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# A conductor formula for completed group algebras

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**Abstract**

Let $\mathfrak{o}$ be the ring of integers in a finite extension of $\mathbb{Q}_p$. If $G$ is a finite group and $\Gamma$ is a maximal order containing the group ring $\mathfrak{o}G$, Jacobinski's conductor formula gives a complete description of the central conductor of $\Gamma$ into $\mathfrak{o}G$ in terms of characters of $G$. We prove a similar result for completed group algebras $\mathfrak{o}[[G]]$, where $G$ is a $p$-adic Lie group of dimension 1. We will also discuss several consequences of this result.

## Introduction

Let $\mathfrak{o}$ be the ring of integers in a number field $K$ and consider the group ring $\mathfrak{o}G$ of a finite group $G$ over $\mathfrak{o}$. The central conductor $\mathcal{F}(\mathfrak{o}G)$ consists of all elements $x$ in the center of $\mathfrak{o}G$ such that $x\Gamma \subset \mathfrak{o}G$, where $\Gamma$ is a chosen maximal order containing $\mathfrak{o}G$, i.e.

$$\mathcal{F}(\mathfrak{o}G) = \{x \in \zeta(\mathfrak{o}G) | x\Gamma \subset \mathfrak{o}G\}.$$

Here, for any ring $\Lambda$, we write $\zeta(\Lambda)$ for the center of $\Lambda$. A result of Jacobinski [Ja66] (see also [CR81, Theorem 27.13]) gives a complete description of the central conductor in terms of the irreducible characters of $G$. More precisely, we have

$$\mathcal{F}(\mathfrak{o}G) = \bigoplus_{\chi} \frac{|G|}{\chi(1)} \mathcal{D}^{-1}(\mathfrak{o}[\chi]/\mathfrak{o}),$$

where the sum runs through all irreducible characters of $G$ modulo Galois action, and $\mathcal{D}^{-1}(\mathfrak{o}[\chi]/\mathfrak{o})$ denotes the inverse different of $\mathfrak{o}[\chi]$, the ring of integers in $K(\chi) := K(\chi(g)|g \in G)$, with respect to $\mathfrak{o}$. Jacobinski's main interest was in determining annihilators of Ext; in fact, he showed that

$$\mathcal{F}(\mathfrak{o}G) \cdot \mathrm{Ext}^1_{\mathfrak{o}G}(M, N) = 0$$

for all $\mathfrak{o}G$-lattices $M$ and $\mathfrak{o}G$-modules $N$. For instance, this implies that $|G|/\chi(1)$ annihilates $\mathrm{Ext}^1_{\mathfrak{o}G}(M_\chi, N)$ if $M_\chi$ is an $\mathfrak{o}G$-lattice such that $K \otimes_{\mathfrak{o}} M_\chi$ is absolutely simple with character $\chi$. Later, Roggenkamp [Ro71] showed that the annihilators achieved in this way are in fact best possible in a certain precise sense.

In this article we consider completed group algebras $\mathfrak{o}[[G]]$, where $\mathfrak{o}$ denotes the ring of integers in a finite extension $K$ of $\mathbb{Q}_p$ and $G$ is a $p$-adic Lie group of dimension 1, i.e.

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$G$ can be written as a semi-direct product $H \rtimes \Gamma$ with finite $H$ and a cyclic pro-$p$-group $\Gamma$, isomorphic to $\mathbb{Z}_p$. We will exclude the special case $p = 2$, as we will make heavily use of results of Ritter and Weiss [RW04] (where the underlying prime is assumed to be odd) on the total ring of fractions $\mathcal{Q}^K(G)$ of $\mathfrak{o}[[G]]$. But it turns out that the results provided by Ritter and Weiss are not sufficient for our purposes such that we have to determine the structure of $\mathcal{Q}^K(G)$ in more detail, thereby generalizing results of Lau [La] (where $K = \mathbb{Q}_p$ and $G$ is pro-$p$). The main result of this first section is that there is always a finite Galois extension $E$ of $K$ such that $\mathcal{Q}^E(G)$ splits. In section 2 and 3 we provide the necessary preparations for our main result which will be stated and proved in section 4. More precisely, if we define the central conductor in complete analogy to the group ring case, then we have an equality

$$\mathcal{F}(\mathfrak{o}[[G]]) = \bigoplus_{\chi/\sim} \frac{|H|w_\chi}{\chi(1)} \cdot \mathcal{D}^{-1}(\mathfrak{o}_\chi/\mathfrak{o})\mathfrak{o}_\chi[[\Gamma_\chi]],$$

where the sum runs through all irreducible characters of $G$ with open kernel up to a certain explicit equivalence relation. Moreover, $w_\chi$ is the index of a certain subgroup (depending on $\chi$) in $G$ and $\mathfrak{o}_\chi$ denotes the ring of integers in $K_\chi := K(\chi(h)|h \in H)$. Finally, $\Gamma_\chi$ is a cyclic pro-$p$-group which has an explicitly determined topological generator.

The proof of Jakobinski's central conductor formula does not carry over unchanged to the present situation for two reasons. First, the completed group algebra is an order over the power series ring $\mathfrak{o}[[T]]$, but there is no canonical choice of embedding of $\mathfrak{o}[[T]]$ into $\zeta(\mathfrak{o}[[G]])$. Secondly and more seriously, the ring $\mathfrak{o}[[T]]$ is a regular local ring, but it is not a Dedekind domain. And even if we localize at a height one prime ideal, the residue field will not be finite. Hence we do not have the well elaborated theory of maximal orders over discrete valuation rings with finite residue field at our disposal. We will overcome this problem by replacing our chosen maximal $\mathfrak{o}[[T]]$-order by a suitable maximal $\mathfrak{o}$-order.

Finally, we draw some consequences in section 5. Especially, we obtain annihilation results for the corresponding Ext-groups in complete analogy to the group ring case. We further apply our main result to the theory of non-commutative Fitting invariants introduced by the author [Ni10]. In fact, this theory may be applied to $\mathfrak{o}[[G]]$-modules even if $G$ is non-abelian. But in contrast to the commutative case, the Fitting invariant of a finitely presented $\mathfrak{o}[[G]]$-module $M$ might not be contained in the annihilator of $M$. To achieve annihilators one has to multiply by a certain ideal of $\zeta(\mathfrak{o}[[G]])$ which is hard to determine in general. But it is easily seen that this ideal always contains the central conductor such that our main theorem provides a method to compute explicit annihilators of a finitely presented $\mathfrak{o}[[G]]$-module, at least, if we are able to compute its Fitting invariant.

