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Algebraische Zahlentheorie

Organised by
Benjamin Howard, Chestnut Hill
Guido Kings, Regensburg
Ramdorai Sujatha, Bombay
Otmar Venjakob, Heidelberg

June 19th – June 25th, 2011

Abstract. The workshop brought together researchers from Europe, Japan and the US, who reported on various recent developments in algebraic number theory and related fields. Dominant topics were Shimura varieties, automorphic forms and Iwasawa theory.

Mathematics Subject Classification (2000): 11R, 11S.

Introduction by the Organisers

The workshop *Algebraische Zahlentheorie*, organised by Benjamin Howard (Chestnut Hill), Guido Kings (Regensburg), Ramdorai Sujatha (Bombay) and Otmar Venjakob (Heidelberg) was well attended with 55 participants from Europe, Japan and the US. In total we had 19 talks on various topics such as p-adic Hodge theory, Galois representations and p-adic representation theory, automorphic forms, Shimura varieties, Iwasawa theory etc.

This time the Algebraic Number Theory Workshop was completely dominated by tremendous results and talks of young mathematicians. To start with, Peter Scholze described the theory of perfectoid spaces, a general framework for questions of changing between equal- and mixed-characteristic local fields, which leads to an improvement on Faltings's almost purity theorem as well as the proof of a new important special case of the weight-monodromy conjecture l-adic cohomology. His work is similarly based on Huber's adic spaces as Eugen Hellmann's new approach to "arithmetic families of filtered ϕ-modules and crystalline representations generalising vastly Kisin's weakly admissible filtered ϕ-modules. Moritz

Kertz contributed a strong result (jointly with Spencer Bloch, Hélène Esnault) on the formal deformation part of p-adic variational Hodge conjecture.

Another group of talks concerned Iwasawa theory. Jonathan Pottharst and Antonio Lei (joint work with Sarah Zerbes and David Löffler) each proposed new approaches for Selmer groups and related main conjectures in the non-ordinary case: while Pottharst considers families of Galois representations and uses (ϕ, Γ)-modules over the Robba-ring, Lei considered Galois representations attached to weight k normalised eigen-newforms using Wach-modules. Takako Fukaya reports on a joint work with Kazuya Kato concerning the proof (in certain cases) of a conjecture by Sharifi, which he had presented in Oberwolfach during an earlier Algebraic Number Theory workshop and which relates a cup-product pairing in Galois cohomology with L-values of certain cusp forms. Ming-Lun Hsieh reported on strong results towards the Iwasawa Main Conjecture for CM fields and their descent implications towards the (p-adic) Birch&Swinnerton-Dyer Conjecture. Thanasis Bouganis used similar automorphic techniques in order to show (the first step) of those congruences among certain abelian p-adic L-functions attached to unitary groups which show up in the work of Kakde and Ritter & Weiss to indicate the existence of non-abelian p-adic L-functions. Finally Cornelius Greither describes joint work with Cristian Popescu on Fitting ideals associated with 1-motives over global fields and related Equivariant Main Conjectures, in particular they obtain explicit constructions of Tate sequences.

With respect to another dominating topic, viz Galois representations and p-adic representation theory (towards p-adic local Langlands), we had first of all Gaëtan Cheneviers report on joint work with Jean Lannes concerning the classification of certain Galois representations of dimension 16 an 24 related to the set of isometry classes of even unimodular lattices in the standard euclidean space of the same dimensions. While Jan Kohlhaase reported on joint work with Benjamin Schraen on "Homological vanishing theorems for locally analytic representations", Florian Herzig reported on "Weights in a Serre-type conjecture for $U(3)$", joint work with Matthew Emerton and Toby Gee on generalisations of Serre's conjectures if the reductive group is an outer form of GL_3.

Concerning p-adic Hodge theory Takeshi Tsuji discussed extensions of the functor D_{crys} to the category of p-adic perverse sheaves with singularities along a simple normal crossing divisor. Jan Brunier described an interpretation of the coefficients of the q-expansion of certain weight 1/2 harmonic weak Maass forms in terms of Heegner divisors.

Also there were a couple of talks concerning Shimura varieties and period spaces beginning with the talk of Michael Rappoport, one of the few senior speakers, on the Arithmetic Fundamental Lemma of (and jointly with) Wei Zhang. Eva Viehmann presented the proof of a conjecture by Harris on the cohomology of Rapoport-Zink spaces being parabolically induced from that of a smaller moduli space while Fritz Hörmann and Keerthi Shyam Madapusi Sampath talked about heights of special cycles on and certain compactifications of Shimura varieties, respectively.

Bianca Viray discussed transcendental elements in Brauer groups of elliptic surfaces.

Finally it is perhaps worth mentioning that the organisers were quite relieved about having postponed the traditional hiking tour (this time to St. Roman) from Wednesday to Thursday afternoon, thereby avoiding heavy thunderstorms.

Workshop: Algebraische Zahlentheorie

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Abstracts

On the Arithmetic Fundamental Lemma of Wei Zhang

Michael Rapoport
(joint work with Wei Zhang)

The Fundamental Lemma (FL) arises in the relative trace formula approach of Jacquet/Rallis to the Gross-Prasad conjecture for the inclusion of a unitary group of size $n-1$ in a unitary group of size n. The AFL arises in the context of the arithmetic Gross-Prasad conjecture for the inclusion of a Shimura variety for a unitary group of size $n-1$ in a Shimura variety for a unitary group of size n. In the talk I stated the FL conjecture and AFL conjecture, and I explained how the FL conjecture is equivalent to a tantalizingly simple identity between two counting functions of certain lattices. In this report I restrict myself to the AFL conjecture.

Let F be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_F$ and residue field k with q elements. Let E/F be an unramified quadratic extension with ring of integers $\mathcal{O}_E$ and residue field k'. Let $\sigma \in \mathrm{Gal}(E/F)$ be the non-trivial element. Let $n \geq 1$.

Consider GL_{n-1} as an algebraic subgroup of GL_n via the embedding $F^{n-1} \hookrightarrow F^n$ (trivial last component). We consider the conjugation action of GL_{n-1} on GL_n. An element $g \in \mathrm{GL}_n(E)$ is called *regular semi-simple* (rs) for this action, if its orbit under GL_{n-1} is Zariski-closed and its stabilizer in GL_{n-1} is trivial. Let

$$S(F) = \{s \in \mathrm{GL}_n(E) \mid s \cdot \sigma(s) = 1\}\,.$$

Then $S(F)$ is stable under $\mathrm{GL}_{n-1}(F)$. We set

$$S(F)_{rs} = S(F) \cap \mathrm{GL}_n(E)_{rs}\,.$$

For $\gamma \in S(F)_{rs}$ and $f \in C_c^\infty(S(F))$ and $s \in \mathbb{C}$ define

$$O_\gamma(f,s) = \int\limits_{\mathrm{GL}_{n-1}(F)} f(h^{-1}\gamma h)\eta(\det h)|\det h|^s dh\,.$$

Here $\eta = \eta_{E/F}$ is the quadratic character of $F^\times$ corresponding to E/F by local classfield theory, and the Haar measure is normalized by $\mathrm{vol}\,(S(\mathcal{O}_F)) = 1$, where $S(\mathcal{O}_F) = S(F) \cap \mathrm{GL}_n(\mathcal{O}_E)$. We put

$$O'_\gamma(f) = \frac{d}{ds} O_\gamma(f,s)_{|s=0}\,.$$

Let $J_1 \in \mathrm{Herm}_{n-1}(E/F)$ be the *non-split* hermitian form of size $n-1$ relative to E/F. By adding on an orthogonal vector u with $(u,u) = 1$, we obtain

$$\mathrm{U}(J_1 \oplus 1)(F) \subset \mathrm{GL}_n\,(E)\,.$$

Put

$$\mathrm{U}(J_1 \oplus 1)(F)_{rs} = \mathrm{U}(J_1 \oplus 1)(F) \cap \mathrm{GL}_n\,(E)_{rs}\,.$$

An element $\gamma \in S(F)_{rs}$ is said to *match* an element $g \in \mathrm{U}(J_1 \oplus 1)(F)_{rs}$ if γ and g are conjugate under $\mathrm{GL}_{n-1}(E)$ (as elements of $\mathrm{GL}_n(E)$). Then g is unique up to conjugation by $\mathrm{U}(J_1)(F)$.

Conjecture 1. (**AFL**) (Wei Zhang [3]): *Assume $\gamma \in S(F)_{rs}$ matches the element $g \in \mathrm{U}(J_1 \oplus 1)(F)_{rs}$. Then*

$$O'_\gamma(1_S) = -\omega(\gamma) \cdot \langle \Delta(N_{n-1}), (1,g)\Delta(N_{n-1}) \rangle \,.$$

Here $1_S = \operatorname{char} S(\mathcal{O}_F)$. And $\omega(\gamma)$ is the following natural sign factor associated to γ,

$$\omega(\gamma) = (-1)^{\mathrm{val}\,(\det\,(\gamma_i v)_{i=0,\ldots,n-1})}\,,$$

where $v = (0,\ldots,0,1) \in E^n$. The bracket denotes an arithmetic intersection number. More precisely, let $\breve{F}$ be the completion of the maximal unramified extension of E, and $\mathcal{O}_{\breve{F}}$ its ring of integers. Let $\mathcal{N}_n$ be a unitary RZ-space of signature $(1, n-1)$, cf. [1]. Then $\mathcal{N}_n$ is formal scheme locally formally of finite type over $\operatorname{Spf} \mathcal{O}_{\breve{F}}$ of arithmetic formal dimension n. For $n = 1$, $\mathcal{N}_1 = \operatorname{Spf} \mathcal{O}_{\breve{F}}$. The group $\mathrm{U}(J_1 \oplus 1)(F)$ acts on $\mathcal{N}_n$. There is a closed embedding of formal schemes over $\operatorname{Spf} \mathcal{O}_{\breve{F}}$,

$$\delta : \mathcal{N}_{n-1} = \mathcal{N}_{n-1} \times_{\operatorname{Spf} \mathcal{O}_{\breve{F}}} \mathcal{N}_1 \longrightarrow \mathcal{N}_n \,.$$

This defines

$$\Delta : \mathcal{N}_{n-1} \xrightarrow{(\mathrm{id},\delta)} \mathcal{N}_{n-1} \times_{\operatorname{Spf} \mathcal{O}_{\breve{F}}} \mathcal{N}_n \,.$$

Then the bracket on the RHS is defined by

$$\chi(\mathcal{O}_{\Delta(\mathcal{N}_{n-1})} \otimes^{\mathbb{L}} \mathcal{O}_{(1,g)\Delta(\mathcal{N}_{n-1})}) \cdot \log q \,.$$

This concludes the statement of the AFL-conjecture.

Theorem 2. (Wei Zhang [3]) *AFL holds for $n = 2$ and $n = 3$.*

In the remainder of my talk I reported on results of joint work in progress with Wei Zhang on AFL, for arbitrary n, but strong restrictions on g, cf. [2]. Recall the unimodular vector $u \in E^n$. Let

$$L_g = \langle u, gu, \ldots, g^{n-1}u \rangle_{\mathcal{O}_E}$$

(a $\mathcal{O}_E$-lattice in E^n) and let L_g^* be its dual lattice. Then $L_g \subset L_g^*$. Our results concern the *minuscule case*, i.e., (L_g^*/L_g) is a k'-vector space.

References

[1] S. Kudla, M. Rapoport, *Special cycles on unitary Shimura varieties, I. Unramified local theory*, Invent. Math. 184 (2011), no. 3, 629–682.

[2] M. Rapoport, W. Zhang, *On the arithmetic fundamental lemma for unitary groups*, in preparation.

[3] W. Zhang, *On arithmetic fundamental lemmas*, preprint 2010.

Perfectoid Spaces

Peter Scholze

We explained the theory of perfectoid spaces, which gives a general framework for questions of changing between equal- and mixed-characteristic local fields.

As an example, let K be the completion of the field $\mathbb{Q}_p(p^{1/p^\infty})$, and K' be the t-adic completion of $\mathbb{F}_p((t^{1/p^\infty}))$. We have the following theorem, due to Fontaine-Wintenberger, [1]:

Theorem 1. *There is a canonical isomorphism of absolute Galois group $G_K \cong G_{K'}$.*

In other words, the categories of finite extensions of K and K' are identified. Our aim is to generalize this theorem to the relative situation. First, we have to find a suitable category of rings.

Definition 2. *A perfectoid K-algebra is a Banach K-algebra R such that the subset of power-bounded elements $R^\circ \subset R$ is open and bounded, and such that Frobenius is surjective on K°/p.*

There is an analogous definition of perfectoid K'-algebras R', where the last condition simply says that R' is perfect. Now we have the following theorem, which we call the tilting equivalence.

Theorem 3. *The categories of perfectoid K-algebras and of perfectoid K'-algebras are equivalent.*

One can explicitly write down the functors in both directions à la Fontaine, e.g.: $R \mapsto R' = (\varprojlim R^\circ/p)[t^{-1}]$. As usual, this allows to define a (multiplicative, but not additive) map $R' \to R$ which we denote $f' \mapsto [f']$. However, one can also give a purely abstract proof of this theorem using (a variant of) the vanishing of the cotangential complex for perfect rings.

In order to pass to spaces, one has to use some language of nonarchimedean geometry, and we choose to work with Huber's adic spaces, as defined in [2]. These spaces are associated to pairs (R, R^+), where $R^+ \subset R^\circ$ is open and integrally closed. In our context, we choose the ring R perfectoid. We call such pairs (R, R^+) affinoid perfectoid K-algebras. In general, Huber defines a topological space $X = \mathrm{Spa}(R, R^+)$ consisting of equivalence classes of continuous valuations on R that are ≤ 1 on R^+. Moreover, there is a natural structure presheaf $\mathcal{O}_X$ on X, which in general is not a sheaf. One also has the subpresheaf $\mathcal{O}_X^+ \subset \mathcal{O}_X$ consisting of functions of valuation at most 1.

Theorem 4. *Let (R, R^+) be an affinoid perfectoid K-algebra. We can canonically form the tilt (R', R'^+). Let $X = \mathrm{Spa}(R, R^+)$, with $\mathcal{O}_X$, and $X' = \mathrm{Spa}(R', R'^+)$, with $\mathcal{O}_{X'}$.*

(i) *There is a canonical homeomorphism $X \cong X'$, given by mapping x to x' defined via $|f'(x')| = |[f'](x)|$. Rational subsets are identified under this homeomorphism.*

(ii) *For any rational subset $U \subset X$, the pair $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is affinoid perfectoid with tilt $(\mathcal{O}_{X'}(U), \mathcal{O}_{X'}^+(U))$.*
(iii) *The presheaves $\mathcal{O}_X$, $\mathcal{O}_{X'}$ are sheaves.*
(iv) *For all $i > 0$, the cohomology group $H^i(X, \mathcal{O}_X) = 0$; even better, the cohomology group $H^i(X, \mathcal{O}_X^+)$ is almost zero, i.e. annihilated by the maximal ideal of the valuation ring of K.*

This allows one to define general perfectoid spaces by gluing affinoid perfectoid spaces, and tilting extends to spaces. Further, one can define etale morphisms of perfectoid spaces, and then etale topoi. This leads to an improvement on Faltings's almost purity theorem:

Theorem 5. *Let R be a perfectoid K-algebra, and let S/R be finite etale; in particular, S gets a natural structure as Banach algebra. Then S is perfectoid and S° is almost finitely presented etale over R°.*

Corollary 6. *For any perfectoid space X over K with tilt X', we have an equivalence of sites $X_{\mathrm{et}} \cong X'_{\mathrm{et}}$.*

Corollary 7. *We have an equivalence of etale topoi of adic spaces: $(\mathbb{P}^n_{K'})^{\mathrm{ad}}_{\mathrm{et}} \cong \varprojlim (\mathbb{P}^n_K)^{\mathrm{ad}}_{\mathrm{et}}$. Here the transition maps are the p-th power map on coordinates.*

As an application, we prove the following special case of the weight-monodromy conjecture by reduction to characteristic p.

Theorem 8. *Let $X \subset T$ be a smooth complete intersection in a proper toric variety, over a p-adic field. Then for $\ell \neq p$, the ℓ-adic cohomology groups of X satisfy the weight-monodromy conjecture.*

References

[1] Fontaine, J.-M.; Wintenberger, J.-P., *Le "corps des normes" de certaines extensions algébriques de corps locaux*, C. R. Acad. Sci. Paris Sér. A-B **288** (1979), no. 6, A367A370.
[2] Huber, R., *A generalization of formal schemes and rigid analytic varieties*, Math. Z. **217** (1994), no. 4, 513551.

Selmer groups of nonordinary motives

Jonathan Pottharst

We describe the problem of interpolating the Selmer groups of the members of a p-adic family of motives [14]. These subtle invariants are defined as follows. Suppose that V is a finite-dimensional $\mathbf{Q}_p$-vector space with a continuous, linear action of $G_{\mathbf{Q},S}$ that is crystalline at p. Its *Selmer groups* $H^*_f(\mathbf{Q}, V)$ are the cohomology of the *Selmer complex* $R\Gamma_f(G_{\mathbf{Q},S}, V)$, obtained by modifying Galois cohomology [5]: one forms the mapping fiber of

$$R\Gamma(G_{\mathbf{Q},S}, V) \oplus \bigoplus_{\ell \in S} R\Gamma_f(G_{\mathbf{Q}_\ell}, V) \to \bigoplus_{\ell \in S} R\Gamma(G_{\mathbf{Q}_\ell}, V),$$

where the first term on the left maps via restriction, and the other terms and maps are as follows. If $\ell \in S$ is not equal to p, then $R\Gamma_f(G_{\mathbf{Q}_\ell}, V) = [V^{I_\ell} \xrightarrow{1-F} V^{I_\ell}]$ (with $F \in G_{\mathbf{F}_\ell}$ a Frobenius element), mapping via the Hochschild–Serre/inflation morphism. If $\ell = p$, then $R\Gamma_f(G_{\mathbf{Q}_p}, V) = [D_{\mathrm{crys}} \xrightarrow{(1-\varphi,1)} D_{\mathrm{crys}} \oplus D_{\mathrm{dR}}/\mathrm{Fil}^0]$, where $D_{\mathrm{crys}} = D_{\mathrm{crys}}(V|_{G_{\mathbf{Q}_p}})$, mapping via the Bloch–Kato exponential. For $G = G_{\mathbf{Q},S}, G_{\mathbf{Q}_\ell}$, a general principle of *base change* shows that if $\mathcal{V}$ is a family of G-representations above over a base X, and $Y \to X$ is a morphism, then there is a canonical isomorphism $R\Gamma(G, \mathcal{V}) \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_Y \cong R\Gamma(G, V \otimes_{\mathcal{O}_X} \mathcal{O}_Y)$. This reduces the problem of interpolation, essentially, to the interpolation of local conditions at p. The main problem is that neither the Frobenius nor the Hodge filtration of D_{crys} vary continuously with V.

Classically, this problem has been satisfactorily overcome only for ordinary families. We call V *ordinary* if there exists $V^+ \subseteq V|_{G_{\mathbf{Q}_p}}$ such that V^+ (resp. V/V^+) has all negative (rep. nonnegative) Hodge–Tate weights, where we take the geometric normalization that $\mathbf{Q}_p(1)$ has Hodge–Tate weight -1. Then, under a generic hypothesis, the natural maps

$$R\Gamma(\mathbf{Q}_p, V^+) \leftarrow R\Gamma_f(\mathbf{Q}_p, V^+) \to R\Gamma_f(\mathbf{Q}_p, V)$$

are quasi-isomorphisms [8], and one can use $R\Gamma(\mathbf{Q}_p, V^+)$ as a substitute for the local condition $R\Gamma_f(\mathbf{Q}_p, V)$. When the V^+ fit into the family along with the V, then one accordingly solves the interpolation problem; the result is the *control theorem of Greenberg and Nekovář* [9, 12]. The main problem now is that, with p fixed, nonordinary motives are rather common.

Our method to handle many of the remaining cases begins with the observation that even if V is nonordinary, one often has φ-stable $D^+_{\mathrm{crys}} \subseteq D_{\mathrm{crys}}$ such that the filtration on D^+_{crys} (resp. $D_{\mathrm{crys}}/D^+_{\mathrm{crys}}$) induced by the Hodge filtration has all negative (resp. nonnegative) weights; for example, in the ordinary case, one can take $D^+_{\mathrm{crys}} = D_{\mathrm{crys}}(V^+)$. Then, D^+_{crys} is of the form $(D^+)_{\mathrm{crys}}$ for a unique subobject $D^+ \subseteq D$ of the (φ, Γ)-module $D = D_{\mathrm{rig}}(V|_{G_{\mathbf{Q}_p}})$ over the Robba ring [3]. The latter objects have local Galois cohomology (via the *Herr complex* [11]) and local conditions, and one has $R\Gamma_{(f)}(\mathbf{Q}_p, V) \cong R\Gamma_{(f)}(\mathbf{Q}_p, D)$. One shows that one can replace $R\Gamma_f(\mathbf{Q}_p, V)$ with $R\Gamma(\mathbf{Q}_p, D^+)$, similarly to above. Our control theorem, generalizing Greenberg and Nekovář's, follows by the same method, presupposing (1) the D^+ interpolate along with the V to begin with, and (2) the Galois cohomology of (φ, Γ)-modules satisfies base change [14].

The hypothesis (2) follows once one knows that the Galois cohomology of families of (φ, Γ)-modules is finitely generated; only partial results are known in this direction [7, 1, 6, 16, 15], and we consider it to be a key open problem. The hypothesis (1) might seem to be a problem in light of the difficulties with D_{crys}, but in the passage from D_{crys} to D the discontinuities in the Frobenius and Hodge filtration seem rather magically to cancel each other, and such families of D^+ seem common. (To begin with, affinoid families of Galois representations give rise to families of (φ, Γ)-modules [4, 10].) The families seem to arise most generally, even universally, as *refined deformation problems* [2, 6] or (on the other side of

the Langlands philosophy) as *local eigenvarieties*. Only partial results in this direction are known, but more might soon be within reach. It is worth noting that the results we do have seem to go through finiteness of Galois cohomology as a fundamental ingredient.

More definitive results are available in the case of the cyclotomic deformation of a fixed motive [15]. Recall that $\Gamma = \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) = \mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p)$, and one puts $\Lambda = \mathbf{Z}_p[\![\Gamma]\!]$, $W = \mathrm{Spf}(\Lambda)^{\mathrm{rig}}$, and $\Lambda_{\mathrm{rig}} = \Gamma(W, \mathcal{O}_W)$. Thus W is a disjoint union of open unit disks, Λ_{rig} is its rigid analytic functions, and Λ is those functions bounded by 1. We consider $\Lambda, \mathcal{O}_W, \Lambda_{\mathrm{rig}}$ with Γ-action via multiplication by inverses in $\Gamma \subset \Lambda^\times$, and one forms cyclotomic deformations in the appropriate categories by tensoring with these rings or $D_{\mathrm{rig}}(\mathcal{O}_W)$. We show that the cyclotomic deformation of any fixed (φ, Γ)-module has finitely generated Galois cohomology over affinoids in W satisfying a version of Tate's local duality and Euler–Poincaré formula. Since base change holds, we may patch over an affinoid covering of W, and the standard recipe now gives Selmer complexes of $\mathcal{O}_W$-modules with coherent cohomology. By the "coadmissible module" formalism of Schneider–Teitelbaum [17], they may be identified to complexes $R\Gamma_{f,\mathrm{rig}}(\mathbf{Q}, V)$ with coadmissible cohomology, satisfying a version of Poitou–Tate's global duality and Euler–Poincaré formula. In particular, all the inputs are in place for the control theorem to hold, and the Bloch–Kato Selmer groups are interpolated. Moreover, we show that, under generic hypotheses, if $H^2_{f,\mathrm{rig}}(\mathbf{Q}, V)$ is a *torsion* Λ_{rig}-module, then Perrin–Riou's *Weak Leopoldt Conjecture* holds for $V, V^*(1)$, and its characteristic ideal agrees with Perrin–Riou's *algebraic p-adic L-function* [13] up to "Gamma factors" depending only on the Hodge–Tate weights of V. The proof involves a theory of *Wach modules* over $B^+_{\mathrm{rig},\mathbf{Q}_p}$ for crystalline (φ, Γ)-modules, giving rise to a big logarithm map, and a variant of the "$\delta(D)$-theorem", which calculates its determinant in terms of the Gamma factors. Finally, in the case where V is associated to an elliptic modular cuspform f, Kato's Euler system shows that our Selmer groups are indeed torsion, and the two characteristic ideals (associated to the two possible φ-eigenspaces in D_{crys}) divide the *classical* p-adic L-functions $L_p(f, \alpha), L_p(f, \beta)$.

