$ if $S' \subset S$, Conjecture 2.2 implies $SKu(L/K,S) \subset \mathcal{I}(G)$ for all admissible sets S.*

(ii) If the sets S and T satisfy $Hyp(S,T)$, the Stickelberger element $\theta_S^T(0)$ is contained in $SKu(L/K,S)$. Hence Conjecture 2.2 predicts that $\theta_S^T(0) \in \mathcal{I}(G)$ which is part of [Ni], Conjecture 2.1.

(iii) In the above definitions, we may replace $\mathbb{Z}$ and $\mathbb{Q}$ by $\mathbb{Z}_p$ and $\mathbb{Q}_p$, respectively. We obtain a local Sinnott-Kurihara ideal $SKu_p(L/K)$ contained in $\zeta(\mathbb{Q}_pG)$ and a $\zeta(\mathbb{Z}_pG)$-module $\mathcal{I}_p(G)$. Since we have an equality

$$\mathcal{I}(G) = \bigcap_p \mathcal{I}_p(G) \cap \zeta(\mathbb{Q}G),$$

we have an equivalence

$$SKu(L/K) \subset \mathcal{I}(G) \iff SKu_p(L/K) \subset \mathcal{I}_p(G)\ \forall p.$$

If G is abelian, we obviously have $\mathcal{I}(G) = \zeta(\mathbb{Z}G) = \mathbb{Z}G$ and the results in [Ba77], [Ca79], [DR80] each imply the following theorem (cf. [Gr07], §2).

Theorem 2.4. *Conjecture 2.2 holds if L/K is an abelian extension.*

We also define a modified version of the Sinnott-Kurihara ideal as follows. For a finite prime $\mathfrak{p}$ of K, define a $\zeta(\mathbb{Z}G_{\mathfrak{P}})$-module $U'_{\mathfrak{p}}$ by

$$U'_{\mathfrak{p}} := \langle \mathrm{nr}(N_{I_{\mathfrak{P}}}), \mathrm{nr}(1 - \varepsilon_{\mathfrak{p}}\phi_{\mathfrak{P}}^{-1})\rangle_{\zeta(\mathbb{Z}G_{\mathfrak{P}})} \subset \zeta(\mathbb{Q}G_{\mathfrak{P}}).$$

If S contains $S_{\mathrm{ram}} \cup S_\infty$, we define

$$U' := \prod_{\mathfrak{p}\in S\setminus S_\infty} U'_{\mathfrak{p}},\ SKu'(L/K,S) := \mathfrak{A}_S \cdot U' \cdot L(0)^\sharp \subset SKu(L/K,S).$$

As before, the definition of U' does not depend on S and all the above remarks remain true if we replace $SKu(L/K,S)$ by $SKu'(L/K,S)$ throughout. We put $SKu'(L/K) := SKu'(L/K, S_{\mathrm{ram}} \cup S_\infty)$. If G is abelian, the reduced norm is just the identity on $\mathbb{Q}G$. Moreover, $\mathfrak{A}_S$ is the whole $\mathbb{Z}G$-annihilator of μ_L, the roots of unity in L, and hence independent of S. This implies the following proposition.

Proposition 2.5. *If L/K is an abelian extension, then*

$$SKu(L/K) = SKu(L/K,S) = SKu'(L/K,S) \subseteq \mathbb{Z}G$$

for all admissible sets S.

Now let $r < 0$ be an integer. We denote the absolute Galois group of L by G_L and put $\mu_{1-r}(L) := (\mathbb{Q}/\mathbb{Z})(1-r)^{G_L}$.

Conjecture 2.6. *Let L/K be a Galois extension of number fields with Galois group G and let $r < 0$. Then for any $x \in \mathrm{Ann}_{\mathbb{Z}G}(\mu_{1-r}(L))$ one has*

$$\mathrm{nr}(x) \cdot \theta_S(r) \in \mathcal{I}(G)$$

for all finite sets S of primes of K containing $S_{\mathrm{ram}} \cup S_\infty$.

Remark 2.7. *(i) As above, it suffices to consider the case, where $S = S_{\mathrm{ram}} \cup S_\infty$.*

(ii) *Since the* $\mathbb{Z}G$*-annihilator of* $\mu_{1-r}(L)$ *is generated by the elements* $\prod_{\mathfrak{p}\in T}(1-\phi_{\mathfrak{P}}N(\mathfrak{p})^{r-1})$, *where* T *runs through all finite sets of primes in* K *such that* $Hyp(S,T)$ *is satisfied (cf. [Co77]), Conjecture 2.6 in particular implies that* $\theta_S^T(r)\in\mathcal{I}(G)$ *for all finite sets of primes* S *and* T *such that* $Hyp(S,T)$ *holds.*

(iii) *Note that Conjecture 2.6 outside its* 2*-primary part implicitly is a part of [Ni11c], Conj. 2.11 if* L/K *is a CM-extension (resp. an extension of totally real fields) for even (resp. odd)* r.

Again, the results in [Ba77], [Ca79], [DR80] each imply the following theorem.