## 1 On the total ring of fractions of a completed group algebra

Let $p$ be an odd prime and let $G$ be a profinite group which contains a finite normal subgroup $H$ such that $G/H \simeq \Gamma$ for a pro-p-group $\Gamma$, isomorphic to $\mathbb{Z}_p$; thus $G$ can be written as a semi-direct product $H \rtimes \Gamma$ and is a $p$-adic Lie group of dimension 1. We denote the completed group algebra $\mathbb{Z}_p[[G]]$ by $\Lambda(G)$. If $K$ is a finite field extension of $\mathbb{Q}_p$ with ring of integers $\mathfrak{o}$, we put $\Lambda^{\mathfrak{o}}(G) := \mathfrak{o} \otimes_{\mathbb{Z}_p} \Lambda(G) = \mathfrak{o}[[G]]$. We fix a topological generator $\gamma$ of $\Gamma$ and choose a natural number $n$ such that $\gamma^{p^n}$ is central in $G$. Since also $\Gamma^{p^n} \simeq \mathbb{Z}_p$, there is an isomorphism $\mathfrak{o}[[\Gamma^{p^n}]] \simeq \mathfrak{o}[[T]]$ induced by $\gamma^{p^n} \mapsto 1 + T$. Here, $R := \mathfrak{o}[[T]]$ denotes the power series ring in one variable over $\mathfrak{o}$. If we view $\Lambda^{\mathfrak{o}}(G)$ as an $R$-module, there is a decomposition

$$\Lambda^{\mathfrak{o}}(G) = \bigoplus_{i=0}^{p^n-1} R\gamma^i[H].$$

Hence $\Lambda^{\mathfrak{o}}(G)$ is finitely generated as an $R$-module and an $R$-order in the separable $L := Quot(R)$-algebra $\mathcal{Q}^K(G) := \bigoplus_i L\gamma^i[H]$. Note that $\mathcal{Q}^K(G)$ is obtained from $\Lambda^{\mathfrak{o}}(G)$ by inverting all regular elements and $\mathcal{Q}^K(G) = K \otimes_{\mathbb{Q}_p} \mathcal{Q}(G)$, where $\mathcal{Q}(G) := \mathcal{Q}^{\mathbb{Q}_p}(G)$.

Let $\mathbb{Q}_p^c$ be an algebraic closure of $\mathbb{Q}_p$ and fix an irreducible $\mathbb{Q}_p^c$-valued character $\chi$ of $G$ with open kernel. Choose a finite Galois extension $E$ of $\mathbb{Q}_p$ such that the character $\chi$ has a realization $V_\chi$ over $E$. Let $\eta$ be an irreducible constituent of $\mathrm{res}_H^G \chi$ and set

$$St(\eta) := \{g \in G : \eta^g = \eta\},\ e_\eta = \frac{\eta(1)}{|H|} \sum_{h \in H} \eta(h^{-1})h,\ e_\chi = \sum_{\eta | \mathrm{res}_H^G \chi} e_\eta.$$

By [RW04, Corollary to Proposition 6] $e_\chi$ is a primitive central idempotent of $\mathcal{Q}^E(G)$. In fact, any primitive central idempotent of $\mathcal{Q}^c(G) := \mathbb{Q}_p^c \otimes_{\mathbb{Q}_p} \mathcal{Q}(G)$ is an $e_\chi$, and $e_\chi = e_{\chi'}$ if and only if $\chi = \chi' \otimes \rho$ for some character $\rho$ of $G$ of type $W$ (i.e. $\mathrm{res}_H^G \rho = 1$). Since the occurring irreducible constituents of $\mathrm{res}_H^G \chi$ are precisely the Galois conjugates of $\eta$ by [CR81, Proposition 11.4], we have an equality

$$e_\chi = \sum_{i=0}^{w_\chi - 1} e_{\eta^{\gamma^i}}, \tag{1}$$

where $w_\chi = [G : St(\eta)]$. By [RW04, Proposition 5] there is a distinguished element $\gamma_\chi \in \zeta(\mathcal{Q}^E(G)e_\chi)$ which generates a procyclic $p$-subgroup $\Gamma_\chi$ of $(\mathcal{Q}^E(G)e_\chi)^\times$ and acts trivially on $V_\chi$. Moreover, $\gamma_\chi$ induces an isomorphism $\mathcal{Q}^E(\Gamma_\chi) \xrightarrow{\simeq} \zeta(\mathcal{Q}^E(G)e_\chi)$ by [RW04, Proposition 6]. Note that we may write $\gamma_\chi = \gamma^{w_\chi} \cdot c = c \cdot \gamma^{w_\chi}$, where $c \in (E[H]e_\chi)^\times$ by [RW04, Proposition 5 and its proof].

**Proposition 1.1.** *Let $G$ be a $p$-adic Lie group of dimension 1 and let $K$ be a finite extension of $\mathbb{Q}_p$. For any irreducible character $\chi$ of $G$ with open kernel put $K_\chi := K(\chi(h)|h \in H)$. Then there is an isomorphism*

$$\zeta(\mathcal{Q}^K(G)) \simeq \bigoplus_{\chi/\sim} \mathcal{Q}^{K_\chi}(\Gamma_\chi),$$

*where the sum runs through all irreducible characters of $G$ with open kernel up to the equivalence relation: $\chi \sim \chi'$ if and only if there is $\sigma \in \mathrm{Gal}(K_\chi/K)$ such that $(\mathrm{res}_H^G \chi)^\sigma = \mathrm{res}_H^G \chi'$.*

*Proof.* Since there are only finitely many central primitive idempotents $e_\chi$ of $\mathcal{Q}^c(G)$, we may choose a finite Galois extension $E$ of $\mathbb{Q}_p$ such that $E[H]$ contains each $e_\chi$. We may also assume that $K$ is a subfield of $E$. Now let $\sigma$ be an element of $\mathrm{Gal}(E/K)$. Then $\sigma$ acts on $\mathcal{Q}^E(G)$ and $e_\chi^\sigma = e_{\chi^\sigma}$. Moreover by the above, $e_{\chi^\sigma} = e_\chi$ if and only if there is a character $\rho$ of type $W$ such that $\chi = \chi^\sigma \otimes \rho$, thus if and only if $\mathrm{res}^G_H \chi = \mathrm{res}^G_H \chi^\sigma$. Since the center of $\mathcal{Q}^K(G)$ coincides with the $\mathrm{Gal}(E/K)$-invariants of $\zeta(\mathcal{Q}^E(G))$, we have an equality

$$\zeta(\mathcal{Q}^K(G)) = \bigoplus_{\chi/\sim} \zeta(\mathcal{Q}^K(G)\varepsilon_\chi), \ \varepsilon_\chi = \sum_{\sigma\in \mathrm{Gal}(K_\chi/K)} e_{\chi^\sigma}.$$

Note that $\chi^\sigma$ depends on a chosen lift of $\sigma$, but $e_{\chi^\sigma}$ does not. Now let

$$\beta = (\beta_\sigma)_\sigma \in \bigoplus_{\sigma\in \mathrm{Gal}(K_\chi/K)} \zeta(\mathcal{Q}^E(G)e_{\chi^\sigma}) = \bigoplus_{\sigma\in \mathrm{Gal}(K_\chi/K)} \mathcal{Q}^E(\Gamma_{\chi^\sigma})$$

be invariant under $\mathrm{Gal}(E/K)$. The uniqueness of $\gamma_\chi$ implies that $\gamma_{\chi^\sigma} = \gamma_\chi^\sigma$; thus $\beta$ is determined by $\beta_1$ and $\beta_1$ lies in $\mathcal{Q}^E(\Gamma_\chi)^{\mathrm{Gal}(E/K_\chi)} = \mathcal{Q}^{K_\chi}(\Gamma_\chi)$ as desired. □

**Remark 1.2.** *In the special case, where $K = \mathbb{Q}_p$ and $G$ is a pro-$p$-group, this was shown by Lau [La] using a different method. The same is true for Corollary 1.5 below.*

**Corollary 1.3.** *Let $K$ be a finite extension of $\mathbb{Q}_p$ with ring of integeres $\mathfrak{o}$. Choose a maximal order $\tilde{\Lambda}^{\mathfrak{o}}(G)$ containing $\Lambda^{\mathfrak{o}}(G)$. Then*

$$\zeta(\tilde{\Lambda}^{\mathfrak{o}}(G)) \simeq \bigoplus_{\chi/\sim} \Lambda^{\mathfrak{o}_\chi}(\Gamma_\chi),$$