References

[1] Joël Bellaïche, Ranks of Selmer groups in an analytic family. To appear in *Trans. Amer. Math. Soc.*

[2] Joël Bellaïche and Gaëtan Chenevier, *Families of Galois representations and Selmer groups*, Astérisque **324** (2009).

[3] Laurent Berger, Équations différentielles p-adiques et (φ, N)-modules filtrés. *Représentations p-adiques de groupes p-adiques I: Représentations galoisiennes et* (φ, Γ)*-modules*, Astérisque **319** (2008), pp. 13–38.

[4] Laurent Berger and Pierre Colmez, Familles de représentations de de Rham et monodromie p-adique. *Représentations p-adiques de groupes p-adiques I: Représentations galoisiennes et* (φ, Γ)*-modules*, Astérisque **319** (2008), pp. 303–337.

[5] Spencer Bloch and Kazuya Kato, L-functions and Tamagawa numbers of motives. *The Grothendieck Festschrift, Vol. I,* Progr. Math. **85** (1990), pp. 333–400.

[6] Gaëtan Chenevier, Sur la densité de représentations cristallines de $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Preprint.

[7] Pierre Colmez, Représentations triangulines de dimension 2. *Représentations p-adiques de groupes p-adiques I: Représentations galoisiennes et* (φ, Γ)*-modules*, Astérisque **319** (2008), pp. 213–258.
[8] Takako Fukaya and Kazuya Kato, A formulation of conjectures on p-adic zeta functions in noncommutative Iwasawa theory. *Proceedings of the St. Petersburg Mathematical Society, Vol. XII*, Amer. Math. Soc Transl. Ser. 2 **219** (2006), pp. 1–85.
[9] Ralph Greenberg, Iwasawa theory for p-adic representations. *Algebraic number theory*, Adv. Stud. Pure Math. **17** (1989), pp. 97–137.
[10] Kiran S. Kedlaya and Ruochuan Liu, On families of (φ, Γ)-modules. *Algebra & Number Theory* **4** (2010), no. 7, pp. 943–967.
[11] Ruochuan Liu, Cohomology and duality for (φ, Γ)-modules over the Robba ring. *Int. Math. Res. Not.* **2008**, no. 3, Art. ID rnm150.
[12] Jan Nekovář, *Selmer complexes*. Astérisque **310** (2006).
[13] Bernadette Perrin-Riou, *p-adic L-functions and p-adic representations*. SMF/AMS Texts and Monographs **3** (2000).
[14] Jonathan Pottharst, Analytic families of finite-slope Selmer groups. Preprint.
[15] Jonathan Pottharst, Cyclotomic Iwasawa theory of motives. Preprint.
[16] Jonathan Pottharst and Liang Xiao, Cohomology of families of (φ, Γ)-modules. Preprint.
[17] Peter Schneider and Jeremy Teitelbaum, Algebras of p-adic distributions and admissible representations. *Invent. math.* **153** (2000), no. 1, pp. 145–196.

p-adic deformation of cycle classes

Moritz Kerz
(joint work with Spencer Bloch, Hélène Esnault)

Aim: Understand formal deformation part of p-adic variational Hodge conjecture.

Notation:

- k perfect field, $\mathrm{ch}(k) = p > 0$,
- W ring of Witt vectors of k, $W_n = W/p^n$,
- $X \to W$ smooth projective scheme, $X_n = X \otimes_W W_n$, $\hat{X}$ formal p-adic completion of X, $\iota : X_1 \to X$ immersion of closed fibre.

Conjecture 1. *Fontaine, Mazur and Emerton suggested that for* $\xi_1 \in \mathrm{CH}^r(X_1)_{\mathbb{Q}}$ *with crystalline cycle class*

$$[\xi_1] \in H^{2r}_{\mathrm{cris}}(X_1/W)_{\mathbb{Q}} \cong H^{2r}_{\mathrm{dR}}(X/W)_{\mathbb{Q}}$$

lying inside the Hodge piece $F^r H^{2r}_{\mathrm{dR}}(X/W)_{\mathbb{Q}}$*, there exists* $\xi \in \mathrm{CH}^r(X)_{\mathbb{Q}}$ *with*

$$[\iota^* \xi] = [\xi_1] \in H^{2r}_{\mathrm{cris}}(X_1/W)_{\mathbb{Q}}.$$

Remark 2. *The case* $r = 1$ *of the conjecture is shown by Berthelot-Ogus* [1].

Ansatz for $p > r \geq 1$:

We construct motivic pro-complex $(\mathbb{Z}_n(r))_{n \geq 1}$ of $\hat{X}$. Then we define Chow groups of the formal scheme $\hat{X}$ to be

$$\mathrm{CH}^r(\hat{X}) = H^{2r}_{\mathrm{cont}}(X_{1,\mathrm{Zar}}, \mathbb{Z}_{\cdot}(r)).$$

We define $(\mathbb{Z}_n(r))_n$ by glueing the following data:

- The Suslin-Voevodsky motivic complex $\mathbb{Z}_{X_1}(r)$ of X_1 in the Zariski topology [3],
- the synotomic complex
$$\sigma_n(r) = \tau_{\leq r}\, R\epsilon_*\, \sigma_n(r)_{\mathrm{et}},$$
where $\sigma_n(r)_{\mathrm{et}}$ is the syntomic complex in the étale topology [2] and $\epsilon : X_{1,\mathrm{et}} \to X_{1,\mathrm{Zar}}$ is the morphism of sites.

Set
$$\mathbb{Z}_n(r) = \mathrm{hofib}(\sigma_n(r) \oplus \mathbb{Z}_{X_1}(r) \to W_n\Omega^r_{X_1,\log}[-r]).$$

Proposition 3.
- *$\mathcal{H}^r(\mathbb{Z}_n(r))$ is isomorphic to the Milnor K-sheaf $\mathcal{K}^{\mathrm{M}}_{r,X_{n+1}}$,*
- *there is an exact triangle of pro-systems*
$$p(r)\Omega^{\leq r-1}_{\hat{X}}[-1] \to \mathbb{Z}_{\cdot}(r) \to \mathbb{Z}_{X_1}(r) \xrightarrow{\alpha} \cdots,$$
where α is related to the crystalline cycle class.

Here $p(r)\Omega^\bullet$ is the de Rham complex
$$p^r\mathcal{O}_{\hat{X}} \to p^{r-1}\Omega 1_{\hat{X}} \to \cdots \to p\Omega^{r-1}_{\hat{X}} \to \Omega^r_{\hat{X}} \to \Omega^{r+1}_{\hat{X}} \to \cdots.$$

Theorem 4. *For $\xi_1 \in \mathrm{CH}^r(X_1)$ the following are equivalent:*
$$[\xi_1] \in \mathrm{im}(H^{2r}(X, \Omega^{\geq r}_X) \to H^{2r}(X, p(r)\Omega^\bullet_X)$$
there is $\xi_1 \in \mathrm{CH}^r(\hat{X})$ with $\iota^*\xi = \xi_1$.

References

[1] Berthelot, P.; Ogus, A. *F-isocrystals and de Rham cohomology I*, Invent. Math. 72 (1983), no. 2, 159–199.

[2] Kato, K. *On p-adic vanishing cycles (application of ideas of Fontaine-Messing)*, Algebraic geometry, Sendai, 1985, 207–251.

[3] Mazza, C.; Voevodsky, V.; Weibel, C. *Lecture notes on motivic cohomology*, Cambridge, MA, 2006.

Homological vanishing theorems for locally analytic representations

Jan Kohlhaase

(joint work with Benjamin Schraen)

Let p be a prime number, let L be a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, let $\mathbb{G}$ be a connected, reductive group over L, let $G := \mathbb{G}(L)$ be its group of L-rational points, and let $\Gamma \subseteq G$ be a discrete and cocompact subgroup of G. The (co)homology of Γ-representations has been an area of research for a long time. One of the most striking results in this direction is the following vanishing theorem due to Garland, Casselman, Prasad, Borel and Wallach.

Theorem (Garland et al.)**.** *If Γ is irreducible, if the L-rank r of $\mathbb{G}$ is at least 2, and if V is a finite dimensional representation of Γ over a field of characteristic zero, then $\mathrm{H}^i(\Gamma, V) = 0$, unless $i \in \{0, r\}$.*

The proof of this theorem uses the full force of the theory of smooth complex representations of G. If $X \subseteq \mathbb{P}^d_L$ denotes Drinfeld's p-adic symmetric space, if $\Gamma \subseteq \mathrm{PGL}_{d+1}(L)$ acts without fixed points on X, and if $X_\Gamma := \Gamma\backslash X$ denotes the quotient of X by Γ, then the same methods were used by Schneider and Stuhler to compute the de Rham cohomology $\mathrm{H}_{\mathrm{dR}}(X_\Gamma)$ of X_Γ from that of X (cf. [3]). The interest in the rigid varieties X_Γ stems from the fact that they uniformize certain Shimura varieties.
The case of trivial coefficients was extended by Schneider who considered finite dimensional algebraic representations M of $\mathrm{SL}_{d+1}(L)$ over L and the induced locally constand sheaf $\mathcal{M}_\Gamma$ on X_Γ (cf. [2]). He formulated several conjectures on the structure of the de Rham cohomology $\mathrm{H}_{\mathrm{dR}}(X_\Gamma, \mathcal{M}_\Gamma)$ which are related to two spectral sequences

$$\begin{aligned} E_2^{p,q} &= \mathrm{H}^p(\Gamma, \mathrm{H}_{\mathrm{dR}}(X) \otimes_L M) \quad \Longrightarrow \quad \mathrm{H}^{p+q}(X_\Gamma, \mathcal{M}_\Gamma) \\ E_1^{p,q} &= \mathrm{H}^q(\Gamma, \Omega^p_X(X) \otimes_L M) \quad \Longrightarrow \quad \mathrm{H}^{p+q}(X_\Gamma, \mathcal{M}_\Gamma). \end{aligned}$$

Whereas $\mathrm{H}_{\mathrm{dR}}(X)$ is the dual of a smooth representation, the global differential forms $\Omega^p_X(X)$ are Fréchet spaces over L which carry a *locally analytic* action of $\mathrm{PGL}_{d+1}(L)$ in the sense of Schneider-Teitelbaum. Representations of this type were intensively studied by Morita, Schneider-Teitelbaum and Orlik. The main motivation for our work [1] was to study the (co)homology of Γ with coefficients in locally analytic representations of p-adic reductive groups, and to apply our results to the conjectures of Schneider.

Let K be a spherically complete valued field containing L, denote by $\mathfrak{o}_L$ the valuation ring of L, and let π be a uniformizer of L. For simplicity we shall only consider the group $G := \mathrm{PGL}_{d+1}(L)$. Let $P = N \cdot T$ be the standard Levi decomposition of the subgroup of upper triangular matrices of G, let $G_0 := \mathrm{PGL}_{d+1}(\mathfrak{o}_L)$, and let B denote the subgroup of G_0 consisting of all matrices whose reduction modulo π is upper triangular. For any positive integer n let $B_n := \ker(G_0 \to \mathrm{PGL}_{d+1}(\mathfrak{o}_L/\pi^n\mathfrak{o}_L))$. We let $T^- := \{\mathrm{diag}(\lambda_1, \ldots, \lambda_{d+1}) \in T \mid |\lambda_1| \geq \ldots \geq |\lambda_{d+1}|\}$ and $t_i \in T^-$ for $1 \leq i \leq d$ be representatives of the fundamental antidominant cocharacters of the root system of (G, T) with respect to P.

Given a locally analytic character $\chi : T \to K^\times$ and a discrete and cocompact subgroup Γ of G, our first goal is to study the homology $\mathrm{H}_*(\Gamma, \mathrm{Ind}_P^G(\chi))$ of Γ with coefficients in the *locally analytic principal series representation*

$$\mathrm{Ind}_P^G(\chi) := \{f \in \mathcal{C}^{\mathrm{an}}(G, K) \mid \forall g \in G\ \forall p \in P : f(gp) = \chi(p)^{-1} f(g)\}.$$

This is done by constructing an explicit Γ-acyclic resolution in the following way. For any positive integer n we denote by $\mathcal{A}_n$ the subspace of $\mathrm{Ind}_P^G(\chi)$ consisting of all functions with support in $B \cdot P$ and whose restriction to $B \cap \overline{N}$ is rigid analytic on every coset modulo $B_n \cap \overline{N}$. Here $\overline{N}$ denotes the group of all lower triangular unipotent matrices. If n is sufficiently large then $\mathcal{A} := \mathcal{A}_n$ is a B-stable K-Banach space inside $\mathrm{Ind}_P^G(\chi)$. By Frobenius reciprocity there exists a unique G-equivariant

map

$$\varphi : \mathrm{c}\text{-}\mathrm{Ind}_B^G(\mathcal{A}) \to \mathrm{Ind}_P^G(\chi),$$

which will be the final term of the desired resolution. In fact, we show that φ is surjective and that there is a homomorphism $K[T^-] \to \mathrm{End}_G(\mathrm{c}\text{-}\mathrm{Ind}_B^G(\mathcal{A}))$, $t \mapsto U_t$, of K-algebras such that $\ker(\varphi) = \sum_{i=1}^d \mathrm{im}(U_{t_i} - \chi(t_i))$ (cf. [1], Proposition 2.4). This suggests to consider the following Koszul complex whose exactness is the main technical result of our work (cf. [1], Theorem 2.5).

Theorem 1. *The augmented Koszul complex*

$$(\bigwedge^\bullet K^d) \otimes_K \mathrm{c}\text{-}\mathrm{Ind}_B^G(\mathcal{A}) \longrightarrow \mathrm{Ind}_P^G(\chi) \xrightarrow{\varphi} 0$$

defined by the endomorphisms $(U_{t_i} - \chi(t_i))_{1 \le i \le d}$ *of* $\mathrm{c}\text{-}\mathrm{Ind}_B^G(\mathcal{A})$ *is a* G*-equivariant exact resolution of* $\mathrm{Ind}_P^G(\chi)$ *by* Γ*-acyclic representations.*

As a corollary one immediately obtains the following result.

Corollary 2. *We have* $\mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi)) \simeq \mathrm{H}_q((\bigwedge^\bullet K^d) \otimes_K \mathrm{c}\text{-}\mathrm{Ind}_B^G(\mathcal{A})_\Gamma)$ *for any integer* $q \ge 0$*. In particular, if* $q > d$ *then* $\mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi)) = 0$*.*

It is a crucial observation that $\mathrm{c}\text{-}\mathrm{Ind}_B^G(\mathcal{A})_\Gamma$ is naturally a K-Banach space and that the operator induced by U_{t_i} is continuous with operator norm ≤ 1 for any i. This leads to the following vanishing theorem (cf. [1], Theorem 3.2).

Theorem 3. *If* $|\chi(t_i)| > 1$ *for some* $1 \le i \le d$ *then* $\mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi)) = 0$ *for all* $q \ge 0$*.*

For the proof one simply refers to Corollary 2 and uses the fact that under the above hypothesis the endomorphism $U_{t_i} - \chi(t_i)$ of the K-Banach space $\mathrm{c}\text{-}\mathrm{Ind}_B^G(\mathcal{A})_\Gamma$ is invertible.
A similarly far-reaching observation is that if $t := t_1 \cdot \ldots \cdot t_d$ then the K-linear endomorphism U_t of $\mathrm{c}\text{-}\mathrm{Ind}_B^G(\mathcal{A})$ is not only continuous but even *compact*, i.e. it is the strong limit of continuous operators with finite rank. A Fredholm argument for $U_t - \chi(t)$ then leads to the following very general finiteness result (cf. [1], Theorem 3.9).

Theorem 4. *For any integer* $q \ge 0$ *the* K*-vector space* $\mathrm{H}_q(\Gamma, \mathrm{Ind}_P^G(\chi))$ *is finite dimensional.*

We finally broaden our point of view and consider locally analytic G-representations V over K possessing a G-equivariant finite resolution

$$0 \longrightarrow V \longrightarrow M_0 \longrightarrow \cdots \longrightarrow M_n \longrightarrow 0,$$

in which all M_i are finite direct sums of locally analytic principal series representations $\mathrm{Ind}_P^G(\chi_{ij})$. Theorems 3 and 4 and a spectral sequence argument lead to vanishing and finiteness theorems for V. Examples to which this procedure applies include *locally algebraic* representations of the form $V = \mathrm{Ind}_P^G(\mathbf{1})^\infty \otimes_K M$, for which the necessary resolution is provided by the locally analytic BGG-resolution

of Orlik-Strauch. Here $\mathrm{Ind}_P^G(\mathbf{1})^\infty$ denotes the smooth principal series representation associated with the trivial character $\mathbf{1}$ and M is a finite dimensional algebraic representation of G. Another example is given by certain subquotients of *p-adic holomorphic discrete series representations*, i.e. representations of the form $\Omega_X^p(X) \otimes_K M$. In fact, our vanishing theorems eventually allow us to prove Schneider's conjectures in several previously unknown cases (cf. [1], Theorem 4.10).

References

[1] J. Kohlhaase, B. Schraen, *Homological vanishing theorems for locally analytic representations*, Mathematische Annalen, to appear.

[2] P. Schneider, *The cohomology of local systems on p-adically uniformized varieties*, Mathematische Annalen **293** (1992), 623–650.

[3] P. Schneider, U. Stuhler, *The cohomology of p-adic symmetric spaces*, Inventiones Mathematicae **105** (1991), 47–122.

On arithmetic families of filtered φ-modules and crystalline representations

Eugen Hellmann

Let K be a finite extension of $\mathbb{Q}_p$ and write $G_K = \mathrm{Gal}(\bar{K}/K)$ for its absolute Galois group. Further let K_0 denote the maximal unramified extension of $\mathbb{Q}_p$ in K with Frobenius automorphism φ. In p-adic Hodge theory filtered φ-modules appear as a category of linear algebra data, describing so called *crystalline* representations of G_K. Recall that a p-adic representations V (that is, a finite dimensional $\mathbb{Q}_p$-vector space with continuous G_K-action) is called *crystalline* if

$$D_{\mathrm{cris}}(V) = \left(V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}\right)^{G_K}$$

has K_0-dimension equal to the $\mathbb{Q}_p$-dimension of V, where B_{cris} is Fontaine's ring of crystalline periods. The extra structures on B_{cris} endow $D_{\mathrm{cris}}(V)$ with a semi-linear automorphism Φ and a filtration $\mathcal{F}^\bullet$ on $D_{\mathrm{cris}}(V) \otimes_{K_0} K$. By work of Fontaine and Colmez-Fontaine, the functor $V \mapsto D_{\mathrm{cris}}(V)$ mapping a crystalline representation to its filtered φ-module is known to be fully faithful, with essential image consisting of the objects that are *weakly admissible* which is a semi-stability condition relating the slopes of the Frobenius Φ with the slopes of the filtration .

Our goal is to study *arithmetic families*[1] of the objects above, that is, we want to replace the $\mathbb{Q}_p$-vector space V by a vector bundle with a continuous G_K-action on a certain space X. It turns out that the right category to study these objects is the category of *adic spaces* (or adic spaces locally of finite type) introduced by Huber [2], which is a generalization of the more classical category of rigid spaces.

On the side of filtered φ-modules we consider coherent $\mathcal{O}_X \otimes_{\mathbb{Q}_p} K_0$-modules D on an adic space X which are locally on X free of finite rank, equipped with an

[1] This terminology is due to Kedlaya and Liu, to distinguish these families from representations of fundamental groups.

id $\otimes\, \varphi$-linear automorphism Φ and a filtration $\mathcal{F}^\bullet$ on $D \otimes_{K_0} K$. Write $\mathfrak{D}$ for the functor

$$X \longmapsto \left\{ \begin{array}{c} \text{isomorphism classes of filtered } \varphi\text{-modules } (D, \Phi, \mathcal{F}^\bullet) \text{ on } X \\ \text{with a trivialization } D \cong (\mathcal{O}_X \otimes_{\mathbb{Q}_p} K_0)^d \end{array} \right\}.$$

This functor is representable by the product

$$\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathrm{GL}_d \times \coprod_\nu \mathrm{Gr}_\nu$$

where $\mathrm{Gr}_\nu = \mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_d/P_\nu$ is the flag variety parametrizing filtrations *of type* ν. Here ν is a (dominant) cocharacter of $\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_d$.

Theorem 1. (i) *Let X be an adic space locally of finite type over $\mathbb{Q}_p$ and $(D, \Phi, \mathcal{F}^\bullet)$ a family of filtered φ-modules over X. The set*

$$X^{\mathrm{wa}} = \{x \in X \mid D \otimes k(x) \text{ is weakly admissible}\} \subset X$$

is open. It is the maximal open subset $Y \subset X$ such that the rigid analytic points of Y are exactly the weakly admissible rigid points of X.
(ii) *The functor*

$$X \longmapsto \{(D, \Phi, \mathcal{F}^\bullet) \in \mathfrak{D}(X) \mid D \otimes k(x) \text{ is weakly admissible}\}$$

is an open subfunctor of $\mathfrak{D}$ (that is, the formation $X \mapsto X^{\mathrm{wa}}$ is compatible with base change).

Remark: The above result fails in the category of Berkovich spaces which can be shown by easy examples. Berkovich spaces have the wrong topology for our purpose, as affinoids are not open.

Let $A \subset \mathrm{GL}_d$ denote the diagonal torus and W the Weyl group of (GL_d, A). Fix a (dominant) cocharacter

$$\nu : \mathbb{G}_m \longrightarrow \mathrm{Res}_{K/\mathbb{Q}_p} A_K$$

defining a filtration type and let $\mathfrak{D}_\nu^{\mathrm{wa}} \subset \mathfrak{D}_\nu = \mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_d \times \mathrm{Gr}_\nu$ be the functor of weakly admissible filtered φ-modules with filtration of type ν. There is a map

$$\alpha : \mathfrak{D}_\nu \longrightarrow A/W \cong \mathbb{A}^{d-1} \times \mathbb{G}_m$$

which maps $(\Phi, \mathcal{F}^\bullet)$ to the coefficients of the characteristic polynomial of Φ^f, where $f = [K_0 : \mathbb{Q}_p]$.

Theorem 2. *Let $x \in A/W$.*
(i) *The weakly admissible locus $\alpha^{-1}(x)^{\mathrm{wa}} \subset \alpha^{-1}(x)$ in the fiber over x is Zariski-open in $\alpha^{-1}(x)$.*
(ii) *The weakly admissible locus in the fiber $\alpha^{-1}(x)$ is non-empty if and only if x lies in a* Newton stratum $(A/W)_{\mu(\nu)}$. *The coweight $\mu(\nu)$ is explicit in terms of ν.*

Remark: The question whether there exists a weakly admissible filtration on a φ-module with fixed semi-simplified Frobenius Φ^{ss} appears in work of Breuil and Schneider on the p-adic Langlands correspondence. In loc. cit. Breuil and Schneider characterize the subset of those Φ^{ss} in the torus. Theorem 2 (ii) gives a more group theoretic characterisation of (the image of) this subset in the adjoint quotient.

On the side of Galois representations we define a sheafified version $\mathcal{O}_X \widehat{\otimes} B_{\mathrm{cris}}$ of Fontaine's period ring B_{cris} on an adic space X and define the families we want to consider as follows.

Definition: Let X be an adic space locally of finite type over $\mathbb{Q}_p$. A family of G_K-representation over X is a vector bundle $\mathcal{E}$ on X endowed with a continuous G_K-action. The family $\mathcal{E}$ is called *crystalline* if the sheaf

$$D_{\mathrm{cris}}(\mathcal{E}) = \left(\mathcal{E} \otimes_{\mathcal{O}_X} (\mathcal{O}_X \widehat{\otimes} B_{\mathrm{cris}})\right)^{G_K}$$

is locally on X free of rank d over $\mathcal{O}_X \otimes_{\mathbb{Q}_p} K_0$, where $d = \mathrm{rk}_{\mathcal{O}_X} \mathcal{E}$.