Theorem 2.8. *Conjecture 2.6 holds if* L/K *is an abelian extension.*

3 A reduction step

In order to prove one of the conjectures of the preceding paragraph, we may henceforth assume that the field K is totally real, as otherwise $\theta_{S_\infty}(r) = L(r)^\sharp = 0$; hence also $SKu(L/K) = 0$ and $\theta_S(r) = 0$ for all finite sets S containing S_∞. By the same reason, we may assume that L is totally complex if r is even. Note that we actually have to exclude the special case, where $r = 0$ and $L/\mathbb{Q}$ is a CM-extension of degree 2. But in this case the occurring Galois group has to be abelian and everything is known by Theorem 2.4.
Let us denote the set of complex places of L by $S_\mathbb{C}(L)$. For any $w\in S_\mathbb{C}(L)$, the decomposition group G_w is cyclic of order two and we denote its generator by j_w. If r is even we define

$$H = H(r) := \langle j_w \cdot j_{w'} \mid w, w' \in S_\mathbb{C}(L)\rangle.$$

If r is odd, we define

$$H = H(r) := \langle j_w \mid w \in S_\mathbb{C}(L)\rangle.$$

In both cases, H is normal in G such that the fixed field L^H is a Galois extension of K with Galois group $\overline{G} := G/H$. Note that L^H/K is a Galois CM-extension if r is even, whereas L^H/K is a Galois extension of totally real fields if r is odd.

Proposition 3.1. *Let* L/K *be a Galois extension of number fields with Galois group* G *and let* p *be an odd prime. Assume that* G *has a unique* 2*-Sylow subgroup. Then the* p*-part of Conjecture 2.2 (resp. Conjecture 2.6) is true for* L/K *if and only if the* p*-part of Conjecture 2.2 (resp. Conjecture 2.6) is true for* L^H/K.

Proof. Since H is normal in G, the group ring element $\varepsilon_H := |H|^{-1}N_H$ is a central idempotent in $\mathbb{Q}_pG$. Let G_2 be the unique 2-Sylow subgroup of G. Then j_w lies in G_2 for any $w\in S_\mathbb{C}(L)$ such that H is a finite 2-group. Since p is odd, this implies that ε_H is actually in $\mathbb{Z}_pG$. Now $\mathbb{Z}_pG$ decomposes into $\mathbb{Z}_pG = \varepsilon_H\mathbb{Z}_pG \oplus (1-\varepsilon_H)\mathbb{Z}_pG$ and the canonical epimorphism $\pi : \mathbb{Z}_pG \twoheadrightarrow \mathbb{Z}_p\overline{G}$ induces an isomorphism $\pi : \varepsilon_H\mathbb{Z}_pG \simeq \mathbb{Z}_p\overline{G}$. But by the definition of H, we have $L_{S_\infty}(r,\chi) = 0$ if H is not contained in the kernel of the irreducible character χ. That means that $SKu(L/K) = \varepsilon_H \cdot SKu(L/K)$ which identifies with $SKu(L^H/K)$ via π. Using $\mu_{1-r}(L)^H = \mu_{1-r}(L^H)$, a similar observation holds in the case $r < 0$. $\square$

Remark 3.2. *Note that Proposition 3.1 in particular applies if* G *is nilpotent.*

4 Integrality of Stickelberger elements

The aim of this section is to prove Theorem 0.1 and Theorem 0.2. We have to make precise what we mean by suitable sets S and T. For this, we introduce the following terminology. If p is a prime, we denote by S_p the set of p-adic places of K. If $r = 0$, we will say that S and T are $(p, 0)$-*admissible* if the following conditions are satisfied:

(i) The union of S and T contains all non-p-adic ramified primes, i.e. $S_{\mathrm{ram}} \setminus (S_{\mathrm{ram}} \cap S_p) \subset S \cup T$,

(ii) S contains all wildly ramified primes in S_p,

(iii) $S \cap T = \emptyset$,

(iv) $E_S^{T_{\mathrm{nr}}}$ is torsionfree, where T_{nr} denotes the set of all unramified primes in T.

If $r < 0$, we will say that S and T are (p, r)-*admissible* if $Hyp(S, T)$ is satisfied. Note that S and T are in fact (p, r)-admissible for all primes p and all $r \leq 0$ if $Hyp(S, T)$ is satisfied.

Now assume that L/K is a finite Galois extension of number fields with Galois group G, where G decomposes as $G = H \times C$ with H nilpotent and C abelian. As in the introduction let $\mathcal{M}(H)$ (resp. $\mathcal{M}_p(H)$) be a maximal order in $\mathbb{Q}H$ (resp. $\mathbb{Q}_pH$) containing $\mathbb{Z}H$ (resp. $\mathbb{Z}_pH$). Then we may view $\mathcal{M}(H)[C]$ (resp. $\mathcal{M}_p(H)[C]$) as an order in $\mathbb{Q}G$ (resp. $\mathbb{Q}_pG$) and we have the following precise version of Theorem 0.2.

