Universität Regensburg
Mathematik

Gupta-Bleuler quantization of the Maxwell field in globally hyperbolic space-times

Felix Finster and Alexander Strohmaier

Preprint Nr. 13/2013

GUPTA-BLEULER QUANTIZATION OF THE MAXWELL FIELD IN GLOBALLY HYPERBOLIC SPACE-TIMES

FELIX FINSTER AND ALEXANDER STROHMAIER

JULY 2013

Abstract. We give a complete framework for the Gupta-Bleuler quantization of the free electromagnetic field on globally hyperbolic space-times. We describe one-particle structures that give rise to states satisfying the microlocal spectrum condition. The field algebra in the so-called Gupta-Bleuler representations satisfies the time-slice axiom, and the corresponding vacuum states satisfy the microlocal spectrum condition. We also give an explicit construction of ground states on ultrastatic space-times. Unlike previous constructions, our method does not require a spectral gap or the absence of zero modes. The only requirement, the absence of zero-resonance states, is shown to be stable under compact perturbations of topology and metric. Usual deformation arguments based on the time-slice axiom then lead to a construction of Gupta-Bleuler representations on a large class of globally hyperbolic space-times. As usual, the field algebra is represented on an indefinite inner product space, in which the physical states form a positive semi-definite subspace. Gauge transformations are incorporated in such a way that the field can be coupled perturbatively to a Dirac field. Our approach does not require any topological restrictions on the underlying space-time.

Contents

F.F. was supported by the Scheme 4 Grant #41122 of the London Mathematical Society.

1. Introduction

The classical approach to perturbative quantum electrodynamics begins with the canonical quantization scheme of the Dirac field and the electromagnetic field. Whereas the free electromagnetic field in Minkowski space-time may well be described by the quantized field algebra of the electromagnetic field strength, minimal coupling to a Dirac field requires the quantization of the vector potential. The gauge group in a coupled theory will act on both the Dirac field and the vector potential and can therefore not be factored out from the unperturbed electromagnetic field before the coupling is introduced. It was realized by Gupta and Bleuler [18, 4] that the field algebra of the vector potential is most conveniently represented in a Poincaré covariant manner on an indefinite inner product space, in which the physical states form a positive semi-definite subspace. Perturbative quantum electrodynamics can then be carried out consistently on the level of formal power series.

In quantum field theory on curved space-time, one considers quantized fields on a classical curved space-time. In a space-time (M^n, g), the classical Maxwell equations can be formulated with differential forms by

$$dF = 0, \qquad \delta F = J\,,$$

where $F \in \Omega^2(M)$ is the field strength, and J is the electromagnetic current. In the so-called potential method for solving these equations, one sets $F = dA$ with a one-form A, the so-called *electromagnetic potential*. Then the equation $dF = 0$ is automatically satisfied, so that Maxwell's equations reduce to

$$\delta dA = J\,. \tag{1.1}$$

The potential A is not uniquely determined. Namely, transforming A according to

$$A(x) \mapsto A(x) - d\Lambda(x) \tag{1.2}$$

with $\Lambda \in \Omega^0(M)$ maps the solution space of (1.1) to itself and leaves F unchanged. The transformations (1.2) are the *classical gauge transformations* of electrodynamics.

The physical requirement of *gauge invariance* implies that all observable quantities should be invariant under gauge transformations. In particular, the electromagnetic potential A is not gauge invariant. The field strength $F = dA$ is, making the electromagnetic field an observable quantity. Another way of forming gauge invariant quantities is to integrate the electromagnetic potential along a closed curve, or more generally a cycle,

$$\int_\gamma A \qquad \text{for a cycle } \gamma\,.$$

By Stokes' theorem, knowing $\int_\gamma A$ for all homologically trivial cycles is equivalent to knowing the field strength $F = dA$. However, as the Aharanov-Bohm experiment shows (see for example [29]), also cohomologically non-trivial cycles correspond to measurable quantities. Thus not the field strength alone, but the integrals of the electromagnetic potential along all cycles, should be regarded as the fundamental physical object of electrodynamics.

This physical significance of the electromagnetic potential becomes clearer in quantum mechanics, where A is needed to describe the coupling of the electromagnetic field to the quantum mechanical particle. For example, in the Dirac equation the coupling is described by the term $\gamma^i A_j(x)\psi(x)$, which can be generated by the so-called *minimal coupling* procedure where one replaces the partial derivatives in the Dirac equation according to $\partial_j \to \partial_j - iA_j(x)$. Thus it is impossible to work with the field strength alone; one must consider the potential $A(x)$ as being the basic object describing the electromagnetic field. In such a coupled situation, the gauge transformations (1.2) extend to transformations on the whole system, which typically describe local phase transformations of the wave functions

$$\psi(x) \mapsto e^{-i\Lambda(x)}\, \psi(x) .$$

In geometric terms, the minimal coupling is best understood as follows. The solutions of the Dirac equation are sections of a Dirac bundle that is twisted by a line bundle. The classical electromagnetic field vector potential should be regarded as a connection on this line bundle, and the Dirac equation is formed using the connection on the bundle. Once a local trivialization of the line bundle and the Dirac bundle is fixed, the connection determines a one-form, the vector potential. Gauge transformations then correspond to different choices of local trivializations.

As a consequence of the classical gauge freedom (1.2), the Cauchy problem for Maxwell's equations (1.1) is ill-posed. In order to circumvent this problem, one typically chooses a specific gauge. A common choice is the *Lorenz gauge*

$$\delta A = 0 . \tag{1.3}$$

Then the Maxwell equations go over to the wave equation

$$\Box A = 0 .$$

When performing a gauge transformation (1.2), the gauge condition (1.3) becomes

$$\delta A = \Box \Lambda , \tag{1.4}$$

and the field equations transform to

$$\Box A = \Box d\Lambda .$$

The goal of this paper is to quantize the electromagnetic field in a curved space-time in such a way that this field can readily be coupled to a quantized Dirac field. We restrict attention to the first step where the electromagnetic field without source terms is quantized. In a second step, the coupling to quantum particles could be described perturbatively. With this in mind, we only consider the homogeneous Maxwell equations. However, as the coupling to other particles and fields requires the electromagnetic potential, we want to construct field operators $\hat{A}$ for the electromagnetic potential. We do not impose any cohomological restrictions on our space-time. When coupling to the Dirac field, we assume that the quantization starts in a fixed topological sector. This means that we assume that we have chosen a fixed line bundle and a fixed connection with respect to which we perturb. Thus the electromagnetic potential to be quantized will consist of globally defined one-forms.

Conceptually, it is best to split the construction of the field operators in canonical quantization into two steps. Step one constructs the so called field algebra, i.e. a $*$-algebra that satisfies the canonical commutation relations. Step two consists of finding representations of this algebra that are physically reasonable. In quantum field theory in Minkowski space-time, there is usually a preferred Poincaré invariant ground state and therefore a physically preferred representation. In this situation, the construction is canonical and is often carried out in one step by employing a procedure which in physics is called frequency splitting. In curved space-time, it was first shown by

Dimock [13] that the algebra of the free scalar field on a globally hyperbolic space-time can be constructed in a functorial manner. Thus the first step can be carried out just as in Minkowski space-time. Dimock later used this procedure to quantize the electromagnetic field strength [14]. The canonical quantization of the electromagnetic vector potential in a curved background in the Gupta-Bleuler framework was first described by Furlani [16], who assumes the space-time to be ultrastatic with compact Cauchy surfaces. We note here, however, that in the presence of zero modes, the construction given in [16] contains gaps (in particular, Theorem III.1 does not hold if $H^1(M) \neq \{0\}$, essentially because when projecting out the zero modes, the locality of the commutation relations is lost). Another series of papers [15, 9, 11, 10] deals with the uncoupled electromagnetic field in curved space-times, where the field algebra for the field strength is constructed under certain cohomological conditions. These papers also deal with the construction of physically reasonable states.

The paper is organized as follows. After a brief mathematical introduction (Section 2), we define the field algebras, introduce gauge transformations and prove the time slice axiom (Section 3). In Section 4, we consider representations of the field algebras and explain the properties which we demand from a physically reasonable representation. More precisely, we define so-called *Gupta-Bleuler representations* as representations on an indefinite inner product space which satisfy a microlocal spectrum condition. Moreover, we demand that applying the observables to the vacuum should generate the positive semi-definite subspace of physical states. Furthermore, the gauge condition $\delta \hat{A} = \Box \Lambda$ should be satisfied for the expectation values of the physical states. In Section 5, we construct Gupta-Bleuler representations for ultrastatic manifolds. Here our main point is to treat the zero resonance states (Section 5.1) and the zero modes (Section 5.2). Using a glueing construction, these representations are then extended to general globally hyperbolic space-times (Section 6). All our constructions are gauge covariant, where we extend the classical gauge transformation law (1.2) to the field operators $\hat{A}$ by

$$\hat{A}(x) \mapsto \hat{A}(x) - d\Lambda(x) , \tag{1.5}$$

and Λ is again a real-valued function. This corresponds to the usual procedure in canonical quantization schemes (see for example [33, Section 8]) in which the gauge freedom described by so-called scalar photons is not quantized. In particular, the gauge transformations (1.5) leave the commutator relations of the field operators. An alternative procedure described in the literature is to fix the gauge with a *gauge parameter*. This leads to modifications of the commutator relations for non-observable quantities. In Appendix A, we show that working with different gauge parameters gives an equivalent description of the physical system.

2. Mathematical Preliminaries

Let (M, g) be a globally hyperbolic Lorentzian space-time of dimension $n \geq 2$, i.e. M is an oriented, time-oriented Lorentzian manifold that admits a smooth global Cauchy surface Σ (see [3]). We assume that the metric has signature $(+1, -1, \ldots, -1)$. Let $\Omega^p(M) \subset C^\infty(M; \Lambda^p T^*M)$ be the space of smooth real-valued p-forms and $\Omega^p_0(M) \subset \Omega^p(M)$ be the forms with compact support. As usual denote by $d : \Omega^p(M) \to \Omega^{p+1}(M)$ the exterior derivative and by $\delta : \Omega^{p+1}(M) \to \Omega^p(M)$ its formal adjoint with respect to the inner product on the space of p-forms

$$\langle f, g\rangle = \int_M f \wedge *g \ ,$$

where $*$ is the Hodge star operator. Note that this inner product is indefinite if $0 < p < n$.

The wave operator $\Box_p : \Omega^p(M) \to \Omega^p(M)$ is defined by $\Box_p = d\delta + \delta d$. It is formally self-adjoint with respect to the above inner product. The wave equation $\Box A = 0$ for p-forms $A \in \Omega^p(M)$ is a normally hyperbolic differential equation. It is well-known that the Cauchy problem for this equation can be solved uniquely, and moreover there exist unique advanced and retarded fundamental solutions $G^p_\pm : \Omega^p_0(M) \to \Omega^p(M)$ such that

(1) $G^p_\pm$ is continuous with respect to the usual locally convex topologies on $\Omega^p_0(M)$ and $\Omega^p(M)$, respectively.
(2) $\Box G^p_\pm f = G^p_\pm \Box f = f$ for all $f \in \Omega^p_0(M)$,
(3) $\operatorname{supp} G^p_\pm f \subset J^\pm(\operatorname{supp} f)$, where $J^\pm(\operatorname{supp} f)$ denotes the causal future respectively past of $\operatorname{supp} f$.

(we refer the reader to the monograph [2] for a detailed general proof in the context of operators on vector bundles). The map G^p is then defined to be the difference of retarded and advanced fundamental solutions $G^p := G^p_+ - G^p_-$. Note that G^p maps onto the space of smooth solutions of the equation $\Box A = 0$ with spatially compact support, i.e. solutions whose support have compact intersection with Σ. The function $G^p(f)$ can be viewed as a distribution and may be paired with a test function $g \in \Omega^p_0(M)$, using the inner product $\langle \cdot, \cdot \rangle$. We will denote $G^p(f)(g) = \langle G^p(f), g\rangle$ by $G^p(f, g)$. The bilinear form $G^p(\cdot, \cdot)$ defines a distribution on $M \times M$ with values in $\Lambda^p T^*M \boxtimes \Lambda^p T^*M$. It is straightforward to verify that $G^p(f, g) = -G^p(g, f)$.

Throughout the paper, we regard the space of p-forms as a subset of the set of distributional p-forms by using the inner product. That is if $A \in \Omega^p(M)$ we may pair A with a test function $f \in \Omega^p_0(M)$

$$A(f) := \int_M A \wedge *f.$$

For example if A is a one-form which in local coordinates is given by $A = \sum_{i=1}^n A_i(x)\, dx^i$, then the corresponding integral in local coordinates is

$$A(f) = \int_M \left(\sum_{i,k=1}^n g^{ik}(x) A_i(x) f_k(x) \right) \sqrt{|g|}\, dx,$$

where $f = \sum_{i=1}^n f_i(x)\, dx^i$. In physics this is often referred to as the field "smeared out" with a test function f. Of course the equation $dA = 0$ is then equivalent to $A(\delta f) = 0$ for all test functions $f \in \Omega^p_0(M)$. Similarly, the wave equation $\Box A = 0$ is equivalent to $A(\Box f) = 0$ for all $f \in \Omega^p_0(M)$. When dealing with quantum fields, we shall always take this "dual" point of view. Note that any cycle γ can be thought of as a co-closed distributional current. This means that knowing $\int_\gamma A$ for all cycles is equivalent to knowing that $A(f)$ for all $f \in \Omega^1_0(M)$ with $\delta f = 0$.

3. The Field Algebras and the Gauge Ideals

In this section, we construct the field algebra of the quantized Maxwell field. Since we will be dealing mostly with one-forms, we shall often omit the subscript p in the case $p = 1$ and simply write $G_\pm$ for the advanced and retarded fundamental solutions and set $G = G_+ - G_-$. The *field algebra* $\mathcal{F}$ is defined to be the unital $*$-algebra generated by symbols $A(f)$ for $f \in \Omega^1_0(M)$ together with the relations

$$\begin{gathered} f \mapsto A(f) \text{ is linear,} \\ A(f)A(g) - A(g)A(f) = -\mathrm{i}\, G(f,g), \\ A(\Box f) = 0, \quad \text{for all } f \in \Omega^1_0(M) \text{ with } \delta f = 0\,, \\ (A(f))^* = A(f). \end{gathered}$$

For every open subset $\mathcal{O} \subset M$, we define the *local field algebra* $\mathcal{F}(\mathcal{O}) \subset \mathcal{F}$ to be the sub-algebra generated by the $A(f)$ with $\mathrm{supp}(f) \in \mathcal{O}$. Inside $\mathcal{F}$, the *algebra of observables* $\mathcal{A}$ is defined as the unital subalgebra generated by $A(f)$ with $\delta f = 0$. The local algebras of observables $\mathcal{A}(\mathcal{O})$ are given by $\mathcal{A}(\mathcal{O}) = \mathcal{A} \cap \mathcal{F}(\mathcal{O})$.

The physical interpretation of the algebra $\mathcal{A}(\mathcal{O})$ is that it consists of all the physical quantities that can be measured in the space-time region $\mathcal{O}$. In particular, if $g \in \Omega^2_0(\mathcal{O})$, then $A(\delta g)$ is an observable. Since $A(\delta g) = dA(g)$, this observable corresponds to the field strength operator smeared out with the test function g. However, as explained at the end of the previous section, the algebra of observables may also contain observables that correspond to smeared out measurements of A along homologically non-trivial cycles. Thus it may be strictly larger than the algebra generated by $dA(g)$.

Definition 3.1. *Let $\Lambda \in C^\infty(M)$. The* **gauge ideal** *$\mathcal{I}_\Lambda \subset \mathcal{F}$ is the two-sided ideal generated by*

$$\big\{ A(\Box f) - \Lambda(\delta \Box f) \;|\; f \in \Omega^1_0(M) \big\}\,.$$

Lemma 3.2. $\mathcal{A} \cap \mathcal{I}_\Lambda = \{0\}$.

Proof. We first observe that

$$\Box\big(\Omega^1_0(M)\big) \cap \ker \delta|_{\Omega^1_0(M)} \subset K := \Box \ker \delta|_{\Omega^1_0(M)}\,. \tag{3.1}$$

Indeed, if f lies in the intersection on the left, then $f = \Box g$ and $0 = \delta f$, and thus $\Box \delta g = 0$. Since δg has compact support and solves the wave equation, it follows that $\delta g = 0$, proving (3.1). We introduce the vector spaces

$$W = \Omega^1_0(M)/K \qquad \text{and} \qquad U = \Box\big(\Omega^1_0(M)\big)/K\,, \quad V = \ker \delta|_{\Omega^1_0(M)}/K\,.$$

Then

$$U \cap V = \{0\}\,. \tag{3.2}$$

We introduce W^*_U and W^*_V as the subspaces of the dual space W^* consisting of elements that vanish on U and V, respectively.

The algebra $\mathcal{F}$ has a natural filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}$, where $\mathcal{F}_n$ is the span of products of the form $A(f_1) \cdots A(f_n)$. Moreover, the linear map

$$\sigma_n \;:\; \mathcal{F}_n \to \bigotimes\nolimits^n_s W\,, \quad A(f_1)\cdots A(f_n) \mapsto f_1 \otimes_s \cdots \otimes_s f_n$$

is well-defined, where $\otimes_s$ denotes the symmetric tensor product.

Let $\mathfrak{f} \in \mathcal{A} \cap \mathcal{I}_\Lambda$. We want to show that $\mathfrak{f}$ vanishes. Thus assume by contradiction that $\mathfrak{f} \neq 0$. Then there is a minimal n such that $\mathfrak{f} \in \mathcal{F}_n$ and $F := \sigma_n(\mathfrak{f}) \neq 0$. Since $\mathfrak{f} \in \mathcal{A}$ is an observable, we know that $F \in \otimes^n_s V$, and thus

$$\iota_\lambda F = 0 \qquad \text{for all } \lambda \in W^*_V\,, \tag{3.3}$$

where ι_λ denotes the contraction with λ. On the other hand, as $\mathfrak{f} \in \mathcal{I}_\Lambda$, we know that

$$(\lambda_1 \otimes_s \cdots \otimes_s \lambda_n)F = 0 \qquad \text{for all } \lambda_i \in W_U^* \,. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain by linearity that

$$(\lambda_1 \otimes_s \cdots \otimes_s \lambda_n)F = 0 \qquad \text{for all } \lambda_i \in W_U^* + W_V^* \,. \tag{3.5}$$

In view of (3.2), the set $W_U^* + W_V^*$ is a point-separating subspace of W^*. Hence (3.5) implies that $F = 0$, a contradiction. ☐

This lemma gives a canonical injective map $\mathcal{A} \to \mathcal{F}/\mathcal{I}_\Lambda$.

Remark 3.3. (gauge transformations) The analog of classical gauge transformations can be realized by the algebra homomorphism

$$\mathfrak{G}_\Lambda \;:\; A(f) \mapsto A(f) - (d\Lambda)(f) \,. \tag{3.6}$$

This algebra homomorphisms leaves the algebra of observables $\mathcal{A}$ invariant. Moreover, it transforms the gauge ideals by

$$\mathfrak{G}_{\Lambda'} \mathcal{I}_\Lambda = \mathcal{I}_{\Lambda+\Lambda'} \,. \tag{3.7}$$

Thus the gauge freedom is described in the algebraic formulation by the freedom in choosing a gauge ideal. ♢

In classical gauge theories, the time evolution is uniquely defined only after a gauge-fixing procedure. In the same way, in our framework the time slice axiom holds only after dividing out the gauge ideal:

Proposition 3.4. (time slice axiom) *Let U be an open neighborhood of a Cauchy surface in M and $\Lambda \in C^\infty(M)$. Then*

$$\mathcal{F}(U)/\mathcal{I}_\Lambda = \mathcal{F}/\mathcal{I}_\Lambda \,.$$

In particular $\mathcal{A}(U) = \mathcal{A}$.

Proof. Since the above gauge transformations leave $\mathcal{F}(U)$ invariant, we can arrange in view of (3.6) and (3.7) that $\Lambda = 0$. Let $f \in \Omega^1_0(M)$. By Lemma 3.5 below there exist forms $h \in \Omega^1_0(M)$ and $g \in \Omega^1_0(U)$ with $\Box h = f - g$. We conclude that $A(f-g) = A(\Box h) \in \mathcal{I}_\Lambda$. ☐

Lemma 3.5. *Suppose that U is an open neighborhood of a Cauchy surface Σ in M. Then for every $f \in \Omega^\bullet_0(M)$ there exists $h \in \Omega^\bullet_0(M)$ such that the form*

$$f - \Box h$$

is compactly supported in U. If $df = 0$, then h can be chosen to be closed. If $\delta f = 0$, then h can be chosen to be co-closed.

Proof. Let $\eta_+ \in C^\infty(M)$ and η_- be non-negative smooth functions such that

- $\eta_+(x)^2 + \eta_-(x)^2 = 1$ for all $x \in M$.
- η_+ has future compact support and η_- has past compact support.
- $\mathrm{supp}(d\eta_\pm) \subset U$.

Now one checks by direct computation that $h := \eta_+^2 G_-(f) + \eta_-^2 G_+(f)$ has the required properties. If f is closed, we may take

$$h := d\big(\eta_+ G_-(\eta_+ G_-(\delta f)) + \eta_- G_+(\eta_- G_+(\delta f))\big).$$

Again one checks that $f - \Box h$ has compact support in U. Moreover, by construction, h is closed. A straightforward modification of this argument shows that h can be chosen to be co-closed if f is co-closed. $\Box$

4. Representations of $\mathcal{F}$

In Minkowski space, there is a unique vacuum state determined by Poincaré invariance and the spectrum condition. In general curved space-times, the lack of such a distinguished vacuum state has led to alternative selection criteria for physical states. The spectrum condition is then replaced by microlocal versions [30]. Before introducing representations, we therefore recall some basic notions of microlocal analysis.

4.1. **Polarization Sets and Wavefront Sets of Bundle-Valued Distributions.** We denote by $\psi\mathrm{DO}^m(M,E)$ the set of properly supported pseudo-differential operators acting on sections of a vector bundle $E \to M$. More precisely, we work with polyhomogeneous symbols, i.e. symbols in the Hörmander classes S^m_{phg} defined in [21, Chapter 18]. The principal symbol σ_A of a pseudo-differential operator $A \in \psi\mathrm{DO}^m(M,E)$ is then a positive homogeneous section of degree m in $C^\infty(\dot{T}^*M, \pi^* \operatorname{End}(E))$ (where $\dot{T}^*M$ denotes the cotangent space with its zero section removed, and $\pi : \dot{T}^*M \to M$ is the canonical projection). Following [12], we define:

Definition 4.1. *Let $u \in \mathcal{D}'(M;E)$ be a distribution with values in E. Then the* **polarization set** $\mathrm{WF}_{\mathrm{pol}}(u)$ *is defined by*

$$\mathrm{WF}_{\mathrm{pol}}(u) = \bigcap_{\substack{A \in \psi\mathrm{DO}^0(M;E),\\ Au \in C^\infty(M;E)}} \mathcal{N}_A\,,$$

where

$$\mathcal{N}_A := \Big\{(x,\xi;v) \in \dot{T}^*M \times E \,\Big|\, v \in E_x \text{ and } \sigma_A(x,\xi)\,v = 0\Big\}.$$

Moreover, the **wave front set** *can be defined by*

$$\mathrm{WF}(u) = \pi\Big(\mathrm{WF}_{\mathrm{pol}}(u) \setminus \dot{T}^*M \times \{0\}\Big)\,,$$

where $\pi : \dot{T}^*M \times E \to \dot{T}^*M$ is the natural projection.

4.2. **Gupta-Bleuler Representations.** As is usual in the Gupta-Bleuler formalism, the representation of the field algebra will not be on a Hilbert space, but rather on a space equipped with an indefinite inner product. Thus we let $(\mathfrak{K}, \langle\cdot,\cdot\rangle)$ be a locally convex topological vector spaced endowed with an indefinite inner product.

For a given real-valued function $\Lambda \in C^\infty(M)$, we let π be a representation of $\mathcal{F}$ on $\mathfrak{K}$ and $\Omega \in \mathfrak{K}$ such that the following hold:

(a) $\overline{\pi(\mathcal{F})\,\Omega} = \mathfrak{K}$, (cyclicity)
(b) $\pi(\mathcal{A})\,\Omega$ is a positive semi-definite subspace $\mathfrak{H}_0 \subset \mathfrak{K}$ and $\langle\Omega,\Omega\rangle = 1$.
(c) $\pi(\mathcal{I}_\Lambda) = 0$,

(d) For any $n \in \mathbb{N}$, the space $\mathfrak{K}_n$ defined by

$$\mathfrak{K}_n = \overline{\mathcal{F}_n \Omega} \subset \mathfrak{K}$$

is a Krein space (endowed with the inner product $\langle \cdot, \cdot \rangle$ and the locally convex topology induced by $\mathfrak{K}$). As before, $\mathcal{F}_n$ is the span of products of the form $A(f_1) \cdots A(f_n)$.

(e) microlocal spectrum condition:

$$\mathrm{WF}\Big(\underbrace{A(\cdot) \cdots A(\cdot)}_{m \text{ factors}} \Omega \Big) \subseteq \Gamma_m \qquad \text{for all } m\,, \tag{4.1}$$

where $A(\cdot) \cdots A(\cdot) \Omega$ is a Krein-space-valued distribution. The sets Γ_m are defined below.

(f) Gupta-Bleuler condition:

$$\big\langle \phi, \pi\big(A(df) - \Lambda(\Box f)\big)\, \phi \big\rangle = 0 \qquad \text{for all } \phi \in \mathfrak{H}_0\,.$$

Then $(\pi, \mathfrak{K}, \Omega)$ is called a **Gupta-Bleuler representation in the Λ-gauge**. The distribution in (4.1) can be expressed in terms the n-point distributions defined by

$$\omega_n(f_1, \ldots, f_n) = \big\langle \Omega, \pi\big(A(f_1) \cdots A(f_n)\big)\, \Omega \big\rangle\,.$$

Note that $\mathfrak{H}_0$ is not a Hilbert space, because its inner product is only positive semi-definite. Dividing out the null subspace

$$\mathcal{N} = \{ \psi \in \mathfrak{H}_0 \mid \langle \psi, \psi \rangle = 0 \}$$

and forming the completion, one gets a Hilbert space, which in the usual Gupta-Bleuler formalism is interpreted as the physical Hilbert space.

Here the sets Γ_m are defined as follows. We denote the closed light cone and its boundary by

$$J^+ = \{ (x, \xi) \mid \mathfrak{g}_x(\xi, \xi) \geq 0 \text{ and } \xi_0 \geq 0 \}$$
$$L^+ = \{ (x, \xi) \mid \mathfrak{g}_x(\xi, \xi) = 0 \text{ and } \xi_0 \geq 0 \}\,.$$

Let $\mathcal{G}_k$ be the set of all finite graphs with vertices $\{1, \ldots, k\}$ such that for every element $G \in \mathcal{G}_k$ all edges occur in both admissible directions. We write $s(e)$ and $r(e)$ for the source and the target of an edge respectively. Following [6], we define an immersion of a graph $G \in \mathcal{G}_k$ into the space-time M as an assignment of the vertices ν of G to points $x(\nu)$ in M, and of edges e of G to piecewise smooth curves $\gamma(e)$ in M with source $s(\gamma(e)) = x(s(e))$ and range $r(\gamma(e)) = x(r(e))$, together with a covariantly constant causal co-vector field ξ_e on γ such that

(1) If e^{-1} denotes the edge with opposite direction as e, then the corresponding curve $\gamma(e^{-1})$ is the inverse of $\gamma(e)$.
(2) For every edge e the co-vector field ξ_e is directed towards the future whenever $s(e) < r(e)$.
(3) $\xi_{e^{-1}} = -\xi_e$.

We set

$$\begin{aligned} \Gamma_m := \Big\{ (x_1, \xi_1; \ldots; x_m, \xi_m) \in T^*M^m \backslash 0 \;\Big|\; & \text{there exists } G \in \mathcal{G}_m \\ & \text{and an immersion } (x, \gamma, \xi) \text{ of } G \text{ in } M \text{ such that} \\ & x_i = x(i) \quad \text{for all } i = 1, \ldots, m \text{ and } \xi_i = -\sum_{e,\, s(e)=i} \xi_e(x_i) \Big\}\,. \end{aligned}$$

The microlocal spectrum condition for states was introduced for scalar fields by Brunetti, Fredenhagen and Köhler in [6], who also showed that for quasi-free representations, it suffices to verify the microlocal spectrum condition for the two-point functions. Quasi-free representations are those which satisfy the Wick rule

$$\omega_m(f_1, \ldots, f_m) = \sum_P \prod_r \omega_2(f_{(r,1)}, f_{(r,2)}) \,,$$

where P denotes a partition of the set $\{1, \ldots, m\}$ into subsets which are pairings of points labeled by r. More precisely, following the arguments in [31] it follows that if the representation is quasi-free, then the microlocal spectrum condition is equivalent to the condition

$$\mathrm{WF}\Big(\pi\big(A(\cdot)\big)\,\Omega\Big) \subset J^+$$

(where $A(\cdot)\,\Omega$ is again understood as a Krein-space-valued distribution).

The microlocal spectrum condition for quasi-free states of the Klein-Gordon field was shown in [30] to be equivalent to the well-known Hadamard condition. Moreover, the microlocal spectrum condition is a sufficient condition for the construction of Wick polynomials (see [6, 19]) and interacting fields (see [5, 20]) in general globally hyperbolic space-times. For this reason, the microlocal spectrum condition is generally recognized to be a useful substitute for the spectrum condition in Minkowski space valid in general globally hyperbolic space-times.

Remark 4.2. Requiring the microlocal spectrum condition on the level of observables only results in a slightly weaker condition that manifests itself in a condition on the polarization set. Namely, assume that $u \in \mathcal{D}'(M, \Lambda^1(M))$ satisfies the condition

$$\mathrm{WF}(du) \subseteq \big\{(x,\xi) \in \dot{T}^*M \,|\, \xi \in J^+\big\} \,. \tag{4.2}$$

Then

$$\mathrm{WF}_{\mathrm{pol}}(u) \subseteq \big\{(x,\xi;v) \in \dot{T}^*M \times T^*M \,|\, \xi \in J^+ \text{ or } \xi \sim v\big\} \,. \tag{4.3}$$

Note that

$$\mathrm{WF}_{\mathrm{pol}}(u) \subseteq \mathrm{WF}_{\mathrm{pol}}(du) + \mathcal{N}_d$$

with

$$\mathcal{N}_d = \big\{(x,\xi;v) \,\big|\, \sigma_d(x,\xi)\cdot v = 0\big\} = \Big\{(x,\xi;v) \,\Big|\, v \sim \xi\Big\} \,.$$

Microlocally, the set $\mathcal{N}_d$ corresponds to the so-called longitudinal photons. Our condition (e) is stronger in that it imposes the microlocal spectrum condition in all directions, including those corresponding to longitudinal photons. We point out that the inverse implication (4.3)⇒(4.2) is in general false, because the set $\mathrm{WF}_{\mathrm{pol}}(u)$ only detects the highest order of the singularities. ◊

Remark 4.3. *(gauge invariance)* Note that property (c) implements the field equation. The property (f), on the other hand, realizes the gauge condition (1.4), but only if we take the inner product with a vector in $\mathfrak{H}_0$. ◊

4.3. **Fock Representations.** In order to construct Gupta-Bleuler-Fock representations of the field algebra $\mathcal{F}$, one can proceed as follows. Let $\kappa : C_0^\infty(M) \to \mathcal{K}$ be a real-linear continuous map into a complex Krein space $\mathcal{K}$ with the following properties:

(i) $\kappa(\Box f) = 0$ for all $f \in \Omega^1_0(M)$
(ii) $\langle \kappa(f), \kappa(f)\rangle \geq 0$ if $\delta f = 0$.
(iii) $\mathrm{Im}\langle \kappa(f), \kappa(g)\rangle = G(f,g)$

(iv) microlocal spectrum condition:
$$\mathrm{WF}(\kappa) \subseteq \big\{(x,\xi) \in \dot{T}^*M \mid \xi \in J^+\big\} .$$
(v) $\mathrm{span}_{\mathbb{C}}\mathrm{Rg}(\kappa)$ is dense in $\mathcal{K}$.
(vi) $\langle \kappa(df), \kappa(g)\rangle = 0$ for all $f \in \Omega^0_0(M)$, $g \in \Omega^1_0(M)$ with $\delta g = 0$.

We introduce the Bosonic Fock space by
$$\mathfrak{K} = \bigoplus_{N=0}^{\infty} \hat{\bigotimes}_s^N \mathcal{K} \,, \tag{4.4}$$
where $\hat{\otimes}$ denotes the completed symmetric tensor products of Krein spaces. Note that $\mathfrak{K}$ is an indefinite inner product space but does not have a canonical completion to a Krein space. For $\psi \in \mathcal{K}$, we let $a(\psi)$ be the annihilation operator and $a^*(\psi)$ be the creation operator, defined as usual by
$$\begin{aligned} a^*(\psi)\, \phi_1 \otimes_s \ldots \otimes_s \phi_N &= \psi \otimes_s \big(\phi_1 \otimes_s \ldots \otimes_s \phi_N\big) \\ a(\psi)\, \phi_1 \otimes_s \ldots \otimes_s \phi_N &= \langle \psi, \phi_1\rangle\, \phi_2 \otimes_s \ldots \otimes_s \phi_{N-1} \,. \end{aligned} \tag{4.5}$$
By construction, we have the canonical commutation relations
$$\big[a(\psi), a^*(\phi)\big] = \langle \psi, \phi \rangle \,. \tag{4.6}$$

For each $f \in \Omega^1_0(M)$ and a given $\Lambda \in C^\infty(M)$, we let $\hat{A}(f)$ be the following endomorphism of $\mathfrak{K}$:
$$\hat{A}(f) = \frac{1}{\sqrt{2}}\Big(a\big(\kappa(f)\big) + a^*\big(\kappa(f)\big)\Big) + d\Lambda(f) \,. \tag{4.7}$$
Then the mapping
$$\pi \;:\; A(f) \mapsto \hat{A}(f)$$
extends to a $*$-representation π of the field algebra $\mathcal{F}$ by operators that are symmetric with respect to the indefinite inner product on $\mathfrak{K}$.

Theorem 4.4. *The representation π is a Gupta-Bleuler representation.*

Proof. We need to check the properties (a)-(f) of a Gupta-Bleuler representation.

(a) Cyclicity: The Fock space is the direct sum of finite particle subspaces. Suppose that $N \geq 1$ and let P_N be the canonical projection onto the N-particle subspace $\hat{\bigotimes}_s^N \mathcal{K}$. Since
$$P_N\, \hat{A}(f_1) \cdots \hat{A}(f_N)\Omega = \Big(\frac{1}{\sqrt{2}}\Big)^N \kappa(f_1) \otimes_s \cdots \otimes_s \kappa(f_N)$$
and the complex span of the range of κ is dense in $\mathcal{K}$, we know that the complex span of $\{P_N \hat{A}(f_1) \cdots \hat{A}(f_N)\Omega\}$ is dense in $\hat{\bigotimes}_s^N \mathcal{K}$. Therefore, if any element in $\bigoplus_{k=0}^{N-1} \hat{\bigotimes}_s^k \mathcal{K}$ can be approximated by elements in $\pi(\mathcal{F})\Omega$, so can be any element in $\bigoplus_{k=0}^{N} \hat{\bigotimes}_s^k \mathcal{K}$. By induction in N, we conclude that $\pi(\mathcal{F})\Omega$ is dense in $\mathfrak{K}$.

(b) is a direct consequence of (ii).

(c) follows from $\kappa(\Box(f)) = 0$ and the definition of $\hat{A}$ by direct computation.

(d) is clear by construction, because the finite tensor product of Krein spaces is a Krein space.

(e) The microlocal spectrum condition can be proved exactly as in [31, Propositions 2.2 and 6.1] and [6, Proposition 4.3].

(f) The space $\mathfrak{H}_0$ is generated by vectors of the form
$$\phi = \hat{A}(f_1) \cdots \hat{A}(f_n)\,\Omega \qquad \text{with } \delta f_i = 0\,.$$
Using (4.7), we obtain
$$\pi\big(A(df) - \Lambda(\Box f)\big) = \frac{1}{\sqrt{2}}\Big(a\big(\kappa(df)\big) + a^*\big(\kappa(df)\big)\Big)\,.$$
As a consequence of the commutation relations (4.6) and the property (vi), the operators $a(\kappa(df))$ and $a^*(\kappa(df))$ commute with all the $\hat{A}(f_k)$. We conclude that $a(\kappa(df))\phi = 0$ and thus
$$\big\langle \phi, \pi\big(A(df) - \Lambda(\Box f)\big)\,\phi\big\rangle = \frac{1}{\sqrt{2}}\,\Big\langle \phi, \Big(a\big(\kappa(df)\big) + a^*\big(\kappa(df)\big)\Big)\,\phi \Big\rangle = 0\,. \qquad \Box$$

4.4. **Generalized Fock Representations.** In order to quantize the zero modes, we need to generalize the previous construction as follows. Let $\kappa : C_0^\infty(M) \to \mathcal{K}$ be a real-linear continuous map into a complex Krein space $\mathcal{K}$ with the following properties:

(i) $\kappa(\Box f) = 0$ for all $f \in \Omega^1_0(M)$
(ii) $\langle \kappa(f), \kappa(f)\rangle \geq 0$ if $\delta f = 0$.
(iii) There is a bilinear form G_Z on $\Omega^1_0(M) \times \Omega^1_0(M)$ with smooth integral kernel and the following properties:
$$\mathrm{Im}\langle \kappa(f), \kappa(g)\rangle + G_Z(f,g) = G(f,g) \tag{4.8}$$
$$\text{The vector space } Z := \Omega^1_0(M)/\{f \,|\, G_Z(f,\cdot) = 0\} \text{ is finite dimensional} \tag{4.9}$$
$$G_Z(f, \delta g) = 0 \qquad \text{for all } f \in \Omega^1_0(M) \text{ and } g \in \Omega^2_0(M) \tag{4.10}$$
(iv) microlocal spectrum condition:
$$\mathrm{WF}(\kappa) \subseteq \big\{(x,\xi) \in \dot{T}^*M \,|\, \xi \in J^+\big\}\,.$$
(v) $\mathrm{span}_{\mathbb{C}}\mathrm{Rg}(\kappa)$ is dense in $\mathcal{K}$.
(vi) $\langle \kappa(df), \kappa(g)\rangle = 0$ for all $f \in \Omega^0_1(M)$, $g \in \Omega^1_0(M)$ with $\delta g = 0$.

We introduce $\mathfrak{K}$ and a as in the previous section (see (4.4) and (4.5)). Let $\nu : \Omega^1_0(M) \to Z$ be the quotient map, and $\tilde{G}_Z$ the induced symplectic form on Z.

We choose a complex structure $\mathfrak{J}$ on Z such that $K(\cdot,\cdot) := -\tilde{G}_Z(\cdot, \mathfrak{J}\cdot)$ is a real inner product. This complex structure then induces a canonical splitting $Z = Y \oplus \tilde{Y}$ into two K-orthogonal Lagrangian subspaces Y and $\tilde{Y}$ such that the symplectic form is given by $\tilde{G}_Z((x_1,x_2),(y_1,y_2)) = K(x_1,y_2) - K(x_2,y_1)$. Let pr_1 and pr_2 be the canonical projections and let $\nu_i := \mathrm{pr}_i \circ \nu$. On the Schwartz space $\mathcal{S}(Y,\mathbb{C})$, we define $\hat{A}_{\mathfrak{J}}(f) \in \mathrm{End}(\mathcal{S}(Y,\mathbb{C}))$ by
$$\big(\hat{A}_{\mathfrak{J}}(f)\phi\big)(x) = K\big(\nu_1(f),x\big)\,\phi(x) + i\,(D_{\nu_2(f)}\phi)(x)\,.$$
A short computation using the identity
$$\begin{aligned} K\big(\nu_1(f),x\big)\,\big(D_{\nu_2(g)}\phi\big)(x) - D_{\nu_2(g)}\Big(K\big(\nu_1(f),x\big)\,\phi(x)\Big) \\ = \big(D_{\nu_2(g)}K(\nu_1(f),x)\big)\,\phi(x) = K\big(\nu_1(f),\nu_2(g)\big)\,\phi(x) \end{aligned}$$
shows that $\hat{A}_{\mathfrak{J}}$ satisfies the canonical commutation relations
$$[\hat{A}_{\mathfrak{J}}(f), \hat{A}_{\mathfrak{J}}(g)] = iG_Z(f,g)\,.$$

We now define $\hat{A}(f)$ on $\mathfrak{K}\otimes\mathcal{S}(Y,\mathbb{C})$ by

$$\hat{A}(f)=\frac{1}{\sqrt{2}}\Big(a\big(\kappa(f)\big)+a^*\big(\kappa(f)\big)\Big)\otimes\mathbb{1}+\mathbb{1}\otimes A_{\mathfrak{J}}(f)+d\Lambda(f)\,.$$

Since by assumption G_Z has a smooth integral kernel, the wave front set of the distribution $\mathbb{1}\otimes A_{\mathfrak{J}}(f)$ is empty. A straightforward modification of the proof of Theorem 4.4 leads to the following result.

Theorem 4.5. *Under the assumptions* (i)–(vi) *stated above, the mapping* $\pi : f\mapsto \hat{A}(f)$ *defines a Gupta-Bleuler representation.*

5. Constructions for Ultrastatic Manifolds

In this section we assume that the manifold M is ultrastatic, i.e. that it is of the form $M=\mathbb{R}\times\Sigma$ with metric of product type $g=dt^2-h$, where h is a complete Riemannian metric on Σ. Then M is globally hyperbolic and each $\Sigma_t:=\{t\}\times\Sigma$ is a Cauchy surface. A one-form $f\in\Omega^1(M)$ can be decomposed as

$$f=f_0\,dt+f_\Sigma\,,$$

where $f_0\in C^\infty(M)$ and $f_\Sigma\in C^\infty(M)\otimes_\pi\Omega^1(\Sigma)$. Here $\otimes_\pi$ denotes the projective tensor product of two locally convex spaces. We can think of f_Σ as a family of one-forms $f_\Sigma(t)$ on Σ that depends smoothly on the parameter t. Let

$$\Psi^f:=\begin{pmatrix} f\\ \dot{f}\end{pmatrix},$$

where $\dot{f}:=\frac{df}{dt}$. The restriction Ψ^f_t of Ψ^f to the hypersurface Σ_t will be viewed as an element in $(C^\infty(\Sigma)\oplus\Omega^1(\Sigma))^2$. We say that a one-form $f\in\Omega^1(M)$ has *spatially compact support* if $\Psi^f_t\in(C_0^\infty(\Sigma)\oplus\Omega_0^1(\Sigma))^2$ for all $t\in\mathbb{R}$. The set of spatially compact one-forms is denoted by $\Omega^1_{\mathrm{sc}}(M)$.

In view of the unique solvability of the Cauchy problem, the set of smooth solutions $\Omega^1_{\mathrm{sc}}(M)\cap\ker(\Box)$ of the wave equation with spatially compact support can be identified with the space of initial data with compact support on Σ_0. Thus, the map $f\mapsto\Psi^f_0$ defines an isomorphism between $\Omega^1_{\mathrm{sc}}(M)\cap\ker(\Box)$ and $(C_0^\infty(\Sigma)\oplus\Omega_0^1(\Sigma))^2$. Since G maps $\Omega^1_0(M)$ onto $\Omega^1_{\mathrm{sc}}(M)\cap\ker(\Box)$, the assignment

$$f\mapsto\Psi_0^{G(f)}$$

defines a surjective map to the Cauchy data space $(C_0^\infty(\Sigma)\oplus\Omega_0^1(\Sigma))^2$. There exists a natural symplectic form σ on the Cauchy data space defined by

$$\sigma\left(\begin{pmatrix}\mathfrak{f}\\ \dot{\mathfrak{f}}\end{pmatrix},\begin{pmatrix}\mathfrak{g}\\ \dot{\mathfrak{g}}\end{pmatrix}\right)=-\int_\Sigma\Big(\mathfrak{f}_0\,\dot{\mathfrak{g}}_0-\dot{\mathfrak{f}}_0\,\mathfrak{g}_0\Big)\,d\mu_\Sigma+\int_\Sigma\Big(\langle\mathfrak{f}_\Sigma,\dot{\mathfrak{g}}_\Sigma\rangle-\langle\dot{\mathfrak{f}}_\Sigma,\mathfrak{g}_\Sigma\rangle\Big)\,d\mu_\Sigma\,, \tag{5.1}$$

where $\langle\cdot,\cdot\rangle$ is the fibrewise inner product on forms on Σ induced by the Riemannian metric h, and μ_Σ is the Riemannian measure on Σ. An elementary computation using Stokes' formula shows that (see for example [2, eq. (4.6)])

$$G(f,g)=\sigma\big(\Psi^{Gf},\Psi^{Gg}\big)\,.$$

5.1. **Absence of Zero Resonance States.** Usually, the construction of ground states on ultrastatic space-times assumes the existence of a spectral gap. In what follows, we shall generalize this construction significantly assuming a weaker condition, which we now formulate. Let $\Omega^p_{(2)}(\Sigma)$ be the space of square-integrable p-forms on Σ. Since Σ is assumed to be complete, the Hodge Laplacian Δ with domain of definition $\Omega^p_0(\Sigma)$ is an essentially self-adjoint operator on $\Omega^p_{(2)}(\Sigma)$. We denote the self-adjoint extension again by Δ with domain $\mathcal{D}(\Delta)$. Let dE_z be the spectral measure of Δ. Moreover, let $\Omega^{p\perp}_{(2)}(\Sigma) \subset \Omega^p_{(2)}(\Sigma)$ be the orthogonal complement of $\ker \Delta$, and $\Omega^{p\perp}(\Sigma) \subset \Omega^{p\perp}_{(2)}(\Sigma) \cap \Omega^p(\Sigma)$ be the projection of $\Omega^p_0(\Sigma)$ onto the orthogonal complement of $\ker \Delta$. Of course, Δ leaves $\Omega^{p\perp}_{(2)}(\Sigma)$ invariant. For simplicity, we denote its restriction to $\Omega^{p\perp}_{(2)}(\Sigma)$ again by Δ. Our constructions rely on the following condition:

$$\begin{aligned} \text{(A)}\ \ & \text{The kernel of } \Delta \text{ is finite dimensional,} \\ & \text{and the domain of the operator } \Delta^{-\frac{1}{4}} \text{ contains } \Omega^{p\perp}(\Sigma). \end{aligned} \tag{5.2}$$

It is remarkable that for a large class of manifolds, this condition can be guaranteed under topological conditions on the boundary at infinity. In fact, this condition is closely related to the absence of zero resonance states. Namely, assume that the resolvent family $(\Delta-\lambda^2)^{-1}$ of the Laplacian on differential forms admits a meromorphic continuation in the following sense. For suitably weighted L^2-spaces

$$\mathcal{H}_1 := L^2(\Sigma, \rho^{-1}\, d\mu_\Sigma) \;\subset\; \Omega^\bullet_{(2)}(\Sigma) \;\subset\; \mathcal{H}_1^* = \mathcal{H}_{-1} := L^2(\Sigma, \rho\, d\mu_\Sigma)$$

with a positive weight function $\rho \in C^\infty(\Sigma)$ that vanishes at infinity, we assume that the family of operators

$$(\Delta - \lambda^2)^{-1} \;:\; \mathcal{H}_1 \to \mathcal{H}_{-1}$$

has a meromorphic extension to a neighborhood of $\lambda = 0$, with the property that the negative Laurent coefficient are operators of finite rank. This assumption is well-known to be satisfied for odd-dimensional manifolds which are isometric to $\mathbb{R}^{2n+1}$ outside compact sets (see for example [25]). Moreover, meromorphic continuations have been established for manifolds with cylindrical ends [24] in the context of the Atiyah-Patodi-Singer index theorem. It follows from standard glueing constructions and the meromorphic Fredholm theorem that the meromorphic properties of the resolvent are stable under compactly supported metric and topological perturbations and therefore only depend on the structure near infinity.

Under these assumptions, there exist finite-rank operators $A, B : \mathcal{H}_1 \to \mathcal{H}_{-1}$ such that for any $f \in \Omega^\bullet_0$, the measure $d\langle f, E_z f\rangle$ has the representation

$$d\langle f, E_z f\rangle = \left(\langle f, Af\rangle\, \delta(z) + \langle f, Bf\rangle\, \frac{\Theta(z)}{\sqrt{z}} + \big\langle f, C(\sqrt{z}) f\big\rangle\, \Theta(z) \right) dz\,, \tag{5.3}$$

where Θ denotes the Heaviside step function, and C is a holomorphic family of operators with values in the bounded operators $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_{-1})$. The operator A is in fact the orthogonal projection onto $\ker \Delta$. Since (5.3) remains true for vectors in $\ker \Delta$ if we set $B = 0$, $C = 0$ and $A = 1$, we can extend this formula by linearity to the domain $\Omega^{\bullet\perp}(\Sigma)$. Using the spectral theorem, one easily sees that the above condition (A) is equivalent to the vanishing of the operator B.

Vectors in the range of B are commonly referred to as *zero resonant states.* The topological significance of these states was first pointed out by Atiyah, Patodi and Singer [1] and elaborated in [27, 24] in the case of manifolds with cylindrical ends (see also the introduction in [28]). For another class of manifolds, referred to as non-parabolic at infinity, it was pointed out by Carron [7] that the existence of certain non-square-integrable harmonic forms depends only on the geometry near infinity, the obstruction being an index of a certain Dirac operator. In many situations, it can be shown that these non-square-integrable harmonic forms correspond to zero resonant states [8, 32].

In order to illustrate that assumption (A) is stable and holds for a large class of manifolds, we now work out the above connections in the case of odd-dimensional manifolds which are isometric to $\mathbb{R}^{2n+1}$ outside compact sets. This covers the physically interesting case of three space dimensions. Our results could be extended to even dimensions by a straightforward analysis of the logarithmic terms that are known to be present in the corresponding expansion (5.3).

Proposition 5.1. *Let (Σ^{2n+1}, g) with $n \geq 1$ be a complete Riemannian manifold which is asymptotically Euclidean in the sense that there exist compact subsets $K_1 \subset \Sigma$ and $K_2 \subset \mathbb{R}^{2n+1}$ such that $\Sigma \setminus K_1$ is isometric to $\mathbb{R}^{2n+1} \setminus K_2$. Then the operator B in* (5.3) *vanishes and condition (A) in* (5.2) *is satisfied.*

Proof. By the above, it suffices to show that $B = 0$. Following [7], we introduce the Sobolev space $W(\Lambda^\bullet T^*\Sigma)$ as the completion of $\Omega^\bullet_0(\Sigma)$ with respect to the quadratic form

$$\int_U |\alpha|^2 \, d\mu_\Sigma + \int_\Sigma \Big(|d\alpha|^2 + |\delta\alpha|^2 \Big) \, d\mu_\Sigma \,, \tag{5.4}$$

where U is a non-empty relatively compact open subset of Σ. Note that for $\Sigma = \mathbb{R}^{2n+1}$, the space $\{\alpha \in W(\Lambda^\bullet T^*\Sigma) \mid d\alpha + \delta\alpha = 0\}$ is zero provided that $n \geq 1$. As shown in [7, Theorem 0.6], the number

$$\dim \frac{\{\alpha \in W(\Lambda^\bullet T^*\Sigma) \mid d\alpha + \delta\alpha = 0\}}{\ker(\Delta)}$$

depends only on the geometry of Σ near infinity. Therefore, it is enough to show that the range of B is contained in $W(\Lambda^\bullet T^*\Sigma)$.

To this end, we must show that for every zero-resonance state $u \in \operatorname{rg} B$, there is a sequence $u_n \in \Omega^\bullet_0(\Sigma)$ which converges to u in $W(\Lambda^\bullet T^*\Sigma)$. If $\chi \in C^\infty_0(\mathbb{R})$ is an even real-valued function with $\int_\mathbb{R} \chi(x)\, dx = 0$, then its Fourier transform $\hat{\chi}(\lambda) \in \mathcal{S}(\mathbb{R})$ is a Schwartz function that vanishes at $\lambda = 0$. We choose the function χ with the additional property that

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\hat{\chi}\big(\sqrt{z}\big)}{\sqrt{z}} \, dz = 1 \,.$$

Moreover, for any $\varepsilon > 0$ we define

$$\chi_\varepsilon(x) = \chi(\varepsilon x) \qquad \text{and thus} \qquad \hat{\chi}_\varepsilon(\lambda) = \frac{1}{\varepsilon}\, \hat{\chi}\Big(\frac{\lambda}{\varepsilon}\Big).$$

Since B has finite rank, there exists a compactly supported section $v \in \Omega^\bullet_0(\Sigma)$ such that $u = Bv$. By finite propagation speed of the operator $\cos(t\Delta^{\frac{1}{2}})$, the section $u_\varepsilon := \sqrt{\varepsilon}\, \hat{\chi}_\varepsilon(\Delta^{\frac{1}{2}})(v)$ is again compactly supported. We want to show that

$$\lim_{\varepsilon \searrow 0} u_\varepsilon = u \qquad \text{with convergence in } W(\Lambda^\bullet T^*\Sigma) \,.$$

First, it follows from (5.3) that u_ε converges in $\mathcal{H}_{-1}$ to u. This implies in particular that
$$\lim_{\varepsilon \searrow 0} \int_U |u_\varepsilon - u|^2 \, d\mu_\Sigma = 0$$
(where U is again the relatively compact set in (5.4)). Next, again using (5.3),
$$\int_\Sigma \big(|du_\varepsilon|^2 + |\delta u_\varepsilon|^2 \big) \, d\mu_\Sigma = \langle u_\varepsilon, \Delta u_\varepsilon \rangle = \varepsilon \int_0^\infty z \, |\hat{\chi}_\varepsilon(\sqrt{z})|^2 \, d\langle v, E_z v\rangle \xrightarrow{\varepsilon \searrow 0} 0 \,,$$
showing that u_ε converges in W. Since it converges in $\mathcal{H}_{-1}$ to u, the limit in W is again u. ◻

5.2. **Construction of κ.** We assume throughout this section that condition (A) in (5.2) holds. We choose the Krein space $\mathcal{K}$ as
$$\mathcal{K} = \big(-\Omega^{0\perp}_{(2)}(\Sigma) \oplus \Omega^{1\perp}_{(2)}(\Sigma) \big) \otimes_{\mathbb{R}} \mathbb{C} \,.$$
Our assumptions imply that the operator Δ^s has a trivial kernel on $\Omega^{p\perp}_{(2)}(\Sigma)$ and is densely defined for all $s \in \mathbb{R}$. We introduce the spaces
$$\mathcal{H}^s = \overline{\mathcal{D}\big(\Delta^{\frac{s}{2}}\big)}^{\|.\|_s} \,,$$
where the bar denotes the completion with respect to the norm $\|\phi\|_s := \|\Delta^{\frac{s}{2}} \phi\|$ of the subspaces $\mathcal{D}(\Delta^{\frac{s}{2}}) \subset \Omega^{p\perp}_{(2)}(\Sigma)$. It follows from the spectral calculus that $\mathcal{H}^s$ is the topological dual of $\mathcal{H}^{-s}$. Moreover, it is obvious that
$$\mathcal{D}(\Delta^s) \subset \mathcal{H}^{2s} \,,$$
with continuous inclusion. Next, the following map is continuous:
$$\Delta^t \;:\; \mathcal{H}^s \to \mathcal{H}^{s-2t} \qquad \text{for all } t \in \mathbb{R} \,.$$
Furthermore, using that Δ commutes with all projections onto the form degree and $\Delta = (d_\Sigma + \delta_\Sigma)^2$, we also have the continuous mappings
$$d_\Sigma, \delta_\Sigma \;:\; \mathcal{H}^s \to \mathcal{H}^{s-1} \,,$$
which commute with Δ^t in the sense that
$$\Delta^t \, d_\Sigma = d_\Sigma \, \Delta^t \qquad \text{and} \qquad \Delta^t \, \delta_\Sigma = \delta_\Sigma \, \Delta^t$$
as continuous operators from $\mathcal{D}\big(\Delta^s\big)$ to $\mathcal{D}\big(\Delta^{s-2t-1}\big)$. Finally, the adjoints with respect to the dual pairings $\mathcal{H}^s$ and $\mathcal{H}^{-s}$ are computed as usual, i.e.
$$(\Delta^t)^* = \Delta^t \,, \qquad d_\Sigma^* = \delta_\Sigma \,, \qquad \delta_\Sigma^* = d_\Sigma \,.$$

In the following computations, by $\langle \cdot, \cdot \rangle$ we denote the dual pairing between the spaces $\mathcal{H}^s$ and $\mathcal{H}^{-s}$. We define the maps τ and κ by
$$\begin{aligned} \tau \;:\; & \big(\Omega^{0\perp}(\Sigma) \oplus \Omega^{1\perp}(\Sigma) \big)^2 \to \mathcal{K} \,, \\ & \begin{pmatrix} \mathfrak{f} \\ \dot{\mathfrak{f}} \end{pmatrix} \mapsto \Big(\Delta^{\frac{1}{4}} \mathfrak{f}_0 + i \Delta^{-\frac{1}{4}} \dot{\mathfrak{f}}_0 \Big) \oplus \Big(\Delta^{\frac{1}{4}} \mathfrak{f}_\Sigma + i \Delta^{-\frac{1}{4}} \dot{\mathfrak{f}}_\Sigma \Big) \end{aligned} \tag{5.5}$$
$$\kappa \;:\; C^\infty_0(M) \to \mathcal{K} \,, \quad f \mapsto \tau(P_\perp \Psi_0^{Gf}) \,, \tag{5.6}$$
where $P_\perp$ is the orthogonal projection onto $\Omega^{\bullet\perp}(\Sigma)$.

Proposition 5.2. *The mapping κ has the following properties:*

(i) $\kappa(\Box f) = 0$ *for all* $f \in \Omega^1_0(M)$

(ii) $\langle \kappa(f), \kappa(f) \rangle \geq 0$ *if* $\delta f = 0$.
(iii) $\mathrm{Im}\left(\langle \kappa f, \kappa g \rangle\right) = G(f,g)$ *for all* $f, g \in \Omega^1_0(M)$ *with* $\Psi^{Gf}_0 \perp \ker \Delta$.
(v) $\mathrm{span}_{\mathbb{C}} \mathrm{Rg}(\kappa)$ *is dense in* $\mathcal{K}$.
(vi) $\langle \kappa(df), \kappa(g) \rangle = 0$ *for all* $f \in \Omega^0_1(M)$, $g \in \Omega^1_0(M)$ *with* $\delta g = 0$.

Proof. The properties (i) and (v) hold by construction. We set

$$\begin{pmatrix} \mathfrak{f} \\ \dot{\mathfrak{f}} \end{pmatrix}(t) = P_\perp \Psi^{Gf}_t \,,$$

and similarly for g. Then the computation

$$\begin{aligned}
&\mathrm{Im}\Big\langle \tau\Big(\begin{pmatrix} \mathfrak{f} \\ \dot{\mathfrak{f}} \end{pmatrix}\Big), \tau\Big(\begin{pmatrix} \mathfrak{g} \\ \dot{\mathfrak{g}} \end{pmatrix}\Big) \Big\rangle \\
&\quad = -\mathrm{Im}\Big\langle \Delta^{\frac{1}{4}} \mathfrak{f}_0 + i \Delta^{-\frac{1}{4}} \dot{\mathfrak{f}}_0,\, \Delta^{\frac{1}{4}} \mathfrak{g}_0 + i \Delta^{-\frac{1}{4}} \dot{\mathfrak{g}}_0 \Big\rangle \\
&\qquad + \mathrm{Im}\Big\langle \Delta^{\frac{1}{4}} \mathfrak{f}_\Sigma + i \Delta^{-\frac{1}{4}} \dot{\mathfrak{f}}_\Sigma,\, \Delta^{\frac{1}{4}} \mathfrak{g}_\Sigma + i \Delta^{-\frac{1}{4}} \dot{\mathfrak{g}}_\Sigma \Big\rangle \\
&\quad = -\big\langle \Delta^{\frac{1}{4}} \mathfrak{f}_0, \Delta^{-\frac{1}{4}} \dot{\mathfrak{g}}_0 \big\rangle + \big\langle \Delta^{-\frac{1}{4}} \dot{\mathfrak{f}}_0, \Delta^{\frac{1}{4}} \mathfrak{g}_0 \big\rangle \\
&\qquad + \big\langle \Delta^{\frac{1}{4}} \mathfrak{f}_\Sigma, \Delta^{-\frac{1}{4}} \dot{\mathfrak{g}}_\Sigma \big\rangle - \big\langle \Delta^{-\frac{1}{4}} \dot{\mathfrak{f}}_\Sigma, \Delta^{\frac{1}{4}} \mathfrak{g}_\Sigma \big\rangle \\
&\quad = -\big\langle \mathfrak{f}_0, \dot{\mathfrak{g}}_0 \big\rangle + \big\langle \dot{\mathfrak{f}}_0, \mathfrak{g}_0 \big\rangle + \big\langle \mathfrak{f}_\Sigma, \dot{\mathfrak{g}}_\Sigma \big\rangle - \big\langle \dot{\mathfrak{f}}_\Sigma, \mathfrak{g}_\Sigma \big\rangle = \sigma\Big(\begin{pmatrix} \mathfrak{f} \\ \dot{\mathfrak{f}} \end{pmatrix}, \begin{pmatrix} \mathfrak{g} \\ \dot{\mathfrak{g}} \end{pmatrix}\Big)
\end{aligned}$$

together with (5.1) yields (iii). To prove (ii), we first note that

$$\begin{aligned}
&\Big\langle \tau\Big(\begin{pmatrix} \mathfrak{f} \\ \dot{\mathfrak{f}} \end{pmatrix}\Big), \tau\Big(\begin{pmatrix} \mathfrak{f} \\ \dot{\mathfrak{f}} \end{pmatrix}\Big) \Big\rangle \\
&\quad = -\big\langle \Delta^{\frac{1}{2}} \mathfrak{f}_0, \mathfrak{f}_0 \big\rangle - \big\langle \Delta^{-\frac{1}{2}} \dot{\mathfrak{f}}_0, \dot{\mathfrak{f}}_0 \big\rangle + \big\langle \Delta^{\frac{1}{2}} \mathfrak{f}_\Sigma, \mathfrak{f}_\Sigma \big\rangle + \big\langle \Delta^{-\frac{1}{2}} \dot{\mathfrak{f}}_\Sigma, \dot{\mathfrak{f}}_\Sigma \big\rangle \,. \qquad (5.7)
\end{aligned}$$

Since Gf solves the wave equation,

$$0 = \Box \mathfrak{f} = (-\ddot{\mathfrak{f}}_0 - \Delta \mathfrak{f}_0)\, dt + (-\ddot{\mathfrak{f}}_\Sigma - \Delta \mathfrak{f}_\Sigma) \qquad (5.8)$$

Moreover, if $\delta f = 0$, we have

$$0 = \delta \mathfrak{f} = -\dot{\mathfrak{f}}_0 - \delta_\Sigma \mathfrak{f}_\Sigma \,. \qquad (5.9)$$

The last equality in (5.9) can be verified with the computation

$$\begin{aligned}
\langle \delta \mathfrak{f}, \varphi \rangle &= \langle f, d\varphi \rangle = \langle \mathfrak{f}_0\, dt + \mathfrak{f}_\Sigma, \dot{\varphi}\, dt + d_\Sigma \varphi \rangle \\
&= \langle \mathfrak{f}_0, \dot{\varphi} \rangle - \langle \mathfrak{f}_\Sigma, d_\Sigma \varphi \rangle \overset{(*)}{=} -\langle \dot{\mathfrak{f}}_0, \varphi \rangle - \langle d_\Sigma \mathfrak{f}_\Sigma, \varphi \rangle = -\langle (\dot{\mathfrak{f}}_0 + d_\Sigma \mathfrak{f}_\Sigma), \varphi \rangle \,,
\end{aligned}$$

where in $(*)$ we integrated by parts. Then

$$\begin{aligned}
&-\big\langle \Delta^{-\frac{1}{2}} \dot{\mathfrak{f}}_0, \dot{\mathfrak{f}}_0 \big\rangle + \big\langle \Delta^{\frac{1}{2}} \mathfrak{f}_\Sigma, \mathfrak{f}_\Sigma \big\rangle \\
&\quad = -\big\langle \Delta^{-\frac{1}{2}} (d_\Sigma \delta_\Sigma) \dot{\mathfrak{f}}_\Sigma, \dot{\mathfrak{f}}_\Sigma \big\rangle + \big\langle \Delta^{-\frac{1}{2}} (d_\Sigma \delta_\Sigma + \delta_\Sigma d_\Sigma) \mathfrak{f}_\Sigma, \mathfrak{f}_\Sigma \big\rangle \\
&\quad = \big\langle \Delta^{-\frac{1}{2}}\, \delta_\Sigma d_\Sigma \mathfrak{f}_\Sigma, \mathfrak{f}_\Sigma \big\rangle = \big\langle \Delta^{-\frac{1}{2}}\, d_\Sigma \mathfrak{f}_\Sigma, d_\Sigma \mathfrak{f}_\Sigma \big\rangle \geq 0 \,.
\end{aligned}$$

Moreover, differentiating (5.9) with respect to t and using (5.8) gives

$$\delta_\Sigma \dot{\mathfrak{f}}_\Sigma = \Delta \mathfrak{f}_0 \,.$$

Hence

$$\begin{aligned}
&-\big\langle \Delta^{\frac{1}{2}} \mathfrak{f}_0, \mathfrak{f}_0 \big\rangle + \big\langle \Delta^{-\frac{1}{2}} \dot{\mathfrak{f}}_\Sigma, \dot{\mathfrak{f}}_\Sigma \big\rangle \\
&\quad = -\big\langle \Delta^{-\frac{3}{2}} \Delta \mathfrak{f}_0, \Delta \mathfrak{f}_0 \big\rangle + \big\langle \Delta^{-\frac{3}{2}} \Delta \dot{\mathfrak{f}}_\Sigma, \dot{\mathfrak{f}}_\Sigma \big\rangle \\
&\quad = -\big\langle \Delta^{-\frac{3}{2}} \delta_\Sigma \dot{\mathfrak{f}}_\Sigma, \delta_\Sigma \dot{\mathfrak{f}}_\Sigma \big\rangle + \big\langle \Delta^{-\frac{3}{2}} (d_\Sigma \delta_\Sigma + \delta_\Sigma d_\Sigma) \dot{\mathfrak{f}}_\Sigma, \dot{\mathfrak{f}}_\Sigma \big\rangle \\
&\quad = \big\langle \Delta^{-\frac{3}{2}} \delta_\Sigma d_\Sigma \dot{\mathfrak{f}}_\Sigma, \dot{\mathfrak{f}}_\Sigma \big\rangle = \big\langle \Delta^{-\frac{3}{2}} d_\Sigma \dot{\mathfrak{f}}_\Sigma, d_\Sigma \dot{\mathfrak{f}}_\Sigma \big\rangle \geq 0\,.
\end{aligned}$$

This shows (ii).

In order to prove (vi), we polarize (5.7) to obtain

$$\begin{aligned}
\langle \kappa(d\varphi), \kappa(g) \rangle &= -\big\langle \Delta^{\frac{1}{2}} \dot{\varphi}, \mathfrak{g}_0 \big\rangle - \big\langle \Delta^{-\frac{1}{2}} \ddot{\varphi}, \dot{\mathfrak{g}}_0 \big\rangle + \big\langle \Delta^{\frac{1}{2}} d_\Sigma \varphi, \mathfrak{g}_\Sigma \big\rangle + \big\langle \Delta^{-\frac{1}{2}} d_\Sigma \dot{\varphi}, \dot{\mathfrak{g}}_\Sigma \big\rangle \\
&= -\big\langle \Delta^{\frac{1}{2}} \dot{\varphi}, \mathfrak{g}_0 \big\rangle + \big\langle \Delta^{\frac{1}{2}} \varphi, \dot{\mathfrak{g}}_0 \big\rangle + \big\langle \Delta^{\frac{1}{2}} d_\Sigma \varphi, \mathfrak{g}_\Sigma \big\rangle + \big\langle \Delta^{-\frac{1}{2}} d_\Sigma \dot{\varphi}, \dot{\mathfrak{g}}_\Sigma \big\rangle \\
&= -\big\langle \Delta^{\frac{1}{2}} \dot{\varphi}, \mathfrak{g}_0 \big\rangle - \big\langle \Delta^{\frac{1}{2}} \varphi, \delta_\Sigma \mathfrak{g}_\Sigma \big\rangle + \big\langle \Delta^{\frac{1}{2}} d_\Sigma \varphi, \mathfrak{g}_\Sigma \big\rangle + \big\langle \Delta^{-\frac{1}{2}} d_\Sigma \dot{\varphi}, \dot{\mathfrak{g}}_\Sigma \big\rangle = 0\,,
\end{aligned}$$

where we have used the wave equations $\Box G\varphi = 0$ and $\Box \mathfrak{g} = 0$ together with the identity $0 = \delta \mathfrak{g} = -\dot{\mathfrak{g}}_0 - \delta_\Sigma \mathfrak{g}_\Sigma$. ☐

In view of (4.8), we define

$$G_Z(f,g) = G(f,g) - \mathrm{Im} \langle \kappa(f), \kappa(g) \rangle\,. \tag{5.10}$$

Proposition 5.2 (iii) has the following implication.

Corollary 5.3. *The symplectic vector space Z defined by* (4.9) *is canonically isomorphic to* $\ker \Delta \oplus \ker \Delta$ *with the standard symplectic structure*

$$\sigma((f_1, f_2), (g_1, g_2)) = \langle f_1, g_2 \rangle_{L^2(\Sigma)} - \langle f_2, g_1 \rangle_{L^2(\Sigma)}\,,$$

associated with the usual inner product on $L^2(\Sigma) \oplus L^2(\Sigma)$ *and the complex structure* $\mathfrak{J}(f_1, f_2) = (-f_2, f_1)$.

Summarizing the results of this section, we come to the following conclusion.

Theorem 5.4. *Let* (Σ, g) *be a Riemannian manifold satisfying assumption* (A) *in* (5.2). *The above mapping κ together with the form G_Z given by* (5.10) *and the complex structure $\mathfrak{J}$ given above defines a generalized Fock representation (see Section 4.4) in the ultrastatic space-time* $(\mathbb{R} \times \Sigma, dt^2 - g)$, *thereby giving rise to a Gupta-Bleuler representation of $\mathcal{F}$ (see Section 4.2).*

5.3. **Positivity of the Energy and the Microlocal Spectrum Condition.** It is obvious from definition 5.5 that τ is injective and that its image is dense in $\mathcal{K}$. We introduce the operator H on $\mathfrak{K}$ by

$$H = \begin{pmatrix} \Delta^{\frac{1}{2}} & 0 \\ 0 & \Delta^{\frac{1}{2}} \end{pmatrix} \tag{5.11}$$

(acting in the first component on functions and on the second component on one-forms). Then

$$\begin{aligned} H\,\tau\begin{pmatrix}\mathfrak{f}\\ \dot{\mathfrak{f}}\end{pmatrix} &= \Big(\Delta^{-\frac{1}{4}}\Delta\mathfrak{f}_0 + i\Delta^{\frac{1}{4}}\dot{\mathfrak{f}}_0\Big) \oplus \Big(\Delta^{-\frac{1}{4}}\Delta\mathfrak{f}_\Sigma + i\Delta^{\frac{1}{4}}\dot{\mathfrak{f}}_\Sigma\Big) \\ &= i\Big(\Delta^{\frac{1}{4}}\dot{\mathfrak{f}}_0 + i\Delta^{-\frac{1}{4}}(-\Delta)\mathfrak{f}_0\Big) \oplus \Big(\Delta^{\frac{1}{4}}\dot{\mathfrak{f}}_\Sigma + i\Delta^{-\frac{1}{4}}(-\Delta)\mathfrak{f}_\Sigma\Big) \\ &= i\tau\begin{pmatrix}0 & 1\\ -\Delta & 0\end{pmatrix}\begin{pmatrix}\mathfrak{f}\\ \dot{\mathfrak{f}}\end{pmatrix} \overset{(5.8)}{=} i\partial_t\,\tau\begin{pmatrix}\mathfrak{f}\\ \dot{\mathfrak{f}}\end{pmatrix}. \end{aligned}$$

Assume that $\delta f = 0$. Then, similar as above,

$$\begin{aligned} &\left\langle \tau\left(\begin{pmatrix}\mathfrak{f}\\ \dot{\mathfrak{f}}\end{pmatrix}\right), H\,\tau\left(\begin{pmatrix}\mathfrak{f}\\ \dot{\mathfrak{f}}\end{pmatrix}\right)\right\rangle \\ &= -\langle\Delta\mathfrak{f}_0, \mathfrak{f}_0\rangle - \langle\dot{\mathfrak{f}}_0, \dot{\mathfrak{f}}_0\rangle + \langle\Delta\mathfrak{f}_\Sigma, \mathfrak{f}_\Sigma\rangle + \langle\dot{\mathfrak{f}}_\Sigma, \dot{\mathfrak{f}}_\Sigma\rangle \\ &= \langle d_\Sigma\mathfrak{f}_\Sigma, d_\Sigma\mathfrak{f}_\Sigma\rangle + \langle\Delta^{-1}d_\Sigma\dot{\mathfrak{f}}_\Sigma, d_\Sigma\dot{\mathfrak{f}}_\Sigma\rangle \geq 0\,. \end{aligned}$$

Proposition 5.5. *The mapping κ satisfies the*

(iv) *microlocal spectrum condition:*

$$\mathrm{WF}(\kappa(.)) \subset J^+$$

Proof. Let T_t be the operator that shifts functions and distributions by $t \in \mathbb{R}$ in time. By construction of H, we know that

$$\kappa(T_{-t}f) = e^{iHt}\,\kappa(f)\,.$$

For a given point $x_0 \in \Sigma$, we choose a chart x and a bundle chart. Let $\chi \in \Omega^1_0(\Sigma)$ be any smooth form supported in our chart with $\chi(x_0) \neq 0$. The following computation will be carried out in local coordinates. Let $u_\xi \in \mathcal{D}'(\mathbb{R})$ be the family of distributions

$$u_\xi(g) = \kappa(\chi\,e^{-i\xi x} \otimes g)\,.$$

This family is polynomially bounded in ξ and, by construction,

$$T_t\,u_\xi = e^{iHt}\,u_\xi\,.$$

As a consequence,

$$u_\xi(\chi * g) = \sqrt{2\pi}\int_0^\infty \hat{\chi}(\lambda)\,dE_\lambda\,u_\xi(g)\,,$$

where dE_λ is the spectral measure of the generator (5.11), considered as a self-adjoint operator on the Hilbert space $L^2(\Sigma) \otimes \mathbb{C}^2$.

Choosing a test function $\eta \in C^\infty_0(\mathbb{R})$, we have

$$\begin{aligned} &\kappa\Big(\chi(x)\,e^{-i\xi x} \otimes (\eta * \eta)(t)\,e^{-i\xi_0 t}\Big) = u_\xi\Big((\eta * \eta)\,e^{-i\xi_0 \cdot}\Big) \\ &= u_\xi\Big(\big(\eta e^{-i\xi_0 \cdot}\big) * \big(\eta e^{-i\xi_0 \cdot}\big)\Big) = \sqrt{2\pi}\int_0^\infty \hat{\eta}(\lambda - \xi_0)\,dE_\lambda\,u_\xi\big(\eta\,e^{-i\xi_0 \cdot}\big)\big). \end{aligned}$$

Taking the Hilbert space norm, one sees that $\|\eta\,e^{-i\xi_0 \cdot})\|$ is polynomially bounded in (ξ, ξ_0), whereas the the operator norm of the spectral integral decays rapidly in ξ_0. We thus obtain rapid decay in (ξ, ξ_0) in a conic neighborhood of any direction $(\tilde{\xi}, \tilde{\xi}_0)$ with $\tilde{\xi}_0 < 0$. □

We remark that this result could also be inferred somewhat less directly from [31, Theorem 2.8].

6. Construction of Gupta-Bleuler representations

In this section, we show that Gupta-Bleuler representations and states exist for a large class of globally hyperbolic space-times. Thus let (M, g) be a globally hyperbolic space-time. According to [3], the manifold admits a smooth foliation $(\Sigma_t)_{t\in\mathbb{R}}$ by Cauchy hypersurfaces. Assume there exists a metric g on Σ_0 such that (Σ_0, g) satisfies condition (A) in (5.2). Then, using the constructions in [26], there is a globally hyperbolic space-time $(\tilde{M}, \tilde{g})$ which is future-isometric to (M, g) and past isometric to the ultrastatic space-time $(\mathbb{R}\times\Sigma_0, dt^2 - g)$. On $\mathbb{R}\times\Sigma$, we choose κ as in Section 5.

By propagation of singularities (see [21, Theorem 23.2.9]) and lemma 3.5 we have the following.

Lemma 6.1. *Suppose that U is an open neighborhood of a Cauchy surface. Assume that $\kappa : C_0^\infty(U) \to \mathcal{K}$ satisfies properties in section 4.4 in U, i.e.*

(i)' $\kappa(\Box f) = 0$ *for all* $f \in \Omega^1_0(U)$.

(ii)' $\langle \kappa(f), \kappa(f)\rangle \geq 0$ *if* $f \in \Omega^1_0(U)$ *with* $\delta f = 0$.

(iii)' *There is a bilinear form G_Z on $\Omega^1_0(U)\times\Omega^1_0(U)$ with smooth integral kernel and the following properties:*

$$\mathrm{Im}\langle \kappa(f), \kappa(g)\rangle + G_Z(f,g) = G(f,g)$$

The vector space $Z_U := \Omega^1_0(U)/\{f \,|\, G_Z(f,\cdot) = 0\}$ *is finite dimensional*

$$G_Z(f, \delta g) = 0 \qquad \text{for all } f\in\Omega^1_0(U) \text{ and } g \in \Omega^2_0(U)$$

(iv)'

$$\mathrm{WF}(\kappa)\cap U \subseteq \left\{(x,\xi)\in \dot{T}^*U \,|\, \xi\in J^+\right\}.$$

(v)' $\mathrm{span}_{\mathbb{C}}\kappa(\Omega^1_0(U))$ *is dense in* $\mathcal{K}$.

(vi)' $\langle\kappa(df), \kappa(g)\rangle = 0$ *for all* $f\in\Omega^0_1(U)$, $g\in\Omega^1_0(U)$ *with* $\delta g = 0$.

Then there exists a unique extension $\tilde{\kappa} : C_0^\infty(M)\to\mathcal{K}$ that satisfying (i) everywhere. This unique extension then satisfies (ii),(iii),(iv), (v) and (vi) everywhere.

One can now construct a Gupta-Bleuler representation on $(\tilde{M}, \tilde{g})$ by constructing the generalized Fock representation on the ultrastatic space-time $(\mathbb{R}\times\Sigma_0, dt^2 - g)$. Since there exists a neighborhood $\tilde{U}_1$ of a Cauchy surface in $(\tilde{M}, \tilde{g})$ that is isometric to a neighborhood U_{us} of a Cauchy surface in $(\mathbb{R}\times\Sigma_0, dt^2 - g)$ one can use the above Lemma to migrate this Gupta-Bleuler representation to $(\tilde{M}, \tilde{g})$ by first restricting κ to U_{us} and then extending from $\tilde{U}$ to $\tilde{M}$. Similarly, since there exists a neighborhood U of a Cauchy surface in (M, g) that is isometric to a neighborhood $\tilde{U}_2$ of a Cauchy surface in $(\tilde{M}, \tilde{g})$ one constructs a Gupta-Bleuler representation on (M, g) by restricting κ to $\tilde{U}_2$ and then extending it to M.

Appendix A. The Gauge Parameter and other Commutator Relations

A common procedure in physics is to add a gauge fixing term to the classical Lagrangian of the electromagnetic field. This leads to the modified wave equation

$$\Box_\xi A = 0 \qquad \text{where} \qquad \Box_\xi = \delta d + \frac{1}{\xi}\, d\delta\,,$$

which involves the *gauge parameter* $\xi\in(0,\infty)$. Choosing $\xi = 1$, the so-called *Feynman gauge*, gives again the ordinary wave equation $\Box A = 0$. Another common gauge is the *Landau gauge* $\xi \searrow 0$ (where the limit is taken after computing expectation values).

Working with a gauge-parameter $\xi \neq 1$ has the unpleasant consequence that the modified wave operator $\Box_\xi$ is not normally hyperbolic. But, in a chosen foliation, the modified wave equation can be written as a symmetric hyperbolic system (see for example [22]), showing that the Cauchy problem is well-posed, and that the propagation speed is finite. But the propagation speed will in general be faster than the speed of light. Moreover, the formulation as a symmetric hyperbolic system depends on the choice of the foliation. In particular, the advanced and retarded fundamental solutions depend on the foliation. The propagation with speed faster than light is not problematic from the physical point of view if one keeps in mind that all observable quantities still propagate at most with the speed of light.

We now show that working with different gauge parameters gives an equivalent description of the physical system. To this end, we will construct a bijection of the corresponding field algebras. In preparation, we relate the solutions of the modified wave equation to the solutions of the ordinary wave equation: We choose an operator $R : C^\infty_{\mathrm{sc}}(M) \to C^\infty_{\mathrm{sc}}(M)$ such that

$$\Box R f = f \qquad \text{for all } f \in C^\infty_{\mathrm{sc}}(M) \cap \ker(\Box_\xi)\,.$$

One method for constructing the operator R is to solve the Cauchy problem $\Box\phi = f$ for vanishing initial data on a Cauchy surface. In Minkowski space, a particular choice of the operator R is discussed in [23] and [17, Exercise 7.3]. There are of course many other choices. Suppose that $\Box_\xi \psi = 0$. Then, of course $\Box\delta\psi = 0$, and the calculation

$$\begin{aligned}
\Box\big(\mathbb{1} + (\xi^{-1}-1)\, dR\delta\big)\psi &= \big(d\delta + \delta d + (\xi^{-1}-1)\, d\delta d R\delta\big)\psi \\
&= \big(d\delta + \delta d + (\xi^{-1}-1)\, d\Box R\delta\big)\psi = \big(d\delta + \delta d + (\xi^{-1}-1)\, d\delta\big)\psi \\
&= \big(\delta d + \frac{1}{\xi} d\delta\big)\psi = \Box_\xi \psi = 0
\end{aligned}$$

shows that the following operator maps solutions of the corresponding wave equations into each other,

$$\mathfrak{I}_R := \mathbb{1} + (\xi^{-1}-1)\, dR\delta \;:\; \Omega^1_{\mathrm{sc}}(M) \cap \ker(\Box_\xi) \to \Omega^1_{\mathrm{sc}}(M) \cap \ker(\Box)\,.$$

One easily checks that $\mathfrak{I}_R$ is invertible with explicit inverse given by

$$\mathfrak{I}_R^{-1} = \mathbb{1} + (1-\xi)\, dR\delta.$$

We thus obtain a one-to-one correspondence between solutions of the ordinary and modified wave equations.

In order to extend this correspondence to the field operators, we use the following dual formulation. Assume that L is a given map $L : C^\infty_0(M) \to C^\infty_0(M)$ such that

$$f - L\Box f \in \mathrm{Im}(\Box) \qquad \text{for all } f \in C^\infty_0(M)\,.$$

Again, there are many possibilities to choose L. A particular choice is

$$L = \eta_- G^0_+ + \eta_+ G^0_-\,,$$

where η_+ and η_- are smooth functions with $\eta_+ + \eta_- = 1$, having past and future compact support, respectively (as before, $G^0_\pm$ denote the causal fundamental solutions for the scalar wave operator). Then the computation

$$\big(1 + (\xi^{-1}-1)\, dL\delta\big)\Box f = \Box_\xi f + (\xi^{-1}-1)\, d\big(L\Box - 1\big)\delta f$$

shows that the map

$$\mathfrak{I}_L := 1 + (\xi^{-1}-1)\, dL\delta$$

has the property

$$\mathfrak{I}_L \Box f \in \operatorname{Im}(\Box_\xi) \qquad \text{for all } f \in \Omega^1_0(M) \, .$$

This allows us to identify the field algebras of the modified and the ordinary wave operators via the relation

$$\tilde{A}(f) = A(\mathfrak{I}_L f) \, .$$

By the above, $\tilde{A}$ satisfies the equation

$$\Box_\xi \tilde{A} = 0$$

as an operator-valued distribution. The commutation relations become

$$\tilde{A}(f)\tilde{A}(g) - \tilde{A}(f)\tilde{A}(g) = -i \, G(\mathfrak{I}_L f, \mathfrak{I}_L g) \, .$$

Acknowledgments: We are grateful to Gilles Carron for helpful conversations. A.S. would like to thank the Vielberth foundation, Regensburg, for generous support.

References

[1] M.F. Atiyah, V.K. Patodi, and I.M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.
[2] C. Bär, N. Ginoux, and F. Pfäffle, *Wave Equations on Lorentzian Manifolds and Quantization*, ESI Lectures in Mathematics and Physics, European Mathematical Society (EMS), Zürich, 2007.
[3] A.N. Bernal and M. Sánchez, *On smooth Cauchy hypersurfaces and Geroch's splitting theorem*, arXiv:gr-qc/0306108, Comm. Math. Phys. **243** (2003), no. 3, 461–470.
[4] K. Bleuler, *Eine neue Methode zur Behandlung der longitudinalen und skalaren Photonen*, Helvetica Phys. Acta **23** (1950), 567–586.
[5] R. Brunetti and K. Fredenhagen, *Interacting quantum fields in curved space: renormalizability of ϕ^4*, arXiv:gr-qc/9701048, Operator Algebras and Quantum Field Theory (Rome, 1996), Int. Press, Cambridge, MA, 1997, pp. 546–563.
[6] R. Brunetti, K. Fredenhagen, and M. Köhler, *The microlocal spectrum condition and Wick polynomials of free fields on curved spacetimes*, arXiv:gr-qc/9510056, Comm. Math. Phys. **180** (1996), no. 3, 633–652.
[7] G. Carron, *Théorèmes de l'indice sur les variétés non-compactes*, J. Reine Angew. Math. **541** (2001), 81–115.
[8] ———, *Le saut en zéro de la fonction de décalage spectral*, J. Funct. Anal. **212** (2004), no. 1, 222–260.
[9] C. Dappiaggi and B. Lang, *Quantization of Maxwell's Equations on Curved Backgrounds and General Local Covariance*, arXiv:1104.1374v2 [gr-qc], Lett. Math. Phys. **101** (2012), no. 3, 265–287.
[10] C. Dappiaggi, K. Sanders, and T.-H. Hack, *Electromagnetism, local covariance, the aharonov-bohm effect and gauss' law*, arXiv:1211.6420 [math-ph] (2012).
[11] C. Dappiaggi and D. Siemssen, *Hadamard states for the vector potential on asymptotically flat spacetimes*, arXiv:1106.5575 [gr-qc] (2011).
[12] N. Dencker, *On the propagation of polarization sets for systems of real principal type*, J. Funct. Anal. **46** (1982), no. 3, 351–372.
[13] J. Dimock, *Algebras of local observables on a manifold*, Comm. Math. Phys. **77** (1980), no. 3, 219–228.
[14] ———, *Quantized electromagnetic field on a manifold*, Rev. Math. Phys. **4** (1992), no. 2, 223–233.
[15] C.J. Fewster and M.J. Pfenning, *A quantum weak energy inequality for spin-one fields in curved space-time*, arXiv:gr-qc/0303106, J. Math. Phys. **44** (2003), no. 10, 4480–4513.
[16] E.P. Furlani, *Quantization of the electromagnetic field on static space-times*, J. Math. Phys. **36** (1995), no. 3, 1063–1079.
[17] W. Greiner and J. Reinhardt, *Field quantization*, Springer-Verlag, Berlin, 1996.
[18] S.N. Gupta, *Theory of longitudinal photons in quantum electrodynamics*, Proc. Phys. Soc. Sect. A. **63** (1950), 681–691.

[19] S. Hollands and R.M. Wald, *Local Wick polynomials and time ordered products of quantum fields in curved spacetime*, arXiv:gr-qc/0103074, Comm. Math. Phys. **223** (2001), no. 2, 289–326.
[20] ———, *Existence of local covariant time ordered products of quantum field in curved spacetime*, arXiv:gr-qc/0111108, Comm. Math. Phys. **231** (2002), no. 2, 309–345.
[21] L. Hörmander, *The Analysis of Linear Partial Differential Operators. III*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 274, Springer-Verlag, Berlin, 1985.
[22] F. John, *Partial Differential Equations*, fourth ed., Applied Mathematical Sciences, vol. 1, Springer-Verlag, New York, 1991.
[23] B. Lautrup, *Canonical quantum electrodynamics in covariant gauges*, Mat. Fys. Medd. Dan. Vid. Selsk. **35** (1966), 11.
[24] R.B. Melrose, *The Atiyah-Patodi-Singer Index Theorem*, Research Notes in Mathematics, vol. 4, A K Peters Ltd., Wellesley, MA, 1993.
[25] ———, *Geometric Scattering Theory*, Stanford Lectures, Cambridge University Press, Cambridge, 1995.
[26] O. Müller, *Asymptotic flexibility of globally hyperbolic manifolds*, arXiv:1110.1037 [math.DG], C. R. Math. Acad. Sci. Paris **350** (2012), no. 7-8, 421–423.
[27] W. Müller, *Eta invariants and manifolds with boundary*, J. Differential Geom. **40** (1994), no. 2, 311–377.
[28] W. Müller and A. Strohmaier, *Scattering at low energies on manifolds with cylindrical ends and stable systoles*, arXiv:0907.3517 [math.AP], Geom. Funct. Anal. **20** (2010), no. 3, 741–778.
[29] M.E. Peskin and A. Tonomura, *The Aharonov-Bohm Effect*, Lecture Notes in Physics, vol. 340, Springer Verlag, Berlin, 1989.
[30] M.J. Radzikowski, *Micro-local approach to the Hadamard condition in quantum field theory on curved space-time*, Comm. Math. Phys. **179** (1996), no. 3, 529–553.
[31] A. Strohmaier, R. Verch, and M. Wollenberg, *Microlocal analysis of quantum fields on curved space-times: analytic wave front sets and Reeh-Schlieder theorems*, arXiv:math-ph/0202003, J. Math. Phys. **43** (2002), no. 11, 5514–5530.
[32] X.P. Wang, *Asymptotic expansion in time of the Schrödinger group on conical manifolds*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 6, 1903–1945.
[33] S. Weinberg, *The Quantum Theory of Fields. Vol. I*, Cambridge University Press, Cambridge, 1996, Foundations, Corrected reprint of the 1995 original.

Fakultät für Mathematik, Universität Regensburg, D-93040 Regensburg, Germany
E-mail address: finster@ur.de

Department of Mathematical Sciences, Loughborough University, Loughborough, LE11 3TU, United Kingdom
E-mail address: a.strohmaier@lboro.ac.uk