We write $\mathfrak{D}^{\mathrm{cris}}$ for the functor

$$X \longmapsto \left\{ \begin{array}{c} \text{isomorphism classes of crystalline representations } \mathcal{E} \text{ over } X \\ \text{with a trivialization of } D_{\mathrm{cris}}(\mathcal{E}) \end{array} \right\}.$$

As a consequence of the definitions and Theorem 1 we obtain a morphism

$$D_{\mathrm{cris}} : \mathfrak{D}^{\mathrm{cris}} \longrightarrow \mathfrak{D}^{\mathrm{wa}} \subset \mathfrak{D}.$$

Theorem 3. *The functor $\mathfrak{D}^{\mathrm{cris}}$ is representable by an open subspace of $\mathfrak{D}^{\mathrm{wa}}$. The inclusion $\mathfrak{D}^{\mathrm{cris}} \hookrightarrow \mathfrak{D}^{\mathrm{wa}}$ is a bijection on the level of rigid analytic points.*

The map D_{cris} cannot be an isomorphism: On the side of G_K-representations we have the obstruction that the semi-simplification of the reduction modulo p of a p-adic Galois representation has to be locally constant on an adic space X.

The main step in the proof of this theorem is to construct an intermediate open subfunctor parametrizing families of integral data for the weakly admissible filtered φ-modules in the sense of Kisin [4]. This space was conjecturally described by Pappas and Rapoport [5] as the image of a certain period map. The proof uses a generalization of work of Kedlaya and Liu [3] on families of φ-modules over the Robba ring to the set up of adic spaces.

References

[1] C. Breuil, P. Schneider, *First steps towards p-adic Langlands functoriality*, J. reine angew. Math. **610** (2007), 149-180.

[2] R. Huber, *A generalization of formal schemes and rigid analytic varieties*, Math. Z. **217** (1994), 513–551.

[3] K. Kedlaya, R. Liu, *On families of* (φ, Γ)*-modules*, to appear in Algebra and number theory.

[4] M. Kisin, *Crystalline representations and F-crystals*, in Algebraic geometry and number theory, Progr. Math. **253** (2006), Birkhäuser, Boston, 459–496.

[5] G. Pappas, M. Rapoport Φ*-modules and coefficient spaces*, Moscow Math. J. **9** no. 3 (2009), 625-664.

Harmonic Maass forms and periods

Jan Hendrik Bruinier

Half-integral weight modular forms play important roles in arithmetic geometry and number theory. Thanks to the theory of theta functions, such forms include important generating functions for the representation numbers of integers by quadratic forms. Among weight 3/2 modular forms, one finds Gauss' function

$$\sum_{x,y,z\in\mathbb{Z}} q^{x^2+y^2+z^2} = 1 + 6q + 12q^2 + 8q^3 + 6q^4 + 24q^5 + \cdots ,$$

which is essentially the generating function for class numbers of imaginary quadratic fields, as well as Gross's theta functions which enumerate the supersingular reductions of CM elliptic curves.

In the 1980s, Waldspurger [12], and Kohnen and Zagier [8] established that half-integral weight modular forms also serve as generating functions of a different type. Using the Shimura correspondence [10], they proved that certain coefficients of half-integral weight cusp forms essentially are square-roots of central values of quadratic twists of modular L-functions.

In analogy with these works, Katok and Sarnak [7] employed a Shimura correspondence to relate coefficients of weight 1/2 Maass forms to sums of values and sums of line integrals of Maass cusp forms. We investigate the arithmetic properties of the coefficients of a different class of Maass forms, the weight 1/2 harmonic weak Maass forms (see [4]).

A *harmonic weak Maass form of weight* $k \in \frac{1}{2}\mathbb{Z}$ *on* $\Gamma_0(N)$ (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$) is a smooth function on $\mathbb{H}$, the upper half of the complex plane, which satisfies:

(i) $f\mid_k \gamma = f$ for all $\gamma \in \Gamma_0(N)$;
(ii) $\Delta_k f = 0$, where Δ_k is the weight k hyperbolic Laplacian on $\mathbb{H}$;
(iii) There is a polynomial $P_f = \sum_{n\le 0} c^+(n)q^n \in \mathbb{C}[q^{-1}]$ such that $f(\tau) - P_f(\tau) = O(e^{-\varepsilon v})$ as $v \to \infty$ for some $\varepsilon > 0$. Analogous conditions are required at all cusps.

Throughout, for $\tau \in \mathbb{H}$, we let $\tau = u + iv$, where $u, v \in \mathbb{R}$, and we let $q := e^{2\pi i\tau}$.

The polynomial P_f, the *principal part of* f *at* ∞, is uniquely determined. If P_f is non-constant, then f has exponential growth at the cusp ∞. Similar remarks apply at all of the cusps.

Spaces of harmonic weak Maass forms include *weakly holomorphic modular forms*, those meromorphic modular forms whose poles (if any) are supported at cusps. We are interested in those harmonic weak Maass forms which do not arise in this way. Such forms have been a source of recent interest due to their connection to Ramanujan's mock theta functions (see e.g. [2, 13]).

A harmonic Maass form f has a Fourier expansion of the form

$$f(\tau) = \sum_{n\gg-\infty} c^+(n)q^n + \sum_{n<0} c^-(n)\Gamma(1-k, 4\pi|n|v)q^n, \tag{1}$$

where $\Gamma(a,x)$ denotes the incomplete Gamma function. We call $\sum_{n\gg-\infty} c^+(n)q^n$ the *holomorphic part* of f, and we call its complement its *non-holomorphic part*. The non-holomorphic part is a period integral of a weight $2-k$ modular form. Equivalently, $\xi_k(f) := 2iv^k\overline{\frac{\partial}{\partial\tau}}$ is a weight $2-k$ modular form on $\Gamma_0(N)$.

Every weight $2-k$ cusp form is the image under ξ_k of a weight k harmonic weak Maass form. The mock theta functions correspond to those forms whose images under $\xi_{1/2}$ are weight $3/2$ unary theta functions. We consider those weight $1/2$ harmonic weak Maass forms whose images under $\xi_{1/2}$ are orthogonal to the unary theta series. We show that the coefficients of their holomorphic parts are given by periods of algebraic differentials of the third kind on modular curves.

Although we treat the general case in [3], to simplify exposition, here we assume that p is prime and that $G \in S_2^{new}(\Gamma_0(p))$ is a normalized Hecke eigenform with rational Fourier coefficients and which is invariant under the Fricke involution. Then the Hecke L-function $L(G,s)$ satisfies and odd functional equation and therefore vanishes at $s=1$, the central critical point.

Using Kohnen's theory of plus-spaces, it can be shown that there is weight $1/2$ harmonic Maass form f on $\Gamma_0(4p)$ in the plus space whose principal part has integer coefficients and vanishing constant term, and such that the Shimura lift of $\xi_{1/2}(f)$ is equal to G. If we denote the Fourier coefficients of f by $c^{\pm}(n)$ as in (1), it follows from Walspurger's theorem and the properties of the differential operator $\xi_{1/2}$ that the squares of the coefficients $c^-(\Delta)$ are given by the central values $L(G,\chi_\Delta,1)$ of the twisted L-function $L(G,\chi_\Delta,s)$, where Δ is a negative fundamental discriminant with $\left(\frac{\Delta}{p}\right)=1$.

The coefficients $c_g^+(n)$ are more mysterious. We describe them in terms of periods of differentials of the third kind associated to Heegner divisors on the modular curve $X := X_0(p)$. Let $d <$ be a discriminant, and let $\Delta > 0$ be a fundamental discriminant, and assume that both are squares modulo p. Let $\mathcal{Q}_{p,d\Delta}$ be the set of integral binary quadratic forms $[a,b,c]$ of discriminant $d\Delta$ for which p divides a. The group $\Gamma_0(p)$ acts on $\mathcal{Q}_{p,d\Delta}$ with finitely many orbits. We consider the Heegner divisor

$$Z_\Delta(d) = \sum_{Q\in\mathcal{Q}_{p,d\Delta}/\Gamma_0(p)} \chi_\Delta(Q)\frac{\alpha_Q}{w_Q}$$

on X, where for a quadratic form $Q=[a,b,c]$ we denote by α_Q the zero of aX^2+bX+c in $\mathbb{H}$ and by w_Q the order of the stabilizer of Q in $\Gamma_0(p)/\{\pm1\}$. Moreover, χ_Δ is the genus character associated to Δ as in [6]. We associate a divisor to f by putting

$$Z_\Delta(f) = \sum_{n<0} c^+(n)Z_\Delta(n).$$

This divisor has degree 0, and by the theory of complex multiplication, it is defined over $\mathbb{Q}(\sqrt{\Delta})$. Using (a generalization of) Borcherds products [1], it is proved in [5] that this divisor defines a point in the G-istypical component of the Jacobian of X.

Recall that a differential of the third kind on X is a meromorphic differential whose poles have at most first order and have integral residues. If η is a differential of the third kind on X that has poles at the points P_j with residues a_j, then $\mathrm{res}(\eta) = \sum_j a_j(P_j)$ defines a degree zero divisor on X, called the residue divisor of η. By the Riemann-Roch theorem, for any degree zero divisor D on X there exists a differential of the third kind with residue divisor D. It is unique up to addition of holomorphic differentials.

Lemma 1. *Let D be a degree zero divisor on X whose class is G-isotypical. Then there exists a unique differential η_D of the third kind on X such that*

(1) $\mathrm{res}(\eta_D) = D$;
(2) *the first Fourier coefficient of η_D in the cusp ∞ vanishes;*
(3) *η_D is G-isotypical, that is, for any Hecke operator T with corresponding eigenvalue $\lambda_G(T)$ on G, the difference $T(\eta_D) - \lambda_G(T)\eta_D$ is the logarithmic derivative of a rational function on X.*

The differential η_D of the lemma is called the *normalized differential of the third kind* associated to D. If D is defined over a number field k, then η_D is also defined over k.

Let $H_1^+(X, \mathbb{R})$ be the invariant subspace of the first homology of X under the involution induced by complex conjugation on X. The Hecke algebra acts on this space, and it is well known that the G-isotypical subspace is one-dimensional. Let C_G be a generator.

Theorem 2. *Let $G \in S_2^{new}(\Gamma_0(p))$ be a newform as above and write ω_G for the corresponding holomorphic differential on X. Let f be a harmonic Mass form of weight $1/2$ associated to G as above, and let $\eta_\Delta(f)$ be the normalized differential of the third kind associated to the divisor $Z_\Delta(f)$. Then*

$$c^+(\Delta) = \frac{\Re\left(\int_{C_G} \eta_\Delta(f)\right)}{\sqrt{\Delta}\int_{C_G} \omega_G}.$$

Note that this can also be described in terms of periods of differentials of the third kind on the elliptic curve of conductor p corresponding to G.

By transcendence results due to Waldschmidt (see [12] [9]) for periods of differentials on algebraic curves, it is easily seen that the right hand side is algebraic if and only if $Z_\Delta(f)$ defines a torsion point of the Jacobian. Employing the Gross-Zagier formula and the modularity of twisted Heegner divisors [5, 6], this happens if and only if $L'(G, \chi_\Delta, 1)$ vanishes, see [5].

The above theorem can be viewed as a refinement of the main result of [5]. Its proof uses the explicit construction of the canonical differential of the third kind associated to $Z_\Delta(f)$ by means of a regularized theta lift (see [5]), combined with a careful study of the action of the Hecke algebra on various spaces of differentials and on the homology of X.

References

[1] R. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. Math. **132** (1998), 491–562.

[2] K. Bringmann and K. Ono, *Dyson's ranks and Maass forms*, Ann. of Math. **171** (2010), 419–449.

[3] J. H. Bruinier, *Harmonic Maass forms and periods*, in preparation.

[4] J. H. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. J. **125** (2004), 45–90.

[5] J. H. Bruinier and K. Ono, *Heegner divisors, L-functions and harmonic weak Maass forms*, Annals of Math. **172** (2010), 2135–2181.

[6] B. Gross, W. Kohnen, and D. Zagier, *Heegner points and derivatives of L-series. II.* Math. Ann. **278** (1987), 497–562.

[7] S. Katok and P. Sarnak, *Heegner points, cycles and Maass forms*, Israel J. Math. **84** (1993), 193–227.

[8] W. Kohnen and D. Zagier, *Values of L-series of modular forms at the center of the critical strip*, Invent. Math. **64** (1981), no. 2, 175–198.

[9] A. J. Scholl, *Fourier coefficients of Eisenstein series on non-congruence subgroups*, Math. Proc. Camb. Phil. Soc. **99** (1986), 11–17.

[10] G. Shimura, *On modular forms of half integral weight*, Ann. of Math. (2) **97** (1973), 440–481.

[11] M. Waldschmidt, *Nombers transcendents et groupes algébraiques*, Astérisque **69–70** (1979).

[12] J.-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier*, J. Math. Pures Appl. (9) **60** (1981), no. 4, 375–484.

[13] S. P. Zwegers, *Mock theta functions*, Ph.D. Thesis, Universiteit Utrecht, 2002.

Kneser neighbours and orthogonal Galois representations in dimensions 16 and 24

Gaëtan Chenevier

(joint work with Jean Lannes)

Let $n \geq 1$ be an integer. Recall that an even unimodular lattice in the standard euclidean space $\mathbb{R}^n$ is a lattice $L \subset \mathbb{R}^n$ of covolume 1 with $x \cdot x \in 2\mathbb{Z}$ for all $x \in L$. Let X_n denote the set of isometry classes of even unimodular lattices in $\mathbb{R}^n$. As is well-known, X_n is a finite set which is non-empty if and only if $n \equiv 0 \bmod 8$. For example, the lattice

$$\mathrm{E}_n = \mathrm{D}_n + \mathbb{Z}\,\frac{e_1 + \ldots + e_n}{2},$$

$\{e_1, \ldots, e_n\}$ denoting the canonical basis of $\mathbb{R}^n$ and D_n the sublattice of index 2 in $\mathbb{Z}^n$ whose elements (x_i) satisfy $\sum_i x_i \equiv 0 \bmod 2$, is even unimodular for $n \equiv 0 \bmod 8$.

The set X_n has been determined in only three cases. One has $\mathrm{X}_8 = \{\mathrm{E}_8\}$ (Mordell), $\mathrm{X}_{16} = \{\mathrm{E}_8 \oplus \mathrm{E}_8, \mathrm{E}_{16}\}$ (Witt) and Niemeier showed that X_{24} has 24 explicit elements (see [V]). The number of numerical coincidences related to Niemeier's lists is quite extraordinary and makes that list still mysterious. For the other values of n the Minkowski-Siegel-Smith mass formula shows that X_n is huge, perhaps impossible to describe. For instance, X_{32} already has more than 80.10^6 elements ([S]).

Let $L \subset \mathbb{R}^n$ be an even unimodular lattice, and let p be a prime; Knesers defines a p-neighbour of L as an even unimodular lattice $M \subset \mathbb{R}^n$ such that $M \cap L$ has

index p in L (hence in M). The relation of being p-neighbours turns X_n into a graph which was shown to be connected by Kneser, providing a theoretical way to compute X_n from the single lattice E_n. This is actually the way Kneser and Niemeier computed X_n for $n \leq 24$, using the prime $p = 2$ and the huge number of symmetries present in those cases.

In this paper, we are interested in giving an explicit formula for the number $\mathrm{N}_p(L, M)$ of p-neighbours of L which are isometric to M. Equivalently, it amounts to determining the $\mathbb{Z}$-linear operator $\mathrm{T}_p : \mathbb{Z}[\mathrm{X}_n] \to \mathbb{Z}[\mathrm{X}_n]$ defined by $\mathrm{T}_p[L] = \sum[N]$, $[-]$ denoting the isometry class of a lattice, the summation being over all the p-neighbours of L.

Before stating our main results, let us mention that the p-neighbours of a given even unimodular lattice L are in canonical bijection with the $\mathbb{F}_p$-points of the projective quadric C_L over $\mathbb{Z}$ defined by the quadratic form $x \mapsto \frac{x \cdot x}{2}$ on L. The quadric C_L is hyperbolic over $\mathbb{F}_p$ for each prime p, thus L has exactly

$$\mathrm{c}_n(p) = |\mathrm{C}_L(\mathbb{F}_p)| = 1 + p + p^2 + \ldots + p^{n-2} + p^{n/2-1}$$

p-neighbours, where $n = \mathrm{rk}_{\mathbb{Z}} L$. We are thus interested in the partition of the quadric $\mathrm{C}_L(\mathbb{F}_p)$ into $|\mathrm{X}_n|$ parts (some of them being possibly empty) given by the isometry classes. Of course, $\mathrm{N}_p(\mathrm{E}_8, \mathrm{E}_8) = \mathrm{c}_8(p)$ as $\mathrm{X}_8 = \{\mathrm{E}_8\}$, thus the first interesting case (perhaps known to specialists!) is $n = 16$.

Theorem A : *Let $n = 16$. In the basis $\mathrm{E}_8 \oplus \mathrm{E}_8, \mathrm{E}_{16}$ the matrix of T_p is*

$$\mathrm{c}_{16}(p) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (1 + p + p^2 + p^3) \frac{1 + p^{11} - \tau(p)}{691} \begin{bmatrix} -405 & 286 \\ 405 & -286 \end{bmatrix},$$

$\sum_{n \geq 1} \tau(n) q^n = q \prod_{n \geq 1} (1 - q^n)^{24}$ denoting Ramanujan's Δ function.

One first makes two well-known observations. First, the operators T_p commute with each others. Second, they are self-adjoint for the scalar product defined by $\langle [L], [M] \rangle = \delta_{[L],[M]} |\mathrm{O}(L)|$, which amounts to saying that for all $L, M \in \mathrm{X}_n$ we have

$$\mathrm{N}_p(L, M) |\mathrm{O}(M)| = \mathrm{N}_p(M, L) |\mathrm{O}(L)|. \tag{1}$$

Our main question (for any n) is thus equivalent to first finding a basis of $\mathbb{R}[\mathrm{X}_n]$ made of eigenvectors common to all of the T_p operators, and then to describing the system of eigenvalues (λ_p) of the (T_p) on each of these eigenvectors. If $n = 16$ it is not difficult to compute T_2, and this was essentially done by Borcherds for $n = 24$ (see [N-V]). In both cases, the eigenvalues of T_2 are distinct integers (this was noticed by Nebe and Venkov [N-V] for $n = 24$). Let us mention the important presence of $(\mathrm{c}_n(p))$ as "trivial" system of eigenvalues : formula (1) shows that $\sum_{L \in X_n} [L] |O(L)|^{-1} \in \mathbb{Q}[X_n]$ is an eigenvector for T_p with eigenvalue $\mathrm{c}_n(p)$.

Assume now $n = 16$. The non-trivial system of eigenvalues is related to Ramanujan's Δ-function in a non-trivial way. Our proof relies on Siegel theta series

$$\vartheta_g : \mathbb{Z}[X_n] \to \mathrm{M}_{\frac{n}{2}}(\mathrm{S}p_{2g}(\mathbb{Z})),$$

the latter space being the space of classical Siegel modular forms of weight $n/2$ and genus g. The generalized Eichler commutation relation ([R], [W]) asserts that ϑ_g intertwines T_p with some explicit Hecke operator on the space of Siegel modular forms. By a classical result of Witt, Kneser and Igusa (see [K]),

$$\vartheta_g(\mathrm{E}_8 \oplus \mathrm{E}_8) = \vartheta_g(\mathrm{E}_{16}) \ \text{ if } \ g \leq 3,$$

whereas $\mathrm{F} = \vartheta_4(\mathrm{E}_8 \oplus \mathrm{E}_8) - \vartheta_4(\mathrm{E}_{16})$ does not vanish, thus the T_p-eigenvalue we are looking for is related to the Hecke eigenvalues of $F \in \mathrm{S}_8(\mathrm{Sp}_8(\mathbb{Z}))$. A result by Poor and Yuen [P-Y] asserts that the latter space is 1-dimensional (generated by the famous Schottky form!). But another non-trivial member of this space is Ikeda's lift of Ramanujan's Δ function (see [I]), whose Hecke eigenvalues are explicitly given in terms of Δ. By unravelling the precise formulae we obtain Theorem A. Actually, we found a direct proof of the existence of Ikeda's lift of Δ that relies on the triality for the reductive group $\mathrm{PGO}^+_{E_8}$ over $\mathbb{Z}$ and two theta series constructions.

Pre-Theorem* B : *There is an explicit formula as well for* T_p *if* $n = 24$.

In this case it is more difficult to find the non trivial systems of eigenvalues on $\mathbb{Q}[\mathrm{X}_{24}]$. Five of them were actually identified as Ikeda lifts in the work of Nebe and Venkov [N-V], with a particular one due to Borcherds-Freitag-Weissauer. We rather rely on Chapter 9 of the book [A] by Arthur ... which is still missing at the moment; hence the * in the statement above. The relation with automorphic forms comes from the canonical identification $\mathrm{X}_n = \mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A}_f)/\mathrm{G}(\widehat{\mathbb{Z}})$ where G is the $\mathbb{Z}$-orhogonal group of E_n, so that $\mathbb{C}[\mathrm{X}_n]$ is canonically the dual of the space of automorphic forms of G of level 1 and trivial coefficients. The quickest way (although perhaps inappropriate!) to state our results is in terms of Galois representations.

Fix a prime ℓ. Thanks to the works of many authors (including [A]), for any system of eigenvalues $\pi = (\lambda_p)$ of (T_p) on $\overline{\mathbb{Q}}_\ell[\mathrm{X}_n]$, there exists a unique continuous, semi-simple representation

$$\rho_{\pi,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GO}(n, \overline{\mathbb{Q}_\ell})$$

which is unramified outside ℓ and such that $\mathrm{Trace}(\rho_{\pi,\ell}(\mathrm{Frob}_p)) = \lambda_p$ for each prime $p \neq \ell$. This Galois representation is furthermore crystalline at ℓ with Hodge-Tate numbers $0, 1, \ldots, n-2$, the number $n/2 - 1$ occuring twice.

For example, if $\pi = (\mathrm{c}_n(p))$ then $\rho_{\pi,\ell} = (\oplus_{i=0}^{n-2}\omega^i) \oplus \omega^{n/2-1}$, where ω is the ℓ-adic cyclotomic character. Moreover, Theorem A implies that if π is the non-trivial system of eigenvalues when $n = 16$, then $\rho_{\pi,\ell} = \rho_{\Delta,\ell}\otimes(\oplus_{i=0}^{3}\omega^i)\oplus\omega^7\oplus(\oplus_{i=0}^{6}\omega^i)\otimes\omega^4$, where $\rho_{\Delta,\ell}$ is Deligne's Galois representation attached to Δ.

Consider now the following collection of orthogonal ℓ-adic Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with similitude factor ω^{22} and distinct Hodge-Tate numbers in $\{0, 2, \ldots, 22\}$. For r odd and $1 \leq r \leq 23$, set

$$[r] = (\oplus_{i=0}^{r-1}\omega^i) \otimes \omega^{\frac{23-r}{2}}. \tag{2}$$

For r even, $k \in \{12, 16, 18, 20, 22\}$, $k + r \leq 24$, set

$$\Delta_k[r] = \rho_{\Delta_k,\ell} \otimes (\oplus_{i=0}^{r-1} \omega^i) \otimes \omega^{\frac{24-(k+r)}{2}}, \tag{3}$$

where Δ_k is a generator of $\mathrm{S}_k(\mathrm{SL}_2(\mathbb{Z}))$. Set

$$\mathrm{Sym}^2\Delta = \mathrm{Sym}^2\rho_{\Delta,\ell}. \tag{4}$$

For $(j, k) \in \{(6, 8), (4, 10), (8, 8), (12, 6)\}$ then the space of vector-valued Siegel modular cusp forms of genus 2 (for $\mathrm{Sp}_4(\mathbb{Z})$) and coefficient $\mathrm{Sym}^j \otimes \det^k$ has dimension one (Tsushima, see [VdG]). Let $\Delta_{j,k}$ be a generator and let $\rho_{\Delta_{j,k},\ell}$ be its associated 4-dimensional ℓ-adic Galois representation (Weissauer) ; it is symplectic with Hodge-Tate numbers $0, k-2, j+k-1$, and $j+2k-3$. Set

$$\Delta_{j,k}[2] = \rho_{\Delta_{j,k},\ell} \otimes (1 \oplus \omega) \otimes \omega^{\frac{24-2k-j}{2}}. \tag{5}$$

Fact: *There are exactly* 24 *representations of dimension* 24 *which are direct sums of representations in the list* (2)–(5) *above and whose Hodge-Tate numbers are* $0, 1, \ldots, 22$ *with* 11 *occuring twice.*

For instance, $\mathrm{Sym}^2\Delta \oplus \Delta_{20}[2] \oplus \Delta_{16}[2] \oplus \Delta[2] \oplus [9]$, $\Delta_{4,10}[2] \oplus \Delta_{18}[2] \oplus \Delta[4] \oplus [3] \oplus [1]$, and $\mathrm{Sym}^2\Delta \oplus \Delta_{6,8}[2] \oplus \Delta_{16}[2] \oplus \Delta[2] \oplus [5]$ are three of them. A more precise form of pre-Theorem B (still conditionnal to Arthur's results) is then the following.