Theorem 4.1. *Let L/K be a finite Galois extension of number fields with Galois group $G = H \times C$, where H is nilpotent and C is abelian. Let p be a prime and $r \in \mathbb{Z}_{\leq 0}$. If S and T are two finite sets of primes of K which are (p, r)-admissible, then*

$$\theta_S^T(r) \in \zeta(\mathcal{M}_p(H)[C]) = \zeta(\mathcal{M}_p(H))[C].$$

In particular, if $Hyp(S, T)$ is satisfied, we have

$$\theta_S^T(r) \in \zeta(\mathcal{M}(H)[C]) = \zeta(\mathcal{M}(H))[C].$$

Proof. We first assume that $G = C$ is abelian. Then $\zeta(\mathcal{M}(H))[C] = \mathbb{Z}G$ and the assertion follows easily from Theorem 2.4 if $r = 0$ and from Theorem 2.8 if $r < 0$ as long as $Hyp(S, T)$ is satisfied. We are left with the case, where $r = 0$ and S and T are $(p, 0)$-admissible. We claim that $\theta_S^T(0)$ lies in $SKu_p(L/K)$ and hence Theorem 2.4 again implies the desired result. To see this, we write $\theta_S^T(0)$ as

$$\theta_S^T(0) = \delta_{T_{\mathrm{nr}}}(0) \cdot \prod_{\mathfrak{p} \in T \setminus T_{\mathrm{nr}}} \delta_{\mathfrak{p}}(0) \prod_{\mathfrak{p} \in S \setminus S_\infty} (1 - \varepsilon_{\mathfrak{p}} \phi_{\mathfrak{P}}^{-1}) \cdot L(0)^\sharp.$$

The set T_{nr} satisfies $Hyp(T_{\mathrm{nr}}, S_{\mathrm{ram}})$ by condition (iv) such that $\delta_{T_{\mathrm{nr}}}(0)$ lies in $\mathfrak{A}_{S_{\mathrm{ram}}}$. Let $\mathfrak{p} \in T \setminus T_{\mathrm{nr}}$ and $q \in \mathbb{Z}$ be the rational prime below $\mathfrak{p}$. If we denote the q-Sylow subgroup of the inertia group $I_{\mathfrak{P}}$ by $R_{\mathfrak{P}}$, the intermediate extension corresponding to $G_{\mathfrak{P}}/R_{\mathfrak{P}}$ is tame above $\mathfrak{p}$. Note that condition (ii) forces that $R_{\mathfrak{P}} = 1$ if $q = p$. Since G is abelian, by

[Ch85], p.369 the ramification index $e_{\mathfrak{p}} = |I_{\mathfrak{P}}|$ divides $q_{\mathfrak{p}} - 1$ exactly if $q = p$ and up to a power of q if $q \neq p$. Hence

$$\delta_{\mathfrak{p}}(0) = 1 - \varepsilon_{\mathfrak{p}} \phi_{\mathfrak{P}}^{-1} q_{\mathfrak{p}} = 1 - \varepsilon_{\mathfrak{p}} \phi_{\mathfrak{P}}^{-1} - \phi_{\mathfrak{P}}^{-1} \frac{q_{\mathfrak{p}} - 1}{e_{\mathfrak{p}}} N_{I_{\mathfrak{P}}} \in \mathbb{Z}_p \otimes U_{\mathfrak{p}}.$$

For the tamely ramified primes above p the element

$$e_{\mathfrak{p}} = (e_{\mathfrak{p}} - N_{I_{\mathfrak{P}}})(1 - \varepsilon_{\mathfrak{p}} \phi_{\mathfrak{P}}^{-1}) + N_{I_{\mathfrak{P}}} \in U_{\mathfrak{p}}$$

lies in $\mathbb{Z}_p^{\times}$, since $p \nmid e_{\mathfrak{p}}$. Therefore, we get $\mathbb{Z}_p \otimes U_{\mathfrak{p}} = \mathbb{Z}_p G_{\mathfrak{P}}$ in this case. Finally, we obviously have $(1 - \varepsilon_{\mathfrak{p}} \phi_{\mathfrak{P}}^{-1}) \in U_{\mathfrak{p}}$ for the primes $\mathfrak{p} \in S \setminus S_{\infty}$.
We now treat the general case, where $G = H \times C$. Since we have to deal with Stickelberger elements corresponding to various subextensions of L/K, we will write $\theta_S^T(L/K, r)$ for $\theta_S^T(r)$ for clarity. Any irreducible character of G may be written as $\chi \cdot \lambda$, where $\chi \in \mathrm{Irr}\,(H)$ and $\lambda \in \mathrm{Irr}\,(C)$. We have the following decomposition

$$\zeta(\mathbb{Q}_p[H \times C]) = \bigoplus_{\chi \in \mathrm{Irr}_p\,(H)/\sim} \mathbb{Q}_p(\chi)[C],$$

where the sum runs over all irreducible characters of H modulo Galois action and $\mathbb{Q}_p(\chi) := \mathbb{Q}_p(\chi(h) | h \in H)$. We fix an irreducible character χ of H. Then the image of $\theta_S^T(L/K, r)^{\sharp}$ in the χ-component of the above decomposition is given by

$$\sum_{\lambda \in \mathrm{Irr}_p\,(C)} L_S^T(L/K, \chi \cdot \lambda, r) e_{\lambda} \in \mathbb{Q}_p(\chi)[C] \tag{2}$$

and we wish to show that it actually lies in $\mathbb{Z}_p(\chi)[C]$, where $\mathbb{Z}_p(\chi)$ denotes the ring of integers in $\mathbb{Q}_p(\chi)$. Since H is a finite nilpotent group, [CR81], Th. 11.3 implies that there is a subgroup U of H and a linear character ψ of U such that χ is induced by ψ, i.e. $\chi = \mathrm{ind}_U^H \psi$. Let us denote the abelianization of U by U^{ab}; as ψ is linear, it is inflated by a character ψ^{ab} of U^{ab} and hence $\chi = \mathrm{ind}_U^H \mathrm{infl}_{U^{\mathrm{ab}}}^U \psi^{\mathrm{ab}}$. Note that ψ^{ab} is irreducible, since χ is. Moreover, if λ is an irreducible character of C, we have

$$\chi \cdot \lambda = (\mathrm{ind}_U^H \psi) \cdot \lambda = \mathrm{ind}_{U \times C}^G (\psi \cdot \lambda) = \mathrm{ind}_{U \times C}^G \mathrm{infl}_{U^{\mathrm{ab}} \times C}^{U \times C} (\psi^{\mathrm{ab}} \cdot \lambda).$$