*where $\mathfrak{o}_\chi$ denotes the ring of integers in $K_\chi$.*

**Remark 1.4.** *Here, $\tilde{\Lambda}^{\mathfrak{o}}(G)$ is an order over $R = \mathfrak{o}[[T]]$, where we have identified $1 + T$ with $\gamma^{p^n}$ for a chosen large $n$. Now let $\chi$ be an irreducible character of $G$ with open kernel. If $n$ is sufficiently large, then $\gamma^{p^n}$ acts trivially on $V_\chi$ and hence $\gamma_\chi^{p^n} = (\gamma^{p^n})^{w_\chi} e_\chi$. As $\gamma^{p^n}$ is central in $G$, the integer $w_\chi$ divides $p^n$. We have shown that the inclusion*

$$R = \mathfrak{o}[[T]] \rightarrowtail \Lambda^{\mathfrak{o}_\chi}(\Gamma_\chi)$$

*is induced by $1 + T \mapsto \gamma_\chi^{p^n/w_\chi}$.*

**Corollary 1.5.** *The algebra $\mathcal{Q}^K(G)$ has Wedderburn decomposition*

$$\mathcal{Q}^K(G) \simeq \bigoplus_{\chi/\sim} (D_\chi)_{n_\chi \times n_\chi},$$

*where $n_\chi \in \mathbb{N}$ and $D_\chi$ is a skewfield with center $\mathcal{Q}^{K_\chi}(\Gamma_\chi)$. If $s_\chi$ denotes the Schur index of $D_\chi$, then we have an equality $\chi(1) = n_\chi s_\chi$.*

*Proof.* All assertions are immediate from Proposition 1.1 apart from the last equality. Let us denote the simple component $(D_\chi)_{n_\chi\times n_\chi}$ by $A_\chi$. With $E$ as in Proposition 1.1 we compute

$$\begin{aligned}
(n_\chi s_\chi)^2 &= \dim_{\mathcal{Q}^{K_\chi}(\Gamma_\chi)}(A_\chi) \\
&= [K_\chi : K]^{-1} \cdot \dim_{\mathcal{Q}^K(\Gamma_\chi)}(A_\chi) \\
&= [K_\chi : K]^{-1} \cdot \dim_{\mathcal{Q}^E(\Gamma_\chi)}(E \otimes_K A_\chi) \\
&\overset{(1)}{=} [K_\chi : K]^{-1} \cdot \sum_{\sigma \in \mathrm{Gal}(K_\chi/K)} \dim_{\mathcal{Q}^E(\Gamma_\chi)}(\mathcal{Q}^E(G)e_{\chi^\sigma}) \\
&\overset{(2)}{=} [K_\chi : K]^{-1} \cdot \sum_{\sigma \in \mathrm{Gal}(K_\chi/K)} \chi^\sigma(1)^2 \\
&= \chi(1)^2
\end{aligned}$$

Here, (1) is implied by the isomorphism $E \otimes_K A_\chi \simeq \bigoplus_{\sigma \in \mathrm{Gal}(K_\chi/K)} \mathcal{Q}^E(G)e_{\chi^\sigma}$ and (2) is shown in the proof of [RW04, Proposition 6]. □

**Theorem 1.6.** *There is a finite Galois extension $E$ of $K$ such that $\mathcal{Q}^E(G)$ splits.*

*Proof.* We choose $E$ as in Proposition 1.1 and let $L' := E \otimes_K L = \mathcal{Q}^E(\Gamma^{p^n})$. As $L'$-vector space, we have a decomposition

$$\mathcal{Q}^E(G) = \bigoplus_{i=0}^{p^n-1} L'[H]\gamma^i.$$

Now let $\chi$ be an irreducible character of $G$ with open kernel. Enlarging $E$ if necessary, we may assume that the group ring $E[H]$ splits. Since $E$ is a subfield of $L'$, we obtain

$$\begin{aligned}
\mathcal{Q}^E(G)e_\chi &= \bigoplus_{i=0}^{p^n-1} L'[H]e_\chi\gamma^i \\
&= \bigoplus_{i=0}^{p^n-1} \bigoplus_{j=0}^{w_\chi-1} L'[H]e_{\eta^{\gamma^j}}\gamma^i \\
&= \bigoplus_{i=0}^{p^n-1} \bigoplus_{j=0}^{w_\chi-1} L'_{\eta(1)\times\eta(1)}\gamma^i,
\end{aligned}$$

where we have used equation (1). We now choose an indecomposable idempotent $f_\eta = f_\eta e_\eta$ of $L'[H]e_\eta = L'_{\eta(1)\times\eta(1)}$. Observe that for any other indecomposable idempotent $f'_\eta$ of $L'[H]e_\eta$ we have an isomorphism $f_\eta L'[H]f'_\eta \simeq L'$. As $\mathcal{Q}^E(G)e_\chi$ is a simple algebra over its center $\mathcal{Q}^E(\Gamma_\chi)$ by Corollary 1.5, and $f_\eta$ is also an idempotent in $\mathcal{Q}^E(G)e_\chi$, it suffices to show that $f_\eta\mathcal{Q}^E(G)e_\chi f_\eta$ is a field, namely $\mathcal{Q}^E(\Gamma_\chi)$. For this, we first observe that for any $0 \le i < p^n$ also $f_{\eta,i} := \gamma^i f_\eta \gamma^{-i}$ is an indecomposable idempotent which belongs to $L'[H]e_{\eta^{\gamma^i}}$ such that

$$f_\eta L'[H]f_{\eta,i} \simeq \begin{cases} L' & \text{if } w_\chi | i \\ 0 & \text{otherwise.} \end{cases}$$

This implies that we have an isomorphism

$$f_\eta \mathcal{Q}^E(G)e_\chi f_\eta \simeq \bigoplus_{\substack{i=0\\ w_\chi|i}}^{p^n-1} L'\gamma^i = \bigoplus_{i=0}^{w_\chi^{-1}p^n-1} L'\gamma^{w_\chi i} = \bigoplus_{i=0}^{w_\chi^{-1}p^n-1} L'\gamma_\chi^i.$$

Here, the last equality holds, as we may write $\gamma_\chi = \gamma^{w_\chi}\cdot c = c\cdot\gamma^{w_\chi}$, where $c$ lies in $(E[H]e_\chi)^\times$. Finally, $L' = \mathcal{Q}^E(\Gamma^{p^n})$ and $\gamma^{p^n}$ identifies with $\gamma_\chi^{p^n/w_\chi}$ by Remark 1.4 such that

$$\bigoplus_{i=0}^{w_\chi^{-1}p^n-1} L'\gamma_\chi^i = \mathcal{Q}^E(\Gamma_\chi)$$

as desired. □

**Corollary 1.7.** *The semi-simple algebra $\mathcal{Q}^c(G)$ splits.*

**Remark 1.8.** *If $A_\chi = (D_\chi)_{n_\chi\times n_\chi}$ is a simple component of $\mathcal{Q}^K(G)$, then the arguments in the proof of Theorem 1.6 can be refined to show that $K(\eta)\otimes_{K_\chi} A_\chi$ splits, where $K(\eta) = K(\eta(h)|h\in H)$. See [La, Theorem 1] for details in the case $K=\mathbb{Q}_p$ and $G$ a pro-p-group. In fact, in this special case one can determine the occurring skewfields very explicitly.*

## 2 Traces and conductors

Recall that $L = Quot(R)$ and denote by $\mathrm{Tr}$ the ordinary trace map from $\mathcal{Q}^K(G)$ to $L$.

**Lemma 2.1.** *The elements $\gamma^i h$, $0\le i<p^n$, $h\in H$ form an $L$-basis of $\mathcal{Q}^K(G)$ such that*

$$\mathrm{Tr}\,(\gamma^i h) = \begin{cases} p^n|H| & \text{if } \gamma^i h = 1\\ 0 & \text{otherwise.}\end{cases}$$

*Its dual basis with respect to* $\mathrm{Tr}$ *is given by $(p^n|H|)^{-1}h^{-1}\gamma^{-i}$, $0\le i<p^n$, $h\in H$.*

*Proof.* Let $0\le i,j<p^n$ and $h,h'\in H$ such that $\gamma^i h\gamma^j h' = \gamma^{i+j}h_j h'$ with $h_j := \gamma^{-j}h\gamma^j\in H$. Assume that $\gamma^i h\gamma^j h' = x\cdot\gamma^j h'$, where $x\in L$. To prove the desired formula for $\mathrm{Tr}$ we have to show that $i=0$ and $h=1$. In fact, we see that $h_j h' = h'$; hence $h=h_j=1$. Moreover, we have $\gamma^{i+j} = x\cdot\gamma^j$ which implies that $x=\gamma^i$. But $x$ belongs to $L$ which does not contain $\gamma^i$ for $0<i<p^n$; hence also $i=0$. It is now easily checked that $(p^n|H|)^{-1}h^{-1}\gamma^{-i}$, $0\le i<p^n$, $h\in H$ is the dual basis. □

**Lemma 2.2.** *Let $K'$ be a finite field extension of $K$ with ring of integers $\mathfrak{o}'$. Consider $\Lambda^{\mathfrak{o}'}(\Gamma)$ as $R$-order via the embedding*

$$R\rightarrowtail \Lambda^{\mathfrak{o}'}(\Gamma),\ 1+T\mapsto\gamma^{p^n}.$$

*Then the inverse different $\mathcal{D}^{-1}(\Lambda^{\mathfrak{o}'}(\Gamma)/R) := \left\{x\in\mathcal{Q}^{K'}(\Gamma)|\mathrm{Tr}_{\mathcal{Q}^{K'}(\Gamma)/L}(x\Lambda^{\mathfrak{o}'}(\Gamma))\subset R\right\}$ is given by*

$$\mathcal{D}^{-1}(\Lambda^{\mathfrak{o}'}(\Gamma)/R) = p^{-n}\mathcal{D}^{-1}(\mathfrak{o}'/\mathfrak{o})\Lambda^{\mathfrak{o}'}(\Gamma),$$

*where $\mathcal{D}^{-1}(\mathfrak{o}'/\mathfrak{o})$ denotes the usual inverse different of $\mathfrak{o}'$ with respect to $\mathfrak{o}$.*

*Proof.* If $K' = K$, the result follows from Lemma 2.1 with $G = \Gamma$. Hence we may assume that $n = 0$. But if $x_1, \dots, x_k$ form an $\mathfrak{o}$-basis of $\mathfrak{o}'$, then $x_1, \dots, x_k$ are also an $\mathfrak{o}[[T]]$-basis of $\mathfrak{o}'[[T]]$ which is isomorphic to $\Lambda^{\mathfrak{o}'}(\Gamma)$ via $1 + T \mapsto \gamma$. Hence its dual basis with respect to the ordinary trace $\mathrm{Tr}_{K'/K}$ of fields, is also a dual basis with respect to $\mathrm{Tr}_{\mathcal{Q}^{K'}(\Gamma)/L}$. $\square$

By Corollary 1.5 we may write $\mathcal{Q}^K(G) = \bigoplus_{\chi/\sim} A_\chi$, where $A_\chi = (D_\chi)_{n_\chi \times n_\chi}$, $n_\chi \in \mathbb{N}$ and $D_\chi$ is a skewfield with Schur index $s_\chi$ and center $\mathcal{Q}^{K_\chi}(\Gamma_\chi)$. By the same Corollary, we have $\chi(1) = s_\chi n_\chi$ such that the ordinary trace may be written as

$$\mathrm{Tr} = \sum_{\chi/\sim} \chi(1)\mathrm{tr}_\chi, \tag{2}$$

where $\mathrm{tr}_\chi$ denotes the reduced trace from $A_\chi$ to $L$. Moreover, we have

$$\mathrm{tr}_\chi = \mathrm{Tr}_{\mathcal{Q}^{K_\chi}(\Gamma_\chi)/L} \circ \mathrm{tr}_{A_\chi/\mathcal{Q}^{K_\chi}(\Gamma_\chi)}, \tag{3}$$

where $\mathrm{Tr}_{\mathcal{Q}^{K_\chi}(\Gamma_\chi)/L}$ denotes the ordinary trace of fields and $\mathrm{tr}_{A_\chi/\mathcal{Q}^{K_\chi}(\Gamma_\chi)}$ denotes the reduced trace from $A_\chi$ into its center. Recall from Remark 1.4 that we have chosen a sufficiently large integer $n \geq 0$ such that $R = \mathfrak{o}[[T]]$ embeds into $\Lambda^{\mathfrak{o}_\chi}(\Gamma_\chi)$ via $1 + T \mapsto \gamma_\chi^{p^n/w_\chi}$. Now Lemma 2.2 implies that the following definition does not depend on $n$.

**Definition 2.3.** *Choose a maximal $R$-order $\tilde{\Lambda}^{\mathfrak{o}}(G)$ containing $\Lambda^{\mathfrak{o}}(G)$. We have a decomposition $\tilde{\Lambda}^{\mathfrak{o}}(G) = \bigoplus_{\chi/\sim} \tilde{\Lambda}^{\mathfrak{o}}_\chi(G)$, where each $\tilde{\Lambda}^{\mathfrak{o}}_\chi(G)$ is a maximal $R$-order in $A_\chi$. For sufficiently large $n$ we call the two-sided $\tilde{\Lambda}^{\mathfrak{o}}_\chi(G)$-lattice*

$$\mathcal{D}_\chi^{-1}(\tilde{\Lambda}^{\mathfrak{o}}(G)) = \mathcal{D}^{-1}_{\mathrm{norm}}(\tilde{\Lambda}^{\mathfrak{o}}_\chi(G)/R) := p^n \cdot \left\{ x \in A_\chi | \mathrm{tr}_\chi(x\tilde{\Lambda}^{\mathfrak{o}}_\chi(G)) \subset R \right\}$$

*the normalized inverse different.*

We point out that this is abuse of notation, since in general $\mathcal{D}_\chi^{-1}(\tilde{\Lambda}^{\mathfrak{o}}(G))$ might not be invertible.

**Definition 2.4.** *Let $\Lambda \subset \tilde{\Lambda}$ be a pair of rings. Then*

$$(\tilde{\Lambda} : \Lambda)_l := \left\{ x \in \tilde{\Lambda} | x\tilde{\Lambda} \subset \Lambda \right\}$$

*is called the left conductor of $\tilde{\Lambda}$ into $\Lambda$. Similarly,*

$$(\tilde{\Lambda} : \Lambda)_r := \left\{ x \in \tilde{\Lambda} | \tilde{\Lambda}x \subset \Lambda \right\}$$

*is called the right conductor of $\tilde{\Lambda}$ into $\Lambda$.*

Using Lemma 2.1 and equation (2), we can adjust the proof of Jacobinski's conductor formula given in [CR81, Theorem 27.8] to show the following result.

**Theorem 2.5.** *Let $\tilde{\Lambda}^{\mathfrak{o}}(G)$ be a maximal $R$-order containing $\Lambda^{\mathfrak{o}}(G)$. Then*

$$(\tilde{\Lambda}^{\mathfrak{o}}(G) : \Lambda^{\mathfrak{o}}(G))_l = (\tilde{\Lambda}^{\mathfrak{o}}(G) : \Lambda^{\mathfrak{o}}(G))_r = \bigoplus_{\chi/\sim} \frac{|H|}{\chi(1)} \mathcal{D}_\chi^{-1}(\tilde{\Lambda}^{\mathfrak{o}}(G)).$$

## 3 Some further preliminaries

Let $\pi$ be a prime element in $\mathfrak{o}$. For any $R$-module $M$ we write $M_{(\pi)}$ for the localization of $M$ at the prime $(\pi)$. In particular, if $\Lambda$ is an $R$-order in the $L$-algebra $A$, then $\Lambda_{(\pi)}$ is an $R_{(\pi)}$-order in $A$.

**Lemma 3.1.** *Let $A$ be a separable $L$-algebra of finite dimension over $L$ and let $\Lambda$ be an $R$-order in $A$. If $A$ is split, then there is an $\mathfrak{o}$-order $\Delta$ in $\Lambda_{(\pi)}$ such that $L \otimes_{\mathfrak{o}} \Delta = A$.*

*Proof.* Choose a maximal $R$-order $\tilde{\Lambda}$ in $A$ containing $\Lambda$. Then $\tilde{\Lambda}_{(\pi)}$ is a maximal $R_{(\pi)}$-order containing $\Lambda_{(\pi)}$ by [Re75, Theorem 11.1]. Note that we have $\pi^N \tilde{\Lambda}_{(\pi)} \subset \Lambda_{(\pi)}$ if $N$ is sufficiently large. There are natural numbers $k > 0$ and $n_i$, $1 \leq i \leq k$ such that $A \simeq \bigoplus_{i=1}^k L_{n_i \times n_i}$. We put $\tilde{\Lambda}_0 := \bigoplus_{i=1}^k R_{n_i \times n_i}$ and observe that this is a maximal $R$-order in $A$ by [Re75, Theorem 8.7]. The global dimension of $\tilde{\Lambda}_0$ is given by

$$\text{gl.dim}\, \tilde{\Lambda}_0 = \text{gl.dim}\, R = 2 < \infty.$$

Here, the second equality follows from [Ei95, Corollary 19.6], and the first equality holds, since the global dimension is invariant under Morita equivalence (cf. [Ra69, Corollary, p. 476]). Recall that a noetherian ring $\Gamma$ is called *quasi-local* if $\Gamma/\text{rad}(\Gamma)$ is a simple artinian ring, where $\text{rad}(\Gamma)$ denotes the Jacobson radical of $\Gamma$. As $R/\text{rad}(R)$ is a field, any component $R_{n_i \times n_i}$ of $\tilde{\Lambda}_0$ is quasi-local. As $R$ is a regular local ring of dimension 2, these observations permit us to apply [Ra69, Theorem 5.4] componentwise which implies the existence of an invertible element $a \in A$ such that $\tilde{\Lambda} = a^{-1}\tilde{\Lambda}_0 a$.

For $1 \leq i \leq k$ and $1 \leq j, l \leq n_i$ let $e_{jl}^i \in \tilde{\Lambda}_0$ be the element which is zero everywhere except for the $i$-th component, where it is equal to the matrix with 1 in position $(j, l)$ and 0 everywhere else. Then the elements 1 and $a^{-1}e_{jl}^i a$ with $(i, j, l) \neq (1, 1, 1)$ form an $R$-basis of $\tilde{\Lambda}$. For $N$ as above, the free $\mathfrak{o}$-module

$$\Delta := \mathfrak{o} \oplus \bigoplus_{(i,j,l)\neq(1,1,1)} \pi^N a^{-1} e_{jl}^i a \cdot \mathfrak{o} \subset \Lambda_{(\pi)}$$

is closed under multiplication, thus is an $\mathfrak{o}$-order in the separable $K$-algebra $K \otimes_{\mathfrak{o}} \Delta$. As the $\mathfrak{o}$-rank of $\Delta$ equals the $L$-dimension of $A$, we have $L \otimes_{\mathfrak{o}} \Delta = A$ as desired. □

**Lemma 3.2.** *Let $A \subset \mathcal{Q}^K(G)$ be a semi-simple component of $\mathcal{Q}^K(G)$, i.e. $A$ is the direct sum of some, but maybe not all $A_\chi$. If $\Lambda$ is an $R$-order in $A$, then there is an $\mathfrak{o}$-order $\Delta$ in $\Lambda_{(\pi)}$ such that $L \otimes_{\mathfrak{o}} \Delta = A$.*

*Proof.* By Theorem 1.6 there is a finite Galois extension $K'$ of $K$ such that $A' := K' \otimes_K A$ splits. Let $\mathfrak{o}'$ be the ring of integers in $K'$ with prime element $\pi'$ and $R' = \mathfrak{o}' \otimes_{\mathfrak{o}} R = \mathfrak{o}'[[T]]$ with field of fractions $L' = K' \otimes_K L$. Then $\Lambda' := \mathfrak{o}' \otimes_{\mathfrak{o}} \Lambda$ is an $R'$-order in $A'$. By Lemma 3.1 there is an $\mathfrak{o}'$-order $\Delta'$ in $\Lambda'_{(\pi')}$ such that $L' \otimes_{\mathfrak{o}'} \Delta' = A'$. We put $\Delta := (\Delta')^{\text{Gal}(K'/K)}$. Then $\Delta$ is contained in $(\Lambda'_{(\pi')})^{\text{Gal}(K'/K)} = \Lambda_{(\pi)}$ and a ring containing $\mathfrak{o}$. Note that, by construction, multiplication in $\Delta$ is induced by multiplication in $A$. As $\Delta$ is an $\mathfrak{o}$-submodule of $\Delta'$ and $\Delta'$ is finitely generated and free over $\mathfrak{o}'$ (and thus over $\mathfrak{o}$), also $\Delta$ is finitely generated and free over $\mathfrak{o}$. Hence $\Delta$ is an $\mathfrak{o}$-order. Finally, $\Delta$ contains a $K'$-basis of $K' \otimes_{\mathfrak{o}'} \Delta'$ by Hilbert's Theorem 90. Thus $L \otimes_{\mathfrak{o}} \Delta = A$, as both sides have the same dimension over $L$. □

**Corollary 3.3.** *There exists a maximal oder $\tilde{\Lambda}$ in $A$ which contains a maximal $\mathfrak{o}$-order $\tilde{\Delta}$ such that $L \otimes_{\mathfrak{o}} \tilde{\Delta} = A$.*

*Proof.* Choose any $\mathfrak{o}$-order $\Delta$ as in Lemma 3.2 and a maximal $\mathfrak{o}$-order $\tilde{\Delta}$ in $K \otimes_{\mathfrak{o}} \Delta$ containing $\Delta$. Then also $L \otimes_{\mathfrak{o}} \tilde{\Delta} = A$. Moreover, $R \otimes_{\mathfrak{o}} \tilde{\Delta}$ is an $R$-order in $A$ and hence contained in a maximal order $\tilde{\Lambda}$. Obviously, $\tilde{\Delta}$ is contained in $\tilde{\Lambda}$. □