Theorem* B : *The Galois representations attached to the* 24 *systems of eigenvalues* π *occuring in* $\mathbb{Q}[\mathrm{X}_{24}]$ *are exactly the Galois representations above.*

Thanks to this list, the only unknown to determine T_p on $\mathbb{Z}[\mathrm{X}_{24}]$ are the Hecke eigenvalues of the four Siegel cusp forms $\Delta_{j,k}$. They have actually been computed by Van der Geer and Faber for all $p \leq 11$, and even for some of them up to $p = 37$: see [VdG]. We checked that their results are consistant with Borcherds's computation of T_2. Better yet, we directly proved that for any even unimodular root lattice $L \subset \mathbb{R}^{24}$ with Coxeter number $\mathrm{h}(L)$ we have

$$\mathrm{N}_p(\mathrm{Leech}, L) = 0 \ \text{ if } p < \mathrm{h}(L).$$

In particular, if $L \in \{\mathrm{E}_{24}, \mathrm{E}_8^3, \mathrm{E}_8 \oplus \mathrm{E}_{16}, \mathrm{A}_{24}^+\}$ and $p \leq 23$ then $\mathrm{N}_p(\mathrm{Leech}, L) = 0$. This simple fact allowed us to confirm all the eigenvalues given in the table of Van der Geer and Faber for $p \leq 23$ and to compute [1] T_p for those p. Using the Ramanujan estimates for the Δ_k and $\Delta_{j,k}$, and with the convention $\mathrm{h}(\mathrm{Leech}) = 1$, we obtain for instance :

Corollary* : *Assume* $p \geq 11$. *If* L *and* M *are two even unimodular lattices in* $\mathbb{R}^{24}$ *with Coxeter numbers* $\mathrm{h}(L) \geq \mathrm{h}(M)$, *then* $\mathrm{N}_p(L, M) \geq 1$ *if, and only if,* $p \geq \mathrm{h}(L)/\mathrm{h}(M)$.

[1] See `http://www.math.polytechnique.fr/~chenevier/niemeier/niemeier.html` for tables.

References

[A] J. Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups*, preprint available at `http://www.claymath.org/cw/arthur/`.

[V] B. Venkov, *On the classification of integral even unimodular* 24*-dimensional quadratic forms*, in J. H. Conway & N. J. Sloane, Sphere packings, lattices and groups, 3. ed., Grundlehren der Math. Wissen. **290**, Springer-Verlag, New York (1999).

[I] T. Ikeda, *On the lifting of elliptic cusp forms to Siegel cusp forms of degree* $2n$, Ann. of Math. (2) **154** (2001), no. 3, 641681.

[K] M. Kneser, *Lineare Relationen zwischen Darstellungsanzahlen quadratischer Formen*, Math. Ann. **168** (1967), 31–39.

[N-V] G. Nebe, B. & Venkov, *On Siegel modular forms of weight 12*, J. Reine Angew. Math. **531** (2001), 49–60.

[P-Y] C. Poor & D. S. Yuen, *Dimensions of spaces of Siegel modular forms of low weight in degree four*, Bull. Austral. Math. Soc. **54** (1996), no. 2, 309–315.

[R] S. Rallis, *Langlands' functoriality and the Weil representation*, Amer. J. Math. **104** (1982), no. 3, 469–515.

[S] J.-P. Serre, *Cours d'arithmétique*, P. U. F., Paris (1970).

[VdG] G. van der Geer, *Siegel modular forms and their applications*, The 1-2-3 of modular forms, 181–245, Universitext, Springer, Berlin (2008).

[W] L. Walling, *Action of Hecke operators on Siegel theta series II*, Int. J. Number Theory **4** (2008), no. 6, 981–1008.

p-adic perverse sheaves and arithmetic $\mathcal{D}$-modules with singularities along a normal crossing divisor

Takeshi Tsuji

For a proper smooth variety X over $\mathbb{C}$, we have an equivalence of categories between the category of perverse sheaves of $\mathbb{C}$ vector spaces on X and the category of regular holonomic $\mathcal{D}_X$-modules, given by M. Kashiwara and Z. Mebkhout. Let K be a finite extension of $\mathbb{Q}_p$ and let X be a proper smooth scheme over the ring of integers O_K of K. As a generalization of the theory of crystalline representations of the absolute Galois group of K by J.-M. Fontaine, G. Faltings ([3]) defined crystalline smooth $\mathbb{Q}_p$-sheaves on the étale site $(X_K)_{\text{ét}}$ of the generic fiber $X_K = X \times_{\mathrm{Spec}(O_K)} K$, and constructed a fully faithful exact functor D_{crys} from the category of crystalline smooth $\mathbb{Q}_p$-sheaves on $(X_K)_{\text{ét}}$ to the category of filtered convergent F-isocrystals on $\mathfrak{X} = \varprojlim_m (X \bmod p^m)$. See also [2]. The functor D_{crys} may be regarded as a p-adic analogue of the equivalence of categories by Kashiwara and Mebkhout, and it is natural to ask whether one can generalize the functor D_{crys} to p-adic pervers sheaves on $(X_K)_{\text{ét}}$. In this talk, we discussed on this question in the case where p-adic preverse sheaves have singularities along a simple normal crossing divisor. The basic strategy is to "glue" the theories of crystalline smooth $\mathbb{Q}_p$-sheaves on the strata (with certain log structures) associated to the normal crossing divisor.

We assume that K is absolutely unramified, i.e. p is a uniformizer of O_K in the following. Let X be a proper smooth scheme over O_K, and let D be a relative simple normal crossing divisor over O_K. Let D_i ($i \in \{1, 2, \ldots, n\}$) be the irreducible components of D, which are smooth over O_K. For a subset $I \subset \{1, 2, \ldots n\}$, put

$X_I = \cap_{i\in I} D_i$ (if $I \neq \emptyset$), $X_I = X$ (if $I = \emptyset$), and $U_I = X_I \backslash \cup_{j\in I^c} (X_I \cap D_j)$, where $I^c = \{1, 2, \ldots, n\}\backslash I$. Let $X^{\log}$ be the scheme X endowed with the log structure associated to D, and let $X_I^{\log}$ (resp. $U_I^{\log}$) be the scheme X_I (resp. U_I) endowed with the inverse image of the log structure of $X^{\log}$. Note that $X_I^{\log}$ and $U_I^{\log}$ are not (log) smooth over O_K unless $I = \emptyset$. We will "glue" the theories of crystalline smooth $\mathbb{Q}_p$-sheaves on $(X_I^{\log})_{\text{Két}}$ (=the Kummer étale site of the log scheme $X_I^{\log}$, cf. [5]). The pull-back functor from the category of smooth $\mathbb{Q}_p$-sheaves on $(X_I^{\log})_{\text{Két}}$ to the category of smooth $\mathbb{Q}_p$-sheaves on $(U_I^{\log})_{\text{Két}}$ is an equivalence of categories, and if we have a rational point $x \in U_I(K)$, then we have the following exact sequence:

$$(*) \qquad 1 \longrightarrow \prod_I \widehat{\mathbb{Z}}(1) \longrightarrow \pi_1(U_I^{\log}, \overline{x}) \longrightarrow \pi_1(U_I, \overline{x}) \longrightarrow 1.$$

Let $\mathrm{Perv}^{\mathrm{unip}}_{\mathbb{Q}_p}(X_K, D_K)$ denote the category of preverse $\mathbb{Q}_p$-sheaves $\mathcal{F}$ on $(X_K)_{\text{ét}}$ such that $\mathcal{H}^q(\mathcal{F})|_{U_I}$ are locally constant and have unipotent local monodromies along $X_I \cap D_j$ for all $j \in I^c$. Let $\mathrm{LC}_{\mathrm{gl},\mathbb{Q}_p}(X_K, D_K)$ denote the category whose objects are $\{\mathcal{F}_I, \mathcal{F}_I|_{X^{\log}_{I\cup\{i\}}} \xrightarrow{c_i} \mathcal{F}_{I\cup\{i\}} \xrightarrow{v_i} \mathcal{F}_I(-1)|_{X^{\log}_{I\cup\{i\}}}\}_{I\subset\{1,2,\ldots,n\}, i\in I^c}$ where $\mathcal{F}_I$ is a smooth $\mathbb{Q}_p$-sheaves on $(X_I^{\log})_{\text{Két}}$ and the morphisms c_i and v_i satisfy the following conditions (cf. [4]): (i) The local monodromies of $\mathcal{F}_I$ along D_i ($i \in \{1, 2, \ldots, n\}$) are unipotent; (ii) The compositions $v_i c_i$ and $c_i v_i$ are the logarithms of the local monodromies along D_i; (iii) $c_i c_j = c_j c_i$, $c_i v_j = v_j c_i$, and $v_i v_j = v_j v_i$. We define $\mathrm{LC}_{\mathrm{gl,crys}}(X_K, D_K)$ to be the full subcategory consisting of $\{\mathcal{F}_I, ...\}$ such that $\mathcal{F}_I$ are crystalline (cf. [6]).

Let $\mathrm{MF}^{\nabla}_{\mathrm{gl,conv}}((X_K, D_K), \varphi)$ be the category whose objects are $\{\mathcal{E}_I, \mathcal{E}_I \otimes_{\mathcal{O}_{\mathfrak{X}_I}} \mathcal{O}_{\mathfrak{X}_{I\cup\{i\}}} \xrightarrow{c_i} \mathcal{E}_{I\cup\{i\}} \xrightarrow{v_i} \mathcal{E}_I(-1) \otimes_{\mathcal{O}_{\mathfrak{X}_I}} \mathcal{O}_{\mathfrak{X}_{I\cup\{i\}}}\}_{I\subset\{1,2,\ldots,n\}, i\in I^c}$ where $\mathcal{E}_I$ is a filtered convergent F-isocrystals on $(\mathfrak{X}_I^{\log}, \mathcal{J}_I)$ (cf. [6]) and the morphisms c_i and v_i satisfy the following conditions: (i) The residues of $\mathcal{E}_I$ along D_i ($i \in \{1, 2, \ldots, n\}$) are nilpotent; (ii) The compositions $v_i c_i$ and $c_i v_i$ are the residues along D_i; (iii) $c_i c_j = c_j c_i$, $c_i v_j = v_j c_i$ and $v_i v_j = v_j v_i$. Here $\mathcal{J}_I$ denotes the ideal of the log structure $M_{\mathfrak{X}_I^{\log}}$ generated locally by x_i ($i \in I$) such that D_i is defined by $x_i = 0$. Let $\mathrm{MF}^{\nabla}_{\mathrm{gl,conv}}((X_K, D_K), \varphi)$ denote the category of the triple $\{(\mathcal{M}, \mathcal{M}^\dagger, \iota)\}$, consisting of a filtered $\mathcal{D}_{\mathfrak{X},\mathbb{Q}}$-module $\mathcal{M}$, F-$\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$-module (cf. [1]) $\mathcal{M}^\dagger$, and an isomorphism $\iota\colon \mathcal{M}^\dagger \cong \mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{X},\mathbb{Q}}} \mathcal{M}$ of $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$-modules.

Now the gluing procedure is described as follows:

$$\begin{array}{ccc}
\mathrm{Perv}^{\mathrm{unip}}_{\mathbb{Q}_p}(X_K, D_K) & \xrightarrow{\mathrm{I}} & \mathrm{LC}_{\mathrm{gl},\mathbb{Q}_p}(X_K, D_K) \\
\cup & & \cup \\
\mathrm{Perv}^{\mathrm{unip}}_{\mathrm{crys}}(X_K, D_K) & \longrightarrow & \mathrm{LC}_{\mathrm{gl,crys}}(X_K, D_K) \\
 & & \mathrm{II}\downarrow \\
\mathrm{MF}(\mathcal{D}_X, \varphi) & \xleftarrow{\mathrm{III}} & \mathrm{MF}^{\nabla}_{\mathrm{gl,conv}}((X_K, D_K), \varphi)
\end{array}$$

The construction of the functor I is not yet completed. The functors II and III are both fully faithful and exact. We will explain how they are (or are expected to be) constructed.

The functor I.

In the case over $\mathbb{C}$, glueing of perverse sheaves was studied by J.-L. Verdier (along divisor), R. MacPherson and K. Vilonen (along stratification), A. Galligo, M. Granger and Ph. Maisonobe (along normal crossing divisor), A. Beilinson (along divisor) ... in 80 's. Beilinson also dealt with étale sheaves. The construction of the functor I, which is expected to be an equivalence of categories, is not yet completed, but it is plausible that one can do it by using the following theorem, which is an analogue of [4] III.2.1.Proposition.

Let i_I, $i_I^{\log}$ and ε be the morphisms of sites $(X_{I,K})_{\text{ét}} \to (X_K)_{\text{ét}}$, $(X_{I,K}^{\log})_{\text{Két}} \to (X_K^{\log})_{\text{Két}}$ and $(X_K^{\log})_{\text{Két}} \to (X_K)_{\text{ét}}$. Let n be a positive integer and put $\Lambda = \mathbb{Z}/n\mathbb{Z}$.

Theorem *Let $\mathcal{F}$ be an element of $D_c^b((X_K)_{\acute{e}t}, \Lambda)$ satisfying the following conditions:*

(i) $\mathcal{H}^q(\mathcal{F})|_{U_{I,K}}$ are locally constant for all q and I;

(ii) $R^q i_I^!(\mathcal{F})|_{U_{I,K}} = 0$ for all I and $q < |I|$;

(iii) $R^q i_I^*(\mathcal{F})|_{U_{I,K}} = 0$ for all I and $q > |I|$.

Then $R^q(i_I^{\log})^!(\varepsilon^*(\mathcal{F}))$ is locally constant if $q = |I|$ and 0 otherwise.

Let $\mathcal{F}$ be as in the theorem, and put $\mathcal{F}_I = R^q(i_I^{\log})^!(\varepsilon^*(\mathcal{F}))$. If $\cap_{i\in I} D_i$ is non-empty and has a rational point x, then letting V_I denote the representation of $\pi_1(X_I^{\log}, \overline{x})$ corresponding to $\mathcal{F}_I$, we obtain homomorphisms $V_{I\cup\{i\}} \xrightarrow{c_i} V_I \xrightarrow{v_i'} V_{I\cup\{i\}}$ for $i \in I^c$ such $c_i v_i' = \gamma_i - 1$, $v_i' c_i = \gamma_i - 1$, and $c_i c_j = c_j c_i$, $v_i' v_j' = v_j' v_i'$, $c_i v_j' = v_j' c_i$. Here we chose a generator t of $\widehat{\mathbb{Z}}(1)$, and, for $i \in I^c$, γ_i denotes the element of $\pi_1(X_{I\cup\{i\}}^{\log}, \overline{x})$ given by t in the i- component of $\prod_{I\cup\{i\}} \widehat{\mathbb{Z}}(1)$ via the exact sequence $(*)$. The homomorphism c_i is $\pi_1(X_{I\cup\{i\}}, \overline{x})$-equivariant and does not depend on the choice of γ_i. The above arguments seem to work also for $\mathbb{Q}_p$-sheaves, and by setting $v_i = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-1} (\gamma_i - 1)^{n-1} \circ v_i' \otimes t^{-1}$, we should obtain a canonical morphism $v_i \colon \mathcal{F}_I|_{X_{I\cup\{i\}}^{\log}} \to \mathcal{F}_{I\cup\{i\}}(-1)$ (defined globally) which satisfy the conditions in the definition of $\mathrm{LC}_{\mathrm{gl},\mathbb{Q}_p}(X_K, D_K)$.

The functor II.

A filtered convergent F-isocrystals on $(\mathfrak{X}_I^{\log}, \mathcal{J}_I)$ is a locally projective $\mathcal{O}_{\mathfrak{X}_I} \otimes K$-module $\mathcal{E}_I$ endowed with a convergent integrable connection $\nabla \colon \mathcal{E}_I \to \mathcal{E}_I \otimes_{\mathcal{O}_{\mathfrak{X}_I}} \Omega 1_{\mathfrak{X}_I^{\log}} = \mathcal{E}_I \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega 1_{\mathfrak{X}^{\log}}$ (or equivalently a $\mathcal{D}^\dagger_{\mathfrak{X}^{\log},\mathbf{Q}}$-module structure), a Frobenius endomorphism, and a filtration satisfying certain conditions. Similarly as a usual proper smooth scheme over O_K, we can define crystalline smooth $\mathbf{Q}_p$-sheaves on $(\mathfrak{X}_I^{\log})_{\text{Két}}$ and a fully faithful functor D_{crys} from the category of crystalline smooth $\mathbb{Q}_p$-sheaves on $(\mathfrak{X}_I^{\log})_{\text{Két}}$ to the category of filtered convergent F-isocrystals on $(\mathfrak{X}_I^{\log}, \mathcal{J}_I)$. By the following proposition, the functors D_{crys} are glued and give the fully faithful functor II.

Proposition *Let $\mathcal{F}_I$ be a crystalline smooth $\mathbb{Q}_p$-sheaves on $(\mathfrak{X}_I^{\log})_{Ét}$. Then for $i \in I^c$, the logarithm of the monodromy of $\mathcal{F}_I$ along D_i: $\mathcal{F}_I \to \mathcal{F}_I(-1)$ induces the residue $D_{crys}(\mathcal{F}_I) \to D_{crys}(\mathcal{F}_I(-1)) = D_{crys}(\mathcal{F}_I)$ along D_i via the functor D_{crys}.*

This proposition follows form the following properties of a period ring. For an affine open $U = \mathrm{Spec}(R) \subset X$ and $U_I = \mathrm{Spec}(R_I) = \mathrm{Spec}(R) \cap X_I$, the residue along D_i ($i \in I^c$) of the period ring $B^+_{\mathrm{dR}}(U_I^{\log})$ is given by $t_p^{-1}\log(\gamma_i)$, where $t \in \widehat{\mathbb{Z}}(1)$ and $\gamma_i \in \pi_1(X_I^{\log})$ is defined as in the explanation on the functor I and t_p denotes the image of t in $\mathbb{Z}_p(1)$.

The functor III.

We first glue $\mathcal{E}_I$ and $\mathcal{E}_{I\cup\{1\}}$ using c_1 and v_1 for each $I \subset \{2,3,\dots,n\}$, obtaining $\mathcal{E}_I^{(1)}$ for $I \subset \{2,3,\dots,n\}$ and $c_i^{(1)}$ and $v_i^{(1)}$ for $i \in \{2,3,\dots,n\}$. Then we glue $\mathcal{E}_I^{(1)}$ and $\mathcal{E}_{I\cup\{2\}}^{(1)}$ for each $I \subset \{3,4,\dots,n\}$ using $c_2^{(1)}$ and $v_2^{(1)}$. Repeating this procedure, we obtain the fully faithful functor III. In the case $n = 1$, the construction goes as follows. Let $\mathcal{I}$ be the ideal of $\mathcal{O}_{\mathfrak{X}}$ defining the divisor $\mathfrak{D} = \varprojlim_m (D \bmod p^m) \subset \mathfrak{X}$. Let $(\mathcal{E}, \mathcal{E}_{\mathfrak{D}}, \overline{\mathcal{E}} := \mathcal{E} \otimes \mathcal{O}_{\mathfrak{D}} \xrightarrow{c} \mathcal{E}_{\mathfrak{D}} \xrightarrow{v} \overline{\mathcal{E}})$ be an object of the category $\mathrm{MF}^{\nabla}_{\mathrm{gl,conv}}((X_K, D_K), \varphi)$. Let $\mathcal{E}'$ be the fiber product of $\mathcal{E} \to \overline{\mathcal{E}} \xleftarrow{v} \mathcal{E}_{\mathfrak{D}}$. Then we have a canonical morphism of $\mathcal{D}_{\mathfrak{X},\mathbb{Q}}$-modules

$$\alpha\colon \mathcal{D}_{\mathfrak{X},\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{X}^{\log},\mathbb{Q}}} (\overline{\mathcal{E}} \otimes \mathcal{I}^{-1}) \to \mathcal{D}_{\mathfrak{X},\mathbb{Q}} \otimes_{\mathcal{D}_{\mathfrak{X}^{\log},\mathbb{Q}}} (\mathcal{E}' \otimes \mathcal{I}^{-1})$$

locally given by $\overline{e} \otimes x^{-1} \mapsto \partial_x \otimes (xe \otimes 1, 0) \otimes x^{-1} - 1 \otimes (\partial_x^{\log}(e), c(\overline{e})) \otimes x^{-1}$ for $e \in \mathcal{E}$ and its image $\overline{e}$ in $\overline{\mathcal{E}}$, where $x, x_1, \dots, x_d$ are local coordinates such that $\mathfrak{D}$ is defined by $x = 0$. Then the underlying $\mathcal{D}_{\mathfrak{X},\mathbb{Q}}$-modules we want is $\mathcal{M} = \mathrm{Cok}(\alpha)$. We then glue filtrations and Frobenii of $\mathcal{E}$ and $\mathcal{E}_{\mathfrak{D}}$. We see that the natural morphism $\mathcal{E}' \otimes \mathcal{I}^{-1} \to \mathcal{M}$ is injective, there exists an exact sequence $0 \to \mathcal{E} \to \mathcal{E}' \otimes \mathcal{I}^{-1} \to \mathcal{E}_{\mathfrak{D}} \otimes \mathcal{I}^{-1} \to 0$, and the homomorphisms v and c are reconstructed from $t\colon \mathcal{E}' \otimes \mathcal{I}^{-1} \to \mathcal{E}$ and $\partial_t\colon \mathcal{E} \to \mathcal{E}' \otimes \mathcal{I}^{-1}$. The submodules $\mathcal{E}' \otimes \mathcal{I}^{-1}$ and $\mathcal{E}$ are determined by using a V-filtration (i.e. the theory of vanishing cycles for $\mathcal{D}$-modules), and this allows us to prove that the functor $(\mathcal{E}, \mathcal{E}_{\mathfrak{D}}, c, v) \mapsto \mathcal{M}$ is fully faithful.

References

[1] P. Berthelot, *Introduction à la théorie arithmétique des $\mathcal{D}$-modules. Cohomologies p-adiques et applications arithmétiques (II).* Astérisque **279** (2002), 1–80.

[2] O. Brinon, *Représentations p-adiques cristallines et de de Rham dans le cas relatif,* Mém. Soc. Math. Fr. (N.S.) **112**, 2008.

[3] G. Faltings, *Crystalline cohomology and p-adic Galois-representations*, *Algebraic analysis, geometry, and number theory (Baltimore, 1988).* Johns Hopkins Univ. Press, Baltimore, 1989, pp. 25–80.

[4] A. Galligo, M. Granger and Ph. Maisonobe, *$\mathcal{D}$-modules et faisceaux pervers dont le support singulier est un croisement normal*, Ann. Inst. Fourier (Grenoble) **35** (1985), 1–48.

[5] C. Nakayama, *Logarithmic étale cohomology,* Math. Ann. **308(3)** (1997), 365–404.

[6] T. Tsuji, *Crystalline sheaves and filtered convergent F-isocrystals on log schemes*, preprint.

Weights in a Serre-type conjecture for U(3)

Florian Herzig
(joint work with Matthew Emerton, Toby Gee)

1. Introduction

If f is a modular eigenform of some weight $k \geq 2$, then by classical results of Eichler–Shimura and Deligne one can associate to it a (unique) continuous, semisimple mod p Galois representation $\bar{r}_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ which encodes the Hecke eigenvalues of f and which is odd (i.e., it sends complex conjugations in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to matrices of determinant -1). Serre's conjecture [5] asserts that, conversely, any irreducible, odd Galois representation $\bar{r} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is modular, that is, isomorphic to $\bar{r}_f$ for some f. Suppose now that $\bar{r} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is irreducible and modular. The *weight part of Serre's conjecture* asks for the possible weights $k \geq 2$ such that $\bar{r} \cong \bar{r}_f$ for some eigenform f of weight k and level prime to p. This problem was solved in work of Serre, Deligne, Fontaine, Gross, Edixhoven, Coleman–Voloch, and others by the early 1990's, whereas Serre's conjecture itself was established only around 2007 by Khare–Wintenberger and Kisin.