But general properties of L-functions imply that

$$\sum_{\lambda \in \mathrm{Irr}_p\,(C)} L_S^T(L/K, \chi \cdot \lambda, r) e_{\lambda} = \sum_{\lambda \in \mathrm{Irr}_p\,(C)} L_S^T(L^{[U,U]}/L^{U \times C}, \psi \cdot \lambda, r) e_{\lambda},$$

where $[U, U]$ denotes the commutator subgroup of U. But the righthand side lies in $\mathbb{Z}_p(\psi)[C]$, since it is the ψ-component of the Stickelberger element $\theta_{S'}^{T'}(L^{[U,U]}/L^{U \times C}, r)^{\sharp}$ attached to the *abelian* subextension $L^{[U,U]}/L^{U \times C}$; here, $S' = S(L^{U \times C})$ and similarly for T'. This and (2) implies that

$$\sum_{\lambda \in \mathrm{Irr}_p\,(C)} L_S^T(L/K, \chi \cdot \lambda, r) e_{\lambda} \in \mathbb{Q}_p(\chi)[C] \cap \mathbb{Z}_p(\psi)[C] = \mathbb{Z}_p(\chi)[C]$$

as desired. In particular, if $Hyp(S,T)$ is satisfied, then $\theta_S^T(r)$ is (p,r)-admissible for all primes p, and hence

$$\theta_S^T(r) \in \bigcap_p \zeta(\mathcal{M}_p(H))[C] \cap \zeta(\mathbb{Q}H)[C] = \zeta(\mathcal{M}(H))[C].$$

$\square$

Now let J be a subset of S_{ram} and put $S_J := S_\infty \cup S \setminus J$. Let $K \subset L_J \subset L$ be the maximal subfield of L which is unramified outside S_J. Then L_J/K is a Galois extension with Galois group $G_J = G/H_J$, where $H_J = \mathrm{Gal}(L/L_J)$. We have the following stronger version of Theorem 0.1.

Theorem 4.2. *Let L/K be a Galois extension of number fields with Galois group G and let J be a subset of S_{ram}. If the Galois group G_J of the subextension L_J/K is nilpotent, then*

$$\prod_{\mathfrak{p}\in J} \mathrm{nr}(N_{I_{\mathfrak{P}}}) \cdot \theta_{S_J}^T(r) \in \zeta(\mathcal{M}(G)),$$

whenever $r \leq 0$ and $Hyp(S_{\mathrm{ram}} \cup S_\infty, T)$ is satisfied. In particular,

$$SKu'(L/K) \subset \zeta(\mathcal{M}(G))$$

if G is nilpotent.

Proof. Since H_J is normal in G, the idempotent $|H_J|^{-1}N_{H_J}$ is central in $\mathbb{Q}G$ and lies in $\mathcal{M}(G)$. If χ is an irreducible character of G, the χ-component of $\prod_{\mathfrak{p}\in J} \mathrm{nr}(N_{I_{\mathfrak{P}}})$ is zero if H_J is not contained in the kernel of χ. Hence we have an equality

$$\prod_{\mathfrak{p}\in J} \mathrm{nr}(N_{I_{\mathfrak{P}}}) \cdot \theta_{S_J}^T(L/K,r) = \prod_{\mathfrak{p}\in J} \mathrm{nr}(N_{I_{\mathfrak{P}}}) \cdot |H_J|^{-1}N_{H_J} \cdot \tilde{\theta}_{S_J}^T(L_J/K,r), \tag{3}$$

where $\tilde{\theta}_{S_J}^T(L_J/K,r)$ denotes any lift of $\theta_{S_J}^T(L_J/K,r)$ in $\zeta(\mathcal{M}(G))$; note that this is possible, since $\theta_{S_J}^T(L_J/K,r)$ lies in $\zeta(\mathcal{M}(G_J))$ by Theorem 4.1 as $Hyp(S_J,T)$ is satisfied for L_J/K. Hence the righthand side of the above equation also lies in $\zeta(\mathcal{M}(G))$. The second part of the theorem is clear by the definition of $SKu'(L/K)$. $\square$

Corollary 4.3. *Let L/K be a Galois extension of number fields with Galois group G. Then*

$$\prod_{\mathfrak{p}\in S_{\mathrm{ram}}} \mathrm{nr}(N_{I_{\mathfrak{P}}}) \cdot \theta_{S_\infty}^T(r) \in \zeta(\mathcal{M}(G)),$$

whenever $r \leq 0$ and $Hyp(S_{\mathrm{ram}} \cup S_\infty, T)$ is satisfied.