**Corollary 3.4.** *Let $\widehat{R_{(\pi)}}$ be the completion of $R_{(\pi)}$ with respect to the prime $(\pi)$ and $\hat{L} = Quot(\widehat{R_{(\pi)}})$. Then any maximal $\widehat{R_{(\pi)}}$-order $\hat{\Lambda}$ in $\hat{A} := \hat{L} \otimes_L A$ contains a maximal $\mathfrak{o}$-order $\hat{\Delta}$ such that $\hat{L} \otimes_{\mathfrak{o}} \hat{\Delta} = \hat{A}$.*

*Proof.* Take a maximal order $\tilde{\Lambda}$ as in Corollary 3.3. Then clearly $\hat{L} \otimes_{\mathfrak{o}} \tilde{\Delta} = \hat{A}$. Moreover, $\tilde{\Delta}$ is contained in the $(\pi)$-adic completion $\widehat{\tilde{\Lambda}_{(\pi)}}$ of $\tilde{\Lambda}_{(\pi)}$ which is a maximal $\widehat{R_{(\pi)}}$-order in $\hat{A}$. Now let $\hat{\Lambda}$ be an arbitrary maximal $\widehat{R_{(\pi)}}$-order in $\hat{A}$. Since $\widehat{R_{(\pi)}}$ is a complete discrete valuation ring, it follows from [Re75, Theorem 17.3] that there is an $a \in \hat{A}^{\times}$ such that $\hat{\Lambda} = a\widehat{\tilde{\Lambda}_{(\pi)}}a^{-1}$. Then $\hat{\Delta} := a\tilde{\Delta}a^{-1}$ has the desired properties. □

## 4 A formula for the central conductor

**Definition 4.1.** *Let $\tilde{\Lambda}^{\mathfrak{o}}(G)$ be a maximal order containing $\Lambda^{\mathfrak{o}}(G)$. Then the central conductor of $\tilde{\Lambda}^{\mathfrak{o}}(G)$ into $\Lambda^{\mathfrak{o}}(G)$ is defined to be*

$$\begin{aligned}\mathcal{F}(\Lambda^{\mathfrak{o}}(G)) = \mathcal{F}(\tilde{\Lambda}^{\mathfrak{o}}(G)/\Lambda^{\mathfrak{o}}(G)) &:= \zeta(\Lambda^{\mathfrak{o}}(G)) \cap (\tilde{\Lambda}^{\mathfrak{o}}(G) : \Lambda^{\mathfrak{o}}(G))_l \\ &= \zeta(\Lambda^{\mathfrak{o}}(G)) \cap (\tilde{\Lambda}^{\mathfrak{o}}(G) : \Lambda^{\mathfrak{o}}(G))_r \\ &= \left\{ x \in \zeta(\Lambda^{\mathfrak{o}}(G)) | x\tilde{\Lambda}^{\mathfrak{o}}(G) \subset \Lambda^{\mathfrak{o}}(G) \right\}.\end{aligned}$$

**Remark 4.2.** *The theorem below shows that the central conductor only depends on $\Lambda^{\mathfrak{o}}(G)$ and not on the maximal order containing $\Lambda^{\mathfrak{o}}(G)$. Hence the notation $\mathcal{F}(\Lambda^{\mathfrak{o}}(G))$ is justified.*

**Theorem 4.3.** *Let $\tilde{\Lambda}^{\mathfrak{o}}(G)$ be a maximal order containing $\Lambda^{\mathfrak{o}}(G)$. Then the central conductor of $\tilde{\Lambda}^{\mathfrak{o}}(G)$ into $\Lambda^{\mathfrak{o}}(G)$ is given by*

$$\mathcal{F}(\Lambda^{\mathfrak{o}}(G)) = \bigoplus_{\chi/\sim} \frac{|H|w_{\chi}}{\chi(1)} \cdot \mathcal{D}^{-1}(\mathfrak{o}_{\chi}/\mathfrak{o})\Lambda^{\mathfrak{o}_{\chi}}(\Gamma_{\chi}).$$

*In particular, the central conductor $\mathcal{F}(\Lambda^{\mathfrak{o}}(G))$ does not depend on the maximal order $\tilde{\Lambda}^{\mathfrak{o}}(G)$.*

*Proof.* According to Corollary 1.5 we write $\mathcal{Q}^K(G) = \bigoplus_{\chi/\sim} A_{\chi}$, where each $A_{\chi} \simeq (D_{\chi})_{n_{\chi} \times n_{\chi}}$ is simple. Similarly, $\tilde{\Lambda}^{\mathfrak{o}}(G)$ decomposes into $\bigoplus_{\chi/\sim} \tilde{\Lambda}^{\mathfrak{o}}_{\chi}(G)$, where each $\tilde{\Lambda}^{\mathfrak{o}}_{\chi}(G)$ is a maximal $R$-order in $A_{\chi}$ with center $\Lambda^{\mathfrak{o}_{\chi}}(\Gamma_{\chi})$. We define

$$\begin{aligned}d_{\chi}^{-1} &:= \left\{ x \in A_{\chi} | \mathrm{tr}_{A_{\chi}/\mathcal{Q}^{K_{\chi}}(\Gamma_{\chi})}(x\tilde{\Lambda}^{\mathfrak{o}}_{\chi}(G)) \subset \Lambda^{\mathfrak{o}_{\chi}}(\Gamma_{\chi}) \right\} \\ \delta_{\chi}^{-1} &:= p^n \mathcal{D}^{-1}(\Lambda^{\mathfrak{o}_{\chi}}(\Gamma_{\chi})/R) = w_{\chi} \cdot \mathcal{D}^{-1}(\mathfrak{o}_{\chi}/\mathfrak{o})\Lambda^{\mathfrak{o}_{\chi}}(\Gamma_{\chi}),\end{aligned}$$

where the last equality follows from Lemma 2.2 and Remark 1.4.

**Lemma 4.4.** *We have an equality* $\mathcal{D}_\chi^{-1}(\tilde{\Lambda}_\chi^{\mathfrak{o}}(G)) = d_\chi^{-1} \cdot \delta_\chi^{-1}$.

*Proof.* This follows easily from the definitions using equation (3). In fact, the proof is similar to the corresponding statement in the proof of [CR81, Theorem 27.13]; note that only $\delta_\chi^{-1}$ has to be invertible. ☐

By Theorem 2.5 and the definition of the central conductor we obtain

$$\mathcal{F}(\Lambda^{\mathfrak{o}}(G)) = \bigoplus_{\chi/\sim} \Lambda^{\mathfrak{o}_\chi}(\Gamma_\chi) \cap \left( \frac{|H|}{\chi(1)} \cdot \mathcal{D}_\chi^{-1}(\tilde{\Lambda}_\chi^{\mathfrak{o}}(G)) \right).$$

Hence it must be shown that

$$\Lambda^{\mathfrak{o}_\chi}(\Gamma_\chi) \cap \left( \frac{|H|}{\chi(1)} \cdot \mathcal{D}_\chi^{-1}(\tilde{\Lambda}_\chi^{\mathfrak{o}}(G)) \right) = \frac{|H|}{\chi(1)} \delta_\chi^{-1} \tag{4}$$

for each irreducible character $\chi$. We note that

$$\frac{|H|}{\chi(1)} \delta_\chi^{-1} \subset \frac{|H|}{\chi(1)} \mathcal{D}_\chi^{-1}(\tilde{\Lambda}_\chi^{\mathfrak{o}}(G)) \subset \tilde{\Lambda}^{\mathfrak{o}}(G);$$

so each element of $\frac{|H|}{\chi(1)} \delta_\chi^{-1}$ is integral over $R$, and thus lies in $\Lambda^{\mathfrak{o}_\chi}(\Gamma_\chi)$. This gives one inclusion in (4). For the reverse inclusion let $y \in \Lambda^{\mathfrak{o}_\chi}(\Gamma_\chi)$. Then by Lemma 4.4 we have

$$y \in \frac{|H|}{\chi(1)} \cdot \mathcal{D}_\chi^{-1}(\tilde{\Lambda}_\chi^{\mathfrak{o}}(G)) \iff y\delta_\chi \subset \frac{|H|}{\chi(1)} d_\chi^{-1} \iff y\delta_\chi \subset \frac{|H|}{\chi(1)} (\mathcal{Q}^{K_\chi}(\Gamma_\chi) \cap d_\chi^{-1}).$$

Hence the theorem follows from the lemma below. ☐

**Lemma 4.5.** *We have* $\mathcal{Q}^{K_\chi}(\Gamma_\chi) \cap d_\chi^{-1} = \Lambda^{\mathfrak{o}_\chi}(\Gamma_\chi)$.