In recent years several generalisations of Serre's conjectures to other reductive groups over number fields have been proposed (by Ash–Doud–Pollack–Sinnott, Buzzard–Diamond–Jarvis, the author, Schein, Gee, ...). Little is known about these conjectures so far, except when the reductive group is a form of GL_2. The goal of this talk was to discuss a recent theorem about the weight part when the reductive group is an outer form of GL_3 (see [3]).

2. Setup

Let F be a CM field such that p splits completely in F. To simplify the notation we will assume that $[F : \mathbb{Q}] = 2$, and we fix a prime $\mathfrak{p}$ of F that divides p. Let G be a compact unitary group of rank n over $\mathbb{Q}$, that is, we assume that G is an outer form of GL_n that splits over F and that $G(\mathbb{R}) \cong \mathrm{U}(n)$ is compact. Fix an isomorphism $G_{/F} \xrightarrow{\sim} \mathrm{GL}_{n/F}$, so from our choice of $\mathfrak{p}$ we get an isomorphism $G_{/\mathbb{Q}_p} \xrightarrow{\sim} \mathrm{GL}_{n/\mathbb{Q}_p}$.

Since G is compact, there is a simple way to describe automorphic forms on G. Fix a compact open subgroup $U = \prod_l U_l \subset G(\mathbb{A}^\infty)$ such that $U_p = \mathrm{GL}_n(\mathbb{Z}_p)$. (Here $\mathbb{A}^\infty$ denotes the ring of finite adeles of $\mathbb{Q}$.) For any $\overline{\mathbb{Z}}_p[U_p]$-module M define

$$S(U, M) = \{f : G(\mathbb{Q})\backslash G(\mathbb{A}^\infty) \to M : f(gu) = u_p^{-1} f(g)\ \forall g \in G, u \in U\}.$$

It is naturally equipped with an action of the (abstract) Hecke algebra $\mathbb{T}$ generated by the Hecke operators at primes $l \neq p$ such that l splits in F and U_l is maximal compact. If U is sufficiently small, $S(U, -)$ is an exact functor, $S(U, M)$ is a finite free $\overline{\mathbb{Z}}_p$-module if M is, and we have $S(U, M \otimes_{\overline{\mathbb{Z}}_p} A) \cong S(U, M) \otimes_{\overline{\mathbb{Z}}_p} A$ for any $\overline{\mathbb{Z}}_p$-algebra A.

Let $T \subset B \subset \mathrm{GL}_{n/\overline{\mathbb{F}}_p}$ denote the diagonal maximal torus and the upper-triangular Borel subgroup. For each dominant weight $\lambda \in X^*(T)$, which we can think of as non-increasing n-tuple of integers $\lambda_1 \geq \cdots \geq \lambda_n$, let $F(\lambda)$ denote the irreducible $\mathrm{GL}_{n/\overline{\mathbb{F}}_p}$-representation of highest weight λ. By restriction we can consider $F(\lambda)$ as representation of $\mathrm{GL}_n(\mathbb{F}_p)$, and hence also of $\mathrm{GL}_n(\mathbb{Z}_p)$. As such it will usually fail to be irreducible, but Steinberg showed that $F(\lambda)$ is an irreducible $\mathrm{GL}_n(\mathbb{F}_p)$-representation provided $\lambda_i - \lambda_{i+1} \leq p-1$ for all i, and that all irreducible $\mathrm{GL}_n(\mathbb{F}_p)$-representations over $\overline{\mathbb{F}}_p$ arise in this way. In the context of Serre-type conjectures, such irreducible representations are usually called *Serre weights*, as they play the role of the weight $k \geq 2$ in Serre's conjecture. Note that, as $\mathrm{GL}_n(\mathbb{F}_p)$ is a finite group, there are only finitely many Serre weights.

Suppose we are given an "eigenform" $f \in S(U, F(\lambda))$, i.e., a $\mathbb{T}$-eigenvector with system of eigenvalues given by an algebra homomorphism $\mathbb{T} \to \overline{\mathbb{F}}_p$, whose kernel we denote by $\mathfrak{m}$. Then by the work of many people (starting with Kottwitz, Clozel, and Harris–Taylor) one can associate to f a (unique) continuous, semisimple Galois representation $\bar{r}_{\mathfrak{m}} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ which encodes the Hecke eigenvalues of f, and which satisfies $\bar{r}_{\mathfrak{m}}^c \cong \bar{r}_{\mathfrak{m}}^\vee \otimes \bar{\epsilon}^{1-n}$, where c denotes the complex conjugation in $\mathrm{Gal}(F/\mathbb{Q})$ and $\bar{\epsilon}$ the mod p cyclotomic character. We say that $\bar{r}_{\mathfrak{m}}$ is *modular* (of Serre weight $F(\lambda)$). Given an irreducible, modular Galois representation $\bar{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, let $W(\bar{r})$ denote the set of Serre weights $F(\lambda)$ such that $S(U, F(\lambda))_{\mathfrak{m}} \neq 0$ for some U as above and some maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ of residue field $\overline{\mathbb{F}}_p$ such that $\bar{r} \cong \bar{r}_{\mathfrak{m}}$. (In fact, $\mathfrak{m}$ is uniquely determined by $\bar{r}$.)

If $\bar{r}|_{G_{F_{\mathfrak{p}}}}$ is tamely ramified, we previously specified a set $W^?(\bar{r})$ of Serre weights in terms of $\bar{r}|_{I_{F_{\mathfrak{p}}}}$ and conjectured that $W^?(\bar{r})$ coincides with the subset of $W(\bar{r})$ consisting of "regular" Serre weights $F(\lambda)$ (those that satisfy $\lambda_i - \lambda_{i+1} < p-1$ for all i); see [4]. The set $W^?(\bar{r})$ is defined in terms of the reduction modulo p of a Deligne–Lusztig representation of $\mathrm{GL}_n(\mathbb{F}_p)$ over $\overline{\mathbb{Q}}_p$ that is naturally associated to $\bar{r}|_{I_{F_{\mathfrak{p}}}}$.

From now on suppose that $n = 3$. In this case, $W^?(\bar{r})$ typically has 9 elements when $\bar{r}|_{G_{F_{\mathfrak{p}}}}$ is tamely ramified. A Serre weight $F(\lambda_1, \lambda_2, \lambda_3)$ is said to be *generic* (resp. *strongly generic*) if

$$\begin{aligned} -1 + \delta < \lambda_1 - \lambda_2 < p - 1 - \delta, \\ -1 + \delta < \lambda_2 - \lambda_3 < p - 1 - \delta, \\ |\lambda_1 - \lambda_3 - (p-2)| > \delta. \end{aligned}$$

with $\delta = 4$ (resp. $\delta = 6$).

Theorem 1. *Suppose that $\bar{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ is a continuous representation such that $\bar{r}|_{G_{F_{\mathfrak{p}}}}$ is irreducible. If $\bar{r}$ is modular of some strongly generic Serre weight, then for all generic Serre weights $F(\lambda)$ we have $F(\lambda) \in W(\bar{r})$ if and only if $F(\lambda) \in W^?(\bar{r})$.*

Note that $\bar{r}|_{G_{F_{\mathfrak{p}}}}$ is tamely ramified, since it is irreducible. The modularity assumption on $\bar{r}$ guarantees, in fact, that $W^?(\bar{r})$ consists of 9 generic Serre weights.

We remark that, using automorphic induction of suitably chosen Hecke characters, we can construct many examples of Galois representations $\bar{r}$ to which our theorem applies.

3. Ideas of the proof

3.1. **Weight elimination.** To show that $F(\lambda) \in W(\bar{r})$ implies $F(\lambda) \in W^?(\bar{r})$ for $F(\lambda)$ generic, we use integral p-adic Hodge theory. The assumption that $\bar{r}$ is modular of a particular Serre weight implies by a standard argument that $\bar{r}$ admits certain automorphic lifts $r : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_3(\overline{\mathbb{Q}}_p)$. For example, there is always an automorphic lift with $r|_{G_{F_\mathfrak{p}}}$ crystalline of Hodge–Tate weights $\lambda + (2, 1, 0)$. Similarly, for any Deligne–Lusztig representation V of $\mathrm{GL}_3(\mathbb{F}_p)$ over $\overline{\mathbb{Q}}_p$ such that $F(\lambda)$ occurs in the reduction modulo p of V, there is an automorphic lift with $r|_{G_{F_\mathfrak{p}}}$ potentially crystalline of Hodge–Tate weights $(2, 1, 0)$ and Galois type determined by V. (Here we use the local-global compatibility results in [1].) Thus we can rule out some Serre weights $F(\lambda)$ by showing that $\bar{r}|_{G_{F_\mathfrak{p}}}$ cannot be the reduction modulo p of such a potentially crystalline Galois representation. It turns out that by using this method with all possible principal series and cuspidal V we get enough information to deduce that $F(\lambda) \in W^?(\bar{r})$. The p-adic Hodge theory calculations use Breuil modules with descent data and results of Caruso about maximal such objects [2].

3.2. **Weight cycling.** It remains to show that $\bar{r}$ is actually modular of all 9 predicted Serre weights. For this we use the smooth representation theory of the group $\mathrm{GL}_3(\mathbb{Q}_p)$ over $\overline{\mathbb{F}}_p$. First we define Hecke operators $\overline{T}_1$, $\overline{T}_2$ on $S(U, F(\lambda))$; these are the analogues of the classical unramified Hecke operators at p and commute with the $\mathbb{T}$-action. The "weight cycling" argument proceeds as follows. We show that the action of $\overline{T}_i$ factors as $S(U, F(\lambda)) \hookrightarrow S(U, M) \to S(U, F(\lambda))$, where M is a certain induced, finite-dimensional $\mathrm{GL}_3(\mathbb{Z}_p)$-representation that maps surjectively to $F(\lambda)$. (This works for any split reductive group over a p-adic field.) If $F(\lambda) \in W(\bar{r})$ and $\overline{T}_i$ has eigenvalue zero on $S(U, F(\lambda))_\mathfrak{m}$ (with $\mathfrak{m}$ determined by $\bar{r}$ as above), then we deduce that $S(U, F(\mu))_\mathfrak{m} \neq 0$, i.e., $F(\mu) \in W(\bar{r})$, for some Serre weight $F(\mu)$ that is a constituent of $\ker(M \to F(\lambda))$. A local-global compatibility argument shows that $\overline{T}_1, \overline{T}_2$ act nilpotently on $S(U, F(\lambda))_\mathfrak{m}$, since $\bar{r}|_{G_{F_\mathfrak{p}}}$ is irreducible. We can explicitly compute the set of implied Serre weights $F(\mu)$ (it has cardinality 2 or 5). The existence of the strongly generic Serre weight implies that all of them are generic. Weight elimination then further cuts down the list; in fact, for each generic $F(\lambda) \in W(\bar{r})$ we can rule out all but one of the implied Serre weights $F(\mu)$ for either $\overline{T}_1$ or $\overline{T}_2$ (or both). A combinatorial argument concludes.

References

[1] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor, *Local-global compatibility for $l = p$, II*, preprint, 2011.

[2] Xavier Caruso, *$\mathbb{F}_p$-représentations semi-stables*, to appear in Ann. Inst. Fourier.

[3] Matthew Emerton, Florian Herzig, and Toby Gee, *Weight cycling and Serre-type conjectures for unitary groups*, preprint, 2011.

[4] Florian Herzig, *The weight in a Serre-type conjecture for tame n-dimensional Galois representations*, Duke Math. J. **149** (2009), no. 1, 37–116.
[5] Jean-Pierre Serre, *Sur les représentations modulaires de degré* 2 *de* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, Duke Math. J. **54**(1987), no. 1, 179–230.

On conjectures of Sharifi

Takako Fukaya
(joint work with Kazuya Kato)

In the proof of Iwasawa main conjecture, Mazur and Wiles [1] study deep relation between arithmetic of cyclotomic fields and modular curves. Sharifi, mainly in his paper [3], studies even stronger relation between these two by formulating some conjectures. The purpose of the talk is to give partial results on his conjectures.

We first set up the notation on ideal class groups and modular curves. In this article, p is always an odd prime number. For a positive integer r, let

$$A_r = \mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))\{p\},$$

the p-power torsion part of the ideal class group of $\mathbb{Q}(\zeta_{p^r})$, where ζ_{p^r} is a primitive p^r-th root of unity. Let

$$X = \varprojlim_r A_r,$$

where the inverse limit is taken with respect to the norm maps of ideal class groups. This X is a finitely generated torsion module over $\Lambda := \mathbb{Z}_p[[\mathrm{Gal}(K/\mathbb{Q})]]$, where $K = \cup_r \mathbb{Q}(\zeta_{p^r})$. Via the action of complex conjugation, we have decompositions $X = X^+ \oplus X^-$ and $\Lambda = \Lambda^+ \times \Lambda^-$.

Let

$$H_r = H^1_{\text{ét}}(X_1(p^r) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Z}_p)^{\mathrm{ord}},$$

where ord represents the ordinary part for the dual Hecke operator $T^*(p)$. We write $\mathfrak{h}_r$ for the ring of dual Hecke operators, that is, the subring of $\mathrm{End}_{\mathbb{Z}_p}(H_r)$ generated over $\mathbb{Z}_p$ by the dual Hecke operators $T^*(n) : H_r \to H_r$ $(n \geq 1)$. We denote by I_r the Eisenstein ideal of $\mathfrak{h}_r$ generated by $1-T^*(p)$ and $1-T^*(\ell)+\ell\langle\ell\rangle^{-1}$ for prime numbers $\ell \neq p$, where $\langle\ell\rangle \in \mathfrak{h}_r$ is the diamond operator. Set

$$H = \varprojlim_r H_r, \quad \mathfrak{h} = \varprojlim_r \mathfrak{h}_r, \quad I = \varprojlim_r I_r.$$

Via the action of complex conjugation, we have a decomposition $H = H^+ \oplus H^-$.

Now we introduce two homomorphisms

$$\Psi : X^- \longrightarrow H^-/IH^-,$$
$$\Phi : H^-/IH^- \longrightarrow X^-.$$

We have a canonical isomorphism

$$H^+/IH^+ \xrightarrow{\simeq} \mathfrak{h}/I \ ; \ e \leftrightarrow (1 \mod I)$$

as $\mathfrak{h}$-modules. We have a homomorphism

$$\mathrm{Gal}(\overline{\mathbb{Q}}/K) \longrightarrow H^-/IH^- \ ; \ \sigma \mapsto \sigma(e) - e.$$

(The group $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ acts on H/IH and it is known that its action on H^-/IH^- is trivial and on H^+/IH^+ is trivial modulo H^-/IH^-. Hence the above map turns out to be a well-defined homomorphism.) This homomorphism factors through X^- and that is Ψ.

For each $r \geq 1$, Sharifi defines a homomorphism $H_r^- \to A_r^-/p^rA_r^-$ which sends modular symbols $[u:v]_r^-$ $(u, v \in \mathbb{Z}/p^r\mathbb{Z} - \{0\})$ generating H_r^- as a $\mathbb{Z}_p$-module, to $\{1-\zeta_{p^r}^u, 1-\zeta_{p^r}^v\}$. Here $\{\ ,\ \} : \mathbb{Z}[1/p, \zeta_{p^r}]^\times \times \mathbb{Z}[1/p, \zeta_{p^r}]^\times \longrightarrow H^2(\mathbb{Z}[1/p, \zeta_{p^r}], \mathbb{Z}/p^r\mathbb{Z}(2)) \cong A_r/p^rA_r(1) \cong A_r/p^rA_r$ is the cup product pairing. And Sharifi conjectures that this homomorphism factors through $H_r^-/I_rH_r^-$ of H_r^-. We prove this, and that we can take $\varprojlim_r$ of these homomorphisms. This $\varprojlim_r$ version is Φ.

Conjecture 1 (Sharifi)**.**

$$\Phi \circ \Psi = 1, \quad \Psi \circ \Phi = 1.$$

Remark 2. *In fact, this Conjecture* 1 *is not stated explicitly in* [3]*, but is discussed as a remark. We call this Sharifi's conjecture.*

We see what is induced from the above Conjecture 1. Firstly we discuss the relation with the conjecture of McCallum and Sharifi [2].

Conjecture 3 (McCallum-Sharifi [2])**.** *The group $A_r^-/p^rA_r^-$ is generated by elements $\{1-\zeta_{p^r}^u, 1-\zeta_{p^r}^v\}$ $(u, v \in \mathbb{Z}/p^r\mathbb{Z} - \{0\})$.*

This conjecture gives a new understanding of $A_r^-/p^rA_r^-$, and Sharifi proves this conjecture for $p < 1000$. Conjecture 3 is deduced from Conjecture 1 as follows. We have a commutative diagram:

$$\begin{array}{ccc} H^- & \xrightarrow[\text{surj. by Conj 1}]{\Phi} & X^- \\ \downarrow & & \downarrow \text{surj.} \\ H_r^- & \longrightarrow & A_r^-/p^rA_r^-. \end{array}$$

Here the vertical arrows are natural projections, and the lower horizontal arrow is by $[u:v]_r^- \mapsto \{1-\zeta_{p^r}^u, 1-\zeta_{p^r}^v\}$. Conjecture 1 implies that the Φ is surjective, and also it is known that the right vertical arrow is surjective. Hence Conjecture 3 follows from Conjecture 1.

Next we see the relation between Conjecture 1 and Iwasawa theory. Iwasawa main conjecture states that

$$X^- \sim \Lambda^-/(\xi),$$

where ξ is the p-adic Riemann zeta function of Kubota-Leopoldt, and $\sim$ represents that the both hands sides have the same characteristic ideals. The method of the proof of the Iwasawa main conjecture due to Mazur-Wiles [1] is, roughly, to show

$$X^- \cong B/IB \sim \mathfrak{h}/I \cong \Lambda^-/(\xi),$$

for some $\mathfrak{h}$-submodule B of H^-. Here the first isomorphism is given by a homomorphism which is similar to our Ψ. Now Conjecture 1 implies that $X^- \cong H^-/IH^-$, and this provides a deeper understanding of X^-.

Now the following is our main theorem.

Theorem 4. *Assume either* (i) *or* (ii): (i) *ξ has no multiple zero,* (ii) *The class of $(1-T^*(p))[0:1] \in H^-$ generates $H^-/IH^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as an $\mathfrak{h}/I \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-module. Then we have*

(1) *$\Phi \circ \Psi = 1$, and $\Psi \circ \Phi = 1$ modulo p-primary torsion of H^-/IH^-.*

(2) *$X^- \cong (H^-/IH^-)/(p\text{-torsion})$, where $(p\text{-torsion})$ denotes the p-primary torsion.*

(3) *Conjecture* 3 *is true.*

Remark 5. (1) *There is no known example of multiple zero of ξ.*

(2) *Assumption* (ii) *is closely related with a result of Sharifi. He proves Conjecture* 3 *under a slightly stronger assumption on $1-T^*(p)$.*

(3) *X^- has no p-torsion by the result of Ferrero-Washington.*

(4) *Results are valid for ω^i-components, for Teichmülar character ω.*

Remark 6. *In* [3], *Sharifi also formulates some conjecture involving two-variable p-adic L function of cusp forms. We can prove that this conjecture is true, under the same assumption as in Theorem* 4.

Now we give a schetch of the proof of Theorem 4. We have a commutative diagram:

$$\begin{tikzcd}
H^- \arrow[r, "1"] & \text{“}K_2'' \arrow[r, "2"] \arrow[d, "3"'] & X^- \arrow[r, "\Psi"] & H^-/IH^- \arrow[d, "\xi'"] \\
& S_\Lambda \arrow[r, "\text{proj.}"] & S_\Lambda/IS_\Lambda \arrow[r, "\cong"] & H^-/IH^-.
\end{tikzcd}$$

Here “K_2'' denotes a certain p-adic completion of the inverse system for r of K_2 of $X_1(p^r)$, and S_Λ is the space of ordinary Λ-adic cusp forms. And also ξ' is the derivative of ξ. The isomorphism on the right lower horizontal arrow is given by a result of Ohta which states S_Λ is isomorphic to H^-. The map 1 is by $[u:v]_r \mapsto \{g_{0,u/p^r}, g_{0,u/p^r}\}$, where $g_{0,u/p^r}, g_{0,u/p^r}$ are Siegel units, and $\{g_{0,u/p^r}, g_{0,u/p^r}\}$ denotes Beilinson-Kato element. Map 2 is the evaluation at ∞-cusp. Map 3 is by Coleman power series. We can show that composition of the maps 1 and 2 coincides with Φ and hence the composition of the upper three horizontal arrows is $\Phi \circ \Psi$. By explicit computation, we obtain that the lower composition $H^- \to H^-/IH^-$ in the diagram coincides with the the natural projection times ξ'. Hence we have $\xi'\Phi \circ \Psi = \xi'$. If ξ has no multiple zero, $(\xi' \bmod (\xi)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is invertible in $\Lambda^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathfrak{h}/I \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and this shows that $\Phi \circ \Psi = 1$ in $H^-/IH^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. From this, and by some other argument, we obtain the result.

Remark 7. *Sharifi formulates conjectures also for $\mathbb{Q}(\zeta_{Np^r})$ and $X_1(Np^r)$ $(r \geq 1)$ for a fixed $N \geq 1$. We have results in this generalized situation.*

Finally the author would like to express her sincere gratitude to the organizers of the conference and the institute for the invitation.

References

[1] B. Mazur, A. Wiles, *Class fields of abelian extensions of* $\mathbb{Q}$, Invent. Math. **76** (1984), 179–330.

[2] W. McCallum, R. Sharifi, *A cup product in the Galois cohomology of number fields*, Duke Math. J. **120** (2003), 269–310.

[3] R. Sharifi, *A reciprocity map and the two-variable p-adic L-function*, Ann. of Math. **173** (2011), 251–300.

Toroidal compactifications of integral models of Shimura varieties of Hodge type

Keerthi Shyam Madapusi Sampath

This talk was concerned with the construction of toroidal compactifications of integral canonical models of Shimura varieties of Hodge type, following [2] and [5]. In general, a Shimura variety $\mathrm{Sh}_K(G,X)$ is associated with a reductive group G over $\mathbb{Q}$, a Hermitian symmetric domain X homogeneous under $G(\mathbb{R})$, and a compact open sub-group $K \subset G(\mathbb{A}_f)$ [1]; the pair (G,X) is called a **Shimura datum**. The variety is $\mathrm{Sh}_K(G,X)$ is defined over a number field $E(G,X) \subset \mathbb{C}$, called the **reflex field** of the Shimura datum (G,X). The most fundamental examples of such varieties are the **Siegel modular varieties**: these are associated with the Shimura datum $(\mathrm{GSp}_{2g}, S_g^{\pm})$, where GSp_{2g} is the symplectic similitude group of rank g, and $S_g^{\pm}$ is the union of the Siegel half-spaces of degree g. A Shimura variety is of **Hodge type** when the associated Shimura datum (G,X) admits an embedding in a Shimura datum of the form $(\mathrm{GSp}_{2g}, S_g^{\pm})$; for $K \subset G(\mathbb{A}_f)$ small enough, such a Shimura variety can be interpreted (non-canonically) as a moduli space of polarized abelian varieties equipped with certain Hodge cycles and level structures [6].

A significant part of the Langlands program over number fields is a correspondence between L-functions associated with l-adic Galois representations and those associated with automorphic representations of reductive groups over $\mathbb{Q}$. Most progress towards such a correspondence has been obtained via a deep understanding of the zeta functions of Shimura varieties. To study the local factors of these zeta functions, and to apply the Grothendieck-Lefschetz trace formula, one would like to have good integral models for the varieties; in fact, one would also like to have good compactifications of these integral models [4]. Such integral models (and their compactifications) are also important to S. Kudla's program relating intersection numbers of certain arithmetic cycles on orthogonal Shimura varieties to Fourier coefficients of Eisenstein series [3]. We note that these orthogonal Shimura varieties are of Hodge type.

