Characteristic polyhedra of idealistic exponents with history

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A purely polyhedral approach to the invariant introduced by Bierstone and Milman to give a constructive proof of Hironaka's resolution of singularities in characteristic zero

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Introduction

Let X be a scheme of finite type over an arbitrary field k or of finite type over the integers $\mathbb{Z}$. In resolution of singularities we consider the question if it is possible to find a proper birational morphism $\pi_X : \widetilde{X} \to X$ such that $\widetilde{X}$ is regular. In the strong version of resolution we want to know if we can additionally achieve π_X by a finite sequence of blow ups in regular centers.

If X is embedded into a regular scheme Z (also of finite type over k or $\mathbb{Z}$), then the aim of strong embedded resolution of singularities is to obtain a finite sequence of blow ups $\pi : \widetilde{Z} \to Z$ as above such that $\widetilde{Z}$ is regular, the strict transform $\widetilde{X}$ of X is regular and the total transform $X' \subset \widetilde{Z}$ of X has at most simple normal crossing singularities. Recall that the total transform consists of the strict transform together with the exceptional divisors which arose due to the blow ups. Further $\pi_X := \pi_{|\widetilde{X}} : \widetilde{X} \to X$ yields a strong resolution of singularities of X.

Whenever we speak of resolution of singularities in this thesis we mean the strong embedded version.

In his celebrated paper [H1] from 1964 Hironaka proved the existence of resolution of singularities for arbitrary dimensional algebraic varieties over fields of characteristic zero. For his remarkable theorem he received the Fields medal a few years later. The original proof is quite complicated and consists of more than 200 very technical pages. Moreover, the result is not constructive. Nowadays there are quite accessible and constructive proofs available, which all are based on Hironaka's work. The first results in this direction were published nearly 25 years ago by Bierstone and Milman [BM2], [BM3] and Villamayor [V1], [V2]. More recent approaches are for example by Bravo, Encinas and Villamayor [BEV], Cutkosky [Cu1], Encinas and Hauser [EHa], Hauser [Ha], Kollár [Ko] and Włodarczyk [W].

In positive characteristic Abhyankar was the first to show resolution of singularities for surfaces and he also proved the case of dimension three if the characteristic of the base field k is not $2, 3$ or 5 (k algebraically closed!). Both results have been simplified by Cutkosky in [Cu2] and [Cu3]. (For the precise references to Abhyankar's original papers see also Cutkosky's articles).

In the appendix of [CGO] Hironaka gave an alternative approach to the resolution of hypersurfaces of dimension 2. There he made intensive use of the characteristic polyhedron of a singularity, which he introduced in [H2]. Following his strategy Cossart, Jannsen and Saito [CJS] extended the proof to arbitrary excellent schemes of dimension at most 2. (This includes in particular the arithmetic case over $\mathbb{Z}$!). Again the characteristic polyhedron played a crucial role.

In 2008/2009 Cossart and Piltant were able to prove the existence of a birational and global resolution in dimension 3, if the base field k is differentially finite over a perfect field k_0 [CP1], [CP2]. Since their result is not given by a resolution algorithm, it is not clear that the resolution is achieved purely by blow ups in regular centers.

There are some more programs which try to tackle the proof for arbitrary characteristic. But up to now none of them succeeded to show resolution in arbitrary dimension (not even in dimension 4). By using so called alterations de Jong was able to prove in [dJ] a weaker form of resolution in positive characteristic for all dimensions (where the term “birational” has to be replaced by “generically finite”).

In this thesis we focus on Bierstone and Milman's approach [BM3] to resolution of singularities in characteristic zero. (See also [BM4] for the hypersurface case).

Let X be a scheme of finite type over a field k of characteristic zero, which is embedded into a regular scheme Z (also of finite type over k). The strategy for the proof of resolution of singularities in characteristic zero is to define an invariant $\mathrm{inv}_X(x)$ for each $x \in X$, which satisfies the following properties:

(1) $\mathrm{inv}_X(x)$ has values in a totally ordered abelian group and is upper semi-continuous.

(2) If T is the locus where the maximal value of $\mathrm{inv}_X(x)$ on Z is attained, then there is a canonical way to deduce from T the center of the next blow up.

(3) This center is regular and has only simple normal crossings with the exceptional divisors obtained by the resolution process.

(4) After each such blow up the invariant decreases strictly and after finitely many steps the singularities are resolved.

In order to define $\mathrm{inv}_X(x)$ some important tools are needed. The most powerful of those used in the proof is the notion of maximal contact. Let $(G, \leq)$ be a totally ordered abelian group and let $\iota : X \to (G, \leq)$ be an upper semi-continuous function on X such that ι does not increase after a blow up with a certain good center. Roughly speaking a subscheme W of Z has maximal contact with X at a point x with respect to ι, if its transform W' after a sequence of blow ups (with certain good centers) contains all the points of the transform of X, where ι didn't drop. This is related to Abhyankar's Tschirnhausen transformation and has been developed in detail in [AHV] and [G2]. In characteristic zero maximal contact locally always exists. In positive characteristic Narasimhan was the first who gave an example (in dimension 3) for the non-existence of maximal contact, see [N] and also [C3]. Another example in dimension 2 can be found in [CJS], Theorem 14.3, p.151. Giraud investigated in [G3] how far the concept of maximal contact can be generalized to this situation. So far there is no appropriate generalization which extends the characteristic zero proof to the case of positive characteristic.

The notion of maximal contact leads to another important tool, the so called coefficient ideal with respect to some regular subscheme W. This is a local construction which yields a restriction to a smaller dimensional ambient scheme so that we can then apply induction on its dimension.

The invariant used in [BM3] has the form

$$\mathrm{inv}_X(x) = (\nu_1, s_1;\, \nu_2, s_2;\, \ldots;\, \nu_t, s_t;\, \nu_{t+1}),$$

$\nu_1 = H_{X,x}$ is the Hilbert-Samuel function of X, $\nu_i \in \mathbb{Q}_0 \cup \{\infty\}$, $i > 2$, are certain higher order multiplicities (sometimes also called residual orders) and $s_i \in \mathbb{Z}_0$ counts certain exceptional divisors. (A good reference for the Hilbert-Samuel function is Bennett's paper [Be]). The starting point for this thesis has been the following problem, which we formulate here as

Main Theorem 1. *There is a purely polyhedral approach for obtaining the numbers $\nu_i \in \mathbb{Q}_0 \cup \{\infty\}$. This means we can get ν_i by only considering certain polyhedra.*

For the proof we have to introduce an appropriate language. For example, Villamayor uses so called basic objects, Bierstone and Milman consider presentations, Włodarczyk defines marked ideals. (In Remark 2.6.4 we explain how these notions are related). We follow Hironaka and work with idealistic exponents, see [H3] and [H4]. These are pairs $\mathbb{E} = (J, b)$, where $J \subset \mathcal{O}_Z$ denotes a quasi-coherent ideal sheaf on Z and $b \in \mathbb{Z}_+$ is a certain positive integer. Two of them can be related via an equivalence relation $\sim$, which detects the behavior under so called permissible blow ups and under certain projections. So, parallel to [H2], we have to define the characteristic polyhedron of an idealistic exponent. In our investigations it turns out that the polyhedron is not invariant under $\sim$. Thus we have to modify the equivalence relation appropriately in order to obtain the correct invariant $\mathrm{inv}_X(x)$ in the end.

Let us briefly discuss the contents. In the first section we define the notion of an idealistic exponent over an *arbitrary* field k and recall some basic properties. Note that in [H3] and [H4] the base field is required to be perfect.

In the second section we define the tangent cone, the directrix and the ridge of an idealistic exponent. The latter two are closely related: for example if the base field is perfect, then the reduced ideal of the ridge and the ideal of the directrix coincide. For a detailed discussion see Remark 1.2.7.

It was already shown in [H3] that the directrices of equivalent idealistic exponents coincide (perfect base field!). For the tangent cone and the ridge this is not necessarily true. But still there are simple examples giving the hint that there must be some relationship. In order to reveal it we introduce the new concept of *idealistic tangent cones and idealistic ridges (and idealistic directrices).* (The author knows no reference, where this already appeared). We show that these are equivalent, if the idealistic exponents are equivalent.

In section 1.3 we define *idealistic coefficient exponents*, which give an idealistic variant of the coefficient ideal with respect to $V(y)$, where $(y) = (y_1, \dots, y_r)$ is part of a regular system of parameters $(u, y) = (u_1, \dots, u_e, y)$ of the local ring $R = \mathcal{O}_{Z,x}$ at $x \in Z$. Its order is by definition the d-invariant of X at x. Our aim is to make this number independent of the choices for (y) with fixed (u). This special number, called the δ-invariant of X at x, then leads to the definition of another invariant, which later yields ν_i. In fact, if $V(y)$ has maximal contact, then the d-invariant achieves already the intrinsic value of the δ-invariant. Of course, we are also interested in the behavior under the equivalence $\sim$. Here we have

Main Theorem 2. *The idealistic coefficient exponents with respect to the same $V(y)$ of two equivalent idealistic exponents are also equivalent.*

Moreover, if we have two choices $V(y)$ and $V(z)$ for a fixed system (u) and a fixed idealistic exponent $\mathbb{E}$ such that $\mathbb{E} \cap (y, 1) \sim \mathbb{E} \cap (z, 1)$, then the idealistic coefficient exponents are equivalent.

In particular, the equivalences imply that the d-invariants coincide.

But, in general, the d-invariant is not independent of the choice of (y). Therefore we have to find a good choice for (y) in order to obtain the δ-invariant. In characteristic zero the property of $V(y)$ to have maximal contact with X at x is a sufficient condition to achieve this. Hence in section 1.4 we recall the notion of maximal contact.

In the second chapter we come to the definition of characteristic polyhedra of idealistic exponents. First, we give a motivation. We introduce the Newton polyhedron associated to an idealistic exponent $\mathbb{E}$. Then we define a non-intrinsic polyhedron $\Delta(\mathbb{E}, u, y)$ with respect to (y) ((y) as above) as the Newton polyhedron of the idealistic coefficient exponent with respect to (y). This has a very concrete description, which at first sight depends on a choice of generators of the ideal. But as it turns out, it is in fact independent of this choice. Further it is a certain projection of the Newton polyhedron and we can relate the polyhedron with the d-invariant. In the subsequent section, we recall Hironaka's definition of the characteristic polyhedron of a singularity and its properties. By using this we construct the characteristic polyhedron of an idealistic exponent in section 2.3. For this we can prove

Main Theorem 3. *The characteristic polyhedron of an idealistic exponent $\mathbb{E}$ coincides with $\Delta(\mathbb{E}, u, y^*)$, where (y^*) is a system which we obtain by preparing (y) in a certain way, and it does not depend on the choice of (y) or (y^*). Further it is a certain projection of the associated Newton polyhedron.*

It follows that the d-invariant of the characteristic polyhedron of an idealistic exponent coincides with the δ-invariant. Together with Main Theorem 2 we get

Main Theorem 4. *The δ-invariant does not depend on the choice of (y) and coincides for equivalent idealistic exponents.*

But still the polyhedra depend on (u) and they do not behave well under the equivalence $\sim$, see Example 2.1.9. The point is that, in the resolution process, we have to take care of the exceptional divisors which arise by blow ups. As we already wrote at the beginning, the aim of (embedded) resolution is to modify X such that the total transform (which contains the exceptional components) has at most simple normal crossing singularities. Hence this a first reason to consider also the exceptional divisor.

Another reason is the general fact that there may not necessarily exist a canonical resolution of X if we neglect the exceptional divisors: for example, the singular locus of $X = V(t^2 + xyz)$ consists of the curves $V(t, x, y)$, $V(t, x, z)$ and $V(t, y, z)$. Since none of them is a better center than the other, we have to blow up the origin in order to obtain a canonical resolution. After blowing up $V(t, x, y, z)$ the situation in the X-, the Y- and the Z-chart is the same as before. Without having in mind that one of the three curves in the singular locus is an exceptional component, we run into a loop.

In order to include the exceptional components in our investigations, we define the notion of exceptional data $\mathcal{E}(x)$ associated to an idealistic exponent (see section 2.5). This leads to the definition of the ν-invariant. As the name indicates, the ν-invariant of certain idealistic exponents coincide with the terms ν_i in $\mathrm{inv}_X(x)$. Since the polyhedron behaves badly under $\sim$, the ν-invariants of equivalent idealistic exponents may differ. Thus we have to extend $\sim$ to $\sim_{\mathcal{E}(x)}$ by fixing the exceptional data, which forces the ν-invariants to coincide under the equivalence relation $\sim_{\mathcal{E}(x)}$. This leads to the concept of idealistic exponents with history. Clearly, they are closely related to the notions (basic objects, marked ideals, ...) in other proofs of resolution in characteristic zero, but to the author there is no reference known with this special approach via idealistic exponents. Taking also Main Theorem 3 into account yields

Main Theorem 5. *The ν-invariant does not depend on the choice of (y) and coincides for two equivalent idealistic exponents with history. Further, in characteristic zero the terms ν_i in the Bierstone-Milman invariant $\mathrm{inv}_X(x)$ are the ν-invariants of certain idealistic exponents with history.*

Except for the second part in the previous theorem the base field k has been arbitrary up to this point.

In the third chapter we recall the definition of $\mathrm{inv}_X(x)$ given in [BM3]. Thus we restrict our attention here to base fields k of characteristic zero. After clarifying the setup, we consider in section 3.2 the special case, where we neglect the exceptional components. In particular this includes the beginning of the resolution process before making any blow ups.

As we explained before, we have to consider the exceptional components. Thus we give the precise construction of $\mathrm{inv}_X(x)$ in the general case (and even starting with some simple normal crossing divisor) and obtain Main Theorem 1 in section 3.3. Using the theory of idealistic exponents with history our goal becomes an immediate consequence of these constructions.

In section 3.4 we investigate the behavior of the polyhedra in each step of the construction of $\mathrm{inv}_X(x)$.

Since the definition of $\mathrm{inv}_X(x)$ is quite complicated we mention a method to abbreviate its construction in section 3.5 and further we explain how the generators behave in these steps.

Let us remark that we are considering only the situation in the local ring. We really want to focus on the construction of $\mathrm{inv}_X(x)$. Hence we neither regard extensions of all these constructions to open neighborhoods of x nor their gluing. Once we have shown Main Theorem 5 all these properties follow by [BM3].

All schemes in this thesis are of finite type over an arbitrary base field k.

As usual $\mathbb{Z}$ denotes the integers, $\mathbb{N}$ the natural, $\mathbb{Q}$ the rational, $\mathbb{R}$ the real and $\mathbb{C}$ the complex numbers. We denote by $\mathbb{Z}_0$ the non-negative integers and by $\mathbb{Z}_+$ (or $\mathbb{N}$) the positive integers. Analogously we define $\mathbb{Q}_0, \mathbb{Q}_+, \mathbb{R}_0$ and $\mathbb{R}_+$.

Further we frequently use multiindex notation without mentioning this every time. For example we abbreviate for $b \in \mathbb{Z}_+$, $(u) = (u_1, \dots, u_n)$ and $A = (A_1, \dots, A_n), B = (B_1, \dots, B_n) \in \mathbb{Z}_0^n$ the following:

$$u^A = u_1^{A_1} \cdots u_n^{A_n}, \quad |B| = B_1 + \ldots + B_n, \quad \frac{A}{b - |B|} = \left(\frac{A_1}{b - |B|}, \dots, \frac{A_n}{b - |B|} \right)$$

and

$$\binom{A}{B} := \binom{A_1}{B_1} \cdots \binom{A_n}{B_n}, \quad \text{where } \binom{A_i}{B_i} = 0 \text{ if } A_i < B_i.$$

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1 Idealistic Exponents

In this chapter we make the reader familiar with the world in which we work by introducing some basic notions.

Following Hironaka we recall the definition and properties of idealistic exponents. After that we define the tangent cone, the directrix and the ridge which provide important local data of the idealistic exponent. Since the tangent cone and the ridge don't behave nicely, we give idealistic interpretations of them.

Another interesting local construction in characteristic zero is the coefficient ideal. We define this in the idealistic setting and deduce from it the d_x-invariant which will be our main tool in order to obtain the invariant of Bierstone and Milman.

Finally we focus on the case of characteristic zero and recall the concept of maximal contact in our setting.

1.1 Definition and first properties

First let us recall the definition of idealistic exponents. For this we follow [H3] and [H4].

Let Z be a regular irreducible scheme of finite type over an arbitrary field k. Note that by the Hilbert basis theorem Z is Noetherian.

Definition 1.1.1. *An idealistic exponent $\mathbb{E} = (J, b)$ on Z is a pair consisting of a quasi-coherent ideal sheaf J on Z and a positive integer $b \in \mathbb{Z}_+$.*

We define its order at a point $x \in Z$ (not necessarily closed) as

$$\mathrm{ord}_x(\mathbb{E}) = \begin{cases} \dfrac{\mathrm{ord}_x(J)}{b} & , \textit{ if } \mathrm{ord}_x(J) \geq b \textit{ and} \\ 0 & , \textit{ else,} \end{cases}$$

where $\mathrm{ord}_x(J) = \sup\{d \in \mathbb{Z}_0 \cup \{\infty\} \mid J_x \subseteq \mathfrak{m}_x^d\}$ *(and $\mathfrak{m}_x$ denotes the maximal ideal in the local ring at x). Further we define the singular locus of $\mathbb{E}$ as*

$$\mathrm{Sing}\,(\mathbb{E}) = \{x \in Z \mid \mathrm{ord}_x(\mathbb{E}) \geq 1\}.$$

We denote the closed subscheme corresponding to J by $X \subseteq Z$.

If $Z = \operatorname{Spec}(R)$ is affine, then we identify J with the corresponding ideal in R and we also say $\mathbb{E}$ is an idealistic exponent on R.

It is also possible to define the order by $\widetilde{\operatorname{ord}_x}(\mathbb{E}) = \dfrac{\operatorname{ord}_x(J)}{b}$ (as it is done in the older literature [H3] section 1, Remark 5, p.56). The difference to the previous definition is that for $x \notin \operatorname{Sing}(\mathbb{E})$ we have $\operatorname{ord}_x(\mathbb{E}) = 0$, but maybe $0 \leq \widetilde{\operatorname{ord}_x}(\mathbb{E}) < 1$. For more details see Remark 1.1.7.

Definition 1.1.2. *Let $\mathbb{E} = (J, b)$ be an idealistic exponent on Z. A blow up $\pi : Z' \to Z$ with center D is called permissible for $\mathbb{E}$, if D is regular and $D \subseteq \operatorname{Sing}(\mathbb{E})$. The transform of $\mathbb{E}$ is then given by $\mathbb{E}' = (J', b)$, where J' is defined via $J\mathcal{O}_{Z'} = J'H^b$, where H denotes the ideal sheaf of the exceptional divisor.*

Remark 1.1.3. *In other literature there is also the notion of permissible centers, see for example Hironaka's appendix in [CGO] or [CJS]. These are regular subschemes $D \subset X$ such that X is normally flat along D. (For the precise definition see [CJS], Definition 2.1, p.30). For idealistic exponents we do not require that the center of a permissible blow up fulfills this extra condition. But later we have to make the additional assumption that the center D has at most simple normal crossing singularities with the exceptional components if there are some. Otherwise, it is not guaranteed that all the exceptional divisors have only simple normal crossings after the blow up with center D.*

Definition 1.1.4. *We define a local sequence of regular blow ups over Z as a sequence of the form*

$$\begin{array}{ccccccccc} Z = Z_0 \supset U_0 & \xleftarrow{\pi_1} & Z_1 \supset U_1 & \xleftarrow{\pi_2} & \ldots & \xleftarrow{\pi_{l-1}} & Z_{l-1} \supset U_{l-1} & \xleftarrow{\pi_l} & Z_l \\ \cup & & \cup & & & & \cup & & \\ D_0 & & D_1 & & \ldots & & D_{l-1} & & \end{array} \tag{1.1}$$

where $l \in \mathbb{Z}_+ \cup \{\infty\}$, each $U_i \subset Z_i$ is an open subscheme, $D_i \subset U_i$ is a regular closed subscheme and $\pi_{i+1} : Z_{i+1} \to U_i$ denotes the blow up with center D_i, $0 \leq i \leq l-1$.

Remark 1.1.5. *Let $\mathbb{E} = (J, b)$ be an idealistic exponent on Z and consider a local sequence of regular blow ups as in (1.1). In Definition 1.1.2 we have introduced when a blow up is permissible for $\mathbb{E}$ and further we have defined the transform of $\mathbb{E}$ under such a blow up. Denote by $\mathbb{E}_i$ the transform of $\mathbb{E}_0 := \mathbb{E}$ in Z_i for $0 \leq i \leq l-1$. Then we say that the local sequence of regular blow ups (1.1) is permissible for $\mathbb{E}$ if each blow up π_{i+1} is permissible for $\mathbb{E}_i$ for $0 \leq i \leq l-1$.*

Let $t = (t_1, \ldots, t_a)$ be a finite system of indeterminates. Then we use the notation

$$Z[t] := Z \times_k \mathbb{A}_k^a = Z \times_{\operatorname{Spec}(k)} \operatorname{Spec}(k[t]).$$

We consider the idealistic exponent $\mathbb{E}[t] = (J[t], b)$, where $J[t] = J\mathcal{O}_{Z[t]}$ (with respect to the canonical projection).

Definition 1.1.6. *Let $\mathbb{E}_1 = (J_1, b_1)$ and $\mathbb{E}_2 = (J_2, b_2)$ be two idealistic exponents on Z. Then we define*

$$\mathbb{E}_1 \subset \mathbb{E}_2$$

if the following condition holds:

$$\text{Let } t = (t_1, \dots, t_a) \text{ be an arbitrary finite system of indeterminates and let } \mathbb{E}_i[t] = (J_i[t], b_i),\ i \in \{1, 2\}. \text{ If any local sequence of regular blow ups over } Z[t] \text{ is permissible for } \mathbb{E}_1[t], \text{ then it is also permissible for } \mathbb{E}_2[t]. \tag{1.2}$$

Further we say $\mathbb{E}_1$ and $\mathbb{E}_2$ are equivalent,

$$\mathbb{E}_1 \sim \mathbb{E}_2,$$

if both $\mathbb{E}_1 \subset \mathbb{E}_2$ and $\mathbb{E}_1 \supset \mathbb{E}_2$. By $\mathbb{E}_1 \cap \mathbb{E}_2 \sim \mathbb{E}_3$ we mean that a local sequence of regular blow ups over $Z[t]$ is permissible for $\mathbb{E}_3[t]$ if and only if it is permissible for $\mathbb{E}_1[t]$ and $\mathbb{E}_2[t]$.

By definition $\operatorname{Sing}(\mathbb{E}_1 \cap \mathbb{E}_2) = \operatorname{Sing}(\mathbb{E}_1) \cap \operatorname{Sing}(\mathbb{E}_2)$ and $\operatorname{ord}_x(\mathbb{E}_1 \cap \mathbb{E}_2) = \min\{\operatorname{ord}_x(\mathbb{E}_1), \operatorname{ord}_x(\mathbb{E}_2)\}$ for $x \in \operatorname{Sing}(\mathbb{E}_1 \cap \mathbb{E}_2)$.

Remark 1.1.7. *As mentioned before, there is an alternative way to define the order of an idealistic exponent $\mathbb{E}$ and the difference appears for points not contained in* $\operatorname{Sing}(\mathbb{E})$ *(see the remarks below Definition 1.1.1). For example let $\mathbb{E}$ be such that* $\operatorname{Sing}(\mathbb{E}) = \emptyset$. *All these idealistic exponents are equivalent, because there exist no permissible local sequence of regular blow ups. But for a fixed point x the orders* $\widetilde{\operatorname{ord}}_x(\mathbb{E}) \in [0; 1) \cap \mathbb{Q}$ *clearly don't have to coincide (e.g. $\mathbb{E}_1 = (y + z, 2)$ and $\mathbb{E}_2 = (y + z, 3)$ and $x = V(y, z)$)*

The point is that the notion of inclusion respectively equivalence defined above involves only the behavior of $\mathbb{E}$ under permissible blow ups. Since the centers of permissible blow ups are contained in the singular locus, non-singular points (i.e. points where $\widetilde{\operatorname{ord}}_x(\mathbb{E}) < 1$*) are not considered.*

We have the following basic properties of idealistic exponents:

Lemma 1.1.8. *Let $\mathbb{E} = (J, b)$ and $\mathbb{E}_i = (J_i, b_i)$, $i \in \{1, 2, 3, 4\}$, be idealistic exponents on Z. Then:*

(i) For every $a \in \mathbb{Z}_+$ we have $(J^a, ab) \sim (J, b)$.

(ii) *Let $m \in \mathbb{Z}_+$ with $b_1 \mid m$ and $b_2 \mid m$. Then*

$$(J_1, b_1) \cap (J_2, b_2) \sim \left(J_1^{\frac{m}{b_1}} + J_2^{\frac{m}{b_2}}, m \right).$$

(iii) *We always have $(J_1 J_2, b_1 + b_2) \supset (J_1, b_1) \cap (J_2, b_2)$. If further $\mathrm{Sing}\,(J_i, b_i + 1) = \emptyset$ for $i \in \{1, 2\}$, then the previous inclusion becomes an equivalence.*

(iv) *If $\mathbb{E}_1 \subset \mathbb{E}_2$ and $\mathbb{E}_3 \subset \mathbb{E}_4$, then $\mathbb{E}_1 \cap \mathbb{E}_3 \subset \mathbb{E}_2 \cap \mathbb{E}_4$. In particular $\mathbb{E}_1 \sim \mathbb{E}_2$ implies by symmetry $\mathbb{E}_1 \cap \mathbb{E}_3 \sim \mathbb{E}_2 \cap \mathbb{E}_3$.*

(v) *Let $\pi : Z' \to Z$ be a permissible blow up for $\mathbb{E}_1$ and $\mathbb{E}_2$. Then we have $(\mathbb{E}_1 \cap \mathbb{E}_2)' \sim \mathbb{E}_1' \cap \mathbb{E}_2'$.*

By *(i)* we may extend the definition of idealistic exponents (J, b) to $b \in \mathbb{Q}_+$. Suppose $b = \dfrac{c}{d} \in \mathbb{Q}_+$, where the greatest common divisor of $c, d \in \mathbb{Z}_+$ is 1. Then we define (J, b) to be an idealistic exponent with assigned number $b \in \mathbb{Q}_+$ which is equivalent to $(J^d, b\,d)$; note that $b\,d = c \in \mathbb{Z}_+$.

Proof. Proof of *(i)*: Suppose $\mathbb{E}_1 = (J, b)$, $\mathbb{E}_2 = (J^a, ab)$ and let $x \in Z$ be an arbitrary point. Then

$$\frac{\mathrm{ord}_x(J^a)}{ab} = \frac{a \cdot \mathrm{ord}_x(J)}{a \cdot b} = \frac{\mathrm{ord}_x(J)}{b}, \tag{$*$}$$

which implies $\mathrm{ord}_x(\mathbb{E}_1) = \mathrm{ord}_x(\mathbb{E}_2)$ and thus $\mathrm{Sing}\,(\mathbb{E}_1) = \mathrm{Sing}\,(\mathbb{E}_2)$. Let (t) be a finite system of indeterminates, then it is clear that $(*)$ is stable under the change from $(\mathbb{E}_i, Z)$ to $(\mathbb{E}_i[t], Z[t])$. Hence it suffices to consider the situation of $(\mathbb{E}_i, Z)$. There condition $(*)$ is stable under permissible blow ups $\pi : Z' \to Z$: The transform $\mathbb{E}_1' = (J', b)$ of (J, b) under π is defined via $J\mathcal{O}_{Z'} = H^b J'$, where H denotes the sheaf of the exceptional divisor. Since $J^a \mathcal{O}_{Z'} = (J\mathcal{O}_{Z'})^a = (H^b J')^a = H^{ab} J'^a$, the transform of (J^a, ab) is $\mathbb{E}_2' = (J'^a, ab)$. As above we get $\mathrm{ord}_{x'}(\mathbb{E}_1') = \mathrm{ord}_{x'}(\mathbb{E}_2')$ for every $x' \in Z'$ which implies $\mathrm{Sing}\,(\mathbb{E}_1') = \mathrm{Sing}\,(\mathbb{E}_2')$. In particular every local sequence of regular blow ups which is permissible for $\mathbb{E}_1$ is so for $\mathbb{E}_2$ and vice versa.

Proof of *(ii)*: By the first part $(J_i, b_i) \sim \left(J_i^{\frac{m}{b_i}}, m \right)$. Hence it suffices to prove

$$(J_1, b) \cap (J_2, b) \sim (J_1 + J_2, b).$$

For any $x \in Z$ we have $\mathrm{ord}_x(J_1 + J_2, b) = \min\{\mathrm{ord}_x(J_1, b), \mathrm{ord}_x(J_2, b)\}$. Further the relation between J_1, J_2 and $J_1 + J_2$ is stable under extensions of Z by $\mathbb{A}_k^a$ for some $a \in \mathbb{Z}_+$ and under permissible blow ups $\pi : Z' \to Z$. The latter follows because the transform of $J_1 + J_2$ is given by $(J_1 + J_2)\mathcal{O}_{Z'} = J_1\mathcal{O}_{Z'} + J_2\mathcal{O}_{Z'} = H^b J_1' + H^b J_2' = H^b(J_1' + J_2')$ (H: sheaf of the exceptional divisor). This yields *(ii)*.

Proof of *(iii)*: By definition

$$\operatorname{ord}_x(J_1J_2, b_1 + b_2) = \frac{b_1}{b_1 + b_2}\operatorname{ord}_x(J_1, b_1) + \frac{b_2}{b_1 + b_2}\operatorname{ord}_x(J_2, b_2)$$

for any $x \in Z$. As before we see easily the stability of the relation between J_1, J_2 and J_1J_2 under extensions with $\mathbb{A}_k^a$ for any $a \in \mathbb{Z}_+$ and under permissible blow ups $\pi : Z' \to Z$. Recall that a blow up is called permissible for $\mathbb{E}$, if the center D is regular and contained in $\operatorname{Sing}(\mathbb{E})$. The latter are those points with $\operatorname{ord}_x(\mathbb{E}) \geq 1$. Hence if a local sequence of regular blow ups is permissible for $\mathbb{E}_1$ and $\mathbb{E}_2$, then also for $(J_1J_2, b_1 + b_2)$. This shows $(J_1, b_1) \cap (J_2, b_2) \subset (J_1J_2, b_1 + b_2)$.

Let us now consider the case where $\operatorname{Sing}(J_i, b_i + 1) = \emptyset$ for $i \in \{1, 2\}$. This implies

$$\operatorname{ord}_x(J_i, b_i + 1) = \frac{\operatorname{ord}_x(J_i)}{b_i + 1} < 1 \qquad \text{for all } i \in \{1, 2\} \text{ and } x \in Z. \tag{$**$}$$

Suppose there exists a local sequence of regular blow ups which is permissible for (J_1J_2, b_1+b_2), but not for $\mathbb{E}_1\cap\mathbb{E}_2$. Since the situation is stable under extensions by $\mathbb{A}_k^a$ and under permissible blow ups, we may assume that the given local sequence of regular blow ups is a single blow up $\pi : Z' \to Z$ with center D. Then D is not permissible for $\mathbb{E}_1$ or $\mathbb{E}_2$; without loss of generality D is not permissible for $\mathbb{E}_1$. So there exists $y \in D$ such that $\operatorname{ord}_y(\mathbb{E}_1) = \dfrac{\operatorname{ord}_y(J_1)}{b_1} < 1$, but $\operatorname{ord}_y(J_1J_2, b_1 + b_2) \geq 1$. We write $\operatorname{ord}_y(\mathbb{E}_i) = \dfrac{m_i}{b_i}$, where we set $m_i := \operatorname{ord}_y(J_i)$ for $i \in \{1, 2\}$. Then $\operatorname{ord}_y(\mathbb{E}_1) < 1$ means $m_1 < b_1$ and this yields

$$\operatorname{ord}_y(J_1J_2, b_1 + b_2) = \frac{m_1}{b_1 + b_2} + \frac{m_2}{b_1 + b_2} < \frac{b_1}{b_1 + b_2} + \frac{m_2}{b_1 + b_2} = 1 + \frac{m_2 - b_2}{b_1 + b_2}.$$

Hence $\dfrac{m_2 - b_2}{b_1 + b_2} > 0$, because $\operatorname{ord}_y(J_1J_2, b_1 + b_2) \geq 1$. Since b_1 and b_2 are positive, we can multiply by $b_1 + b_2$ and get $m_2 > b_2$ or equivalently $m_2 \geq b_2 + 1$ (Note that $m_2 = \operatorname{ord}_y(J_2) \in \mathbb{Z}_+ \cup \{\infty\}$). But then $\operatorname{ord}_y(J_2, b_2 + 1) = \dfrac{m_2}{b_2 + 1} \geq 1$ and this contradicts $(**)$.

Proof of *(iv)*: A local sequence of regular blow ups is permissible for $\mathbb{E}_1 \cap \mathbb{E}_3$ if and only if it is permissible for $\mathbb{E}_1$ and $\mathbb{E}_3$. Since $\mathbb{E}_1 \subset \mathbb{E}_2$ (resp. $\mathbb{E}_3 \subset \mathbb{E}_4$) we know: if a local sequence of regular blow ups is permissible for $\mathbb{E}_1$ (resp. $\mathbb{E}_3$) then it is so for $\mathbb{E}_2$ (resp. $\mathbb{E}_4$). Together we get the first part. The rest follows by symmetry.

Proof of *(v)*: Set $m := b_1 b_2$. By *(ii)* we have $\mathbb{E}_1 \cap \mathbb{E}_2 \sim (J_1^{b_2} + J_2^{b_1}, m)$. The transform $\mathbb{E}' = (J', b)$ of an idealistic exponent $\mathbb{E} = (J, b)$ under a permissible blow up $\pi : Z' \to Z$ is given by $J\mathcal{O}_{Z'} = H^b J'$ (H: sheaf of the exceptional divisor). Hence $\mathbb{E}'_1 \cap \mathbb{E}'_2 \sim ({J'}_1^{b_2} + {J'}_2^{b_1}, m)$ and the claim follows, since $(J_1^{b_2} + J_2^{b_1})\mathcal{O}_{Z'} = (J_1\mathcal{O}_{Z'})^{b_2} + (J_2\mathcal{O}_{Z'})^{b_1} = H^{b_1 b_2}{J'}_1^{b_1} + H^{b_1 b_2}{J'}_2^{b_2} = H^m({J'}_1^{b_1} + {J'}_2^{b_2})$. □

Remark 1.1.9. *The following is a strategy how to construct an example, where the inclusion $(J_1, b_1) \cap (J_2, b_2) \subset (J_1 J_2, b_1 + b_2)$ is strict, i.e. where we have additionally $(J_1, b_1) \cap (J_2, b_2) \not\sim (J_1 J_2, b_1 + b_2)$:*

Let $b_1, b_2, m_1, m_2 \in \mathbb{Z}_+$ with $m_2 < b_2$ and $m_1 + m_2 \geq b_2$. Let $Z = \mathbb{A}_k^n = \mathrm{Spec}\,(R)$ with $R = k[u_1, \ldots, u_n]$. Denote by $x \in Z$ the generic point of a regular irreducible closed subscheme $D \subset Z$. Choose ideals $J_1, J_2 \subset R$ with $\mathrm{ord}_x(J_1) = b_1 + m_1$ and $\mathrm{ord}_x(J_2) = m_2$. Since $m_2 < b_2$, the point x is not contained in $\mathrm{Sing}\,(J_2, b_2)$ and hence not in $\mathrm{Sing}\,(\mathbb{E})$, where $\mathbb{E}$ denotes the idealistic exponent $(J_1, b_1) \cap (J_2, b_2)$. But $x \in \mathrm{Sing}\,(J_1 J_2, b_1 + b_2)$, because $\mathrm{ord}_x(J_1 J_2) = b_1 + m_1 + m_2 \geq b_1 + b_2$. This means the blow up with center D is a local sequence of regular blow ups which is permissible for $(J_1 J_2, b_1 + b_2)$, but not for $\mathbb{E}$.

For example take $b_1 = b_2 = 2$, $m_1 = m_2 = 1$, $J_1 = \langle x^3 + y^5 \rangle$, $J_2 = \langle xz^2 + y^3 \rangle$ and $D = V(x, y)$.

Another important result is the following

Proposition 1.1.10 (Numerical Exponent Theorem)**.** *Let $\mathbb{E}_1 = (J_1, b_1)$ and $\mathbb{E}_2 = (J_2, b_2)$ be idealistic exponents on Z. If $\mathbb{E}_1 \subset \mathbb{E}_2$, then $\mathrm{ord}_x(\mathbb{E}_1) \leq \mathrm{ord}_x(\mathbb{E}_2)$ for all $x \in Z$.*

By symmetry $\mathbb{E}_1 \sim \mathbb{E}_2$ implies $\mathrm{ord}_x(\mathbb{E}_1) = \mathrm{ord}_x(\mathbb{E}_2)$ for all $x \in Z$.

Proof. We follow Hironaka's idea in [H3], section 2, Proposition 8, p.68; but we give a slightly modified version.

Let $x \in Z$ and set $m_i = \mathrm{ord}_x J_i$ for $i \in \{1, 2\}$. Then $\mathrm{ord}_x(\mathbb{E}_i) = \dfrac{m_i}{b_i}$. Consider the local situation at x; let $R = \mathcal{O}_{Z,x}$ be the regular local ring, $\mathfrak{m} \subset R$ the maximal ideal, $K = R/\mathfrak{m}$ the residue field and $I_i = (J_i)_x$. We extend the situation to $R_0 := R \times_K K[t]$ and set $U_0 := \mathrm{Spec}\,(R_0)$ and $I_{i,0} := I_i \cdot R_0$. If $(u) = (u_1, \ldots, u_n)$ denotes a regular system of parameters for R, then (u, t) is one for R_0. Let $L_0 \subset U_0$ be the line $V(u) \subset \mathrm{Spec}\,(R_0)$ and $x_0 \in L_0 \subset U_0$ the origin $V(u, t)$. We now consider the following local sequence of regular blow ups for $\alpha \in \mathbb{Z}_+$:

$$\begin{array}{ccccccccc}
L_0 & \cong & L_1 & \cong & \ldots & \cong & L_{\alpha-1} & \cong & L_\alpha \\
\cap & & \cap & & & & \cap & & \cap \\
U_0 & \xleftarrow{\pi_1} & Z_1 \supset U_1 & \xleftarrow{\pi_2} & \ldots & \xleftarrow{\pi_{\alpha-1}} & Z_{\alpha-1} \supset U_{\alpha-1} & \xleftarrow{\pi_\alpha} & Z_\alpha \supset U_\alpha \\
\Cup & & \Cup & & & & \Cup & & \Cup \\
x_0 & & x_1 & & \ldots & & x_{\alpha-1} & & x_\alpha,
\end{array} \tag{1.3}$$

where $\pi_i : Z_i \to U_{i-1}$ is the blow up with center $x_{i-1} \in U_{i-1}$, $L_i \cong L_0$ is the strict transform of L_0, x_i denotes the unique intersection point of L_i with the exceptional divisor of the blow up π_i and $U_i = \mathrm{Spec}\,(R_i) \subset Z_i$ is an open affine neighborhood of x_i, $i \in \{1, \ldots, \alpha\}$. Since $L_0 = V(u)$, it is clear that x_1 is the origin of the T-chart of the blow up. Hence we can choose U_i as the T-chart of π_i and then $x_i = V\left(\frac{u}{t^i}, t\right)$ and $L_i = V\left(\frac{u}{t^i}\right)$. Note that $\left(\frac{u}{t^i}, t\right)$ is a regular system of parameters for R_i.

Let $\mathbb{E} = (J, b)$ be an idealistic exponent on Z, $I_0 = J_x \subset R \subset R[t]$ and set $m = \mathrm{ord}_x(J)$. We expand an arbitrary $f \in I_0$ in the $\mathfrak{m}_0$-adic completion $\widehat{R_0}$ of R_0 ($\mathfrak{m}_0 := \langle u, t \rangle$) as $f = \sum_A C_A\, u^A$, where $C_A \in R_0^\times \cup \{0\}$ and it is only non-zero if $|A| \geq m$. The transform f_α of f in $\widehat{R_\alpha}$ is given by

$$f_\alpha = \frac{f}{t^{\alpha b}} = \sum C_A \left(\frac{u}{t^\alpha}\right)^A t^{(|A|-b)\alpha} = t^{(m-b)\alpha} \cdot \widetilde{f_\alpha}.$$

Hence if $(m-b)\alpha \geq b$ or equivalently if $\left(\frac{m}{b} - 1\right)\alpha \geq 1$, then $V(t) = \pi_\alpha^{-1}(x_{\alpha-1})$ is a permissible center for (I_α, b), where $I_\alpha = \langle f_\alpha \mid f \in I_0 \rangle \subset R_\alpha$ defines the transform of (I_0, b) in R_α. Since $V(t)$ is a divisor, the blow up with center $V(t)$ is an isomorphism. Let $S_0 = R_\alpha$ and $V_0 = \mathrm{Spec}\,(S_0) = U_\alpha$. Then we continue (1.3) by $\mathbb{Z}_0 \ni \beta$-times blowing up $V(t)$,

$$V_0 \xleftarrow{\tau_1} V_1 \xleftarrow{\tau_2} \ldots \xleftarrow{\tau_b} V_\beta,$$

and call the extended sequence $S(\alpha, \beta)$.

The transform of f_α in V_β is then given by $f_{\alpha,\beta} = t^{(m-b)\alpha - b\beta} \cdot \widetilde{f_\alpha}$. The local sequence of regular blow ups $S(\alpha, \beta)$ is permissible for $\mathbb{E}$ if and only if $(m-b)\alpha \geq \beta b$ or equivalently if

$$\left(\frac{m}{b} - 1\right)\alpha \geq \beta \tag{$*$}$$

Consider $\mathbb{E}_1$ and $\mathbb{E}_2$ with $\mathbb{E}_1 \subset \mathbb{E}_2$. Assume there exists a $x \in Z$ with

$$\mathrm{ord}_x(\mathbb{E}_1) = \frac{m_1}{b_1} > \frac{m_2}{b_2} = \mathrm{ord}_x(\mathbb{E}_2).$$

Choose $\alpha = b_1 b_2$ and $\beta = m_1 b_2 - b_1 b_2$. Then

$$\left(\frac{m_1}{b_1} - 1\right)\alpha = m_1 b_2 - b_1 b_2 = \beta,$$

hence $S(\alpha, \beta)$ is permissible for $\mathbb{E}_1$. But on the other hand, since $\frac{m_1}{b_1} > \frac{m_2}{b_2}$, we have

$$\left(\frac{m_2}{b_2} - 1\right)\alpha < \left(\frac{m_1}{b_1} - 1\right)\alpha = \beta$$

and this means $S(\alpha, \beta)$ is not permissible for $\mathbb{E}_2$. Together we see $\mathbb{E}_1 \not\subset \mathbb{E}_2$, which is a contradiction and thus proves the proposition. $\square$

Hironaka shows the statement directly: (sketch) In (1.3) he blows up one more time. If $\mathbb{E}_1 \subset \mathbb{E}_2$, then with the use of the modified relation $(*)$ divided by $\alpha \in \mathbb{Z}_+$, we have

$$\left\lfloor \frac{m_1}{b_1} - 1 + \frac{1}{\alpha}\left(\frac{m_1}{b_1} - 1\right)\right\rfloor \leq \left\lfloor \frac{m_2}{b_2} - 1 + \frac{1}{\alpha}\left(\frac{m_2}{b_2} - 1\right)\right\rfloor,$$

where $\lfloor(.)\rfloor$ denotes the integral part of $(.)$. If we take the limit $\alpha \to \infty$ we get $\frac{m_1}{b_1} - 1 \leq \frac{m_2}{b_2} - 1$ and hence $\frac{m_1}{b_1} \leq \frac{m_2}{b_2}$.

Corollary 1.1.11. *If* $\mathbb{E}_1 \subset \mathbb{E}_2$*, then* $\mathrm{Sing}\,(\mathbb{E}_1) \subseteq \mathrm{Sing}\,(\mathbb{E}_2)$*. If* $\mathbb{E}_1 \sim \mathbb{E}_2$*, then* $\mathrm{Sing}\,(\mathbb{E}_1) = \mathrm{Sing}\,(\mathbb{E}_2)$*.*

Proof. By definition $\mathrm{Sing}\,(\mathbb{E}) = \{x \in Z \mid \mathrm{ord}_x(\mathbb{E}) \geq 1\}$. Hence Proposition 1.1.10 immediately implies the assertion. $\square$

Remark 1.1.12. *The converse of the Numerical Exponent theorem and its corollary is in general false. More precisely the condition* $\mathrm{ord}_x(\mathbb{E}_1) \leq \mathrm{ord}_x(\mathbb{E}_2)$ *for all* $x \in Z$ *is not stable under permissible blow ups.*

Let us give an example. Consider the idealistic exponents $\mathbb{E}_1 = (y^2 + x^3, 2)$ *and* $\mathbb{E}_2 = (x^2 + y^3, 2)$ *over* $\mathbb{A}_k^2$ *for an arbitrary field* k*. Then the order coincides in all points (recall that by definition the order is zero for non-singular points) and thus* $\mathrm{Sing}\,(\mathbb{E}_1) = \mathrm{Sing}\,(\mathbb{E}_2)$*. The singular locus is in both cases the origin, but it is easy to construct a local sequence of regular blow ups over* $\mathbb{A}_k^2$ *which is permissible only for one of the idealistic exponents.*

Notation: Let $m \in \mathbb{Z}_0$ be a non-negative integer and Z as usual a regular scheme (resp. let R be a regular ring). Then we denote by $\mathrm{Diff}_{\mathbb{Z}}^{\leq m}(Z)$ (resp. $\mathrm{Diff}_{\mathbb{Z}}^{\leq m}(R)$) the (absolute) differential operators of $\mathcal{O}_Z$ (resp. R) on itself.

Proposition 1.1.13 (Diff Theorem)**.** *Let* $\mathbb{E} = (J, b)$ *be an idealistic exponent on* Z*. Let* $\mathcal{D}$ *be a left* $\mathcal{O}_Z$*-submodule of* $\mathrm{Diff}_{\mathbb{Z}}^{\leq m}(Z)$*. Then*

$$(J, b) \subset (\mathcal{D}J, b - m)$$

or equivalently $(J, b) \sim (\mathcal{D}J, b - m) \cap (J, b)$*.*

If $m \geq b$, then the assigned number of $(\mathcal{D}J, b - m)$ is not positive and hence is a priori not defined. Let us define the singular locus as $\{x \in Z \mid \mathrm{ord}_x(\mathcal{D}J) \geq b - m\}$ (which is equivalent to our definition before, but works in the excluded case of non-positive assigned numbers). Then $\mathrm{Sing}\,(\mathcal{D}J, b-m) = Z$ and the claim follows immediately.

We also use this proposition in the case of a single differential operator $\mathcal{D} \in \mathrm{Diff}_{\mathbb{Z}}^{\leq m}(Z)$; here we identify $\mathcal{D}$ with the submodule of $\mathrm{Diff}_{\mathbb{Z}}^{\leq m}(Z)$ generated by $\mathcal{D}$.

Proof. We follow the proof of [H3] section 8, Theorem 1, p.104. To abbreviate notation we write $\mathbb{D}$ instead of $(\mathcal{D}J, b-m)$. We want to show

$$\mathbb{E} = (J, b) \subset (\mathcal{D}J, b-m) = \mathbb{D}.$$

Let $t = (t_1, \ldots, t_a)$ be a finite system of indeterminates and $\mathbb{E}[t] = (J[t], b)$ resp. $\mathbb{D}[t] = ((\mathcal{D}J)[t], b-m)$ the extended idealistic exponents on $Z[t] = Z \times_k \mathbb{A}_k^a$ (k: base field). Clearly $\mathcal{D}$ may be considered as a left $\mathcal{O}_{Z[t]}$-submodule of $\mathrm{Diff}_{\mathbb{Z}}^{\leq m}(Z[t])$ (define a differential operator of $\mathcal{D}$ to be linear on (t)). In order to prove the proposition we have to show that a local sequence of regular blow ups which is permissible for $\mathbb{E}[t]$ automatically is also permissible for $\mathbb{D}[t]$. Since $(\mathcal{D}J)[t] = \mathcal{D}(J[t])$ we see that the relation of $\mathbb{E}[t]$ to $\mathbb{D}[t]$ is the same as for $\mathbb{E}$ and $\mathbb{D}$. Hence it suffices to show that every local sequence of regular blow ups which is permissible for $\mathbb{E}$, is so for $\mathbb{D}$. Let $D \subset Z$ be an arbitrary closed regular subscheme. Let $\pi : Z' \to Z$ be the blow up with center D. We show:

(i) If π is permissible for $\mathbb{E}$, then so it is for $\mathbb{D}$.

(ii) Let $\mathbb{E}'$ and $\mathbb{D}'$ denote the transforms under π. Then the relation between them is the same as before.

Obviously these two properties imply the assertion.

Let $y \in Z$ be a generic point of D. Let $R = \mathcal{O}_{Z,y}$ be the regular local ring at y, denote by $\mathfrak{m}$ the maximal ideal and by K the residue field. Suppose $J_y \subset \mathfrak{m}^l$ for some $l \in \mathbb{Z}_+$. Then it is known that $\mathcal{D}J_y \subset \mathfrak{m}^{l-m}$ and this implies

$$\mathrm{ord}_y(\mathcal{D}J) \geq \mathrm{ord}_y(J) - m. \qquad (*)$$

For (i): Suppose D is permissible for $\mathbb{E}$ and recall that y denotes a generic point of D. Then $D \subseteq \mathrm{Sing}\,(\mathbb{E})$ and therefore $\mathrm{ord}_y(J) \geq b$. This together with $(*)$ yields

$$\mathrm{ord}_y(\mathcal{D}J, b-m) = \frac{\mathrm{ord}_y(\mathcal{D}J)}{b-m} \geq \frac{\mathrm{ord}_y(J) - m}{b-m} \geq \frac{b-m}{b-m} = 1,$$

and we get $D \subseteq \mathrm{Sing}\,(\mathbb{D})$. This proves (i).

For (ii) we have to show that there exists an $\mathcal{O}_{Z'}$ -submodule $\mathcal{D}'$ of $\mathrm{Diff}_{\mathbb{Z}}^{\leq m}(Z')$ such that

$$(\mathcal{D}J)' = \mathcal{D}'J' \qquad (**)$$

with $\mathbb{D}' = ((\mathcal{D}J)', b-m)$ and $\mathbb{E}' = (J', b)$. *Caution:* Do not forget that the transformations are given by different laws, namely $(\mathcal{D}J)\mathcal{O}_{Z'} = H^{b-m}(\mathcal{D}J)'$ and $J\mathcal{O}_{Z'} = H^b J'$, where $H \subset \mathcal{O}_{Z'}$ denotes the ideal sheaf of the exceptional divisor. Let further $Q \subset \mathcal{O}_Z$ denote the ideal sheaf corresponding to the center D and set

$$\mathcal{D}' := H^{-b+m} \cdot \mathcal{D} \cdot Q^b$$

(viewed as an $\mathcal{O}_{Z'}$- left module in the function field of Z). Since $Q\mathcal{O}_{Z'} = H$, we get by using the (different) transformation laws the property $(**)$.

It is left to verify $\mathcal{D}' \subset \mathrm{Diff}_{\mathbb{Z}}^{\leq m}(Z')$. The commutator of two differential operators has order smaller than the sum of their orders. This implies

$$Diff_{\mathbb{Z}}^{\leq m}(Z) \cdot Q^b \subset Q \cdot \mathrm{Diff}_{\mathbb{Z}}^{\leq m}(Z) \cdot Q^{b-1} + \mathrm{Diff}_{\mathbb{Z}}^{\leq m-1}(Z) \cdot Q^{b-1}.$$

Hence we can make an induction on $m \geq 0$. The case $m = 0$ is clear and by the last formula and the induction hypothesis it suffices to prove

$$H^m \cdot \mathrm{Diff}_{\mathbb{Z}}^{\leq m}(Z) \subset \mathrm{Diff}_{\mathbb{Z}}^{\leq m}(Z')$$

(note that $Q\mathcal{O}_{Z'} = H$). This is a local question at every closed point $x' \in Z'$ with $x = \pi(x') \in D$. Let $(u) = (u_1, \ldots u_n)$ be a regular system of parameters for $(R = \mathcal{O}_{Z,x}, \mathfrak{m}, K)$ such that $Q_x = \langle u_1, \ldots, u_d \rangle_R$ and $(u') = (u'_1, \ldots u'_n) = \left(u_1, \frac{u_2}{u_1} \ldots, \frac{u_d}{u_1}, u_{d+1}, \ldots, u_n\right)$ is a regular system of parameters for $R' = \mathcal{O}_{Z',x'}$.