Proof. We may apply Theorem 4.2 for $J = S_{\mathrm{ram}}$, since the extension $L_{S_{\mathrm{ram}}}/K$ is unramified and hence contained in the Hilbert class field of K. This implies that $L_{S_{\mathrm{ram}}}/K$ is abelian and thus nilpotent. $\square$

Corollary 4.4. *Let L/K be an abelian Galois extension of number fields with Galois group G. Then*

$$\prod_{\mathfrak{p}\in J} \mathrm{nr}(N_{I_{\mathfrak{P}}}) \cdot \theta_{S_J}^T(r) \in \mathbb{Z}G,$$

whenever $r \leq 0$ and $Hyp(S_{\mathrm{ram}} \cup S_\infty, T)$ is satisfied.

Proof. If G is abelian, the righthand side of equation (3) equals

$$\prod_{\mathfrak{p}\in J} N_{I_{\mathfrak{P}}} \cdot |H_J|^{-1} N_{H_J} \cdot \tilde{\theta}_{S_J}^{T}(L_J/K, r) = z \cdot N_{H_J} \cdot \tilde{\theta}_{S_J}^{T}(L_J/K, r),$$

where $z = |H_J|^{-1} \cdot \prod_{\mathfrak{p}\in J} |I_{\mathfrak{P}}|$ is an integer. The assertion follows, since $\theta_{S_J}^{T}(L_J/K, r)$ lies in $\mathbb{Z}G_J$ by Theorem 2.8. □

Remark 4.5. *These results may tempt us to state a conjecture similar to Conjecture 2.2 also in case $r < 0$. We have not done so, since the author is not aware of a convincing reason, why this should be true in general.*

5 The ETNC in almost tame extensions

Let us fix a finite Galois extension L/K of number fields with Galois group G and a finite set S of places of K which contains $S_{\mathrm{ram}} \cup S_\infty$. In [Bu01] the author defines the following element of $K_0(\mathbb{Z}G, \mathbb{R})$:

$$T\Omega(L/K, 0) := \psi_G^*(\chi_{G,\mathbb{R}}(\tau_S, \lambda_S^{-1}) + \hat{\partial}_G(L_S^*(0)^\sharp)).$$

Here, ψ_G^* is a certain involution on $K_0(\mathbb{Z}G, \mathbb{R})$ which is not important for our purposes, since we will be only interested in the nullity of $T\Omega(L/K, 0)$. Furthermore, $\tau_S \in \mathrm{Ext}_G^2(E_S, \Delta S)$ is Tate's canonical class (cf. [Ta66]), where ΔS is the kernel of the augmentation map $\mathbb{Z}S(L) \twoheadrightarrow \mathbb{Z}$ which maps each $\mathfrak{P} \in S(L)$ to 1. Finally, λ_S denotes the negative of the usual Dirichlet map, so $\lambda_S : \mathbb{R} \otimes E_S \to \mathbb{R} \otimes \Delta S$, $u \mapsto -\sum_{\mathfrak{P}\in S(L)} \log |u|_{\mathfrak{P}} \mathfrak{P}$, and $\chi_{G,\mathbb{R}}(\tau_S, \lambda_S^{-1})$ is the refined Euler characteristic associated to the perfect 2-extension whose extension class is τ_S, metrised by λ_S^{-1}. For more precise definitions we refer the reader to [Bu01]. The ETNC for the motive $h^0(\mathrm{Spec}(L))$ with coefficients in $\mathbb{Z}G$ in this context asserts that the element $T\Omega(L/K, 0)$ is zero. Note that this statement is also equivalent to the Lifted Root Number Conjecture formulated by Gruenberg, Ritter and Weiss [GRW99] (cf. [Bu01], Th. 2.3.3).

It is also proven in loc.cit. that $T\Omega(L/K, 0)$ lies in $K_0(\mathbb{Z}G, \mathbb{Q})$ if and only if Stark's conjecture holds. In this case the ETNC decomposes into local conjectures at each prime p by means of the isomorphism

$$K_0(\mathbb{Z}G, \mathbb{Q}) \simeq \bigoplus_{p\nmid\infty} K_0(\mathbb{Z}_pG, \mathbb{Q}_p).$$

Now let L/K be a Galois CM-extension and p be an odd prime. Since Stark's conjecture is known for odd characters (cf. [Ta84], Th. 1.2, p. 70), $T\Omega(L/K, 0)$ has a well defined image $T\Omega(L/K, 0)_p^-$ in $K_0(\mathbb{Z}_pG_-, \mathbb{Q}_p)$. Recall that $j \in G$ denotes complex conjugation. We will say that L/K is *almost tame* above p if j lies in $G_{\mathfrak{P}}$ for any prime $\mathfrak{p}$ of K above p which is wildly ramified in L/K.

We have the following connection to the integrality conjecture 2.2 (cf. [Ni], proof of Th. 5.1 and Cor. 5.6):

Theorem 5.1. *Let p be an odd prime and L/K a Galois CM-extension and assume that $T\Omega(L/K, 0)_p^-$ vanishes. If the p-part of the roots of unity of L is a cohomologically trivial G-module or if L/K is almost tame above p, then the p-part of Conjecture 2.2 holds, i.e. $SKu_p(L/K) \subset \mathcal{I}_p(G)$.*

Now let T consist of a prime $\mathfrak{p}_0 \nmid p$ and all finite places of K which ramify in L/K and do not lie above p; we may choose $\mathfrak{p}_0$ such that E_S^T is torsionfree for any finite set S of places of K which contains S_∞ and is disjoint to T. We have the following reformulation of [Ni11a], Th. 2 using non-commutative Fitting invariants as introduced in [Ni10]; though, we will not make much use of this notion in this paper.