Let (G,X) be a Shimura datum of Hodge type. Let p be a prime, let $v|p$ be a finite place of $E(G,X)$, and let E_v be the completion of $E(G,X)$ along v. As noted earlier, for K small enough, $\mathrm{Sh}_K(G,X)$ has a moduli interpretation. The main obstacle to constructing integral model for $\mathrm{Sh}_K(G,X)$ over $\mathcal{O}_{E_v}$ is that there is no obvious way to re-interpret this moduli interpretation so as to make

sense of it over bases that are not in characteristic 0: indeed, there is no positive characteristic analogue of a general Hodge cycle[1]. The method of construction is then necessarily indirect. One embeds the Shimura variety inside a suitable Siegel modular variety; the latter, being the moduli space of polarized abelian varieties with level structure, has a natural integral model over $\mathcal{O}_{E_v}$. Then, one considers the normalization of the Zariski closure of $\mathrm{Sh}_K(G,X)$ inside such an integral model: this is our candidate $\mathcal{S}_K(G,X)_{\mathcal{O}_{E_v}}$ for a good integral model for the Shimura variety $\mathrm{Sh}_K(G,X)$

Suppose now that the level $K_p \subset G(\mathbb{Q}_p)$ at p is **hyperspecial**; that is, we have $K_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$, for some reductive model $G_{\mathbb{Z}_{(p)}}$ of G over $\mathbb{Z}_{(p)}$. In this situation, we expect that the model $\mathcal{S}_K(G,X)_{\mathcal{O}_{E_v}}$ is smooth. Moreover, it is shown in [7] that the smoothness of $\mathcal{S}_K(G,X)_{\mathcal{O}_{E_v}}$ will canonically single out $\mathcal{S}_K(G,X)_{\mathcal{O}_{E_v}}$ from among all integral models of $\mathrm{Sh}_K(G,X)$; in particular, $\mathcal{S}_K(G,X)_{\mathcal{O}_{E_v}}$ will not depend on the choice of embedding of $\mathrm{Sh}_K(G,X)$ into a Siegel modular variety.

The main result we talked about is the following

Theorem 1. *Kisin* [2] *Let $\ddot{\mathcal{S}}$ be the Zariski closure of $\mathrm{Sh}_K(G,X)$ inside the natural integral model of a suitable Siegel Shimura variety as above. Then every irreducible component of $\ddot{\mathcal{S}}$ is smooth over $\mathcal{O}_{E_v}$.*

Clearly, this result is stronger than simply showing that the normalization $\mathcal{S}_K(G,X)_{\mathcal{O}_{E_v}}$ of $\ddot{\mathcal{S}}$ is smooth. The main idea of the proof is to interpret the complete local ring at any closed point of an irreducible component of $\ddot{\mathcal{S}}$ as a deformation ring of a p-divisible group equipped with crystalline Tate cycles.

Extending the methods used in the proof of the theorem above, the speaker, in his University of Chicago Ph. D. Thesis [5], constructed toroidal compactifications of $\mathcal{S}_K(G,X)_{\mathcal{O}_{E_v}}$. In particular, we now know that the integral canonical model $\mathcal{S}_K(G,X)_{\mathcal{O}_{E_v}}$ is proper whenever $\mathrm{Sh}_K(G,X)$ is a complete variety.

References

[1] P. Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, Proc. Sympos. Pure Math. **33** (1979), 247–289.

[2] M. Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. **23** (2010), 967–1012.

[3] S.S. Kudla, *Special cycles and derivatives of Eisenstein series*, Math. Sci. Res. Inst. Publ. **49** (2004), 243–270.

[4] R. Langlands, *Some contemporary problems with origins in the Jugendtraum*, Proc. Sympos. Pure Math. **28** (1974), 401–418.

[5] K. Madapusi Sampath, *Toroidal compactifations of integral canonical models of Shimura varieties of Hodge type*, Thesis (Ph. D.)–University of Chicago (2011).

[6] J.S. Milne, *Shimura varieties and Motives*, Proc. Sympos. Pure Math. **55** (1994), 447–523.

[7] J.S. Milne, *The points on a Shimura variety modulo a prime of good reduction*, The zeta functions of Picard modular surfaces (1994), 151–253.

[1] Conjecturally, however, any such cycle should be generated by algebraic cycles

Non abelian p-adic L-functions and Eisenstein series of unitary groups

Thanasis Bouganis

In [6, 7] a vast generalization of the Main Conjecture of the classical (abelian) Iwasawa theory to a non-abelian setting was proposed. However, the evidences for this non-abelian Main Conjecture are still very modest. One of the central difficulties of the theory seems to be the construction of non-abelian p-adic L-functions. Actually, the only known results in this direction are mainly restricted to the Tate motive thanks to the works of Ritter and Weiss in [14, 15] and Kakde [13].

For other motives besides the Tate motive not much is known (but see [1, 5]). Our aim is to tackle the question of the existence of non-abelian p-adic L-functions for "motives", whose classical L-functions can be studied through L-functions of automorphic representations of definite unitary groups. Our first goal is to prove the so called "torsion congruences" (to be explained below) for these motives and then use our approach to tackle the so called Möbious-Wall congruences (see [15]).

Let p be an odd prime number. We write F for a totally real field and F' for a totally real Galois extension with $\Gamma := Gal(F'/F)$ of order p. We assume that the extension is unramified outside p. We write $G_F := Gal(F(p^\infty)/F)$ where $F(p^\infty)$ is the maximal abelian extension of F unramified outside p (may be ramified at infinity). We make the similar definition for F'. Our assumption on the ramification of F'/F implies that there exist a transfer map $ver : G_F \to G_{F'}$, which induces also a map $ver : \mathbb{Z}_p[[G_F]] \to \mathbb{Z}_p[[G_{F'}]]$, between the Iwasawa algebras of G_F and $G_{F'}$, both of them taken with coefficients in $\mathbb{Z}_p$.

Let us now consider a motive M/F (by which we really mean the usual realizations of it and their compatibilities) defined over F such that its p-adic realization has coefficients in $\mathbb{Z}_p$. Then under some assumptions on the critical values of M and some ordinarity assumptions at p (to be made more specific later) it is conjectured that there exists an element $\mu_F \in \mathbb{Z}_p[[G_F]]$ that interpolates the critical values of M/F twisted by characters of G_F. Similarly we write $\mu_{F'}$ for the element in $\mathbb{Z}_p[[G_{F'}]]$ associated to M/F', the base change of M/F to F'. Then the so-called torsion congruences read

$$ver(\mu_F) \equiv \mu_{F'} \mod T,$$

where T is the trace ideal in $\mathbb{Z}_p[[G'_F]]^\Gamma$ generated by the elements $\sum_{\gamma\in\Gamma} \alpha^\gamma$ with $\alpha \in \mathbb{Z}_p[[G'_F]]$. These congruences have been proved by Ritter and Weiss [14] for the Tate motive and under some assumptions by the author [2] for M/F equal to the motive associated to an elliptic curve with complex multiplication. In this work we prove these congruences for motives that their L-functions can be studied by automorphic representations of definite unitary groups.

We now write K for a totally imaginary quadratic extension of F, that is K is a CM field. On our prime number p we put the following ordinary assumption: all primes above p in F are split in K. We write $K' := F'K$, a CM field with $K'^+ = F'$. Next we pick an ordinary CM type Σ of K and denote this pair by

(Σ, K). We consider the inflated type Σ' of Σ to K' and write (Σ', K') for this CM type which is also an ordinary CM type.

The motives M/F that we will consider are of the form $M(\psi)/F \otimes M(\pi)/F$ where $M(\psi)/F$ and $M(\pi)/F$ are to be defined as next. Let ψ be a Hecke character of K and assume that its infinite type is $k\Sigma$ for some integer $k \geq 1$. We write $M(\psi)/F$ for the motive over F that is obtained by "Weil Restriction" to F from the rank one motive over K associated to ψ. We consider now a hermitian space (W, θ) over K and write n for its dimension. We write $U(\theta)/F$ for the corresponding unitary group. We consider now a motive $M(\pi)/F$ over F such that there exists an automorphic representation π of some unitary group $U(\theta)(\mathbb{A}_F)$ with the property that the L-function $L(M(\pi)/K, s)$ of $M(\pi)/K$ over K is equal to $L(\pi, s)$.

We prove the torsion congruences for the motive $M/F := M(\psi)/F \otimes M(\pi)/F$ under the following three assumptions: (1) The p-adic realizations of $M(\pi)$ and $M(\psi)$ have $\mathbb{Z}_p$-coefficients, (2) the motive $M(\pi)/F$ is associated to an automorphic representation π of a definite unitary group in n variables and we have $k \geq n$.

Theorem: Assume that the prime p is unramified in F (but may ramify in F'). Then we have: (1) For $n = 1$: The torsion congruences hold true. (2) For $n = 2$: We write (π, π) for the standard normalized Peterson inner product of π. If (π, π) has trivial valuation at p then the torsion congruences hold true.

The key idea of the proof is the following: Special values of L functions of unitary representations can be realized with the help of the doubling method of Shimura, Piateski-Shapiro and Rallis [16, 17, 8] either (i) as values of hermitian Siegel-type Eisenstein series on CM points of Hermitian domains or (ii) as constant terms of hermitian Klingen-type Eisenstein series for some proper Fourier-Jacobi expansion. In the first approach (see [4]) we consider Siegel-type Eisenstein series of the group $U(n, n)$ with the property that their values at particular CM points are equal to the special L-values that we want to study. The CM points are obtained from the doubling method as indicated by the embedding $U(n, 0) \times U(0, n) \hookrightarrow U(n, n)$. Then we make use of the fact that the CM pairs (K, Σ) and (K', Σ') that we consider are closely related (i.e. the second is induced from the first) which allows us to relate the various CM points over K and K'. Then we use the diagonal embedding, induced from the embedding $K \hookrightarrow K'$, between the symmetric space of $U(n, n)_{/F}$ and that of $Res_{F'/F} U(n, n)_{/F'}$ to relate the Eisenstein series over the different fields.

In the other approach (see [3]) we obtain Klingen-type Eisenstein series of the group $U(n+1, 1)$ with the property that the constant term of their Fourier-Jacobi expansion is related with the special values that we want to study. Then again we use the embedding $K \hookrightarrow K'$ to relate these Klingen-type Eisenstein series over the different fields and hence also to obtain a relation between their constant terms. The main difficulty here is that the Klingen-type Eisenstein series have a rather complicated Fourier-Jacobi expansion, which makes hard the direct study of the arithmetic properties of these Eisenstein series. However the Klingen-type

Eisenstein series are obtained with the help of the pull-back method from Siegel-type Eisenstein series of the group $U(n+1, n+1)$ using the embedding $U(n+1, 1) \times U(0, n) \hookrightarrow U(n+1, n+1)$. The Siegel-type Eisenstein series have a much better understood Fourier expansion, which turns out it suffices to study also the Klingen-type Eisenstein series.

Of course both approaches should happen in a p-adic setting. The needed theory for all these has been developed in the papers [9, 10] of Harris, Li and Skinner on the Eisenstein measure on unitary groups and in the works of Ming-Lun Hsieh [11, 12].

References

[1] Th. Bouganis, Special values of L-fucntions and false Tate curve extenions (with an appendix by Vl. Dokchitser), *J. Lond. Math. Soc.* vol 82 (3) (2010) .

[2] Th. Bouganis, Non-abelian congruences between special values of L-fucntions of elliptic curves; the CM case, to appear in the *International Journal of Number Theory* (IJNT).

[3] Th Bouganis, Non abelian p-adic L-functions and Eisenstein series of unitary groups; the Constant Term Method. *In preparation.*

[4] Th Bouganis, Non abelian p-adic L-functions and Eisenstein series of unitary groups; the CM-method. *preprint.*

[5] Th. Bouganis and O. Venjakob, On the non-commutative Main Conjecture for Elliptic Curves with Complex Multiplication, *Asian J. Math.* Vol. 14, No. 3, (2010), pp 385-416

[6] J. Coates, T. Fukaya, K. Kato, R. Sujatha and O. Venjakob, The GL_2-main conjecture for elliptic curves without complex multiplication, Publ. Math. IHES. 101 (2005), no. 1, 163-208

[7] T. Fukaya and K. Kato, A formulation of conjectures on p-adic zeta functions in non-commutative Iwasawa theory, Proceedings of the St. Petersburg Mathematical Society, vol XII (Providence RI), Amer. Math. Soc. Transl. Ser. 2 vol 219, 2006

[8] S. Gelbart, I. Piatetski-Shapiro and S. Rallis, Explicit Constructions of Automorphic L-function, *Lecture Notes in Mathematics 1254*, Springer-Verlag 1987.

[9] M. Harris, J.-S. Li, and C. Skinner, The Rallis inner product formula and p-adic L-functions, *Automorphic Representations, L-Functions and Applications: Progress and Prospects*, Volume in honor of S.Rallis, Berlin:de Gruyter, 225-255 (2005)

[10] M. Harris, J.-S. Li, and C. Skinner, p-adic L-functions for unitary Shimura varieties, I: Construction of the Eisenstein measure, *Documenta Math.*, Extra Volume: John H.Coates' Sixtieth Birthday 393-464 (2006)

[11] M-L. Hsieh, Eisenstein congruence on unitary groups and Iwasawa main conjectures for CM fields, preprint.

[12] M-L. Hsieh, p-adic oridnary Eisenstein series and p-adic L-functions for unitary groups, *Annales de l'Institut Fourier*, Vol. 60 (2010)

[13] M. Kakde, Proof of the main conjecture of non-commutative Iwasawa theory of totally real number fields, *preprint* 2011

[14] J. Ritter and A. Weiss, Congruences between abelian pseudomeasures, *Math. Res. Lett.* 15 (2008) 715-725

[15] J. Ritter and A. Weiss, Congruences between abelian pseudomeasures II, available at the arXiv.

[16] G. Shimura, Euler Products and Eisenstein Series, *CBMS Regional Conference Series in Mathematics*, No. 93, Amer. Math. Soc. 1997

[17] G. Shimura, Arithmeticity in the Theory of Automorphic Forms, *Mathematical Surveys and Monographs*, Vol 82, Amer. Math. Soc. 2000

On Iwasawa main conjecture for CM fields

Ming-Lun Hsieh

We report our recent work in [7] on Iwasawa main conjecture for CM fields. Let $\mathcal{K}$ be a totally imaginary quadratic extension of a totally real number field $\mathcal{F}$. Let $p > 2$ be a rational prime. We assume that $\mathcal{K}$ is p-ordinary, namely every prime of $\mathcal{F}$ above p splits in $\mathcal{K}$. Let Σ be a p-ordinary CM-type of $\mathcal{K}$. Fix an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Then Σ and ι_p give rise to p-adic CM-type Σ_p such that Σ_p and its complex conjugation Σ_p^c give a partition of the set of places of $\mathcal{K}$ lying above p. Let $d = [\mathcal{F} : \mathbb{Q}]$. Let $\mathcal{K}_\infty$ be the compositum of the cyclotomic $\mathbb{Z}_p$-extension and the anticyclotomic $\mathbb{Z}_p^d$-extension of $\mathcal{K}$ and let $\Gamma_\mathcal{K} = \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K})$. By class field theory, $\Gamma_\mathcal{K}$ is a free $\mathbb{Z}_p$-module of rank $1 + d$. Let $\mathcal{K}' \supset \mathcal{K}(\mu_p)$ be a finite abelian extension of $\mathcal{K}$ which is linear disjoint from $\mathcal{K}_\infty$ and let $\Delta = \mathrm{Gal}(\mathcal{K}'/\mathcal{K})$. Let $\psi : \Delta \to \mathbb{C}_p^\times$ be a character of Δ, which is called a *branch* character. Let W_p be the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$ and let $\Lambda := W_p[[\Gamma_\mathcal{K}]]$ be a $(d+1)$-variable Iwasawa algebra over W_p.

Let $\mathcal{K}'_\infty = \mathcal{K}'\mathcal{K}_\infty$ and let M_Σ be the maximal p-abelian Σ_p-ramified extension of $\mathcal{K}'_\infty$ and let X_Σ be the Galois group $\mathrm{Gal}(M_\Sigma/\mathcal{K}'_\infty)$. Then M_Σ is also Galois over $\mathcal{K}$, and Δ acts on X_Σ by the usual conjugation action. Define the Iwasawa module $X_\Sigma^{(\psi)}$ to be the maximal ψ-isotypic quotient of X_Σ. It is proved by Hida and Tilouine that $X_\Sigma^{(\psi)}$ is a finitely generated and torsion Λ-module. Therefore to $X_\Sigma^{(\psi)}$ we can attached a characteristic power series $F_{\psi,\Sigma} \in \Lambda$. On the other hand, Katz [1] and Hida and Tilouine [4] have constructed a p-adic L-function $L_{\psi,\Sigma} \in \Lambda$, which interpolates p-adically Hecke L-values for $\mathcal{K}$. The $(d+1)$-variable main conjecture for CM fields is stated as follows.

Conjecture 1 (Iwasawa main conjecture for CM fields)**.** *We have the following equality between ideals in* Λ

$$(F_{\psi,\Sigma}) = (L_{\psi,\Sigma}).$$

This conjecture is a theorem of Rubin [8] if $\mathcal{F} = \mathbb{Q}$. Rubin uses the technique of Euler system constructed from elliptic units to prove the one-sided divisibility $(F_{\psi,\Sigma}) \supset (L_{\psi,\Sigma})$ and then appeals to the p-adic class number formula to conclude the equality. For general CM fields, the Euler system technique seems not applicable currently. In [7], we prove the reverse divisibility relation $(F_{\psi,\Sigma}) \subset (L_{\psi,\Sigma})$ under certain assumptions. To state our result precisely, we need to introduce some notation. Let $\mathfrak{m}_\Lambda$ be the maximal ideal of Λ. For a number field L, we let $G_L = \mathrm{Gal}(\overline{\mathbb{Q}}/L)$ and let $\omega_L : G_L \to \mathrm{Gal}(L(\mu_p)/L) \subset \mu_{p-1}$ be the Teichmüller character. Let $D_{\mathcal{K}/\mathcal{F}}$ be the relative discriminant of $\mathcal{K}/\mathcal{F}$ and let $\mathfrak{c}(\psi)$ be the prime-to-p conductor of the branch character ψ. Let $\psi_+ : G_\mathcal{F}^{ab} \to \mathbb{C}_p^\times$ be the composition $\psi \circ V$, where $V : G_\mathcal{F}^{ab} \to G_\mathcal{K}^{ab}$ is the *Verschiebung* map.

Theorem 2. *Suppose that*

(1) $p \nmid 3 \cdot h_\mathcal{K}/h_\mathcal{F} \cdot D_\mathcal{F} \cdot \#(\Delta)$,
(2) *ψ is unramified at Σ_p^c, and $\psi\omega_\mathcal{K}^{-a}$ is unramified at Σ_p for some integer $a \not\equiv 2 \pmod{p-1}$,*

(3) *either* $\psi_+ \neq \omega_{\mathcal{F}}\tau_{\mathcal{K}/\mathcal{F}}$ *or*

$$\psi_+ = \omega_{\mathcal{F}}\tau_{\mathcal{K}/\mathcal{F}} \text{ and } W_\Sigma(\psi) \equiv 1 \pmod{\mathfrak{m}_\Lambda},$$

where $W_\Sigma(\psi)$ *is the root number of the Katz* p*-adic function* $L_{\psi,\Sigma}$.

Then we have the following inclusion between ideals of Λ

$$(F_{\psi,\Sigma}) \subset (L_{\psi,\Sigma}).$$

The assumptions in the above theorem are in particular responsible for the application of results on the non-vanishing modulo p of Hecke L-values and the vanishing of μ-invariant of anticyclotomic Katz p-adic L-functions in [5] and [6] (based on works of Hida in [3] and [2]). The above result combined with the control theorems of Selmer groups for CM elliptic curves due to Perrin-Riou yields the following consequence.

Theorem 3. *Let* $E_{/\mathcal{F}}$ *be an elliptic curve over* $\mathcal{F}$ *with the complex multiplication by the ring of integer of an imaginary quadratic field* $\mathcal{M}$. *Let* $\mathcal{K} = \mathcal{F}\mathcal{M}$. *Suppose that* $p \not| 3 \cdot h_{\mathcal{K}}/h_{\mathcal{F}} \cdot D_{\mathcal{F}}$ *and that* $E_{/\mathcal{F}}$ *satisfies the following conditions:*

(N1) *E has good ordinary reduction at all places above p,*

(N2) *the root number* $W(E/\mathcal{F})$ *of* $E_{/\mathcal{F}}$ *is* $+1$.

Then we have

(a) *If* $L(E_{/\mathcal{F}}, 1) = 0$, *then the p-primary Selmer group* $\mathrm{Sel}_{\mathcal{F}}(E)$ *has positive* $\mathbb{Z}_p$*-corank.*

(b) *If* $L(E_{/\mathcal{F}}, 1) \neq 0$, *then*

$$ord_p\left(\frac{L(E_{/\mathcal{F}}, 1)}{\Omega_E}\right) \leq length_{\mathbb{Z}_p}(\mathrm{Sel}_{\mathcal{F}}(E)),$$

where Ω_E *is the period of a Néron differential of* E *over* $\overline{\mathbb{Z}}_{(p)}$.

References

[1] Katz, Nicholas M., *p-adic L-functions for CM fields*, Invent. Math. 49 (1978), no. 3, 199–297.

[2] Hida, H., *The Iwasawa μ-invariant of p-adic Hecke L-functions*, Ann. of Math. (2) 172 (2010), no. 1, 41–137.

[3] Hida, H., *Non-vanishing modulo p of Hecke L-values.* Geometric aspects of Dwork theory. Vol. I, II, 735–784, Walter de Gruyter GmbH & Co. KG, Berlin, 2004.

[4] Hida, H.; Tilouine, J. *Anti-cyclotomic Katz pp-adic LL-functions and congruence modules*, Ann. Sci. Ecole Norm. Sup. (4) 26 (1993), no. 2, 189–259.

[5] Hsieh, Ming-Lun, *On the non-vanishing of Hecke L-values modulo p*, 24pp. Preprint is available at "http://www.math.ntu.edu.tw/ mlhsieh/research.htm".

[6] Hsieh, Ming-Lun, *On the μ-invariant of anticyclotomic p-adic L-functions for CM fields*, 21pp. Preprint is available at "http://www.math.ntu.edu.tw/ mlhsieh/research.htm".

[7] Hsieh, Ming-Lun, *Eisenstein congruence on unitary groups and the main conjecture for CM fields*, 79pp. Preprint is available at "http://www.math.ntu.edu.tw/ mlhsieh/research.htm".

[8] Rubin, Karl, *The "main conjectures" of Iwasawa theory for imaginary quadratic fields*, Invent. Math. 103 (1991), no. 1, 25–68.

Descent on elliptic surfaces and transcendental Brauer elements

Bianca Viray

Let X be a smooth projective geometrically integral variety over a field k and let $\operatorname{Br} X$ denote $\mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_{\mathrm{m}})$, the Brauer group of X. We introduce the following notation.

$$\operatorname{Br}_0 X := \operatorname{im}(\operatorname{Br} k \to \operatorname{Br} X), \quad \operatorname{Br}_1 X := \ker\left(\operatorname{Br} X \to \operatorname{Br} \overline{X}\right)$$

Elements in $\operatorname{Br}_0 X$ are termed *constant*, and elements in $\operatorname{Br}_1 X$ are termed *algebraic*. If $\mathcal{A} \in \operatorname{Br} X$ is not algebraic, then we say it is *transcendental*.

Transcendental elements of the Brauer group are notoriously difficult to compute. In fact, since 1972, there have only been a handful of explicit examples of transcendental elements [1, 2, 5, 6, 7, 9, 10]

Techniques developed by Wittenberg [10] and Ieronymou [7] can be used to compute 2-torsion transcendental elements of a surface with a genus 1 fibration, assuming that the Jacobian fibration has rational 2-torsion and that the original fibration obtains a section after a quadratic extension of the base. In this talk, we use techniques from classical descent to remove the assumption on the 2-torsion of the Jacobian fibration.

First let us fix some notation. From now on, we work over an algebraically closed field k of characteristic 0. Let X be a smooth projective geometrically integral surface, and assume we have a generically smooth morphism $\pi\colon X \to W$, where W is 1-dimensional. We denote the function field of W, $\mathbf{k}(W)$, by K. Let C denote the generic fiber of π and let J denote the Jacobian of C.