Denote by $\mathcal{D}_{M,u}$, $M \in \mathbb{Z}_0^n$, the differential operator on the $\mathfrak{m}$-adic completion $\widehat{R}$ of R which is defined by $\mathcal{D}_{M,u}(Cu^A) = \binom{A}{M} Cu^{A-M}$ for $C \in K$ and $A \in \mathbb{Z}_0^n$. Then it suffices to show that $\mathcal{D}_{M,u}$ is a linear combination of $\mathcal{D}_{N,u'}$, $N \in \mathbb{Z}_0^n$ and $|N| \leq |M|$, with coefficients in $K[u']$, for every $M \in \mathbb{Z}_0^n$. By an easy computation one sees

$$\frac{\partial}{\partial u_1} = \frac{\partial}{\partial u'_1} - \frac{1}{u'_1} \sum_{i=2}^{d} u'_i \frac{\partial}{\partial u'_i},$$

$$\frac{\partial}{\partial u_i} = \frac{1}{u'_1} \frac{\partial}{\partial u'_i}, \quad \text{for } 2 \leq i \leq d, \quad \text{and} \quad \frac{\partial}{\partial u_j} = \frac{\partial}{\partial u'_j}, \quad \text{for } d+1 \leq j \leq n.$$

Further in characteristic zero we have for $M = (M_1, \ldots, M_e)$

$$\mathcal{D}_{M,u} = \frac{1}{M_1! \cdot M_2! \cdots M_e!} \left(\frac{\partial}{\partial u}\right)^M. \qquad (***)$$

In the case of characteristic $p > 0$ the differential operator $\mathcal{D}_{M,u}$ is formally given by $(***)$. For $f \in R$ we obtain $\mathcal{D}_{M,u}(f)$ by first applying $\mathcal{D}_{M,u}$ formally on f and then considering the residue class of this formal $\mathcal{D}_{M,u}(f)$ modulo p. □

Let $(f_1, \dots, f_m)$ denote a set of generators of the ideal J and let $\mathcal{D}$ be as before. Instead of $\mathcal{D}J$ we want to apply the Diff Theorem for the ideal generated by $(\mathcal{D}f_1, \dots, \mathcal{D}f_m)$. In general, these two ideals do not coincide. We frequently use the Diff Theorem in the following slightly modified version:

Corollary 1.1.14. *Let* $\mathbb{E} = (J, b)$ *on* Z *and* $\mathcal{D} \subset \mathrm{Diff}_{\mathbb{Z}}^{\leq m}(Z)$ *be as in the previous theorem. Further let* $(f_1, \dots, f_m)$ *be a set of generators of the ideal* J*. Then*

$$(J, b) \subset (\, \langle \mathcal{D}f_1, \dots, \mathcal{D}f_m \rangle,\, b - m\,)$$

or equivalently $(J, b) \sim (\, \langle \mathcal{D}f_1, \dots, \mathcal{D}f_m \rangle,\, b - m\,) \cap (J, b)$*.*

Proof. By the Diff Theorem, Proposition 1.1.13, we have $(J, b) \subset (\mathcal{D}J, b - m)$. Clearly, $\langle \mathcal{D}f_1, \dots, \mathcal{D}f_m \rangle \subseteq \mathcal{D}J$. Hence every local sequence of regular blow ups which is permissible for $(\mathcal{D}J, b - m)$ is so for $(\, \langle \mathcal{D}f_1, \dots, \mathcal{D}f_m \rangle,\, b - m\,)$.

Alternatively: $\langle \mathcal{D}f_1, \dots, \mathcal{D}f_m \rangle \subseteq \mathcal{D}J$ implies together with Lemma 1.1.8 *(ii)*

$$\begin{aligned} &(\mathcal{D}J, b - m) = (\mathcal{D}J + \langle \mathcal{D}f_1, \dots, \mathcal{D}f_m \rangle, b - m) \sim \\ &\sim (\mathcal{D}J, b - m) \cap (\, \langle \mathcal{D}f_1, \dots, \mathcal{D}f_m \rangle,\, b - m\,) \subset (\, \langle \mathcal{D}f_1, \dots, \mathcal{D}f_m \rangle,\, b - m\,). \end{aligned}$$

This shows the assertion. □

Since this is an immediate consequence of the Diff Theorem, we do not distinguish between the corollary and the proposition. If we use them, then we refer only to the Diff Theorem, Proposition 1.1.13.

1.2 Tangent cone, directrix and ridge

Let $x \in Z$ be an arbitrary point and let $R = \mathcal{O}_{Z,x}$ be the regular local ring with maximal ideal $\mathfrak{m}$ and residue field $K = R/\mathfrak{m}$. Therefore we can associate the tangent space of Z at x

$$T_x(Z) := \operatorname{Spec}(gr_x(Z)),$$

where $gr_x(Z) = \bigoplus_{a\geq 0} \mathfrak{m}^a/\mathfrak{m}^{a+1}$.

Let further $\mathbb{E} = (J, b)$ be an idealistic exponent on Z. By abuse of notation we neglect in $\mathbb{E}_x = (J_x, b)$ the index x and write also $\mathbb{E} = (J, b)$. In the following we introduce the tangent cone, the directrix and the ridge of $\mathbb{E}$ at x and we discuss the aspect of their uniqueness up to equivalence. In order to get the last point we give an interpretation of these objects as idealistic exponents.

Recall that $\operatorname{ord}_x(J) = \sup\{d \in \mathbb{Z}_0 \cup \{\infty\} \mid J \subseteq \mathfrak{m}^d\}$ is the order of J at x. For an element $f \in R$ we define the order at x as the order of the principal ideal generated by f or equivalently $\operatorname{ord}_x(f) = \sup\{d \in \mathbb{Z}_0 \cup \{\infty\} \mid f \in \mathfrak{m}^d\}$.

Definition 1.2.1. *Let $f \in R$ and $b \in \mathbb{Q}_+$ with $b \leq \operatorname{ord}_x(f)$. We define the b-initial form of f (with respect to $\mathfrak{m}$) as*

$$in(f, b) := \begin{cases} f \mod \mathfrak{m}^{b+1}, & \text{if } b \in \mathbb{Z}_+, \\ 0, & \text{if } b \notin \mathbb{Z}_+. \end{cases}$$

Note that $b < \operatorname{ord}_x(f)$ implies $in(f, b) = 0$.

Definition 1.2.2. *Let $\mathbb{E} = (J, b)$ be an idealistic exponent on Z and $x \in \operatorname{Sing}(\mathbb{E})$. Then we define the tangent cone $T_x(\mathbb{E})$ of $\mathbb{E}$ at x in the vector space $T_x(Z)$ generated by the homogeneous ideal $In_x(J, b) \subset gr_x(Z)$, where*

$$In_x(J, b) := In_x(\mathbb{E}) := \begin{cases} \langle J \mod \mathfrak{m}^{b+1}\rangle = \langle in(f, b) \mid f \in J\rangle, & \text{if } b \in \mathbb{Z}_+, \\ \langle 0\rangle, & \text{if } b \notin \mathbb{Z}_+. \end{cases}$$

Let $\mathbb{E}_1 = (J_1, b_1), \mathbb{E}_2 = (J_2, b_2)$ be two idealistic exponent on Z. Then we set

$$In_x(\mathbb{E}_1 \cap \mathbb{E}_2) = In_x(\mathbb{E}_1) + In_x(\mathbb{E}_2)$$

Remark 1.2.3. *(1) By the assumption $x \in \operatorname{Sing}(\mathbb{E})$ we know $\operatorname{ord}_x(J) \geq b$ and hence $In_x(J, b) \subset gr_x(Z)$ is well-defined and generated by homogeneous element of degree b.*

(2) The tangent cone $T_x(\mathbb{E})$ is not *invariant under the equivalence relation $\sim$. An easy example is $\mathbb{E}_1 = (y, 1) \sim (y^2, 2) = \mathbb{E}_2$ over any field K. The ideals of the corresponding cones are $In_x(\mathbb{E}_1) = \langle Y\rangle$ and $In_x(\mathbb{E}_2) = \langle Y^2\rangle \subset K[Y]$,*

where Y denotes the image of y in $\mathfrak{m}/\mathfrak{m}^2$. Another example is given next. We overcome this later by using an idealistic interpretation of the tangent cone. Then we can show that all tangent cones are equivalent.

(3) *If $\mathbb{E}_2$ is an idealistic exponent with $\mathrm{ord}_x(\mathbb{E}_2) > 1$, then $In_x(\mathbb{E}_2) = \langle 0 \rangle$ and*

$$T_x(\mathbb{E}_1 \cap \mathbb{E}_2) = T_x(\mathbb{E}_1).$$

Example 1.2.4. Consider the idealistic exponents

$$(x^2y + z^3, 3) \sim (x^2, 2) \cap (x^2y + z^3, 3) \sim (\,\langle x^6, (x^2y + z^3)^2\rangle, 6\,)$$

over any field K. The first equivalence follows by applying the Diff Theorem 1.1.13 for the differential operator $\dfrac{\partial}{\partial y}$ and the second by Lemma 1.1.8*(ii)*. The ideals generating the tangent cones are $\langle X^2Y + Z^3\rangle$ resp. $\langle X^2, X^2Y + Z^3\rangle = \langle X^2, Z^3\rangle$ resp. $\langle X^6, (X^2Y + Z^3)^2\rangle$, where the capital letters denote the images in $\mathfrak{m}/\mathfrak{m}^2$ of the corresponding small letters.

Let us for the moment consider a more general situation: Let K be a field, consider the polynomial ring $S = K[U] = K[U_1, \ldots, U_n]$ as a graded ring and let $I \subset S$ be a homogeneous ideal. Then I defines a cone $C = \mathrm{Spec}\,(S/I)$. The directrix and the ridge are now defined as follows.

Definition 1.2.5 (Hironaka)**.** *The directrix $\mathrm{Dir}(C)$ of the cone C is the smallest K-subvectorspace $T = \bigoplus_{j=1}^{r} KY_j \subset S_1 = \bigoplus_{i=1}^{n} KU_i$ generated by elements $Y_1, \ldots Y_r \in S_1$ (homogeneous of degree one) such that*

$$(\,I \cap K[Y_1, \ldots, Y_r]\,)\,S = I.$$

Hence $T = \bigoplus_{j=1}^{r} KY_j$ is the minimal K-subspace such that I is generated by elements in $K[Y_1, \ldots, Y_r]$. (i.e. $(Y_1, \ldots, Y_r)$ is the smallest list of variables to describe the generators of I).

We also say $(Y) = (Y_1, \ldots, Y_r)$ defines the directrix and we implicitly assume that r is minimal. By abuse of notation we denote the vector space in $\mathbb{A}_k^n = \mathrm{Spec}\,(S)$ corresponding to $\mathrm{Dir}(C)$ also by $\mathrm{Dir}(C)$. Further we call $I\mathrm{Dir}(C) := \langle Y_1, \ldots Y_r\rangle$ the ideal of the directrix.

Recall that a polynomial $\varphi \in K[U] = S$ is called additive if for any $x, y \in K^n$ we have $\varphi(x + y) = \varphi(x) + \varphi(y)$.

Definition 1.2.6 ([G1], Ch. I, §5)**.** *The ridge (or faîte in French) $\mathrm{Rid}(C)$ of the cone C is the smallest additive subspace $K[\varphi_1, \ldots, \varphi_l] \subset S$ generated by additive homogeneous polynomials $\varphi_1, \ldots, \varphi_l \in S$ such that*

$$(\,I \cap K[\varphi_1, \ldots, \varphi_l]\,)\,S = I.$$

As above we say $(\varphi_1, \dots, \varphi_l)$ defines the ridge, identify $\mathrm{Rid}(C)$ with the group subscheme which it defines in $\mathbb{A}_K^N$ and we call $I\mathrm{Rid}(C) := \langle \varphi_1, \dots, \varphi_l \rangle$ the ideal of the ridge.

Remark 1.2.7. *In the case of $char(K) = 0$ the additive polynomials are those homogeneous of degree one. Thus the previous definitions coincide in this situation,* $\mathrm{Dir}(C) = \mathrm{Rid}(C)$*.*

If $p = char(K) > 0$ is positive, then the additive homogeneous polynomials are of the form $\varphi = \sum_{i=1}^n \lambda_i U_i^q$, $\lambda \in K$ and $q = p^e$, $e \in \mathbb{Z}_0$. If moreover K is perfect, then $\varphi = \psi^q$ for some $\psi \in K[U]$ homogeneous of degree one. Hence the directrix is the reduction of the ridge, $\mathrm{Dir}(C) = (\mathrm{Rid}(C))_{red}$*, if K is perfect.*

For arbitrary K and $\lambda \in K$, we do not know if there is an element $\rho \in K$ such that $\rho^q = \lambda$, $q = p^e$ as before. But there is a purely inseparable finite extension $K(\lambda)/K$ such that this property holds in $K(\lambda)$; e.g. $K(\lambda) = K[X]/\langle X^q - \lambda \rangle$. Since $\left\{ \lambda_i^{(j)} \in K \mid \varphi_j = \sum_{i=1}^n \lambda_i^{(j)} U_i^{q_j},\ q_j = p^{e_j}, e_j \in \mathbb{Z}_0,\ j \in \{1, \dots, l\} \right\}$ is a finite set, there exists a purely inseparable finite extension K'/K such that $\mathrm{Dir}(C)_{K'} = (\mathrm{Rid}(C)_{K'})_{red}$*, where $(.)_{K'} = (.) \times_K K'$.*

Remark 1.2.8. *The above definition of the ridge is only the explicit version. This is enough for our purposes, but for completeness we briefly sketch the formal definition: (For more details on this see [G1] and [BHM]). Let us consider*

$$\mathbb{A}_K^n : (Sch/K) \to (Sets), \qquad X \mapsto \mathbb{A}_K^n(X) = Hom(X, \mathbb{A}_K^n)$$

as its functor of points from the category of schemes over K to the category of sets. Then we define the sub-functor $\mathfrak{F} : (Sch/K) \to (Sets)$ by

$$\mathfrak{F}(X) = \{ v \in \mathbb{A}^n(X) \mid \forall c \in C(X) : v + c \in C(X) \},$$

where $X \in ob(Sch/K)$. One can show that $\mathfrak{F}(X)$ is a group scheme and $\mathfrak{F}$ is representable by a subscheme F of C. (For details see [BHM], section 2.1). Then the ridge of the cone C is defined to be the scheme F which represents $\mathfrak{F}$.

One can prove that the naive and the formal definition of the ridge coincide (see [G1], Ch. I, §5, or [BHM], section 2.2) and further Berthomieu, Hivert and Mourtada give in [BHM] an effective algorithm to compute the additive generators of the ridge.

Coming back to our situation ($S = gr_x(Z)$, $C = T_x(\mathbb{E}) = \mathrm{Spec}\,(S/In_x(\mathbb{E}))$), we have

Definition 1.2.9. *Let $\mathbb{E} = (J, b)$ be an idealistic exponent on Z. Then we define*

(1) the directrix of $\mathbb{E}$ at x by

$$\mathrm{Dir}_x(\mathbb{E}) := \mathrm{Dir}_x(T_x(\mathbb{E}))$$

(resp. $\mathrm{Dir}_x((J, b)) := \mathrm{Dir}_x(T_x((J, b)))$*),*

(2) and the ridge of $\mathbb{E}$ *at* x *by*

$$\mathrm{Rid}_x(\mathbb{E}) := \mathrm{Rid}_x(T_x(\mathbb{E}))$$

(resp. $\mathrm{Rid}_x((J,b)) := \mathrm{Rid}_x(T_x((J,b)))$*).*

Let $\mathbb{E}_1 = (J_1, b_1)$ *and* $\mathbb{E}_2 = (J_2, b_2)$ *be two idealistic exponent on* Z*. Then we set*

$$\mathrm{Dir}_x(\mathbb{E}_1 \cap \mathbb{E}_2) = \mathrm{Dir}_x(\mathbb{E}_1) \cap \mathrm{Dir}_x(\mathbb{E}_2) \text{ and } \mathrm{Rid}_x(\mathbb{E}_1 \cap \mathbb{E}_2) = \mathrm{Rid}_x(\mathbb{E}_1) \cap \mathrm{Rid}_x(\mathbb{E}_2)$$

If $\mathbb{E}_2$ is an idealistic exponent with $\mathrm{ord}_x(\mathbb{E}_2) > 1$, then $\mathrm{Dir}_x(\mathbb{E}_1 \cap \mathbb{E}_2) = \mathrm{Dir}_x(\mathbb{E}_1)$ and $\mathrm{Rid}_x(\mathbb{E}_1 \cap \mathbb{E}_2) = \mathrm{Rid}_x(\mathbb{E}_1)$, because $T_x(\mathbb{E}_2) = T_x(Z)$.

As we mentioned before, the tangent cones of equivalent idealistic exponents are not equal in general. Hence it is not clear if there is a relation between the corresponding directrices and ridges. First of all, $(y, 1) \sim (y^p, p)$, $p = char(K)$, yield two different ridges, namely the ridge of the first is given by (Y) and the second by (Y^p). We will overcome this by introducing new definitions for the tangent cone, the directrix and the ridge. More precisely, we develop idealistic versions of them. Then we can show that the idealistic tangent cones of equivalent idealistic exponents are equivalent and this yields the equivalence of the idealistic ridges — for both result see Proposition 1.2.19.

For the directrix we find already in [H3], Proposition 19.2, p.60, the following result (if the base field k is perfect).

Proposition 1.2.10. *If* $\mathbb{E}_1 \sim \mathbb{E}_2$*, then* $\mathrm{Dir}_x(\mathbb{E}_1) = \mathrm{Dir}_x(\mathbb{E}_2)$*. Hence the directrix* $\mathrm{Dir}_x(\mathbb{E})$ *and thus its dimension are uniquely determined by* x *and the equivalence class of* $\mathbb{E}$*.*

Instead of recalling Hironaka's proof, we give later an alternative one with the help of an idealistic interpretation, see Proposition 1.2.19.

Remark 1.2.11. *Since there is no such result for the ridge, Hironaka introduces the following two objects (see [H3], p.60 and p.107), which are forced to be uniquely determined by* x *and the equivalence class of* $\mathbb{E}$*. The tangent additive group scheme of* $\mathbb{E}$ *at* x *is defined by*

$$A_x(\mathbb{E}) := \bigcap_{\mathbb{D} \sim \mathbb{E}} \mathrm{Rid}_x(\mathbb{D})$$

and further he considers also the following additive group subscheme of $T_x(Z)$

$$B_x(\mathbb{E}) := \bigcap_{\mathbb{D} \supset \mathbb{E}} \mathrm{Rid}_x(\mathbb{D}) = \bigcap_{\mathbb{D} \supset \mathbb{E}} A_x(\mathbb{D}).$$

The latter plays an important role in the definition of so called Tschirnhausen idealistic exponents and the Tschirnhausen decomposition of an idealistic exponent, *which is a certain kind of generalization of Abhyankar's Tschirnhausen transformation.*

We don't need this for our approach. Therefore we don't recall these notions in detail and refer to [H3] beginning with Definition 2, p.106 ff.

We now come to the *idealistic interpretation* of the tangent cone $T_x(\mathbb{E})$, the directrix $\mathrm{Dir}_x(\mathbb{E})$ and the ridge $\mathrm{Rid}_x(\mathbb{E})$ of $\mathbb{E}$ at $x \in \mathrm{Sing}\,\mathbb{E}$.

Observation 1.2.12. Before we start, we want to point out the following:

(1) Consider an idealistic exponent $\mathbb{E} = (J, b)$ on $\mathbb{A}_k^n = \mathrm{Spec}\,(R)$, $R = k[u] = k[u_1, \dots, u_n]$. By Lemma 1.1.8 *(i)* $\mathbb{E}$ is equivalent to $\mathbb{E}^a := (J^a, ab)$ for all $a \in \mathbb{Z}_+$. Let $x \in \mathbb{A}_k^n$ be the origin and suppose $x \in \mathrm{Sing}\,(\mathbb{E}) = \mathrm{Sing}\,(\mathbb{E}^a)$. We consider the situation locally at x. Then $In_x(\mathbb{E}) = \langle in(f, b) \mid f \in J \rangle$ and $gr_x(\mathbb{A}_k^n) \cong k[U] = k[U_1, \dots, U_n]$, where U_i denotes the image of u_i in $\mathfrak{m}/\mathfrak{m}^2$ ($\mathfrak{m} = \langle u_1, \dots, u_n \rangle$). We want to show the following equality of ideals in $gr_x(\mathbb{A}_k^n)$:

$$In_x(\mathbb{E}^a) = (In_x(\mathbb{E}))^a.$$

Clearly, $in(f + g, b) = in(f, b) + in(g, b)$ for $f, g \in J$. Consider an element $g \in J^a$ which is of the form $g = g_1 \cdots g_a$ for $g_1, \dots, g_a \in J$. Since $x \in \mathrm{Sing}\,(\mathbb{E})$ the initials $in(g_i, b)$ are either zero or homogeneous of degree b ($\mathrm{ord}_x(g_i) \geq b$) for all $i \in \{1, \dots, a\}$. Thus $in(g, ab) = \prod_{i=1}^{a} in(g_i, b)$ and we get the desired equality $In_x(\mathbb{E}^a) = (In_x(\mathbb{E}))^a$. If we put $\mathbb{T}_x(\mathbb{E}) := (In_x(\mathbb{E}), b)$ and $\mathbb{T}_x(\mathbb{E}^a) := (In_x(\mathbb{E}^a), ab)$, then the last equation implies that these are equivalent idealistic exponents on $T_x(\mathbb{A}_k^n) = \mathrm{Spec}\,(gr_x(\mathbb{A}_k^n))$.

Let $I\mathrm{Dir}_x(\mathbb{E}) = \langle Y_1, \dots, Y_r \rangle$ be the ideal of the directrix with elements $Y_j \in k[U], 1 \leq j \leq r$, which are homogeneous of degree one. By definition of the directrix, the generators of $In_x(\mathbb{E})$ are contained in $k[Y_1, \dots, Y_r]$ and (Y) is minimal with this condition. This implies that the generators of $In_x(\mathbb{E}^a)$ are contained in $k[Y_1, \dots, Y_r]$ and (Y) is also minimal: Suppose this is wrong, say they are contained in $k[Z_1, \dots, Z_s]$ for some $s < r$. Then the same is true for the generators of $In_x(\mathbb{E})$ which is a contradiction. This shows

$$\mathrm{Dir}_x(\mathbb{E}) = \mathrm{Dir}_x(\mathbb{E}^a).$$

In particular, $\mathbb{D}\mathrm{ir}_x(\mathbb{E}) := (I\mathrm{Dir}_x(\mathbb{E}), 1)$ and $\mathbb{D}\mathrm{ir}_x(\mathbb{E}^a) := (I\mathrm{Dir}_x(\mathbb{E}^a), 1)$ are equivalent idealistic exponents on $T_x(\mathbb{A}_k^n)$.

Now let $I\mathrm{Rid}_x(\mathbb{E}) = \langle \varphi_1, \ldots, \varphi_l \rangle$ be the ideal of the ridge with additive homogeneous polynomials $\varphi_i \in k[Y_1, \ldots, Y_r] \subset k[U], 1 \leq i \leq l$. Since φ_i is additive, the order is some p-power, say p^{d_i} for some $d_i \in \mathbb{Z}_0$ ($p = char(k)$). Let p^c ($c \in \mathbb{Z}_0$) be the maximal p-power dividing a. Then

$$I\mathrm{Rid}_x(\mathbb{E}^a) = \langle \varphi_1^{p^c}, \ldots, \varphi_l^{p^c} \rangle.$$

Hence $\mathbb{R}\mathrm{id}_x(\mathbb{E}) = \bigcap_{i=1}^{l} (\varphi_i, p^{d_i})$ and $\mathbb{R}\mathrm{id}_x(\mathbb{E}^a) = \bigcap_{i=1}^{l} (\varphi_i^{p^c}, p^{d_i+c})$ are equivalent idealistic exponents on $T_x(\mathbb{A}_k^n)$.

(2) Let $\mathbb{E}_1 = (J_1, b)$ and $\mathbb{E}_2 = (J_2, b)$ be two idealistic exponents on Z and $x \in \mathrm{Sing}\,(\mathbb{E}_1 \cap \mathbb{E}_2)$. By definition $In_x(\mathbb{E}_1 \cap \mathbb{E}_2) = In_x(\mathbb{E}_1) + In_x(\mathbb{E}_2)$ and this is equal to

$$In_x(J_1, b) + In_x(J_2, b) = In_x(J_1 + J_2, b).$$

Hence $T_x((J_1, b) \cap (J_2, b)) = T_x(J_1 + J_2, b)$ and this implies the equality of the corresponding directrices and ridges.

This observation gives the hint that the tangent cone (resp. the ridge) of equivalent idealistic exponents might be related if we use an idealistic interpretation. Hence we introduce the following completely new definitions of the tangent cone, the directrix and the ridge as idealistic exponents. To the author there is no reference know, where this idea already appeared.

Definition 1.2.13. *Let $\mathbb{E} = (J, b)$ be an idealistic exponent on Z and $x \in \mathrm{Sing}\,(\mathbb{E})$. Recall that K denotes the residue field of Z at x and $p = char(K) \geq 0$. Let further $I\mathrm{Dir}_x(\mathbb{E}) = \langle Y_1, \ldots, Y_r \rangle$ and $I\mathrm{Rid}_x(\mathbb{E}) = \langle \varphi_1, \ldots, \varphi_l \rangle$ for elements Y_j homogeneous of degree one, $1 \leq j \leq r$, and additive homogeneous polynomials φ_i of order p^{d_i}, $1 \leq i \leq l$. Then we define the following idealistic exponents on $T_x(Z) = \mathrm{Spec}\,(gr_x(Z))$:*

$$\begin{aligned}
&\mathbb{T}_x(\mathbb{E}) = (\, In_x(\mathbb{E}),\, b\,) && \textit{(idealistic tangent cone of } \mathbb{E} \textit{ at } x\textit{),}\\
&\mathbb{D}\mathrm{ir}_x(\mathbb{E}) = (\, I\mathrm{Dir}_x(\mathbb{E}),\, 1\,) && \textit{(idealistic directrix of } \mathbb{E} \textit{ at } x\textit{),}\\
&\mathbb{R}\mathrm{id}_x(\mathbb{E}) = \textstyle\bigcap_{i=1}^{l} (\, \varphi_i,\, p^{d_i}\,) && \textit{(idealistic ridge of } \mathbb{E} \textit{ at } x\textit{) .}
\end{aligned}$$

If we have two idealistic exponent on Z, say $\mathbb{E}_1 = (J_1, b_1)$ and $\mathbb{E}_2 = (J_2, b_2)$, and $x \in \mathrm{Sing}\,(\mathbb{E}_1 \cap \mathbb{E}_2)$, then we set

$$\begin{aligned}
&\mathbb{T}_x(\mathbb{E}_1 \cap \mathbb{E}_2) = \mathbb{T}_x(\mathbb{E}_1) \cap \mathbb{T}_x(\mathbb{E}_2) = (\, In_x(\mathbb{E}_1),\, b_1\,) \cap (\, In_x(\mathbb{E}_2),\, b_2\,),\\
&\mathbb{D}\mathrm{ir}_x(\mathbb{E}_1 \cap \mathbb{E}_2) = \mathbb{D}\mathrm{ir}_x(\mathbb{E}_1) \cap \mathbb{D}\mathrm{ir}_x(\mathbb{E}_2) = (\, I\mathrm{Dir}_x(\mathbb{E}) + I\mathrm{Dir}_x(\mathbb{E}),\, 1\,),\\
&\mathbb{R}\mathrm{id}_x(\mathbb{E}_1 \cap \mathbb{E}_2) = \mathbb{R}\mathrm{id}_x(\mathbb{E}_1) \cap \mathbb{R}\mathrm{id}_x(\mathbb{E}_2).
\end{aligned}$$

Remark 1.2.14. *(1) Note that in the case $p = char(K) = 0$ the generators of the ridge are homogeneous of degree $p^{d_i} = 1$. This means $d_i = 0$ for all i. Alternatively one could use in the definition of $\mathbb{R}\mathrm{id}_x(\mathbb{E})$ the characteristic exponent of K instead of the characteristic of K. Recall that the characteristic exponent of K is defined to be 1 if $char(K) = 0$ and it is p if $char(K) = p > 0$.*

(2) If $\mathbb{E}_2$ is an idealistic exponent on Z with $\mathrm{ord}_x(\mathbb{E}_2) > 1$, then we have $\mathbb{T}_x(\mathbb{E}_2) = (\langle 0\rangle, b_2)$ and hence $\mathbb{T}_x(\mathbb{E}_1 \cap \mathbb{E}_2) = \mathbb{T}_x(\mathbb{E}_1)$ and the same results for the idealistic directrix and ridge.

(3) By Observation 1.2.12 we have

$$\mathbb{T}_x(J,b) \sim \mathbb{T}_x(J^a, ab),\ \mathbb{D}\mathrm{ir}_x(J,b) = \mathbb{D}\mathrm{ir}_x(J^a, ab),\ \mathbb{R}\mathrm{id}_x(J,b) \sim \mathbb{R}\mathrm{id}_x(J^a, ab)$$

for an arbitrary idealistic exponent and a positive integer $a \in \mathbb{Z}_+$.

Further we have seen in the observation that for two idealistic exponents with the same assigned number $\mathbb{T}_x(J_1, b) \cap \mathbb{T}_x(J_2, b) = \mathbb{T}_x(J_1 + J_2, b)$, which implies the equalities of the corresponding idealistic directrices and ridges.

Before we come to the properties of $\mathbb{T}_x(\mathbb{E})$, $\mathbb{D}\mathrm{ir}_x(\mathbb{E})$ and $\mathbb{R}\mathrm{id}_x(\mathbb{E})$, let us have another look at Example 1.2.4, where the tangent cone differed for equivalent idealistic exponents

Example 1.2.15. Consider the idealistic exponents $\mathbb{E}_1 = (x^2y + z^3, 3), \mathbb{E}_2 = (x^2, 2) \cap (x^2y + z^3, 3)$ and $\mathbb{E}_3 = (\langle x^6, (x^2y + z^3)^2\rangle, 6)$ on $\mathbb{A}^3_K = \operatorname{Spec} K[x, y, z]$ for any field K. As we already pointed out before $\mathbb{E}_1 \sim \mathbb{E}_2 \sim \mathbb{E}_3$ and the ideals of the tangent cone (at the origin x) are given by

$$\begin{aligned} In_x(\mathbb{E}_1) &= \langle X^2Y + Z^3\rangle, \\ In_x(\mathbb{E}_2) &= \langle X^2, X^2Y + Z^3\rangle \text{ and} \\ In_x(\mathbb{E}_3) &= \langle X^6, (X^2Y + Z^3)^2\rangle, \end{aligned}$$

where the capital letters denote the images in $\mathfrak{m}/\mathfrak{m}^2$ of the the corresponding small letters ($\mathfrak{m} = \langle x, y, z\rangle$). In the idealistic version of the tangent cone, we have to take care of the degree of each generator, hence

$$\begin{aligned} \mathbb{T}_x(\mathbb{E}_1) &= (X^2Y + Z^3, 3), \\ \mathbb{T}_x(\mathbb{E}_2) &= (X^2, 2) \cap (X^2Y + Z^3, 3) \quad \text{and} \\ \mathbb{T}_x(\mathbb{E}_3) &= (X^6, 6) \cap (X^2Y + Z^3)^2, 6). \end{aligned}$$

It is clear that these are equivalent idealistic exponents on $T_x(\mathbb{A}^3_K)$.

Lemma 1.2.16. *Let $\mathbb{E} = (J, b)$ be an idealistic exponent on Z and $x \in \mathrm{Sing}(\mathbb{E})$. Then we have*

(i) $\mathbb{D}\mathrm{ir}_x(\mathbb{E}) \subset \mathbb{R}\mathrm{id}_x(\mathbb{E}) \subset \mathbb{T}_x(\mathbb{E})$.

(ii) $\mathrm{Dir}_x(\mathbb{E}) = \mathrm{Sing}(\mathbb{D}\mathrm{ir}_x(\mathbb{E})) \subseteq \mathrm{Sing}(\mathbb{R}\mathrm{id}_x(\mathbb{E})) \subseteq \mathrm{Sing}(\mathbb{T}_x(\mathbb{E})) \subseteq T_x(Z)$.

Proof. Let $(Y) = (Y_1, \dots, Y_r)$ be the elements (homogeneous of degree one) which determine $\mathrm{Dir}_x(\mathbb{E})$ and extend these by $(U) = (U_1, \dots, U_e)$ such that $gr_x(Z) = K[U, Y]$. Further let $(\varphi) = (\varphi_1, \dots, \varphi_l)$ be the additive homogeneous polynomials which yield $\mathrm{Rid}_x(\mathbb{E})$.

Since the generators of all three idealistic exponents are homogeneous, the extension of the base by $K[T_1, \dots, T_a]$ doesn't change the situation. Hence, we may consider the idealistic exponents on $K[U, Y]$. Since by definition the generators of $In_x(\mathbb{E})$ are contained in $K[\varphi] \subset K[Y]$, any center which is permissible for $\mathbb{D}\mathrm{ir}_x(\mathbb{E})$ (resp. $\mathbb{R}\mathrm{id}_x(\mathbb{E})$) is so for $\mathbb{R}\mathrm{id}_x(\mathbb{E})$ (resp. $\mathbb{T}_x(\mathbb{E})$). After blowing up either $\mathbb{D}\mathrm{ir}_x(\mathbb{E})$ (resp. $\mathbb{R}\mathrm{id}_x(\mathbb{E})$) is resolved or the situation is still the same. This shows *(i)*.

The first equality and the last inclusion of *(ii)* follow by definition and *(i)* implies the rest via Corollary 1.1.11. □

In characteristic zero or if the characteristic $p > 0$ is greater than b, we have the following

Corollary 1.2.17. *Let $\mathbb{E} = (J, b)$ be an idealistic exponent on Z and $x \in \mathrm{Sing}(\mathbb{E})$. Assume $char(K) = 0$ or $b < char(K)$, where K denotes the residue field of Z at x. Then*

$$\mathbb{D}\mathrm{ir}_x(\mathbb{E}) \sim \mathbb{R}\mathrm{id}_x(\mathbb{E}) \sim \mathbb{T}_x(\mathbb{E}).$$

In particular $\mathrm{Dir}_x(\mathbb{E}) = \mathrm{Sing}(\mathbb{D}\mathrm{ir}_x(\mathbb{E})) = \mathrm{Sing}(\mathbb{R}\mathrm{id}_x(\mathbb{E})) = \mathrm{Sing}(\mathbb{T}_x(\mathbb{E}))$.

Proof. By Lemma 1.2.16 we only have to show $\mathbb{T}_x(\mathbb{E}) \subset \mathbb{D}\mathrm{ir}_x(\mathbb{E})$. Let $(R = \mathcal{O}_{Z,x}, \mathfrak{m}, K)$ be the local ring of Z at x and let $(u, y) = (u_1, \dots, u_e, y_1, \dots, y_r)$ be a regular system of parameters for R such that $I\mathrm{Dir}_x(\mathbb{E}) = \langle Y_1, \dots, Y_r \rangle$, where Y_j denotes the image of y_j in $\mathfrak{m}/\mathfrak{m}^2$. Then in particular

$$\mathbb{D}\mathrm{ir}_x(\mathbb{E}) = (\langle Y_1, \dots, Y_r \rangle, 1) \sim (Y_1, 1) \cap \ldots \cap (Y_r, 1).$$

Recall that $\mathbb{T}_x(\mathbb{E}) = (In_x(\mathbb{E}), b)$. By definition of the directrix, the generators of $In_x(\mathbb{E})$ are contained in $K[Y]$ and each Y_j appears to a non-zero power. Hence they lie in $\langle Y \rangle^b \setminus \langle Y \rangle^{b+1}$ and every generator $F \in In_x(\mathbb{E})$ can be written as

$$F = \sum_{\substack{B \in \mathbb{Z}_0^r \\ |B| = b}} C_B \, Y^B,$$

for some $C_B \in K$. Further for every $1 \leq j \leq r$ there exists a generator $F(j) \in In_x(\mathbb{E})$ such that there is a $B(j) = (B_1, \ldots, B_r) \in \mathbb{Z}_0^r$ with $C_{B(j)} \neq 0$ and $B_j \geq 1$. Therefore this is an element of $In_x(\mathbb{E})$, where Y_j appears. Set $M(j) := B(j) - e_j \in \mathbb{Z}_0^r$. (Here $e_j \in \mathbb{Z}_0^r$ denotes the j-th unit vector with zero everywhere except the j-th place, there is a one). Note that $|M(j)| = b - 1$. Let $\mathcal{D}_{M(j)} \in \mathrm{Diff}_K^{\leq b-1}(K[Y])$ be the differential operator which is defined via

$$\mathcal{D}_{M(j)}(C\,Y^B) = \binom{B}{M(j)}\, C\,Y^{B-M(j)}.$$

for $C \in K$ and $B \in \mathbb{Z}_0^r$. In particular, $\mathcal{D}_{M(j)}(C\,Y^{B(j)}) = C\,\binom{B(j)}{M(j)}\,Y_j = C\,B_j\,Y_j$ and $\mathcal{D}_{M(j)}(C\,Y^B) = 0$ if $|B| = b$ and $B \notin M(j) + \mathbb{Z}_+^r$; consequently

$$\mathcal{D}_{M(j)}(F(j)) = C_{B(j)}\,B_j\,Y_j + \sum_{B'(i)} C_{B'(i)}\,B'_i\,Y_i,$$

where $B'(i) = (B'_1, \ldots, B'_r) \in \{M(j) + e_i \mid i \in \{1, \ldots, r\} \setminus \{j\}\}$. Since we have $1 \leq B_j \leq b$ and $char(K) = 0$ or $b < char(K)$, we get that B_j (and thus $C_{B(j)}\,B_j$) is a unit in K. We set

$$Y_j^* := (C_{B(j)}\,B_j)^{-1}\,\mathcal{D}_{M(j)}\,F(j) = Y_j + \sum_{B'(i)} (C_{B(j)}\,B_j)^{-1} C_{B'(i)}\,B'_i\,Y_i \in K[Y].$$

We choose in R a system of representatives of $K = R/\mathfrak{m}$ and define with this $y_1^* \in R$ by replacing (Y) by (y) in the Y_1^*. The system $(y_1^*, y_2, \ldots, y_r)$ fulfills the same properties as (y). So we may consider the regular system of parameters $(u; y_1^*, y_2, \ldots, y_r)$ instead of (u, y) and put $\mathcal{D}_1 := \mathcal{D}_{M(1)}$. (Note that $\mathcal{D}_1$ is defined in terms of (Y)). Then we repeat the above procedure to obtain y_2^* and $\mathcal{D}_2$. After that we determine y_3^* and $\mathcal{D}_3$ ... We continue until we have $(y^*) = (y_1^*, \ldots, y_r^*)$. Then the Diff Theorem 1.1.13 yields for all $j \in \{1, \ldots, r\}$ that $(F(j), b) \subset (\mathcal{D}_j F(j), 1) = (Y_j^*, 1)$. This implies

$$\mathbb{T}_x(\mathbb{E}) = (In_x(\mathbb{E}), b) \subset (Y_1^*, 1) \cap \ldots \cap (Y_r^*, 1) = \mathbb{D}\mathrm{ir}_x(\mathbb{E}).$$

□

Remark 1.2.18. *For the arbitrary case the equivalences need not hold. One point is in particular that B_j may be zero in K. Therefore the assumption $char(K) = 0$ or $b < char(K)$ is essential.*

If the residue field K is perfect, then we can also say something on the directrix and the ridge: $\mathbb{D}\mathrm{ir}_x(\mathbb{E}) = (Y_1, 1) \cap \ldots \cap (Y_r, 1) \sim (Y_1^{p^{d_1}}, p^{d_1}) \cap \ldots \cap (Y_r^{p^{d_r}}, p^{d_r}) = \mathbb{R}\mathrm{id}_x(\mathbb{E})$ for certain d_j.

We have seen that there is not necessarily a relation between the tangent cones $T_x(\mathbb{E})$ of equivalent idealistic exponents. For idealistic interpretations we have the following strong result.

Proposition 1.2.19. *Let $\mathbb{E}_1 = (J_1, b_1)$ and $\mathbb{E}_2 = (J_2, b_2)$ be two idealistic exponents on Z with $\mathbb{E}_1 \subset \mathbb{E}_2$ and $x \in \mathrm{Sing}\,(\mathbb{E}_1) \subseteq \mathrm{Sing}\,(\mathbb{E}_2)$. Then we have*

(i) $\mathbb{T}_x(\mathbb{E}_1) \subset \mathbb{T}_x(\mathbb{E}_2)$.

(ii) $\mathrm{Dir}_x(\mathbb{E}_1) \subseteq \mathrm{Dir}_x(\mathbb{E}_2)$ *and hence* $\mathbb{D}\mathrm{ir}_x(\mathbb{E}_1) \subset \mathbb{D}\mathrm{ir}_x(\mathbb{E}_2)$.

(iii) $\mathbb{R}\mathrm{id}_x(\mathbb{E}_1) \subset \mathbb{R}\mathrm{id}_x(\mathbb{E}_2)$.

By symmetry we get equivalence $\sim$ and equality if $\mathbb{E}_1 \sim \mathbb{E}_2$.

This implies Proposition 1.2.10 and further it yields that the idealistic version of the tangent cone, the directrix and the ridge are uniquely determined by x and the equivalence class of $\mathbb{E}$.

Proof. By Observation 1.2.12 we already have the result in the case $\mathbb{E} = (J, b) \sim (J^a, ab)$ for some $a \in \mathbb{Z}_+$. Hence it suffices to consider $b_1 = b_2 = b$. Recall $R = \mathcal{O}_{Z,x}$ with maximal ideal $\mathfrak{m}$ and residue field K. Let further $(u) = (u_1, \dots, u_n)$ be a regular system of parameters for R.

Let $\mathbb{E} = (J, b) \in \{(\mathbb{E}_1)_x, (\mathbb{E}_2)_x\}$. First we extend the base R to $R_0 = R \times_K K[t]$, where t is an independent indeterminate. Then (u, t) is a regular system of parameters for R_0. We use the notation $\mathbb{E}_0 = (J^{(0)} = J[t], b)$ and $V_0 = \mathrm{Spec}\,(R_0)$. Similar to the proof of the Numerical Exponent Theorem 1.1.10 we perform some auxiliary blow ups. Let $L_0 \subset V_0$ denote the line $V(u)$ and $x_0 \in L_0 \subset Z_0$ the origin $V(u, t)$. We now consider for $\alpha \in \mathbb{Z}_+$ the following local sequence of regular blow ups, which is permissible for $\mathbb{E}$ (since $x \in \mathrm{Sing}\,(\mathbb{E})$),

$$\begin{array}{ccccccccc}
L_0 & \cong & L_1 & \cong & \dots & \cong & L_{\alpha-1} & \cong & L_\alpha \\
\cap & & \cap & & & & \cap & & \cap \\
V_0 & \xleftarrow{\pi_1} & Z_1 \supset V_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_{\alpha-1}} & Z_{\alpha-1} \supset V_{\alpha-1} & \xleftarrow{\pi_\alpha} & Z_\alpha \supset V_\alpha \\
\Cup & & \Cup & & & & \Cup & & \Cup \\
x_0 & & x_1 & & \dots & & x_{\alpha-1} & & x_\alpha,
\end{array} \tag{1.4}$$

where $\pi_i : Z_i \to V_{i-1}$ is the blow up with center $x_{i-1} \in V_{i-1}$, $L_i \cong L_0$ is the strict transform of L_0, x_i denotes the unique intersection point of L_i with the exceptional divisor of the blow up π_i and $V_i = \mathrm{Spec}\,(R_i) \subset Z_i$ is the T-chart of the blow up, $i \in \{1, \dots, \alpha\}$. Since $L_0 = V(u)$, it is clear that x_i is the origin of V_i. Hence $x_i = V\left(\frac{u}{t^i}, t\right)$, $L_i = V\left(\frac{u}{t^i}\right)$ and $\left(\frac{u}{t^i}, t\right)$ is a regular system of parameters for R_i.

Let $f \in J^{(0)}$ be an arbitrary element. In the $\mathfrak{m}$-adic completion of R we have $f(u) = f_0(u) + h(u)$, where f_0 denotes the part homogeneous of degree b and $h(u) \in \langle u \rangle^{b+1}$. Let $d = d(f) = \mathrm{ord}_x(h)$, $d > b$. The transform of f in V_α is given by

$$f^{(\alpha)}\left(\frac{u}{t^\alpha}, t\right) = f_0^{(\alpha)}\left(\frac{u}{t^\alpha}\right) + t^{\alpha\cdot(d-b)} \cdot h_*, \qquad \text{for some } h_* \in \left\langle \frac{u}{t^\alpha} \right\rangle^{b+1} + \langle t \rangle. \quad (1.5)$$

Recall that $x_\alpha = V(\frac{u}{t^\alpha}, t)$. It is clear that the generators of the ideal of the tangent cone (and thus also its idealistic version) at x didn't change under the extension of the base and by the previous we see that the tangent cone at x_α is the same as the one before the permissible local sequence of regular blow ups (1.4); just replace in $In_x(J^{(0)}, b)$ the coordinates (u) by $(\frac{u}{t^\alpha})$ in order to get $In_{x_\alpha}(J^{(\alpha)}, b)$.

Hence given $\mathbb{E}_1 \subset \mathbb{E}_2$, then we can perform the above permissible local sequence of regular blow ups and get $\mathbb{E}_1^{(\alpha)} \subset \mathbb{E}_2^{(\alpha)}$ on V_α. Further every $f^{(\alpha)} \in J_1^{(\alpha)}$ and $g^{(\alpha)} \in J_2^{(\alpha)}$ is of the form (1.5). Now choose α so large that

$$\alpha \cdot (d(f) - b) \geq b \quad \text{and} \quad \alpha \cdot (d(g) - b) \geq b \qquad (*)$$

for every $f^{(\alpha)} \in J_1^{(\alpha)}$ and $g^{(\alpha)} \in J_2^{(\alpha)}$.

For simplicity let us drop the indices and assume from the very beginning that $\mathbb{E}_1 \subset \mathbb{E}_2$ on V_0 are of the special type described above. By the previous discussion this is justified. As usual capital letters (U, T) denote the images of (u, t) in $\langle u, t \rangle / \langle u, t \rangle^2$. We want to point out that by (1.5) the generators of $In_x(\mathbb{E}_1)$ and $In_x(\mathbb{E}_2)$ are contained in $K[U]$, because t doesn't appear in the part homogeneous of degree b. Hence we consider $\mathbb{T}_x(\mathbb{E}_1)$ and $\mathbb{T}_x(\mathbb{E}_2)$ as idealistic exponents on $\mathrm{Spec}\,(K[U])$.

Since the tangent cones are generated by homogeneous elements, an extension by some independent indeterminates $(t') = (t'_1, \ldots, t'_a)$ for some $a \in \mathbb{Z}_+$ doesn't affect the situation. So it suffices to consider the case without an extension of the base.

For *(i)* we first want to show

$$\mathbb{T}_x(\mathbb{E}_1) \subset \mathbb{T}_x(\mathbb{E}_2). \qquad (1.6)$$

Suppose this is wrong. Then there exists a local sequence of regular blow ups $(\diamondsuit)$ over $K[U]$ which is permissible for $\mathbb{T}_x(\mathbb{E}_1)$, but not for $\mathbb{T}_x(\mathbb{E}_2)$. By (1.5) $In_x(\mathbb{E}_1)$ is generated by the f_0 and $In_x(\mathbb{E}_2)$ by the g_0 (for $f \in J_1$ and $g \in J_2$). We can lift the centers of $(\diamondsuit)$ back to $K[u]$ (by using (u) instead of (U)) and we can further intersect them with $V(t)$. Because of the special form (1.5) we get by blowing up these modified centers a local sequence of regular blow ups $(\widetilde{\diamondsuit})$ over V_0, which is permissible for $\mathbb{E}_1$ by the permissibility of $(\diamondsuit)$ and property $(*)$ of α. But since

$(\diamondsuit)$ is not permissible for $\mathbb{T}_x(\mathbb{E}_2) = (In_x(\mathbb{E}_2), b)$, we also have that $(\widetilde{\diamondsuit})$ is *not* permissible for $\mathbb{E}_2$. This contradicts $\mathbb{E}_1 \subset \mathbb{E}_2$ and proves *(i)*; thus (1.6) holds.

Now we come to *(ii)*, $\mathrm{Dir}_x(\mathbb{E}_1) \subseteq \mathrm{Dir}_x(\mathbb{E}_2)$. By Lemma 1.2.16 $\mathrm{Dir}_x(\mathbb{E}_i) \subseteq \mathrm{Sing}\,(\mathbb{T}_x(\mathbb{E}_i))$ and by definition of the directrix it is a permissible center for $\mathbb{T}_x(\mathbb{E}_i)$, $i \in \{1, 2\}$. Further (1.6) implies that $\mathrm{Dir}_x(\mathbb{E}_1)$ is a permissible center for $\mathbb{T}_x(\mathbb{E}_2)$, which contains the origin $V(U, Y)$. By the minimality of the directrix $\mathrm{Dir}_x(\mathbb{E}_2)$ any permissible center containing the origin must lie in $\mathrm{Dir}_x(\mathbb{E}_2)$. This implies $\mathrm{Dir}_x(\mathbb{E}_1) \subseteq \mathrm{Dir}_x(\mathbb{E}_2)$. The second part of *(ii)* is clear.

Part *(iii)*, $\mathbb{R}\mathrm{id}_x(\mathbb{E}_1) \subset \mathbb{R}\mathrm{id}_x(\mathbb{E}_2)$, is similar to *(i)*. Assume $\mathbb{R}\mathrm{id}_x(\mathbb{E}_1) \not\subset \mathbb{R}\mathrm{id}_x(\mathbb{E}_2)$, then there exists a local sequence of regular blow ups over $K[U]$ which is permissible for $\mathbb{R}\mathrm{id}_x(\mathbb{E}_1)$, but not for $\mathbb{R}\mathrm{id}_x(\mathbb{E}_2)$. By the definition of the ridge, this local sequence of regular blow ups is permissible for $\mathbb{T}_x(\mathbb{E}_1)$, but not for $\mathbb{T}_x(\mathbb{E}_2)$. This is a contradiction to (1.6).
(Alternatively one could lift the local sequence of regular blow ups as in the proof of *(i)* to one over R_0 and this yields a contradiction to $\mathbb{E}_1 \subset \mathbb{E}_2$ as before). $\square$

Remark 1.2.20. *Let $\mathbb{E}$ be an idealistic exponent on Z and $x \in \mathrm{Sing}\,(\mathbb{E})$. Suppose the ideal of the ridge be given by $I\mathrm{Rid}_x(\mathbb{E}) = \langle \varphi_1, \dots, \varphi_l \rangle \subset gr_x(R)$ for additive homogeneous polynomials φ_i of order p^{d_i}, $1 \leq i \leq l$. By definition, we have $\mathbb{R}\mathrm{id}_x(\mathbb{E}) = \bigcap_{i=1}^{l} (\,\varphi_i,\, p^{d_i}\,)$. Let $\varphi \in \{\varphi_1, \dots, \varphi_l\}$. Then φ is of order p^d for some $d \in \mathbb{Z}_0$. If there is a $\psi \in gr_x(R)$ such that $\varphi = \psi^a$ for some $a \in \mathbb{Z}_+$, then we get by Lemma 1.1.8(i) that $(\varphi, p^d) \sim \left(\psi, \dfrac{p^d}{a}\right)$. Clearly, this may only happen for $a = p^{d'}$ with $d' \leq d$. If we choose $a \in \mathbb{Z}_+$ maximal and apply this for all φ_i, then we get $\mathbb{R}\mathrm{id}_x(\mathbb{E}) \sim \bigcap_{i=1}^{l} (\,\psi_i,\, p^{e_i}\,)$ for certain $\psi_i \in gr_x(R)$ and $e_i \in \mathbb{Z}_0$, $e_i \leq d_i$ such that $\psi_i^{a_i} = \varphi_i$, for $a_i := p^{d_i - e_i}$ and $i \in \{1, \dots, l\}$. Further $I\mathrm{Rid}_x(\mathbb{E})_{red} = \langle \psi_1, \dots, \psi_l \rangle$, since the a_i are maximal.*

In particular, if K is perfect, then there exist $\psi_1, \dots \psi_l \in gr_x(R)$ such that $\varphi_i = \psi_i^{p^{d_i}}$ for every i. This implies that the ψ_i are homogeneous of degree one and hence $I\mathrm{Rid}_x(\mathbb{E})_{red} = I\mathrm{Dir}_x(\mathbb{E})$. Therefore $\mathbb{R}\mathrm{id}_x(\mathbb{E}) \sim \mathbb{D}\mathrm{ir}_x(\mathbb{E})$ (if K is perfect).