Theorem 5.2. *Let p be an odd prime and L/K a Galois CM-extension which is almost tame above p. Then*

$$T\Omega(L/K,0)_p^- = 0 \iff \mathrm{Fitt}_{\mathbb{Z}_pG_-}(A_L^T(p)) = [\langle \theta_{S_1}^T(0)\rangle]_{\mathrm{nr}(\mathbb{Z}_pG_-)},$$

where S_1 denotes the set of all wildly ramified primes above p.

Note that $A_L^T(p)$ is a cohomologically trivial G-module by [Ni11a], Th. 1 and that $\theta_{S_1}^T(0)$ is $(p,0)$-admissible. For a natural number n let ζ_n be a primitive nth root of unity and let us denote the normal closure of L over $\mathbb{Q}$ by L^{cl}; note that L^{cl} is again a CM-field. To ease notation, we will call a prime p *exceptional* if at least one of the following holds:

(i) $p = 2$,

(ii) there is a prime $\mathfrak{p}$ in K above p which ramifies wildly in L and $j \notin G_{\mathfrak{P}}$, i.e. L/K is not almost tame above p,

(iii) $L^{\mathrm{cl}} \subset (L^{\mathrm{cl}})^+(\zeta_p)$.

Note that there are only finitely many exceptional primes, since such a prime has to ramify in $L^{\mathrm{cl}}/\mathbb{Q}$ or equals 2. We now prove the following theorem which is Theorem 0.3 of the introduction.

Theorem 5.3. *Let L/K be a Galois CM-extension of number fields with Galois group G and let p be a non-exceptional prime. If the Iwasawa μ-invariant attached to the cyclotomic $\mathbb{Z}_p$-extension of L^+ vanishes, then the p-minus part of the ETNC for the pair $(h^0(\mathrm{Spec}(L))(0), \mathbb{Z}G)$ is true.*

Lemma 5.4. *Let $N > 0$ be a natural number. Then there are infinitely many primes $r \in \mathbb{Z}$ such that*

(i) $r \equiv 1 \bmod p^N$.

(ii) $j \in G_{\mathfrak{R}}$ *for all primes* $\mathfrak{R}$ *in* L *above* r.

(iii) The Frobenius automorphism Frob_p *at p in* $\mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q})$ *generates* $\mathrm{Gal}(k_r/\mathbb{Q})$*, where k_r denotes the unique subfield of $\mathbb{Q}(\zeta_r)$ of degree p^N over $\mathbb{Q}$.*

Proof. This is [Ni11b], Lemma 6.5. But the proof of [Gr00], Prop. 4.1 carries over unchanged to the present situation. □

Proof of Theorem 5.3. Let $N \in \mathbb{N}$ be large and choose a prime r as in Lemma 5.4 which does not ramify in $L^{\mathrm{cl}}/\mathbb{Q}$. We put $L_N := Lk_r$, $K_N = Kk_r$ and $G_N = \mathrm{Gal}(L_N/K) = G \times C_N$, where $C_N \simeq \mathrm{Gal}(k_r/\mathbb{Q})$ is cyclic of order p^N, generated by Frob_p. Note that p is still a non-exceptional prime for L_N/K. Moreover, we define $T_N := T \cup S_r$, where

S_r denotes the set of places in K above r. The proof of [Ni11b], Th. 6.8 shows that it suffices to prove that the denominators of the Stickelberger elements $\theta_{S_1}^T(L_N/K,0)$ are bounded, independently of N (see loc.cit., Remark 3). Now assume that G is nilpotent. The Stickelberger element $\theta_{S_1}^{T_N}(L_N/K,0)$ is $(p,0)$-admissible and thus lies in $\zeta(\mathcal{M}_p(G))[C_N]$ by Theorem 4.1. But $\theta_{S_1}^T(L_N/K,0)$ differs from $\theta_{S_1}^{T_N}(L_N/K,0)$ by $\frac{1-j}{2}\delta_{S_r}(0) \in \mathrm{nr}((\mathbb{Z}_p G_N)_-^\times)$, since $j \in G_{\mathfrak{R}}$ for all $\mathfrak{R} \mid r$; more precisely, it follows from the arguments in [Ni11a], following Prop. 9 that the epimorphism $\sigma : A_{L_N}^{T_N}(p) \twoheadrightarrow A_{L_N}^T(p)$ is in fact an isomorphism. Since $\frac{1-j}{2}\delta_{S_r}(0)$ is a generator of $\mathrm{Fitt}_{(\mathbb{Z}_p G_N)_-}(\ker(\sigma))$ which is the non-commutative Fitting invariant of 0, it has to lie in $\mathrm{nr}((\mathbb{Z}_p G_N)_-^\times)$. But $\mathrm{nr}((\mathbb{Z}_p G_N)_-^\times) \subset \mathcal{I}_p(G_N) \subset \zeta(\mathcal{M}_p(G))[C_N]$ (the second inclusion follows from the proof of [Ni10], Lemma 6.6), and thus $\theta_{S_1}^T(L_N/K,0)$ lies in $\zeta(\mathcal{M}_p(G))[C_N]$ for all N. It follows that $|G| \cdot \theta_{S_1}^T(L_N/K,0) \in \mathbb{Z}_p G_N$ for all N as desired.
For arbitrary G, note that we already now that $T\Omega(L/K,0)_p^-$ is torsion, since the strong Stark conjecture holds by [Ni11a], Cor. 2. But in this case, a general induction argument (see [GRW99], §8, especially Prop. 9 and Th. 5') shows that we may assume that G is l-elementary for a prime l, i.e. G is the direct product of an l-group and a cyclic group of order prime to l. In particular, G is nilpotent and we also obtain the general case. □

We now give the proofs of the corollaries mentioned in the introduction.