By the purity theorem [4, Prop 2.1], we can write

$$\operatorname{Br} X = \bigcap_{\substack{V \text{ prime} \\ \text{vertical divisor}}} \ker\left(\operatorname{Br} C \xrightarrow{\partial_V} \mathrm{H}^1(\kappa(V), \mathbb{Q}/\mathbb{Z})\right)$$

So if we can explicitly compute a finite number of elements in $(\operatorname{Br} C)[2]$ that contains the elements of $(\operatorname{Br} X)[2]$, we will be able to compute all of the transcendental 2-torsion elements of $\operatorname{Br} X$.

To this end, we prove

Theorem 1 (V.)**.** *Assume that C has a model $y^2 = f(x)$ where $\deg(f) = 4$. Let L be the degree 4 étale algebra $K[\alpha]/(f(\alpha))$. The following diagram commutes,*

$$\begin{tikzcd}
\frac{J(K)}{2J(K)} \arrow[r, hook] \arrow[dr, "x-\alpha"] & \mathrm{H}^1(G_K, J[2]) \arrow[r, two heads] \arrow[d] & \mathrm{H}^1(G_K, J)[2] \arrow[d] \\
& \ker\left(N : \frac{L^\times}{L^{\times 2} K^\times} \to \frac{K^\times}{K^{\times 2}}\right) \arrow[r, "h"] & (\operatorname{Br} C)[2],
\end{tikzcd}$$

where $h\colon \ell \mapsto \operatorname{Cor}_{\mathbf{k}(C_L)/\mathbf{k}(C)}((\ell, x-\alpha)_2)$.

Moreover, for any set S that contains the places of bad reduction of $y^2 = f(x)$, there is a finite group $(\ker N)_{\text{S-unr}}$ *such that*

$$h^{-1}(\mathrm{Br}X) \subseteq (\ker N)_{\text{S-unr}}\,.$$

This theorem should be thought of as a geometric analogue of descent theorems found in [8]; in fact, many ideas in the proof of Theorem 1 come from [8]. However, in contrast to classical descent where one tries obtain the rank by controlling the Tate-Shafarevich group, we seek to use our knowledge of the rank to understand the Brauer group, which is the geometric analogue of the Tate-Shafarevich group.

The finite group $(\ker N)_{\text{S-unr}}$ should be thought of as an geometric analogue of the Selmer group, but one where the unramified conditions are not imposed at the bad places. For an analogue where conditions are imposed at all places, see [3].

We note that we may not obtain all of $\mathrm{Br}X$ with this method, as h is not necessarily surjective. However, we know that the cokernel of h has order at most 2, so we do obtain at least half of the Brauer group, and we are missing at most 1 generator.

References

[1] Michael Artin, David Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London Math. Soc. (3) **25** (1972), 75–95.

[2] Jean-Louis Colliot-Thélène, Manuel Ojanguren, *Variétés unirationnelles non rationnelles: au-delà de l'exemple d'Artin et Mumford*, Invent. Math. **97** (1989), no. 1, 141–158.

[3] Jean-Louis Colliot-Thélène, Alexei N. Skorobogatov, Peter Swinnerton-Dyer, *Hasse principle for pencils of curves of genus one whose Jacobians have rational 2-division points*, Invent. Math. **134** (1998), no. 3, 579–650.

[4] Alexander Grothendieck *Le groupe de Brauer. III. Exemples et compléments*, 1968 Dix Exposés sur la Cohomologie des Schémas pp. 88–188 North-Holland, Amsterdam; Masson, Paris.

[5] David Harari, Alexei N. Skorobogatov, *Non-abelian descent and the arithmetic of Enriques surfaces*, Int. Math. Res. Not. 2005, no. 52, 3203–3228.

[6] Brendan Hassett, Anthony Várilly-Alvarado, Patrick Varilly, *Transcendental obstructions to weak approximation on general K3 surfaces*, To appear Adv. Math., `arXiv:10051879`

[7] Evis Ieronymou, *Diagonal quartic surfaces and transcendental elements of the Brauer groups*, J. Inst. Math. Jussieu **9** (2010), no. 4, 769–798.

[8] Bjorn Poonen, Edward F. Schaefer, *Explicit descent for Jacobians of cyclic covers of the projective line*, J. Reine Angew. Math. **488** (1997), 141–188.

[9] Alexei N. Skorobogatov, Peter Swinnerton-Dyer, *2-descent on elliptic curves and rational points on certain Kummer surfaces*, Adv. Math. **198** (2005), no. 2, 448–483.

[10] Olivier Wittenberg, *Transcendental Brauer-Manin obstruction on a pencil of elliptic curves*, Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), 259–267, Progr. Math., **226**, Birkhäuser Boston, Boston, MA, 2004.

A reformulation of Kato's main conjecture for modular forms

Antonio Lei
(joint work with David Loeffler, Sarah Livia Zerbes)

Let $f = \sum a_n q^n \in S_k(\Gamma_0(N), \theta)$ be a normalised eigen-newform and $F = \mathbb{Q}(a_n : n \geq 1)$. We fix an odd prime $p \nmid N$ and a prime v above p in F. We write K for the completion of F at v. By [4], we may associate to f a 2-dimensional K-linear representation V_f with Hodge-Tate weights 0 and $1-k$. We fix a $G_{\mathbb{Q}}$-stable $\mathcal{O}_K$-lattice T_f in V_f. For simplicity, we assume that the reduced representation $T_f/\mathfrak{M}_K T_f$ is irreducible and that the roots to $X^2 - a_p X + \theta(p)p^{k-1}$ are not in $p^{\mathbb{Z}}$. For $T = T_f(k-1)$ and $i \geq 1$, define

$$\mathbb{H}^i(T) = \varprojlim H^i_{\text{ét}}(\mathbb{Z}[\mu_{p^n}, 1/p], T);$$
$$H^i_{\text{Iw}}(\mathbb{Q}_p, T) = \varprojlim H^i(\mathbb{Q}_p(\mu_{p^n}), T).$$

Let $\Gamma = \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$ and $\Lambda = \mathcal{O}_K[[\Gamma]]$. Kato has showed in [5] that $\mathbb{H}^2(T)$ is Λ-torsion and that $\mathbb{H}^1(T)$ is free of rank 1 over Λ. Moreover, $\mathbb{H}^1(T)$ contains an element z_f such that the image of $\text{loc}(z_f)$ under the Perrin-Riou map $\mathcal{L}_\alpha$ is the p-adic L-function L_α of $\bar{f} = \sum \bar{a}_n q^n$ associated to α. Here, $\text{loc} : \mathbb{H}^1(T) \to H^1_{\text{Iw}}(\mathbb{Q}_p, T)$ is the localisation map and α satisfies

$$\alpha^2 - \bar{a}_p \alpha + \bar{\theta}(p)p^{k-1} = 0 \quad \text{with} \quad v_p(\alpha) < k-1. \tag{1}$$

Kato's main conjecture is as follows.

Conjecture 1. *If η is a character on Γ_{tor}, then*

$$\text{Char}\, \mathbb{H}^2(T)^\eta = \text{Char}\left(\mathbb{H}^1(T)/\Lambda z_f\right)^\eta.$$

Here, $(\star)^\eta$ denotes the η-isotypical component and $\text{Char}(\star)$ *denotes the characteristic ideal of a torsion $\mathcal{O}_K[[\Gamma/\Gamma_{\text{tor}}]]$-module.*

When f is ordinary at p (i.e. $v_p(a_p) = 0$), there exists a unique p-adic unit α satisfying (1). Via the Poitou-Tate exact sequence, we see that $\text{Sel}(\bar{f}/\mathbb{Q}(\mu_{p^\infty}))$ is Λ-cotorsion and that Conjecture 1 is equivalent to asserting that

$$\text{Char}\, \text{Sel}(\bar{f}/\mathbb{Q}(\mu_{p^\infty}))^{\vee,\eta} = (\Lambda L_\alpha)^\eta$$

where $(\star)^\vee$ denotes the Pontryagin dual.

In the non-ordinary case, we follow the strategy of Kobayashi's work on supersingular elliptic curves with $a_p = 0$ in [6] to obtain a similar reformulation of Conjecture 1. Let $\text{Col}_\star : H^1_{\text{Iw}}(\mathbb{Q}_p, T) \to \Lambda$ be a Λ-homomorphism and write $\mathbb{H}^1_\star(T) = \ker \text{Col}_\star$. Define $H^1(\mathbb{Q}(\mu_{p^\infty}), T^\vee)^\star \subset H^1(\mathbb{Q}(\mu_{p^\infty}), T^\vee)$ to be the exact annihilator of $\mathbb{H}^1_\star(T)$. Recall that $T^\vee \cong V_{\bar{f}}/T_{\bar{f}}(1)$. We define the modified Selmer group for $\bar{f}$ by

$$\text{Sel}^\star(\bar{f}/\mathbb{Q}(\mu_{p^\infty})) = \ker\left(\text{Sel}(\bar{f}/\mathbb{Q}(\mu_{p^\infty})) \to \frac{H^1(\mathbb{Q}(\mu_{p^\infty}), T^\vee)}{H^1(\mathbb{Q}(\mu_{p^\infty}), T^\vee)^\star}\right).$$

Let $L_\star = \mathrm{Col}_\star(\mathrm{loc}(z_f))$, which we assume to be non-zero at all isotypical components. The Poitou-Tate exact sequence then implies that $\mathrm{Sel}^\star(\bar{f}/\mathbb{Q}(\mu_{p^\infty}))$ is Λ-cotorsion and that Conjecture 1 is equivalent to asserting that

$$\mathrm{Char}\,\mathrm{Sel}^\star(\bar{f}/\mathbb{Q}(\mu_{p^\infty}))^{\vee,\eta} = (\mathrm{im}\,\mathrm{Col}_\star\,/L_\star)^\eta.$$

The main work is therefore to construct $\mathrm{Col}_\star$ and to determine its image.

Let $\mathbb{N}(T)$ be the Wach module of T. We write $\varphi^*\mathbb{N}(T)$ for the $\mathcal{O}_K \otimes \mathbb{A}^+_{\mathbb{Q}_p}$-module generated by $\varphi\mathbb{N}(T)$. By works of Berger [1, 2], there is a Λ-isomorphism c_T from $H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T)$ to $\mathbb{N}(T)^{\psi=1}$. Moreover, $(\varphi^*\mathbb{N}(T))^{\psi=0}$ is a free Λ-module of rank 2. We observe that we have an inclusion $(1-\varphi)\mathbb{N}(T)^{\psi=1} \subset (\varphi^*\mathbb{N}(T))^{\psi=0}$. Therefore, on choosing a basis n_1, n_2 for the latter, we may define two Λ-homomorphisms $\mathrm{Col}_\star : H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T) \to \Lambda$, $\star = 1, 2$, via the relation

$$(1-\varphi) \circ c_T(z) = \mathrm{Col}_1(z)n_1 + \mathrm{Col}_2(z)n_2.$$

We give the details of the above construction in [7] and describe the images in [8] by the following theorem.

Theorem 2. *Let χ be the p-adic cyclotomic character. Fix a topological generator γ of $\Gamma/\Gamma_{\mathrm{tor}}$. Then, for $\star = 1, 2$, there exists $I^\eta_\star \subset [0, k-1]$ such that $\mathrm{im}(\mathrm{Col}^\eta_\star)$ is pseudo-isomorphic to*

$$\prod_{i \in I^\eta_\star} (\gamma - \chi(\gamma)^i)\mathcal{O}_K[[\Gamma/\Gamma_{\mathrm{tor}}]].$$

We remark that the index sets $I^\eta_\star$ depend on the choice of n_1 and n_2. It is possible to choose n_1 and n_2 so that $I^\eta_1 = I^\eta_2 = \emptyset$ for any η.

Corollary 3. *Kato's main conjecture is equivalent to asserting that*

$$\mathrm{Char}\,\mathrm{Sel}^\star(\bar{f}/\mathbb{Q}(\mu_{p^\infty}))^{\vee,\eta} = \Lambda L^\eta_\star / I^\eta_\star$$

where $\star = 1$ or 2.

We now consider the case when $a_p = 0$. We may define $\mathrm{Col}_\star$ using a basis coming from the one constructed in [3] and let v_1, v_2 be the associated basis for $\mathbb{D}_{\mathrm{cris}}(V)$. Recall from [2] that $\mathbb{N}(T) \subset K \otimes \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes \mathbb{D}_{\mathrm{cris}}(V)$. If $x \in \mathbb{N}(T)$, we may write $x = x_1v_1 + x_2v_2$ for some $x_1, x_2 \in K \otimes \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$. We show in [7] that

$$\mathbb{H}^1_\star(T) = \ker \mathrm{Col}_\star = \{x \in \mathbb{N}(T)^{\psi=1} : \varphi(x_\star) = -p\psi(x_\star)\}.$$

for $\star = 1, 2$. By working out their exact annihilators, we recover the definition of plus and minus Selmer groups over $\mathbb{Q}(\mu_{p^\infty})$ for supersingular elliptic curves in [6]. We now explain how to extend this definition to general extensions of $\mathbb{Q}$ which contain $\mathbb{Q}(\mu_{p^\infty})$.

Let $E/\mathbb{Q}_p$ be a finite extension. By a result of Colmez and Cherbonnier, we have $H^1_{\mathrm{Iw}}(E, T) \cong \mathbb{D}^\dagger_E(T)^{\psi=1}$ where $\mathbb{D}^\dagger_E(T)$ denotes the overconvergent (φ, Γ_E)-module of T. If $x \in \mathbb{D}^\dagger_E(T)$, $x = x_1v_1 + x_2v_2$ for some $x_1, x_2 \in \mathbb{B}^\dagger_{\mathrm{rig}}$. We may therefore extend the definition of $\mathbb{H}^1_\star(T)$ by

$$\mathbb{H}^1_\star(E, T) = \{x \in \mathbb{D}^\dagger_E(T)^{\psi=1} : \varphi(x_\star) = -p\psi(x_\star)\}.$$

As before, let $H^1(E(\mu_{p^\infty}), T^\vee)^\star \subset H^1(E(\mu_{p^\infty}), T^\vee)$ be its exact annihilator. If L is a finite extension of $\mathbb{Q}$, we may then define

$$\mathrm{Sel}^\star(\bar{f}/L(\mu_{p^\infty})) = \ker\left(\mathrm{Sel}(\bar{f}/L(\mu_{p^\infty})) \to \prod_{\nu|p} \frac{H^1(L_\nu(\mu_{p^\infty}), T^\vee)}{H^1(L_\nu(\mu_{p^\infty}), T^\vee)^\star}\right).$$

On taking direct limit, we may define $\mathrm{Sel}^\star$ over a p-adic Lie extension that contains $\mathbb{Q}(\mu_{p^\infty})$.

In [9], we analyse the analogous construction in the ordinary case. In particular, we construct a basis for the Wach module such that

$$\ker \mathrm{Col}_1 = \{x \in \mathbb{N}(T)^{\psi=1} : \varphi(x_1) = \alpha x_1\}$$

where α is the unique p-adic unit that satisfies (1). If $E/\mathbb{Q}_p$ is a finite extension, the exact annihilator of $\{x \in \mathbb{D}_E^\dagger(T)^{\psi=1} : \varphi(x_1) = \alpha x_1\}$ turns out to be $H^1_f(E(\mu_{p^\infty}), T^\vee)$. This recovers the classical definition of the usual Selmer group. It therefore seems plausible to conjecture that the Selmer groups we construct in the non-ordinary case should satisfy some $\mathfrak{M}_H(G)$-conjecture similar to the one formulated by Coates et al. in the ordinary case.

References

[1] L. Berger, *Bloch and Kato's exponential map: three explicit formulas*, Kazuya Kato's fiftieth birthday, Doc. Math. (2003), Extra Vol., 99–129.

[2] L. Berger, *Limites de représentations cristallines*, Compos. Math. **140** (2004), no. 6, 1473–1498.

[3] L. Berger, H. Li, H. J. Zhu, *Construction of some families of 2-dimensional crystalline representations*, Math. Ann. **329** (2004), no. 2, 365–377.

[4] P. Deligne, *Formes modulaires et représentations l-adiques*, Séminaire Bourbaki (1968/69), Exp. No. 355, 139–172.

[5] K. Kato, *p-adic Hodge theory and values of zeta functions of modular forms*, Astérisque **295** (2004), 117–290.

[6] S. Kobayashi, *Iwasawa theory for elliptic curves at supersingular primes*, Invent. Math. **152** (2003), no. 1, 1–36.

[7] A. Lei, D. Loeffler, S. L. Zerbes, *Wach modules and Iwasawa theory for modular forms*, Asian J. Math. **14** (2010), no. 4, 475–528.

[8] A. Lei, D. Loeffler, S. L. Zerbes, *Coleman maps and the p-adic regulator*, to appear in Algebra Number Theory, arXiv:1006.5163v2.

[9] A. Lei, S. L. Zerbes, *Signed Selmer Groups over p-adic Lie Extensions*, preprint, arXiv:1104.2168v1.

Cohomology of moduli spaces of non-basic p-divisible groups

Eva Viehmann

Moduli spaces of p-divisible groups have been constructed by Rapoport and Zink who show that these spaces uniformize corresponding Shimura varieties along Newton strata. Conjecturally, their cohomology realizes local Langlands correspondences. In view of this expectation and the relation to the cohomology of Shimura varieties Harris conjectured that in the so-called non-basic case the cohomology

of Rapoport-Zink spaces is parabolically induced from that of a smaller moduli space. In some special cases this conjecture, or an analog over function fields, was shown by Boyer, Harris and Taylor, and Mantovan. In this talk I report on work in progress [9] which corrects and proves the general conjecture.

Let $\mathbb{X}$ be a p-divisible group over $\overline{\mathbb{F}}_p$. Let n be its height. We consider the following functor on the category $\mathcal{N}ilp$ of schemes over $\mathbb{Z}_p$ on which p is locally nilpotent.

$$\begin{aligned} \mathcal{M}_G : \mathcal{N}ilp &\to \mathcal{S}ets \\ S &\mapsto \{(X,\rho)\}/\sim \end{aligned}$$

where X is a p-divisible group over S and where $\rho : \mathbb{X}_{\overline{S}} \to X_{\overline{S}}$ is a quasi-isogeny. Here $\overline{S}$ denotes the closed subscheme of S defined by the ideal sheaf $p\mathcal{O}_S$. Two pairs (X_1,ρ_1) and (X_2,ρ_2) are equivalent if $\rho_1 \circ \rho_2^{-1}$ lifts to an isomorphism $X_2 \to X_1$ over S.

Remark 1. (1) *In* [7] *Rapoport and Zink prove that $\mathcal{M}_G$ is pro-represented by a formal scheme which is locally formally of finite type over $\mathbb{Z}_p$.*

(2) *Let (N,F) denote the rational Dieudonné module of $\mathbb{X}$. Here N is a* $\mathrm{Quot}(W(\overline{\mathbb{F}}_p))$*-vector space of dimension n and $F : N \to N$ is a σ-linear isomorphism. The index G of $\mathcal{M}_G$ refers to the group $G = GL_n$ of automorphisms of the vector space N.*

(3) *More generally Rapoport and Zink consider moduli spaces of p-divisible groups together with extra structure of (EL) or (PEL) type, i.e. equipped with a polarization and/or endomorphisms and level structures. The results presented in this talk have generalizations for these more general moduli spaces, provided that the associated group G is unramified at p. In Remark 3 we comment on the correct statement of our main result for non-split groups G. This corrects a conjecture by Harris, see* [3], *Conjecture 5.2. The required techniques are direct analogs of the ones needed for p-divisible groups without additional structure, but are more tedious to present.*

Let $\mathcal{M}_G^{\mathrm{rig}}$ be the generic fiber of $\mathcal{M}_G$, a Berkovich space over $E_0 = W(\overline{\mathbb{F}}_p)[1/p]$, the completion of the maximal unramified extension of $\mathbb{Q}_p$. One considers a tower of finite étale coverings of $\mathcal{M}_G^{\mathrm{rig}}$: To each open subgroup K of $K_0 = G(\mathbb{Z}_p)$ one associates a covering of $\mathcal{M}_G^{\mathrm{rig}} = \mathcal{M}_G^{K_0}$ parametrizing trivializations of the Tate module of the universal p-divisible group up to multiplication by K. This can be further generalized to spaces $\mathcal{M}_G^K$ for all bounded open subgroups $K \subset G(\mathbb{Q}_p)$.

We consider the ℓ-adic cohomology of the towers $(\mathcal{M}_G^K)_K$ which is defined as follows. Let $\ell \neq p$ be a prime and let $\hat{\overline{\mathbb{Q}}}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$. Let

$$H^i(\mathcal{M}_G^K) = \varinjlim_U H_c^i(U \times_{\mathbb{Q}_p} \hat{\overline{\mathbb{Q}}}_p, \mathbb{Q}_\ell(\dim \mathcal{M}_G^K)) \tag{1}$$

where U varies among the open subspaces of $\mathcal{M}_G^K$ of finite type and $i \in \mathbb{N}$.

These cohomology groups carry an action of three groups:

- The group $J(\mathbb{Q}_p)$ of self quasi-isogenies of $\mathbb{X}$ acts on $\mathcal{M}_G$ and thus on each $\mathcal{M}_G^K$ and on $H^i(\mathcal{M}_G^K)$.
- $G(\mathbb{Q}_p)$ acts on the tower of coverings $\mathcal{M}_G^K$ via Hecke correspondences on the level structures.
- the cohomology carries in addition an action of the Weil group $W_{\mathbb{Q}_p}$. The inertia subgroup acts via the coefficients $\hat{\overline{\mathbb{Q}}}_p$, an action of Frobenius is induced by the existence of a (non-effective) Weil descent datum.

For $\rho \in \mathrm{Groth}(J(\mathbb{Q}_p)$ let

$$H^{i,j}(\mathcal{M}_G^\infty)_\rho = \varinjlim_K \mathrm{Ext}^j_{J(\mathbb{Q}_p)}(H^i(\mathcal{M}_G^K), \rho). \tag{2}$$

By [5], Section 8.2, $H^{i,j}(\mathcal{M}_G^\infty)_\rho \in \mathrm{Groth}(G(\mathbb{Q}_p) \times W_E)$ and vanishes for almost all $i, j \geq 0$. Let

$$H^\bullet(\mathcal{M}_G^\infty)_\rho = \sum_{i,j\geq 0} (-1)^{i+j} H^{i,j}(\mathcal{M}_G^\infty)_\rho \in \mathrm{Groth}(G(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}). \tag{3}$$

From now on we assume that the fixed p-divisible group $\mathbb{X}$ has a decomposition $\mathbb{X} = \mathbb{X}_1 \times \cdots \times \mathbb{X}_r$ where the $\mathbb{X}_i$ are isoclinic and $\nu_i > \nu_{i+1}$ for all i where ν_i is the Newton slope of $\mathbb{X}_i$. Then we consider in addition the functor $\mathcal{M}_M$ on the category $\mathcal{N}ilp$ with

$$\mathcal{M}_M : S \mapsto \{(X = X_1 \times \cdots \times X_r, \rho)\}/\sim$$

where $(X, \rho) \in \mathcal{M}_G(S)$ and where ρ induces quasi-isogenies $\rho|_{\mathbb{X}_i} : \mathbb{X}_i \to X_i$ for all i. As before it is pro-represented by a formal scheme which is locally formally of finite type over E_0.

Our assumption $\nu_i > \nu_{i+1}$ implies that the group $J(\mathbb{Q}_p)$ still acts on the moduli space $\mathcal{M}_M$. In the same way as for $\mathcal{M}_G$ one constructs a tower of coverings of the associated rigid space, this time indexed by bounded open subgroups of $M(\mathbb{Q}_p)$ where M is the Levi subgroup of G stabilizing the decomposition of the Dieudonné module of $\mathbb{X}$ corresponding to the decomposition of $\mathbb{X}$. As above we assign to each $\rho \in \mathrm{Groth}(J(\mathbb{Q}_p))$ a virtual representation $H^\bullet(\mathcal{M}_M^\infty)_\rho \in \mathrm{Groth}(M(\mathbb{Q}_p) \times W_{\mathbb{Q}_p})$. Let further P denote the corresponding parabolic subgroup stabilizing the filtration corresponding to $\mathbb{X}_1 \subseteq \mathbb{X}_1 \times \mathbb{X}_2 \subseteq \cdots \subseteq \mathbb{X}$.