*Proof.* To show the non-trivial inclusion let $y \in \mathcal{Q}^{K_\chi}(\Gamma_\chi) \cap d_\chi^{-1}$. Then in particular

$$\operatorname{tr}_{A_\chi / \mathcal{Q}^{K_\chi}(\Gamma_\chi)}(y) = y \cdot \chi(1) \in \Lambda^{\mathfrak{o}_\chi}(\Gamma_\chi).$$

Let $\pi_\chi$ be a prime element in $\mathfrak{o}_\chi$. As $\chi(1)$ is an integer, the above equation shows that we may localize and even complete at the prime $(\pi_\chi)$. More precisely, let $\hat{\Lambda}_\chi$ and $\hat{R}_\chi$ be the completions of $\tilde{\Lambda}_\chi^{\mathfrak{o}}(G)$ and $\Lambda^{\mathfrak{o}_\chi}(\Gamma_\chi)$ at the prime $(\pi_\chi)$, respectively. Then $\hat{\Lambda}_\chi$ is a maximal $\hat{R}_\chi$-order in $\hat{A}_\chi := \hat{L}_\chi \otimes_{\mathcal{Q}^{K_\chi}(\Gamma_\chi)} A_\chi$, where $\hat{L}_\chi$ denotes the quotient field of $\hat{R}_\chi$. Now $y \in \hat{L}_\chi$ is such that

$$\operatorname{tr}_{\hat{A}_\chi / \hat{L}_\chi}(y\hat{\Lambda}_\chi) \subset \hat{R}_\chi$$

and we wish to show that $y$ belongs to $\hat{R}_\chi$. As the reduced trace is $\hat{L}_\chi$-linear, we may alter $y$ by a unit in $\hat{R}_\chi$ such that we may assume that $y = \pi_\chi^k$ for an appropriate integer $k$. By Corollary 3.4 there is a maximal $\mathfrak{o}_\chi$-order $\Delta_\chi$ contained in $\hat{\Lambda}_\chi$ such that $\hat{L}_\chi \otimes_{\mathfrak{o}_\chi} \Delta_\chi = \hat{A}_\chi$. Hence an $\mathfrak{o}_\chi$-basis of $\Delta_\chi$ is also a $\hat{L}_\chi$-basis of $\hat{A}_\chi$ which we may use to compute the reduced trace. Hence $y \in K_\chi$ is such that

$$\operatorname{tr}_{K_\chi \otimes_{\mathfrak{o}_\chi} \Delta_\chi / K_\chi}(y\Delta_\chi) \subset \hat{R}_\chi \cap K_\chi = \mathfrak{o}_\chi.$$

But $\mathfrak{o}_\chi$ is a complete discrete valuation ring with finite residue field such that we may conclude as in the proof of [CR81, Theorem 27.13] that $y \in \mathfrak{o}_\chi$ as desired. ☐

# 5 Consequences of the central conductor formula

In this section we derive several consequences of our main Theorem 4.3.

**Corollary 5.1.** *Let $\chi$ be an irreducible character of $G$ with open kernel. Then*

$$\frac{|H|w_\chi}{\chi(1)} \in \mathbb{Z}_p.$$

*In particular, $n_\chi$ and the Schur index $s_\chi$ are divisors of $|H|w_\chi$ in $\mathbb{Z}_p$.*

*Proof.* By Theorem 4.3 the quotient $\frac{|H|w_\chi}{\chi(1)} \in \mathbb{Q}_p$ belongs to the central conductor $\mathcal{F}(\Lambda^{\mathfrak{o}}(G)) \subset \zeta(\Lambda^{\mathfrak{o}}(G))$, so it is integral. The second assertion is clear, since $\chi(1) = n_\chi s_\chi$ by Corollary 1.5. □

## 5.1 Annihilation of Ext

**Corollary 5.2.** *Let $M$ be a $\Lambda^{\mathfrak{o}}(G)$-lattice and let $N$ be a $\Lambda^{\mathfrak{o}}(G)$-module. Then*

$$\left(\bigoplus_{\chi/\sim} \frac{|H|w_\chi}{\chi(1)} \cdot \mathcal{D}^{-1}(\mathfrak{o}_\chi/\mathfrak{o})\Lambda^{\mathfrak{o}_\chi}(\Gamma_\chi)\right) \cdot \mathrm{Ext}^i_{\Lambda^{\mathfrak{o}}(G)}(M,N) = 0$$

*for all integers $i \geq 1$.*

*Proof.* We do induction on the integer $i$. The case $i = 1$ is immediate from Theorem 4.3 and [CR81, Theorem 29.4]. For $k$ sufficiently large, there is an exact sequence

$$M' \rightarrowtail \Lambda^{\mathfrak{o}}(G)^k \twoheadrightarrow M.$$

As $M$ and $\Lambda^{\mathfrak{o}}(G)$ are projective (in fact free) as $R$-modules, so is $M'$, that is $M'$ is a $\Lambda^{\mathfrak{o}}(G)$-lattice. Applying $\mathrm{Hom}_{\Lambda^{\mathfrak{o}}(G)}(\_, N)$ to the above exact sequence gives isomorphisms

$$\mathrm{Ext}^j_{\Lambda^{\mathfrak{o}}(G)}(M',N) \simeq \mathrm{Ext}^{j+1}_{\Lambda^{\mathfrak{o}}(G)}(M,N)$$

for all integers $j \geq 1$. The case $j = i-1$ gives the induction step. □

Now we put $\mathfrak{d}_\chi^{-1} := (\mathcal{D}^{-1}(\mathfrak{o}_\chi/\mathfrak{o}) \cap K) \cdot R \supseteq R$. Then a proof similar to that of [CR81, Theorem 27.13 (ii)] shows the following result.

**Corollary 5.3.** *We have*

$$R \cap \mathcal{F}(\Lambda^{\mathfrak{o}}(G)) = \bigcap_{\chi/\sim} \frac{|H|w_\chi}{\chi(1)} \cdot \mathfrak{d}_\chi^{-1}.$$

For a $\Lambda^{\mathfrak{o}}(G)$-lattice $M$ let $\Upsilon(M) := \left\{e_\chi | e_\chi \cdot \mathcal{Q}^K(G) \otimes M = 0\right\}$.

**Corollary 5.4.** *Let $M$ and $N$ be $\Lambda^{\mathfrak{o}}(G)$-lattices. Then $\bigcap_{e_\chi \notin \Upsilon(M)} (|H|w_\chi/\chi(1))\mathfrak{d}_\chi^{-1}$ annihilates $\mathrm{Ext}^i_{\Lambda^{\mathfrak{o}}(G)}(M,N)$. In particular,*

$$\left(\bigcap_{\chi/\sim} \frac{|H|w_\chi}{\chi(1)} \cdot \mathfrak{d}_\chi^{-1}\right) \cdot \mathrm{Ext}^i_{\Lambda^{\mathfrak{o}}(G)}(M,N) = 0$$

*for any $\Lambda^{\mathfrak{o}}(G)$-lattices $M$ and $N$ and any integer $i \geq 1$.*

*Proof.* The last assertion is an immediate consequence of Corollary 5.2 and Corollary 5.3. The first assertion is also easy and is shown exactly in the same way as [CR81, Theorem 29.9]. □

**Corollary 5.5.** *Let $M$ and $N$ be $\Lambda^o(G)$-lattices and assume that $\mathcal{Q}^K(G)\otimes M$ is absolutely simple. Then there is a unique idempotent $e_\chi \notin \Upsilon(M)$ and for any integer $i \geq 1$ we have*

$$\frac{|H|w_\chi}{\chi(1)} \cdot \mathrm{Ext}^i_{\Lambda^o(G)}(M,N) = 0.$$

In fact, the annihilation results above are in some sense best possible. A proof along the lines of the proof [CR81, Theorem 29.22] gives the following analogue of a result of Roggenkamp [Ro71].

**Corollary 5.6.** *Let $x \in R$. Then*

1. *The element $x$ annihilates $\mathrm{Ext}^1_{\Lambda^o(G)}(M,N)$ for all $\Lambda^o(G)$-lattices $M$ and $N$ if and only if $x \in \bigcap_{\chi/\sim} \frac{|H|w_\chi}{\chi(1)} \cdot \mathfrak{d}_\chi^{-1}$.*
2. *For each central idempotent $e$ of $\mathcal{Q}^K(G)$ there exists a $\Lambda^o(G)$-lattice $M$ (namely $M = \tilde{\Lambda}^o(G)e$) such that $eM = M$ and*

$$x \cdot \mathrm{Ext}^1_{\Lambda^o(G)}(M,N) = 0 \text{ for all } N \iff x \in \bigcap_{e_\chi | e} \frac{|H|w_\chi}{\chi(1)} \mathfrak{d}_\chi^{-1}.$$

## 5.2 Non-commutative Fitting invariants

For the following we refer the reader to [Ni10]. Let $A$ be a separable $L$-algebra and $\Lambda$ be an $R$-order in $A$, finitely generated as $R$-module, where $R$ is an integrally closed complete commutative noetherian local domain with field of quotients $L$. Let $N$ and $M$ be two $\zeta(\Lambda)$-submodules of an $R$-torsionfree $\zeta(\Lambda)$-module. Then $N$ and $M$ are called $\mathrm{nr}(\Lambda)$-*equivalent* if there exists an integer $n$ and a matrix $U \in \mathrm{Gl}_n(\Lambda)$ such that $N = \mathrm{nr}(U) \cdot M$, where $\mathrm{nr} : A \to \zeta(A)$ denotes the reduced norm map which extends to matrix rings over $A$ in the obvious way. We denote the corresponding equivalence class by $[N]_{\mathrm{nr}(\Lambda)}$. We say that $N$ is $\mathrm{nr}(\Lambda)$-contained in $M$ if for all $N' \in [N]_{\mathrm{nr}(\Lambda)}$ there exists $M' \in [M]_{\mathrm{nr}(\Lambda)}$ such that $N' \subset M'$. We will say that $x$ is contained in $[N]_{\mathrm{nr}(\Lambda)}$ (and write $x \in [N]_{\mathrm{nr}(\Lambda)}$) if there is $N_0 \in [N]_{\mathrm{nr}(\Lambda)}$ such that $x \in N_0$. Now let $M$ be a finitely presented (left) $\Lambda$-module and let

$$\Lambda^a \xrightarrow{h} \Lambda^b \twoheadrightarrow M \tag{5}$$

be a finite presentation of $M$. We identify the homomorphism $h$ with the corresponding matrix in $M_{a\times b}(\Lambda)$ and define $S(h) = S_b(h)$ to be the set of all $b \times b$ submatrices of $h$ if $a \geq b$. The Fitting invariant of $h$ over $\Lambda$ is defined to be

$$\mathrm{Fitt}_\Lambda(h) = \begin{cases} [0]_{\mathrm{nr}(\Lambda)} & \text{if} \quad a < b \\ \left[\langle \mathrm{nr}(H) | H \in S(h)\rangle_{\zeta(\Lambda)}\right]_{\mathrm{nr}(\Lambda)} & \text{if} \quad a \geq b. \end{cases}$$

We call $\mathrm{Fitt}_\Lambda(h)$ a Fitting invariant of $M$ over $\Lambda$. One defines $\mathrm{Fitt}^{\max}_\Lambda(M)$ to be the unique Fitting invariant of $M$ over $\Lambda$ which is maximal among all Fitting invariants of $M$ with respect to the partial order "$\subset$".

We now specialize to the situation in this article, where $\Lambda$ is $\Lambda^{\mathfrak{o}}(G)$. Then Theorem 4.3 and [Ni10, Lemma 4.1 and Theorem 4.2] imply the following result.

**Corollary 5.7.** *Let $M$ be a finitely presented $\Lambda^{\mathfrak{o}}(G)$-module. Then*

$$\left(\bigoplus_{\chi/\sim} \frac{|H|w_\chi}{\chi(1)} \cdot \mathcal{D}^{-1}(\mathfrak{o}_\chi/\mathfrak{o})\Lambda^{\mathfrak{o}_\chi}(\Gamma_\chi)\right) \cdot \mathrm{Fitt}^{\max}_{\Lambda^{\mathfrak{o}}(G)}(M) \subset \mathrm{Ann}_{\Lambda^{\mathfrak{o}}(G)}(M).$$

Together with Corollary 5.3 and [Ni10, Lemma 3.4] this yields the following corollary.

**Corollary 5.8.** *Let $M$ be a finitely presented $\Lambda^{\mathfrak{o}}(G)$-module. Then*

$$\left(\bigcap_{e_\chi \notin \Upsilon(M)} \frac{|H|w_\chi}{\chi(1)} \cdot \mathfrak{d}_\chi^{-1}\right) \cdot \mathrm{Fitt}^{\max}_{\Lambda^{\mathfrak{o}}(G)}(M) \subset \mathrm{Ann}_{\Lambda^{\mathfrak{o}}(G)}(M).$$

**Remark 5.9.** *Note that in fact $\mathcal{H}(\Lambda^{\mathfrak{o}}(G)) \cdot \mathrm{Fitt}^{\max}_{\Lambda^{\mathfrak{o}}(G)}(M) \subset \mathrm{Ann}_{\Lambda^{\mathfrak{o}}(G)}(M)$, where $\mathcal{H}(\Lambda^{\mathfrak{o}}(G))$ is a certain ideal of $\zeta(\Lambda^{\mathfrak{o}}(G))$ which always contains the central conductor. In general, however, this containment is not an equality. Though the ideal $\mathcal{H}(\Lambda^{\mathfrak{o}}(G))$ is hard to determine in general, considerable progress is made in [JN]; in particular, by [JN, Corollary 4.15] one knows that $\mathcal{H}(\Lambda^{\mathfrak{o}}(G))$ equals $\zeta(\Lambda^{\mathfrak{o}}(G))$ (to wit: is best possible) if and only if $p$ does not divide the order of the commutator subgroup of $G$ (which is finite).*

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