1.3 Idealistic coefficient exponents and the d-invariant

An important tool in the study of singularities in characteristic zero is the coefficient ideal with respect to a closed subscheme of maximal contact. (We recall the concept of maximal contact in section 1.4).

We now give the precise definition of the coefficient ideal in the idealistic setting. But we don't restrict our attention to characteristic zero and admit an arbitrary residue field of Z at x. It is known that the concept of maximal contact does not work in full generality, therefore we define the idealistic coefficient exponent with respect to any regular subvariety $W = V(z) = V(z_1, \ldots, z_n)$ containing x; we only want to assume that (z) is part of a regular system of parameters for the local ring R of Z at x. (The interesting case for us is, when $W = V(y_1, \ldots, y_s)$ $(s \leq r)$, where $(y) = (y_1, \ldots, y_r)$ is such that the image of (y) in $gr_x(Z)$ defines the directrix $\mathrm{Dir}_x(\mathbb{E})$).

Further we introduce in this section the d-invariant $d_x(\mathbb{E}, u, z)$ of $\mathbb{E}$ at x with respect to some regular system of parameters (u, z) for R. This is the first step towards the invariant used in [BM3].

Definition 1.3.1. *Let $\mathbb{E} = (J, b)$ be an idealistic exponent on Z and $x \in Z$. Let $(R = \mathcal{O}_{Z,x}, \mathfrak{m}, K)$ be the regular local ring of Z at x. We consider a fixed system of elements $(u) = (u_1, \ldots, u_d)$ which can be extended to a regular system of parameters for R. Let $(z) = (z_1, \ldots, z_s)$ be elements of R such that (u, z) is a regular system of parameters for R. We define the idealistic coefficient exponent $\mathbb{D}_x(\mathbb{E}, u, z)$ of $\mathbb{E}$ at x with respect to (z) as the idealistic exponent on $W = \mathrm{Spec}\,(K[[u]])$ which is given by the following construction: Any $f \in J_x$ may be written (in the $\mathfrak{m}$-adic completion $\widehat{R}$) as*

$$f = f(u, z) = \sum_{B \in \mathbb{Z}_0^s} f_B(u)\, z^B$$

with $f_B(u) \in K[[u]]$. Then we set $\mathbb{D}(f, u, z) := \bigcap_{\substack{B \in \mathbb{Z}_0^s \\ |B| < b}} (f_B(u),\, b - |B|)$ and define further

$$\mathbb{D}_x(\mathbb{E}, u, z) := \bigcap_{f \in J_x} \mathbb{D}(f, u, z) = \bigcap_{l=0}^{b-1} \;(\; I(l, u, z),\; b - l \;),$$

where $I(l, u, z) = \langle\, f_B \mid f \in J_x, B \in \mathbb{Z}_0^s : |B| = l \,\rangle$.

The idea of coefficient ideals goes back to Hironaka (in the context of idealistic exponents this appears in [H4] Theorem 1.3, p.908, and [H3] section 8, Theorem 5, p.111) and was developed by Villamayor (for basic objects) and Bierstone-Milman (for presentations). One can even give a definition which is independent of the choice of (u,z), but for our purposes we need the above variant. Further let us point out that in our definition the residue field is not necessarily perfect (and may have positive characteristic), whereas Hironaka requires the base field to be perfect in the articles cited above.

We may consider $\mathbb{D}_x(\mathbb{E},u,z)$ as an idealistic exponent on $\widehat{R}$. Then we have $\mathbb{E}_x \subset \mathbb{D}_x(\mathbb{E},u,z)$ by construction.

In our context one of the first questions coming into one's mind may be the following: Are the idealistic coefficient exponents of equivalent idealistic exponents also equivalent? For the idealistic approach there is no reference known to the author where this is proven. Hence we give the answer in

Theorem 1.3.2. *Let $\mathbb{E}_1 \subset \mathbb{E}_2$ be two idealistic exponents on Z and $x \in Z$. Further let $(u,z) = (u_1,\dots,u_d;z_1,\dots,z_s)$ be a regular system of parameters for $(R = \mathcal{O}_{Z,x},\mathfrak{m},K)$. Then we have*

$$\mathbb{D}_x(\mathbb{E}_1,u,z) \subset \mathbb{D}_x(\mathbb{E}_2,u,z).$$

By symmetry, $\mathbb{E}_1 \sim \mathbb{E}_2$ implies $\mathbb{D}_x(\mathbb{E}_1,u,z) \sim \mathbb{D}_x(\mathbb{E}_2,u,z)$.

Proof. Let $\mathbb{E} = (J,b) \in \{\mathbb{E}_1,\mathbb{E}_2\}$. We consider $\mathbb{E}_x = (J_x,b)$ on R. In order to simplify the notation we suppress the local index x and write $J = J_x$ and $\mathbb{E} = \mathbb{E}_x$. First of all let us mention the following easy observation:

> Consider $(J,b)\cap(z,1)$. By Lemma 1.1.8*(ii)* we may then assume that in the expansion of all $g \in J$, $g = \sum_B g_B(u)z^B$, we have $g_B(u) = 0$ for all $B \in \mathbb{Z}_0^s$ with $|B| \geq b$. $(*)$

For $M \in \mathbb{Z}_0^s$ let $\mathcal{D}_M \in \mathrm{Diff}_K^{\leq j}(\widehat{R})$, $j = |M|$, be the differential operator which is defined by

$$\mathcal{D}_M(C_{A,B}\,u^A z^B) = \binom{B}{M} C_{A,B}\,u^A z^{B-M}$$

for $C_{A,B} \in K$. In particular $\mathcal{D}_M(C_{A,M}\,u^A z^M) = C_{A,M}\,u^A$. Further, we define for $j \in \{1,\dots,b-1\}$ the finite sets

$$S(j) = \{M \in \mathbb{Z}_0^s \mid |M| = j\}.$$

Let $M \in S(b-1)$. By the Diff Theorem 1.1.13 we have $(J,b) \sim (J,b)\cap(\mathcal{D}_M J,1)$ and hence

$$(z,1)\cap(J,b) \sim (z,1)\cap(J,b)\cap(\mathcal{D}_M J,1).$$

In here, we have for $g = \sum_{B\in\mathbb{Z}_0^s} g_B(u)\, z^B \in J$ (where the expansion is considered in $\widehat{R}$)

$$\mathcal{D}_M(g) = g_M(u) + \sum_{\substack{B\in M+\mathbb{Z}_0^s \\ |B|>|M|=b-1}} \binom{B}{M} g_B(u)\, z^{B-M} = g_M(u),$$

where the last equality follows by $(*)$. If we apply this to all $M \in S(b-1)$ and all $g \in J$, we get

$$(z,1)\cap \mathbb{E} \sim (z,1)\cap (J,b)\cap \bigcap_{M\in S(b-1)} (\mathcal{D}_M J, 1) \sim (z,1)\cap \mathbb{E}\cap \mathbb{D}^{(1)}(\mathbb{E}), \tag{1.7}$$

where we define $\mathbb{D}^{(1)}(\mathbb{E}) := (I^{(1)}, 1)$ with

$$I^{(1)} := \langle\, g_M \mid M \in S(b-1), g\in J\,\rangle \subset K[[u]].$$

(Note that $\mathbb{D}^{(1)}(\mathbb{E})$ is an idealistic exponent on $\widehat{R}' := K[[u]]$). The ideal $I^{(1)}$ is generated by those $g_M(u)$ which appear in expansions of elements g of J in front of some power z^M with $|M| = b-1$.

By Lemma 1.1.8 *(i)* and *(iii)*,

$$(I^{(1)},1)\cap (z^{b-1}, b-1) \sim (I^{(1)},1)\cap (z,1)\cap (\langle z\rangle^{b-1} I^{(1)}, b).$$

Let us consider $(z,1)\cap (J,b)\cap \mathbb{D}^{(1)}(\mathbb{E})$. By part *(ii)* of the Lemma we may assume that in the expansion of all $g \in J$, $g = \sum_B g_B(u) z^B$, we have $g_B(u) = 0$ for all $B \in \mathbb{Z}_0^s$ with $|B| = b-1$. Together with $(*)$ we get $g_B(u) = 0$ for all $B \in \mathbb{Z}_0^s$ with $|B| \geq b-1$. $(**)$

Now let $M \in S(b-2)$. The Diff Theorem 1.1.13 yields $(J,b) \sim (J,b)\cap(\mathcal{D}_M J, 2)$ and therefore

$$(z,1)\cap (J,b)\cap \mathbb{D}^{(1)}(\mathbb{E}) \sim (z,1)\cap (J,b)\cap (\mathcal{D}_M J, 2)\cap \mathbb{D}^{(1)}(\mathbb{E}).$$

In here, we have for $g = \sum_{B\in\mathbb{Z}_0^s} g_B(u)\, z^B \in J$

$$\mathcal{D}_M(g) = g_M(u) + \sum_{\substack{B\in M+\mathbb{Z}_0^s \\ |B|>|M|=b-2}} \binom{B}{M} g_B(u)\, z^{B-M} = g_M(u),$$

where the last equality follows by $(**)$. If we apply this to all $M \in S(b-2)$ and all $g \in J$, we get

$$(z,1)\cap \mathbb{E}\cap \mathbb{D}^{(1)}(\mathbb{E}) \sim (z,1)\cap \mathbb{E}\cap \mathbb{D}^{(1)}(\mathbb{E})\cap \mathbb{D}^{(2)}(\mathbb{E}), \tag{1.8}$$

where we define

$$\mathbb{D}^{(2)}(\mathbb{E}) := (I^{(2)}, 2) = \bigcap_{M \in S(b-2)} (\mathcal{D}_M J, 2)$$

with $I^{(2)} := \langle\, g_M \mid M \in S(b-2), g \in J \,\rangle \subset K[[u]]$. (Again $\mathbb{D}^{(2)}(\mathbb{E})$ is an idealistic exponent on $\widehat{R}' = K[[u]]$). Putting (1.7) and (1.8) together gives

$$(z, 1) \cap \mathbb{E} \;\sim\; (z, 1) \cap \mathbb{E} \cap \mathbb{D}^{(1)}(\mathbb{E}) \cap \mathbb{D}^{(2)}(\mathbb{E}). \tag{1.9}$$

We go on with this procedure and get at the end

$$(z, 1) \cap \mathbb{E} \sim (z, 1) \cap \mathbb{E} \cap \bigcap_{l=1}^{b-1} \mathbb{D}^{(l)}(\mathbb{E}) \sim (z, 1) \cap \bigcap_{l=1}^{b} \mathbb{D}^{(l)}(\mathbb{E}), \tag{1.10}$$

where for $l \in \{1, \ldots, b-1\}$ we have $\mathbb{D}^{(l)}(\mathbb{E}) = (I^{(l)}, l)$ and

$$I^{(l)} = \langle\, g_M \mid M \in S(b-l), g = \sum g_M(u) y^M \in J \,\rangle \subset K[[u]].$$

By extending $(**)$ we may assume that in the expansion of an element $g \in J$, $g = \sum_M g_M(u) z^M$, we have $g_M = 0$ for all $M \in \mathbb{Z}_0^s$ with $|M| \geq 1$. So we set

$$I^{(b)} := \langle\, g_{(0,\ldots,0)} \mid g = \sum g_M(u) z^M \in J \,\rangle \subset K[[u]]$$

and $\mathbb{D}^{(b)}(\mathbb{E}) := (I^{(b)}, b)$.

By construction $\bigcap_{l=1}^{b} \mathbb{D}^{(l)}(\mathbb{E}) = \mathbb{D}_x(\mathbb{E}, u, z)$ is an idealistic exponent on $K[[u]]$ and therefore does not involve any element of (z).

Hence we get for $\mathbb{E}_1$ and $\mathbb{E}_2$ (recall $\mathbb{E}_1 \subset \mathbb{E}_2$)

$$(z, 1) \cap \mathbb{D}_x(\mathbb{E}_1, u, z) \sim (z, 1) \cap \mathbb{E}_1 \subset (z, 1) \cap \mathbb{E}_2 \sim (z, 1) \cap \mathbb{D}_x(\mathbb{E}_2, u, z). \tag{1.11}$$

Since $\mathbb{D}_x(\mathbb{E}_1, u, z)$ and $\mathbb{D}_x(\mathbb{E}_2, u, z)$ are idealistic exponents on $K[[u]]$, this already implies

$$\mathbb{D}_x(\mathbb{E}_1, u, z) \subset \mathbb{D}_x(\mathbb{E}_2, u, z),$$

which proves the theorem. □

Corollary 1.3.3. *We want to point out, that (1.10) implies*

$$(z, 1) \cap \mathbb{E} \sim (z, 1) \cap \mathbb{D}_x(\mathbb{E}, u, z)$$

(Keep in mind that we have here the local situation at a point x).

By the last theorem, $\mathbb{D}_x(\mathbb{E}, u, z)$ is invariant under the equivalence $\sim$ if we fix (u, z). But we might also consider various choices for (z). In this case we have

Proposition 1.3.4. *Let $\mathbb{E}$ be an idealistic exponent on Z and $x \in Z$. Fix a system of elements $(u) = (u_1, \dots, u_d)$ which can be extended to a regular system of parameters for $(R = \mathcal{O}_{Z,x}, \mathfrak{m}, K)$. Let $(z) = (z_1, \dots, z_s)$ and $(y) = (y_1, \dots, y_s)$ be two possible extensions of (u). Assume $(z, 1) \cap \mathbb{E} \subset (y, 1) \cap \mathbb{E}$. Then*

$$\mathbb{D}_x(\mathbb{E}, u, z) \subset \mathbb{D}_x(\mathbb{E}, u, y).$$

By symmetry, $(z, 1) \cap \mathbb{E} \sim (y, 1) \cap \mathbb{E}$ implies $\mathbb{D}_x(\mathbb{E}, u, z) \sim \mathbb{D}_x(\mathbb{E}, u, y)$.

Proof. First of all, Corollary 1.3.3 and the assumption imply

$$(z, 1) \cap \mathbb{D}_x(\mathbb{E}, u, z) \sim (z, 1) \cap \mathbb{E} \subset (y, 1) \cap \mathbb{E} \sim (y, 1) \cap \mathbb{D}_x(\mathbb{E}, u, y). \qquad (*)$$

Let $(\diamondsuit)$ be a local sequence of regular blow ups over $K[[u]]$ which is permissible for $\mathbb{D}_x(\mathbb{E}, u, z)$. We can lift it to a local sequence of regular blow ups $(\widetilde{\diamondsuit})$ over $K[[u]][z]$ just by intersecting the centers with $V(z)$. Then $(\widetilde{\diamondsuit})$ is permissible for $(z, 1) \cap \mathbb{D}_x(\mathbb{E}, u, z)$ and by $(*)$ it is so for $(y, 1) \cap \mathbb{D}_x(\mathbb{E}, u, y)$. In particular, it is permissible for $\mathbb{D}_x(\mathbb{E}, u, y)$ and since the latter lives on $K[[u]]$, the local sequence of regular blow ups $(\diamondsuit)$ is permissible for $\mathbb{D}_x(\mathbb{E}, u, y)$. This shows the assertion. $\square$

Therefore under the special assumption $(z, 1) \cap \mathbb{E} \sim (y, 1) \cap \mathbb{E}$ the idealistic coefficient exponent for a fixed system (u) does not depend on the choice of (z). In particular this holds,

- if $(z, 1) \sim (y, 1)$ or
- if $(y, 1) \cap \mathbb{E} \sim \mathbb{E} \sim (z, 1) \cap \mathbb{E}$.

We see later that the second condition is valid if (y) and (z) have maximal contact (see Lemma 1.3.7 and section 1.4).

Note that Theorem 1.3.2 and Proposition 1.3.4 imply Main Theorem 2.

Definition 1.3.5. *Let $\mathbb{E}$ be an idealistic exponent on Z, $x \in Z$ and as before $(u, z) = (u_1, \dots, u_d; z_1, \dots, z_s)$ denotes a regular system of parameters for the local ring at x. We define the d-invariant of $\mathbb{E}$ at x with respect to (z) as*

$$d_x(\mathbb{E}, u, z) := \mathrm{ord}_x(\, \mathbb{D}_x(\mathbb{E}, u, z)\,).$$

More general: Since $\mathbb{D}_x(\mathbb{E}, u, z)$ is an idealistic exponent on $K[[u]]$, we define the d_x-invariant at any point $w \in$ Spec $(K[[u]])$ by $d_x(\mathbb{E}, u, z)(w) := \mathrm{ord}_w(\, \mathbb{D}_x(\mathbb{E}, u, z)\,)$.

Theorem 1.3.2, Proposition 1.3.4 and the Numerical Exponent Theorem 1.1.10 imply

Corollary 1.3.6. *Let* $\mathbb{E}_1$ *and* $\mathbb{E}_2$, $\mathbb{E}_1 \subset \mathbb{E}_2$, *be two idealistic exponents on* Z, $x \in Z$ *and* $(u, z) = (u_1, \ldots, u_d; z_1, \ldots, z_s)$ *be a regular system of parameters for the local ring at* x. *Then we have for every* $w \in \operatorname{Spec}(K[[u]])$

$$d_x(\mathbb{E}_1, u, z)(w) \leq d_x(\mathbb{E}_2, u, z)(w).$$

Further if $(y) = (y_1, \ldots, y_s)$ *is another choice to extend* (u) *and assume* $(z, 1) \cap \mathbb{E}_1 \subset (y, 1) \cap \mathbb{E}_1$, *then*

$$d_x(\mathbb{E}_1, u, z)(w) \leq d_x(\mathbb{E}_1, u, y)(w).$$

The important cases for us are $\mathbb{E}_1 \sim \mathbb{E}_2$ or $(z, 1) \cap \mathbb{E}_1 \sim (y, 1) \cap \mathbb{E}_1$. Here we get $d_x(\mathbb{E}_1, u, z)(w) = d_x(\mathbb{E}_2, u, z)(w)$ and $d_x(\mathbb{E}_1, u, z)(w) = d_x(\mathbb{E}_1, u, y)(w)$. Therefore the d_x-invariant does not depend on the choice of (z) with respect to the condition $(z, 1) \cap \mathbb{E}_1 \sim (y, 1) \cap \mathbb{E}_1$ and it is invariant under $\sim$.

But in the arbitrary case $d_x(\mathbb{E}, u, z)$ depends on the choice of $(z) = (z_1, \ldots, z_s)$ (recall (u) is fixed).

From now on we focus on the case that (z) is related to the directrix of $\mathbb{E}$ at x:

More precisely, let $(u, y) = (u_1, \ldots, u_e, y_1, \ldots, y_r)$ be a regular system of parameters for $(R = \mathcal{O}_{Z,x}, \mathfrak{m}, K)$ such that the images of (y) in $\mathfrak{m}/\mathfrak{m}^2$ define the $\operatorname{Dir}_x(\mathbb{E})$. (As usual r is as small as possible). For the later use let $(z) = (z_1 \ldots, z_s)$, $s \leq r$, be elements in R such that the images of z_j and y_j in $\mathfrak{m}/\mathfrak{m}^2$ coincide. Further, we set $(\widetilde{u}) := (u, y_{s+1}, y_{s+2}, \ldots, y_r)$. (1.12)

We want to find a criterion to decide if $d_x(\mathbb{E}, \widetilde{u}, z)$ is independent of (z) and further it would be useful to have a procedure to calculate this number. Up to now the assumption that (z) is related to the directrix is not enough, see Example 1.4.1. In order to achieve our aim, we introduce in the next chapter a polyhedral approach to the d-invariant, where the notion of vertex preparation and normalization give the desired procedure. Further we deduce from this the invariant which is used by Bierstone and Milman to simplify Hironaka's original proof for resolution of singularities in characteristic zero.

As in Corollary 1.2.17 we have a very useful result in characteristic zero or for large characteristic $p > 0$ which is essential in our later considerations.

Lemma 1.3.7. *Let* $\mathbb{E} = (J, b)$ *be an idealistic exponent on* Z *and* $x \in \operatorname{Sing}(\mathbb{E})$. *Further* $(R = \mathcal{O}_{Z,x}, \mathfrak{m}, K)$ *denotes the local ring at* x. *Assume* $char(K) = 0$ *or* $b < char(K)$. *Let* (u, y) *be as in* (1.12). *Then there exists for every* $s \in \mathbb{Z}_+$, $s \leq r$, *a system* (z) *with the property of* (1.12) *and*

$$\mathbb{E}_x \sim (z, 1) \cap \mathbb{D}_x(\mathbb{E}, \widetilde{u}, z),$$

where $(\widetilde{u})$ *is defined as in* (*1.12*)*. More precisely, for every* $j \in \{1,\dots,s\}$ *there exist* $\mathcal{D}_j \in \mathrm{Diff}_K^{\leq b-1}(K[Y])$ *and* $F(j) \in In_x(\mathbb{E})$ *such that* $\mathcal{D}_j(F(j)) = \epsilon_j\, Z_j$*, where* $\epsilon_j \in R$ *denotes a unit. Further there is an* $f(j) \in J\widehat{R}$ *which maps in* $gr_x(Z)$ *to* $F(j)$ *and* $(\mathcal{D}'_j(f(j)),1) \sim (z_j,1)$*, where* $\mathcal{D}'_j$ *denotes the differential operator on* $\widehat{R}$ *induced by* $\mathcal{D}_j$*.*

Proof. By (1.12) we have $I\mathrm{Dir}_x(\mathbb{E}) = \langle Y_1,\dots,Y_r\rangle$, where Y_j denotes the image of y_j in $\mathfrak{m}/\mathfrak{m}^2$. Recall the following from the proof of Corollary 1.2.17:

Every generator $F \in \langle Y\rangle^b \setminus \langle Y\rangle^{b+1}$ of the ideal $In_x(\mathbb{E})$ can be written as $F = \sum_{B\in\mathbb{Z}_0^r:|B|=b} C_B\, Y^B$ for some $C_B \in K$. Further for $j=1$ there exists a generator $F(j)$ of $In_x(\mathbb{E})$ such that there is a $B(j) = (B_1,\dots,B_r) \in \mathbb{Z}_0^r$ with $C_{B(j)} \neq 0$ and $B_j \geq 1$ (Y_j appears). Set $M(j) := B(j) - e_j \in \mathbb{Z}_0^r$, $|M(j)| = b-1$. Let $\mathcal{D}_j := \mathcal{D}_{M(j)} \in \mathrm{Diff}_K^{\leq b-1}(K[Y])$ the differential operator which is defined via $\mathcal{D}_{M(j)}(C\,Y^B) = \binom{B}{M(j)}\, C\, Y^{B-M(j)}$. Consequently

$$\mathcal{D}_{M(j)}(F(j)) = C_{B(j)}\, B_j\, Y_j + \textstyle\sum_{B'(i)} C_{B'(i)}\, B'_i\, Y_i,$$

where $B'(i) = (B'_1,\dots,B'_r) \in \{M(j)+e_i \mid i \in \{1,\dots,r\}\setminus\{j\}\}$. The assumptions on $char(K)$ imply that B_j (and thus $C_{B(j)}\, B_j$) is a unit in K. Set

$$Y_j^* := (C_{B(j)}\, B_j)^{-1}\, \mathcal{D}_{M(j)}\, F(j) = Y_j + \sum_{B'(i)} (C_{B(j)}\, B_j)^{-1} C_{B'(i)}\, B'_i\, Y_i \in K[Y].$$

We choose as system of representatives of $K = R/\mathfrak{m}$ in R and define with this $y_1^* \in R$ by replacing (Y) by (y) in the Y_1^*. The system $(y_1^*, y_2,\dots,y_r)$ fulfills the same properties as (y). So we may consider the regular system of parameters $(u; y_1^*, y_2,\dots,y_r)$ instead of (u,y) and put $\mathcal{D}_1 := \mathcal{D}_{M(1)}$. Then we repeat the above procedure for $j=2$ to obtain y_2^* and $\mathcal{D}_2$. ...We continue until we have $(z^*) = (z_1^*,\dots,z_s^*) := (y_1^*,\dots,y_s^*)$.

Denote by $\mathcal{D}'_j$ the differential operator on $\widehat{R}$ induced by $\mathcal{D}_j$, $1 \leq j \leq s$. ($\mathcal{D}_j$ extends by acting trivially on (u)). Further there exist $f(j) \in J\widehat{R}$, which are mapped to $F(j) \in gr_x(Z)$ and $\mathcal{D}'_j(f(j)) = \epsilon_j z_j^* + h_j$ for some units $\epsilon_j \in R$ and further terms $h_j \in R$, which do not involve z_j^*. Set $z_j := z_j^* + \epsilon_j^{-1} h_j$. Then $\mathcal{D}_j(F(j)) = \epsilon_j\, Z_j$, $(\mathcal{D}'_j(f(j)),1) = (z_j,1)$ and by the Diff Theorem 1.1.13 we have $\mathbb{E}_x \sim \mathbb{E}_x \cap (z,1)$. Together with Corollary 1.3.3 we get $\mathbb{E}_x \sim (z,1) \cap \mathbb{D}_x(\mathbb{E},u,z)$. □

Remark 1.3.8. *(1) We do not necessarily need the assumption that the images of* z_j *and* y_j *in* $\mathfrak{m}/\mathfrak{m}^2$ *coincide. Instead of* $\mathcal{D}_j$ *we could consider any differential operator* $\mathcal{D}$ *of order* $b-1$ *such that there is an* $f \in J$ *for which* $\mathcal{D}f$ *is linear. (*$\mathcal{D}$ *exists if* $\mathrm{ord}_x(\mathbb{E}) = 1$*). Set* $z_1 := \mathcal{D}f$*. As above we get* $\mathbb{E}_x \sim (z_1,1) \cap \mathbb{D}_x(\mathbb{E},\widetilde{u}',z_1)$*, where* $(\widetilde{u}')$ *denotes the remaining part of the* *regular system of parameters for R. Then go on and consider $\mathbb{D}_x(\mathbb{E}, \widetilde{u}', z_1)$, which is independent of z_1. If we do this until the order of the idealistic coefficient exponent is bigger than one, then we get $(z) = (z_1, \ldots, z_r)$ such that $\mathbb{E}_x \sim (z, 1) \cap \mathbb{D}_x(\mathbb{E}, u, z)$ and the images of (z) in $\mathfrak{m}/\mathfrak{m}^2$ define $\mathrm{Dir}_x(\mathbb{E})$.*

(2) *Later we see that $V(z)$ has maximal contact with $\mathbb{E}$ at x. As we mentioned in Remark 1.2.18, the failure of maximal contact may occur when $b \geq p$, where differential operators may behave badly. Therefore the assumption $char(K) = 0$ or $b < char(K)$ is crucial.*

1.4 Maximal contact

We already mentioned two results for the special case of characteristic zero or for large characteristic, see Corollary 1.2.17 and Lemma 1.3.7. In this section we introduce the concept of maximal contact, which is an important tool in the proof of resolution of singularities in characteristic zero. Classical references for this are [G2] and [AHV].

As we pointed out at the end of the previous section, we are interested in the idealistic coefficient exponent with respect to special $(z) = (z_1, \dots, z_s) := (y_1, \dots, y_s)$, where $(u, y) = (u_1, \dots, u_e, y_1, \dots, y_r)$, $r \le s$, is a regular system of parameters for the local ring of Z at a point x such that the images of (y) in $\mathfrak{m}/\mathfrak{m}^2$ give $\mathrm{Dir}_x(\mathbb{E})$. By the example below this condition does not suffice to guarantee that the d_x-invariant is independent of the choice of (z).

In characteristic zero and for large characteristic maximal contact is the right tool to obtain this independence, which follows by Lemma 1.3.7 and Corollary 1.3.6.

Example 1.4.1. Consider the idealistic exponent $\mathbb{E} = (z^2 + 2zu_1^2 + u_1^4 + u_1^7u_2^3, 2)$ over any field K. Let $x \in \mathbb{A}_K^2$ be the origin. (If $char(K) = 2$ we may omit the monomial $2zu_1^2$). The directrix of $\mathbb{E}$ at x is given by the initial form of (z) and the idealistic coefficient exponent at x with respect to (z) is

$$\mathbb{D}_x(\mathbb{E}; u_1, u_2; z) = (2u_1^2, 1) \cap (u_1^4 + u_1^7u_2^3, 2).$$

Therefore $d_x(\mathbb{E}, u, z) = 2$.

But for $y := z + u_1^2$ we have $\mathbb{E} = (y^2 + u_1^7u_2^3, 2)$, $\mathbb{D}_x(\mathbb{E}, u, y) = (u_1^7u_2^3, 2)$ and $d_x(\mathbb{E}, u, y) = 5$. Note that the initial form of y yields also the directrix.

Recall that we denoted by X the subscheme of Z corresponding to $J \subset \mathcal{O}_Z$. A map $\iota : X \to G$ into a totally ordered abelian group $(G, \le)$ is called upper semi-continuous if the set

$$X_{\ge\nu} := X_{\ge\nu}^{\iota} := \{x \in X \mid \iota(x) \ge \nu\}$$

is closed for all $\nu \in G$. In particular, if $\nu \in G$ is a maximal value of ι (X is Noetherian!), then the set $X_{\ge\nu} = X_\nu := X_\nu^{\iota} := \{x \in X \mid \iota(x) = \nu\}$ is closed.

One can show that this is equivalent to the following two conditions.

(1) If $x, y \in X$ and $x \in \overline{\{y\}}$, then $\iota(x) \ge \iota(y)$.

(2) For all $y \in X$ there is a dense open subset $U \subset \overline{\{y\}}$ such that $\iota(x) = \iota(y)$ for all $x \in U$.

For a proof of this see [CJS] Lemma 1.34 (a), p.26.

Examples for ι are the order of an idealistic exponent $\mathbb{E}$ and the Hilbert-Samuel function of the corresponding scheme X, where the latter is given as follows: Let $x \in X$ and denote by $(\mathcal{O}_{X,x}, \mathfrak{m}_x, K_x)$ the local ring of X at x. Then $H_{X,x}(l) := length(\mathcal{O}_{X,x}/\mathfrak{m}_x^{l+1})$, for $l \in \mathbb{N}$, defines $H_X : X \to \mathbb{N}^{\mathbb{N}}$.

Definition 1.4.2. *Let $\mathbb{E} = (J, b)$ be an idealistic exponent on Z and $x \in X \subset Z$, where X is given by $J \subset \mathcal{O}_Z$. Let $W \subset Z$ be a closed subscheme and $\iota : X \to (G, \leq)$ an upper semi-continuous map. Suppose ι is an appropriate measure for the singularities of $\mathbb{E}$ and does not increase under blow ups, which are permissible for $\mathbb{E}$. We say W has maximal contact with $\mathbb{E}$ at x with respect to ι, if*

(1) $x \in W$ and

(2) Take any local sequence of regular blow ups over Z which is permissible for $\mathbb{E}$, say

$$\begin{array}{ccccccccc} Z = Z_0 \supset U_0 & \xleftarrow{\pi_1} & Z_1 \supset U_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_{l-1}} & Z_{l-1} \supset U_{l-1} & \xleftarrow{\pi_l} & Z_l \\ \cup & & \cup & & & & \cup & & \\ D_0 & & D_1 & & \dots & & D_{l-1} & & \end{array}$$

($l \in \mathbb{Z}_+ \cup \{\infty\}$, $U_i \subset Z_i$ is an open subscheme, $D_i \subset U_i$ a regular closed subscheme and $\pi_{i+1} : Z_{i+1} \to U_i$ the blow up with center D_i, $0 \leq i \leq l-1$). Assume there is a sequence of points $x_i \in D_i$ ($i \in \{0, \dots, l\}$) such that $x_0 = x$, $\pi_{i+1}(x_{i+1}) = x_i$ and $\iota(x_{i+1}) = \iota(x_i)$ for $i \geq 0$. Then we have $D_i \subset W_i$ for all $i \geq 0$, where W_i denotes the strict transform of W in Z_i.

By the assumption that ι is an appropriate measure for the singularities of $\mathbb{E}$, we exclude trivial cases like $\iota(x) = 0$ for all $x \in X$.

In particular, W_i contains the set of points above $x = x_0$, for which ι didn't decrease, $\{x_i \in X_i \mid \iota(x_i) = \iota(x_0) \wedge \pi^{(i)}(x_i) = x_0\}$ (where $X_i \subset Z_i$ denotes the closed subscheme associated to the transform of $\mathbb{E}$ in Z_i and $\pi^{(i)} = \pi_i \circ \cdots \circ \pi_1$).

If there is no confusion possible, we omit the reference to ι and say only that W has maximal contact with $\mathbb{E}$ at x.

In resolution of singularities we want to measure an improvement of the singularity after a permissible blow up. One possible way to achieve this is to construct an ι which drops or at least does not increase after every blow up with center contained in the maximum locus of ι. For the non-increasing case we then have to find an argument that equality may not happen infinitely many times. Finally, if the set of values G of ι is discrete, then the singularities can be resolved by finitely many of these blow ups.

For this reason we suppose above that ι does not increase under permissible blow ups.

Lemma 1.4.3. *Let $\mathbb{E} = (J, b)$, X, x and $\iota : X \to (G, \leq)$ be as in the previous definition. Let $(z) = (z_1, \dots, z_s)$ be a system of elements in $(R = \mathcal{O}_{Z,x}, \mathfrak{m})$ which can be extended to a regular system of parameters for R. Assume $\mathbb{E}$ is locally at x equivalent to*

$$\mathbb{E}_x = (J_x, b) \sim (z, 1) \cap (J_x, b).$$

Then $W := V(z)$ has maximal contact with $\mathbb{E}$ at x and moreover the images of (z) in $\mathfrak{m}/\mathfrak{m}^2$ are part of a minimal generating system for the directrix $\mathrm{Dir}_x(\mathbb{E})$.

Proof. We prove something more: $\mathrm{Sing}\,(\mathbb{E}_x) \subset W$ and this condition is stable under permissible blow ups. Clearly, this implies the claim (and moreover it is independent of ι).

Let (u, z) be a regular system of parameters for R, $(u) = (u_1, \dots, u_d)$ and K the residue field of R. By Corollary 1.3.3 we have $\mathbb{E}_x \sim (z, 1) \cap \mathbb{D}_x(\mathbb{E}, u, z)$ on $\widehat{R}$. Therefore $\mathrm{Sing}\,(\mathbb{E}_x) \subset W = V(z)$ and any permissible center D is contained in W, say $D = V(z, u_1, \dots, u_c)$ for some $c \leq d$. In the Z_j-charts $\mathbb{E}_x$ is resolved because the transform of $(z_j, 1)$ is $(1, 1)$. So it suffices to consider the remaining charts. There the transform is of the same type as before, namely $\mathbb{E}'_x = (z', 1) \cap (J', b)$. This implies $\mathrm{Sing}\,(\mathbb{E}'_x) \subseteq V(z') = W'$ and hence W has maximal contact.

The second part follows immediately by $\mathbb{E}_x \sim (z, 1) \cap \mathbb{D}_x(\mathbb{E}, u, z)$. $\square$

We have already seen that the idealistic exponents of the form considered in the previous lemma are not rare. Let us reformulate Lemma 1.3.7 in this new context.

Lemma 1.4.4. *Let $\mathbb{E} = (J, b)$ be an idealistic exponent on Z and $x \in \mathrm{Sing}\,(\mathbb{E})$. Let $(u, y) = (u_1, \dots, u_e, y_1, \dots, y_r)$ be a regular system of parameters for $(R = \mathcal{O}_{Z,x}, \mathfrak{m}, K)$ such that the images of (y) in $\mathfrak{m}/\mathfrak{m}^2$ define the $\mathrm{Dir}_x(\mathbb{E})$. Assume $char(K) = 0$ or $b < char(K)$.*

Then there exists a system $(z) = (z_1, \dots, z_r)$ of elements in R such that for every $j \in \{1, \dots, r\}$:

(i) The images of z_j and y_j in $\mathfrak{m}/\mathfrak{m}^2$ coincide.

(ii) If we set $(\widetilde{u}^{(j)}) := (u, z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_r)$, then we have the equivalence $\mathbb{E}_x \sim (z_j, 1) \cap \mathbb{D}_x(\mathbb{E}, \widetilde{u}^{(j)}, z_j)$. In particular, $\mathbb{E}_x \sim (z, 1) \cap \mathbb{D}_x(\mathbb{E}, u, z)$.

(iii) Each $V(z_j)$ (and thus $V(z_1, \dots, z_r)$) has maximal contact with $\mathbb{E}$ at x; here we mean maximal contact in the sense that $\mathrm{Sing}\,(\mathbb{E}_x) \subset V(z_j)$ and this is stable under blow ups which are permissible for $\mathbb{E}_x$.

(iv) *There exist* $\mathcal{D}_j \in \mathrm{Diff}_K^{\leq b-1}(K[Y])$ *and* $F(j) \in In_x(\mathbb{E})$ *such that* $\mathcal{D}_j(F(j)) = \epsilon_j Z_j$ *for some units* $\epsilon_j \in R$. *Further there are* $f(j) \in J\widehat{R}$ *which map in* $gr_x(Z)$ *to* $F(j)$ *and* $(\mathcal{D}'_j(f(j)), 1) \sim (z_j, 1)$, *where* $\mathcal{D}'_j$ *denotes the differential operator on* $\widehat{R}$ *induced by* $\mathcal{D}_j$.

Remark 1.4.5. *If we restrict our attention to* $char(K) = 0$ *or* $b < char(K)$, *then we already know the following: Fix* (u) *as above and let* (y) *and* (z) *be two extensions of* (u) *to a regular system of parameters such that* $V(y)$ *and* $V(z)$ *have maximal contact. Then we have by the previous lemma and Corollary 1.3.3*

$$(y,1) \cap \mathbb{E}_x \sim (y,1) \cap \mathbb{D}_x(\mathbb{E},u,y) \sim \mathbb{E}_x \sim (z,1) \cap \mathbb{D}_x(\mathbb{E},u,z) \sim (z,1) \cap \mathbb{E}_x$$

and Proposition 1.3.4 implies $\mathbb{D}_x(\mathbb{E},u,y) \sim \mathbb{D}_x(\mathbb{E},u,z)$. *Corollary 1.3.6 yields* $d_x(\mathbb{E},u,y) = d_x(\mathbb{E},u,z)$ *(recall that the* d_x*-invariant is defined by the order of the idealistic coefficient exponent). This means* $d_x(\mathbb{E},u,y)$ *is independent of the choice of the maximal contact variables* (y).

Moreover, we have shown in Theorem 1.3.2 that $\mathbb{D}_x(\mathbb{E},u,y)$ *is invariant under the equivalence relation* $\sim$. *Hence for fixed* (u) *the number* $\delta_x(\mathbb{E},u) := d_x(\mathbb{E},u,y)$ *is an invariant of* x, *the equivalence class of* $\mathbb{E}$ *and the condition that* $V(y)$ *has maximal contact with* $\mathbb{E}$ *at* x.

We want to point out that so far we did not make any choice of generators for J_x. *Therefore the previous is also independent of this choice. This follows also by the invariance under* $\sim$.

We see later that we can easily relate $d_x(\mathbb{E},u,y)$ *with the polyhedron associated to* $(\mathbb{E},u,y)$. *This implies that the invariant used by Bierstone and Milman in [BM3] to prove resolution of singularities in characteristic zero can be deduced from certain polyhedra. In the next chapter we introduce a more general polyhedron, which is independent of* (y) *and the notion of maximal contact. We show then that a certain more generally defined number* $\delta_x(\mathbb{E},u)$ *is intrinsic with no assumption on the base field* k *and further if* $V(y)$ *has maximal contact then the two definitions coincide.*

2 Characteristic Polyhedra and idealistic exponents with history

In this chapter we work in the following local situation:

Setup A. Let $\mathbb{E}$ be an idealistic exponent on Z and $x \in \mathrm{Sing}\,(\mathbb{E})$. Denote as usual by $(R = \mathcal{O}_{Z,x}, \mathfrak{m}, K)$ the local ring of Z at x. By abuse of notation we skip the index x and write $\mathbb{E} = (J, b)$ instead of $\mathbb{E}_x$. Fix a system $(u) = (u_1, \dots, u_e)$ of elements in $\mathfrak{m}$ which can be extended to a regular system of parameters for R. We consider various choices of a system $(y) = (y_1, \dots, y_r)$ such that

$$(u, y) \text{ is a regular system of parameters for } R. \tag{2.1}$$

By the assumption $x \in \mathrm{Sing}\,(\mathbb{E})$ it is guaranteed that $\mathrm{ord}_x(\mathbb{E}) \geq 1$ which is equivalent to $\mathrm{ord}_x(J) \geq b > 0$.

First we define the Newton polyhedron $\Delta^N(f, b; u, y)$, where $(f) = (f_1, \dots, f_m)$ denotes a set of generators of J. After that we show how to obtain the d_x-invariant $d_x(\mathbb{E}, u, y)$ from $\Delta^N(f, b; u, y)$. As we already remarked, the d_x-invariant is not necessarily independent of the choices for (y). This motivates the definition of a characteristic polyhedron $\Delta_x(\mathbb{E}, u)$ associated to $\mathbb{E}$ and (u). For this we have to recall Hironaka's characteristic polyhedron of a singularity.

Moreover, we prove that $\Delta_x(\mathbb{E}, u)$ is a suitable projection of the Newton polyhedron and we construct the ν-invariant of $\mathbb{E}$ and (u). For suitable $\mathbb{E}$ this coincides with the term ν_i in the invariant of Bierstone and Milman if the characteristic of K is zero.

But for equivalent idealistic exponents the polyhedra $\Delta_x(\mathbb{E}, u)$ need not be equal. This leads to the definition of idealistic exponents with history.

2.1 Motivation — A first approach via polyhedra

First let us explain why polyhedra are useful in the context of resolution of singularities. For this we introduce the Newton polyhedron of an idealistic exponent and deduce how this yields the d_x-invariant.

Let $\mathbb{E} = (J, b)$, $x \in \mathrm{Sing}\,(\mathbb{E})$, $(R, \mathfrak{m}, K)$ and $(u, y) = (u_1, \ldots, u_e; y_1 \ldots, y_r)$ be as in Setup A. Set $n = e + r$. Let $(f) = (f_1, \ldots, f_m)$ be a set of generators of J. In the $\mathfrak{m}$-adic completion of R we can write each element $g \in J$ as

$$g = \sum_{(A,B)\in\mathbb{Z}_0^n} C_{A,B}\, u^A\, y^B \tag{2.2}$$

with coefficients $C_{A,B} \in R^\times \cup \{0\}$. Denote by $C_{A,B,i}$ the coefficients of the expansion of f_i, $1 \le i \le m$.

Definition 2.1.1. *For the given data we introduce the following objects.*

(1) The Newton polyhedron $\Delta^N(\mathbb{E}, u, y)$ (or $\Delta^N_x(\mathbb{E}, u, y)$) of $\mathbb{E} = (J, b)$ at x with respect to (u, y) is defined to be the smallest closed convex subset of $\mathbb{R}_0^n$ containing all elements of the set

$$S(f, b; u, y) := \left\{ \frac{(A, B)}{b} + \mathbb{R}_0^n \;\middle|\; 1 \le i \le m \,\wedge\, C_{A,B,i} \ne 0 \,\wedge\, |B| \le b \right\}.$$

Let $\mathbb{E}'$ be another idealistic exponent on Z which is singular at x. Then $\Delta^N(\mathbb{E} \cap \mathbb{E}', u, y) \subset \mathbb{R}_0^n$ denotes the smallest closed convex subset containing $\Delta^N(\mathbb{E}, u, y)$ and $\Delta^N(\mathbb{E}', u, y)$.

(2) Using this we define the polyhedron $\Delta(\mathbb{E}, u, y) = \Delta_x(\mathbb{E}, u, y)$ of $\mathbb{E} = (J, b)$ at x with respect to (u, y) as the Newton polyhedron of the idealistic coefficient exponent with respect to (y);

$$\Delta(\mathbb{E}, u, y) := \Delta^N(\mathbb{D}_x(\mathbb{E}, u, y), u) \subseteq \mathbb{R}_0^e.$$

Further $\Delta(\mathbb{E} \cap \mathbb{E}', u, y) \subset \mathbb{R}_0^e$ denotes the smallest closed convex subset containing $\Delta(\mathbb{E}, u, y)$ and $\Delta(\mathbb{E}', u, y)$.

If there is no confusion possible, we just say $\Delta^N(\mathbb{E}, u, y)$ is the Newton polyhedron of $\mathbb{E}$ and $\Delta(\mathbb{E}, u, y)$ is the polyhedron of $\mathbb{E}$.

These polyhedra are not necessarily invariant under the equivalence relation $\sim$, see Example 2.1.9. But they are independent of the choice of the generators $(f) = (f_1, \ldots, f_m)$ of J. We could define $\Delta^N(\mathbb{E}, u, y)$ to be the smallest closed convex subset of $\mathbb{R}_0^n$ containing all the elements of the set

$$\widetilde{S}(\mathbb{E}, u, y) := \left\{ \frac{(\widetilde{A}, \widetilde{B})}{b} + \mathbb{R}_0^n \;\middle|\; \exists\, g = \sum_{(A,B)} C_{A,B}\, u^A\, y^B \in J \,:\, C_{\widetilde{A},\widetilde{B}} \ne 0 \,\wedge\, |\widetilde{B}| \le b \right\}.$$

In fact, denote by $\Delta(S)$ the polyhedron generated by some set $S \subset \mathbb{R}_0^n$. Then we have:

Lemma 2.1.2. *The Newton polyhedron does not depend on the choice of the generating set $(f) = (f_1, \ldots, f_m)$ of J. More precisely,*

$$\Delta(\, S(f, b; u, y)\,) = \Delta(\, \widetilde{S}(\mathbb{E}, u, y)\,).$$

Proof. Since $f_1, \ldots, f_m \in J$, we get the inclusion $\Delta(\, S(f, b; u, y)\,) \subseteq \Delta(\, \widetilde{S}(\mathbb{E}, u, y)\,)$. On the other hand, let $g \in J = \langle f_1, \ldots, f_m \rangle$. Then $g = \sum_{i=1}^{m} \lambda_i f_i$ for $\lambda_i \in R$ and therefore we get that in the expansion of g every $(A, B) \in \mathbb{Z}_0^n$ with non-zero coefficient and $|B| \leq b$ is contained in $\Delta(\, S(f, b; u, y)\,)$. This yields $\Delta(\, S(f, b; u, y)\,) = \Delta(\, \widetilde{S}(\mathbb{E}, u, y)\,)$. □

The definition of the idealistic coefficient exponent implies that $\Delta(\mathbb{E}, u, y)$ is the smallest convex subset of $\mathbb{R}_0^e$ containing

$$S_*(f, b; u, y) := \left\{ \frac{A}{b - |B|} + \mathbb{R}_0^e \;\middle|\; 1 \leq i \leq m \,\wedge\, C_{A,B,i} \neq 0 \,\wedge\, |B| < b \right\}. \tag{2.3}$$

Proposition 2.1.3. *The polyhedron $\Delta(\mathbb{E}, u, y)$ associated to an idealistic exponent $\mathbb{E} = (J, b) = (\langle f \rangle, b)$ on R is a certain projection of the corresponding Newton polyhedron $\Delta^N(\mathbb{E}, u, y)$.*

Proof. Let $v = (v_1, \ldots, v_n) := \dfrac{(A, B)}{b} \in S(f, b; u, y)$ with

$$(A, B) = (A_1, \ldots, A_e; B_1, \ldots B_r) \in \mathbb{Z}_0^n \quad \text{and} \quad |B| = B_1 + \ldots + B_r < b.$$

We project v from the point $C^{(1)} = (0, \ldots, 0, 1) \in \mathbb{R}_0^n$ onto $\mathbb{R}^{n-1} \times \{0\}$. and denote the corresponding projection map by

$$\pi_n : \{w = (w_1, \ldots, w_n) \in \mathbb{R}_0^n \mid w_n < 1\} \to \mathbb{R}_0^{n-1}.$$

Since $B_r \leq |B| < b$, we have $v_n < 1$ and the projection makes sense. Then the image point is

$$\begin{aligned} \pi_n(v) &= \left(\frac{v_1}{1 - v_n}, \ \ldots, \ \frac{v_{n-1}}{1 - v_n} \right) = \\ &= \left(\frac{A_1}{b - B_r}, \ \ldots, \ \frac{A_e}{b - B_r}, \ \frac{B_1}{b - B_r}, \ \ldots, \ \frac{B_{r-1}}{b - B_r} \right). \end{aligned} \tag{$*$}$$

One gets this as follows: The projection line $\mathcal{L} : \mathbb{R}_0 \to \mathbb{R}_0^n$ from $C^{(1)} = (0, \ldots, 0, 1)$ through v is given by $\mathcal{L}(\lambda) = C^{(1)} + \lambda \cdot (v - C^{(1)})$. We are at the point $\pi_n(v)$, if the n-th coordinate $(\mathcal{L}(\lambda_0))_n = C_n^{(1)} + \lambda_0 \cdot (v_n - C_n^{(1)}) = 1 + \lambda_0 \cdot (v_n - 1)$ is zero. Since $v_n < 1$ this is only the case for $\lambda_0 = \dfrac{1}{1 - v_n}$ and this shows $(*)$. (The second part is just putting in the definition of the v_i).

We have

$$(\pi_n(v))_{n-1} = \frac{B_{r-1}}{b - B_r} \leq \frac{B_1 + \ldots + B_{r-1}}{b - B_r} = \frac{|B| - B_r}{b - B_r} < \frac{b - B_r}{b - B_r} = 1.$$

Hence we can do the last step again: We project from $C^{(2)} = (0, \ldots, 0, 1) \in \mathbb{R}_0^{n-1}$ onto $\mathbb{R}^{n-2} \times \{0\}$, get $\pi_{n-1} : \{w = (w_1, \ldots, w_{n-1}) \in \mathbb{R}_0^{n-1} \mid w_{n-1} < 1\} \to \mathbb{R}_0^{n-2}$ and

$$\pi_{n-1}(\pi_n(v)) = \left(\frac{A_1}{b - (B_{r-1} + B_r)}, \quad \ldots, \quad \frac{B_{r-2}}{b - (B_{r-1} + B_r)} \right) \in \mathbb{R}_0^{n-2}.$$

By the same argument as above we have $(\pi_{n-1}(\pi_n(v)))_{n-2} < 1$ and we go on.

After r steps we have $\pi^{(r)} := \pi_{n-r+1} \circ \cdots \circ \pi_n$ and

$$\pi^{(r)}(v) = \left(\frac{A_1}{b - |B|}, \quad \ldots, \quad \frac{A_e}{b - |B|} \right) = \frac{A}{b - |B|} \in \mathbb{R}_0^{n-r} = \mathbb{R}_0^e.$$

Hence after these r step-by-step projections, we get a point of the generating set $S_*(f, b; u, y)$ of the polyhedron $\Delta(\mathbb{E}; u, y)$. Further those points with $|B| \geq b$ can be ignored in the projection $\pi^{(r)}$, because they don't map to $\mathbb{R}_0^e$. Therefore we have seen

$$\pi^{(r)} \left(\Delta^N(\mathbb{E}, u, y) \right) = \Delta(\mathbb{E}, u, y).$$

□

Up to now we characterized the projection above via several step-by-step projections. Later we give a direct description of $\pi^{(r)}$ in only one projection, see Lemma 2.4.1.

Corollary 2.1.4. *The polyhedron $\Delta(\mathbb{E}, u, y)$ of an idealistic exponent $\mathbb{E} = (J, b)$ is independent of the chosen set of generators $(f) = (f_1, \ldots, f_m)$.*

Proof. This is an immediate consequence of Lemma 2.1.2 and Proposition 2.1.3. □

Corollary 2.1.5. *Let $(x) = (u_1, \ldots, u_d)$, $d < e$, be a subsystem of (u) and denote $(z) = (z_1, \ldots, z_t) = (u_{d+1}, \ldots, u_e, y_1, \ldots, y_r)$. Then $\Delta(\mathbb{E}, x, z)$ is a projection of $\Delta(\mathbb{E}, u, y)$*

Proof. Let $(D, C, B) = (D_1, \ldots, D_d; C_1, \ldots, C_{e-d}; B_1, \ldots, B_r) \in \mathbb{Z}_0^n$. Suppose $|C| + |B| < b$. Then the proof of Proposition 2.1.3, this time with $v := \dfrac{(D, C)}{b - |B|}$, shows the claim. □

Let us now come to the connection between the d_x-invariant and the polyhedron associated to an idealistic exponent.

Definition 2.1.6. *Let $\Delta \subset \mathbb{R}_0^n$ be any subset. We define*

$$\delta(\Delta) := \inf\{\, |v| = v_1 + \ldots + v_n \mid v = (v_1, \ldots, v_n) \in \Delta \,\}.$$

If $\Delta = \Delta(\mathbb{E}, u, y)$, then we set $\delta_x(\Delta(\mathbb{E}, u, y)) := \delta(\Delta(\mathbb{E}, u, y))$.

Recall that the d_x-invariant is defined to be the order of the idealistic coefficient exponent (see Definition 1.3.5). Therefore we have

Lemma 2.1.7. *Let $\mathbb{E} = (J, b)$ and (u, y) be as in Setup A. Then*

$$\delta_x(\Delta(\mathbb{E}, u, y)) = d_x(\mathbb{E}, u, y).$$

Hence we get the connection between the d_x-invariant and the polyhedra defined above. Although the polyhedra $\Delta(\mathbb{E}, u, y)$ may change under $\sim$, we have by Corollary 1.3.6 that $\delta_x(\Delta(\mathbb{E}, u, y))$ is invariant under the equivalence.

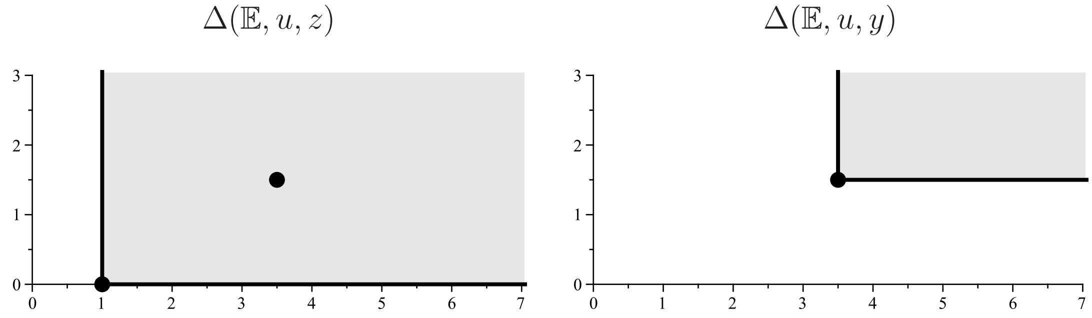
As we explained in Remark 1.4.5 $d_x(\mathbb{E}, u, y)$ and therefore $\delta_x(\Delta(\mathbb{E}, u, y))$ does not depend on the choice of the maximal contact hypersurface $V(y)$, if we consider the case $char(K) = 0$ or $b < char(K)$. Later we see how we can deduce from $\delta_x(\Delta(\mathbb{E}, u, y))$ the invariant of Bierstone and Milman. Therefore the previous considerations already suffice to prove Main Theorem 1.

But we want to show something more. In the following two examples we see that $\delta_x(\Delta(\mathbb{E}, u, y))$ depends on the choice of (y) and further the polyhedra (and thus the Newton polyhedra) of equivalent idealistic exponents may differ. Nevertheless, we want to prove that *for arbitrary characteristic* we are able to maximize $\delta_x(\Delta(\mathbb{E}, u, y))$ with respect to the choices for (y), so that the obtained number depends only on $\mathbb{E}$, x and (u). For this we introduce the intrinsic polyhedron $\Delta_x(\mathbb{E}, u)$ in section 2.3.

Example 2.1.8. The number $\delta_x(\Delta(\mathbb{E}, u, y))$ is not necessarily independent of (y). In Example 1.4.1 we considered the idealistic exponent

$$\mathbb{E} = (y^2 + u_1^7 u_2^3, 2) = (z^2 + 2zu_1^2 + u_1^4 + u_1^7 u_2^3, 2)$$

over any field K. Recall that $y := z + u_1^2$ and x was the origin. Then we get $\delta_x(\Delta(\mathbb{E}, u, y)) = 5$ and $\delta_x(\Delta(\mathbb{E}, u, z)) = 1$. The picture looks as follows:



Example 2.1.9. We show by example that the Newton polyhedron and the polyhedron of $\mathbb{E}$ may change under the equivalence $\sim$. The origin of this example is [BM5], Example 5.14, p.788 and it has been slightly modified and worked out for our setting together with Vincent Cossart.

Let $K = \mathbb{C}$, $d \in \mathbb{Z}_+, d \geq 2$. We look at the origin of $\mathbb{A}^4_{\mathbb{C}}$. Consider the two idealistic exponents

$$\begin{aligned}\mathbb{E}_1 &= (z^d - x^{d-1}y^{d-1},\, d) \,\cap\, (t,\, 1)\\ \mathbb{E}_2 &= (z^d - x^{d-1}y^{d-1},\, d) \,\cap\, (t^{d-1} - x^{d-2}y^{d-1},\, d-1)\end{aligned}$$

First, (t, z) yields the directrix in both cases; therefore $(u) = (x, y)$ and $(y) = (t, z)$.

Claim: $\mathbb{E}_1 \sim \mathbb{E}_2$, *but the polyhedra differ.*

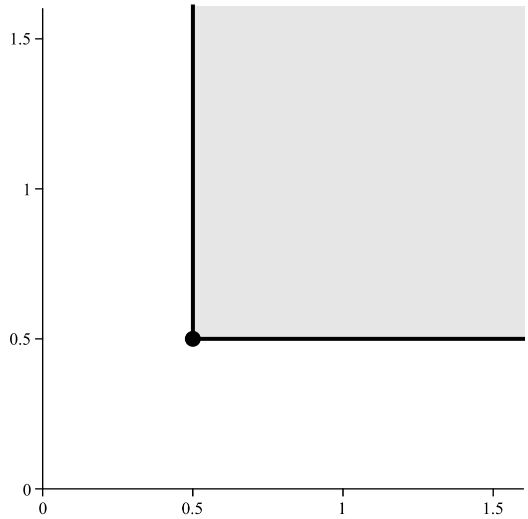
We see immediately that the generating set of the polyhedron associated to $\mathbb{E}_1$ is

$$V_1 = \left\{ \left(\frac{d-1}{d}, \frac{d-1}{d} \right) \right\}$$

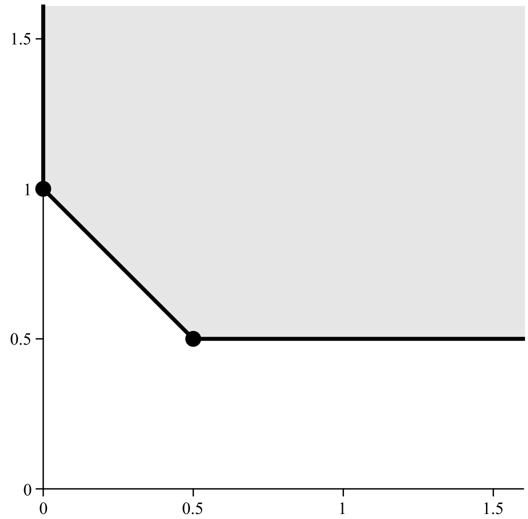
and the one for $\mathbb{E}_2$ is

$$V_2 = \left\{ \left(\frac{d-1}{d}, \frac{d-1}{d} \right); \left(\frac{d-2}{d-1}, 1 \right) \right\}.$$

Clearly the polyhedra are different. This implies that the Newton polyhedra differ. The picture for $d = 2$ looks as follows:



$\Delta(\mathbb{E}_1; t, z; x, y)$



$\Delta(\mathbb{E}_2; t, z; x, y)$

Now we show $\mathbb{E}_1 \sim \mathbb{E}_2$. For this we use the Diff-Theorem 1.1.13. Let $\mathcal{D} = \dfrac{\partial}{\partial x}$, then we get

$$(z^d - x^{d-1}y^{d-1}, d) \sim (z^d - x^{d-1}y^{d-1}, d) \cap (x^{d-2}y^{d-1}, d-1).$$

Further $\mathcal{D} = \dfrac{\partial^{d-2}}{\partial t^{d-2}}$ yields

$$(t^{d-1} - x^{d-2}y^{d-1}, d-1) \sim (t^{d-1} - x^{d-2}y^{d-1}, d-1) \cap (t, 1),$$

where also $(t^{d-1} - x^{d-2}y^{d-1}, d-1) \cap (t, 1) \sim (x^{d-2}y^{d-1}, d-1) \cap (t, 1)$. If we apply this for $\mathbb{E}_1$ and $\mathbb{E}_2$, then we see that both are equivalent to

$$(z^d - x^{d-1}y^{d-1},\, d) \cap (x^{d-2}y^{d-1},\, d-1) \cap (t,\, 1)$$

It is still possible to simplify the above: with the Diff Theorem for $\mathcal{D} = \dfrac{\partial^{d-1}}{\partial z^{d-1}}$ we get $(z, 1)$ and since $(x^{d-1}y^{d-1},\, d) \cap (x^{d-2}y^{d-1},\, d-1)$ is equivalent to $(x^{d-1}y^{d-1},\, d)$ (use $\mathcal{D} = \dfrac{\partial}{\partial x}$), the above idealistic exponents are equivalent to

$$(t,\, 1) \cap (z,\, 1) \cap (x^{d-1}y^{d-1},\, d).$$

This proves the claim and shows further that $V(t, z)$ has maximal contact. So it is possible that the polyhedra differ.

The last example plays also a crucial role if there exist exceptional components of a resolution process. It forces us to introduce idealistic exponents with history, which take care of the exceptional components (see section 2.6).

2.2 Hironaka's characteristic polyhedron

Now we recall a slightly modified version of Hironaka's definition of the characteristic polyhedron associated to (J, u), where J is an ideal in the local ring R and (u) denotes a system of elements in R as in Setup A; for references see [H2], [C1] or [CJS].

In this section there does not appear any assigned number $b \in \mathbb{Q}_+$, but we explain later how to deduce from this a polyhedron corresponding to an idealistic exponent $\mathbb{E} = (J, b)$, which contains all the necessary informations.

Let $L : \mathbb{R}^e \to \mathbb{R}$ be a linear form. This means there exist $c_1, \ldots, c_e \in \mathbb{R}$ such that for $A = (A_1, \ldots, A_e) \in \mathbb{R}^e$

$$L(A) = \sum_{i=1}^{e} c_i A_i.$$

It is called *positive* (resp. *semi-positive*), if it takes only positive (resp. non-negative) values on $\mathbb{R}^e_0 \setminus \{0\}$ or equivalently, if we have $c_i > 0$ (resp. $c_i \geq 0$) for all $i \in \{1, \ldots, e\}$. The set of all positive (resp. semi-positive) linear forms on $\mathbb{R}^e$ is denoted by $\mathbb{L}_+$ (resp. $\mathbb{L}_0$).

Definition 2.2.1. *Let $L \in \mathbb{L}_+$ be a positive linear form on $\mathbb{R}^e$. Then we define the valuation v_L on R by*

$$v_L(f) := \sup\{\, L(A) + |B| \mid f \in u^A y^B R \,\}$$

for $f \in R$. Further we set for $c \in \mathbb{R}_+$

(1) $I(L;\, c)_{u,y} := \langle\, u^A y^B \mid L(A) + |B| \geq c \,\rangle_R = \{f \in R \mid v_L(f) \geq c\}$,

(2) $I^+(L;\, c)_{u,y} := \langle\, u^A y^B \mid L(A) + |B| > c \,\rangle_R = \{f \in R \mid v_L(f) > c\}$,

and

$$gr_L(R) := \bigoplus_{c \in \mathbb{R}_+} I(L;\, c)_{u,y} / I^+(L;\, c)_{u,y}.$$

One can show easily that v_L is a valuation, in particular we have $v_L(fg) = v_L(f) + v_L(g)$ and $v_L(f + g) \geq \inf\{v_L(f), v_L(g)\}$ for $f, g \in R$.

Moreover, $\{c \in \mathbb{R}_+ \mid I(L;\, c)_{u,y}/I^+(L;\, c)_{u,y} \neq 0\}$ is a discrete subset of $\mathbb{R}_+$.

Definition 2.2.2. *Let $L \in \mathbb{L}_+$ be a positive linear form. Let $f \in R$ with $v_L(f) = c \in \mathbb{R}_+$. The class of f in $I(L;\, c)_{u,y}/I^+(L;\, c)_{u,y}$ is called the initial form of f with respect to L (or L-initial form of f for short), denoted by $in(f; L)_{u,y}$. Moreover we write $In(J; L)_{u,y}$ for the homogeneous ideal in $gr_L(R)$ given by*

$$In(J; L)_{u,y} := \langle in(f; L)_{u,y} \mid f \in J \rangle_{gr_L(R)}.$$

Let $U_i = in(u_i; L)_{u,y}$ and $Y_j = in(y_j; L)_{u,y}$, $1 \leq i \leq e$ and $1 \leq j \leq r$. Then we can identify $gr_L(R)$ with the graded K-algebra $K[U,Y] = \bigoplus_{c\in\mathbb{R}} \mathcal{A}_c$, where for each $c \in \mathbb{R}$ the monomials $\{U^A Y^B \mid v_L(U^A Y^B) = L(A) + |B| = c\}$ build a base of the part $\mathcal{A}_c$, which is homogeneous of degree c (see [H2]).

Let $f \in R$ and $c = v_L(f)$. We can write f (at least in the $\mathfrak{m}$-adic completion of R) as $f = \sum_{(A,B)} C_{A,B}\, u^A\, y^B$ with coefficients $C_{A,B} \in R^\times \cup \{0\}$ and $(A,B) \in \mathbb{Z}_0^{r+e}$. Then

$$in(f; L) := in(f; L)_{u,y} = \sum_{\substack{(A,B)\\ L(A)+|B|=c}} \overline{C_{A,B}}\, U^A\, Y^B, \tag{2.4}$$

where $\overline{C_{A,B}}$ denotes the image of $C_{A,B}$ in $gr_L(R)$.

One sees easily that for $f, g, h \in R$ with $v_L(f) = v_L(g)$ we have

$$in(f;L) + in(g;L) = \begin{cases} 0 & \text{, if } in(f;L) = -in(g;L), \\ in(f+g;L) & \text{, else,} \end{cases}$$
$$in(f;L) \cdot in(h;L) = in(f \cdot h; L).$$

If $L = L_0$ is given by $L(A) = |A|$, then $in(g, L_0)$ coincides with the initial form $in_{\mathfrak{m}}(g) = g \mod \mathfrak{m}^{d+1}$ with respect to $\mathfrak{m}$, where d denotes the order of g in $\mathfrak{m}$.

Definition 2.2.3. *Let $R' := R/\langle u\rangle$, $\mathfrak{m}' = \mathfrak{m}R'$ and $J' = JR'$. We may identify (y) with its image in R' and $\mathfrak{m}' = \langle y\rangle$. Then we denote by $In_{\mathfrak{m}'}(J')$ the initial ideal of J' with respect to $\mathfrak{m}'$. This is the homogeneous ideal in $gr_{\mathfrak{m}'}(R') = K[Y]$ given by*

$$In_{\mathfrak{m}'}(J') = \langle\, in_{\mathfrak{m}'}(g') \mid g' \in J' \,\rangle = \bigoplus_{c\in\mathbb{Z}_0} (\,(J' \cap (\mathfrak{m}')^c) + (\mathfrak{m}')^{c+1}\,)/(\mathfrak{m}')^{c+1}.$$

Definition 2.2.4. *Let $\Delta \subset \mathbb{R}_0^e$ be any subset and $L \in \mathbb{L}_0$ a semi-positive linear form on $\mathbb{R}_0^e$. We define*

$$\delta_L(\Delta) := \inf\{\, L(v) \mid v \in \Delta \,\}.$$

Further we set

$$\Delta(L) := \{\, v \in \mathbb{R}_0^e \mid L(v) \geq 1 \,\}.$$

If $L = L_0$ is the linear form given $L(v) = |v|$, then $\delta_{L_0}(\Delta) = \delta(\Delta)$ (for $\delta(\Delta)$ see Definition 2.1.6).

Lemma 2.2.5. *Let $L \in \mathbb{L}_+$ be a positive linear form on $\mathbb{R}^e$ and $c \in \mathbb{R}_+$. We can associate to L the positive linear form $L^{(n)}$ on $\mathbb{R}^n$ defined by $L^{(n)}(A,B) := L(A) + |B|$ for some $(A,B) \in \mathbb{R}^e \times \mathbb{R}^{n-e} = \mathbb{R}^n$. Then*

$$g \in I(L;\, c)_{u,y} \;\Leftrightarrow\; \Delta^N((g,c),u,y) \subset \Delta(L^{(n)}).$$

Proof. This follows immediately from the equivalence

$$L(A) + |B| \geq c \ \Leftrightarrow \ L^{(n)}\left(\frac{(A,B)}{c}\right) \geq 1.$$

□

Definition 2.2.6. *Let $J \subset R$ and (u, y) as in Setup A.*

(1) $\Delta(J; u, y)$ is defined to be the intersection of all $\Delta(L)$ for all $L \in \mathbb{L}_+$ satisfying the condition

$$In(J; L)_{u,y} = (In_{\mathfrak{m}'}(J'))\, K[U, Y]. \tag{2.5}$$

Thus

$$\Delta(J; u, y) = \bigcap_{\substack{L \in \mathbb{L}_+ \\ (2.5)}} \Delta(L).$$

(2) $\Delta(J; u)$ is defined to be the intersection of all $\Delta(J; u, y)$ for all (y) satisfying (2.1), i.e. which extend (u) to a regular system of parameters for R;

$$\Delta(J; u) = \bigcap_{\substack{(y) \\ (u,y) \text{ RSP for } R}} \Delta(J; u, y) = \bigcap_{\substack{(y) \\ (2.1)}} \Delta(J; u, y).$$

We call $\Delta(J; u)$ the (first) characteristic polyhedron of J with respect to (u).

Let us now come to the *concrete description of the characteristic polyhedron.* Here we follow [CJS], section 7.

Definition 2.2.7. *A closed convex subset $\Delta \subset \mathbb{R}_0^e$ is called an F-subset if for every $v \in \Delta$ the set $v + \mathbb{R}_0^e$ is also contained in Δ.*

The essential boundary of an F-subset Δ is given by

$$\partial\Delta := \{v \in \Delta \mid \forall\, w \in \Delta : v \in w + \mathbb{R}_0^e \Rightarrow w = v\}.$$

Further we set $\Delta^+ := \Delta \setminus \partial\Delta$.

A point $v \in \Delta$ is called a vertex of Δ if there exists a positive linear form $L_v \in \mathbb{L}_+$ such that $\Delta \cap \{w \in \mathbb{R}_0^e \mid L_v(w) = \delta_{L_v}(\Delta)\} = \{v\}$. The set of vertices of Δ is denoted by $Vert(\Delta)$.

The Newton polyhedron and the polyhedron of an idealistic exponent are examples for F-subsets.

Definition 2.2.8. *Let $(R, \mathfrak{m}, K)$ and (u, y) be as in Setup A. Let $f \in \mathfrak{m}$ with $f \notin \langle u \rangle \subset R$ and consider an expansion of f in $\widehat{R}$ as in (2.2),*

$$f = \sum_{(A,B)\in\mathbb{Z}_0^n} C_{A,B}\, u^A\, y^B \tag{2.6}$$

for some coefficients $C_{A,B} \in R^\times \cup \{0\}$.

Denote $R' = R/\langle u \rangle$, $\mathfrak{m}' = \mathfrak{m}R'$ and $f' = f \mod \langle u \rangle$. Let $n_{(u)}(f)$ be the multiplicity of f' in $\mathfrak{m}'$, i.e. $f' \in \langle \mathfrak{m}' \rangle^{n_{(u)}(f)} \setminus \langle \mathfrak{m}' \rangle^{n_{(u)}(f)+1}$.

(1) We define the polyhedron $\Delta(f, u, y)$ as the polyhedron associated to the idealistic exponent $(f, n_{(u)}(f))$; by (2.3) it is the smallest F-subset containing the points

$$S^*(f, u, y) := \left\{ \frac{A}{n_{(u)}(f) - |B|} \;\middle|\; C_{A,B} \neq 0 \,\wedge\, |B| < n_{(u)}(f) \right\}.$$

(2) Let $v \in \mathbb{R}_0^e \setminus \Delta(f, u, y)^+$. The v-initial of f is defined as

$$in_v(f) := in_v(f)_{u,y} := in_0(f)_{u,y} + in_v(f)^+_{u,y} \in K[U, Y],$$

where we set (using (2.6))

$$in_0(f) := in_0(f)_{u,y} := \sum_{\substack{B\in\mathbb{R}_0^r \\ |B| = n_{(u)}(f)}} \overline{C_{0,B}}\, Y^B \in K[Y],$$

$$in_v(f)^+ := in_v(f)^+_{u,y} := \sum_{(A,B)} \overline{C_{A,B}}\, U^A\, Y^B \in K[U, Y],$$

and the last sum ranges over those (A, B) with $\dfrac{A}{n_{(u)}(f) - |B|} = v$.

(3) Let $L : \mathbb{R}^e \to \mathbb{R}$ be a semi-positive linear form, then we write $\delta_L(f, u, y) := \delta_L(\Delta(f, u, y))$ and we have

$$\delta_L(f, u, y) = \inf \left\{ \frac{L(A)}{n_{(u)}(f) - |B|} \;\middle|\; C_{A,B} \neq 0 \wedge |B| < n_{(u)}(f) \right\}.$$

Further $E_L := \Delta(f, u, y) \cap \{v \in \mathbb{R}_0^e \mid L(v) = \delta_L(f, u, y)\}$ is a face of the polyhedron $\Delta(f, u, y)$ with slope L and we define the E_L-initial of f by

$$in_{E_L}(f) := in_{E_L}(f)_{u,y} := in_0(f) + \sum_{(A,B)} \overline{C_{A,B}}\, U^A\, Y^B,$$

where the last sum ranges over those (A, B) with $|B| < n_{(u)}(f)$ and

$$\frac{L(A)}{n_{(u)}(f) - |B|} = \delta_L(f, u, y).$$

Remark 2.2.9. *(1) The expression $in_v(f)_{u,y}$ is independent of (2.6).*
If $w \notin \Delta(f, u, y)$, then $in_w(f)_{u,y} = in_0(f)_{u,y}$. On the other hand if $v \in Vert(\Delta(f, u, y))$, then $in_v(f)_{u,y} \neq in_0(f)_{u,y}$.

(2) If E_L is bounded, then $in_{E_L}(f)_{u,y}$ is independent of (2.6), $in_{E_L}(f) \in K[U, Y]$ and

$$in_{E_L}(f) = in_0(f) + \sum_{v \in E_L} in_v(f)^+$$

(3) For a vertex $v \in Vert(\Delta(f, u, y)) \subseteq \partial\Delta(f, u, y)$ there exists a positive linear form $L \in \mathbb{L}_+$ on $\mathbb{R}^e$ such that $in_v(f) = in_{E_L}(f)$; for example one could take the linear form which is used to characterize the point as a vertex.
In general, $in_{E_L}(f) \neq in(f, L)$, see Definition 2.2.2 and the remark after that.

(See [CJS], Lemma 7.3, p.97, for part *(1)* and *(2)*)

Since we are not only interested in hypersurfaces, we have to extend the previous definitions to a finite system of elements in R.

Definition 2.2.10. *Let $(f) = (f_1, \ldots, f_m)$ be a finite system of elements in the maximal ideal $\mathfrak{m} = \langle u, y\rangle$ of R with $f_i \notin \langle u\rangle$.*

(1) The polyhedron $\Delta(f, u, y) = \Delta((f_1, \ldots, f_m), u, y)$ is defined to be the smallest F-subset containing the union $\bigcup_{i=1}^m \Delta(f_i, u, y)$.

(2) Let $v \in \mathbb{R}^e_0 \setminus \Delta(f, u, y)^+$. The v-initial of (f) is defined as

$$in_v(f) := in_v(f)_{u,y} := (in_v(f_1), \ldots, in_v(f_m)).$$

Similarly we define $in_{E_L}(f)$ for a face E_L of $\Delta(f, u, y)$.

(3) The set of essential vertices of $\Delta(f, u, y)$ is defined by

$$\widetilde{Vert}(f, u, y) := \{v \in \mathbb{R}^e_0 \setminus \Delta(f, u, y)^+ \mid \exists\, i \in \{1, \ldots, m\} : in_v(f_i) \neq in_0(f_i)\}.$$

We have

$$Vert(f, u, y) \subseteq \widetilde{Vert}(f, u, y) \subseteq \partial\Delta(f, u, y).$$

Lemma 2.2.11. *Let the situation be as in the previous definition. Let* $\beta \in \mathbb{Z}_+$ *be a natural number which is divisible by all* $n_{(u)}(f_i)$ *for* $1 \leq i \leq m$. *Then*

$$\widetilde{Vert}(f,u,y) \subset \frac{1}{\beta!} \cdot \mathbb{Z}_0^e.$$

In particular, the set of essential vertices (and therefore the set of vertices) of $\Delta(f,u,y)$ *is finite.*

Proof. The first part follows immediately by construction of $\Delta(f,u,y)$. Therefore $\beta! \cdot \widetilde{Vert}(f,u,y) = \{\beta! \cdot v \mid v \in \widetilde{Vert}(f,u,y)\} \subset \mathbb{Z}_0^e$ and by [H1], Ch. III, §7, p.244, the closed convex hull of $\beta! \cdot \widetilde{Vert}(f,u,y)$ in $\mathbb{Z}_0^e$ has a finite base, say $E_1, \ldots, E_s$. This implies that $\Delta(f,u,y)$ is generated by $\{\frac{1}{\beta!}E_1, \ldots, \frac{1}{\beta!}E_s\}$. □

Corollary 2.2.12. *For any* $(f) = (f_1, \ldots, f_m)$ *and* (u,y) *as before, there exist finitely many semi-positive linear forms* $L_1, \ldots, L_t \in \mathbb{L}_0$ *on* $\mathbb{R}^e$ *such that*

$$\Delta(f,u,y) = \Delta(L_1) \cap \ldots \cap \Delta(L_t).$$

Moreover, the coefficients of these linear forms are rational.

Proof. By the previous lemma there are only finitely many vertices and by definition of $\Delta(f,u,y)$ they are contained in $\mathbb{Q}$. This implies the corollary. □

We now have two polyhedra:

- the characteristic polyhedron $\Delta(J,u)$ associated to an ideal $J \subset R$ and a system of elements (u) that can be extended to a regular system of parameters for R;
- on the other hand we introduced the concrete polyhedron $\Delta(f,u,y)$ given by generators $(f) = (f_1, \ldots, f_m)$ of J and a chosen regular system of parameters (u,y).

The task is to find a suitable choice for (f) and (y) such that we get the equality $\Delta(J,u) = \Delta(f,u,y)$.

We start with a good set of generators $(f) = (f_1, \ldots, f_m)$, a so called (u)-standard base of J, and then introduce the procedure of vertex preparation in order to minimize the associated polyhedra.

Definition 2.2.13. *Let* $(R, \mathfrak{m}, K)$ *be a regular local ring,* $J \subset R$ *an ideal and* (u) *a system of elements in* R *as in Setup A. A system of non-zero elements* $(f) = (f_1, \ldots, f_m)$ *in* J *is called a* (u)*-standard base of* J*, if there exist* $(y) = (y_1, \ldots, y_r)$*, which extend* (u) *to a regular system of parameters, and a positive linear form* $L \in \mathbb{L}_+$ *such that* $in(f_i, L) = in_0(f_i) \in K[Y]$ *for all* $1 \leq i \leq m$ *and the following properties hold*

(1) $In(J;L)_{u,y} = \langle in_0(f_1), \ldots, in_0(f_m)\rangle \subset gr_{\mathfrak{m}}(R)$,

(2) *if* $n_i := \deg(in_0(f_i))$*, then* $n_1 \leq n_2 \leq \ldots \leq n_m$ *and*

(3) *for all* $i \geq 1$ *we have* $in_0(f_i) \notin \langle in_0(f_1), \ldots, in_0(f_{i-1})\rangle$.

The datum (y, L) *is called a reference datum of the* (u)*-standard base.*

The conditions *(2)* and *(3)* mean that $(in_0(f_1), \ldots, in_0(f_m))$ is a standard base of $In(J;L)_{u,y}$. For a more detailed discussion on (u)-standard bases see [CJS], section 6 or [H2], beginning with Definition (2.20), p.264.

Endow $\mathbb{R}_0^r$ and $\mathbb{Z}_0^r$ with the ordering defined by $B \leq B'$ if $|B| < |B'|$ or if $|B| = |B'|$ and $B \leq B'$ in the lexicographical order. This is the lexicographical order of the vector $(|B|, B)$.

Note that this is a different ordering as the one used in [H1], Ch. III, §7, p.244f and [CJS], Definition 7.10, p.99. But the same proofs work in our setting and we need this slightly modified version in order to get the connection to [BM3], subsection “the diagram of initial exponents”, p.238f and (7.1), p.261f. (It is also used in [C1], Définition 7, p.15).

Definition 2.2.14. *Let* $g = \sum g_B Y^B \in K[Y]$ *be a polynomial. The exponent of* g *is defined by*

$$\exp(g) := \inf\{\, B \in \mathbb{Z}_0^r \mid g_B \neq 0 \,\}.$$

For an ideal $I \subset K[Y]$ *the exponent of* I *is the set of the exponents of all non-zero elements in* I,

$$\exp(I) := \{\, \exp(g) \mid 0 \neq g \in I \,\}.$$

Definition 2.2.15. *Let* $(F) = (F_1, \ldots, F_m)$ *be a system of elements in* $K[[U]][Y]$ *such that*

$$F_i = G_i(Y) + \sum_{|B|<n_i} P_{B,i}(U)\, Y^B$$

where $G_i(Y) = \sum_B C_{B,i}\, Y^B \in K[Y]$ *is homogeneous of degree* n_i *(*$C_{B,i} \in K$*) and* $P_{B,i}(U) \in K[[U]]$.

(1) *We say* $(G) = (G_1, \ldots, G_m)$ *is normalized if* $C_{B,i} = 0$ *for every* $B \in \exp(\langle G_1, \ldots, G_{i-1}\rangle)$.

(2) *The system* (F) *is called normalized if* (G) *is normalized and* $P_{B,i} = 0$ *for every* $B \in \exp(\langle G_1, \ldots, G_{i-1}\rangle)$.

Definition 2.2.16. *Let* $(f) = (f_1, \ldots, f_m)$ *be a finite system of elements in the maximal ideal* $\mathfrak{m} = \langle u, y\rangle$ *of* R *with* $f_i \notin \langle u\rangle$.

(1) (f, u, y) *is* 0*-normalized if*

$$(in_0(f_1), \dots, in_0(f_m))$$

is normalized in the sense of Definition 2.2.15 (1).

(2) (f, u, y) *is normalized at* $v \in \mathbb{R}_0^e \setminus \Delta(f, u, y)^+$ *if*

$$(in_v(f_1), \dots, in_v(f_m))$$

is normalized in the sense of Definition 2.2.15 (2).

(3) (f, u, y) *is normalized along a face* E_L *of* $\Delta(f, u, y)$ *if*

$$(in_{E_L}(f_1), \dots, in_{E_L}(f_m))$$

is normalized in the sense of Definition 2.2.15 (2).

(4) Suppose (f) *is a* (u)*-standard base of the ideal which it generates in* R*. Then we say* (f) *is a normalized* (u)*-standard base, if* (f, u, y) *is normalized in the sense of Definition 2.2.15 (2).*

Since $in_v(f) = in_0(f) + in_v(f)^+$ with $in_v(f)^+ \in K[U, Y] \setminus K[Y]$, the property for (f, u, y) of being normalized at v implies that it is also 0-normalized.

Let (f) be a normalized (u)-standard base. Then (f, u, y) is 0-normalized, because $G_i(Y) = in_0(f_i)$ (with the notation of Definition 2.2.15).

Definition 2.2.17. *Let* $(f) = (f_1, \dots, f_m)$ *be a finite system of elements in the maximal ideal* $\mathfrak{m} = \langle u, y\rangle$ *of* R *with* $f_i \notin \langle u\rangle$ *and* $v \in Vert(\Delta(f, u, y))$*. We say* (f, u, y) *is solvable at* v *if there exist* $\lambda_1, \dots, \lambda_r \in K[U]$ *such that*

$$in_v(f_i)_{u,y} = F_i(Y + \lambda),$$

where $F_i(Y) = in_0(f_i)_{u,y}$*,* $(Y + \lambda) = (Y_1 + \lambda_1, \dots, Y_r + \lambda_r)$ *and* $1 \le i \le m$*; in this case* $\lambda = (\lambda_1, \dots, \lambda_r)$ *is called a solution for* (f, u, y) *at* v*.*

If v is a vertex of the polyhedron, then the v-initial of f_i can not lie in $K[Y]$ for all i. Therefore a solution is non-zero if it exists.

Definition 2.2.18. *Let* $(f) = (f_1, \dots, f_m)$ *be a finite system of elements in the maximal ideal* $\mathfrak{m} = \langle u, y\rangle$ *of* R *with* $f_i \notin \langle u\rangle$*.*

(1) (f, u, y) *is prepared at* $v \in Vert(\Delta(f, u, y))$ *if* (f, u, y) *is normalized at* v *and not solvable at* v*.*

(2) (f, u, y) *is well-prepared if is prepared at any* $v \in Vert(\Delta(f, u, y))$*.*

With all this notation we can now give Hironaka's theorem, where he relates the polyhedra $\Delta(J,u)$ and $\Delta(f,u,y)$ (see [H2] Theorem (4.8), p.291).

Theorem 2.2.19. *Let $(R,\mathfrak{m},K)$ be a regular local ring with regular system of parameters (u,y). Let $(f)=(f_1,\dots,f_m)$ be a (u)-standard basis of the ideal $J\subset R$ with $f_i\notin\langle u\rangle$. Denote $R'=R/\langle u\rangle$, $\mathfrak{m}'=\mathfrak{m}R'$ and $J'=JR'$. Suppose:*

$$\left.\begin{array}{c}\text{There is no proper } K\text{-submodule } T\subset gr^1_{\mathfrak{m}'}(R') \text{ such that}\\ (\,In_{\mathfrak{m}'}(J')\cap K[T]\,)\,gr_{\mathfrak{m}'}(R') \;=\; In_{\mathfrak{m}'}(J').\end{array}\right\} \tag{2.7}$$

Let v be a prepared vertex of $\Delta(f,u,y)$. Then v is also a vertex of $\Delta(J,u)$. In particular, if (f,u,y) is well-prepared, then $\Delta(J,u)=\Delta(f,u,y)$.

The assumption (2.7) is in particular fulfilled if the system (Y) generates the ideal of the directrix of $In_{\mathfrak{m}'}(J')$.

Corollary 2.2.20. *The characteristic polyhedron $\Delta(J,u)$ has only a finite number of vertices.*

Proof. This follows from the theorem by Lemma 2.2.11. □

It is not obvious that we can find some well-prepared (f,u,y). In [H2] there is also a procedure given how to obtain this nice situation if R is complete.

Theorem 2.2.21 (Normalization)**.** *Let $(f)=(f_1,\dots,f_m)$ be a finite system of elements in the maximal ideal $\mathfrak{m}=\langle u,y\rangle$ of R with $f_i\notin\langle u\rangle$.*

Assume $(in_0(f_1),\dots,in_0(f_m))$ is a minimal base of the ideal which it generates. Let $v\in Vert(\Delta(f,u,y))$. Then there exist $x_{ij}\in\langle u\rangle\subset R$ $(1\le j<i\le m)$ such that $h=(h_1,\dots,h_m)$ with

$$h_i := f_i-\sum_{j=1}^{i-1}x_{ij}f_j$$

fulfills

(i) $\Delta(h,u,y)\subseteq\Delta(f,u,y)$.

(ii) If $v\in\Delta(h,u,y)$, then $v\in Vert(h,u,y)$ and (h,u,y) is normalized at v.

(iii) $Vert(f,u,y)\setminus\{v\}\subseteq Vert(h,u,y)$.

(iv) $in_{v'}(f)_{u,y}=in_{v'}(h)_{u,y}$ *for every* $v'\in Vert(f,u,y)\setminus\{v\}$.

Proof. See [H2], Lemma (3.14), p.281, and Lemma (3.15), p.281/282 (resp. [CJS], Theorem 7.19, p.102). □

Let us give an example that the vertex v may really vanish in the above theorem.

Example 2.2.22. Let $n_1, n_2 \in \mathbb{Z}_+$ be two positive integers with $n_1 < n_2$ and $v = (1,1) \in \mathbb{Z}_0^2$. Consider $f_1 := y_1^{n_1} + u_1^{3n_1}$ and $f_2 := y_2^{n_2} - (u_1u_2)^{v(n_2-n_1)}y^{n_1} + u_2^{4n_2}$. By an easy computation we see $Vert(\Delta(f,u,y)) = \{(3,0);\, v;\, (0,4)\}$. Set $h_1 := f_1$ and

$$h_2 := f_2 + (u_1u_2)^{v(n_2-n_1)} f_1 = y_2^{n_2} + u_2^{4n_2} + u_1^{2n_1+n_2}\, u_2^{n_2-n_1}.$$

This implies that $\Delta(h,u,y)$ is generated by $\left\{(3,0);\,(0,4);\, \dfrac{(2n_1+n_2,\, n_2-n_1)}{n_2}\right\}$ and therefore $v = (1,1) \notin \Delta(h,u,y)$.

Nevertheless, in the following special case the equality always holds. This is also the crucial case for our construction of the polyhedron of an idealistic exponent.

Lemma 2.2.23. *Let $(f) = (f_1,\dots,f_m)$ be as in the previous theorem and in addition let it be a (u)-standard base of the ideal which it generates in R. Let (u,y) be a regular system of parameters for R, $v \in Vert(\Delta(f,u,y))$ and let $(h) = (h_1,\dots h_m)$ be as in the theorem above.*
Assume $n_{(u)}(f_1) = \ldots = n_{(u)}(f_m) =: b$. Then $v \in \Delta(h,u,y)$.

Proof. By the assumption we have $n_{(u)}(h_i) = n_{(u)}(f_i) = b$ for all i, and thus

$$\Delta(h,u,y) = \Delta(h,b;u,y).$$

Moreover, $\langle h\rangle = \langle f\rangle$. Thus the Lemma is a consequence of Corollary 2.1.4, namely

$$\Delta(h,u,y) = \Delta(h,b;u,y) = \Delta(f,b;u,y) = \Delta(f,u,y).$$

$\square$

Similar to the Theorem 2.2.21 we have

Theorem 2.2.24 (Dissolution)**.** *Let $(f) = (f_1,\dots,f_m)$ be a finite system of elements in the maximal ideal $\mathfrak{m} = \langle u,y\rangle$ of R with $f_i \notin \langle u\rangle$. Let $v \in Vert(\Delta(f,u,y))$ and let $(d) = (d_1,\dots,d_r)$, $d_i \in \langle u\rangle \subset R$, be a solution for (f,u,y) at v. Set $z = y - d = (y_1 - d_1, \dots y_r - d_r)$. Then we have*

(i) $\Delta(f,u,z) \subseteq \Delta(f,u,y)$.

(ii) $v \notin \Delta(f,u,z)$ *and* $Vert(f,u,y) \setminus \{v\} \subseteq Vert(f,u,z)$.

(iii) $in_{v'}(f)_{u,z} = in_{v'}(f)_{u,y}|_{Y=Z} \in K[U,Z]$ *for every* $v' \in Vert(f,u,y)\setminus\{v\}$, *where* $Z = in_{\mathfrak{m}}(z)$ *and* $|_{Y=Z}$ *shall indicate that we use Z instead of Y without changing any of the exponents.*

Proof. See [H2], Lemma (3.10), p.279 (resp. [CJS], Theorem 7.22, p.103). □

We have already seen an example of vertex solution in Example 2.2.22, there we solved the vertex $v = (1, 1)$. Another one is Example 1.4.1, there we solved the vertex $(2, 0)$.

Additionally one could also show that the solution is of a very special form (see [CJS] Theorem 7.22(a), p.103). Namely, we have $d_i \in \langle u^v \rangle$ and the initial form of d_i is $\lambda_i = c_i U^v$ for some $c_i \in K$. Therefore a vertex v is only solvable if $v \in \mathbb{Z}_0^e$ and (as we already have remarked after the definition of solvable) if a solution exists, then it is always non-trivial.

We equip $\mathbb{R}_0^e$ with the order which is for $v \in \mathbb{R}_0^e$ given by the lexicographical order of $(|v|, v_1, \dots, v_e)$. Let $(f) = (f_1, \dots, f_m)$ be a finite system of elements in the maximal ideal $\mathfrak{m} = \langle u, y \rangle$ of R with $f_i \notin \langle u \rangle$. Assume $(in_0(f_1), \dots, in_0(f_m))$ is a minimal base of the ideal which it generates. Let $v(1) := \min\{v \in Vert(\Delta(f, u, y)) \subset \mathbb{R}_0^e\}$. We apply normalization (Theorem 2.2.21) on $v(1)$ and get (h, u, z). If $v(1)$ is still contained in the polyhedron, we try to solve this vertex (Theorem 2.2.24). Either we delete $v(1)$ from the polyhedron or (h, u, z) is prepared at $v(1)$ (i.e. normalized and not-solvable at $v(1)$). Then start again: let $v(2) := \min\{v \in Vert(\Delta(h, u, z)) \setminus \{v(1)\} \subset \mathbb{R}_0^e\}$ …and so on. We get:

Theorem 2.2.25 (Preparation)**.** *Let $(f) = (f_1, \dots, f_m)$ be a finite system of elements in the maximal ideal $\mathfrak{m} = \langle u, y \rangle$ of R with $f_i \notin \langle u \rangle$. Assume that $(in_0(f_1), \dots, in_0(f_m))$ is a minimal base of the ideal which it generates.*

For any positive integer $M \in \mathbb{Z}_+$, there exist $x_{ij}, d_l \in \langle u \rangle$ ($1 \leq j < i \leq m$ and $1 \leq l \leq r$) such that if we set $(z) := (y - d)$ with $z_l = y_l - d_l$ and $(g) = (g_1, \dots, g_m)$ with $g_i = f_i - \sum_{j=1}^{i-1} x_{ij} f_j$, then we have

(i) $\Delta(g, u, z) \subseteq \Delta(f, u, y)$ and

(ii) (g, u, z) is prepared at every vertex contained in $\{v \in \mathbb{R}_0^e \mid |v| < M\}$.

If we assume moreover that R is complete, then we can drop the restriction given by M and get that (g, u, z) is well-prepared.

Proof. See [H2], Theorem (3.17), p.283 (resp. [CJS], Theorem 7.24, p.104). □

Corollary 2.2.26. *If in Theorem 2.2.25 (f) is a (u)-standard base of the ideal $J \subset R$, then so is (g).*

Proof. See [H2], Corollary (3.17.4), p.285 (resp.[CJS], Corollary 7.26 (1), p.104). □

Remark 2.2.27. *In the above considerations the completeness of R seems to be necessary. But recently Cossart and Piltant have proved in [CP3] that this assumption on R can be dropped in the case of hypersurfaces.*

2.3 The characteristic polyhedron of an idealistic exponent

In this section we deduce the characteristic polyhedron of an idealistic exponent. Recall Setup A:

- ◇ $\mathbb{E} = (J, b)$ is an idealistic exponent on some regular scheme Z,
- ◇ $x \in \operatorname{Sing}(\mathbb{E})$,
- ◇ $(R = \mathcal{O}_{Z,x}, \mathfrak{m}, K)$ denotes the local ring of Z at x,
- ◇ $(u) = (u_1, \ldots, u_e)$ is a fixed system of elements in $\mathfrak{m}$ which can be extended to a regular system of parameters for R,
- ◇ $(y) = (y_1, \ldots, y_r)$ is a possible choice such that (u, y) is a regular system of parameters for R.

By abuse of notation we neglect the index x and write $\mathbb{E} = (J, b)$ in the local situation at x. Moreover, the crucial situation for us is if

$$\left.\begin{array}{l} \diamond\ (y) \text{ is part of a system which yields the directrix of } \mathbb{E} \text{ at } x, \\ \diamond\ (f) = (f_1, \ldots, f_m) \text{ is a } (u)\text{-standard base of } J, \\ \diamond \text{ assume } b \leq \operatorname{ord}_x(f_i)(\ \leq n_{(u)}(f_i)\ ) \text{ for all } i \in \{1, \ldots, m\}. \end{array}\right\} \tag{2.8}$$

We have already defined in section 2.1 a polyhedron associated to $(\mathbb{E}, u, y)$. More precisely, $\Delta(f, b; u, y) = \Delta^N(\mathbb{D}_x(\mathbb{E}, u, y), u) \subseteq \mathbb{R}_0^e$ is the smallest F-subset containing

$$S^*(f, b; u, y) := \left\{ \frac{A}{b - |B|} \;\middle|\; 1 \leq i \leq m \,\wedge\, C_{A,B,i} \neq 0 \,\wedge\, |B| < b \right\},$$

where we expand each $f_i = \sum_{(A,B)} C_{A,B,i}\, u^A\, y^B$ in $\widehat{R}$ as in (2.2).

Note the following extreme case: If $b < \operatorname{ord}_x(f_i)$ for all $i \in \{1, \ldots, m\}$, then the initial forms with respect to the number b are all zero. This means the directrix is the full tangent cone and the system (y) is empty. Then the generators $S^*(f, b; u, y)$ are of the form $\dfrac{A}{b}$ and we obtain the Newton polyhedron, which will in this case be our characteristic polyhedron. Since (y) is empty, there is no choice and the characteristic polyhedron will automatically be independent of (y). So the following discussions are trivial in this extreme case.

In general, the concrete polyhedron clearly depends on the choice of (y) and we would like to know if there is an intrinsic definition which can be achieved by some preparations of the vertices. The first attempt would be to deduce it from $\Delta(f,u,y)$, where the denominator of the generating points is given by $n_{(u)}(f_i)-|B|$ and not $b-|B|$. But in general preparedness of $\Delta(f,u,y)$ does not imply that of $\Delta(f,b;u,y)$ — the main obstruction is that we only have $b \leq n_{(u)}(f_i)$ for all i and especially $b < n_{(u)}(f_i)$ is possible for some i.

Example 2.3.1. Let K be a field of characteristic three and set $b=2$. Let $f_1 = z_1^2 + u_1^3$ and $f_2 = z_2^3 + z_2^2 u_2^2 + u_2^9$ and $J = \langle f_1, f_2 \rangle \subset K[u,z]$. Clearly condition (2.7) holds. Further (f_1, f_2) is a (u)-standard base of J with reference datum (z, L_0), where $L_0 : \mathbb{R}^2 \to \mathbb{R}$ is defined by $L_0(v) = |v|$ (see Definition 2.2.13). The polyhedron $\Delta(f,u,z)$ is generated by $\left\{ \frac{(3,0)}{2}, \frac{(0,2)}{1}, \frac{(0,9)}{3} \right\}$ and the vertices are

$$Vert(\Delta(f,u,z)) = \left\{ v := \left(\frac{3}{2}, 0\right), w := (0,2) \right\}.$$

We have $in_v(f) = (Z_1^2 + U_1^3, Z_2^3)$ and $in_w(f) = (Z_1^2, Z_2^3 + Z_2^2 U_2^2)$. Obviously both vertices are not solvable and further (f,u,z) is also normalized at v and w (recall Definition 2.2.16 and Definition 2.2.15 carefully). Hence $\Delta(f,u,z) = \Delta(J,u)$.

On the other hand, the vertices of $\Delta(f,b;u,z)$ $(b=2)$ are

$$Vert(\Delta(f,b;u,z)) = \left\{ v = \left(\frac{3}{2}, 0\right), \widetilde{v} := \left(0, \frac{9}{2}\right) \right\}.$$

We have $in_{\widetilde{v}}(f) = (Z_1^2, Z_2^3 + U_2^9)$ and since the characteristic is three, we can solve this vertex via $(x_1, x_2) := (z_1, z_2 + u_2^3)$. Thus not all vertices of $\Delta(f,b;u,z)$ are prepared.

Even worse, (z) does not fulfill the extra conditions (2.8) which we stated at the beginning of this section: Since the order of f_2 at the origin is three and thus bigger than $b=2$, the directrix of $\mathbb{E} = (\langle f_1, f_2 \rangle, 2)$ is only given by Z_1! If we set $y_1 := z_1$ and $u_3 := z_2$, then $f_2 \in \langle u_1, u_2, u_3 \rangle$, which means that the assumption $f_i \notin \langle u \rangle$ of Theorem 2.2.19 doesn't hold and it is not clear that we have $\Delta(J, u_1, u_2, u_3) = \Delta(f_1, f_2; u_1, u_2, u_3; y_1)$.

Therefore there is an essential difference between the polyhedron of the ideal J and the polyhedron of the idealistic exponent $\mathbb{E} = (J,b)$.

In particular, we have seen in the previous Example that the vertices of $\Delta(f,u,y)$ and $\Delta(f,b;u,y)$ need not be related. Of course, if

$$v := \frac{A}{n_{(u)}(f_i) - |B|} \in \Delta(f,u,y)$$

(for some i) is coming from some (A, B) with $|B| < b \leq n_{(u)}(f_i)$, then also the point $v' := \dfrac{A}{b - |B|}$ appears in $\Delta(f, b; u, y)$. But $v \in Vert(\Delta(f, u, y))$ does not necessarily imply $v' \in Vert(\Delta(f, b; u, y))$. Further if we have $b \leq |B| < n_{(u)}(f_i)$, then there is no corresponding point in $\Delta(f, b; u, y)$.

Recall condition (2.7) of Theorem 2.2.19: Let $I \subset R$ be any ideal. As before $R' = R/\langle u\rangle$, $\mathfrak{m}' = \mathfrak{m}R'$, $I' = IR'$, $In_{\mathfrak{m}'}(I') = \langle\, in_{\mathfrak{m}'}(g') \mid g' \in I' \,\rangle$ and we identify (y) with its image in R'. Then (2.7) is satisfied for (I, u, y) if there is no proper K-submodule $T \subset gr^1_{\mathfrak{m}'}(R')$ such that $(\, In_{\mathfrak{m}'}(I') \cap K[T]\,)\, gr_{\mathfrak{m}'}(R') \;=\; In_{\mathfrak{m}'}(I')$.

Construction 2.3.2. Let $\mathbb{E} = (J, b)$ on R, $(u) = (u_1, \ldots, u_e)$, $(y) = (y_1, \ldots, y_r)$ and $(f) = (f_1, \ldots, f_m)$ be as in Setup A and (2.8) (for both see p.67). Let

$$(g) := (g_1, \ldots, g_l) := (f_{i_1}, \ldots f_{i_l}),$$

$l \leq m$ and $1 \leq i_1 < i_2 < \ldots < i_l \leq m$, be those elements of the (u)-standard base (f) which fulfill

$$n_{(u)}(f_{i_\alpha}) = b \qquad \text{for all } \alpha \in \{1, \ldots, l\}. \tag{$*$}$$

Set $I := \langle g\rangle \subset R$. By the assumptions on (y) in (2.8) there is a system $(z) = (z_1, \ldots, z_s) := (y_{j_1}, \ldots, y_{j_s})$ with $s \leq r$ and $1 \leq j_1 < j_2 < \ldots < j_s \leq r$ which is a minimal generating set of the directrix $\mathrm{Dir}_x(I, b)$. Let $(w) = (w_1, \ldots, w_d)$ be the elements $\{u, y\} \setminus \{z\}$, $d = r + e - s \geq e$. By definition $g_{i_\alpha} \notin \langle w\rangle$ for all $1 \leq \alpha \leq l$. Further (2.7) is satisfied for (I, w, z) and (g) is a (w)-standard base of I with reference datum (z, L_0) $(L_0(v) = |v|)$. Hence we can apply vertex preparation (Theorem 2.2.25) and get by Theorem 2.2.19 and $(*)$

$$\Delta(I, w) = \Delta(g, w, z^*) = \Delta(g, b; w, z^*),$$

where (z^*) denotes the system of elements which we obtain from (z) by the preparation process. We denote by (y^*) the corresponding modified system (y). (The previous equality does not imply $\Delta(g, u, y^*) = \Delta(I, u)$, because (2.7) needs not to be satisfied for (I, u, y^*)). Consider $\Delta(f, b; u, y^*)$ with the well-prepared (g, w, z^*) and the remaining (unchanged) elements $\{f\} \setminus \{g\}$. By Corollary 2.1.4 the normalization process (Theorem 2.2.21) doesn't change $\Delta(f, b; u, y^*)$. Thus we may normalize and finally, we set

$$\Delta^*(\mathbb{E}, u, y^*) := \Delta^*(J, b; u, y^*) := \Delta(f, b; u, y^*).$$

Remark 2.3.3. *(1) Denote by $(f^{(b)}) = (f_1, \ldots, f_q)$, $l \leq q \leq m$, the elements of (f) with $\mathrm{ord}_x(f_i) = b$ for $i \in \{1, \ldots, q\}$. $(\mathrm{ord}_x(f_i) \geq b$ for $1 \leq i \leq m$ by*

(2.8)). The tangent cone $\mathbb{T}_x(\mathbb{E})$ (and thus the directrix $\mathbb{D}\mathrm{ir}_x(\mathbb{E})$) is completely determined by $(f^{(b)})$, since the b-initial forms of the other elements in (f) are zero. The system (g) is contained in $(f^{(b)})$, but they are not necessarily equal. (For example let $b = 2$, $f_1 = y_1^2 + y_2^5 + u_1^4$ and $f_2 = y_2 u_2$ (over any field K), then $(g) = (f_1)$ and $(f^{(b)}) = (f_1, f_2)$). Thus condition (2.7) need not hold.

(2) *If (y) yields the whole directrix, then $(g) = (f^{(b)})$ and $(z) = (y)$.*

(3) *Note that by Lemma 2.2.23 the normalization steps (which we do in order to obtain $\Delta(I, w) = \Delta(g, w, z)$) don't change the polyhedron.*
Further it follows from Corollary 2.1.5 that $\Delta(f, b; u, y)$ can be deduced from $\Delta(f, b; w, z)$ with the help of a suitable projection.

(4) *The normalization of the vertices in the last step is useful in the next chapter, where we relate this to the invariant of Bierstone and Milman.*

We can now prove the first part of Main Theorem 3. The precise formulation is the following

Theorem 2.3.4. *Let $\mathbb{E} = (J, b)$ be an idealistic exponent on R, $(f) = (f_1, \ldots, f_m)$ a (u)-standard base of J and $(u, y) = (u_1, \ldots, u_e; y_1, \ldots, y_r)$ a regular system of parameters for R such that the initial forms of (y) yield the whole directrix $\mathrm{Dir}_x(\mathbb{E})$. Let (y^*) be a system obtained by Construction 2.3.2. Then the polyhedron*

$$\Delta^*(\mathbb{E}, u, y^*) = \Delta(f, b; u, y^*)$$

does neither depend on the choice of the (u)-standard base $(f) = (f_1, \ldots, f_m)$ of J nor on the choice of (y) or (y^*).

Definition 2.3.5. *In the situation of Theorem 2.3.4 we call*

$$\Delta_x^*(\mathbb{E}, u) := \Delta^*(\mathbb{E}, u) := \Delta^*(\mathbb{E}, u, y^*)$$

the (first) characteristic polyhedron of the idealistic exponent $\mathbb{E}$ with respect to (u).

This polyhedron may not behave well, if we consider equivalent idealistic exponents, see Example 2.1.9.

Proof of Theorem 2.3.4. Let $(g; z^*) = (g_1, \ldots, g_m; z_1^*, \ldots, z_r^*)$ be another choice for fixed $(u) = (u_1, \ldots, u_e)$. (Do not confuse this with the systems (g) and (z^*) which appear in Construction 2.3.2 — these are different objects!)

By abuse of notation we write in the following only (y) (resp. (z)) instead of (y^*) (resp. (z^*)).

By Corollary 2.1.4 the polyhedra are independent of the choice of the generators, thus we have

$$\Delta(f,b;u,y) = \Delta(g,b;u,y) \quad \text{and} \quad \Delta(f,b;u,z) = \Delta(g,b;u,z).$$

It is left to show $\Delta(J,b;u,y) = \Delta(J,b;u,z)$.

Consider $h \in J$ and let $h = \sum_{(A,B)} C_{A,B}\, u^A y^B$ be an expansion as in (2.2). Set

$$S(h,b;u,y) := \left\{ \frac{A}{b-|B|} \;\middle|\; C_{A,B} \neq 0 \wedge |B| < b \right\}.$$

and denote by $\Delta(h,b;u,y)$ the smallest F-subset containing $S(h,b;u,y)$. Then $\Delta(J,b;u,y)$ is the smallest F-subset containing $\bigcup_{h\in J}\Delta(h,b;u,y)$ (see the remark before Lemma 2.1.2).

Further by the assumption we have for every $j \in \{1,\dots,r\}$

$$y_j = L_j(z) + Q_j(u) + H_j(u,z), \qquad \text{where}$$

- $\diamond$ $L_j(z) \in K[z]$ are polynomials homogeneous of degree one such that

$$\langle\, L_1(z),\dots,L_r(z)\,\rangle = \langle z_1,\dots,z_r\rangle \subset R.$$

- $\diamond$ $Q_j(u) \in K[[u]]$ are contained in $\langle u\rangle^2$,
- $\diamond$ $H_j(u,z) \in K[[u,z]]$ are contained in $\langle u,z\rangle^2$ and $H_j(u,0) = 0$.

We split the substitution from (y) to (z) in two steps: first we assume $Q_j(u) \equiv 0$ for all j and after that we consider only the change by $Q_j(u)$,

$$y_j \overset{(1)}{\longmapsto} x_j(u,z) := L_j(z) + H_j(u,z) \overset{(2)}{\longmapsto} y_j(u,z) = x_j(u,z) + Q_j(u),$$

for $1 \leq j \leq r$. We show that the polyhedra after each step coincide with $\Delta(J,b;u,y) \subset \mathbb{R}^e_0$.

The first equality can be shown directly: (sketch) replace in the expansion $h = \sum_{(A,B)} C_{A,B}\, u^A y^B$ the system (y) by (x) and put in the definition of x_j. Then use that the (z)-exponent in an expansion of $H_j(u,z)$ is never zero if the coefficient is non-zero (this follows by $H_j(u,0) = 0$).

We give another (more detailed) proof by using combinatorial blow ups; this is inspired by [C2]. Consider the idealistic exponent (h,b) on $R[t]$, where t is an arbitrary independent new variable. Write (on $\widehat{R}$)

$$h(u,y) = h_b(y) + \sum_{|B|<b} C_{A,B}\, u^A y^B + h^*(u,y), \tag{$\star$}$$

where $h_b(y) \in K[y]$ is zero or homogeneous of degree b and $h^*(u,y)$ has order greater or equal b in $\langle y \rangle$. This expansion always exists, because the initial forms of (y) generate the ideal of the directrix $\mathbb{D}\mathrm{ir}_x(\mathbb{E})$. Set

$$\delta_y := \delta(h,u,y) = \min\left\{ \frac{|A|}{b-|B|} \;\middle|\; C_{A,B} \neq 0 \wedge |B| < b \right\}.$$

Recall that R is the local ring of some regular scheme Z at a singular point of (J,b). Hence the origin $V(t,u,y)$ is a permissible center for the idealistic exponent (h,b).

We can do the same after the first substitution: Insert $L_j(z) + H_j(u,z)$ for y_j in $(\star)$, $1 \leq j \leq r$. Then reorder the terms such that we get an expansion of $h(u, x(u,z))$ which is of the analogous form as $(\star)$ and define $\delta_z := \delta(h,u,x(u,z))$.

We claim now

Lemma 2.3.6. $\delta_y = \delta_z$.

Proof. Suppose the claim is wrong; without loss of generality $\delta_y > \delta_z$. (If $\delta_y < \delta_z$, then just interchange the role of both in the following argumentation).

If we blow up the origin and consider the T-chart, then the new origin is again permissible. Let $\alpha \in \mathbb{Z}_+$ such that

$$\alpha(\delta_y - 1) \in \mathbb{Z}_+.$$

We blow up the origin and consider the T-chart α-times. This is a permissible local sequence of regular blow ups for (h,b) and the transform of $h(u,y)$ is given by

$$h'\left(t, \frac{u}{t^\alpha}, \frac{y}{t^\alpha}\right) = h_b\left(\frac{y}{t^\alpha}\right) + \sum_{|B|<b} C_{A,B}\, t^{\alpha(|A|+|B|-b)} \left(\frac{u}{t^\alpha}\right)^A \left(\frac{y}{t^\alpha}\right)^B + h^*\left(t, \frac{u}{t^\alpha}, \frac{y}{t^\alpha}\right).$$

We get for the t-exponent of the middle term

$$\alpha(|A|+|B|-b) = \alpha(b-|B|)(\delta_y - 1) + \alpha(b-|B|)\left(\frac{A}{b-|B|} - \delta_y\right)$$

By definition of δ_y we have $\dfrac{A}{b-|B|} - \delta_y \geq 0$ and there is a monomial $u^A y^B$ with non-zero coefficient and $\dfrac{A}{b-|B|} - \delta_y = 0$. The choice of α implies $\alpha(\delta_y - 1) \geq 1$ and thus

$$\alpha(b-|B|)(\delta_y - 1) + |B| \geq b - |B| + |B| = b.$$

This means $V\left(t, \frac{y}{t^\alpha}\right)$ is permissible for (h', b). If we blow up with this center and look into the T-chart, then there are no essential changes in the first and the last part. In the middle term the t-exponent becomes

$$(b - |B|)\,(\,\alpha(\delta_y - 1) - 1\,) + \alpha\,(b - |B|)\left(\frac{A}{b - |B|} - \delta_y\right).$$

If $V\left(t, \frac{y}{t^{\alpha+1}}\right)$ is permissible (this holds if $\alpha(\delta_y - 1) \geq 2$), then we may repeat the last step. After $\beta \in \mathbb{Z}_+$, $\beta \leq \alpha(\delta_y - 1)$, such steps the exponent of t in the middle term of the transform of $h(u, y)$ is

$$(b - |B|)\,(\,\alpha(\delta_y - 1) - \beta\,) + \alpha\,(b - |B|)\left(\frac{A}{b - |B|} - \delta_y\right).$$

In particular, we see that in the case $\beta = \alpha(\delta_y - 1)$ the component $V\left(t, \frac{y}{t^{\alpha+\beta}}\right)$ is not permissible for the transform of (h, b).

Let us see how the idealistic exponent $(\,y(u, z), 1) = (x, 1)$ (step (1) of the substitution, $x_j = L_j(z) + H_j(u, z)$ for $1 \leq j \leq r$) has transformed under the previous blow ups. Denote by d_j the order of $H_j(u, z)$ in $\langle u, z\rangle = \langle u, y\rangle$. The definition of $H_j(u, z)$ implies $d_j \geq 2$. The transform of $(x, 1)$ under the first α blow ups above is given by

$$\frac{y}{t^\alpha} = x'\left(t, \frac{u}{t^\alpha}, \frac{z}{t^\alpha}\right) = L_j\left(\frac{z}{t^\alpha}\right) + t^{\alpha(d_j - 1)} \cdot \widetilde{H_j}\left(t, \frac{u}{t^\alpha}, \frac{z}{t^\alpha}\right)$$

for a certain $\widetilde{H_j}\left(t, \frac{u}{t^\alpha}, \frac{z}{t^\alpha}\right) \in K\left[\left[t, \frac{u}{t^\alpha}, \frac{z}{t^\alpha}\right]\right]$ which fulfills again the property $\widetilde{H_j}\left(t, \frac{u}{t^\alpha}, 0\right) = 0$. Since $\alpha(d_j - 1) \geq \alpha \geq 1$ and $\langle\, L_1(z), \ldots, L_r(z)\,\rangle = \langle z\rangle$, we get

$$V\left(t, \frac{y}{t^\alpha}\right) = V\left(t, x'\left(t, \frac{u}{t^\alpha}, \frac{z}{t^\alpha}\right)\right) = V\left(t, \frac{z}{t^\alpha}\right). \tag{$\star\star$}$$

Thus the blow up with center $V\left(t, \frac{z}{t^\alpha}\right)$ after the substitution is the same as the blow up with center $V\left(t, \frac{y}{t^\alpha}\right)$. Since the order of $\widetilde{H_j}\left(t, \frac{u}{t^\alpha}, \frac{z}{t^\alpha}\right)$ in $\left\langle \frac{z}{t^\alpha}\right\rangle$ is greater or equal one ($\widetilde{H_j}\left(t, \frac{u}{t^\alpha}, 0\right) = 0$), the same is true in the T-chart of such a blow up.

We write h'' for the transform of $h = h(u, x(u, z))$ after the $\alpha + \beta$ blow ups described above. Consider an expansion of h'' (with respect to (u, z)) analogous to $(\star)$. By the same arguments as before the t-exponent of the middle term is given by

$$(b - |B|)\,(\,\alpha(\delta_z - 1) - \beta\,) + \alpha\,(b - |B|)\left(\frac{A}{b - |B|} - \delta_z\right).$$

If $\beta \leq \alpha(\delta_z - 1) < \beta + 1 \leq \alpha(\delta_y - 1)$, then $V\left(t, \frac{y}{t^{\alpha+\beta}}\right) = V\left(t, \frac{z}{t^{\alpha+\beta}}\right)$ is not permissible for (h'', b). But this is a contradiction to $\beta < \alpha(\delta_y - 1) \in \mathbb{Z}_+$ and the claim follows. $\square$

By Lemma 2.3.6 we can now drop the index and write

$$\delta := \delta_y = \delta_z.$$

Let $L \in \mathbb{L}_0$ be a semi-positive linear form on $\mathbb{R}^e$ with rational coefficients. Say $L(v) = \sum_{i=1}^{e} \lambda_i v_i$ for $v \in \mathbb{R}_0^e$ and we have $\lambda_i \in \mathbb{Q}_0$ for all $1 \leq i \leq e$. Recall expansion $(\star)$ of $h(u, y) \in J$,

$$h(u,y) = h_b(y) + \sum_{|B|<b} C_{A,B}\, u^A y^B + h^*(u,y).$$

We set

$$\delta_{L,y} := \delta_L(h,u,y) = \min \left\{ \frac{L(A)}{b - |B|} \;\middle|\; C_{A,B} \neq 0 \wedge |B| < b \right\}.$$

Analogous we define $\delta_{L,z} := \delta_L(h, u, x(u,z))$ via the $(\star)$-expansion of $h(u, x(u, z))$.

Lemma 2.3.7. $\delta_{L,y} = \delta_{L,z}$.

Proof. Suppose the claim is wrong; without loss of generality $\delta_{L,y} > \delta_{L,z}$. (If $\delta_{L,y} < \delta_{L,z}$, then change the role of both in the following argumentation).

Let $\rho \in \mathbb{Z}_+$ be a positive integer such that

$$\rho\,\lambda_1,\ \ldots,\ \rho\,\lambda_e,\ \rho\,\delta_{L,y} \in \mathbb{Z}_+$$

and we set $\gamma_i := \rho\,\lambda_i \in \mathbb{Z}_+$ for all $i \in \{1, \ldots, e\}$.

We start similarly to the proof of Lemma 2.3.6. Let $\alpha \in \mathbb{Z}_+$ such that

$$\alpha(\delta - 1) \in \mathbb{Z}_+ \quad \textit{and} \quad \alpha(\delta - 1) \geq \sum_{i=1}^{e} \gamma_i.$$

Blow up the origin and consider the T-chart — do this α-times. Recall that the t-exponent of the middle term in the $(\star)$-expansion of the transform of (h, b) is

$$\alpha(b - |B|)(\delta - 1) + \alpha(b - |B|)\left(\frac{A}{b - |B|} - \delta\right)$$

and that $V\left(t, \frac{y}{t^\alpha}\right)$ is permissible. Next we blow up with center $V\left(t, \frac{u_1}{t^\alpha}, \frac{y}{t^\alpha}\right)$ and consider the T-chart. By doing this we have to add in the t-exponent the term

$$A_1 + |B| - b = (b - |B|)\left(\frac{A_1}{b - |B|} - 1\right)$$

$(A = (A_1, \ldots, A_e))$. We repeat the previous blow up in total γ_1-times. After that we do the same for $\frac{u_i}{t^\alpha}$ and γ_i, $2 \le i \le e$. Then the extra term in the t-exponent after all these blow ups is

$$(b - |B|)\left(\sum_{i=1}^{e} \frac{\gamma_i A_i}{b - |B|} - \sum_{i=1}^{e} \gamma_i\right) =$$

$$= (b - |B|)\left(\rho\, \delta_{L,y} - \sum_{i=1}^{e} \gamma_i\right) + (b - |B|) \cdot \rho \cdot \left(\frac{L(A)}{b - |B|} - \delta_{L,y}\right).$$

By definition of $\delta_{L,y}$ we have $\frac{L(A)}{b - |B|} - \delta_{L,y} \ge 0$ and there is a monomial $u^A y^B$ with non-zero coefficient and $\frac{L(A)}{b - |B|} - \delta_{L,y} = 0$. We put

$$\theta := \alpha(b - |B|)\left(\frac{A}{b - |B|} - \delta\right) + (b - |B|) \cdot \rho \cdot \left(\frac{L(A)}{b - |B|} - \delta_{L,y}\right).$$

(θ depends on the given data, but since this is not important for us, we don't write $\theta(A, B, \ldots)$). Altogether the t-exponent is after these $\alpha + \gamma_1 + \ldots + \gamma_e$ blow ups given by

$$(b - |B|)\left(\rho\, \delta_{L,y} + \alpha(\delta - 1) - \sum_{i=1}^{e} \gamma_i\right) + \theta \ge (b - |B|) \cdot \rho \cdot \delta_{L,y}$$

(use the special choice of α and $\theta \ge 0$). Let $\beta := \rho\, \delta_{L,y} + \alpha(\delta - 1) - \sum_{i=1}^{e} \gamma_i \in \mathbb{Z}_+$. With the same arguments as in the proof of Lemma 2.3.6 we can now deduce a contradiction and the assertion of the lemma follows. $\square$

The polyhedra $\Delta(h(u, y), u, y)$ and $\Delta(h(u, x(u, z), u, z)$ have by construction rational vertices. Hence Lemma 2.3.6 and Lemma 2.3.7 imply that they coincide and step (1) is finished.

Now we come to substitution (2): In order to avoid too long notation we set $\Delta(y) := \Delta(J(u, y), b; u, y) = \Delta(J, b; u, y)$ and $\Delta(z) := \Delta(J(u, y(u, z)), b; u, z) = \Delta(J, b; u, z)$. Our goal is to show $\Delta(y) = \Delta(z)$. By the first step we know $\Delta(y) = \Delta(J(u, x(u, z)), u, z)$. So we may assume that the first substitution is trivial — $y_j = x_j(u, z) = z_j$. Hence we get

$$y_j = z_j + Q_j(u), \quad 1 \le j \le r,$$

for some $Q_j(u) = \sum_{C\in\mathbb{Z}_0^e} D_{C,j}\, u^C \in \langle u\rangle^2 \subset K[[u]]$. Recall that $(f) = (f_1 \dots, f_m)$ denotes a (u)-standard base of J and as in Construction 2.3.2, $(f^{(b)}) = (f_1, \dots, f_q)$, $1 \le q \le m$, are those elements of (f) with $\mathrm{ord}_x(f_i) = b$ for all $i \in \{1, \dots, q\}$. The construction of $\Delta(f, b; u, y)$ implies the minimality of the characteristic polyhedron $\Delta(f^{(b)}, b; u, y) = \Delta(f^{(b)}, u, y)$ with respect to $(f^{(b)}; y)$. Pick $C \in \mathbb{Z}_0^e$ with $D_{C,j} \neq 0$ for some $j \in \{1, \dots, r\}$. Let us consider the substitution

$$y_j \overset{(2_C)}{=} w_j + D_{C,j}\, u^C, \quad 1 \le j \le r.$$

First of all this cannot delete any of the vertices of $\Delta(f^{(b)}, u, y)$ — otherwise we get a contradiction to the minimality of this polyhedron. Further (2_C) creates the point $C \in \Delta(f^{(b)}, u, w)$: Suppose $D_{C,1} = 1 \neq 0$ and $D_{C,j} = 0$ for $j \ge 2$. Consider the monomial $y^B = y_1^{B_1} y^+(B)$, where we use the notation $y^+(B) := y_2^{B_2} \cdots y_r^{B_r}$. Under (2_C) this is mapped to

$$(w_1 + u^C)^{B_1} \cdot w^+(B) = w^B + w^+(B) \cdot \sum_{M=1}^{B_1} (u^C)^M w_1^{B_1 - M}.$$

Hence if $|B| = b$, then the sum yields in $\Delta(f^{(b)}, u, w)$ for every M the point

$$\frac{C \cdot M}{b - (B_2 + \ldots + B_r) - (B_1 - M)} = \frac{C \cdot M}{b - |B| + M} = C.$$

The same argument works also in the case without the restriction on the coefficients $D_{C,j}$; there only has to be at least one which is non-zero. An easy computation analogous to the previous shows

$$y^B = \sum_{M_1=0}^{B_1} \cdots \sum_{M_r=0}^{B_r} D_{M,C}\, (u^C)^{M_1+\ldots+M_r}\, w^{B-(M_1,\ldots,M_r)} = \sum_{M=0}^{B} D_{M,C}\, (u^C)^{|M|}\, w^{B-M}, \tag{$*$}$$

where we set $D_{M,C} := \prod_{j=1}^r \binom{B_j}{M_j} D_{C,j}^{M_j}$, and as before all the monomials correspond to C if $|B| = b$. So $C \in \Delta(f^{(b)}, u, w) \subseteq \Delta(f, b; u, w)$.

Let us see how $\Delta(f, b; u, y)$ behaves under the change from (y) to (w). By $(*)$ we have for arbitrary $A \in \mathbb{Z}_0^e$ and $B \in \mathbb{Z}_0^r$

$$C_{A,B}\, u^A\, y^B = \sum_{M=0}^{B} C_{A,B}\, D_{M,C}\, u^{C\cdot|M|+A}\, w^{B-M}.$$

The corresponding points are

$$\begin{aligned} \frac{C \cdot |M| + A}{b - |B| + |M|} &= \frac{|M|}{b - |B| + |M|} \cdot C + \frac{A}{b - |B| + |M|} \\ &= \frac{|M|}{b - |B| + |M|} \cdot C + \frac{b - |B|}{b - |B| + |M|} \cdot \frac{A}{b - |B|} \end{aligned} \tag{$**$}$$

for $M = (M_1, \dots, M_r) \in \mathbb{Z}_0^r$ and $0 \le M_j \le B_j$ for all j. If $|B| \ge b$, then $\dfrac{|M|}{b-|B|+|M|} \ge 1$ and the first line of $(**)$ implies $\dfrac{C\cdot|M|+A}{b-|B|+|M|} \in C + \mathbb{R}_0^e$.

So suppose $|B| < b$. For $|M| = 0$ we get $\dfrac{A}{b-|B|}$ back and the coefficient of $u^A\, w^B$ is $C_{A,B}$. The factors before $\dfrac{A}{b-|B|}$ and C in the last line of $(**)$ are both non-negative, they are smaller or equal one and their sum is

$$\frac{|M|}{b-|B|+|M|} + \frac{b-|B|}{b-|B|+|M|} = 1.$$

Therefore every point $\dfrac{C\cdot|M|+A}{b-|B|+|M|}$ is contained in the connecting line between $\dfrac{A}{b-|B|}$ and C (for $|B| < b$ and $M \in \mathbb{Z}_0^r$ with $0 \le M_j \le B_j$ for all j).

The conclusion is:

(i) Either $C \in \Delta(f, b; u, y)$ is already contained in the polyhedron. Then we do not create under the change from (y) to (w) a new vertex C. Further we have seen that all points which appear newly are contained in the line between the original point and C and thus they are in the interior of $\Delta(f, b; u, y)$. In particular the vertices are not touched and we get $\Delta(f, b; u, y) = \Delta(f, b; u, w)$.

(ii) Or $C \notin \Delta(f, b; u, y)$ and C becomes a vertex of $\Delta(f, b; u, w)$. Moreover by the last argument $\Delta(f, b; u, w)$ is the smallest F-subset containing C and $\Delta(f, b; u, y)$.

Together we see that in both cases $\Delta(f, b; u, y) \subseteq \Delta(f, b; u, w)$.

Up to now we have considered only a part of the substitution

$$y_j = z_j + Q_j(u) = z_j + \sum_{C \in \mathbb{Z}_0^e} D_{C,j}\, u^C.$$

But we apply this for each C with non-zero coefficients and get

$$\Delta(f, b; u, y) \subseteq \Delta(f, b; u, z).$$

By Corollary 2.1.4 $\Delta(f, b; u, z) = \Delta(g, b; u, z)$. The arguments from above with $(g; z)$ instead of $(f; y)$ show $\Delta(g, b; u, z) \subseteq \Delta(g, b; u, y) = \Delta(f, b; u, y)$. Finally, putting this together yields the desired equality

$$\Delta(f, b; u, y) = \Delta(g, b; u, z)$$

and completes the proof of Theorem 2.3.4. □

The assumption in Theorem 2.3.4 that the initial forms of $(y) = (y_1, \dots, y_r)$ yield the *whole* directrix $\mathrm{Dir}_x(\mathbb{E})$ is crucial.

Example 2.3.8. Consider the idealistic exponent $\mathbb{E} = (\langle f_1, f_2\rangle, 2)$ over a field K of characteristic $p \neq 2$, where

$$f_1(u,y) = y_1^2 + h_1(u_1) \quad \text{and} \quad f_2(u,y) = u_3y_2 + (y_2 + u_2^n)^p + h_2(u_1).$$

for some $h_1, h_2 \in K[u_1]$ and an integer $n \in \mathbb{Z}_+$, $n \geq 2$. The system (y_1, y_2, u_3) generates the directrix $\mathbb{D}\mathrm{ir}_x(\mathbb{E})$ and the elements with $n_{(u)}(f_i) = b = 2$ are $(g) = (f_1)$. Further let $h_1(u_1)$ be such that $\Delta(f_1, u, y) = \Delta(\langle f_1\rangle, u)$ coincides with the characteristic polyhedron. Assume Theorem 2.3.4 would hold in this case. Then $\Delta(f, 2; u, y)$ should be independent of the choice of (y). For $(z) = (z_1, z_2) = (y_1, y_2 + u_2^n)$ we get

$$f_1(u,z) = z_1^2 + h_1(u_1) \quad \text{and} \quad f_2(u,z) = u_3z_2 - u_2^n u_3 + z_2^p + h_2(u_1)$$

and still $\Delta(f_1, u, z) = \Delta(f_1, u, y)$. Set $v := \left(0, \frac{np}{2}, 0\right)$ and $w := \left(0, \frac{n}{2}, \frac{1}{2}\right)$. Obviously $(0,0,1), v \in \Delta(f, 2; u, y)$ and $(0,0,1), w \in \Delta(f, 2; u, z)$. The assumption $p \neq 2$ implies $p > 2$ and thus $\frac{np}{2} > n$. Therefore $w \notin \Delta(f, 2; u, y)$ and further $v \notin \Delta(f, 2; u, z)$.

The polyhedra $\Delta(f, 2; u, y)$ and $\Delta(f, 2; u, z)$ are essentially different.

The previous example illustrates that in general it is not possible to make $\Delta(f, b; u, y)$ (with our definitions) independent of the choice of the system (y). But still we can say something in the previous case, where (y) doesn't give the whole directrix. Namely, in both cases of Example 2.3.8 the point $(0,0,1)$ appears in the polyhedra. Hence $\delta(\Delta(f, 2; u, y)) = \delta(\Delta(f, 2; u, z)) = 1$. For the general statement see Lemma 2.5.3.

To end this section let us give the following result which will later be very useful.

Proposition 2.3.9. *Let $\mathbb{E} = (J, b)$ be an idealistic exponent on $(R, \mathfrak{m})$. Fix a system of elements $(x) = (x_1, , \dots, x_{n-1})$ which can be extended to a regular system of parameters for R. Let $y \in R$ such a possible extension and suppose further that $V(y)$ has maximal contact with $\mathbb{E}$ at the origin $V(\mathfrak{m})$. Then the polyhedron $\Delta(\mathbb{E}, x, y)$ is independent of the choice of (y) with these properties. This means if $z \in R$ is another extension of (x) and $V(z)$ has maximal contact, then $\Delta(\mathbb{E}; x, y) = \Delta(\mathbb{E}; x, z)$.*

Proof. Since both have maximal contact with $\mathbb{E}$ at the origin, we have by Lemma 1.4.3

$$\mathbb{E} \sim \mathbb{E} \cap (y, 1) \cap (z, 1) \tag{$*$}$$

Since (x, y) is a regular system of parameters for R, we can express z by these elements, say

$$z = \epsilon\, y - h(x, y)$$

for some unit $\epsilon \in R^\times$ and some element $h(x, y) \in \widehat{R}$; without loss of generality we may assume that y is not appearing in $h(x, y)$. (If this is not true, we may modify the unit ϵ in order to obtain this). Hence we can write $h(x) = h(x, y)$. Let $g \in J$ and consider an $\mathfrak{m}$-adic expansion of this element

$$g = \sum_{A,B} C_{A,B}\, x^A z^B.$$

As we already have seen in the proof of Theorem 2.3.4 (substitution (**2**), see p.75) we do not change the polyhedron if we insert $z = \epsilon\, y - h(x)$. The vertices are fixed and the points coming from $h(x)$ appear by $(*)$ already before the change from z to y. All other points, which may occur, lie on the connecting line between some point of the generating set of $\Delta(\mathbb{E}, x, z)$ and some point coming from $h(x)$. $\square$

Note in Example 2.3.8 $V(y_2)$ has maximal contact, whereas $V(z_2)$ does not.

2.4 Relation to the Newton polyhedron and further properties

We want to show some properties of the polyhedron $\Delta(f, b; u, y)$ associated to a family of elements $(f) = (f_1, \ldots, f_m)$ in the regular local ring $(R, \mathfrak{m}, K)$ (with regular system of parameters $(u, y) = (u_1, \ldots, u_e; y_1, \ldots, y_r)$ as before).

In Proposition 2.1.3 we have seen that $\Delta(f, b; u, y)$ is a certain (step-by-step) projection of the Newton polyhedron $\Delta^N(f, b; u, y)$. We now give the proof that this can also be realized by one big projection step.

Lemma 2.4.1. *Consider the set*

$$T := \{\, w = (0, \ldots, 0, w_1, \ldots, w_r) \in \mathbb{R}_0^n \mid |w| = 1 \,\}.$$

Then for every $v = (v_1, \ldots, v_n) \in \mathbb{R}_0^n$ with $0 < |v_{e,n}| < 1$, $v_{e,n} := (v_{e+1}, \ldots, v_n)$, there exists a unique $w \in T$ such that the line through w and v intersects $\mathbb{R}_0^e \times \{0\}^r$.

This defines a projection $\pi : \mathbb{R}_0^n \setminus (T + \mathbb{R}_0^n) \to \mathbb{R}_0^e$ from T to $\mathbb{R}_0^e$, where

$$\pi(v) := \left(\frac{v_1}{1 - |v_{e,n}|}, \ \ldots, \ \frac{v_e}{1 - |v_{e,n}|} \right),$$

and the unique intersection points above are given by $\pi(v)$.

If $v = \dfrac{(A, B)}{b} \in \Delta^N(f, b; u, y)$ with $|B| < b$, then

$$\pi(v) = \pi\left(\frac{A}{b}, \frac{B}{b}\right) = \frac{A}{b - |B|}$$

and hence $\pi\left(\Delta^N(f, b; u, y)\right) = \Delta(f, b; u, y)$. This yields $\pi = \pi^{(r)}$, where $\pi^{(r)}$ denotes the composition of the step-by-step projections.

Note that Proposition 2.1.3 resp. Lemma 2.4.1 imply the second part of Main Theorem 3 which claimed that the characteristic polyhedron is a projection of the Newton polyhedron.

Proof of the lemma. We fix $v = (v_1, \ldots, v_n)$ and $w = (0, \ldots, 0, w_1, \ldots, w_r) \in T$ varies. Since we want to project on $\mathbb{R}_0^e$, we may assume $v_{e+j} < w_j$ for every $j \in \{1, \ldots, r\}$. Note, if $v_{e+j_0} = w_{j_0} = 0$ for some $j_0 \in \{1, \ldots, r\}$, then we can ignore the j_0-th coordinate in the following. Denote by $\mathcal{L} : \mathbb{R}_0 \to \mathbb{R}_0^n, \lambda \mapsto w + \lambda \cdot (v - w)$ the map which defines the half-line from w through v. Then $im(\mathcal{L}) \cap (\mathbb{R}_0^e \times \{0\}^r) \neq \emptyset$ if and only if

$$\exists\, \lambda_0 \geq 1 \ : \ \forall\, j \in \{1, \ldots, r\} \ : \ w_j + \lambda_0 (v_{e+j} - w_j) = 0. \qquad (*)$$

If there is an $l \in \{1, \ldots, r\}$ such that $v_{l+e} = 0$, then the equation becomes $(1 - \lambda_0)w_l = 0$. Hence $w_l = 0$ or $\lambda_0 = 1$. In the last case we get for every j

$$0 = w_j + \lambda_0(v_{e+j} - w_j) = v_{e+j}.$$

This means $v = (v_1, \ldots, v_e, 0, \ldots, 0) \in \mathbb{R}_0^e \times \{0\}^r$.

Therefore we may assume $0 < v_{j+e} < w_j$ for all j.

Existence of w: Set $w_j := \dfrac{v_{e+j}}{|v_{e,n}|}$ for $1 \leq j \leq r$ and $\lambda_0 := \dfrac{1}{1 - |v_{e,n}|}$. Then $w = (0, \ldots, 0, w_1, \ldots, w_r) \in T$ and $(*)$ holds since

$$\begin{aligned} w_j + \lambda_0(v_{e+j} - w_j) &= \frac{v_{e+j}}{|v_{e,n}|} + \frac{1}{1 - |v_{e,n}|}\left(v_{e+j} - \frac{v_{e+j}}{|v_{e,n}|}\right) \\ &= \frac{v_{e+j}}{|v_{e,n}|} + \frac{1}{1 - |v_{e,n}|} \cdot v_{e+j} \cdot \frac{|v_{e,n}| - 1}{|v_{e,n}|} = 0. \end{aligned}$$

Uniqueness: Let $\tau_0 \geq 1$ be another solution. By assumption $0 < v_{j+e} < w_j$ and in particular $v_{j+e} - w_j \neq 0$ for every j. Take any of the equalities in $(*)$,

$$w_j + \lambda_0(v_{e+j} - w_j) = 0 = w_j + \tau_0(v_{e+j} - w_j).$$

Then $v_{j+e} - w_j \neq 0$ implies $\lambda_0 = \tau_0$.

The second part follows from $\mathcal{L}(\lambda_0) = (\pi(v), 0, \ldots, 0)$. □

We have seen in Example 2.1.9 that the characteristic polyhedron of equivalent idealistic exponents do not necessarily coincide. But with the use of the following lemma, we can deduce some positive result in this direction.

Lemma 2.4.2. *Let $f, g \in R$ be two elements in the local ring and (u, y) an arbitrary regular system of parameters for R. Consider the idealistic exponents (f, b) and (g, d) on R. Then we have*

$$\Delta(\, f \cdot g,\, b + d;\, u,\, y\,) \subseteq \Delta((f, b) \cap (g, d), u, y).$$

(Recall that $\Delta((f, b) \cap (g, d), u, y)$ is the smallest F-subset of $\mathbb{R}_0^e$ containing the polyhedra $\Delta(f, b; u, y)$ and $\Delta(g, d; u, y)$).

Proof. Let $u^A y^B$ (resp. $u^C y^D$) be a monomial which appears with non-zero coefficient in the $\mathfrak{m}$-adic expansion of f (resp. g). Then we get in $f \cdot g$ the monomial $u^{A+C} y^{B+D}$. It may happen that the coefficient of this monomial in the expansion of $f \cdot g$ becomes zero. Nevertheless, these determine all the points which *may* appear in $\Delta(f \cdot g, b + d; u, y)$ (if necessarily $|B| + |D| < b + d$) and

$$\frac{A + C}{b + d - |B| - |D|} = \frac{b - |B|}{b + d - |B| - |D|} \cdot \frac{A}{b - |B|} + \frac{d - |D|}{b + d - |B| - |D|} \cdot \frac{C}{d - |D|}.$$

If $|B| \geq b$, then we must have $|D| < d$ and further $\dfrac{d - |D|}{b + d - |B| - |D|} \geq 1$. Hence

$$\frac{A + C}{b + d - |B| - |D|} \in \frac{C}{d - |D|} + \mathbb{R}_0^e \quad \text{and} \quad \frac{C}{d - |D|} \in \Delta(g, d; u, y).$$

The analogous statement holds if $|D| \geq d$.

In the case that $|B| < b$ and $|D| < d$ is satisfied, we see that $\dfrac{b - |B|}{b + d - |B| - |D|}$ and $\dfrac{d - |D|}{b + d - |B| - |D|}$ are both non-negative, smaller equal to one and their sum is one. Therefore $\dfrac{A + C}{b + d - |B| - |D|}$ is contained in the F-subset defined by $\dfrac{A}{b - |B|}$ and $\dfrac{C}{d - |D|}$. Together this implies the claim. ☐

Corollary 2.4.3. *Let $a \in \mathbb{Z}_+$ be a positive integer and (f, b) an idealistic exponent on R. Then*

$$\Delta(f^a, ab;\ u, y) = \Delta(f, b; u, y).$$

Proof. (Induction on $a \geq 1$). The case $a = 1$ is trivial. So let $a > 1$. We have by the induction hypothesis $\Delta(f^{a-1}, (a-1)b;\ u, y) = \Delta(f, b; u, y)$. By applying Lemma 2.4.2 ($g = f^{a-1}$ and $d = (a-1) \cdot b$) we get

$$\Delta(f^a, ab; u, y) \subseteq \Delta(\,(\,f, b\,) \cap (\,f^{a-1}, (a-1)b\,); u, y) = \Delta(f, b; u, y).$$

Assume the inclusion is strict. Then there is a vertex $v \in Vert(\Delta(f, b; u, y))$ which is not contained in $\Delta(f^a, ab; u, y)$. But this cannot happen: Write $f = f_v + f_*$ (in $\widehat{R}$), where f_v are those monomials giving v and $f_* := f - f_v$. Then $f^a = (f_v + f_*)^a = f_v^a + \sum_{l=1}^a \binom{a}{l} f_v^{a-l} f_*^l$, the terms in f_v^a yield v again and the points defined by the sum lie in $\Delta(f, b; u, y) \setminus \{v\}$ — the latter correspond to monomials with exponent $\sum_{i=1}^{a-l}(A^{(i)}, B^{(i)}) + \sum_{j=1}^{l}(A^{(j)}, B^{(j)})$, where $\dfrac{A^{(i)}}{b - |B^{(i)}|} = v \neq \dfrac{A^{(j)}}{b - |B^{(j)}|}$ for all i and j. If $l = a$, then the first sum is zero and since v is a vertex, the points corresponding to the second sum are contained in $\Delta(f, b; u, y) \setminus \{v\}$. So suppose $l < a$. The two sums yield in the polyhedron the point

$$w := \frac{\sum_{i=1}^{a-l} A^{(i)} + \sum_{j=1}^{l} A^{(j)}}{b - \sum_{i=1}^{a-l} |B^{(i)}| - \sum_{j=1}^{l} |B^{(j)}|} = \frac{\sum_{i=1}^{a-l} (b - |B^{(i)}|) \cdot v + \sum_{j=1}^{l} A^{(j)}}{b - \sum_{i=1}^{a-l} |B^{(i)}| - \sum_{j=1}^{l} |B^{(j)}|},$$

where we use $\dfrac{A^{(i)}}{b - |B^{(i)}|} = v$ for $1 \leq i \leq a - l$. Since $a - l \geq 1$ (by $l < a$), we have

$$\frac{\sum_{i=1}^{a-l} (b - |B^{(i)}|)}{b - \sum_{i=1}^{a-l} |B^{(i)}| - \sum_{j=1}^{l} |B^{(j)}|} = \frac{b(a - l - 1) + b - \sum_{i=1}^{a-l} |B^{(i)}|}{b - \sum_{i=1}^{a-l} |B^{(i)}| - \sum_{j=1}^{l} |B^{(j)}|} \geq 1.$$

This implies together with $\sum_{j=1}^{l} A^{(j)} \neq 0$ that $w \in v + \mathbb{R}_0^e \setminus \{v\}$.

Therefore $v \in \Delta(f^a, ab; u, y)$ and this contradicts the assumption that the inclusion $\Delta(f^a, ab; u, y) \subseteq \Delta(f, b; u, y)$ is strict. $\square$

Lemma 2.4.4. *Let $\mathbb{E} = (J, b)$ and $\mathbb{E}_i = (J_i, b_i)$, $i \in \{1, 2\}$, be idealistic exponents on some regular scheme Z and $x \in \operatorname{Sing}(\mathbb{E})$. As usual $(R, \mathfrak{m}, K)$ denotes the regular local ring of Z at x and $(t) = (t_1, \ldots, t_n) = (u, y)$ is a regular system of parameters for R. We consider the situation at x and abbreviate the notation by $\Delta(J, b) := \Delta(J, b; u, y)$.*

(i) If $a \in \mathbb{Z}_+$, then $\Delta(J, b) = \Delta(J^a, ab)$.

(ii) Suppose $b_1, b_2 \in \mathbb{Z}_+$ and let $m \in \mathbb{Z}_+$ with $b_1 \mid m$ and $b_2 \mid m$. Then

$$\Delta((J_1, b_1) \cap (J_2, b_2)) = \Delta\left(J_1^{\frac{m}{b_1}} + J_2^{\frac{m}{b_2}}, m\right).$$

Proof. The polyhedron $\Delta(J, b)$ is the smallest F-subset containing all $\Delta(f, b; u, y)$ for $f \in J$. Let $g \in J^a$. If $g = f^a$ for some $f \in J$, then by Corollary 2.4.3 $\Delta(g, ab) = \Delta(f, b)$. Thus $\Delta(J, b) \subseteq \Delta(J^a, ab)$.

We have $\Delta(g_1 + g_2, ab) \subseteq \Delta(\,(\langle g_1, g_2\rangle, ab)\,)$ for every $g_1, g_2 \in J^a$, because clearly $g_1 + g_2 \in \langle g_1, g_2\rangle$. Hence it suffices to consider only elements which are of the form $g = h_1 \cdots h_a \in J^a$ for some $h_1, \ldots, h_a \in J$. We apply Lemma 2.4.2 several times and get

$$\Delta(g, ab) \subseteq \Delta(\,(h_1, b) \cap \ldots \cap (h_a, b)\,) \subseteq \Delta(J, b).$$

This implies the other inclusion and proves *(i)*.

The polyhedron $\Delta((J_1, b_1) \cap (J_2, b_2))$ is the smallest F-subset containing $\Delta(J_1, b_1)$ and $\Delta(J_2, b_2)$. By *(i)* we may suppose $b_1 = b_2 = m = b$ and we only have to show

$$\Delta((J_1, b) \cap (J_2, b)) = \Delta\left(J_1 + J_2, b\right).$$

The inclusion $\subseteq$ is clear, because J_1 and J_2 are contained in $J_1 + J_2$. On the other hand, every element $h \in J_1 + J_2$ can be written as $h = h_1 + h_2$ for some $h_1 \in J_1$ and $h_2 \in J_2$. Therefore $\Delta(h, b)$ is contained in $\Delta((h_1, b) \cap (h_2, b))$ and this implies the other inclusion. $\square$

Lemma 2.4.5. *Let $\mathbb{E} = (J, b)$, $x \in \operatorname{Sing}(\mathbb{E})$, $(R, \mathfrak{m}, K)$ and $(t) = (u, y)$ be as in the previous lemma. Let $M \in \mathbb{Z}_0^n$ and $m := |M|$. Recall that $\mathcal{D}_M \in \operatorname{Diff}_K^{\leq m}\left(\widehat{R}\right)$ denotes the differential operator defined by $\mathcal{D}_M\left(C_D\, t^D\right) = \binom{D}{M} C_D\, t^{D-M}$. We set $\mathcal{D}_M^{log} := t^M \mathcal{D}_M \in \operatorname{Diff}_K^{\leq m}\left(\widehat{R}\right)$. Then*

$$\Delta\big(\,(J, b) \cap (\mathcal{D}_M^{log} J, b - m);\; u, y\,\big) = \Delta(J, b; u, y).$$

Proof. Let $f \in J \subset R$ with $\mathfrak{m}$-adic expansion $f = \sum_{(A,B)\in\mathbb{Z}_0^n} C_{A,B}\, u^A\, y^B$. Then $\mathcal{D}_M^{log}(f) = \sum_{(A,B)\in M+\mathbb{Z}_0^n} \binom{(A,B)}{M} C_{A,B}\, u^A\, y^B$ and the points which *may* occur in $\Delta((J,b)\cap(\mathcal{D}_M^{log}J, b-m),\ u, y)$ because of $\mathcal{D}_M^{log}(f)$ are of the form

$$\frac{A}{(b-m)-|B|} = \frac{b-|B|}{b-m-|B|}\cdot\frac{A}{b-|B|} \in \frac{A}{b-|B|} + \mathbb{R}_0^e$$

for $|B| < b-m$, where we use $\dfrac{b-|B|}{b-m-|B|} \geq 1$. This already implies the lemma. □

Lemma 2.4.6. *Let $\mathbb{E} = (J,b)$ be as before and $x \in Z$. Further (u,y) denotes a regular system of parameters for the regular local ring $(R = \mathcal{O}_{Z,x}, \mathfrak{m}, K)$ at x. Consider $f \in J \subset R$ with expansion $f = \sum_{(A,B)} C_{A,B}\, u^A\, y^B \in \widehat{R}$ as in (2.2). Recall that*

$$\delta(\,\Delta(f,b;u,y)\,) = \min\left\{ \frac{|A|}{b-|B|} \;\middle|\; C_{A,B} \neq 0 \wedge |B| < b \right\}.$$

Then we have

(i) $\delta(\,\Delta(f,b;u,y)\,) \geq 1$ *if and only if* $\mathrm{ord}_x(f) \geq b$.

(ii) $\mathrm{ord}_x(f) > b$ *implies* $in(f,b) = 0$ *and* $\delta(\,\Delta(f,b;u,y)\,) > 1$.

(iii) For $\mathrm{ord}_x(f) \geq b$ *the following three conditions are equivalent*

(a) $\delta(\,\Delta(f,b;u,y)\,) = 1$,

(b) $in(f,b) \neq in_0(f,b) := \sum_{|B|=b} \overline{C_{0,B}}\, Y^B \in K[Y]$,

(c) $in(f,b) = in_\delta(f,b)$, *where* $in_\delta(f,b) := in_0(f,b) + \sum_{(A,B)} \overline{C_{A,B}}\, U^A Y^B$ *and the sum ranges over those* $(A,B) \in \mathbb{Z}_0^n$ *with* $|B| < b$ *and*

$$\frac{|A|}{b-|B|} = \delta(\,\Delta(f,b;u,y)\,).$$

(iv) Suppose $\mathrm{ord}_x(f) \geq b$. *Then* $\delta(\,\Delta(f,b;u,y)\,) > 1$ *if and only if* $in(f,b) = in_0(f,b) \in K[Y]$.

The analogous statements are true if we consider a system of elements in J, say $(f) = (f_1,\ldots,f_m)$. (We only have to use f_i whenever f appears and add the words "for all $i \in \{1,\ldots,m\}$"; except for $\Delta(f,b;u,y)$,).

Proof. We abbreviate the notation and set $\delta := \delta(\,\Delta(f,b;u,y)\,)$.

(i): By definition $\delta \geq 1$ is equivalent to

$$\forall\,(A,B) : C_{A,B} \neq 0 \;\wedge\; |B| < b \;\Rightarrow\; |A| + |B| \geq b. \tag{$*$}$$

This is obviously equivalent to $\mathrm{ord}_x(f) \geq b$. Therefore *(i)* holds.

Let $\mathrm{ord}_x(f) > b$. The first equality of *(ii)*, $in(f,b) = 0$, is clear. If we change in $(*)$ the $\geq$ to $>$, then this modified condition is equivalent to $\delta > 1$. By the assumption the following inequality is true for all (A,B) with $C_{A,B} \neq 0$

$$|A| + |B| \geq \mathrm{ord}_x(f) > b.$$

Especially the modified condition $(*)$ holds, i.e. $\delta > 1$.

(iii) (a) $\Leftrightarrow$ (b): By definition, $\delta = 1$ means that there exists some (A,B) with $C_{A,B} \neq 0, |B| < b$ and $|A| + |B| = b$. But this corresponds to a term $C_{A,B}\, u^A y^B$ in f with $|A| \geq 1$. The existence of this is equivalent to

$$in(f,b) = f \mod \mathfrak{m}^{b+1} \neq in_0(f,b) = \sum_{|B|=b} \overline{C_{0,B}} Y^B.$$

(iii) (a) $\Rightarrow$ (c): Let $\delta = 1$. Then the sum $in_\delta(f,b) = in_0(f,b) + \sum_{(A,B)} \overline{C_{A,B}}\, U^A Y^B$ ranges over those (A,B) with $|B| < b$ and

$$\frac{|A|}{b - |B|} = \delta \;\overset{\delta=1}{\Longleftrightarrow}\; |A| + |B| = b.$$

These are obviously all terms of order b and since $\mathrm{ord}_x(f) \geq b$, we get $in_\delta(f,b) = in(f,b)$.

(iii) (a) $\Leftarrow$ (c): Suppose $in(f,b) = in_\delta(f,b)$. By the definition of $in_\delta(f,b)$ we have $in_\delta(f,b) \neq in_0(f,b)$ and thus there exists some (A,B) such that $C_{A,B} \neq 0$, $|B| < b$ and $|A| = \delta(b - |B|)$. Further $|A| + |B| = b$, because of the assumed equality. Thus

$$b - |B| = |A| = \delta(b - |B|)$$

and since $|B| < b$ it follows that $\delta = 1$.

(iv) is a consequence of *(i)* and the equivalence *(iii) (a) $\Leftrightarrow$ (b)*. $\square$

2.5 The δ-invariant and exceptional data

In section 1.3 we introduced the d_x-invariant of $(\mathbb{E}, u, y)$, where $\mathbb{E} = (J, b)$ is an idealistic exponent on Z, $x \in \operatorname{Sing}(\mathbb{E})$ and (u, y) is a regular system of parameters for the regular local ring $R = \mathcal{O}_{Z,x}$. More precisely, $d_x(\mathbb{E}, u, y)$ is the order of the idealistic coefficient exponent $\mathbb{D}_x(\mathbb{E}, u, y)$ (Definition 1.3.5). We have seen in Corollary 1.3.6 that the d_x-invariant of equivalent idealistic exponents coincide and if (z) is another choice for (y) such that $(y, 1) \cap \mathbb{E} \sim (z, 1) \cap \mathbb{E}$, then we also have $d_x(\mathbb{E}, u, y) = d_x(\mathbb{E}, u, z)$. Further we gave in Lemma 2.1.7 a polyhedral interpretation,

$$d_x(\mathbb{E}, u, y) = \delta_x(\Delta(\mathbb{E}, u, y)),$$

where $\Delta(\mathbb{E}, u, y)$ denotes the non-intrinsic polyhedron of $\mathbb{E}$ (Definition 2.1.1). But still this number depends on (y).

In order to achieve the desired independence we have defined the characteristic polyhedron $\Delta^*_x(\mathbb{E}, u)$ of an idealistic exponent $\mathbb{E}$ for a fixed system (u). (Construction 2.3.2 and Theorem 2.3.4).

Definition 2.5.1. *Let $(f, b; u, y)$ be suitable data (as in Theorem 2.3.4) for which the equality $\Delta^*_x(\mathbb{E}, u) = \Delta(f, b; u, y)$ holds. Then we define the δ-invariant of $(\mathbb{E}, u)$ by*

$$\delta_x(\mathbb{E}, u) := d_x(\mathbb{E}, u, y) = \delta_x(\Delta^*_x(\mathbb{E}, u)) > 1.$$

More generally, we define the δ_x-invariant at any point $w \in \operatorname{Spec}(K[[u]])$ by $\delta_x(\mathbb{E}, u)(w) := d_x(\mathbb{E}, u, y)(w)$ (see Definition 1.3.5).

With this notation we obtain Main Theorem 4:

Proposition 2.5.2. *The rational number $\delta_x(\mathbb{E}, u)$ does not depend on (y) and is invariant under the equivalence $\sim$. Therefore $\delta_x(\mathbb{E}, u)$ is an invariant of the equivalence class of $\mathbb{E}$ and (u). (The same is true for $\delta_x(\mathbb{E}, u)(w)$).*

Proof. With the above explanations we see that this follows by Theorem 2.3.4 and Corollary 1.3.6. □

If we drop the assumption on (y) to give the directrix, then we do not know if there is a polyhedron which is independent of the system (y); we have shown in Example 2.3.8 that we are not able to make $\Delta(f, b; u, y)$ independent of this choice.

But still we can say something in the case, where $(y) = (y_1, \ldots, y_s)$ can only be extended to a system $(y_1, \ldots, y_r), r > s$, which yields the directrix. Namely, in both cases of the example the point $(0, 0, 1)$ appears and therefore we have $\delta(\Delta(f, 2; u, y)) = \delta(\Delta(f, 2; u, z)) = 1$.

Lemma 2.5.3. *Let $\mathbb{E} = (J, b)$ be an idealistic exponent on Z and $x \in \mathrm{Sing}\,(\mathbb{E})$ as before (thus* $\mathrm{ord}_x(J) \geq b$*). Fix a system of elements $(u_1, \ldots, u_d)$ in $R = \mathcal{O}_{Z,x}$ which can be extended to a regular system of parameters for R. Let $(y) = (y_1, \ldots, y_s)$ be such an extension of (u). Assume further that $(y, u_{e+1}, \ldots, u_d)$, $e < d$, gives the directrix. Then we have*

(i) $\delta_x(\,\Delta_x(\,\mathbb{E};\, u_1, \ldots, u_d;\, y_1, \ldots, y_s\,)\,) = 1$. *In particular this is independent of the choice of (y).*

(ii) For $1 \leq i \leq e$, let $L^{(i)} \in \mathbb{L}_0$ be the semi-positive linear form on $\mathbb{R}^d$ defined by $L^{(i)}(v_1, \ldots, v_d) = v_i$. Then

$$\delta_{L^{(i)}}(\,\Delta_x(\,\mathbb{E};\, u_1, \ldots, u_d;\, y_1, \ldots, y_s\,)\,) = 0 \qquad \textit{(Definition 2.2.4).}$$

Proof. By assumption there is an $f \in J$ with $in(f, b) \notin K[Y_1, \ldots, Y_s]$. Hence in the $\mathfrak{m}$-adic expansion $f = \sum_{(A,B)} C_{A,B}\, u^A\, y^B$ there is an (A, B) such that

$$C_{A,B} \neq 0, \quad |A| \neq 0 \quad \text{and} \quad |A| + |B| = b.$$

Since $I\mathrm{Dir}_x(\mathbb{E}) = \langle Y_1, \ldots, Y_s, U_{e+1}, \ldots U_d \rangle$, we can choose (A, B) such that the corresponding monomial cannot be deleted by any coordinate changes. Then (A, B) yields in $\Delta_x(\,\mathbb{E};\, u_1, \ldots, u_d;\, y_1, \ldots, y_s\,)$ the point $v := \dfrac{A}{b - |B|}$ with $|v| = 1$. Further $\mathrm{ord}_x(J) \geq b$ implies by Lemma 2.4.6*(i)*

$$\delta_x(\,\Delta_x(\,\mathbb{E};\, u_1, \ldots, u_d;\, y_1, \ldots, y_s\,)\,) \geq 1.$$

Together this yields *(i)*.

For *(ii)*: By definition $\delta_{L^{(i)}}(\,\Delta_x(\,\mathbb{E};\, u_1, \ldots, u_d;\, y_1, \ldots, y_s\,)\,)$ coincides with

$$\min \left\{ \frac{A_i}{b - |B|} \in \mathbb{R}_0^d \;\middle|\; \exists\, f \in J : f = \sum_{(A',B')} C_{A',B'}\, u^{A'}\, y^{B'} \in \widehat{R} \,\wedge\, C_{A,B} \neq 0 \right\}$$

$(A = (A_1, \ldots, A_d))$. Since $e < d$, there is an $f \in J$ with $in(f, b) \neq 0$ and $in(f, b) \notin K[Y_1, \ldots, Y_s]$, but $in(f, b) \in K[U_{e+1}, \ldots, U_d, Y_1, \ldots, Y_s]$. This means there exists an $A = (0, \ldots, 0, A_{e+1}, \ldots, A_d) \neq 0$ such that $C_{A,B} \neq 0$ in the expansion of f and $|A| + |B| = b$. In particular $|B| < b$ and $\dfrac{A}{b - |B|} \in \Delta_x(\,\mathbb{E};\, u_1, \ldots, u_d;\, y_1, \ldots, y_s\,)$. Hence we get for $1 \leq i \leq e$ the claim,

$$\delta_{L^{(i)}}(\,\Delta_x(\,\mathbb{E};\, u_1, \ldots, u_d;\, y_1, \ldots, y_s\,)\,) = 0.$$

$\square$

The lemma leads to the following extension of the definition of the δ-invariant:

Definition 2.5.4. *Let $\mathbb{E} = (J, b)$ be an idealistic exponent on Z and $x \in \mathrm{Sing}\,(\mathbb{E})$ as before. Fix a system of elements $(u_1, \ldots, u_d)$ in $R = \mathcal{O}_{Z,x}$ which can be extended to a regular system of parameters $(u, y) = (u, y_1, \ldots, y_s)$ for R. Assume further that $(y, u_{e+1}, \ldots, u_d)$, $e \leq d$, gives the directrix. We define the δ-invariant of $(\mathbb{E}; u_1, \ldots, u_d)$ by*

$$\delta_x(\mathbb{E}; u_1, \ldots, u_d) := \begin{cases} \delta_x(\Delta_x^*(\mathbb{E}, u)) > 1, & \text{if } d = e, \\ 1, & \text{if } d < e. \end{cases}$$

Note that also in the case $\delta_x(\mathbb{E}; u_1, \ldots, u_d) = 1$ this is coming from the polyhedra. One sees easily with the geometric picture that

$$\delta_x(\mathbb{E}; u_1, \ldots, u_d) - \sum_{i=1}^{d} \delta_{L^{(i)}}(\, \Delta_x(\, \mathbb{E};\ u_1, \ldots, u_d;\ y_1, \ldots, y_s\,)\,) \geq 0.$$

Thus $\delta_{L^{(i)}}(\, \Delta_x(\, \mathbb{E};\ u_1, \ldots, u_d;\ y_1, \ldots, y_s\,)\,) \leq 1$ for $e < i \leq d$. But, in general, we can't say anything on the exact value.

Example 2.5.5. Recall Example 2.3.8. Suppose $h_1(u_1) = 0$ and $h_2(u_2) = 0$. Then

$$f_1 = y_1^2 = z_1^2 \quad \text{and} \quad f_2 = u_3 y_2 + (y_2 + u_2^n)^p = u_3 z_2 - u_2^n u_3 + z_2^p,$$

where $p = char(K) \neq 2$ and $n \in \mathbb{Z}_+$, $n \geq 2$. We consider the origin x. Then

$$Vert(\, \Delta_x(f, 2;\ u_1, u_2, u_3;\ y_1, y_2)\,) = \left\{\ (0,0,1),\ v := \left(0, \frac{np}{2}, 0\right)\ \right\},$$
$$Vert(\, \Delta_x(f, 2;\ u_1, u_2, u_3;\ z_1, z_2)\,) = \left\{\ (0,0,1),\ w := \left(0, \frac{n}{2}, \frac{1}{2}\right)\ \right\}.$$

We have already seen that by the assumption $p \neq 2$ the two polyhedra are different. In particular,

$$\delta_{L^{(3)}}(\Delta_x(f, 2;\ u_1, u_2, u_3;\ y_1, y_2)) = 0,$$
$$\delta_{L^{(3)}}(\Delta_x(f, 2;\ u_1, u_2, u_3;\ z_1, z_2)) = \frac{1}{2}.$$

Nevertheless, we can define a certain number ν, which depends in general on all the given data $(\mathbb{E}; u_1, \ldots, u_d; y_1, \ldots, y_s)$. In the special case, where $(u_{e+1}, \ldots, u_d)$ are not exceptional, ν is independent of (y). Further we can say a bit more if $V(y)$ has maximal contact.

Let $\mathbb{E} = (J, b)$ be an idealistic exponent on Z and consider a permissible local sequence of regular blow ups

$$\begin{array}{ccccccccc} Z = Z_0 \supset U_0 & \xleftarrow{\pi_1} & Z_1 \supset U_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_{l-1}} & Z_{l-1} \supset U_{l-1} & \xleftarrow{\pi_l} & Z_l \\ \cup & & \cup & & & & \cup & & \\ D_0 & & D_1 & & \dots & & D_{l-1} & & \end{array} \qquad (\star)$$

where $l \in \mathbb{Z}_+$, each $U_i \subset Z_i$ is an open subscheme, $D_i \subset U_i$ is a regular closed subscheme and $\pi_{i+1} : Z_{i+1} \to U_i$ denotes the blow up with center D_i, $0 \leq i \leq l-1$. We write H_i for the exceptional divisor on Z_i of the blow up π_i, $1 \leq i \leq l$.

By abuse of notation we denote the strict transform of H_i in Z_l also by H_i. We further assume that the centers D_i have only simple normal crossings with the exceptional components $\{H_1, \dots H_{i-1}\}$. Therefore the scheme defined by the set of exceptional divisors $E := \{H_1, \dots, H_l\}$ has only simple normal crossing singularities. The transform of $\mathbb{E} := \mathbb{E}_0 := (J_0, b)$ in U_i is denoted by $\mathbb{E}_i = (J_i, b)$.

Let $x \in \operatorname{Sing}(\mathbb{E}_l)$ and denote by $R := \mathcal{O}_{Z_l,x}$ the regular local ring of Z_l at x. We define $E(x)$ to be the ordered set of exceptional components containing x. The normal crossing condition on E implies that there is a regular system of parameters $(t) = (t_1, \dots, t_n) = (u, y) = (u_1, \dots, u_d, y_1, \dots, y_s)$ for R which fulfills the property: For every $H \in E(x)$ there is a $j \in \{1 \dots, n\}$ such that $H = V(t_j)$. Let

$$E_y(x) := (H_{i(1)}, H_{i(2)}, \dots, H_{i(q)}) \quad \text{with} \quad 1 \leq i(1) < i(2) < \dots < i(q) \leq l$$

be the ordered set of those $H \in E(x)$ such that $H \cap V(y) \subsetneq V(y)$. Hence these $H_{i(j)}$ are given by elements in (u) and without loss of generality we may assume that $H_{i(j)} = V(u_{i(j)})$.

We assign to each component $H_i \in E$, $1 \leq i \leq l$, the rational number $d_{H_i} := d_{H_i}(\mathbb{E}_l, u, y) := d_{H_i}(\mathbb{E}_l, u, y)(x)$ which is given by

$$d_{H_i}(\mathbb{E}_l, u, y) := \begin{cases} \delta_{L^{(i)}}(\, \Delta_x(\, \mathbb{E}_l;\, u_1, \dots, u_d;\, y_1, \dots, y_s\,)\,), & \text{if } i \in \{i(1), \dots, i(q)\}, \\ 0, & \text{otherwise,} \end{cases}$$

where $L^{(i)} \in \mathbb{L}_0$ denotes as before the semi-positive linear form on $\mathbb{R}^d$ which is given by $L^{(i)}(v_1, \dots, v_d) = v_i$. (We have seen in Example 2.5.5 that this number may depend on the choice of (y)).

Definition 2.5.6. *Then we define the exceptional data (or the history) of the idealistic exponent $\mathbb{E}_l$ (and the local sequence of regular blow ups $(\star)$) on $V(y)$ at the point x to be the ordered set of the exceptional divisors (which have at most simple normal crossings) together with the assigned numbers constructed above,*

$$\mathcal{E}_x^{(\star)}(\, \mathbb{E}_l\,) := \mathcal{E}_x^{(\star)}(\, \mathbb{E}_l,\, u,\, y\,) := \Big(\, (H_1,\, d_{H_1})\,;\, \dots\,;\, (H_l,\, d_{H_l})\,\Big).$$

The definition can also be given inductively: $\mathcal{E}^{(\star)}_{x_0}(\mathbb{E}_0) := \emptyset$ *for* $x_0 \in Z$. *Further let* $\pi_l : Z_l \to Z_{l-1}$ *be a permissible blow up for* $\mathbb{E}_{l-1}$ *and let* $x = x_l \in Z_l$ *be above* x_{l-1}, *so* $\pi_l(x_l) = x_{l-1}$. *Then we get* $\mathcal{E}^{(\star)}_{x_l}(\mathbb{E}_l, v, z)$ *from* $\mathcal{E}^{(\star)}_{x_{l-1}}(\mathbb{E}_{l-1}, u, y)$ *by adding the pair* (H_l, d_l) *given by the divisor of* π_l *and by modifying the assigned numbers of the others as follows: If the transform of* H_i *under the blow up does not contain the point* $x = x_l$, *then the assigned number is send to zero; otherwise the transform of* H_i *inherits the assigned number of* H_i.

Since x_l lies above x_{l-1}, we have $E(x_l) = (E(x_{l-1}))' \cup E_l$, where $(E(x_{l-1}))'$ denotes the transform under the blow up π_l and E_l the exceptional divisor. Thus we can choose the regular system of parameters (v, z) at x_l such that those exceptional components in $\mathcal{E}^{(\star)}_{x_{l-1}}(\mathbb{E}_{l-1})$, which do not vanish under the blow up, say for example $V(u_i)$ (where (u, y) denotes a regular system of parameters at x_{l-1}), are given by $V(v_i)$ for the same i.

Definition 2.5.7. *Let the situation be as in the previous definition. Consider the idealistic exponent* $\mathbb{E}_l = (J_l, b)$ *on* Z_l *with exceptional data*

$$\mathcal{E}^{(\star)}_x(\, \mathbb{E}_l,\, u,\, y\,) = \Big(\, (H_1,\, d_{H_1})\,;\; \ldots\;;\; (H_l,\, d_{H_l})\,\Big),$$

where $x \in \operatorname{Sing}(\mathbb{E}_l)$ *and* $(u, y) = (u_1, \ldots, u_d; y_1, \ldots, y_s)$ *is a regular system of parameters for the local ring at* x *such that* $(y, u_{e+1}, \ldots, u_d)$, $e \le d$, *defines the directrix.*

We define the ν*-invariant of* $\mathbb{E}_l$ *and* (u, y) *by*

$$\nu_x(\mathbb{E}_l, u, y) := \delta_x(\mathbb{E}_l, u) - \sum_{i=1}^{l} d_{H_i}(\mathbb{E}_l, u, y) = \delta_x(\mathbb{E}_l, u) - \sum_{j=1}^{q} d_{H_{i(j)}}(\mathbb{E}_l, u, y).$$

Remark 2.5.8. *The* ν*-invariant is completely determined by the polyhedron*

$$\Delta_x(\, \mathbb{E}_l;\, u_1, \ldots, u_d;\, y_1, \ldots, y_s\,).$$

As we have already pointed out, it may depend on (y). *Further the exceptional data and hence* $\nu_x(\mathbb{E}_l, u, y)$ *may change under the equivalence* $\sim$*; see Example 2.1.9. If* $V(x)$ *is exceptional, then we get two different exceptional data for the equivalent idealistic exponents.*

We overcome this problem in the next section by refining the definition of being equivalent.

Lemma 2.5.9. *Suppose that either*

(a) $d = e$ *or*

(b) $d > e$ *and no element of* $(u_{e+1}, \ldots, u_d)$ *is exceptional.*

Then $\nu_x(\mathbb{E}_l, u, y)$ *is independent of* (y). *Moreover, we have in the second case* $\nu_x(\mathbb{E}_l, u, y) = 1$.

Proof. If *(a)* holds, then $\nu_x(\mathbb{E}_l, u, y)$ is determined by the characteristic polyhedron $\Delta^*(\mathbb{E}_l, u)$ and by Theorem 2.3.4 it is independent of (y).

The claim for *(b)* is a direct consequence of Lemma 2.5.3. $\square$

Let us now consider the situation that $V(y)$ has maximal contact with $\mathbb{E}$ at x; recall that then $\mathbb{E} \sim (y,1) \cap \mathbb{E}$.

Observation 2.5.10. Let the situation be as before — $\mathbb{E} = (J, b)$, $x \in \mathrm{Sing}\,(\mathbb{E})$ and $(u, y) = (u_1, \ldots, u_d; y_1, \ldots, y_s)$ a regular system of parameters for $(R, \mathfrak{m}, K)$ such that $(y, u_{e+1}, \ldots, u_d)$, $e \leq d$, defines the directrix. Suppose further that $V(y)$ has maximal contact with $\mathbb{E}$ at x. (This is a crucial assumption here).

Let $D := V(y, u_{i(1)}, \ldots, u_{i(c)})$, $1 \leq i(1) < i(2) < \ldots < i(c) \leq d$, $(c \leq d)$ be a permissible center for $\mathbb{E}$. Denote by $L_D \in \mathbb{L}_0$ the semi-positive linear form on $\mathbb{R}^d$ which is given by $L_D(w_1, \ldots, w_d) = w_{i(1)} + \ldots + w_{i(c)}$ for $w \in \mathbb{R}^d_0$. Set

$$\delta := \delta_D(\mathbb{E}, u, y) := \delta_{L_D}(\Delta(\mathbb{E}, u, y)) \qquad \text{(see Definition 2.2.4).}$$

Since D is permissible for $\mathbb{E}$, we have $\delta \geq 1$. Set $(v) = (v_1, \ldots, v_{d-c})$ for the elements of $(u_1, \ldots, u_d)$ which are not contained in $(u_{i(1)}, \ldots, u_{i(c)})$.

Convention: In the following we mean $(u_{i(1)}, \ldots, u_{i(c)})$ if we write (u).

Every $f \in J$ has an expansion in $\widehat{R}$ of the form

$$f(u, v, y) = f_b(u, v, y) + \sum_{(A,D,B) \in \mathbb{Z}_0^n} C_{A,D,B}\, u^A\, v^D\, y^B + f_*(u, v, y),$$

where $f_b(u, v, y)$ is homogeneous of degree b, the sum ranges over those (A, D, B) with $|B| < b$ and $|A| + |D| + |B| > b$ and $f_*(u, v, y) \subset \langle y \rangle^b$. (Note that we even have $f_b(u, v, y) \in K[[u_{e+1}, \ldots, u_d, y]]$).

We blow up with center D and consider the $U_{i(1)}$-chart. Suppose $u_{i(1)}$ does not appear in $f_b(u, v, y)$. (Otherwise the order at the origin of the $U_{i(1)}$ drops below b after the blow up and thus all remaining singularities are contained in the other charts). By abuse of notation we denote the transforms of (u, v, y) again by (u, v, y). The transform of f is $f'(u, v, y) = f_b(u, v, y) + \sum \ldots + f'_*(u, v, y)$ with some $f'_*(u, v, y) \subset \langle y \rangle^b$ and the sum in the middle is

$$\sum_{(A,D,B) \in \mathbb{Z}_0^n} C_{A,D,B}\, u_{i(1)}^{(A_{i(1)} + \ldots + A_{i(c)}) + |B| - b}\, u_{i(2)}^{A_{i(2)}} \cdots u_{i(c)}^{A_{i(c)}}\, v^D\, y^B.$$

By definition $V(y)$ still has maximal contact at the singular points above the center. The polyhedron $\Delta(J', b; u, y)$ is determined by the sum in the middle. The $u_{i(1)}$-component of an associated point is

$$\frac{(A_1 + \ldots + A_d) + |B| - b}{b - |B|} = \delta - 1 + \left(\frac{A_1 + \ldots + A_d}{b - |B|} - \delta \right).$$

Moreover, by the definition of δ there exists $(A, D, B) \in \mathbb{Z}_0^n$ such that $C_{A,D,B} \neq 0$ and $\dfrac{A_1 + \ldots + A_d}{b - |B|} = \delta$. Denote by $L^{(1)} \in \mathbb{L}_0$ the semi-positive linear form on $\mathbb{R}^d$ which is defined by $L^{(1)}(w_1, \ldots, w_d) := w_{i(1)}$ for $w \in \mathbb{R}_0^d$. Denote by $\mathbb{E}' = (J', b)$ the transform of $\mathbb{E} = (J, b)$ under the blow up. Then

$$\delta_{L^{(1)}}(\, \Delta_x(\mathbb{E}'; u_1, \ldots u_d; y)\,) = \delta - 1 \ \ (= \delta_D(\mathbb{E}; u_1, \ldots u_d; y) - 1).$$

The exceptional divisor of this one blow up is locally given by $H = V(u_{i(1)})$ and we have shown that the assigned number (see the remarks before Definition 2.5.6) is

$$d_H(\mathbb{E}'; u_1, \ldots u_d; y) = \delta_D(\mathbb{E}; u_1, \ldots u_d; y) - 1. \tag*{$\widetilde{(2.6)}$}$$

Do not forget that the (u, y) on the left hand side live in the situation after the blow up and those on the right hand side before the blow up. In order to express this difference let us rewrite this equality with the correct notation. Then we get up to some reordering of (u, v)

$$d_H\left(\mathbb{E}';\, u_{i(1)},\, \frac{u_{i(2)}}{u_{i(1)}}, \ldots, \frac{u_{i(c)}}{u_{i(1)}},\, v;\, \frac{y}{u_{i(1)}}\right) = \delta_D(\, \mathbb{E}; u, v; y\,) - 1. \tag{2.9}$$

We may go on and do some more permissible blow ups with centers that have at most simple normal crossing singularities with the exceptional components. Set $\mathbb{E}_0 := \mathbb{E}$ and $\mathbb{E}_1 := \mathbb{E}'$. The transform after $l \in \mathbb{Z}_+$ blow ups is denoted by $\mathbb{E}_l$ and the ambient scheme by Z_l. On Z_l we have the exceptional divisors $E = (H_1, \ldots, H_l)$, where H_i originates from the i-th blow up. Let $x_l \in \mathrm{Sing}\,(\mathbb{E}_l)$ be a singular point above the original $x_0 := x \in Z_0 = Z$. The exceptional data is

$$\mathcal{E}_x^{(\star)}(\, \mathbb{E}_l,\, u,\, y\,) = \Big(\, (H_1,\, d_{H_1})\,;\, \ldots\,;\, (H_l,\, d_{H_l})\,\Big).$$

Let $E_y(x) = \{H_{i(1)}, H_{i(2)}, \ldots, H_{i(q)}\}$, $1 \leq i(1) < i(2) < \ldots < i(q) \leq l$, be the ordered set of exceptional divisors H containing x such that $V(y) \not\subseteq H$. By iterating the above we get

$$d_{H_i}(\, \mathbb{E}_l, u, y\,) = \begin{cases} \delta_{D(i)}(\, \mathbb{E}_{i-1}, u, y\,) - 1, & \text{if } i \in \{i(1), \ldots, i(q)\}, \\ 0, & \text{if } i \notin \{i(1), \ldots, i(q)\}, \end{cases}$$

where $D(i) \subset Z_i$ denotes the center of the i-th blow up and (u, y) denote the corresponding transforms of the original regular system of parameters (u, y) at the beginning.

It is important that we fix the maximal contact variables $(y) = (y_1, \ldots, y_s)$. If we choose others after the blow up, then we might get a different ν-invariant — see the example below. But the fixing of (y) does no harm, because by definition their transform still has maximal contact with the points above the center or the order dropped and the idealistic exponent is resolved.

Example 2.5.11. Consider the idealistic exponent $\mathbb{E} = (f, 2)$ over any field K with $char(K) \neq 2$ and where

$$f = y^2 + u_1 u_2^2$$

Clearly the initial of (y) generates the ideal of the directrix $\mathrm{Dir}_x(\mathbb{E})$ and

$$\nu_x(\mathbb{E}; u_1, u_2) = \delta_x(\mathbb{E}; u_1, u_2) = \frac{3}{2}.$$

We choose as permissible center the origin $x = V(y, u_1, u_2)$ and look into the U_2-chart.

$$f' = y'^2 + u_1' u_2.$$

Consider the origin $x' = V(y', u_1', u_2)$ of the U_2-chart. The situation changed. Namely, the directrix $\mathrm{Dir}_{x'}(\mathbb{E}')$ is only a point. The ν-invariant depends on the choice of the maximal contact variable. (Note that $char(K) \neq 2$). First, we choose y' again and get $\delta_x(\mathbb{E}', u_1', u_2; y') = 1$. We see on $V(y')$ the exceptional divisor $H = V(u_2)$ and $d_H(\mathbb{E}'; u_1', u_2; y') = \dfrac{1}{2}$. Thus $\nu_x(\mathbb{E}'; u_1', u_2; y') = 1 - \frac{1}{2} = \frac{1}{2}$.

On the other hand, $V(z)$, $z := u_2$, has also maximal contact with $\mathbb{E}'$ at x'. Set $(w_1, w_2) = (y', u_1')$. We do not see H on $V(z)$ and therefore $\nu_x(\mathbb{E}'; w_1, w_2; z) = \delta_x(\mathbb{E}'; w_1, w_2; z) = 1 \neq \frac{1}{2} = \nu_x(\mathbb{E}'; u_1', u_2; y')$.

The reason for this is that the exceptional data differ. In order to avoid this in the previous observation, we had to fix the maximal contact variable from the very beginning. Therefore in general the ν-invariant depends on the exceptional data on $V(y)$ — exceptions are listed in Lemma 2.5.9.

2.6 Idealistic exponents with history

Up to now we considered the polyhedra only for a fixed representative $\mathbb{E} = (J, b)$ under $\sim$. But we are also interested on the behavior under these changes. In general, there may occur problems. We have already seen in Example 2.1.9 that the polyhedra — and therefore the ν-invariant — may differ for equivalent idealistic exponents. So there is no hope to get an invariant polyhedra. Nevertheless, our goal is to show that the ν-invariant does not change under a modified equivalence relation, which we achieve by restricting the equivalence of idealistic exponents. More precisely, we want to require that the given exceptional data also coincides. This leads to the definition of idealistic exponents with history.

Definition 2.6.1. *Let $\mathbb{E} = (J, b)$ be an idealistic exponent on some regular scheme Z and let $E = \{H_1, \ldots, H_l\}$, $l \in \mathbb{Z}_+$, be a set of irreducible divisors on Z such that $H_1 \cup \ldots \cup H_l$ has at most simple normal crossing singularities. As usual we denote for $x \in \mathrm{Sing}\,(\mathbb{E})$ by $(u, y) = (u_1, \ldots, u_d; y_1, \ldots, y_s)$ a regular system of parameters for the regular local ring $(R = \mathcal{O}_{Z,x}, \mathfrak{m}, K)$ such that $(y, u_{e+1}, \ldots, u_d)$, $e \leq d$, defines the directrix.*

(1) We define the exceptional data map of $\mathbb{E}$ associated to E

$$\mathcal{E} := \mathcal{E}_E : \mathrm{Sing}\,(\mathbb{E}) \to (E \times \mathbb{R}_0)^l$$

by sending $x \in \mathrm{Sing}\,(\mathbb{E})$ to the exceptional data of $\mathbb{E}$ on $V(y)$ at x which is induced by E, $\mathcal{E}(x) := \mathcal{E}_x^{(\star)}(\,\mathbb{E},\, u,\, y\,) = \{\, (H_1, d_1),\, \ldots,\, (H_l, d_l)\,\}$. Recall that this is an ordered set irreducible divisors (which have at most simple normal crossings) together with assigned numbers $d_i \in \mathbb{R}_0$ which can be read off the polyhedron $\Delta(\mathbb{E}, u, y)$. (Here we identify the ordered set with an element of $(E \times \mathbb{R}_0)^l$).

(2) We call the pair $(\mathbb{E}, \mathcal{E}) = ((J, b), \mathcal{E})$ an idealistic exponent with history on Z.

(3) Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be two equivalent idealistic exponents on Z and E as above. Then the induced idealistic exponents $(\mathbb{E}_1, \mathcal{E}_1)$ and $(\mathbb{E}_2, \mathcal{E}_2)$ are defined to be equivalent at $x \in \mathrm{Sing}\,(\mathbb{E}_1) = \mathrm{Sing}\,(\mathbb{E}_2)$ with respect to (u, y) if

$$\mathcal{E}_1(x) = \mathcal{E}_2(x), \quad \text{in particular the assigned numbers } d_j \text{ coincide.}$$

In this case we write $\mathbb{E}_1 \sim_{\mathcal{E}(x)}^{(u,y)} \mathbb{E}_2$ or if there is no confusion possible we use only $\mathbb{E}_1 \sim_{\mathcal{E}(x)} \mathbb{E}_2$. Further we say $(\mathbb{E}_1, \mathcal{E}_1)$ and $(\mathbb{E}_2, \mathcal{E}_2)$ are equivalent, if they are equivalent at any $x \in \mathrm{Sing}\,(\mathbb{E}_1) = \mathrm{Sing}\,(\mathbb{E}_2)$ and we write $\mathbb{E}_1 \sim_{\mathcal{E}} \mathbb{E}_2$.

The context to resolution of singularities is the following: We start with some idealistic exponent $\mathbb{E}_0$ on Z_0 and we denote by $\mathbb{E}_l$ the transform of $\mathbb{E}_0$ after $l \in \mathbb{Z}_+$ permissible blow ups. Then in the above definition $\mathbb{E} := \mathbb{E}_l$ and E denotes the set of exceptional divisors which arose during the blow ups.

Note that $\mathbb{E}_{0,1} \sim \mathbb{E}_{0,2}$ on Z_0 implies $\mathbb{E}_{l,1} \sim_{\mathcal{E}} \mathbb{E}_{l,2}$ on Z_l and in particular we get $\mathbb{E}_{l,1} \sim_{\mathcal{E}(x)} \mathbb{E}_{l,2}$ for all $x \in \operatorname{Sing}(\mathbb{E}_{l,1}) = \operatorname{Sing}(\mathbb{E}_{l,2})$.

For applications it is sometimes important to consider at the beginning of the resolution process an idealistic exponent $\mathbb{E}_0$ together with a set E_0 of irreducible divisors which have at most simple normal crossings. Then we get already here an idealistic exponent with history $(\mathbb{E}_0, \mathcal{E}_{E_0})$ with non-trivial exceptional data.

Since we focus on the construction of the Bierstone-Milman invariant locally at a point x, it suffices for our purposes to consider the equivalence $\sim_{\mathcal{E}(x)}$ at x. By abuse of notation we call the pair $(\mathbb{E}_x, \mathcal{E}(x))$ (local situation at x) also an idealistic exponent with history.

Definition 2.6.2. *Let $(\mathbb{E}, \mathcal{E} = \mathcal{E}_E)$ (with $E = \{H_1, \ldots, H_l\}$) be an idealistic exponent with history on Z as in the previous definition. A blow up $\pi : Z' \to Z$ with center $D \subset Z$ is called permissible for $(\mathbb{E}, \mathcal{E})$, if the following conditions hold:*

(1) π is permissible for $\mathbb{E}$ in the sense of Definition 1.1.2 (i.e. D is regular and $D \subseteq \operatorname{Sing}(\mathbb{E})$).

(2) $D \cup H_1 \cup \cdots \cup H_l \subset Z$ has at most simple normal crossing singularities.

The transform of $(\mathbb{E}, \mathcal{E})$ under a permissible blow up π is given by $(\mathbb{E}', \mathcal{E}')$, where $\mathbb{E}'$ denotes the transform of $\mathbb{E}$ under π and the exceptional data map $\mathcal{E}'$ is defined by $E' := \{H'_1, \ldots, H'_l, H_{l+1}\}$ — here H'_i is the transform of H_i under the blow up π and H_{l+1} is the exceptional divisor corresponding to π.

Analogous to Remark 1.1.5 this leads to the definition of a local sequence of regular blow ups which is permissible for a given idealistic exponent with history.

Remark 2.6.3. *Let us have another look at Example 2.1.9. Suppose $V(x)$ is exceptional. Then we get that the assigned number is $\dfrac{d-1}{d}$ for $\mathbb{E}_1$ and it is $\dfrac{d-2}{d-1}$ for $\mathbb{E}_2$. (Thus the ν-invariants at the origin differ for these equivalent idealistic exponents). Hence they are not equivalent as idealistic exponents with history, because they have different exceptional data.*

Therefore the Diff Theorem, as it is stated in Proposition 1.1.13, is in general not true for the equivalence $\sim_{\mathcal{E}(x)}$. A weaker version, which is still valid for idealistic exponents with history, is given in Lemma 2.6.7 (iv).

Remark 2.6.4. *Let us briefly explain the relationship between idealistic exponents with history, presentations, basic objects and marked ideals. First, all of them have their common origin in Hironaka's idealistic exponents. We already discussed these notion in Chapter 1 of this thesis. (Alternatively see [H3] or [H4]).*

As an additional aspect basic objects ([BEV], Definition 3.1, p.356) take also the set of exceptional components into account, which are created by the blow ups in the resolution process.

Presentations ([BM3], (4.1), p.241) are more analytic (Bierstone and Milman use the language of manifolds instead of schemes; see also below, where we cite a sentence of their introduction), and besides the set of exceptional components, they also regard the order of the exceptional components. They do this by allowing test blow ups (in our context local sequences of regular blow ups) in the definition of the equivalence relation of presentations, with centers coming from the exceptional components and being not necessarily permissible (loc. cit. (4.3), Definition 4.6, p.242, and Definition 4.10, p.243/244). Hence the equivalence classes of presentations are smaller than those of basic objects. This is also nicely explained in [BM5].

Marked ideals ([W], Definition 2.1.1, p.782/783) are a variant of basic objects, where additionally in the definition of equivalence (loc. cit. Definition 3.1.1, p.791) the orders of the exceptional components are considered (without the use of non-permissible blow ups), and moreover Włodarczyk is able to avoid extensions by finite systems $(t_1, \ldots, t_a)$ *(as they are used in Definition* 1.1.6 *and which appear also for basic objects and presentations). Note that his multiple test blow ups (loc. cit. Definiton 2.1.3 and Definition 2.1.4, p.783) are similar to the notion of local sequence of regular blow ups.*

Idealistic exponents with history are one possible way to translate Bierstone-Milman's presentations into the language of schemes. (In [BM3], p.208, they already write in the introduction: "But our work neither was conceived nor is written in the modern language of algebraic geometry"). As for marked ideals we are able to keep clear of non-permissible blow ups in the definition of equivalence. Since we started with idealistic exponents, we use extensions by finite systems $(t_1, \ldots, t_a)$*, and they are crucial for our proofs.*

It should be noted that in more recent papers Bierstone and Milman use the notion of marked ideals in a slightly modified version as well. So this is another variant to translate their presentations into the language of schemes. But still they use test transformations which are not permissible. See for example [BM6] (in particular section 2), where they show that the algorithm for resolution of marked ideals of [W], [Ko] and [BM3] coincide, [BM6], Corollary 1.4.

Furthermore Villamayor and his students began to consider Rees algebras in their investigations on resolution in positive characteristic. (See [V3] and [EV])

Proposition 2.6.5. *Let $(\mathbb{E},\mathcal{E}) = ((J,b),\mathcal{E})$ be an idealistic exponent with history on some regular scheme Z, $x \in \operatorname{Sing}(\mathbb{E})$, $(u,y) = (u_1,\dots,u_d;y_1,\dots,y_s)$ a regular system of parameters as in the previous definition and $\mathcal{E} := \mathcal{E}(x) := \mathcal{E}_x^{(\star)}(\mathbb{E},u,y)$ some fixed exceptional data of $\mathbb{E}$ on $V(y)$ at x. Then $\nu_x(\mathbb{E},u,y)$ is independent of (y) and invariant under $\sim_{\mathcal{E}(x)}$. Therefore we may also write*

$$\nu_x(\,\mathbb{E},\,\mathcal{E};\,u\,) := \nu_x(\,\mathbb{E},\,u,\,y\,).$$

Proof. By Proposition 2.5.2 and Lemma 2.5.3 $\delta_x(\mathbb{E},u)$ is independent of (y) and invariant under $\sim$. So it is also invariant under $\sim_{\mathcal{E}(x)}$. Further $\sim_{\mathcal{E}(x)}$ fixes the exceptional data $\mathcal{E} = \{\,(H_1,d_1),\,\dots,\,(H_l,d_l)\,\}$. The ν-invariant is defined via $\nu_x(\mathbb{E},u,y) = \delta_x(\mathbb{E},u) - \sum_{i=1}^{l} d_i$. Thus we get the assertion. □

This proves the first part of Main Theorem 5.

Lemma 2.6.6. *Let $\mathbb{E}_1 = (J_1,b_1)$ and $\mathbb{E}_2 = (J_2,b_2)$ be two equivalent idealistic exponents on Z and $x \in \operatorname{Sing}(\mathbb{E}_1) = \operatorname{Sing}(\mathbb{E}_2)$. Let $(t) = (u,y)$ be a regular system of parameters for the local ring $R = \mathcal{O}_{Z,x}$ such that $(y,u_{e+1},\dots,u_d)$, $e \le d$, defines the directrix $\operatorname{Dir}_x(\mathbb{E}_1) = \operatorname{Dir}_x(\mathbb{E}_2)$. Denote by $\mathcal{E}^{(j)} = \{\,(H_1^{(j)},d_1^{(j)}),\,\dots,\,(H_l^{(j)},d_l^{(j)})\,\}$ some fixed exceptional data of $\mathbb{E}_j$ on Z at x, $j \in \{1,2\}$. Then*

$$\big(\mathbb{E}_1,\mathcal{E}^{(1)}\big) \sim_{\mathcal{E}(x)}^{(t,\emptyset)} \big(\mathbb{E}_2,\mathcal{E}^{(2)}\big) \quad \textit{implies} \quad \Big(\mathbb{E}_1,\mathcal{E}^{(1)}_{V(y)}\Big) \sim_{\mathcal{E}(x)}^{(u,y)} \Big(\mathbb{E}_2,\mathcal{E}^{(2)}_{V(y)}\Big),$$

where $\mathcal{E}^{(j)}_{V(y)}$ denotes the induced exceptional data on $V(y)$. This is the data given by those (H_i,d_i) for which $V(y) \not\subseteq H_i$.

Proof. By the first equivalence we know $\mathcal{E}^{(1)} = \mathcal{E}^{(2)}$. Thus the induced data coincide. The claim follows by Theorem 1.3.2. □

In the previous lemma we can choose (u) such that each H_i of the induced exceptional data on $V(y)$ is given by $V(u_i)$.

Let us see which of the properties shown in chapter 1 for $\sim$ survive under the refined equivalence $\sim_{\mathcal{E}(x)}$.

Lemma 2.6.7. *Let $\mathbb{E} = (J,b)$ be an idealistic exponent on some regular scheme Z, $x \in \operatorname{Sing}(\mathbb{E})$, (u,y) a regular system of parameters for $(R = \mathcal{O}_{Z,x},\mathfrak{m},K)$ as in the previous lemma and*

$$\mathcal{E}(x) = \mathcal{E}_x^{(\star)}(\,\mathbb{E},\,u,\,y\,) = \{\,(H_1,d_1),\,\dots,\,(H_l,d_l)\,\}$$

be some fixed exceptional data of $\mathbb{E}$ on $V(y)$ at x.

(i) For every $a \in \mathbb{Z}_+$ we have $(J^a,ab) \sim_{\mathcal{E}(x)} (J,b)$.

(ii) *Suppose there is another choice for* (y)*, say* $(z) = (z_1, \dots, z_s)$*, such that* $(z,1) \cap \mathbb{E} \sim_{\mathcal{E}(x)} (y,1) \cap \mathbb{E}$*. Then*

$$\mathbb{D}_x(\mathbb{E}, u, z) \sim_{\mathcal{E}(x)} \mathbb{D}_x(\mathbb{E}, u, y)$$

(both times with the induced exceptional data on $V(y)$ *and* $V(z)$*).*

(iii) *If* $char(K) = 0$ *or if* $b < char(K)$*, then there exists a choice for the system* $(y) = (y_1, \dots, y_s)$ *such that*

$$\mathbb{E}_x \sim_{\mathcal{E}(x)} (y,1) \cap \mathbb{D}_x(\mathbb{E}, u, y)$$

(with the induced exceptional data on $V(y)$*).*

(iv) *Let* $\mathcal{D}^{log}_{M,u} = u^M \mathcal{D}_{M,u} \in \mathrm{Diff}^{\leq m}_K (K[[u,y]])$*,* $M = (M_1, \dots, M_d) \in \mathbb{Z}_0^d$ *with* $|M| = m$*, be the logarithmic differential operators given by*

$$\mathcal{D}^{log}_{M,u} \left(C_{A,B}\, u^A y^B \right) = \binom{A}{M} C_{A,B}\, u^A y^B$$

(see also Lemma 2.4.5). Then

$$(J,b) \cap (\mathcal{D}^{log}_{M,u} J, b-m) \sim_{\mathcal{E}(x)} (J,b)$$

(with the induced exceptional data on $V(y)$*). Moreover, if* $M_i = 0$ *for all* $i \in \{1, \dots, d\}$ *with* $d_i \neq 0$ *in* $\mathcal{E}(x)$*, then the analogous statement is true for* $\mathcal{D}_{M,u}$*.*

Let $\mathbb{E}_1 = (J_1, b_1)$*,* $\mathbb{E}_2 = (J_2, b_2)$ *be two idealistic exponents on* Z *such that both are equipped with the same exceptional data* $\mathcal{E}_1(x) = \mathcal{E}_2(x) = \mathcal{E}(x)$ *on* $V(y)$ *at* x *. Suppose* $x \in \mathrm{Sing}\,(\mathbb{E}_1) \cap \mathrm{Sing}\,(\mathbb{E}_2)$*.*

(v) *Assume* $b_1, b_2 \in \mathbb{Z}_+$ *and let* $m \in \mathbb{Z}_+$ *be a positive integer such that* $b_1 \mid m$ *and* $b_2 \mid m$*. Then* $(J_1, b_1) \cap (J_2, b_2) \sim_{\mathcal{E}(x)} \left(J_1^{\frac{m}{b_1}} + J_2^{\frac{m}{b_2}}, m \right)$*.*

(vi) $\mathbb{E}_1 \sim_{\mathcal{E}(x)} \mathbb{E}_2$ *implies*

- *(a)* $\mathbb{E}_1 \cap \mathbb{E} \sim_{\mathcal{E}(x)} \mathbb{E}_2 \cap \mathbb{E}$*.*
- *(b)* $\mathrm{ord}_z(\mathbb{E}_1) = \mathrm{ord}_z(\mathbb{E}_2)$ *for all* $z \in Z$*. In particular,* $\mathrm{ord}_x(\mathbb{E}_1) = \mathrm{ord}_x(\mathbb{E}_2)$*.*
- *(c)* $\mathrm{Sing}\,(\mathbb{E}_1) = \mathrm{Sing}\,(\mathbb{E}_2)$*.*
- *(d)* $\mathbb{T}_x(\mathbb{E}_1) \sim \mathbb{T}_x(\mathbb{E}_2)$*,* $\mathbb{D}\mathrm{ir}_x(\mathbb{E}_1) \sim \mathbb{D}\mathrm{ir}_x(\mathbb{E}_2)$ *and* $\mathbb{R}\mathrm{id}_x(\mathbb{E}_1) \sim \mathbb{R}\mathrm{id}_x(\mathbb{E}_2)$*.*
- *(e)* $\mathbb{D}_x(\mathbb{E}_1, u, y) \sim_{\mathcal{E}(x)} \mathbb{D}_x(\mathbb{E}_2, u, y)$ *and* $(y,1) \cap \mathbb{E} \sim (y,1) \cap \mathbb{D}_x(\mathbb{E}, u, y)$ *(both with the induced exceptional data on* $V(y)$ *at* x*).*

Proof. *(i)* follows by Lemma 1.1.8 *(i)* and Lemma 2.4.4 *(i)*.

(ii) First, (u) is a regular system of parameters for the local ring at x of both $V(y)$ and $V(z)$. Therefore it makes sense to say $(z,1)\cap\mathbb{E}\sim_{\mathcal{E}(x)}(y,1)\cap\mathbb{E}$ holds with the induced exceptional data. By the equivalence we know that these data coincide. So this is also true for $\mathbb{D}_x(\mathbb{E},u,z)$ and $\mathbb{D}_x(\mathbb{E},u,y)$, because we consider the same data for them. Further we have shown in Proposition 1.3.4 that $\mathbb{D}_x(\mathbb{E},u,z)\sim\mathbb{D}_x(\mathbb{E},u,y)$ and this implies the claim.

(iii) By restricting the exceptional data to $V(y)$ every divisor H, which appears with non-zero assigned number, fulfills $H\cap V(y)\subsetneq V(y)$. Thus the data are the same on both sides. Lemma 1.3.7 yields the rest.

(iv) Clearly the factoring of the exceptional components in J is preserved by these differential operators and the Diff Theorem, Proposition 1.1.13, provides $(J,b)\cap(\mathcal{D}^{log}_{M,u}J,b-m)\sim(J,b)$ resp. $(J,b)\cap(\mathcal{D}_{M,u}J,b-m)\sim(J,b)$. This proves the assertion.

(v) follows by Lemma 1.1.8 *(ii)* and Lemma 2.4.4 *(ii)*.

(vi) The exceptional data coincide for $\mathbb{E}_1$ and $\mathbb{E}_2$. Hence the same is true for $\mathbb{E}_1\cap\mathbb{E}$ and $\mathbb{E}_2\cap\mathbb{E}$. Lemma 1.1.8 *(iv)* gives the usual equivalence $\sim$ and this shows *(a)*.

$\mathbb{E}_1\sim_{\mathcal{E}(x)}\mathbb{E}_2$ implies in particular $\mathbb{E}_1\sim\mathbb{E}_2$. Hence *(b)* follows by the Numerical Exponent Theorem, Proposition 1.1.10, *(c)* by Corollary 1.1.11 and *(d)* by Proposition 1.2.19.

The restriction on the exceptional data to $V(y)$ forces them to be the same for the given idealistic exponents in *(e)*. Thus *(e)* is a consequence of Theorem 1.3.2 and Corollary 1.3.3. □

Further the behavior under permissible blow ups is interesting for us.

Lemma 2.6.8. *Let $\mathbb{E}_1=(J_1,b_1)$, $\mathbb{E}_2=(J_2,b_2)$ be two idealistic exponents on Z, $x\in\operatorname{Sing}(\mathbb{E}_1)\cap\operatorname{Sing}(\mathbb{E}_2)$ and (u,y) a regular system of parameters for $(R=\mathcal{O}_{Z,x},\mathfrak{m},K)$. Suppose both have the same exceptional data*

$$\mathcal{E}(x)=\{\,(H_1,d_1),\,\ldots,\,(H_l,d_l)\,\}$$

on $V(y)$ at x. Let $\pi:Z'\to Z$ be a blow up which is permissible for both idealistic exponents and $x'\in\operatorname{Sing}(\mathbb{E}_1')\cap\operatorname{Sing}(\mathbb{E}_2')$ with $\pi(x')=x$. Then we have

(i) $(\mathbb{E}_1\cap\mathbb{E}_2)'\sim_{\mathcal{E}(x')}\mathbb{E}_1'\cap\mathbb{E}_2'$.

(ii) $\mathbb{E}_1\sim_{\mathcal{E}(x)}\mathbb{E}_2$ *implies* $\mathbb{E}_1'\sim_{\mathcal{E}(x')}\mathbb{E}_2'$.

(iii) $\mathbb{E}_1\sim_{\mathcal{E}(x)}\mathbb{E}_2$ *is stable under extensions by $\mathbb{A}^a_k$ ($a\in\mathbb{Z}_+$), i.e. stable under extensions of the regular system of parameters by systems $(t)=(t_1,\ldots,t_a)$ corresponding to $\mathbb{A}^a_k$*

Proof. If x is not contained in the center of the blow up π, then the situation at x' did not change. In this case the lemma is trivially true. Thus let us assume that the center contains x. Since the exceptional data are equal for $\mathbb{E}_1$ and $\mathbb{E}_2$, the same is true for $\mathbb{E}_1[t]$ and $\mathbb{E}_2[t]$. Further it follows that the exceptional data for $\mathbb{E}_1'$, $\mathbb{E}_2'$, $(\mathbb{E}_1 \cap \mathbb{E}_2)'$ and $\mathbb{E}_1' \cap \mathbb{E}_2'$ are always given by the transform of $\mathcal{E}(x)$. (For the transform see Definition 2.5.6). Hence *(i)* follows by Lemma 1.1.8 *(v)*. The definition of $\sim$ (Definition 1.1.1) implies *(ii)* and *(iii)*. $\square$

3 The invariant of Bierstone and Milman in characteristic zero

Let X be a singular scheme embedded in some regular scheme Z and $x \in X$. In [BM3] Bierstone and Milman simplify Hironaka's original proof for resolution of singularities in characteristic zero. In order to do this, they construct an invariant

$$\mathrm{inv}_X(x) = (\nu_1, s_1;\ \nu_2, s_2;\ \ldots;\ \nu_t, s_t;\ \nu_{t+1}),$$

(see below), and show that it strictly decreases under blow ups with centers being the maximal locus of this invariant. Here they use as the first term $\nu_1 = H_{X,x}$ the Hilbert-Samuel function. Further the ν_{i+1} are certain higher multiplicities and the $s_i \in \mathbb{Z}_0$ count certain exceptional divisors, $1 \leq i \leq t$.

As an application of the theory of idealistic exponents with history we show in this chapter that these higher multiplicities can be deduced with a purely polyhedral approach. More precisely, we see that ν_{i+1} is the ν-invariant of a certain idealistic exponent with history.

In the upcoming considerations the base field k has always characteristic zero.

3.1 The setup

Before we come to the construction of $\mathrm{inv}_X(x)$, we want to recall the setup which is used in [BM3] at the beginning.

Setup B. Let X be a scheme contained in a regular scheme Z over k, $char(k) = 0$, and $x \in X$. Denote by $(R = \mathcal{O}_{Z,x}, \mathfrak{m}, K)$ the regular local ring at x and by $J \subset R$ the ideal defining X locally at x. Let $(t) = (t_1, \ldots, t_n) = (u, y) = (u_1, \ldots, u_e; y_1, \ldots, y_r)$ be a regular system of parameters for R such that the images of (y) in $\mathfrak{m}/\mathfrak{m}^2$, $(Y) = (Y_1, \ldots, Y_r)$, define $\mathrm{Dir}_x(X)$. ($\mathrm{Dir}_x(X)$ denotes the directrix associated to the tangent cone $T_x(X)$ whose defining ideal is $\langle in_{\mathfrak{m}}(g) \mid g \in J \rangle$). Let $(f) = (f_1, \ldots, f_m)$ be a normalized (u)-standard base of J (Definition 2.2.16*(4)*) and $b_i = \mathrm{ord}_x(f_i)$, $1 \leq i \leq m$. We associate to this the idealistic exponent $\mathbb{E}$ on R,

$$\mathbb{E} := (f_1, b_1) \cap \ldots (f_m, b_m).$$

Further we choose (y) such that $V(y)$ has maximal contact with $\mathbb{E}$ at x and we may assume that R is complete — if not, then we pass to the $\mathfrak{m}$-adic completion $\widehat{R}$ of R.

We say also that $\mathrm{Dir}_x(X)$ is the directrix associated to J.

By construction $\mathrm{Dir}_x(\mathbb{E}) = \mathrm{Dir}_x(X)$ and thus $(Y) = (Y_1, \ldots, Y_r)$ yields $\mathrm{Dir}_x(\mathbb{E})$. Another consequence is the following generalization of the result in [C1] to our setting.

Theorem 3.1.1. *Let $J \subset R$, (u, y) and $\mathbb{E} = (f_1, b_1) \cap \ldots (f_m, b_m)$ be as in Setup B. In particular $V(y)$ has maximal contact with $\mathbb{E}$. Then the characteristic polyhedron of $\mathbb{E}$ is already minimal and coincides with Hironaka's polyhedron,*

$$\Delta^*(\mathbb{E}, u) = \Delta(\mathbb{E}, u, y) = \Delta(f, u, y)$$

Proof. The first equality follows by Proposition 2.3.9 and the second follows by considering the definitions. □

Recall that the exponent of an element $g = \sum_B g_B t^B \in K[[t]]$ is defined to be

$$\exp(g) := \inf\{ B \in \mathbb{Z}_0^n \mid g_B \neq 0 \}, \qquad \text{(Definition 2.2.14)},$$

where $\mathbb{Z}_0^n$ is endowed with the lexicographical order of $(|B|, B)$. Further we define the support of g to be

$$\mathrm{supp}(g) := \{ B \in \mathbb{Z}_0^n \mid g_B \neq 0 \}.$$

Definition 3.1.2 ([BM3], before Theorem 3.17, p.238)**.** *Let $(g) = (g_1, \ldots, g_m)$ ($m \in \mathbb{Z}_+$) be an arbitrary, finite system of elements in R, where R is as in Setup B. The diagram of initial exponents associated to (g) is defined by*

$$\mathfrak{N}(g) := \mathfrak{N}(g, t) := \bigcup_{i=1}^{m} (\, \alpha^i + \mathbb{Z}_0^n \,) \subset \mathbb{Z}_0^n, \qquad \alpha^i := \exp(g_i).$$

The subset $\mathfrak{N}(g) \subset \mathbb{Z}_0^n$ can be decomposed as

$$\mathfrak{N}(g) = \bigcup_{i=1}^{m} \Delta_i = \bigcup_{i=1}^{m} (\, \alpha^i + \Box_i \,), \tag{3.1}$$

where $\Delta_1 := \alpha^1 + \mathbb{Z}_0^n$ and for $2 \leq i \leq m$ $\Delta_i := (\alpha^i + \mathbb{Z}_0^n) \setminus \bigcup_{j=1}^{i-1} \Delta_j$. Further we put $\Box_0 := \mathbb{Z}_0^n \setminus \bigcup_{i=1}^{m} \Delta_i$ and $\Box_i \subseteq \mathbb{Z}_0^n$ is defined via $\Delta_i = \alpha^i + \Box_i$ if $\Delta_i \neq \emptyset$ and $\Box_i = \emptyset$ is $\Delta_i = \emptyset$, $1 \leq i \leq m$.

With decomposition (3.1) we can state the following division theorem which is formulated for our situation in [BM3], Theorem 3.17, p.238. (The proof and references to the origins can be found in [BM1], Theorem 6.2, p.207).

Theorem 3.1.3. *Let $(g) = (g_1, \ldots, g_m)$ $(m \in \mathbb{Z}_+)$ be an arbitrary, finite system of elements in R, where R is as in Setup B. (In particular R is complete). For every $h \in R$ there exist unique $q_i \in R$, $1 \leq i \leq m$, and $r \in R$ such that*

$$\operatorname{supp}(q_i) \subset \square_i, \quad \operatorname{supp}(r) \subset \square_0 \quad \text{and} \quad h = \sum_{i=1}^{m} q_i g_i + r.$$

Observation 3.1.4 ([BM3], (7.1), p.261f)**.** Let $\mathfrak{N} := \mathfrak{N}(f) \subset \mathbb{Z}_0^n$ be the diagram of initial exponents associated to a system $(f) = (f_1, \ldots, f_m)$ as in Setup B. Recall that we have put $\alpha^i = \exp(f_i)$, $1 \leq i \leq m$. By the choice of (y) and the definition of $\mathbb{E}$ we have $\alpha^i \in \{0\}^e \times \mathbb{Z}_0^r$ for all i. Hence α^i corresponds to a monomial $y^{\widetilde{\alpha}^i}$, where $\widetilde{\alpha}^i \in \mathbb{Z}_0^r$ is defined via $\alpha^i = (0, \ldots, 0; \widetilde{\alpha}^i) \in \{0\}^e \times \mathbb{Z}_0^r$.

Since (f) is a (u)-standard base, we have $|\alpha^1| \leq |\alpha^2| \leq \ldots \leq |\alpha^m|$. Moreover, the condition on (f) to be normalized (Definition 2.2.15) implies $\alpha^i \notin \Delta_j$ for all $i \neq j$ (or equivalently $\alpha^i \notin \mathfrak{N}(f_j)$). Therefore $\{\alpha^1, \ldots \alpha^m\}$ is the smallest possible subset needed to define $\mathfrak{N}(f)$. For $\kappa \in \mathbb{Z}_+$ we define

$$\begin{aligned} m(\kappa) &:= \max\{\, i \in \{1, \ldots, m\} \mid |\alpha^i| \leq \kappa \,\}, \\ \mathfrak{N}(\kappa) &:= \textstyle\bigcup_{i=1}^{m(\kappa)} (\, \alpha^i + \mathbb{Z}_0^n \,) \subset \mathbb{Z}_0^n. \end{aligned}$$

Thus we can order the α^i into blocks: There are integers $0 < \kappa_1 < \kappa_2 < \ldots < \kappa_p$ such that

$$\begin{array}{cl} \alpha^1, \ \ldots, \ \alpha^{m(\kappa_1)} & \text{are those with } |\alpha^i| = \kappa_1, \\ \alpha^{m(\kappa_1)+1}, \ \ldots, \ \alpha^{m(\kappa_2)} & \text{are those with } |\alpha^i| = \kappa_2, \\ \vdots & \qquad \vdots \\ \alpha^{m(\kappa_{l-1})+1}, \ \ldots, \ \alpha^{m(\kappa_l)} & \text{are those with } |\alpha^i| = \kappa_l, \\ \vdots & \qquad \vdots \\ \alpha^{m(\kappa_{p-1})+1}, \ \ldots, \ \alpha^{m(\kappa_p)} & \text{are those with } |\alpha^i| = \kappa_p. \end{array}$$

Set $m_l := m(\kappa_l)$, $1 \leq l \leq p$. Then $1 \leq m_1 < m_2 < \ldots < m_p = m$. By permutation of the elements in $(y) = (y_1, \ldots, y_r)$, we may suppose that the last s elements are those which appear with non-zero component in at least one of the $\exp(f_1), \ldots, \exp(f_m)$. In order to avoid complicated indices we use for the last s elements of (y) the notation

$$(z) = (z_s,\, z_{s-1},\, \ldots,\, z_2,\, z_1) := (y_{r-s+1},\, y_{r-s+2},\, \ldots,\, y_r)$$

(we assume that s is as small as possible for any permutation) and the remaining elements of (u, y) are denoted by

$$(w) = (w_1,\, w_2,\, \ldots,\, w_d) := (u;\, y_1,\, y_2,\, \ldots,\, y_{r-s}).$$

Therefore there exists an $\mathfrak{N}^* \subset \mathbb{Z}_0^s$ such that $\mathfrak{N}^* + \mathbb{Z}_0^s = \mathfrak{N}^*$ and $\mathfrak{N} = \mathbb{Z}_0^d \times \mathfrak{N}^*$. In particular, $\alpha^i \in \{0\}^d \times \mathfrak{N}^*$. Let $1 \le s_1 \le s_2 \le \ldots \le s_p = s$ be those positive integers such that the last s_l variables, $(z^l) := (z_{s_l},\, z_{s_l-1}, \ldots,\, z_1)$, are those which appear with non-zero component in $\alpha^1, \ldots, \alpha^{m_l}$, $1 \le l \le p$. Thus for every $j \in \{1, \ldots, s\}$ with $j \le s_l$ there is at least one $i(j) \in \{1, \ldots, m_l\}$ such that

$$\alpha^{i(j)} = \beta^j + (0, e^j) \in \{0\}^{n-s_l} \times \mathbb{Z}_0^{s_l}, \tag{3.2}$$

where $\beta^j \in \{0\}^{n-s_l} \times \mathbb{Z}_0^{s_l}$ and $e^j = (e^j_s, e^j_{s-1}, \ldots, e^j_2, e^j_1) \in \mathbb{Z}_0^s$ fulfils $e^j_j = 1$ and $e^j_h = 0$ for every $h \neq j$. Take into account that we ordered $(z) = (z_s, \ldots, z_1)$, backwards. This means: For every z_j there is an $\alpha^{i(j)}$ such that, if z_j is one of the last s_l elements of (z), then we can choose $\alpha^{i(j)}$ among $\alpha^1, \ldots, \alpha^{m_l}$ and the z_j-component of $\alpha^{i(j)}$ is non-zero.

Further there is for each $\mathfrak{N}(\kappa_l) = \bigcup_{i=1}^{m_l} (\, \alpha^i + \mathbb{Z}_0^n \,)$ an $\mathfrak{N}^l := \mathfrak{N}(\kappa_l)^* \subset \mathbb{Z}_0^{s_l}$ such that

$$\mathfrak{N}(\kappa_l) = \mathbb{Z}_0^{n-s_l} \times \mathfrak{N}^l$$

and each $\Box_i$ is of the form $\Box_i = \mathbb{Z}_0^{n-s_l} \times \Box_i^l$ for some $\Box_i^l \subset \mathbb{Z}_0^{s_l}$.

Now we can state the five properties which are the assumptions needed in [BM3], see loc. cit. (7.2), p.262. We show that these already hold in our given Setup B.

Lemma 3.1.5. *Let the assumptions be as in Setup B, i.e.*

(A1) $(R, \mathfrak{m}, K)$, *char*$(K) = 0$, *is a complete regular local ring and* $J \subset R$ *is an ideal in* R.

(A2) $(u, y) = (u_1, \ldots, u_e; y_1, \ldots, y_r)$ *is a regular system of parameters for* R *such that* $(Y) = (Y_1, \ldots, Y_r)$ *yields the directrix associated to* J.

(A3) $(f) = (f_1, \ldots, f_m)$ *is a normalized* (u)*-standard base of* J *and we consider the idealistic exponent* $\mathbb{E} = \bigcap_{i=1}^m (f_i, b_i)$, $b_i = \mathrm{ord}_x(f_i)$, *where* x *denotes the origin of* Spec (R).

(A4) $V(y)$ *has maximal contact with* $\mathbb{E}$ *at* x *and as described in the previous observation we set* $(u, y) = (w_1, \ldots, w_d; z_s, z_{s-1}, \ldots, z_1) = (w, z)$.

Moreover, let $\mathfrak{N} = \mathfrak{N}(f) \subset \mathbb{Z}_0^n$ *be the diagram of initial exponents associated to* $(f) = (f_1, \ldots, f_m)$ *which is defined by* $\{\alpha^i = \exp(f_i) \mid 1 \leq i \leq m\}$. *Fix* $K_0 \in \mathbb{Z}_0$ *with* $K_0 \geq \max\{|\alpha^i| \mid 1 \leq i \leq m\} - 1$. *Then the following properties are true:*

(i) $\mathrm{ord}_x(f_i) = |\alpha^i|$ *for all* $i \in \{1, \ldots, m\}$.

(ii) Let $\kappa \in \mathbb{Z}_0$. *Let* $H \in gr_x(R) \cong K[U, Y]$ *be a homogeneous polynomial of degree* d. *Then there are unique homogeneous polynomials* $Q_i(H)$ *of degree* $d - |\alpha^i|$, $1 \leq i \leq m(\kappa)$ *(*$Q_i(H) = 0$ *if* $d - |\alpha^i| < 0$*), and* $R(H)$ *homogeneous of degree* d *such that*

$$H = \sum_{i=1}^{m(\kappa)} Q_i(H) \cdot in_{\mathfrak{m}}(f_i) + R(H),$$

$\mathrm{supp}(\, Q_i(H)\,) \subset \Box_i$, $1 \leq i \leq m(\kappa)$, *and* $\mathrm{supp}(\, R(H)\,) \cap \mathfrak{N}(\kappa) = \emptyset$.

(iii) For every $h \in J$, *there are* $q_i(h) \in R$, $1 \leq i \leq m$, *such that* $h = \sum_{i=1}^m q_i(h) f_i$ *and* $\mathrm{supp}(q_i(h)) \subset \Box_i$ *for all* i.

(iv) We define $(g) = (g_s, g_{s-1}, \ldots, g_1)$ *by putting* $g_j := \mathcal{D}_{\beta^j} f_{i(j)}$, *where* $\mathcal{D}_{\beta^j}$ *denotes as usually the differential operator on* R *associated to* $\beta^j \in \mathbb{Z}_0^n$. *Let* $l \in \{1, \ldots, p\}$. *If* $1 \leq j \leq s_l$, *then* $g_j \in \langle z^l \rangle$ *and* $\det\big(\partial g^l / \partial z^l\big)(0) \neq 0$, *where* $\big(\partial g^l / \partial z^l\big)$ *denotes the Jacobian matrix of* $(g^l) := (g_{s_l}, g_{s_l - 1}, \ldots, g_1)$ *with respect to* $(z^l) = (z_{s_l}, z_{s_l - 1}, \ldots, z_1)$ $(= (y_{r - s_l + 1}, y_{r - s_l + 2}, \ldots, y_r))$.

(v) Let $l \in \{1, \ldots, p\}$. *If* $i > m_l$, *then* $\mathcal{D}_{(0,\beta)} f_i \in \langle z^l \rangle$, *for every* $(0, \beta) \in \mathfrak{N}(\kappa_l)$ *with* $\beta \in \mathfrak{N}^l \subset \mathbb{Z}_0^{s_l}$ *and* $|\beta| \leq K_0$.

Proof. By definition $\alpha^i = \exp(f_i)$. This already shows *(i)*, $\mathrm{ord}_x(f_i) = |\alpha^i|$.

Part *(ii)* and *(iii)* are immediate consequences of Theorem 3.1.3. For the first one use $(in_{\mathfrak{m}}(f_1), \ldots, in_{\mathfrak{m}}(f_m))$ and $\exp(in_{\mathfrak{m}}(f_i)) = \exp(f_i)$ for all $1 \leq i \leq m$. For the second use $(f_1, \ldots, f_m)$ and $\langle f \rangle = J$.

Property (3.2) implies that we have in $\widehat{R}$

$$f_{i(j)} = \epsilon \cdot z^{\alpha^{i(j)}} + \varphi(w, z)$$

for a unit $\epsilon \in \widehat{R}^\times$ and a certain $\varphi(w, z) \in \widehat{R}$. (In order to make the notation not too complicated, we drop for these elements the index $i(j)$; but in general one has to use $\epsilon_{i(j)}, \ldots$). Therefore we have

$$g_j = \mathcal{D}_{\beta^j} f_{i(j)} = \epsilon \binom{\alpha^{i(j)}}{\beta^j} z_j + \mathcal{D}_{\beta^j}(\varphi(w, z)) =: \epsilon' \, z_j + \psi(w, z)$$

where we define the unit $\epsilon' \in \widehat{R}^\times$ and the element $\psi(w,z) \in \widehat{R}$ such that z_j does not appear in $\psi(w,z)$. In general, it is not clear that $\psi(w,z) \in \langle z^l \rangle$. Since $|\beta^j| = b_{i(j)} - 1$ (use $|\alpha^{i(j)}| = b_{i(j)}$ and the definition of β^j), we get by the Diff Theorem, Proposition 1.1.13,

$$(f_{i(j)}, b_{i(j)}) \sim (f_{i(j)}, b_{i(j)}) \cap (\mathcal{D}_{\beta^j} f_{i(j)}, 1) = (f_{i(j)}, b_{i(j)}) \cap (\epsilon' z_j + \psi(w,z), 1).$$

Set $\widetilde{z}_j := z_j + (\epsilon')^{-1}\psi(w,z)$. Thus we see that $V(\widetilde{z}_j)$ has maximal contact with $\mathbb{E}$ at x. Proposition 2.3.9 yields that we can replace z_j by $\widetilde{z}_j$ without changing the properties of Setup B. We do so and get $g_j = \mathcal{D}_{\beta^j} f_{i(j)} = \epsilon \binom{\alpha^{i(j)}}{\beta^j} \widetilde{z}_j \in \langle z^l \rangle$.

Convention: From now on we suppose that we already have $z_j = \widetilde{z}_j$ from the beginning.

Using $g_j = \epsilon' \cdot z_j$ for all j yields that

$$\left(\partial g^l / \partial z^l\right) = \begin{pmatrix} \frac{\partial g_s}{\partial z_s} & \cdots & \frac{\partial g_s}{\partial z_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial z_s} & \cdots & \frac{\partial g_1}{\partial z_1} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_s}{\partial z_s} & & 0 \\ & \ddots & \\ 0 & & \frac{\partial g_1}{\partial z_1} \end{pmatrix}$$

is a diagonal matrix with units on the diagonal. Therefore the assertion

$$\det \left(\partial g^l / \partial z^l\right)(0) \neq 0$$

follows. This proves *(iv)*.

Let $(0,\beta) \in \mathfrak{N}(\kappa_l)$. The normalizedness of (f) implies $\mathcal{D}_{(0,\beta)} f_i = 0$ without any reference to K_0: First, $\mathfrak{N}(\kappa_l)$ is given by $\alpha^1 = \exp(f_1), \ldots, \alpha^{m_l} = \exp(f_{m_l})$. Thus $(0,\beta) \in \mathfrak{N}(\kappa_l)$ implies $(0,\beta) + \mathbb{Z}_0^n \subset \exp(\langle f_1, \ldots, f_{m_l} \rangle)$. We have already seen that $\{\alpha^1, \ldots, \alpha^m\} \subset \{0\}^e \times \mathbb{Z}_0^r$. Hence let us identify these points with their projection to $\mathbb{Z}_0^r$. We can expand f_i,

$$f_i = \sum_{B \in \mathbb{Z}_0^r} C_{B,i}\, y^B + \sum_{B \in \mathbb{Z}_0^r} P_{B,i}(u)\, y^B,$$

where $C_{B,i} \in K$ and $P_{B,i} \in K[[u]]$. Since (f,u,y) is normalized we have

$$C_{B,i} = 0 \quad \text{and} \quad P_{B,i}(u) = 0, \qquad \text{for all } B \in \exp(\langle f_1, \ldots, f_{i-1} \rangle).$$

Clearly $i > m_l$ implies $\exp(\langle f_1, \ldots, f_{m_l} \rangle) \subseteq \exp(\langle f_1, \ldots, f_{i-1} \rangle)$.

Set $\beta^* := (0,\beta) \in \mathbb{Z}_0^r$. (This point corresponds to $(0,\beta) \in \mathbb{Z}_0^n$). Then

$$\mathcal{D}_\beta f_i = \sum_{B \in \mathbb{Z}_0^r} C_{B,i} \binom{B}{\beta^*} y^{B-\beta^*} + \sum_{B \in \mathbb{Z}_0^r} P_{B,i}(u) \binom{B}{\beta^*} y^{B-\beta^*}.$$

Since $\begin{pmatrix} B \\ \beta^* \end{pmatrix} = 0$ for $B \notin \beta^* + \mathbb{Z}_0^r$, we can replace the index $B \in \mathbb{Z}_0^r$ in the sum by $B \in \beta^* + \mathbb{Z}_0^r$. But $\beta^* + \mathbb{Z}_0^r \subset \exp(\langle f_1, \dots, f_{i-1} \rangle)$ and the coefficients $C_{B,i}$ and $P_{B,i}$ are all zero, i.e. $\mathcal{D}_\beta f_i = 0$. Thus part *(v)* is also true. $\square$

3.2 The case without exceptional divisors

Let k, $X \subset Z$, $x \in X$, $J \subset R$, $(t) = (t_1, \ldots, t_n) = (u, y) = (u_1, \ldots, u_e; y_1, \ldots, y_r)$, $(f) = (f_1, \ldots, f_m)$ and $\mathbb{E} = (f_1, b_1) \cap \ldots (f_m, b_m)$ be as in Setup B.

Now we come to the description of the procedure which Bierstone and Milman use to determine their invariant $\mathrm{inv}_X(x)$. (For the hypersurface case see [BM4] and for the general case see [BM3]).

First, we do the easier case without considering the exceptional components and after that we investigate the general case.

Construction 3.2.1. Let the situation be as in Setup B. For the moment let us forget about $(t) = (u, y)$ and consider an arbitrary regular system of parameters $(w) = (w_1, \ldots, w_n)$ for R. Locally at x the scheme X is given by the idealistic exponent $\mathbb{E}$. We define

$$\mathcal{G}_1(x) := \mathbb{E} = (f_1, b_1) \cap \ldots (f_m, b_m).$$

Choose $i_0 \in \{1, \ldots, m\}$. Then $\mathrm{ord}_x(f_{i_0}) = b_{i_0}$ and for simplicity we write (f, b) instead of (f_{i_0}, b_{i_0}). After a linear coordinate change we may assume that $\dfrac{\partial^b f}{\partial w_n^b} \neq 0$. Set $N_1(x) := V(z_1)$, where

$$z_1 := \frac{\partial^{b-1} f}{\partial w_n^{b-1}}.$$

Then by the Diff Theorem, Proposition 1.1.13,

$$\mathcal{G}_1(x) \sim \mathcal{G}_1(x) \cap (z_1, 1)$$

and $N_1(x)$ has maximal contact with $\mathcal{G}_1(x)$ at x. After another coordinate change we may suppose that $w_n = z_1$.

In the next step we consider the situation on $N_1(x)$, where we define the idealistic exponent $\mathcal{H}_1(x)$ (on $N_1(x)$) by

$$\mathcal{H}_1(x) := \bigcap_{i=1}^{m} \bigcap_{l:=l(i)=0}^{b_i - 1} \left(\frac{\partial^l f_i}{\partial w_n^l}\Big|_{V(z_1)}, \, b_i - l \right).$$

This is the idealistic coefficient exponent of $\mathcal{G}_1(x)$ with respect to (z_1) (Definition 1.3.1). Then set

$$\mu_2 := \mu_2(x) := \mathrm{ord}_x(\, \mathcal{H}_1(x)\,).$$

and in the case without looking at exceptional components $\nu_2 := \nu_2(x) := \mu_2(x)$. Further we define

$$\mathcal{G}_2(x) \; := \; \bigcap_{j=1}^{m^{(2)}} (g_j, b_j) \; := \bigcap_{(h, b_h) \supset \mathcal{H}_1(x)} (h, b_h \nu_2)\,.$$

This completes the first step of the process (without exceptional divisors). Then we start again with $\mathcal{G}_2(x)$ instead of $\mathcal{G}_1(x)$. Note that by construction there exists $j_0 \in \{1, \dots, m^{(2)}\}$ such that $\mathrm{ord}_x(g_{j_0}) = b_{j_0}$.

Remark 3.2.2. *(1) It is possible and happens that $b_j < \mathrm{ord}_x(g_j)$ for some $j \in \{1, \dots, m^{(2)}\}$.*

(2) Set above $h_{i,l} := \frac{\partial^l f_i}{\partial w_n^l}\Big|_{V(z_1)}$ for $i \in \{1, \dots, m\}$ and $l = l(i) \in \{0, \dots, b_i - 1\}$. Then

$$\mu_2(x) = \min\left\{ \frac{\mathrm{ord}_x(h_{i,l})}{b_i - l} \;\Big|\; i \in \{1, \dots, m\} \wedge l \in \{0, \dots, b_i - 1\} \right\}.$$

(3) If we start with $(t) = (u, y)$ and not an arbitrary regular system of parameters (w), then we can make our choices such that after r steps in the process $z_j = y_j$ for all $1 \le j \le r$. By construction $\mathcal{H}_r(x) = \mathbb{D}_x(\mathbb{E}, u, y)$ is the idealistic coefficient exponent of $\mathbb{E}$ with respect to (y) and thus $\nu_{r+1} = \mu_{r+1} = \delta_x(\mathbb{E}, u)$. Further all the nice properties, which have been proven in the previous chapter, hold also for $\mathcal{H}_r(x)$ and $\mu_2(x)$.
Since $\mu_{s+1}(x) = \mathrm{ord}_x(\,\mathcal{H}_s(x)\,) = \delta_x(\mathcal{G}_1(x); w_1, \dots, w_{n-s}; z_1, \dots, z_s)$, we also get $\mu_{s+1}(x) = 1 = \delta_x(\mathbb{E}; w_1, \dots, w_{n-s})$ if $s < r$ or equivalently if $d := n - s > e$ (Definition 2.5.4).
After r steps we start over with $\mathcal{G}_{r+1}(x)$ instead of $\mathcal{G}_1(x)$. We determine the directrix $\mathrm{Dir}_x(\mathcal{G}_{r+1}(x))$, distinguish $(u) = (u_1, \dots, u_e)$ into

$$(u) = (\, u_1,\, \dots,\, u_{e^{(2)}};\, y_{r+1},\, \dots,\, y_{r^{(2)}}\,),$$

and so on.

This leads to

Observation 3.2.3. Let the situation be as in Setup B. As in Construction 3.2.1 set $\mathcal{G}_1(x) = (f_1, b_1) \cap \dots (f_m, b_m)$. In the case without exceptional components the invariant of Bierstone and Milman has the following form

$$\begin{aligned} \mathrm{inv}_X(x) &= (\nu_1, s_1;\, \nu_2, s_2;\, \dots) = \\ &= (\nu_1, 0;\, 1, 0;\, \dots;\, 1, 0;\, \nu_{r^{(1)}+1}, 0;\, 1, 0;\, \dots;\, 1, 0;\, \nu_{r^{(2)}+1}, 0;\, 1, 0;\, \dots), \end{aligned}$$

where $r^{(1)} := r$ and $r^{(q)}$, $q \ge 3$, is defined in the analogous way to $r^{(2)}$. Set $r^{(0)} := 0$. Then we have $0 = r^{(0)} < r^{(1)} < r^{(2)} < \ldots \le n$ and, for all $q \ge 1$, $(Y_{r^{(q-1)}+1}, \dots, Y_{r^{(q)}})$ yields $\mathrm{Dir}_x(\,\mathcal{G}_{r^{(q-1)}+1}(x)\,)$ and with $e^{(q)} := n - r^{(q)}$ we get

$$\nu_{r^{(q)}+1} \,=\, \delta_x(\,\mathcal{G}_{r^{(q-1)}+1}(x);\, u_1, \dots, u_{e^{(q)}}\,) \,>\, 1.$$

Recall that we have already shown that $\delta_x(\mathcal{G}_{r^{(q-1)}+1}(x); u_1, \ldots, u_{e^{(q)}})$ is coming from some polyhedra and does neither depend on the choice of the representative for $\mathcal{G}_{r^{(q-1)}+1}(x)$ as idealistic exponent nor on the choice of (y). For the exact references see Proposition 2.5.2 and the remarks before Definition 2.5.1.

Putting everything together we get the following proposition, which implies Main Theorem 1 in the special case without exceptional divisors.

Proposition 3.2.4. *Let the data be as in Setup B and use the notation of Observation 3.2.3. Let $J_{r^{(q-1)}+1} \subset K[[u_1, \ldots, u_{e^{(q-1)}}]]$ be the ideal corresponding to $\mathcal{G}_{r^{(q-1)}+1}(x)$, $q \geq 1$. Let $(g) = (g_1, \ldots, g_l)$ ($l \in \mathbb{Z}_+$) denote the generators of $J_{r^{(q-1)}+1}$ which we get from $(f) = (f_1, \ldots, f_m)$ via Construction 3.2.1. Set $d_i := \mathrm{ord}_x(g_i)$ for $1 \leq i \leq l$.*

For every $i \in \{1, \ldots, l\}$, g_i has an expansion of the form

$$g_i = g_i(u^{(q)}, y^{(q)}) = G_i(y^{(q)}) + \sum_{|B|<d_i} G_{B,i}(u^{(q)})\,(y^{(q)})^B + g_i^*(u^{(q)}, y^{(q)}), \tag{3.3}$$

where $(u^{(q)}, y^{(q)}) = (u_1, \ldots, u_{e^{(q)}}; y_{r^{(q-1)}+1}, \ldots, y_{r^{(q)}})$, $B \in \mathbb{Z}_0^{r^{(q)}-r^{(q-1)}}$ and with some $g_i^*(u, y) \in \langle y^{(q)} \rangle^{d_i+1}$,

(i) $G_i(y^{(q)}) \in K[y^{(q)}]$ is a polynomial homogeneous of degree d_i and

(ii) $G_{B,i}(u^{(q)}) \in K[[u^{(q)}]]$ has order $\mathrm{ord}_x(G_{B,i}) > d_i - |B|$ at x.

Further we have the properties (always $1 \leq i \leq l$ and $B := B(i) \in \mathbb{Z}_0^{r^{(q)}-r^{(q-1)}}$)

(iii) $\mathcal{H}_{r^{(q)}}(x) = \{ \left(G_{B,i}(u^{(q)}),\ d_i - |B| \right) \mid i, B : |B| < d_i \}$,

$\mathcal{G}_{r^{(q)}+1}(x) = \{ \left(G_{B,i}(u^{(q)}),\ (d_i - |B|) \cdot \delta^{(q)} \right) \mid i, B : |B| < d_i \}$,

$\nu_{r^{(q)}+1} = \min \left\{ \dfrac{\mathrm{ord}_x(G_{B,i})}{d_i - |B|} \;\middle|\; i, B : |B| < d_i \right\} = \delta^{(q)} > 1$,

where $\delta^{(q)} := \delta_x(\mathcal{G}_{r^{(q-1)}+1}(x),\ u^{(q)}) = \delta(\Delta_x^*(\mathcal{G}_{r^{(q-1)}+1}(x),\ u^{(q)}))$.

(iv) The polyhedron $\Delta_x^*(\mathcal{G}_{r^{(q)}+1}(x), u^{(q+1)}) = \Delta_x^*(\mathcal{G}_{r^{(q)}+1}(x); u_1, \ldots, u_{e^{(q+1)}})$ is a projection of $\Delta_x^*(\mathcal{G}_{r^{(q-1)}+1}(x), u^{(q)}) = \Delta_x^*(\mathcal{G}_{r^{(q-1)}+1}(x); u_1, \ldots, u_{e^{(q)}})$.

Let $s \in \mathbb{Z}_+$ with $r^{(q-1)} < s < r^{(q)}$. We set $(u^{(q,s)}) := (u^{(q)}, y_{s+1}, \ldots, y_{r^{(q)}})$ and $(y^{(q,s)}) := (y_{r^{(q-1)}+1}, \ldots, y_s)$. Then the statements analogous to (3.3) and (i)–(iv) are true for $(u^{(q,s)}, y^{(q,s)})$ instead of $(u^{(q)}, y^{(q)})$. The only major modification, which we have to do, is in (ii): $\mathrm{ord}_x(G_{B,i}(u^{(q,s)})) \geq d_i - |B|$ and there exist at least one $1 \leq i \leq l$ and $B := B(i) \in \mathbb{Z}_0^{s-r^{(q-1)}}$ such that equality holds. By Definition 2.5.4 resp. Lemma 2.5.3 we have $\delta^{(q,s)} := \delta(\Delta_x(\mathcal{G}_{r^{(q-1)}+1}(x),\ u^{(q,s)},\ y^{(q,s)})) = 1$.

Note: For $q = 1$ we set $J_{r^{(q-1)}+1} = J_1 := J \subset K[[u_1, \ldots, u_{e^{(0)}}]] = R$. (Recall that $e^{(0)} = n$ and we put $(u_1, \ldots, u_n) := (w_1, \ldots, w_n)$).

Proof. Assertion *(i)*, *(ii)* and $\delta^{(q)} > 1$ follow since $(Y^{(q)}) = (Y_{r^{(q-1)}+1}, \ldots, Y_{r^{(q)}})$ yields $\mathrm{Dir}_x(\, \mathcal{G}_{r^{(q-1)}+1}(x)\,)$.

Part *(iii)* is a consequence of the definition of the idealistic coefficient exponent (Definition 1.3.1) and the construction of $\mathcal{H}_{r^{(q)}}, \mathcal{G}_{r^{(q)}+1}(x)$ and $\nu_{r^{(q)}+1}$ (Construction 3.2.1).

Since $V(y^{(q)})$ has maximal contact, the polyhedron

$$\Delta_x(\, \mathcal{G}_{r^{(q)}+1}(x),\, u^{(q+1)};\, y_{r^{(q-1)}+1},\, \ldots,\, y_{r^{(q)}}\,) = \Delta_x^*(\, \mathcal{G}_{r^{(q)}+1}(x),\, u^{(q+1)}\,)$$

is minimal (Theorem 3.1.1). Then Corollary 2.1.5 implies *(iv)*.

The proof of the last part with $s \in \mathbb{Z}_+$, $r^{(q-1)} < s < r^{(q)}$, is clear. □

The statement on $\nu_{r^{(q)}+1}$ depends only on the polyhedra $\Delta_x(\, \mathcal{G}_{r^{(q-1)}+1}(x),\, u^{(q)}\,)$. By Corollary 2.1.4 these polyhedra are independent of the choice of the generators (g). Further we do not necessarily need the assumption that (g) is coming from (f) as in Construction 3.2.1. Therefore it is no restriction to assume in the previous proposition that (g) is a normalized (u)-standard base of $J_{r^{(q-1)}+1}$.

Remark 3.2.5. *A similar description of $\mathcal{H}_r(x)$ as above (with $\nu_1(x) = H_{X,x}$ the Hilbert-Samuel function of X) has already been proven in [BM3], see loc. cit. Construction 4.18, p.246, and Theorem 9.4 (p.277). Further they show how to get their invariant in the case without exceptional components by using "weighted initial exponents" and the "weighted diagram of initial exponents", see loc. cit. Remark 3.25, p.240. But note that they do not give a polyhedral approach in the general case, where exceptional divisors also have to be considered.*

So far we have to determine the generators (g) of the ideal $J_{r^{(q-1)}+1}$ step-by-step and apply the previous proposition. By introducing weights on the regular system of parameters (u, y) we are (at least in this special case) able to extend this result such that we get similar statements only with the use of the generators (f) of J.

Remark 3.2.6. *Let the situation be as in Setup B and as in Construction 3.2.1 let $\mathcal{G}_1(x) = (f_1, b_1) \cap \ldots (f_m, b_m)$. Recall that, if we don't consider exceptional components, then we have*

$$\mathrm{inv}_X(x) = (\nu_1, 0;\, 1, 0;\, \ldots;\, 1, 0;\, \nu_{r^{(1)}+1}, 0;\, 1, 0;\, \ldots;\, 1, 0;\, \nu_{r^{(2)}+1}, 0;\, 1, 0;\, \ldots)$$

and with the notation of Proposition 3.2.4 $\nu_{r^{(1)}+1} = \delta^{(1)} > 1$. At the beginning we separated the regular system of parameters of the regular local ring R into (u, y), *where the initial forms of $(y) = (y_1, \dots, y_r)$ build a minimal generating set for the ideal of* $\mathrm{Dir}_x(\mathcal{G}_1(x))$*. The latter is the directrix associated to the homogeneous ideal*

$$I^{(0)} := \langle in(f_i, b_i) \mid 1 \le i \le m \rangle.$$

Let $L^{(0)} := L_0 \in \mathbb{L}_+$ be the positive linear form on $\mathbb{R}^e$ which is given by $L^{(0)}(v) = |v| = v_1 + \ldots + v_e$ for $v = (v_1, \dots, v_e) \in \mathbb{R}^e$. We associated to such a linear form the valuation $v_{L^{(0)}}$ on R (Definition 2.2.1), where

$$v_{L^{(0)}}(g) := \sup\{\, L^{(0)}(A) + |B| \mid g \in u^A y^B R \,\}$$

for $g \in R$. Since $v_{L^{(0)}}(f_i) = b_i$, we get $in(f_i, b_i)_{u,y} = in(f_i, L^{(0)})_{u,y}$. (For the definition of the second initial form see (2.4)).

Up to now the images of (u, y) in the graded ring $gr_{\mathfrak{m}}(R)$ were equipped with the standard grading. So the values are determined by the valuation $v^{(0)}$ on R with

$$v^{(0)}(y_j) = v^{(0)}(u_i) = 1 \quad \text{and} \quad v^{(0)}(\lambda) = 0$$

for $j \in \{1, \dots, r\}$, $i \in \{1, \dots, e\}$ and $\lambda \in R^\times$. Recall that $r^{(1)} = r$ and $e^{(1)} = e$. We define the valuation $v^{(1)}$ on R which assigns weights to the (u, y) as follows

$$v^{(1)}(y_j) = 1, \quad v^{(1)}(u_i) = \frac{1}{\delta^{(1)}} \quad \text{and} \quad v^{(1)}(\lambda) = 0$$

for $j \in \{1, \dots, r^{(1)}\}$, $i \in \{1, \dots, e^{(1)}\}$ and $\lambda \in R^\times$.

Let $L^{(1)} \in \mathbb{L}_+$ be the positive linear form on $\mathbb{R}^{e^{(1)}}$ given by $L^{(1)}(v_1, \dots, v_e) = \dfrac{|v|}{\delta^{(1)}}$ for $v \in \mathbb{R}^{e^{(1)}}$. Then $v^{(1)}(u^A y^B) = c$, for some $c \in \mathbb{Z}_+$, if and only if

$$L^{(1)}(A) + |B| = \frac{|A|}{\delta^{(1)}} + |B| = c. \tag{$*$}$$

The last condition is equivalent to $\dfrac{|A|}{c - |B|} = \delta^{(1)}$ if $c - |B| \neq 0$. Therefore we get together with $v^{(1)}(f_i) = b_i$ that

$$in_{\delta^{(1)}}(f_i, b_i)_{u,y} = in(f_i, b_i)_{u,y} + \sum_{(A,B)} \overline{C_{A,B}}\, U^A Y^B = in(f_i, L^{(1)})_{u,y},$$

where the sum ranges over those $(A, B) \in \mathbb{Z}^{e+r}$ which fulfill $(*)$. Consider the quasi-homogeneous ideal $I^{(1)}$ in the graded ring associated to $v^{(1)}$,

$$I^{(1)} := \langle\, in(f_i, L^{(1)})_{u,y} \mid 1 \le i \le m \,\rangle.$$

The directrix $\mathrm{Dir}_x(I^{(1)})$ *corresponding to* $I^{(1)}$ *is defined in the same way as for a homogeneous ideal; we only have to be careful with the grading. Modify* $(y_1, \dots, y_{r^{(2)}})$ *such that their initial forms with respect to* $L^{(1)}$ *define* $\mathrm{Dir}_x(I^{(1)})$. *In the same way as we determine at the beginning* $(y^{(1)}) = (y_1, \dots, y_{r^{(1)}})$ $(r^{(1)} = r)$, *we can compute now* $(y^{(2)}) = (y_{r^{(1)}+1}, \dots, y_{r^{(2)}})$, $r^{(2)} > r^{(1)}$. *Note that* $v_{L^{(1)}}(y_j) = \dfrac{1}{\delta^{(1)}}$ *for all elements in* $(y^{(2)})$.

Let $M^{(1)} \in \mathbb{L}_+$ *be the positive linear form on* $\mathbb{R}^{r^{(2)}}$ *defined by*

$$M^{(1)}(v,w) = M^{(1)}(v_1, \dots, v_{r^{(1)}}, w_1, \dots, v_{r^{(2)}-r^{(1)}}) = |v| + \frac{|w|}{\delta^{(1)}}$$

for $(v,w) \in \mathbb{R}^{r^{(2)}}$. *We expand* f_i *with respect to* $(u^{(2)}; y^{(1)}, y^{(2)})$, $i \in \{1, \dots, m\}$, *as in (3.3)*

$$f_i = f_i(u^{(2)}; y^{(12)}) = F_i(y^{(12)}) + \sum_{M^{(1)}(B) < b_i} F_{B,i}(u^{(2)})\,(y^{(12)})^B + f_i^*(u^{(2)}; y^{(12)}),$$

where we write $(y^{(12)})$ *for* $(y^{(1)}, y^{(2)})$ *and with some* $f_i^*(u^{(2)}; y^{(12)}) \in \langle y^{(12)} \rangle^{b_i+1}$. *Further the following properties hold*

(i) $F_i(y^{(12)}) \in K[y^{(12)}]$ *is a polynomial, quasi-homogeneous of degree* b_i.

(ii) $F_{B,i}(u^{(2)}) \in K[[u^{(2)}]]$ *has order* $\mathrm{ord}_x(\, G_{B,i}\,) > b_i - |B|$ *at* x.

(iii) $\mathcal{H}_{r^{(2)}}(x) = \{\, \big(\, F_{B,i}(u^{(2)})),\ b_i - M^{(1)}(B) \big) \ \mid\ i, B : M^{(1)}(B) < b_i \,\}$,

$\mathcal{G}_{r^{(2)}+1}(x) = \{\, \big(\, F_{B,i}(u^{(2)})),\ (b_i - M^{(1)}(B)) \cdot \delta^{(2)} \big) \ \mid\ i, B : M^{(1)}(B) < b_i \,\}$,

$$\nu_{r^{(2)}+1} = \min \left\{ \frac{\mathrm{ord}_x(\, F_{B,i}\,)}{b_i - M^{(1)}(B)} \ \middle|\ i, B : M^{(1)}(B) < b_i \right\} = \delta^{(2)} > 1,$$

where $1 \leq i \leq m$ *and* $B := B(i) \in \mathbb{Z}_0^{r^{(2)}}$ *and*

$$\delta^{(2)} := \delta_x(\, \mathcal{G}_1(x),\, u^{(2)},\, y^{(12)}\,) = \delta(\, \Delta_x(\, \mathcal{G}_1(x),\, u^{(2)},\, y^{(12)}\,)\,).$$

A slight modification of Proposition 2.1.3 shows that the polyhedron

$$\Delta_x(\, \mathcal{G}_1(x),\, u^{(2)},\, y^{(12)}\,) \qquad \textit{(Definition 2.1.1)}$$

is a projection of the Newton polyhedron $\Delta^N(\mathcal{G}_1(x),\, (u^{(2)}, y^{(12)})\,)$ *resp. of the characteristic polyhedron* $\Delta(\mathcal{G}_1(x),\, u^{(1)}, y^{(1)}\,) = \Delta_x^*(\mathcal{G}_1(x),\, u^{(1)})$. *(We only have to project from* $\widetilde{C}^{(l)} := (0, \dots, 0, \delta^N)$ *instead of* $C^{(l)} = (0, \dots, 0, 1)$, *where we set* $\delta^N := \delta(\, \Delta^N\big(\mathcal{G}_1(x),\, (u^{(2)}, y^{(12)})\,)\,\big)$ *).*

Further, one can prove

$$\Delta_x(\,\mathcal{G}_1(x),\,u^{(2)},\,y^{(12)}\,) = \Delta_x(\,\mathcal{G}_{r^{(1)}+1}(x),\,u^{(2)},\,y^{(2)}\,) = \Delta_x^*(\,\mathcal{G}_{r^{(1)}+1}(x),\,u^{(2)}\,).$$

In particular this implies that $\Delta_x(\,\mathcal{G}_1(x),\,u^{(2)},\,y^{(12)}\,)$ *is minimal with respect to to the choices for* $(y^{(12)})$*.*

Note that $\mathcal{H}_{r^{(2)}}(x)$ *is not the idealistic coefficient exponent of* $\mathcal{G}_1(x)$ *with respect to* $(y^{(12)})$ *(Definition 1.3.1), because in the definition of the latter we don't take care of the non-standard valuation* $v^{(1)}$*. Of course, it is easy to extend the definition to this more general case. (But then the notation is getting more complicated ...).*

Then we go on and define the new valuation $v^{(2)}$ *on* R *by*

$$\begin{aligned} v^{(2)}(y_j) &= 1, && \text{if } j \in \{1,\ldots,r^{(1)}\},\\ v^{(2)}(y_j) &= \frac{1}{\delta^{(1)}}, && \text{if } j \in \{r^{(1)}+1,\ldots,r^{(2)}\},\\ v^{(2)}(u_i) &= \frac{1}{\delta^{(1)}\,\delta^{(2)}}, && \text{for } i \in \{1,\ldots,e^{(2)}\},\\ v^{(2)}(\lambda) &= 0, && \text{for } \lambda \in K. \end{aligned}$$

Let $I^{(2)}$ *be the quasi-homogeneous ideal (in the graded ring associated to* $v^{(2)}$*) given by the initial forms of* $(f_1,\ldots,f_m)$ *with respect to* $v^{(2)}$*. Via its directrix we distinguish* $(u^{(2)}) = (u^{(3)}, y^{(3)})$*,* $r^{(3)} > r^{(2)}$*. The further procedure and the resulting statements are now clear.*

Thus we get a new version of Proposition 3.2.4, where we only use the generators (f) *of* J*. We achieve the result for* $s \in \mathbb{Z}_+$ *with* $r^{(q-1)} < s < r^{(q)}$ *in the same way as in the first version.*

3.3 Construction of the invariant in the general case

Now we come to the general definition of the invariant introduced by Bierstone and Milman, where exceptional components are involved, too.

Let X be a scheme embedded in some regular scheme Z over a field k, $char(k) = 0$. In the arbitrary case we have to consider the exceptional components, which arose during the resolution of X so far. Suppose we are in the year j. Then we have a sequence

$$\begin{array}{ccccccccccccc}
\emptyset = E_0 & & E_1 & & \dots & & E_i & & \dots & & E_{j-1} & & E_j \\
Z = Z_0 & \xleftarrow{\pi_1} & Z_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_i} & Z_i & \xleftarrow{\pi_{i+1}} & \dots & \xleftarrow{\pi_{j-1}} & Z_{j-1} & \xleftarrow{\pi_j} & Z_j \\
\bigcup & & \bigcup & & & & \bigcup & & & & \bigcup & & \bigcup \\
X = X_0 & \longleftarrow & X_1 & \longleftarrow & \dots & \longleftarrow & X_i & \longleftarrow & \dots & \longleftarrow & X_{j-1} & \longleftarrow & X_j \\
 & & & & & & \Cup & & & & \Cup & & \Cup \\
 & & & & & & x_i & \longleftarrow\!\shortmid & \dots & \longleftarrow\!\shortmid & x_{j-1} & \longleftarrow\!\shortmid & x_j
\end{array} \tag{3.4}$$

where each $\pi_{i+1} : Z_{i+1} \to Z_i$ is a blowing up in a regular center which is contained in the singular locus of X_i and has only normal crossings with E_i, E_i denotes the set of exceptional divisors on Z_i corresponding to the former blow ups and X_i is the transform of X in Z_i. (The last line is needed later).

Let $x \in X_j$. We want to determine

$$\mathrm{inv}_X(x) = (\nu_1, s_1;\, \nu_2, s_2;\, \dots).$$

We denote by $\mathrm{inv}_r(x)$ resp. $\mathrm{inv}_{r+\frac{1}{2}}(x)$ the invariant which is truncated after s_r resp. ν_{r+1}. Hence

$$\mathrm{inv}_r(x) = (\nu_1, s_1;\, \dots;\, \nu_r, s_r) \quad \text{and} \quad \mathrm{inv}_{r+\frac{1}{2}}(x) = (\nu_1, s_1;\, \dots;\, \nu_r, s_r;\, \nu_{r+1}).$$

Further we denote by $E_j(x)$ the set of exceptional components passing through x. The first invariant $\nu_1(x) = H_{X,x}$ is the Hilbert-Samuel function of X at x. (Alternatively one could choose some other upper semi-continuous invariant of X which is an appropriate measure for the singularities of $\mathbb{E}_j$ and which does not increase under permissible blow ups). Since we are mainly interested in the terms $\nu_i(x)$, we first introduce how we get the terms $s_i(x)$ and explain afterwards the precise construction of the $\nu_i(x)$.

Construction 3.3.1 ($\boldsymbol{s_i(x)}$)**.** In order to define $s_1(x)$ we need the following notation: For $i \leq j$ we denote by $\pi_{ij} : Z_j \to Z_i$ the composition map, $\pi_{ij} = \pi_{i+1} \circ \pi_{i+2} \circ \dots \circ \pi_{j-1} \circ \pi_j$ ($\pi_{jj} := id_{Z_j}$), and $x_i := \pi_{ij}(x)$ is the image of $x = x_j \in Z_j$ in Z_i. Let

$$i_1 := \min \left\{ \, k \in \{0, \dots, j\} \;\middle|\; \mathrm{inv}_{\frac{1}{2}}(x) = \nu_1(x) = \nu_1(x_k) \, \right\}.$$

We define $E^1(x) \subseteq E_j(x)$ to be the set of those exceptional components which are the strict transform of an exceptional component in $E_{i_1}(x_{i_1})$, i.e.

$$E^1(x) = \{\, H \in E_j(x) \mid \exists\, H_0 \in E_{i_1}(x_{i_1}) \,:\, H \text{ is the strict transform of } H_0 \,\}\,.$$

Then we set

$$s_1(x) := \#E^1(x),$$
$$\mathcal{E}_1(x) := E_j(x) \setminus E^1(x).$$

Suppose we know $\mathrm{inv}_{r+\frac{1}{2}}(x) = (\nu_1, s_1; \ldots; \nu_r, s_r; \nu_{r+1})$ for some $r \geq 1$. Let

$$i_{r+1} := \min\Big\{\; k \in \{0, \ldots, j\} \mid \mathrm{inv}_{r+\frac{1}{2}}(x) = \mathrm{inv}_{r+\frac{1}{2}}(x_k) \;\Big\} \geq i_r.$$

Then we define $E^{r+1}(x) \subseteq \mathcal{E}_r(x) \big(= \mathcal{E}_{r-1}(x) \setminus E^r(x) = E_j(x) \setminus \bigcup_{k=1}^{r} E^k(x)\big)$ to be the set of those exceptional components coming from the year i_{r+1} and which we didn't already count in $E^1(x), \ldots, E^r(x)$,

$$E^{r+1}(x) = \Big\{\; H \in \mathcal{E}_r(x) \mid \exists\, H_0 \in E_{i_{r+1}}(x_{i_{r+1}}) \,:\, H \text{ strict transform of } H_0 \;\Big\}\,.$$

As above

$$s_{r+1}(x) := \#E^{r+1}(x),$$
$$\mathcal{E}_{r+1}(x) = \mathcal{E}_r(x) \setminus E^{r+1}(x) = E_j(x) \setminus \bigcup_{k=1}^{r+1} E^k(x).$$

The exceptional components in $E^k(x)$ are old, because they all arose before or in the year i_k. The set $\mathcal{E}_k(x)$ consists of new or young exceptional components which occurred after the year i_k. The sets $E^k(x)$ and $\mathcal{E}_k(x)$ play an important role in the construction for ν_i.

Construction 3.3.2 ($\boldsymbol{\nu_i(x)}$). As already mentioned, the first term of the invariant $\nu_1(x) = H_{X,x}$ is the Hilbert-Samuel function of X at x.

Let $(f) = (f_1, \ldots, f_m)$ and (u, y) be as in Setup B. As in Construction 3.2.1 the scheme X_j is locally at x given by the idealistic exponent on $R = \mathcal{O}_{Z_j,x}$

$$\mathcal{G}(x) = (f_1, b_1) \cap \ldots (f_m, b_m).$$

(In order to avoid too many indices, we don't refer to the year j). For the definition of $\nu_i = \nu_i(x)$, $i \in \mathbb{Z}_+$, it is important to know exactly what the ambient scheme and corresponding exceptional components are. In [BM3] this is done by considering triples $(\, N_{i-1}(x),\, \mathcal{G}_i(x),\, \mathcal{E}_{i-1}(x) \,)$, where $N_{i-1}(x)$ is a regular ambient scheme contained in Z_j, $\mathcal{G}_i(x)$ is a local description of X_j on $N_{i-1}(x)$ and $\mathcal{E}_{i-1}(x)$

is an ordered set of exceptional divisors on Z_j which have simultaneously only normal crossing with $N_{i-1}(x)$. In our language this means $(\mathcal{G}_i(x), \mathcal{E}_{i-1}(x))$ is an idealistic exponent with history on $N_{i-1}(x)$ (Definition 2.6.1), where we identify $\mathcal{E}_{i-1}(x)$ with the exceptional data which it defines together with $\mathcal{G}_i(x)$ on $N_{i-1}(x)$.

At the beginning $N_0(x) = \mathrm{Spec}\,(R)$ is the germ of Z_j at x ($R = \mathcal{O}_{Z_j,x}$), $\mathcal{G}_1(x) = \mathcal{G}(x)$ and $\mathcal{E}_0(x) = E_j(x)$. (*Attention:* In [BM3] $\mathcal{E}_0(x) = \emptyset$, but it seems to be more convenient to put $\mathcal{E}_0(x) = E_j(x)$, because $\mathcal{E}_1(x) \supseteq \mathcal{E}_2(x) \supseteq \ldots$).

So let us start with the idealistic exponent with history

$$(\mathcal{G}_1(x), \mathcal{E}_0(x)) = (\mathcal{G}(x), E_j(x)) \quad \text{on } N_0(x) \text{ (resp. on } R)\,.$$

We determine $E^1(x)$ and $\mathcal{E}_1(x)$ as described before and set

$$\mathcal{F}_1(x) := \mathcal{G}_1(x) \cap \big(E^1(x), 1\big)$$

where $(E^1(x), 1) = \bigcap_{H \in E^1(x)} (x_H, 1)$ and x_H denotes a local generator of H. Thus we get the idealistic exponent with history

$$(\mathcal{F}_1(x), \mathcal{E}_1(x)) \quad \text{on } R.$$

Note that also the exceptional data has changed. Not only that there are maybe less components, but also the assigned numbers may differ from those of the previous exceptional data. (For example, if $E^1(x) \neq \emptyset$, then all the assigned numbers in $\mathcal{E}_1(x)$ are zero, because $E(x)$ defines a simple normal crossing divisor).

Using the method of Construction 3.2.1, we choose the maximal contact hypersurface $V(y_1)$ (without loss of generality let y_1 be as in Setup B). Let

$$\mathcal{H}_1(x) = \mathbb{D}_x(\mathcal{F}_1(x);\, u_1, \ldots, u_e, y_2, \ldots, y_r;\, y_1)$$

be the idealistic coefficient exponent of $\mathcal{F}_1(x)$ with respect to (y_1).

If $\mathcal{F}_1(x) = (f_1, b_1) \cap \ldots \cap (f_q, b_q)$ ($q \in \mathbb{Z}_+$, $q \geq m$ and $(f_1, \ldots f_m)$ as in Setup B), then

$$\mathcal{H}_1(x) = \bigcap_{i=1}^{q} \bigcap_{l:=l(i)=0}^{b_i-1} \left(\frac{\partial^l f_i}{\partial y_1^l}\Big|_{V(y_1)},\, b_i - l \right).$$

We set $N_1(x) := V(y_1)$ and get the idealistic exponent with history

$$(\mathcal{H}_1(x), \mathcal{E}_1(x)) \quad \text{on } V(y_1).$$

Again the exceptional data has changed, because we have to consider here $\mathcal{E}_1(x)$ as data on $N_1(x) = V(y_1)$.

We put $h_{i,l} := \frac{\partial^l f_i}{\partial y_1^l}\Big|_{V(y_1)}$ for $1 \le i \le q$ and $0 \le l \le b_i - 1$. Then we define (always $i \in \{1, \dots, q\}$ and $l := l(i) \in \{0, \dots, b_i - 1\}$)

$$\left.\begin{aligned} &\mu_2(x) := \min\left\{ \frac{\mathrm{ord}_x(h_{i,l})}{b_i - 1} \;\Big|\; i,\, l \right\}, \\ &\mu_{2,H}(x) := \min\left\{ \frac{\mathrm{ord}_{H,x}(h_{i,l})}{b_i - l} \;\Big|\; i,\, l \right\}, \quad \text{for } H \in \mathcal{E}_1(x), \\ &\nu_2(x) := \mu_2(x) \;-\; \sum_{H \in \mathcal{E}_1(x)} \mu_{2,H}(x), \end{aligned}\right\} \tag{3.5}$$

where $\mathrm{ord}_{H,x}(h_{i,l})$ denotes the multiplicity of $h_{i,l}$ along H, i.e. if g_H is a local generator of $H \in \mathcal{E}_1(x)$, then

$$\mathrm{ord}_{H,x}(h_{i,l}) = \max\left\{ k \in \mathbb{Z}_0 \cup \{\infty\} \;\middle|\; g_H^k \text{ divides } h_{i,l} \right\}.$$

Clearly, $\mu_{2,H}(x)$ coincides with the assigned number of H in the exceptional data of $(\mathcal{H}_1(x), \mathcal{E}_1(x))$. Further we have

$$\Delta_x^N(\mathcal{H}_1(x), u_1, \dots, u_e, y_2, \dots, y_e) = \Delta_x(\, \mathcal{F}_1(x);\, u_1, \dots, u_e, y_2, \dots, y_e;\, y_1\,)$$

and $\mu_2(x) = \delta(\Delta_x(\, \mathcal{F}_1(x);\, u_1, \dots, u_e, y_2, \dots, y_e;\, y_1\,))$.

If $\nu_2(x) \in \{0, \infty\}$, then the process ends and the invariant is defined as

$$\mathrm{inv}_X(x) := \mathrm{inv}_{1\frac{1}{2}}(x) = (\nu_1, s_1;\, \nu_2).$$

Suppose $0 < \nu_2(x) < 1$. We consider

$$D_2(x) := \prod_{H \in \mathcal{E}_1(x)} g_H^{\mu_{2,H}(x)},$$

where g_H denotes a local generator of $H \in \mathcal{E}_1(x)$. (We allow here fractional exponents; see also the remark below). Then by definition of the terms $\mu_{2,H}(x)$, each $h_{i,l}$, $1 \le i \le q$ and $0 \le l \le b_i - 1$, can be written as

$$h_{i,l} = D_2^{b_i - l} \cdot g_{i,l}$$

for some element $g_{i,l}$. (Recall that $b_i - l = b_{h_{i,l}}$ is the number assigned to $h_{i,l}$ in $\mathcal{H}_1(x)$). We define the new idealistic exponent

$$\mathcal{G}_2(x) := \left(\bigcap_{i=1}^{q} \bigcap_{l=l(i)=0}^{b_i - 1} (\, g_{i,l},\, (b_i - l) \cdot \nu_2\,) \right) \cap (D_2(x), 1 - \nu_2) \quad \text{on } V(y_1). \tag{3.6}$$

This is the idealistic version of the so called companion ideal, thus we call it the idealistic companion exponent. Clearly, the exceptional data has changed again.

If $1 \leq \nu_2(x) < \infty$, then the assigned number of the $D_2(x)$-component is not positive and hence can be omitted, $\mathcal{G}_2(x) := \bigcap_{i=1}^{q} \bigcap_{l=0}^{b_i-1} (\, g_{i,l},\ (b_i - l) \cdot \nu_2\,)$.

Together we get for $0 < \nu_2(x) < \infty$ the idealistic exponent with history

$$(\,\mathcal{G}_2(x),\ \mathcal{E}_1(x)\,) \quad \text{on } N_1(x) = V(y_1).$$

This completes the first step in the general procedure. Then we start again with $(\,\mathcal{G}_2(x),\ \mathcal{E}_1(x)\,)$ instead of $(\,\mathcal{G}_1(x),\ \mathcal{E}_0(x)\,)$.

Lemma 3.3.3. *Let $\mathcal{G}_1(x)$ and $\mathcal{G}_1'(x)$ be two equivalent idealistic exponent with history. Then*

$$\mathcal{F}_1(x) \sim_{\mathcal{E}(x)} \mathcal{F}_1'(x), \qquad \mathcal{H}_1(x) \sim_{\mathcal{E}(x)} \mathcal{H}_1'(x) \qquad \text{and} \qquad \mathcal{G}_2(x) \sim_{\mathcal{E}(x)} \mathcal{G}_2'(x),$$

where we have to consider the induced exceptional data.

Proof. The first (resp. the second) equivalence follows by Lemma 2.6.7*(vi)(a)* (resp. *(vi)(e)*).

The equivalence $\mathcal{G}_2(x) \sim_{\mathcal{E}(x)} \mathcal{G}_2'(x)$ is clear for the cases

- ⋄ $\mathcal{H}_1(x) = (J, b)$ and $\mathcal{H}_1'(x) = (J^a, ab)$ for some $a \in \mathbb{Z}_+$.
- ⋄ $\mathcal{H}_1(x) = (J_1, b) \cap (J_2, b)$ and $\mathcal{H}_1'(x) = (J_1 + J_2, b)$.

Thus we may assume $\mathcal{H}_1(x) = (J, b)$ and $\mathcal{H}_1'(x) = (J', b)$ (with the same assigned number $b \in \mathbb{Z}_+$). For an element $h \in J$ we have defined $g = g(h)$ via $h = D_2^b g$. Set $I := \langle\, g(h) \mid h \in J\,\rangle$, then $J = D_2^b \cdot I$. (Here we identify D_2^b with the ideal which it generates in R). Clearly, $\mathcal{G}_2(x) = (I, \nu_2 b) \cap (D_2, 1 - \nu_2)$. We can do the same for $\mathcal{H}_1'(x)$ and obtain the ideal I' with the analogous property.

If we can show $(I, \nu_2 b) \sim_{\mathcal{E}(x)} (I', \nu_2 b)$ (as idealistic exponents with history on R), then the assertion follows. Since we have factored D_2, the assigned numbers in the induced exceptional data are all zero. Thus we only have to prove

$$(I, \nu_2 b) \sim (I', \nu_2 b).$$

An extension of the regular system of parameters by further independent elements does not change the situation. Hence we may assume that the extension is trivial. Further we have for any point $x_0 \in \mathrm{Spec}\,(R)$

$$\mathrm{ord}_{x_0}(I) = \mathrm{ord}_{x_0}(J) - \mathrm{ord}_{x_0}(D^b) = \mathrm{ord}_{x_0}(J') - \mathrm{ord}_{x_0}(D^b) = \mathrm{ord}_{x_0}(I').$$

For the first (resp. third) equality we use $J = D_2^b \cdot I$ (resp. $J' = D_2^b \cdot I'$) and the second follows by $(J,b) \sim (J',b)$. Therefore $\mathrm{Sing}\,(I,b) = \mathrm{Sing}\,(I',b)$. After a permissible blow up $\pi : \widetilde{Z} \to \mathrm{Spec}\,(R)$ the transform $(\widetilde{I}, \nu_2 b)$ of $(I, \nu_2 b)$ is determined by $I\mathcal{O}_{\widetilde{Z}} = H^{\nu_2 b}\widetilde{I}$, where H denotes the ideal sheaf of the exceptional divisor. For the transform of J we have $J\mathcal{O}_{\widetilde{Z}} = H^b \widetilde{J} = H^{(1-\nu_2)b}\widetilde{D_2^b} \cdot H^{\nu_2 b}\widetilde{I}$. ($\widetilde{D_2^b}$ denotes the transform of D_2^b). Thus the situation is the same as before the blow up, $\widetilde{J} = \widetilde{D_2^b}\widetilde{I}$ and this is also true for J' and I'. Together we get the desired equivalence $(I, \nu_2 b) \sim (I', \nu_2 b)$. □

Main Theorem 1 and the second part of Main Theorem 5 boil down to

Proposition 3.3.4. *Let $r \in \mathbb{Z}_+$, $r \geq 1$. Let $\mathcal{E}_r(x) = \{\,(H_1, d_1), \ldots, (H_j, d_j)\,\}$ be the exceptional data of the idealistic exponent with history $(\mathcal{H}_r(x), \mathcal{E}_r(x))$ on $V(y_1, \ldots, y_r)$. Let $(u) = (u_1, \ldots, u_e)$ be the remaining part of the regular system of parameters for the local ring R of Z at x. Then*

$$\mu_{r+1}(x) = \delta(\, \Delta_x(\, \mathcal{F}_r(x); u; y_r\,)\,) =: \delta_{r+1}$$

and $\nu_{r+1}(x) = \delta_{r+1} - \sum_{i=1}^{j} d_i$. Hence $\nu_{r+1}(x)$ coincides with the ν-invariant $\nu_x(\, \mathcal{F}_r(x), \mathcal{E}_r(x)_{\mathcal{H}}; u\,)$ of the idealistic exponent with history $(\mathcal{F}_r(x), \mathcal{E}_r(x)_{\mathcal{H}})$, where the index $\mathcal{H}$ should indicate that the exceptional data is the one of $\mathcal{H}_r(x)$.

Proof. This follows by the definition of $\mu_{r+1}(x)$, $\mu_{r+1,H}(x)$ and ν_{r+1} (for $H \in \mathcal{E}_r(x)$) (see Construction 3.3.2). □

Thus the invariant $\nu_{r+1}(x)$ can be achieved by purely considering polyhedra. By Proposition 2.6.5 the ν-invariant is independent of the choice of a representative as idealistic exponent with history and also of the choice of (y) (for fixed (u)), thus the same is true for $\nu_{r+1}(x)$. Moreover, equivalent idealistic exponents with history determine the same invariant $\mathrm{inv}_X(x)$ by Lemma 3.3.3.

All this seems now to be obvious. But keep in mind that before coming to this point we had to work hard in order to develop the theory of idealistic exponents with history and to prove important results for them.

Remark 3.3.5. *(1) As we have already mentioned $E^1(x) \subseteq E_j(x)$ is a set of old exceptional components. We lost some information on them. Of course, $E_j(x)$ defines a normal crossing divisor, but it is not clear why $E^1(x)$ should have only normal crossings with an arbitrary hypersurface of maximal contact $V(y_1)$. Hence we have to add them to $\mathcal{G}_1(x)$.*
The young exceptional components in $\mathcal{E}_1(x)$ have only normal crossings with $V(y_1)$ (see [BM3], Lemma 4.15, p.245).

(2) *By adding* $(E^1(x), 1)$ *to* $\mathcal{G}_1(x)$ *we change the ideal of the tangent cone. Namely, the initial forms of the local generators of these exceptional components appear in the defining ideal. So the ideal of the tangent cone becomes possibly bigger and the associated scheme becomes smaller. Thus the same is true for the ridge and the directrix.*

(3) *Since* $(f, b) \sim_{\mathcal{E}(x)} (f^d, b\,d)$ *for every* $d \in \mathbb{Z}_+$ *(Lemma 2.6.7(i)), it is possible to introduce an equivalent idealistic exponent with history for* $(D_2(x), 1-\nu_2)$ *which has only integral exponents; choose for example* $d = \prod_{i=1}^{q} \prod_{l=0}^{b_i-1} b_i - l$. *Further* $D_2(x)$ *is the greatest common divisor of* $(h_{i,l})_{i,l}$ *with respect to their assigend numbers* $b_i - l$. *If we change the representatives as idealistic exponents with history such that every assigned number is* d *(for example take* $d = \prod_{i=1}^{q} \prod_{l=0}^{b_i-1} b_i - l$*), then every modified* $h_{i,l}$, *say* $\widetilde{h}_{i,l}$, *can be written in the form* $\widetilde{h}_{i,l} = D_2^d \cdot \widetilde{g}_{i,l}$.

(4) *We have* $\mu_2(x) \geq 1$. *If* $\mu_2(x) = 1$, *then* $1 - \nu_2 = 1 - (\mu_2 - \sum_{H \in \mathcal{E}_1(x)} \mu_{2,H}) = \sum_{H \in \mathcal{E}_1(x)} \mu_{2,H}$ *and*

$$(D_2(x), 1-\nu_2) = (D_2(x), \sum_{H \in \mathcal{E}_1(x)} \mu_{2,H}) \sim_{\mathcal{E}(x)} \bigcap_{\substack{H \in \mathcal{E}_1(x) \\ \mu_{2,H}(x) \neq 0}} (g_H, 1),$$

where g_H *denotes a local generator of* H. *Hence, as described before, the tangent cone and the ridge may become smaller. In the case* $\mu_2(x) > 1$ *the adding of* $(D_2(x), 1-\nu_2)$ *has at first no effect on the tangent cone or the ridge. But still it may affect the further procedure.*

(5) *If* $E_j(x) = \emptyset$, *then we have* $s_i(x) = 0$ *for all* i *and the above procedure coincides with the year zero case. In particular, if* $\mathcal{E}_k(x) = \emptyset$ *for some* k, *then the remaining process coincides with the case described in section 3.2.*

(6) *In [BM3], Remark 9.15, p.282, they slightly modify the construction of the invariant* $\mathrm{inv}_X(x)$ *if* $(\mathcal{F}_1(x), \mathcal{E}_1(x))$ *can be embedded in a lower dimension ambient scheme, say for example into* $V(z_1, \ldots, z_a)$ *instead of* $N_0(x) = \mathrm{Spec}\,(R)$. *In this case* $\mathrm{inv}_{r+1}(x) := (\nu_1, 0;\ 1, 0;\ \ldots;\ 1, 0)$. *After this shift, they consider* $(\mathcal{F}_1(x), \mathcal{E}_1(x))$ *as an idealistic exponent with history on* $V(z_1, \ldots, z_a)$ *(with the induced exceptional data) and continue as usual. This does not affect our considerations seriously.*

(7) *In the construction we are not forced to start with* $E_0 = \emptyset$ *(see (3.4)). We could also require that there is additionally to* X *a simple normal crossing divisor* E_0 *on* Z_0 *given. This is important for possible applications.*

Remark 3.3.6. *Recall that by construction we have*

$$\mathcal{H}_1(x) = \bigcap_{i=1}^{q} \bigcap_{l=l(i)=0}^{b_i-1} (\, h_{i,l},\, b_i - l\,)$$

and $\mu_2(x) \geq 1$*. If* $0 < \nu_2(x) < 1$*, then the transformation law (under permissible blow ups) of the idealistic exponent* $\bigcap_{i=1}^{q} \bigcap_{l=0}^{b_i-1}(\, g_{i,l},\, (b_i - l)\cdot \nu_2\,)$ *is not consistent with that of* $\bigcap_{i=1}^{q} \bigcap_{l=0}^{b_i-1}(\, h_{i,l},\, (b_i-l)\cdot\nu_2\,)$*. Therefore we have to add* $(D_2(x), 1-\nu_2)$*. (Recall that* $D_2 := D_2(x) = \prod_{H\in\mathcal{E}_1(x)} g_H^{\mu_{2,H}(x)}$ *is the greatest common divisor of the* $(h_{i,l}, b_i - l)_{i,l}$*, which is a monomial in the new exceptional components* $\mathcal{E}_1(x)$*). More precisely,* $h_{i,l} = D_2^{b_i-l} \cdot g_{i,l}$ *for every* i, l *and we defined*

$$\mathcal{G}_2(x) = \left(\bigcap_{i=1}^{q} \bigcap_{l=l(i)=0}^{b_i-1} (\, g_{i,l},\, (b_i - l)\cdot \nu_2\,) \right) \cap (D_2(x), 1-\nu_2).$$

In the last part we have $(D_2, 1-\nu_2) \sim_{\mathcal{E}(x)} (D_2^d, (1-\nu_2)\, d)$ *for all* $d \in \mathbb{R}_+$*. So*

$$\mathcal{G}_2(x) \sim_{\mathcal{E}(x)} \bigcap_{i=1}^{q} \bigcap_{l=l(i)=0}^{b_i-1} \Big((\, g_{i,l},\, (b_i - l)\cdot \nu_2\,) \cap (\, D_2^{b_i-l}, (1-\nu_2)(b_i-l)\,) \Big).$$

Suppose $0 < \nu < 1$ *(*$\nu := \nu_2$*). If a blow up is permissible for the idealistic exponent* $(g, d\cdot\nu) := (\, g_{i,l},\, (b_i - l)\cdot\nu_2\,)$*, then we know* $\mathrm{ord}_x(g) \geq d\nu$*. But* $0 < \nu < 1$ *implies* $d\nu < d$ *and hence* $\mathrm{ord}_x(g) < d$ *might be possible! This means a center which is permissible for* $(g, d\cdot\nu)$ *is not necessarily permissible for* $(h, d) := (h_{i,l}, b_i - l)$ *(*$h = D_2^d \cdot g$*).*

Further the transform of (h, d) *after a permissible blow up is locally given by* $(z_{exc}^{-d} \cdot \widetilde{h}, d)$*, where* z_{exc} *denotes a local generator of the exceptional component and* $\widetilde{h}$ *denotes the total transform. By using* $h = D_2^d \cdot g$ *we get*

$$\left(z_{exc}^{-d} \cdot \widetilde{h} = \left(z_{exc}^{-(1-\nu)d} \cdot \widetilde{D_2}^{d} \right) \cdot \left(z_{exc}^{-\nu d} \cdot \widetilde{g} \right),\ d \right),$$

where $\widetilde{D_2}$ *denotes the total transform of* D_2 *under the blow up. If we consider only* $(g, d\nu)$*, then the transformations are not consistent.*
In the case $\nu \geq 1$ *we get* $d\nu \geq d$ *and the transform of* h *is determined by the terms* $z_{exc}^{-d} \cdot g = z_{exc}^{(\nu-1)d} \cdot z_{exc}^{-d\nu} \cdot g$*, where* $(\nu-1)d \geq 0$*; thus we have not to add* D_2*.*

Remark 3.3.7. *If* $\nu_t \in \{0, \infty\}$ *for some* $t \in \mathbb{Z}_+$*, then*

$$\mathrm{inv}_X(x) = \mathrm{inv}_{t-\frac{1}{2}}(x) = (\nu_1, s_1;\ \ldots;\ \nu_{t-1}, s_{t-1};\ \nu_t).$$

In the case $\nu_t(x) = \infty$ *the center of the upcoming blow up is*

$$N_{t-1}(x) = V(y_1, \ldots, y_{t-1}).$$

In every chart the invariant decreases, because all elements of $(y_1, \ldots, y_{t-1})$ *are coming from certain initial forms.*

If $\nu_t(x) = 0$*, then we set* $\mathcal{G}_t(x) = (D_t(x), 1)$*. (This fits into the definition of these terms; the assigned numbers of the first part in (3.6) are* 0*, because* $\nu_t(x) = 0$*, hence we can ignore it and get* $\mathcal{G}_t(x) = (D_t(x), 1)$*).*

This is the monomial case. For completeness let us recall how the center is chosen in this case (see [BM4], Remark 3.6 (p.66/67)). $D_t(x)$ *is a monomial with rational exponents in the young exceptional components* $H \in \mathcal{E}_{t-1}(x)$*. Since these components have simultaneously normal crossings with* $N_{t-1}(x)$*, we can choose the coordinates* $(u) = (u_1, \ldots, u_{n-t+1})$ *of* $N_{t-1}(x)$ *such that for each* $H \in \mathcal{E}_1(x)$ *the local generator* $g_H = u_i$ *for some* $i \in \{1, \ldots, n-t+1\}$*. Let*

$$S_{\mathrm{inv}_X}(x) := \text{ germ at } x \text{ of } \left\{ y \in Z_j \;\middle|\; \mathrm{inv}_X(y) \geq \mathrm{inv}_X(x) \right\},$$

then every component M *of* $S_{\mathrm{inv}_X}(x)$ *equals*

$$M = S_{\mathrm{inv}_X}(x) \cap \bigcap \{\, H \in E_j(x) \mid M \subseteq H \,\}$$

and we write $M = M_I$ *for* $I = \{H \in E_j(x) \mid M \subseteq H\}$*. In order to get a canonical resolution, Bierstone and Milman extend their invariant to*

$$\mathrm{inv}^e_X(x) := (\mathrm{inv}_X(x), J(x)).$$

For the definition we have to introduce a total ordering on the set

$$W := \{\, I \mid I \subseteq E_j(x) \,\}:$$

Let $E_j(x) = \{H_1^j, \ldots, H_j^j\}$*, where* H_i^j *is the strict transform in* Z_j *of the exceptional component which occurred in the year* i*, i.e. if* $\pi_i : Z_i \to Z_{i-1}$ *denotes the blow up with center* C_{i-1}*, then* $H_i^i = \pi_i^{-1}(C_{i-1}) \subseteq Z_i$*, for all* $i \in \{1, \ldots, j\}$*. One possible total ordering on* W *is given by the lexicographical ordering of* $\mathbb{Z}_0^j$ *via the mapping* $W \to \mathbb{Z}_0^j$*,* $I \mapsto (\rho_1, \ldots, \rho_j)$ *with*

$$\rho_i := \rho_i(I) := \begin{cases} 0 & \text{if } H_i^j \notin I, \\ 1 & \text{if } H_i^j \in I. \end{cases}$$

Then we set

$$J(x) := \max \{\, I \in W \mid M_I \text{ is a component of } S_{\mathrm{inv}_X}(x) \,\}.$$

3.4 Behavior of the polyhedra in the construction of the invariant

In this section we observe what in each step of the general process by Bierstone and Milman happens to our polyhedra. Let $r \in \mathbb{Z}_+$ with $0 < r \leq n$ and let $e = n - r$.

From $\mathcal{G}_r(x)$ to $\mathcal{F}_r(x) = \mathcal{G}_r(x) \cap (E^r(x), 1)$: In this step we add

$$\big(E^1(x), 1\big) = \bigcap_{H \in E^1(x)} (x_H, 1),$$

where x_H denotes a local generator of H. Recall that $s_r = s_r(x) = \# E^r(x)$. By construction $E^r(x) \subseteq \mathcal{E}_{r-1}(x)$ has only normal crossings with $N_{r-1}(x)$. Thus we can choose the regular system of parameters $(u) = (u_1, \ldots, u_{e+1})$ for $N_{r-1}(x)$ such that for all $H \in E^r(x)$ the local generator is $g_H = u_k$ for $k \in I_r := \{k_1, \ldots, k_{s_r}\} \subseteq \{1, \ldots, e+1\}$. (In fact we can choose the regular system of parameters such that the analogous condition holds for every $H \in \mathcal{E}_{r-1}(x)$).

Adding the old exceptional components $(E^r(x), 1)$ corresponds to adding points to the generators of the polyhedron $\Delta_x^N(\mathcal{G}_r(x), u)$. More precisely, the new points are

$$\big\{(\delta_{\alpha k})_{\alpha \in \{1, \ldots, e+1\}} = (0, \ldots, 0, \underset{\substack{\uparrow \\ k}}{1}, 0, \ldots, 0) \;\big|\; k \in I_r = \{k_1, \ldots, k_{s_r}\}\big\},$$

where $\delta_{\alpha k}$ denotes the usual Kronecker delta. Obviously, the number of new points for the polyhedron is s_r.

The idealistic exponents $\bigcap_{H \in E^r(x)} (x_H, 1)$ and

$$\left(\prod_{H \in E^r(x)} x_H, \sum_{H \in E^r(x)} 1 = s_r \right)$$

are equivalent. Thus we can also consider the latter one. But then we add only one point to the generators of the polyhedron, namely the one given by

$$\left(\sum_{k \in I_r} \delta_{\alpha k} \cdot \frac{1}{s_r} \right)_{\alpha \in \{1, \ldots, e+1\}} = \left(0, \ldots, 0, \underset{\substack{\uparrow \\ k_1}}{\frac{1}{s_r}}, 0, \ldots, 0, \underset{\substack{\uparrow \\ k_2}}{\frac{1}{s_r}}, 0, \ldots, 0, \underset{\substack{\uparrow \\ k_{s_r}}}{\frac{1}{s_r}}, 0, \ldots, 0 \right)$$

(without loss of generality we may assume $1 \leq k_1 < k_2 < \ldots < k_{s_r} \leq e + 1$).

In both cases the same variables are involved. Hence the ideal of the tangent cone (the directrix and the ridge) behaves in both bases the same way. This change is described by the initial forms of $(E^r(x), 1)$.

Further observe: $V(u_{k_1}, \ldots, u_{k_{s_r}})$ has maximal contact with $\mathcal{F}_r(x)$ at x. Clearly, the idealistic coefficient exponents with respect to $(u_{k_1}, \ldots, u_{k_{s_r}})$ coincide in both cases. Thus the projections of the polyhedra with respect to $(u_{k_1}, \ldots, u_{k_{s_r}})$ coincide, too.

From $\mathcal{F}_r(x)$ to $\mathcal{H}_r(x)$: Suppose $\mathcal{F}_r(x) = (f_1, b_1) \cap \ldots \cap (f_q, b_q)$, then there is at least one $i \in \{1, \ldots, q\}$ such that $b_i = \mathrm{ord}_x(f_i)$. We assume without loss of generality that $y_r := u_{e+1}$ has maximal contact with $\mathcal{F}_r(a)$ at x. Hence in this step we project the polyhedron $\Delta_x^N(\mathcal{F}_r(x); u_1, \ldots, u_e, u_{e+1}) \subset \mathbb{R}_0^{e+1}$ from the point $(0, \ldots, 0, 1) \in \mathbb{Z}_0^{e+1}$ to $\mathbb{R}_0^e$. The resulting polyhedron is

$$\Delta_x(\mathcal{F}_r(x); u_1, \ldots, u_e; y_r) = \Delta_x^N(\mathcal{H}_r(x); u_1, \ldots, u_e) \subset \mathbb{R}_0^e.$$

From $\mathcal{H}_r(x)$ to $\mathcal{G}_{r+1}(x)$: Suppose $\mathcal{H}_r(x) = (h_1, b_1) \cap \ldots \cap (h_p, b_p)$. The last step is rather consisting of three smaller steps. We determine $D_{r+1}(x)$ and write each (h_i, b_i) as $h_i = D_{r+1}^{b_i} \cdot g_i$. We set

$$\widetilde{\mathcal{H}_r}(x) := \bigcap_{i=1}^{p} (\, g_i,\, b_i\,) \qquad \text{and} \qquad \widetilde{\mathcal{G}_{r+1}}(x) := \bigcap_{i=1}^{p} (\, g_i,\, b_i\, \nu_{r+1}\,).$$

Further recall that $\mathcal{G}_{r+1}(x) = \widetilde{\mathcal{G}_{r+1}}(x) \cap (\, D_{r+1}(a), 1 - \nu_{r+1}\,)$. Then the smaller steps are the following

(1) ***From $\mathcal{H}_r(x)$ to $\widetilde{\mathcal{H}_r}(x)$:*** Since $N_r(x)$ and $\mathcal{E}_r(x)$ have simultaneously only normal crossings, we can choose the coordinates $(u_1, \ldots, u_e)$ of $N_r(x)$ such that for all $H \in \mathcal{E}_r(x)$ the local generator is $g_H = u_l$ for some $l \in \{1, \ldots, e\}$. In this situation we set $\mu_{r+1,l} := \mu_{r+1,H}(x)$. Put

$$I_r := \{\, l_1,\, \ldots,\, l_{m_r}\,\} := \{\, l \in \{1, \ldots, e\} \;\mid\; \mu_{r+1,l} \neq 0\,\} \subseteq \{1, \ldots, e\}.$$

Further we denote by $T_r : \mathbb{R}^e \to \mathbb{R}^e$ the translation in the negative direction by the vector

$$w^{(r)} := \Big(0,\, \ldots,\, 0,\, \underset{\substack{\uparrow \\ l_1}}{\mu_{r+1,1}},\, 0,\, \ldots,\, 0,\, \underset{\substack{\uparrow \\ l_2}}{\mu_{r+1,2}},\, 0,\, \ldots,\, 0,\, \underset{\substack{\uparrow \\ l_{m_r}}}{\mu_{r+1,m_r}},\, 0,\, \ldots,\, 0\Big).$$

This means a point $v \in \mathbb{R}^e$ is sent to $T_r(v) = v - w^{(r)}$. Then we have for the Newton polyhedra

$$T_r \left(\Delta_x^N (\mathcal{H}_r(x), u) \right) = \Delta_x^N (\widetilde{\mathcal{H}_r}(x), u) \subseteq \mathbb{R}_0^e.$$

(2) ***From $\widetilde{\mathcal{H}_r}(x)$ to $\widetilde{\mathcal{G}_{r+1}}(x)$:*** In this step we multiply each point of the polyhedron $\Delta_x^N (\widetilde{\mathcal{H}_r}(x), u)$ by the factor $\dfrac{1}{\nu_{r+1}}$ and get $\Delta_x^N (\widetilde{\mathcal{G}_{r+1}}(x), u)$.

This corresponds to the change of the valuation on the regular system of parameters (u) for the regular local ring $K[[u]]$ corresponding to $V(y)$ (resp. of the regular system of parameters (u, y) for R), see Remark 3.2.6.

(3) ***From $\widetilde{\mathcal{G}_{r+1}}(x)$ to $\mathcal{G}_{r+1}(x)$:*** The last step is similar to "*From $\mathcal{G}_r(x)$ to $\mathcal{F}_r(x)$*". By definition $\mathcal{G}_{r+1}(x) = \widetilde{\mathcal{G}_{r+1}}(x) \cap \{(D_{r+1}(x), 1 - \nu_{r+1})\}$. Thus we add to the generators of $\Delta_x^N (\widetilde{\mathcal{G}_{r+1}}(x), u)$ the points associated to

$$\left(D_{r+1}(x) = \prod_{H \in \mathcal{E}_r(x)} g_H^{\mu_{r+1,H}(x)},\ 1 - \nu_{r+1} \right),$$

where g_H is a local generator of $H \in \mathcal{E}_r(x)$. (Recall that we defined $\nu_{r+1}(x) = \mu_{r+1}(x) - \sum_{H \in \mathcal{E}_r(x)} \mu_{r+1,H}(x)$). As in *(1)* we choose $(u) = (u_1, \ldots, u_e)$ such that for all $H \in \mathcal{E}_r(x)$ the local generator is $g_H = u_l$ for some $l \in \{1, \ldots, e\}$. Again we set $\mu_{r+1,l} := \mu_{r+1,H}(x)$ and

$$I_r := \{ l_1, \ldots, l_{m_r} \} := \{ l \in \{1, \ldots, e\} \mid \mu_{r+1,l} \neq 0 \} \subseteq \{1, \ldots, e\}.$$

Then $(D_{r+1}(x), 1 - \nu_{r+1})$ yields in $\Delta_x^N (\mathcal{G}_{r+1}(x), u)$ the point

$$\Big(0, \ldots, 0, \underset{\substack{\uparrow \\ l_1}}{\frac{\mu_{r+1,1}}{1 - \nu_{r+1}}}, 0, \ldots, 0, \underset{\substack{\uparrow \\ l_2}}{\frac{\mu_{r+1,2}}{1 - \nu_{r+1}}}, 0, \ldots, 0, \underset{\substack{\uparrow \\ l_{m_r}}}{\frac{\mu_{r+1,m_r}}{1 - \nu_{r+1}}}, 0, \ldots, 0 \Big)$$

As we already have mentioned we get in the case $\mu_2(x) = 1$ that $1 - \nu_2 = \sum_{H \in \mathcal{E}_1(x)} \mu_{2,H}$ and $(D_2(x), 1 - \nu_2) \sim_{\mathcal{E}(x)} \bigcap_{i=1}^{m_r} (u_{l_i}, 1)$. Similarly as before we get instead of just one point with this description m_r points. These points are everywhere 0 except at the l_j-th place, where the entry is 1, $j \in \{1, \ldots, m_r\}$ (see also "*From $\mathcal{G}_r(x)$ to $\mathcal{F}_r(x)$*").

Remark 3.4.1. *By definition $\delta(\Delta_x^N (\widetilde{\mathcal{H}_r}(x), u)) = \mu_{r+1}(x)$. By going from $\mathcal{H}_r(x)$ to $\widetilde{\mathcal{H}_r}(x)$ we send the assigned numbers in the exceptional data to zero. Therefore $\delta(\Delta_x^N (\widetilde{\mathcal{H}_r}(x), u)) = \Delta_x(\mathcal{F}_r(x); u; y_r) = \nu_{r+1}(x)$.*

3.5 Possible simplifications for the construction

The construction of the invariant of Bierstone and Milman is quite complicated. Therefore it is hard to formulate a step-by-step result on the behavior of the generators of the ideal J as we did in Proposition 3.2.4 and Remark 3.2.6. But we show now that in certain good situations the procedure becomes easier. In particular, we can sometimes make bigger steps.

For this we introduce the following

Notation 3.5.1. Fix $r \in \mathbb{Z}_+$. Let $\mathcal{I}_r \in \{\mathcal{G}_r(x), \mathcal{F}_r(x), \mathcal{H}_{r+1}(x)\}$ and $s \in \mathbb{Z}_+$, $s < r$.

(1) We define the $\mathcal{G}_s$-part of $\mathcal{I}_r$ to be the part of $\mathcal{I}_r$ which is by the construction coming from $\mathcal{G}_s(x)$.

(2) By the $E^{(s)}$-part (resp. $D^{(s)}$-part) of $\mathcal{I}_r$ we denote the part which occurred by adding $E^s(x), \dots, E^r(x)$ (resp. $D_{s+1}(x), \dots, D_r(x)$).

If $s = 1$, then we speak also of the $\mathcal{G}$-part (resp. E-part, resp. D-part) of $\mathcal{I}_r$ instead of the $\mathcal{G}_1$-part (resp. $E^{(1)}$-part, resp. $D^{(1)}$-part) of $\mathcal{I}_r$.

(For $\mathcal{I}_r = \mathcal{G}_r(x)$ we neglect $E^r(x)$ in the definition of the $E^{(s)}$-part, because it has not been added yet.)

Observation 3.5.2 (Big steps with the old exceptional part $(E^q(x), 1)$). In the definition of $\mathcal{F}_1(x)$ we add the old exceptional components $(E^1(x), 1)$ to $\mathcal{G}_1(x)$. This enables us to make sometimes more than one step in Construction 3.3.2: First, this may change the separation of the regular system of parameters into (u, y) as in Setup B. Thus let us consider an arbitrary regular system of parameters $(t) = (t_1, \dots, t_n)$ for R. Further $E^1(x)$ is a normal crossing divisor on $N_0(x) = Z_j$. Hence we can choose the regular system of parameters $(t) = (t_1, \dots, t_n)$ for R such that every $H \in E^1(x)$ is locally given by some $t_l = 0$ for $l \in \{1, \dots, n\}$, say $E^1(x)$ is given by $(t_{l_1}, \dots, t_{l_{s_1}})$. Suppose $s_1 = \#E^1(x) \geq 1$. Set $(z) = (z_1, \dots, z_{s_1}) = (t_{l_1}, \dots, t_{l_{s_1}})$. Then $V(z)$ has maximal contact with $\mathcal{F}_1(x)$ at x. (Recall that locally x is given by the maximal ideal of R). So this is a possible choice for the first s_1 steps in definition of $\nu_i(x)$. (After that we consider $\mathcal{H}_{s_1}(x)$ which determines ν_{s_1+1}). Since $\mathcal{E}_1(x)$ and $E^1(x)$ have simultaneously only normal crossings, we can require the additional property on (t) that $\mathcal{E}_1(x)$ is given by $(t_{m_1}, \dots, t_{m_p})$, where $t_\iota \neq t_\rho$ for $\iota \in \{m_1, \dots, m_p\}$ and $\rho \in \{l_1, \dots, l_{s_1}\}$. Thus we get for every $i \in \{2, \dots, s_1\}$ (If $s_1 = 1$ then the previous set is empty):

(1) $\mu_{i,H}(x) = 0$ for every $H \in \mathcal{E}_i(x)$, thus $D_i(x) = 1$ and

(2) $\nu_i(x) = \mu_i(x) = 1$.

The first assertion holds, because we can not factor t_ι from t_ρ (ι and ρ as above). The second part follows from the condition that $V(z)$ has maximal contact with $\mathcal{F}_1(x)$ at x.

Therefore we already know ν_i up to the step $i = s_1$. Set $d := s_1$. In the procedure we also added $E^2(x), \ldots, E^d(x) \subset \mathcal{E}_1(x)$. Further $s_q = \#E^q(x)$ for $q \in \{1, \ldots, d\}$ and $\mathcal{E}_d(x) = E_j(x) \setminus \bigcup_{l=1}^{d} E^q(x)$. If the condition

$$s_1 + \ldots + s_d - d \geq 1 \;\Leftrightarrow\; s_2 + \ldots + s_d \geq 1$$

holds, then $D_{d+1}(x) = 1$, $\nu_{d+1}(x) = \mu_{i+1}(x) = 1$ and we can choose the next maximal contact $V(z_{d+1})$ in the E-part of $\mathcal{H}_d(x)$.

Convention: We choose the maximal contact variables in the E-part until we get to the stage $r > d$, where $s_1 + \ldots + s_r - r = 0$.

This means the E-part of $\mathcal{H}_r(x)$ is empty. (Recall that $\mathcal{H}_r(x)$ determines $\nu_{r+1}(x)$). As above it follows for every $i \in \{2, \ldots, r\}$:

(1') $\mu_{i,H}(x) = 0$ for every $H \in \mathcal{E}_i(x)$, thus $D_i(x) = 1$ and

(2') $\nu_i(x) = \mu_i(x) = 1$.

In particular $\mathcal{H}_r(x)$ is only given by the $\mathcal{G}_1$-part. This means, $\mathcal{H}_r(x)$ is the coefficient idealistic exponent of $\mathcal{G}_1(x)$ with respect to $(z_1, \ldots, z_r)$.

In general, we cannot assume $s_1 > 0$. So we set

$$d := \min \{ q \in \mathbb{Z}_+ \mid s_q \neq 0 \}.$$

Then $E^d(x) \neq \emptyset$ and $\mathcal{F}_d(x) = \mathcal{G}_d(x) \cap (E^d(x), 1)$. We choose the maximal contact $V(z_d)$ such that there is some $H \in E^d(x)$ which is locally given by $V(z_d)$.

If $s_d \geq 2$, then the $E^{(d)}$-part of $\mathcal{H}_d(x)$ is non-empty. This implies $\nu_{d+1}(x) = \mu_{d+1}(x) = 1$. In the next step of the procedure we multiply the assigned numbers by $\nu_{d+1} = 1$, thus $\mathcal{G}_{d+1}(x) = \mathcal{H}_d(x)$ and then we add $E^{d+1}(x)$ in order to obtain $\mathcal{F}_{d+1}(x)$. We choose the maximal contact in the $E^{(d)}$-part and so on. This continues until we are at the step

$$r := \min \{ l \in \mathbb{Z}_+ \mid l \geq d \;\wedge\; s_d + \ldots + s_l - (l - d + 1) = 0 \}.$$

Putting everything together yields

Proposition 3.5.3. *Let $d, r \in \mathbb{Z}_+$ be as above. For every $i \in \{d+1, \ldots, r\}$ we get*

(i) $\mu_{i,H}(x) = 0$ *for every* $H \in \mathcal{E}_i(x)$*, thus* $D_i(x) = 1$ *and*

(ii) $\nu_i(x) = \mu_i(x) = 1$,

(iii) *the $E^{(d)}$-part of $\mathcal{H}_r(x)$ (and $\mathcal{G}_{r+1}(x)$) is empty,*

(iv) *hence $\mathcal{H}_r(x)$ is the idealistic coefficient exponent of $\mathcal{G}_d(x)$ with respect to $(z_d, \ldots, z_r)$ and $\mu_{r+1}(x) = \delta(\,\Delta_x(\mathcal{G}_d(x), u, (z_d, \ldots, z_r)\,)$, where (u) denotes the remaining elements of the regular system of parameters $(t) = (u, z)$.*

Further $\nu_{r+1}(x)$ is determined by $\mu_{r+1}(x)$ and the assigned numbers in the exceptional data of $\mathcal{H}_r(x)$.

If $s_d = 1$ then $r = d$ and the above statement is empty except for part *(iv)*.

Recall that we have constructed $\mathcal{G}_2(x)$ from $\mathcal{H}_1(x)$ by factoring out $D_2(x)$, $h = D_2^{b_h} g$ (where $\mathcal{H}_1(x) \subset (h, b_h)$). If $D_2 = D_2(x) = 1$ is trivial, i.e. if the assigned numbers in the exceptional data are all zero, then $\mathcal{G}_2(x) = \mathcal{H}_1(x)$. Together with the previous this leads to

Observation 3.5.4 (Big steps if $D^q(x) = 1$). Set

$$d := \min\{\, q \in \mathbb{Z}_+ \mid D_q = 1\,\} \quad \text{and} \quad r := \min\{\, l \in \mathbb{Z}_+ \mid l > d \ \wedge \ D_q \neq 1\,\}\,.$$

(For the steps before d we have to apply the usual procedure). Consider $\mathcal{G}_d(x)$. Since $D_d(x) = 1$, we have $\mathcal{G}_d(x) = \mathcal{H}_{d-1}(x)$. If $s_d = \#E^d(x) = 0$, then the next step is as without exceptional divisors. On the other hand, if $s_d \geq 1$, then we can apply Observation 3.5.2 until the E-part is empty. Note that we have during this process $D_q = 1$. Thus we have good control on these steps.

This works until we come to $\mathcal{H}_{r-1}(x)$. There $D_r(x) \neq 1$. By the convention of choosing first the exceptional components in the E-part, the $E^{(d)}$-part of $\mathcal{H}_{r-1}(x)$ has to be empty. This implies that $\mathcal{H}_{r-1}(x)$ is only given by the $\mathcal{G}_d$-part. (But it is not necessarily the idealistic coefficient exponent of $\mathcal{G}_d(x)$ with respect to $(z_d, \ldots, z_{r-1})$, because maybe not all $\nu_i(x)$ are equal 1 for $d < i < r$; nevertheless the situation is similar to Remark 3.2.6 — see also Remark 3.5.6 below).

We modify $\mathcal{H}_{r-1}(x)$ as described in Construction 3.3.2 (factor out $D_r(x)$ and then add $(D_r(x), 1 - \nu_r)$) and obtain $\mathcal{G}_r(x)$.

We have already seen that if $\mu_r(x) = 1$, then $(D_r(x), 1 - \nu_r) \sim_{\mathcal{E}(x)} \bigcap_H (g_H, 1)$, where the intersection is over those $H \in \mathcal{E}_r(x)$ with $\mu_{r,H}(x) \neq 0$ and g_H denotes a local generator of H. Then the same procedure as in the previous observation can be applied: First we choose the maximal contact only in the part coming from $D_r(x)$ and after that we consider the $E^{(r)}$-part.

We can apply this until we get to the point, where $D_{r'}(x) \neq 1$ and $\mu_{r'}(x) > 1$. Then we have to apply the full procedure to construct $\nu_{r'}$ and we go back to the beginning of this observation.

Also recall that if the exceptional data $\mathcal{E}_{r'}(x) = \emptyset$ is empty, then the general procedure is the same as in the easy case without exceptional divisors.

So let us recall the result.

Proposition 3.5.5. *Let $d, r, r' \in \mathbb{Z}_+$ be as in the previous observation. (Not to be confused with the d, r in Proposition 3.5.3; these are different integers). Then the case without exceptional divisors and Proposition 3.5.3 determine completely the procedure of Construction 3.3.2 for the steps $i \in \{d, \ldots, r'-1\}$.*

Note that Proposition 3.5.3 and Proposition 3.5.5 depend on the convention that we choose the maximal contact first in the E-part of the given idealistic exponent with history.

Remark 3.5.6. *For the proof of our main theorem we did not need concrete formulas for $\mathcal{G}_r(x)$, $\mathcal{F}_r(x)$ resp. $\mathcal{H}_{r+1}(x)$. Let us now briefly mention some results in this direction.*

In order to simplify the situation we assume that $D_2 = \ldots = D_r = 1$ and $D_{r+1} \neq 1$ for some $r \in \mathbb{Z}_+$. After r steps in the procedure we have distinguished the regular system of parameters for $R = \mathcal{O}_{Z_j,x}$ as $(u, z) = (u_1, \ldots, u_e; z_1, \ldots, z_r)$ and further we know the terms $\nu_2 = \mu_2 \geq 1, \ldots, \nu_r = \mu_r \geq 1$ and ν_{r+1}. ($D_{r+1} \neq 1$ implies $\nu_{r+1} < \mu_{r+1}$).

Define $\beta_1 := 1$ and $\beta_j := (\nu_2 \cdots \nu_j)^{-1}$ for $j > 1$. Recall that we choose (by the convention) the next maximal contact components in the E-part until it is empty. Since the E-part and $\mathcal{E}_r$ have simultaneously only normal crossings, it follows that the E-part of $\mathcal{H}_r(x)$ is empty. ($D_{r+1}(x)$ is determined by $\mathcal{H}_r(x)$). Together with the definition of r this yields that $\mathcal{H}_r(x)$ is completely determined by the $\mathcal{G}_1$-part. Suppose $\mathcal{G}_1(x) = \bigcap_{i=1}^m (f_i, b_i)$ for some $f_i \in R$. Then we can write for every i

$$f_i(u,z) = F_{b_i,i}(z) + \sum_{L_{\nu(r)}(B)<b} F_{B,i}(u) z^B + f_i^*(u,z),$$

for some $f_i^*(u,z) \in \langle z \rangle^{b+1}$ and $F_{b_i,i}(z) \in K[z]$ is (with respect to $L_{\nu(r)}(B) := \sum_{j=1}^r \beta_j\, B_j$) quasi-homogeneous of degree b_i. Further let $i \in \{1, \ldots, m\}$ and $B = B(i) \in \mathbb{Z}_0^r$ be such that $L_\nu(B) < b_i$. Then

$$\mathcal{H}_s(x) = \bigcap_{i,B \text{ as above}} \left(F_{B,i}(u),\, (b - L_\nu(B)) \cdot \frac{1}{\beta_s} \right).$$

By definition, $D_{r+1} \neq 1$ and thus $\nu_{r+1} < \mu_{r+1}$. Further any element $(h_{B,i}, b_h) := (F_{B,i}(u),\, (b - L_\nu(B)) \cdot (\beta_s)^{-1})$ can be written in the form $h_{B,i} = D_{r+1}^{b_h} \cdot g_{B,i}$ and

$$\mathcal{G}_{r+1}(x) = \begin{cases} \bigcap_{i,B} (g_{B,i}, b_h \cdot \nu_{r+1}), & \text{if } 1 \leq \nu_{r+1} < \infty, \\ \left(\bigcap_{i,B} (g_{B,i}, b_h \cdot \nu_{r+1}) \right) \cap (D_{r+1}, 1 - \nu_{r+1}), & \text{if } 0 < \nu_{r+1} < 1, \\ (D_{r+1}, 1 - \nu_{r+1}), & \text{if } \nu_{r+1} = 0. \end{cases}$$

(In the case $\nu_{r+1} = \infty$ the center of the next blow up is $N_r(x) = V(z_1, \dots, z_r)$).

Then we start again with $\mathcal{G}_{r+1}(x)$ instead of $\mathcal{G}_1(x)$. We define

$$s' := \min\{\, l \in \mathbb{Z}_+ \mid l \geq r+1 \,\wedge\, D_{l+1} \neq 1 \,\}$$

and we use for the formulas $(g_{B,i}, b_h \cdot \nu_{r+1})$ (and $(D_{r+1}, 1-\nu_{r+1})$) instead of (f_i, b_i).

In general, let $(f) = (f_1, \dots, f_m)$ be generators of J. Then $\mathcal{G}_1(x) = \bigcap_{i=1}^m (f_i, b_i)$. Set $(f, b) = (f_i, b_i)$ for some i. After r steps in Construction 3.3.2, we have determined $(z) = (z_1, \dots, z_r)$, $\nu_1, \nu_2, \dots, \nu_{r+1}$ and $D_2(x), \dots D_{r+1}(x)$. By the definitions, $(b - b_1)\nu_2 - b_2 = \nu_2\big(b - L_{\nu(2)}(b_1, b_2)\big)$. One can check that f can be written as $f(u, z) =$

$$= F_b(z, D) + \sum_{L_{\nu(r)}(B) < b} z^B \cdot D_2^{b-b_1} \cdot D_3^{(b-b_1)\nu_2 - b_2} \cdots D_{r+1}^{(\nu_2 \cdots \nu_r)(b - L_{\nu(r)}(B))} \cdot F_B(u) + f^*,$$

for some $f^* = f^*(u, z) \in \langle z \rangle^{b+1}$ and $F_b(z, D)$ is (with respect to $L_{\nu(r)}(B) = \sum_{i=1}^r \beta_i\, b_i$) quasi-homogeneous of degree b in the variables z . But the exceptional components $D_2, \dots, D_{r+1}$ are also involved in $F_b(z, D)$. With the above formula we can give a description of the $\mathcal{G}_1$-part of $\mathcal{G}_r(x)$, $\mathcal{F}_r(x)$, $\mathcal{H}_r(x)$ resp. $\mathcal{G}_{r+1}(x)$ similar to the one in Lemma 3.2.4 resp. Remark 3.2.6. But still there may be also an E- and a D-part.

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