Proof of Corollary 0.4. This is an immediate consequence of Theorem 5.3 and Theorem 5.1. □

Proof of Corollary 0.5. Theorem 5.3 and [Ni], Th. 5.3 implies that L/K fulfills the (non-abelian) strong Brumer-Stark property at p. This in turn implies (ii) by loc.cit. Prop. 3.8 and (i) by loc.cit. Lemma 2.9. Since the condition $L^{\mathrm{cl}} \not\subset (L^{\mathrm{cl}})^+(\zeta_p)$ forces $\zeta_p \notin L$, the p-part of the roots of unity is trivial and thus cohomologically trivial as G-module. Therefore Theorem 5.3 and [Bua], Th. 4.1.1 imply (iii). Finally, as already mentioned above, the vanishing of $T\Omega(L/K,0)$ is equivalent to the Lifted Root Number Conjecture of Gruenberg, Ritter and Weiss as formulated in [GRW99] (cf. [Bu01], Th. 2.3.3). Thus Theorem 5.3 also implies (iv). □

Proof of Corollary 0.6. The central conjecture (Conj. 2.4.1) of [BB07] states that a certain element $T\Omega(L/K,1) \in K_0(\mathbb{Z}G,\mathbb{R})$ vanishes. By loc.cit., Th. 5.2 one has an equality

$$\psi_G^*(T\Omega(L/K,0)) - T\Omega(L/K,1) = T\Omega^{loc}(L/K,1),$$

and the vanishing of the righthand side is equivalent to a conjecture of Bley and Burns [BB03] by [BB07], remark 5.4. But this conjecture is known if L/K is at most tamely ramified by [BB03], Cor. 6.3 (i). Finally, [BB10], Th. 1.1 and Cor. 1.2 imply the desired connection to the ETNC for the pair $(h^0(\mathrm{Spec}(L))(1),\mathbb{Z}G)$. □

For completeness, we include the following result which is an easy consequence of [Ni11c], Th. 4.1 and [Bub], Cor. 2.10.

Theorem 5.5. *Let L/K be a Galois extension of number fields with Galois group G and let $r < 0$. Assume that L is totally real if r is odd (resp. that L/K is CM if r is even). If p is an odd prime such that the p-part (resp. minus p-part) of the ETNC for the pair*

$(h^0(\mathrm{Spec}(L))(r), \mathbb{Z}G)$ holds, then the p-part of Conjecture 2.6 is true. In particular, this applies if the Iwasawa μ-invariant attached to the cyclotomic $\mathbb{Z}_p$-extension of L (resp. L^+) vanishes.

Corollary 5.6. *Let L/K be a Galois extension of number fields with Galois group G and let $r < 0$. Let p be an odd prime and assume that G has a unique 2-Sylow subgroup. Then the p-part of Conjecture 2.6 holds provided that the Iwasawa μ-invariant attached to the cyclotomic $\mathbb{Z}_p$-extension of the maximal real subfield of L vanishes.*

Proof. This immediately follows from Theorem 5.5 and Proposition 3.1. □

References

[Ba77] Barsky, D.: *Fonctions zêta p-adique d'une classe de rayon des corps de nombres totalement réels*, Groupe d'Etude d'Analyse Ultramétrique (1977/78), Exp. No. **16**

[Bl06] Bley, W.: *On the equivariant Tamagawa number conjecture for abelian extensions of a quadratic imaginary field*, Documenta Math. **11** (2006), 73-118

[BB03] Bley, W., Burns, D.: *Equivariant epsilon constants, discriminants and étale cohomology*, Proc. Lond. Math. Soc. **87** (2003), 545-590

[BB07] Breuning, M., Burns, D.: *Leading terms of Artin L-functions at s = 0 and s = 1*, Compos. Math. **143** (2007), 1427-1464

[BB10] Breuning, M., Burns, D.: *On Equivariant Dedekind Zeta-functions at s=1*, Doc. Math. Extra Vol.: Andrei A. Suslin's Sixtieth Birthday (2010), 119-146

[Bu01] Burns, D.: *Equivariant Tamagawa numbers and Galois module theory I*, Compos. Math. **129**, No. 2 (2001), 203-237

[Bua] Burns, D.: *On derivatives of Artin L-series*, to appear in Invent. Math., see http://www.mth.kcl.ac.uk/staff/dj_burns/newdbpublist.html

[Bub] Burns, D.: *On main conjectures in non-commutative Iwasawa theory and related conjectures*, preprint, see http://www.mth.kcl.ac.uk/staff/dj_burns/newdbpublist.html

[BF01] Burns, D., Flach, M.: *Tamagawa numbers for motives with (non-commutative) coefficients*, Doc. Math. **6** (2001), 501-570

[BF03] Burns, D., Flach, M.: *Tamagawa numbers for motives with (non-commutative) coefficients II*, Amer. J. Math. **125** (3) (2003), 475-512

[BG03] Burns, D., Greither, C.: *On the equivariant Tamagawa number conjecture for Tate motives*, Invent. Math. **153** (2003), 305-359

[BJ11] Burns, D., Johnston, H.: *A non-abelian Stickelberger Theorem*, Compos. Math. **147**, No. 1 (2011), 35-55

[Ca79] Cassou-Noguès, P.: *Valeurs aux entiers négatifs des fonctions zêta et fonctions zêta p-adiques*, Invent. Math. **51** (1979), 29-59

[Ch85] Chinburg, T.: *Exact sequences and Galois module structure*, Ann. Math. **121** (1985), 351-376

[Co77] Coates, J.: *p-adic L-functions and Iwasawa's theory*, Algebr. Number Fields (Proc. Symp. London math. Soc., Univ. Durham 1975) (1977), 269-353

[CR81] Curtis, C. W., Reiner, I.: *Methods of Representation Theory with applications to finite groups and orders*, Vol. 1, John Wiley & Sons, (1981)

[CR87] Curtis, C. W., Reiner, I.: *Methods of Representation Theory with applications to finite groups and orders*, Vol. 2, John Wiley & Sons, (1987)

[DR80] Deligne, P., Ribet, K.: *Values of abelian L-functions at negative integers over totally real fields*, Invent. Math. **59** (1980), 227-286

[FW79] Ferrero, B., Washington, L.: *The Iwasawa invariant μ_p vanishes for abelian number fields*, Ann. Math. **109** (1979), 377-395

[Fl04] Flach, M.: *The equivariant Tamagawa number conjecture: a survey.* in Burns, D., Popescu, C., Sands, J., Solomon, D. (eds.): *Stark's Conjectures: Recent work and new directions*, Papers from the international conference on Stark's Conjectures and related topics, Johns Hopkins University, Baltimore, August 5-9, 2002, Contemporary Math. **358** (2004), 79-125, updated version available at http://www.math.caltech.edu/people/flach.html

[Gr00] Greither, C.: *Some cases of Brumer's conjecture for abelian CM extensions of totally real fields*, Math. Z. **233** (2000), 515-534

[Gr07] Greither, C.: *Determining Fitting ideals of minus class groups via the Equivariant Tamagawa Number Conjecture*, Compos. Math. **143**, No. 6 (2007), 1399-1426

[GRW99] Gruenberg, K. W., Ritter, J., Weiss, A.: *A Local Approach to Chinburg´s Root Number Conjecture*, Proc. London Math. Soc. (3) **79** (1999), 47-80

[HK03] Huber, A., Kings, G.: *Bloch-Kato conjecture and main conjecture of Iwasawa theory for Dirichlet characters*, Duke Math. J. **119** (2003), 393-464

[Ka] Kakde, M.: *The main conjecture of Iwasawa theory for totally real fields*, preprint, see arXiv:1008.0142v3

[Na06] Navilarekallu, T.: *On the equivariant Tamagawa number conjecture for A_4-extensions of number fields*, J. Number Theory **121** (1) (2006), 67-89

[Ni10] Nickel, A.: *Non-commutative Fitting invariants and annihilation of class groups*, J. Algebra **323** (10) (2010), 2756-2778

[Ni11a] Nickel, A.: *On the equivariant Tamagawa number conjecture in tame CM-extensions*, Math. Z. **268** (1) (2011), 1-35

[Ni11b] Nickel, A.: *On the equivariant Tamagawa number conjecture in tame CM-extensions, II*, Compos. Math. **147** (4) (2011), 1179-1204

[Ni11c] Nickel, A.: *Leading terms of Artin L-series at negative integers and annihilation of higher K-groups*, Math. Proc. Camb. Philos. Soc. **151** (1) (2011), 1-22

[Ni] Nickel, A.: *On non-abelian Stark-type conjectures*, to appear in Ann. Ins. Fourier - see http://www.mathematik.uni-regensburg.de/Nickel/english.html

[RW02] Ritter, J., Weiss, A.: *Toward equivariant Iwasawa theory*, Manuscr. Math. **109** (2002), 131-146

[RW03] Ritter, J., Weiss, A.: *Representing $\Omega_{(l\infty)}$ for real abelian fields*, J. of Algebra and its Appl. **2** (2003), 237-276

[RWa] Ritter, J., Weiss, A.: *On the 'main conjecture' of equivariant Iwasawa theory*, to appear in J. Am. Math. Soc., see arXiv:1004.2578v2

[Si70] Siegel, C.L.: *Über die Fourierschen Koeffizienten von Modulformen*, Nachr. Akad. Wiss. Gött., II. Math.-Phys. Kl. 1970 (1970), 15-56

[Si80] Sinnott, W.: *On the Stickelberger ideal and the circular units of an abelian field*, Invent. Math. **62** (1980), 181-234

[Sw68] Swan, R.G.: *Algebraic K-theory*, Springer Lecture Notes **76** (1968)

[Ta66] Tate, J.: *The cohomology groups of tori in finite Galois extensions of number fields*, Nagoya Math. J. **27** (1966), 709-719

[Ta84] Tate, J.: *Les conjectures de Stark sur les fonctions L d'Artin en $s = 0$*, Birkhäuser, (1984)

[Wi90] Wiles, A.: *The Iwasawa conjecture for totally real fields*, Ann. Math. **131**, 493-540 (1990)

Andreas Nickel andreas.nickel@mathematik.uni-regensburg.de
Universität Regensburg, Fakultät für Mathematik, Universitätsstr. 31, 93053 Regensburg, Germany