Then our main result is

Theorem 2. *For each $\rho \in \mathrm{Groth}(J(\mathbb{Q}_p))$ we have*

$$H^\bullet(\mathcal{M}_G^\infty)_\rho = \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(H^\bullet(\mathcal{M}_M^\infty)_\rho)$$

in $\mathrm{Groth}(G(\mathbb{Q}_p) \times W_{\mathbb{Q}_p})$.

Remark 3. *This proves a conjecture of Harris,* [3] *Conjecture 5.2. For non-split groups G instead of the group GL_n one has to make a correction to Harris' conjecture in order to generalize the above result. Let us briefly outline this modification. For more general moduli spaces of p-divisible groups with extra structure of (EL) or*

(PEL) type one has to be careful when defining the associated moduli space $\mathcal{M}_M$. In [3] *all moduli spaces were associated with a group (in this case the group M), a σ-conjugacy class of elements of $M(E_0)$, and an M-conjugacy class of coweights. In the case where $G = GL_n$ this yields precisely the moduli space above, and the σ-conjugacy class and the coweight are determined by $\mathbb{X}$. For non-split groups G, Harris' construction does not yield the correct moduli space $\mathcal{M}_M$: In general, the p-divisible group $\mathbb{X}$ with extra structure of (EL) or (PEL) type and a decomposition does not determine the M-conjugacy class of coweights uniquely. The correct moduli space needed for the generalization of Theorem 2 is defined in a similar way as above as a moduli space of p-divisible groups with extra structure of (EL) or (PEL) type and a decomposition which respects this additional structure. It is then a disjoint union of finitely many moduli spaces as defined by Harris.*

References

[1] P. Boyer, *Mauvaise réduction des variétés de Drinfeld et correspondance de Langlands locale*, Invent. Math. **138** (1999), 573–629.

[2] L. Fargues, *Cohomologie des espaces de modules de groupes p-divisibles et correspondances de Langlands locales*, in: Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales, Astérisque **291** (2004), 1–199.

[3] M. Harris, *Local Langlands correspondences and vanishing cycles on Shimura varieties*, in: European Congress of Mathematics, Vol. I (Barcelona, 2000), 407–427, Progr. Math., 201, Birkhäuser, Basel, 2001.

[4] M. Harris, R. Taylor, *The geometry and cohomology of some simple Shimura varieties.* With an appendix by V. G. Berkovich. Annals of Mathematics Studies, 151. Princeton University Press, 2001.

[5] E. Mantovan, *On certain unitary group Shimura varieties*, in: Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales, Astérisque **291** (2004), 201–331.

[6] E. Mantovan, *On non-basic Rapoport-Zink spaces*, Ann. Sci. École Norm. Sup. **41** (2008), 671–716.

[7] M. Rapoport, Th. Zink, *Period spaces for p-divisible groups*, Princeton Univ. Press, 1996.

[8] S. W. Shin, *On the cohomology of Rapoport-Zink spaces of EL-type*, preprint, `http://www.math.uchicago.edu/~swshin/RZspacesELtype.pdf`.

[9] E. Viehmann, *The cohomology of non-basic Rapoport-Zink spaces*, in preparation.

[10] T. Zink, *The display of a formal p-divisible group*, in: Cohomologies p-adiques et applications arithmétiques, I. Astérisque **278** (2002), 127–248.

Heights of special cycles on orthogonal Shimura varieties

Fritz Hörmann

A Shimura variety of orthogonal type arises from the Shimura datum $\mathbf{O}(L)$ consisting of the orthogonal group $\mathrm{SO}(L_{\mathbb{Q}})$ of a quadratic space $L_{\mathbb{Q}}$ of signature $(m-2,2)$ and the Hermitian symmetric domain

$$\mathbb{D} := \{N \subseteq L_{\mathbb{R}} \mid N \text{ oriented negative definite plane}\}.$$

For any compact open subgroup $K \subseteq \mathrm{SO}(L_{\mathbb{A}_f})$, we can form the Shimura variety (stack):

$$[\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D} \times \mathrm{SO}(L_{\mathbb{A}_f})/K].$$

It is a smooth manifold if K is sufficiently small, has a canonical integral model $\mathrm{M}({}^K\mathbf{O}(L))$ over $\mathbb{Z}_{(p)}$ for p of good reduction.

For an isometry $x : M \hookrightarrow L$, where M is positive definite, we can form

$$\mathbb{D}_x := \{N \in \mathbb{D} \mid N \perp x(M)\}.$$

For a K-inv. Schwartz function $\varphi \in S(M^*_{\mathbb{A}_f} \otimes L_{\mathbb{A}_f})$, the special cycle $Z(M, L, \varphi; K)$ on the Shimura variety is a quotient of the union of the $\mathbb{D}_x$ over all isometries in the support of φ in a weighted way. Analogous cycles exist for singular lattices M. Consider a toroidal compactification $\mathrm{M}({}^K_\Delta\mathbf{O}(L))$ of the Shimura variety (which we will discuss in detail below) and form the generating series

$$\Theta_n(L) = \sum_{Q \in \mathrm{Sym}^2((\mathbb{Z}^n)^*)} [Z(\mathbb{Z}^n_Q, L, \varphi; K)] \exp(2\pi i Q\tau)$$

with values in its algebraic Chow group $\mathrm{CH}^n(\mathrm{M}({}^K_\Delta\mathbf{O}(L)))_{\mathbb{C}}$. Assume for the moment that $\mathrm{M}({}^K_\Delta\mathbf{O}(L))$ is even defined over $\mathbb{Z}$ in a "reasonably canonical" way. Kudla proposes a definition of arithmetic (Arakelov) cycles $\widehat{Z}(M, L, \varphi; K, \nu)$, depending on the imaginary part ν of τ, too, such that

$$\widehat{\Theta}_n(L) = \sum_{Q \in \mathrm{Sym}^2((\mathbb{Z}^n)^*)} [\widehat{Z}(\mathbb{Z}^n_Q, L, \varphi; K, \nu)] \exp(2\pi i Q\tau)$$

should have values in a suitable Arakelov Chow group $\widehat{\mathrm{CH}}^n(\mathrm{M}({}^K_\Delta\mathbf{O}(L)))_{\mathbb{C}}$. He proposes specific Greens functions, which have singularities at the boundary. The orthogonal Shimura varieties come equipped with a natural Hermitian automorphic line bundle $\Xi^*\overline{\mathcal{E}}$ (whose metric also has singularities along the boundary). Multiplication with a suitable power of its first Chern class and taking push-forward provides us with geometric (resp. arithmetic) degree maps $\deg : \mathrm{CH}^p(\cdots) \to \mathbb{Z}$ (resp. $\widehat{\deg} : \widehat{\mathrm{CH}}^p(\cdots) \to \mathbb{R}$). Assuming that an Arakelov theory can be set up to deal with all different occurring singularities, Kudla conjectures[1]:

(K1) Θ_n and $\widehat{\Theta}_n$ are (holomorphic, resp. non-holomorphic) Siegel modular forms of weight $\frac{m}{2}$ and genus n;

(K2) $\deg(\Theta_n)$ and $\widehat{\deg}(\widehat{\Theta}_n)$ are equal to a special value of a *normalized* standard Eisenstein series of weight $\frac{m}{2}$ associated with the Weil representation of L [5, section 4], resp. its special derivative at the same point;

(K3) $\Theta_{n_1}(\tau_1) \cdot \Theta_{n_2}(\tau_2) = \Theta_{n_1+n_2}(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix})$ and similarly for $\widehat{\Theta}$;

(K4) $\widehat{\Theta}_{m-1}$ can be defined, with coefficients being zero-cycles on the arithmetic model and satisfies the properties above.

Kudla shows that these conjectures imply almost formally vast generalizations of the formula of Gross-Zagier. In full generality they are known only for Shimura curves [9]. The geometric part of (K2) is well-known, and Borcherds showed the geometric part of (K1) for $n = 1$.

[1] If $m - r \leq n + 1$, the statement has to be modified. Here r is the Witt-rank of L.

I reported on results of my thesis, which prove the "good reduction part" of the *arithmetic case* of (K2) for all Shimura varieties of orthogonal type. The approach — after an idea of Bruinier, Burgos, and Kühn [1] — is predominantly Archimedean in nature using Borcherds products. More precisely it is shown:

Main Theorem. *Let D' be the product of primes p such that $p^2|D$, where D is the discriminant of $L_{\mathbb{Z}}$. We have*

$$\begin{array}{lrcll} 1. & \mathrm{vol}_{\mathcal{E}}(\mathrm{M}({}^K_\Delta\mathbf{O}(L))) & = & 4\widetilde{\lambda}^{-1}(L_{\mathbb{Z}};0) & \\ 2. & \widehat{\mathrm{vol}}_{\overline{\mathcal{E}}}(\mathrm{M}({}^K_\Delta\mathbf{O}(L))) & \equiv & \frac{d}{ds}4\widetilde{\lambda}^{-1}(L_{\mathbb{Z}};s)|_{s=0} & \text{in } \mathbb{R}_{2D'} \end{array}$$

Let $M_{\mathbb{Z}}$ be a lattice of dimension n with positive definite $Q_M \in Sym^2(M^*_{\mathbb{Q}})$. Let D'' be the product of primes p such that $M_{\mathbb{Z}_p} \not\subseteq M^*_{\mathbb{Z}_p}$ or $M^*_{\mathbb{Z}_p}/M_{\mathbb{Z}_p}$ is not cyclic. Under some mild conditions on L and M, we have:

$$\begin{array}{lrcl} 3. & \mathrm{vol}_{\mathcal{E}}(Z(L_{\mathbb{Z}},M_{\mathbb{Z}},\kappa;K)) & = & 4\widetilde{\lambda}^{-1}(L_{\mathbb{Z}};s)\widetilde{\mu}(L_{\mathbb{Z}},M_{\mathbb{Z}},\kappa;0), \\ 4. & \mathrm{ht}_{\overline{\mathcal{E}}}(Z(L_{\mathbb{Z}},M_{\mathbb{Z}},\kappa;K)) & \equiv & \frac{\mathrm{d}}{\mathrm{d}s}\,4\left(\widetilde{\lambda}^{-1}(L_{\mathbb{Z}};s)\widetilde{\mu}(L_{\mathbb{Z}},M_{\mathbb{Z}},\kappa;s)\right)\Big|_{s=0} \\ & & & \text{in } \mathbb{R}_{2DD''}. \end{array}$$

Here $\mathbb{R}_N$ is $\mathbb{R}$ modulo $\mathbb{Q}\log(p)$ for $p|N$, and the $\widetilde{\lambda}$ and $\widetilde{\mu}$ are functions in $s \in \mathbb{C}$, given by certain Euler products associated with representation densities of $L_{\mathbb{Z}}$ and $M_{\mathbb{Z}}$. Moreover $\pm\widetilde{\mu}$ appears as the "holomorphic part" of a Fourier coefficient of the standard Eisenstein series associated with the Weil representation of $L_{\mathbb{Z}}$, and K is the discriminant kernel. The proof uses four ingredients:

(1) an interpolated "orbit equation" [5], which gives a recursive relation between the functions λ and μ above of the shape:

$$\lambda^{-1}(L_{\mathbb{Z}};s)\mu(L_{\mathbb{Z}},M_{\mathbb{Z}};s) = \sum_{\text{orbits of isometries } \alpha\,:\,M\hookrightarrow L} \lambda^{-1}(\alpha(M)^{\perp}_{\mathbb{Z}};s);$$

(2) relations between Arakelov invariants on Shimura varieties to Borcherds products — this involves a computation of integrals of Borcherds forms, a q-expansion principle, and the construction of special input forms for the Borcherds lift — the relations match strikingly with the first and second derivative of the "orbit equation" in (1);

(3) a functorial theory of integral models of toroidal compactifications of mixed Shimura varieties and their automorphic vector bundles;

(4) an Arakelov theory able to deal with "automorphic" singularities along the boundary [2, 3].

I reported in more detail about (3), which was established in my thesis [4] under a certain hypothesis that has recently been proven to hold true by Madapusi in [10] (cf. his abstract in this report).

The models are constructed locally, i.e. over an extension of $\mathbb{Z}_{(p)}$. Input data for the theory are p-integral mixed Shimura data (p-MSD) $\mathbf{X} = (P_{\mathbf{X}}, \mathbb{D}_{\mathbf{X}}, h_{\mathbf{X}})$ consisting of an affine group scheme $P_{\mathbf{X}}$ over $\mathrm{spec}(\mathbb{Z}_{(p)})$ of a certain rigid type and a set $\mathbb{D}_{\mathbf{X}}$, which comes equipped with a finite covering $h_{\mathbf{X}} : \mathbb{D}_{\mathbf{X}} \to \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbf{X},\mathbb{C}})$ onto a conjugacy class in the latter group, subject to some axioms which are roughly

Pink's mixed extension of Deligne's axioms for a pure Shimura datum. Then call a compact open subgroup $K \subseteq P_{\mathbf{X}}(\mathbb{A}_f)$ admissible, if it is of the form $K^{(p)} \times P_{\mathbf{X}}(\mathbb{Z}_p)$ for a compact open subgroup $K^{(p)} \subseteq P_{\mathbf{X}}(\mathbb{A}_f^{(p)})$. For the toroidal compactification a certain rational polyhedral cone decomposition Δ is needed. We call the collection ${}^K\mathbf{X}$ (resp. ${}^K_\Delta\mathbf{X}$) extended (compactified) p-integral mixed Shimura data (p-EMSD, resp. p-ECMSD). These form categories where morphisms ${}^K_\Delta\mathbf{X} \to {}^{K'}_{\Delta'}\mathbf{Y}$ are pairs (α, ρ) of a morphism α of Shimura data and $\rho \in P_{\mathbf{Y}}(\mathbb{A}_f^{(p)})$ satisfying compatibility with the K's and Δ's. The theory is then a functor M from p-ECMSD to the category of Deligne-Mumford stacks over reflex rings (above $\mathbb{Z}_{(p)}$), which over $\mathbb{C}$ and restricted to p-MSD becomes naturally isomorphic to the one given by the analytic mixed Shimura variety construction. It is characterized uniquely by Deligne's canonical model condition, Milne's extension property (integral canonicity) and a stratification of the boundary into mixed Shimura varieties, together with boundary isomorphisms of the formal completions along the latter with similar completions of other (more mixed) Shimura varieties. There is also a functor 'compact' dual from the category of p-MSD to the category of schemes over reflex rings. The duals come equipped with an action of the group scheme $P_{\mathbf{X}}$, and we have morphisms

$$\Xi_{\mathbf{X}} : \mathrm{M}({}^K_\Delta\mathbf{X}) \to \left[\mathrm{M}^\vee(\mathbf{X})/P_{\mathbf{X}}\right],$$

which form a pseudo-natural transformation of functors with values in Artin stacks over reflex rings, and which are a model of the usual construction over $\mathbb{C}$ if Δ is trivial. This is the theory of integral automorphic vector bundles. It is compatible with boundary isomorphisms. In particular, it gives a well-defined (and functorial) pull-back of equivariant Arakelov bundles defined on the dual to the Shimura variety.

References

[1] J. H. Bruinier, J. I. Burgos Gil, U. Kühn, *Borcherds products and arithmetic intersection theory on Hilbert modular surfaces*, Duke Math. J. **139**(1): 1–88, 2007.

[2] J. I. Burgos Gil, J. Kramer, U. Kühn, *Cohomological arithmetic Chow rings*, J. Inst. Math. Jussieu **6**(1), 1–172, 2007

[3] J. I. Burgos Gil, J. Kramer, U. Kühn, *Arithmetic characteristic classes of automorphic vector bundles*, Doc. Math. **10**, 619–716, 2005.

[4] F. Hörmann, *The arithmetic volume of Shimura varieties of orthogonal type*, PhD thesis, Humboldt-Universität zu Berlin, 2010.

[5] F. Hörmann, *On recursive properties of certain p-adic Whittaker functions*, 2010, preprint arXiv:1010.1009.

[6] F. Hörmann, *The arithmetic and geometric volume of Shimura varieties of orthogonal type*, 2011, preprint arXiv:1105.5357.

[7] M. Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. **23**(4), 967–1012, 2010

[8] S. S. Kudla, *Special cycles and derivatives of Eisenstein series*, in Heegner points and Rankin L-series, volume 49 of Math. Sci. Res. Inst. Publ., pages 243–270. Cambridge Univ. Press, Cambridge, 2004.

[9] S. S. Kudla, M. Rapoport, T. Yang, *Modular forms and special cycles on Shimura curves*, Annals of Mathematics Studies **161**, Princeton University Press, 2006

[10] K. S. Madapusi, *Toroidal compactifications of integral models of Shimura varieties of Hodge type*, PhD thesis, The University of Chicago, 2011.

1-motives, yet another EMC, and Galois module structure

Cornelius Greither
(joint work with Cristian Popescu)

We introduce "abstract ℓ-adic 1-motives", which are a slight generalisation of 1-motives, as used by Deligne in order to prove the Brumer-Stark conjecture for function fields. To each such 1-motive $\mathcal{M}$ one can attach its ℓ-adic realisation $T_\ell(\mathcal{M})$. Both in the number field case and in the function field case one obtains a 1-motive $\mathcal{M} = \mathcal{M}_{S,T}$ for every G-Galois extension K/k (and suitable auxiliary sets S, T, which are familiar from the theory of L-functions). Our first main result says that $T_\ell(\mathcal{M})$ is cohomologically trivial over G (a point that has not been made earlier, apparently). Secondly, we can show that its Fitting ideal is given by an equivariant p-adic L-series, very much in the style of other and earlier EMC's (equivariant Main Conjectures).

A main application is a very explicit construction of Tate sequences. In order to achieve this, one has to interpret $T_\ell(\mathcal{M})$ via étale cohomology and link it to Tate's canonical class. In doing so, we need to use fundamental work of Burns and Flach [1]. In the talk we moreover gave two geometric illustrations of the underlying principle that cohomological triviality generates congruences betweeen L-functions, and that these congruences have arithmetic meaning: first, we show that the degree zero class number of the Fermat curve $x^\ell + y^\ell = 1$ over a finite field with q elements is either 1 or at least divisible by $\ell^{\ell-2}$. Second, we reprove the congruences for the Hasse-Weil L-functions of certain elliptic curves over the global field $\mathbb{F}(t)$ which were discovered by Chris Hall [3] a few years ago.

References

[1] D. Burns and M. Flach, *On Galois structure invariants associated to Tate motives,* Am. J. Math. **120** (1998), 1343-1397
[2] C. Greither and C. Popescu, *The Galois module structure of ℓ–adic realizations of Picard 1–motives and applications,* to appear in International Math. Research Notices
[3] C. Hall, *L-functions of twisted Legendre curves* J. Number Theory **119** (2006), 128-147

Reporter: Christian Rüschoff

Participants

Prof. Adebisi Agboola
Department of Mathematics
University of California at
Santa Barbara
South Hall
Santa Barbara , CA 93106
USA

Dr. Chandrakant S. Aribam
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Prof. Dr. Laurent Berger
Dept. de Mathematiques, U.M.P.A.
Ecole Normale Superieure de Lyon
46, Allee d'Italie
F-69364 Lyon Cedex 07

Prof. Dr. Gebhard Böckle
Interdisziplinäres Zentrum
für Wissenschaftliches Rechnen
Universität Heidelberg
Im Neuenheimer Feld 368
69120 Heidelberg

Dr. Thanasis Bouganis
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Prof. Dr. Jan Hendrik Bruinier
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt

Prof. Dr. Gaetan Chenevier
Centre de Mathematiques
Ecole Polytechnique
Plateau de Palaiseau
F-91128 Palaiseau Cedex

Prof. Dr. Mirela Ciperiani
Department of Mathematics
The University of Texas at Austin
1 University Station C1200
Austin , TX 78712-1082
USA

Prof. Dr. John H. Coates
Dept. of Pure Mathematics and
Mathematical Statistics
University of Cambridge
Wilberforce Road
GB-Cambridge CB3 0WB

Prof. Dr. Pierre Colmez
Institut de Mathematiques de Jussieu
Theorie des Nombres, Case 247
Universite de Paris VI
4, Place Jussieu
F-75252 Paris

Dr. Amir Dzambic
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Str. 6-10
60325 Frankfurt am Main

Sandra Eisenreich
Fakultät für Mathematik
Universität Regensburg
Universitätsstr. 31
93053 Regensburg

Prof. Dr. Matthias Flach
Department of Mathematics
California Institute of Technology
Pasadena , CA 91125
USA

Prof. Dr. Takako Fukaya
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago , IL 60637-1514
USA

Philipp Graf
NWF-I Mathematik
Universität Regensburg
93040 Regensburg

Prof. Dr. Cornelius Greither
Fakultät für Informatik
Universität der Bundeswehr München
Werner-Heisenberg-Weg 39
85579 Neubiberg

Prof. Dr. Elmar Große-Klönne
Institut für Mathematik
Humboldt-Universität zu Berlin
Rudower Chaussee 25
12489 Berlin

Prof. Dr. Walter Gubler
Fakultät für Mathematik
Universität Regensburg
Universitätsstr. 31
93053 Regensburg

Dr. Eugen Hellmann
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn

Dr. Florian Herzig
Princeton University
School of Mathematics
Institute for Advanced Study
Einstein Drive
Princeton NJ 08540
USA

Dr. Fritz Hörmann
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg

Dr. Benjamin V. Howard
Department of Mathematics
301 Carney Hall
Boston College
Chestnut Hill , MA 02467-3806
USA

Prof. Dr. Ming-Lun Hsieh
Department of Mathematics
National Taiwan University
New Math Building Room 400
No1, Sec. 4, Roosevelt Road
Taipei 10617
TAIWAN

Prof. Dr. Annette Huber-Klawitter
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg

Prof. Dr. Uwe Jannsen
Fakultät für Mathematik
Universität Regensburg
Universitätsstr. 31
93053 Regensburg

Dr. Moritz Kerz
Fakultät für Mathematik
Universität Duisburg-Essen
45117 Essen

Prof. Dr. Guido Kings
NWF-I Mathematik
Universität Regensburg
93040 Regensburg

Dr. Jan Kohlhaase
Fachbereich Mathematik
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Klaus Künnemann
NWF-I Mathematik
Universität Regensburg
93040 Regensburg

Prof. Dr. Antonio Lei
School of Mathematical Sciences
Monash University
Clayton Victoria 3800
AUSTRALIA

Dr. Keerthi Shyam Madapusi Sampath
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago , IL 60637-1514
USA

Prof. Dr. Elena Mantovan
Department of Mathematics
California Institute of Technology
Pasadena , CA 91125
USA

Prof. Dr. Niko Naumann
NWF-I Mathematik
Universität Regensburg
93040 Regensburg

Prof. Dr. Wieslawa Niziol
Department of Mathematics
University of Utah
155 South 1400 East
Salt Lake City , UT 84112-0090
USA

Prof. Dr. Bjorn Poonen
Department of Mathematics
Massachusetts Institute of
Technology
77 Massachusetts Avenue
Cambridge , MA 02139-4307
USA

Dr. Jonathan Pottharst
Department of Mathematics
Boston University
111 Cummington Street
Boston , MA 02215-2411
USA

Prof. Dr. Sujatha Ramdorai
F-002, Renaissance Jagliti, Varthur Road
Ramagondanakalli Whitefield
Bangalore 560 - 066
INDIA

Prof. Dr. Michael Rapoport
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn

Andreas Riedel
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Christian Rüschoff
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Prof. Dr. Alexander Schmidt
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Dr. Tobias Schmidt
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Peter Schneider
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Torsten Schoeneberg
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Anthony J. Scholl
Dept. of Pure Mathematics and
Mathematical Statistics
University of Cambridge
Wilberforce Road
GB-Cambridge CB3 0WB

Peter Scholze
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn

Dr. Jakob Stix
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Prof. Dr. Michael Stoll
Mathematisches Institut
Universität Bayreuth
95440 Bayreuth

Dr. Ulrich Terstiege
Institut f. Experimentelle Mathematik
Universität Essen
Ellernstr. 29
45326 Essen

Prof. Dr. Takeshi Tsuji
Department of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Prof. Dr. Otmar Venjakob
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Dr. Eva Viehmann
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn

Dr. Bianca Viray
Department of Mathematics
Brown University
Box 1917
Providence , RI 02912
USA

Prof. Dr. Annette Werner
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Str. 8
60325 Frankfurt a. Main

Prof. Dr. Jean-Pierre Wintenberger
Institut de Mathematiques
Universite de Strasbourg
7, rue Rene Descartes
F-67084 Strasbourg Cedex

Dr. Malte Witte
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg