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of fluidic membranes and vesicles

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# Numerical computations of the dynamics of fluidic membranes and vesicles

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Vesicles and many biological membranes are made of two monolayers of lipid molecules and form closed lipid bilayers. The dynamical behaviour of vesicles is very complex and a variety of forms and shapes appear. Lipid bilayers can be considered as a surface fluid and hence the governing equations for the evolution include the surface (Navier–)Stokes equations, which in particular take the membrane viscosity into account. The evolution is driven by forces stemming from the curvature elasticity of the membrane. In addition, the surface fluid equations are coupled to bulk (Navier–)Stokes equations.

We introduce a parametric finite element method to solve this complex free boundary problem, and present the first three dimensional numerical computations based on the full (Navier–)Stokes system for several different scenarios. For example, the effects of the membrane viscosity, spontaneous curvature and area difference elasticity (ADE) are studied. In particular, it turns out, that even in the case of no viscosity contrast between the bulk fluids, the tank treading to tumbling transition can be obtained by increasing the membrane viscosity. In addition, we study the tank treading, tumbling and trembling behaviour for different spontaneous curvatures. We also study how features of equilibrium shapes in the ADE and spontaneous curvature models, like budding behaviour or starfish forms, behave in a shear flow.

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## I. INTRODUCTION

Lipid membranes consist of a bilayer of molecules, which have a hydrophilic head and two hydrophobic chains. These bilayers typically spontaneously form closed bag-like structures, which are called vesicles. It is observed that vesicles can attain a huge variety of shapes and some of them are similar to the biconcave shape of red blood cells. Since membranes play a fundamental role in many living systems, the study of vesicles is a very active research field in different scientific disciplines, see e.g. [1–4]. It is the goal of this paper to present a numerical approach to study the evolution of lipid membranes. We present several computations showing quite different shapes, and the influence of fluid flow on the membrane evolution.

Since the classical papers of Canham [5] and Helfrich [6], there has been a lot of work with the aim of describing equilibrium membrane shapes with the help of elastic membrane energies. Canham [5] and Helfrich [6] introduced a bending energy for a non-flat membrane, which is formulated with the help of the curvature of the membrane. In the class of fixed topologies the relevant energy density, in the simplest situation, is proportional to the square of the mean curvature  $\varkappa$ . The resulting energy functional is called the Willmore energy.

When computing equilibrium membrane shapes one has to take constraints into account. Lipid membranes have a very small compressibility, and hence can safely be modelled as locally incompressible. In addition, the presence of certain molecules in the surrounding fluid, for which the membrane is impermeable, leads to an osmotic pressure, which results in a constraint for the volume enclosed by the membrane. The minimal energetic model for lipid membranes consists of the Willmore mean curvature functional together with enclosed volume and surface area constraints. Already this simple model leads to quite different shapes including the biconcave red blood cell shapes, see [1].

Helfrich [6] introduced a variant of the Willmore energy, with the aim of modelling a possible asymmetry of the bilayer membrane. Helfrich [6] studied the functional  $\int (\varkappa - \bar{\varkappa})^2$ , where  $\bar{\varkappa}$  is a fixed constant, the so-called spontaneous curvature. It is argued that the origin of the spontaneous curvature is e.g. a different chemical environment on both sides of the membrane. We refer to [7] and [8] for a recent discussion, and for experiments in situations which lead to spontaneous curvature effects due to the chemical structure of the bilayer.

Typically there is yet another asymmetry in the bilayer leading to a signature in the membrane architecture. This results from the fact that the two membrane layers have a different number of molecules. Since the exchange of molecules between the layers is difficult, an imbalance is conserved during a possible shape change. The total

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area difference between the two layers is proportional to  $M = \int \kappa$ . Several models have been proposed, which describe the difference in the total number of molecules in the two layers with the help of the integrated mean curvature. The bilayer coupling model, introduced by Svetina and coworkers [9–11], assumes that the area per lipid molecule is fixed and assumes that there is no exchange of molecules between the two layers. Hence the total areas of the two layers are fixed, and on assuming that the two layers are separated by a fixed distance, one obtains, to the order of this distance, that the area difference can be approximated by the integrated mean curvature, see [9–11]. We note that a spontaneous curvature contribution is irrelevant in the bilayer coupling model as this would only add a constant to the energy as the area and integrated mean curvature are fixed.

Miao *et al.* [12] noted that in the bilayer coupling model budding always occurs continuously which is inconsistent with experiments. They hence studied a model in which the area of the two layers are not fixed but can expand or compress under stress. Given a relaxed initial area difference  $\Delta A_0$ , the total area difference  $\Delta A$ , which is proportional to the integrated mean curvature, can deviate from  $\Delta A_0$ . However, the total energy now has a contribution that is proportional to  $(\Delta A - \Delta A_0)^2$ . This term describes the elastic area difference stretching energy, see [1, 12], and hence one has to pay a price energetically to deviate from the relaxed area difference.

It is also possible to combine the area difference elasticity model (ADE-model) with a spontaneous curvature assumption, see Miao *et al.* [12] and Seifert [1]. However, the resulting energetical model is equivalent to an area difference elasticity model with a modified  $\Delta A_0$ , see [1] for a more detailed discussion.

It has been shown that the bilayer coupling model (BC-model) and the area difference elasticity model (ADE-model) lead to a multitude of shapes, which also have been observed in experiments with vesicles. Beside others, the familiar discocyte shapes (including the “shape” of a red blood cell), stomatocyte shapes, prolate shapes and pear-like shapes have been observed. In addition, the budding of membranes can be described, as well as more exotic shapes, like starfish vesicles. Moreover, higher genus shapes appear as global or local minima of the energies discussed above. We refer to [1, 12–15] for more details on the possible shapes appearing, when minimizing the energies in the ADE- and BC-models.

Configurational changes of vesicles and membranes cannot be described by energetical considerations alone, but have to be modelled with the help of appropriate evolution laws. Several authors considered an  $L^2$ -gradient flow dynamics of the curvature energies discussed above. Pure Willmore flow has been studied in [16–21], where the last two papers use a phase field formulation of the Willmore problem. Some authors also took other aspects, such as constraints on volume and area [19, 22, 23], as well as a constraint on the integrated mean curvature [19, 20], into account. The effect of different lipid com-

ponents in an  $L^2$ -gradient flow approach of the curvature energy has been studied in [24–30].

The above mentioned works considered a global constraint on the surface area. The membrane, however, is locally incompressible and hence a local constraint on the evolution of the membrane molecules should be taken into account. Several authors included the local inextensibility constraint by introducing an inhomogeneous Lagrange multiplier for this constraint on the membrane. This approach has been used within the context of different modelling and computational strategies such as the level set approach [31–34], the phase field approach [35–37], the immersed boundary method [38–40], the interfacial spectral boundary element method [41] and the boundary integral method [42].

The physically most natural way to consider the local incompressibility constraint makes use of the fact that the membrane itself can be considered as an incompressible surface fluid. This implies that a surface Navier–Stokes system has to be solved on the membrane. The resulting set of equations has to take forces stemming from the surrounding fluid and from the membrane elasticity into account. In total, bulk Navier–Stokes equations coupled to surface Navier–Stokes equations have to be solved. As the involved Reynolds numbers for vesicles are typically small one can often replace the full Navier–Stokes equations by the Stokes systems on the surface and in the bulk. The incompressibility condition in the bulk (Navier–)Stokes equations naturally leads to conservation of the volume enclosed by the membrane and the incompressibility condition on the surface leads a conservation of the membrane’s surface area. A model involving coupled bulk-surface (Navier–)Stokes equations has been proposed by Arroyo and DeSimone [43], and it is this model that we want to study numerically in this paper.

Introducing forces resulting from membrane energies in fluid flow models has been studied numerically before by different authors, [31–33, 36, 37, 40]. However, typically these authors studied simplified models, and either volume or surface constraints were enforced by Lagrange multipliers. In addition, either just the bulk or just the surface (Navier–)Stokes equations have been solved. The only work considering simultaneously bulk and surface Navier–Stokes equations are Arroyo *et al.* [44] and Barrett *et al.* [45, 46], where the former work is restricted to axisymmetric situations. In the present paper we are going to make use of the numerical method introduced in [45, 46].

The paper is organized as follows. In the next section we precisely state the mathematical model, consisting of the curvature elasticity model together with a coupled bulk-surface (Navier–)Stokes system. In Section III we introduce our numerical method which consists of an unfitted parametric finite element method for the membrane evolution. The curvature forcing is discretized and coupled to the Navier–Stokes system in a stable way using the finite element method for the fluid unknowns. Numerical computations in Section IV demonstrate that

we can deal with a variety of different membrane shapes and flow scenarios. In particular, we will study what influence the membrane viscosity, the area difference elasticity (ADE) and the spontaneous curvature have on the evolution of bilayer membranes in shear flow. We finish with some conclusions.

## II. A CONTINUUM MODEL FOR FLUIDIC MEMBRANES

We consider a continuum model for the evolution of biomembranes and vesicles, which consists of a curvature elasticity model for the membrane and the Navier–Stokes equations in the bulk and on the surface. The model is based on a paper by Arroyo and DeSimone [43], where in addition we also allow the curvature energy model to be an area difference elasticity model. We first introduce the curvature elasticity model and then describe the coupling to the surface and bulk Navier–Stokes equations.

The thickness of the lipid bilayer in a vesicle is typically three to four orders of magnitude smaller than the typical size of the vesicle. Hence the membrane can be modelled as a two dimensional surface  $\Gamma$  in  $\mathbb{R}^3$ . Given the principal curvatures  $\varkappa_1$  and  $\varkappa_2$  of  $\Gamma$ , one can define the mean curvature

$$\varkappa = \varkappa_1 + \varkappa_2$$

and the Gauß curvature

$$K = \varkappa_1 \varkappa_2$$

(as often in differential geometry we choose to take the sum of the principal curvatures as the mean curvature, instead of its mean value). The classical works of Canham [5] and Helfrich [6] derive a local bending energy, with the help of an expansion in the curvature, and they obtain

$$\int_{\Gamma} \left( \frac{\alpha}{2} \varkappa^2 + \alpha_G K \right) ds \quad (1)$$

as the total energy of a symmetric membrane. The parameters  $\alpha, \alpha_G$  have the dimension of energy and are called the bending rigidity  $\alpha$  and the Gaussian bending rigidity  $\alpha_G$ . If we consider closed membranes with a fixed topology, the term  $\int_{\Gamma} K ds$  is constant and hence we will neglect the Gaussian curvature term in what follows.

As discussed above, the total area difference  $\Delta A$  of the two lipid layers is, to first order, proportional to

$$M(\Gamma) = \int_{\Gamma} \varkappa ds.$$

Taking now into account that there is an optimal area difference  $\Delta A_0$ , the authors in [47–49] added a term proportional to

$$(M(\Gamma) - M_0)^2$$

to the curvature energy, where  $M_0$  is a fixed constant which is proportional to the optimal area difference.

For non-symmetric membranes a certain mean curvature  $\bar{\varkappa}$  can be energetically favourable. Then the elasticity energy (1) is modified to

$$\int_{\Gamma} \left( \frac{\alpha}{2} (\varkappa - \bar{\varkappa})^2 + \alpha_G K \right) ds.$$

The constant  $\bar{\varkappa}$  is called spontaneous curvature. Taking into account that  $\int_{\Gamma} \alpha_G K ds$  does not change for an evolution within a fixed topology class, the most general bending energy that we use in this paper is given by  $\alpha E(\Gamma)$  with the dimensionless energy

$$E(\Gamma) = \frac{1}{2} \int_{\Gamma} (\varkappa - \bar{\varkappa})^2 ds + \frac{\beta}{2} (M(\Gamma) - M_0)^2, \quad (2)$$

where  $\beta$  has the dimension  $(\frac{1}{\text{length}})^2$ .

We now consider a continuum model for the fluid flow on the membrane and in the bulk, inside and outside of the membrane. We assume that the closed, time dependent membrane  $(\Gamma(t))_{t \geq 0}$  lies inside a spatial domain  $\Omega \subset \mathbb{R}^3$ . For all times the membrane separates  $\Omega$  into an inner domain  $\Omega_-(t)$  and an outer domain  $\Omega_+(t) = \Omega \setminus \overline{\Omega_-(t)}$ . Denoting by  $\vec{u}$  the fluid velocity and by  $p$  the pressure, the bulk stress tensor is given by  $\underline{\underline{\sigma}} = 2\mu \underline{\underline{D}}(\vec{u}) - p \underline{\underline{Id}}$ , with  $\underline{\underline{D}}(\vec{u}) = \frac{1}{2}(\nabla \vec{u} + (\nabla \vec{u})^T)$  being the bulk rate-of-strain tensor. We assume that the Navier–Stokes system

$$\rho(\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u}) - \nabla \cdot \underline{\underline{\sigma}} = 0, \quad \nabla \cdot \vec{u} = 0$$

holds in  $\Omega_-(t)$  and  $\Omega_+(t)$ . Here  $\rho$  and  $\mu$  are the density and dynamic viscosity of the fluid, which can take different (constant) values  $\rho_{\pm}, \mu_{\pm}$  in  $\Omega_{\pm}(t)$ . Arroyo and DeSimone [43] used the theory of interfacial fluid dynamics, which goes back to Scriven [50], to introduce a relaxation dynamics for fluidic membranes. In this model the fluid velocity is assumed to be continuous across the membrane, the membrane is moved in the normal direction with the normal velocity of the bulk fluid and, in addition, the surface Navier–Stokes equations

$$\rho_{\Gamma} \partial_t^{\bullet} \vec{u} - \nabla_s \cdot \underline{\underline{\sigma}}_{\Gamma} = [\underline{\underline{\sigma}}]_{\Gamma}^+ \vec{\nu} + \alpha \vec{f}_{\Gamma}, \quad \nabla_s \cdot \vec{u} = 0$$

have to hold on  $\Gamma(t)$ . Here  $\rho_{\Gamma}$  is the surface material density,  $\partial_t^{\bullet}$  is the material derivative and  $\nabla_s, \nabla_s \cdot$  are the gradient and divergence operators on the surface. The surface stress tensor is given by

$$\underline{\underline{\sigma}}_{\Gamma} = 2\mu_{\Gamma} \underline{\underline{D}}_s(\vec{u}) - p_{\Gamma} \underline{\underline{P}}_{\Gamma},$$

where  $p_{\Gamma}$  is the surface pressure,  $\mu_{\Gamma}$  is the surface shear viscosity,  $\underline{\underline{P}}_{\Gamma}$  is the projection onto the tangent space and

$$\underline{\underline{D}}_s(\vec{u}) = \frac{1}{2} \underline{\underline{P}}_{\Gamma} (\nabla_s \vec{u} + (\nabla_s \vec{u})^T) \underline{\underline{P}}_{\Gamma}$$

is the surface rate-of-strain tensor. Furthermore, the term  $[\underline{\underline{\sigma}}]_{\Gamma}^+ \vec{\nu} = \underline{\underline{\sigma}}_+ \vec{\nu} - \underline{\underline{\sigma}}_- \vec{\nu}$  is the force exerted by the

bulk on the membrane, where  $\vec{\nu}$  denotes the exterior unit normal to  $\Omega_-(t)$ . The remaining term  $\alpha \vec{f}_\Gamma$  denotes the forces stemming from the elastic bending energy. These forces are given by the first variation of the bending energy  $\alpha E(\Gamma(t))$ , see [1, 43]. It turns out that  $\vec{f}_\Gamma$  points in the normal direction, i.e.  $\vec{f}_\Gamma = f_\Gamma \vec{\nu}$ , and we obtain, see [1, 51],

$$f_\Gamma = -\Delta_s \varkappa - (\varkappa - \bar{\varkappa}) |\nabla_s \vec{\nu}|^2 + \frac{1}{2} (\varkappa - \bar{\varkappa})^2 \varkappa + \beta (M(\Gamma) - M_0) (|\nabla_s \vec{\nu}|^2 - \varkappa^2) \quad \text{on } \Gamma(t). \quad (3)$$

Here  $\Delta_s$  is the surface Laplace operator,  $\nabla_s \vec{\nu}$  is the Weingarten map and  $|\nabla_s \vec{\nu}|^2 = \varkappa_1^2 + \varkappa_2^2$ . Assuming e.g. no-slip boundary conditions on  $\partial\Omega$ , the boundary of  $\Omega$ , we obtain that the total energy can only decrease, i.e.

$$\begin{aligned} & \frac{d}{dt} \left( \int_\Omega \frac{\rho}{2} |\vec{u}|^2 dx + \frac{\rho_\Gamma}{2} \int_\Gamma |\vec{u}|^2 ds + \alpha E(\Gamma) \right) \\ &= -2 \left( \int_\Omega \mu |\underline{D}(\vec{u})|^2 dx + \mu_\Gamma \int_\Gamma |\underline{D}_s(\vec{u})|^2 ds \right) \leq 0. \quad (4) \end{aligned}$$

We now non-dimensionalize the problem. We choose a time scale  $\tilde{t}$ , a length scale  $\tilde{x}$  and the resulting velocity scale  $\tilde{u} = \tilde{x}/\tilde{t}$ . Then we define the bulk and surface Reynolds numbers

$$\text{Re} = \tilde{x} \rho_+ \tilde{u} / \mu_+ \quad \text{and} \quad \text{Re}_\Gamma = \tilde{x} \rho_\Gamma \tilde{u} / \mu_\Gamma,$$

the bulk and surface pressure scales

$$\tilde{p} = \mu_+ / \tilde{t} \quad \text{and} \quad \tilde{p}_\Gamma = \mu_+ \tilde{x} / \tilde{t} = \mu_+ \tilde{u},$$

and

$$\begin{aligned} \tilde{\rho} &= \rho / \rho_+ = \begin{cases} 1 & \text{in } \Omega_+ \\ \rho_- / \rho_+ & \text{in } \Omega_- \end{cases}, \\ \tilde{\mu} &= \mu / \mu_+ = \begin{cases} 1 & \text{in } \Omega_+ \\ \mu_- / \mu_+ & \text{in } \Omega_- \end{cases}, \\ \tilde{\mu}_\Gamma &= \mu_\Gamma / (\mu_+ \tilde{x}), \end{aligned}$$

as well as the new independent variables  $\hat{x} = x/\tilde{x}$ ,  $\hat{t} = t/\tilde{t}$ . For the unknowns

$$\vec{\hat{u}} = \vec{u}/\tilde{u}, \quad \hat{p} = p/\tilde{p}, \quad \hat{p}_\Gamma = p_\Gamma/\tilde{p}_\Gamma$$

we now obtain the following set of equations (on dropping the  $\hat{\cdot}$ -notation for the new variables for ease of exposition)

$$\begin{aligned} & \text{Re} \tilde{\rho} (\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u}) - \tilde{\mu} \Delta \vec{u} + \nabla p = 0 \quad \text{in } \Omega_\pm(t), \\ & \text{Re}_\Gamma \tilde{\mu}_\Gamma \partial_t^* \vec{u} - \nabla_s \cdot (2 \tilde{\mu}_\Gamma \underline{D}_s(\vec{u}) - p_\Gamma \underline{P}_\Gamma) \\ &= [2 \tilde{\mu} \underline{D}(\vec{u}) - p \underline{\text{Id}}]_+^+ \vec{\nu} + \alpha \vec{f}_\Gamma^{\text{new}} \quad \text{on } \Gamma(t), \quad (5) \end{aligned}$$

with  $\vec{f}_\Gamma^{\text{new}} = f_\Gamma^{\text{new}} \vec{\nu}$ ,

$$f_\Gamma^{\text{new}} = -\Delta_s \varkappa - (\varkappa - \bar{\varkappa}^{\text{new}}) |\nabla_s \vec{\nu}|^2 + \frac{1}{2} (\varkappa - \bar{\varkappa}^{\text{new}})^2 \varkappa + \beta^{\text{new}} (M(\Gamma) - M_0^{\text{new}}) (|\nabla_s \vec{\nu}|^2 - \varkappa^2),$$

$\alpha^{\text{new}} = \alpha / (\mu_+ \tilde{u} \tilde{x}^2)$  and  $\bar{\varkappa}^{\text{new}} = \tilde{x} \bar{\varkappa}$ ,  $M_0^{\text{new}} = M_0 / \tilde{x}$ ,  $\beta^{\text{new}} = \tilde{x}^2 \beta$ . We remark that the Reynolds numbers for the two regions in the bulk are given by  $\text{Re}$  and  $\text{Re} \tilde{\rho} / \tilde{\mu}$ , respectively, and that they will in general differ in the case of a viscosity contrast between the inner and outer fluid. In addition to the above equations, we of course also require that  $\vec{u}$  has zero divergence in the bulk and that the surface divergence of  $\vec{u}$  vanishes on  $\Gamma$ .

Typical values for the bulk dynamic viscosity  $\mu$  are around  $10^{-3} - 10^{-2} \frac{\text{kg}}{\text{s m}}$ , see [4, 43, 52], whereas the surface shear viscosity typically is about  $10^{-9} - 10^{-8} \frac{\text{kg}}{\text{s}}$ , see [4, 30, 53]. The bending modulus  $\alpha$  is typically  $10^{-20} - 10^{-19} \frac{\text{kg m}^2}{\text{s}^2}$ , see [30, 52, 53].

The term  $\tilde{\mu}_\Gamma = \mu_\Gamma / (\mu_+ \tilde{x})$  in (5) suggests to choose the length scale

$$\tilde{x} = \mu_\Gamma / \mu_+ \quad \iff \quad \tilde{\mu}_\Gamma = 1.$$

As  $\alpha^{\text{new}} = \alpha / (\mu_+ \tilde{u} \tilde{x}^2) = \alpha \tilde{t} / (\mu_+ \tilde{x}^3)$  appears in (5), we choose the time scale

$$\tilde{t} = \mu_+ \tilde{x}^3 / \alpha.$$

Choosing

$$\mu_\Gamma = 5 \cdot 10^{-9} \frac{\text{kg}}{\text{s}}, \quad \mu_+ = 10^{-3} \frac{\text{kg}}{\text{s m}}, \quad \alpha = 10^{-19} \frac{\text{kg m}^2}{\text{s}^2},$$

see e.g. [43], we obtain the length scale  $5 \cdot 10^{-6} \text{m}$  and the time scale 1.25s, which are typical scales in experiments. With these scales for length and time together with values of  $\sim 10^3 \text{kg/m}^3$  for the bulk density and  $\sim 10^{-6} \text{kg/m}^2$  for the surface densities, we obtain for the bulk and surface Reynolds numbers

$$\text{Re} \approx 10^{-5} \quad \text{and} \quad \text{Re}_\Gamma \approx 10^{-8},$$

and hence we will set the Reynolds numbers to zero in this paper. We note that it is straightforward to also consider positive Reynolds numbers in our numerical algorithm, see [45, 46] for details. Together with the other observations above, we then obtain the following reduced set of equations (on dropping the  $\cdot^{\text{new}}$  superscripts).

$$\begin{aligned} & -\tilde{\mu} \Delta \vec{u} + \nabla p = 0 \quad \text{in } \Omega_\pm(t), \\ & -2 \nabla_s \cdot \underline{D}_s(\vec{u}) + \nabla_s \cdot (p_\Gamma \underline{P}_\Gamma) \\ &= [2 \tilde{\mu} \underline{D}(\vec{u}) - p \underline{\text{Id}}]_+^+ \vec{\nu} + \alpha \vec{f}_\Gamma \quad \text{on } \Gamma(t). \quad (6) \end{aligned}$$

A downside of the scaling used to obtain (6) is that the surface viscosity no longer appears as an independent parameter. However, studying the effect of the surface viscosity, e.g. on the tank treading to tumbling transition in shearing experiments, is one of the main focuses of this paper. It is for this reason that we also consider the following alternative scaling, when suitable length and velocity scales are at hand. For example, we may choose the length scale  $\tilde{x}$  based on the (fixed) size of the membrane and a velocity scale  $\tilde{u}$  based on appropriate boundary velocity values. In this case we obtain from (5), for

small Reynolds numbers, the following set of equations (on dropping the  $\cdot^{\text{new}}$  superscripts)

$$\begin{aligned} -\tilde{\mu} \Delta \vec{u} + \nabla p &= 0 \quad \text{in } \Omega_{\pm}(t), \\ -\nabla_s \cdot (2\tilde{\mu}_{\Gamma} \underline{\underline{D}}_s(\vec{u}) - p_{\Gamma} \underline{\underline{P}}_{\Gamma}) \\ &= [2\tilde{\mu} \underline{\underline{D}}(\vec{u}) - p \underline{\underline{Id}}]_{-}^{+} \vec{\nu} + \alpha \vec{f}_{\Gamma} \quad \text{on } \Gamma(t). \end{aligned} \quad (7)$$

Note that here three non-dimensional parameters remain:  $\tilde{\mu}_{\Gamma}$ ,  $\tilde{\mu}$  and  $\alpha$ . Here  $\tilde{\mu}_{\Gamma}$  compares the surface shear viscosity to the bulk shear viscosity,  $\tilde{\mu}$  is the bulk viscosity ratio and  $\alpha$  is an inverse capillary number, which describes the ratio of characteristic membrane stresses to viscous stresses. Clearly, the system (6) corresponds to (7) with  $\tilde{\mu}_{\Gamma} = 1$ . Hence from now on, we will only consider the scaling (7) in detail.

Of course, the system (7) needs to be supplemented with a boundary condition for  $\vec{u}$  or  $\underline{\underline{g}}$ , and with an initial condition for  $\Gamma(0)$ . For the former we partition the boundary  $\partial\Omega$  of  $\Omega$  into  $\partial_1\Omega$ , where we prescribe a fixed velocity  $\vec{u} = \vec{g}$ , and  $\partial_2\Omega$ , where we prescribe the stress-free condition  $\underline{\underline{g}} \vec{n} = \vec{0}$ , with  $\vec{n}$  denoting the outer normal to  $\Omega$ .

### III. NUMERICAL APPROXIMATION

The numerical computations in this paper have been performed with a finite element approximation introduced by the authors in [45, 46]. The approach discretizes the bulk and surface degrees of freedom independently. In particular, the surface mesh is not a restriction of the bulk mesh. The bulk degrees of freedoms  $\vec{u}$  and  $p$  are discretized with the lowest order Taylor–Hood element, P2–P1, in our numerical computations. The evolution of the membrane is tracked with the help of parametric meshes  $\Gamma^h$ , which are updated by the fluid velocity. Since the membrane surface is locally incompressible, it turns out that the surface mesh has good mesh properties during the evolution. This is in contrast to other fluid problems with interfaces in which the mesh often deteriorates during the evolution when updated with the fluid velocity, see e.g. [54].

The elastic forcing by the membrane curvature energy,  $\vec{f}_{\Gamma}$ , is discretized with the help of a weak formulation by Dziuk [18], which is generalized by Barrett *et al.* [46] to take spontaneous curvature and area difference elasticity effects into account. A main ingredient of the numerical approach is the fact that one can use a weak formulation of (3) that can be discretized in a stable way. In fact, defining  $A = \beta(M(\Gamma) - M_0)$  and  $\vec{y} = \vec{z} + (A - \bar{\kappa}) \vec{\nu}$  the following identity, which has to hold for all  $\vec{\chi}$  on  $\Gamma$ ,

characterizes  $\vec{f}_{\Gamma}$ :

$$\begin{aligned} \langle \vec{f}_{\Gamma}, \vec{\chi} \rangle &= \langle \nabla_s \vec{y}, \nabla_s \vec{\chi} \rangle + \langle \nabla_s \cdot \vec{y}, \nabla_s \cdot \vec{\chi} \rangle \\ &\quad - 2 \langle \nabla_s \vec{y}, \underline{\underline{D}}_s(\vec{\chi}) \nabla_s \text{id} \rangle + (A - \bar{\kappa}) \langle \vec{z}, [\nabla_s \vec{\chi}]^T \vec{\nu} \rangle \\ &\quad - \frac{1}{2} \langle [|\vec{z} - \bar{\kappa} \vec{\nu}|^2 - 2(\vec{y} \cdot \vec{z})] \nabla_s \text{id}, \nabla_s \vec{\chi} \rangle \\ &\quad - A \langle (\vec{z} \cdot \vec{\nu}) \nabla_s \text{id}, \nabla_s \vec{\chi} \rangle. \end{aligned}$$

Here  $\langle \cdot, \cdot \rangle$  is the  $L^2$ -inner product on  $\Gamma$ . Roughly speaking the above identity shows that  $\vec{f}_{\Gamma}$  has a divergence structure. We remark here that similar divergence structures have been derived with the help of Noether’s theorem, see [55, 56].

The numerical method of Barrett *et al.* [46] has the feature that a semi-discrete, i.e. continuous in time and discrete in space, version of the method obeys a discrete analog of the energy inequality (4). In addition, this semi-discrete version has the property that the volume enclosed by the vesicle and the membrane’s surface area are conserved exactly. After discretization in time these properties are approximately fulfilled to a high accuracy, see Section IV. The fully discrete system is linear and fully coupled in the unknowns. The overall system is reduced by a Schur complement approach to obtain a reduced system in just velocity and pressure unknowns. For this resulting linear system well-known solution techniques for finite element discretizations for the standard Navier–Stokes equations can be used, see Barrett *et al.* [57].

### IV. NUMERICAL COMPUTATIONS

In shearing experiments the tilt angle of the vesicle in the shear flow direction is often of interest. Here we will always consider shear flow in the  $x_1$  direction with  $x_3$  being the flow gradient direction. Let  $(\vec{p}, \vec{q}) \in \arg \max_{(\vec{p}, \vec{q}) \in \Gamma^h \times \Gamma^h} \{(p_1 - q_1)^2 + (p_3 - q_3)^2\}$ , and set  $\hat{\theta} = \arg(p_1 - q_1 + i(p_3 - q_3))$ , where  $\arg : \mathbb{C} \rightarrow (-\pi, \pi]$ . Then we define the tilt angle  $\theta \in (\pi/2, \pi/2]$  as

$$\theta = \begin{cases} \hat{\theta} + \pi & \hat{\theta} \in (-\pi, -\pi/2], \\ \hat{\theta} & \hat{\theta} \in (-\pi/2, \pi/2], \\ \hat{\theta} - \pi & \hat{\theta} \in (\pi/2, \pi]. \end{cases}$$

The tilt angle  $\theta$  is important for the classification of different types of dynamics in the shear flow experiments that we will present. The classical deformation dynamics for vesicles are the tank treading (TT) and the tumbling (TU) motions. In the tank treading motion the vesicle adopts a constant tilt angle in the flow, while the surface fluid rotates on the membrane surface. This motion is observed for small viscosity contrasts between the inner and the outer fluid and, as we will see later, at low surface membrane viscosity. At large viscosity contrasts or large membrane viscosity the tumbling motion occurs. In the

tumbling regime the membrane rotates as a whole, and the tilt angle oscillates in the whole interval  $(-\pi/2, \pi/2]$ . In the last ten years new dynamic regimes for vesicles in shear flow have been identified. In all these regimes the tilt angle is neither constant nor does it oscillate in the whole interval  $(-\pi/2, \pi/2]$ . This regime was first observed experimentally in [58], and subsequently has been studied by different groups, see e.g. [53, 59–63] for more details. Following [62] we will refer to these new regimes as the transition (TR) mode.

Depending on how irregular the evolution in this transition regime is, different types of motion have been classified. In a vacillating-breathing motion the tilt angle oscillates around the flow direction in a regular way, and the vesicle undergoes periodic shape deformations. Another transition regime is called trembling. Here strongly asymmetric shape perturbations can occur, and quite irregular evolutions of the vesicle orientation are possible. Farutin and Misbah [63] also called such an irregular motion the squaring mode and identified that the inclination angle undergoes discontinuous jumps during the time evolution. Different notions for similar motions appear in the literature, and hence we will give a precise definition of the regimes we consider. We separate the different types of evolutions into tank treading (TT, tilt angle is constant), tumbling (TU, tilt angle oscillates in the full interval  $(-\pi/2, \pi/2]$ ) and the transition regime (TR, tilt angle is non-constant and takes values in an interval strictly smaller than the interval  $(-\pi/2, \pi/2]$ ). The transition motions (TR) we subdivide further into  $(\text{TR}_c)$ , where the tilt angle evolves continuously, and  $(\text{TR}_{dc})$ , where the tilt angle evolves discontinuously.  $(\text{TR}_c)$  would be called vacillating-breathing by most authors, and the trembling and squaring motions discussed above fall into the class  $(\text{TR}_{dc})$ .

In our numerical simulations we will only consider the scaling (7). For all the presented simulations we will state the reduced volume as a characteristic invariant. It is defined as  $v_r = 6\pi^{1/2}v(0)/a^{3/2}(0)$ , see e.g. [15]. Here  $v(t)$  and  $a(t)$  denote the volume of the discrete inner phase and the discrete surface area, respectively, at time  $t$ . Moreover, if nothing else is specified, then our numerical simulations are for no-slip boundary conditions, i.e.  $\partial_1\Omega = \partial\Omega$  and  $\vec{g} = \vec{0}$ . Moreover, in all our experiments it holds that  $\overline{\alpha}\beta = 0$ , and we will only report the values of  $\overline{\alpha}$  and  $\beta$  for simulations where they are nonzero. Here we recall that, as stated in the introduction, the energy  $E(\Gamma(t))$  in (2) for  $\overline{\alpha}\beta \neq 0$  is equivalent to (2) with  $\overline{\alpha} = 0$ , the same value of  $\beta > 0$ , and a modified value of  $M_0$ . Finally, we stress that our sign convention for curvature is such that spheres have negative mean curvature.

### A. Effect of area difference elasticity

We consider  $\overline{\Omega} = [-4, 4]^3$  and set  $\tilde{\mu} = \tilde{\mu}_\Gamma = \alpha = 1$ . The parameters for  $\vec{f}_\Gamma$  are  $\beta = 0.053$  and  $M_0 = -48.24$ . For the vesicle we use a cup-like stomatocyte initial shape

with  $v_r = 0.65$  and  $a(0) = 82.31$ . See Figure 1 for a numerical simulation. As a comparison, we show the same simulation with  $\beta = 0$  in Figure 2.

In Figure 3 we show the evolutions of the discrete volume of the inner phase and the discrete surface area over time. Clearly these two quantities are preserved almost exactly for our numerical scheme. This behaviour is generic for all the simulations presented in this paper.

In our next simulation, we let  $\overline{\Omega} = [-2.5, 2.5]^3$  and set  $\tilde{\mu} = \alpha = 1$ , as well as  $\beta = 0.46$  and  $M_0 = -33.5$ . As initial vesicle we take a varying-diameter cigar-like shape that has  $v_r = 0.75$  and  $a(0) = 9.65$ . A simulation can be seen in Figure 4. As a comparison, we show the simulation with  $\beta = 0$  in Figure 5. Similarly to previous studies, where an energy involving area difference elasticity terms was minimized, we also observe in our hydrodynamic model that less symmetric shapes occur when the ADE-energy contributions are taken into account.

### B. Flow through a constriction

The numerical simulation of a vesicle flowing through a constriction can be seen in Figure 6. This example shows that membranes can drastically deform in order to pass through a constriction. This resembles the remarkable properties of red blood cells, which show a similar behaviour when flowing through capillaries. Here we choose the initial shape of the interface to be a biconcave discocyte surface resembling a human red blood cell, with a reduced volume of  $v_r = 0.568$  and a surface area of  $a(0) = 2.24$ . As the computational domain we choose  $\overline{\Omega} = [-2, -1] \times [-1, 1]^2 \cup [-1, 1] \times [-0.5, 0.5]^2 \cup [1, 2] \times [-1, 1]^2$ . We define  $\partial_2\Omega = \{2\} \times (-1, 1)^2$  and on  $\partial_1\Omega$  we set no-slip conditions, except on the left hand part  $\{-2\} \times [-1, 1]^2$ , where we prescribe the inhomogeneous boundary conditions  $\vec{g}(\vec{z}) = ([1 - z_2^2 - z_3^2]_+, 0, 0)^T$  in order to model a Poiseuille-type flow. The remaining parameters are chosen as  $\tilde{\mu} = \tilde{\mu}_\Gamma = 1$  and  $\alpha = 0.1$ . In the larger domain  $\overline{\Omega} = [-3, -1] \times [-1, 1]^2 \cup [-1, 1] \times [-0.5, 0.5]^2 \cup [1, 4] \times [-1, 1]^2$ , with the analogous boundary conditions, we also show the flow of four vesicles through a constriction, see Figure 7.

### C. Shearing for elliptical shapes

We conducted the following shearing experiments in  $\overline{\Omega} = [-2, 2]^3$  for an initial shape that is convex, with reduced volume  $v_r = 0.83$  and  $a(0) = 6.95$ . The initial shape represents a local minimizer of the curvature energy  $\int_\Gamma \kappa^2 ds$  among all shapes with the same reduced volume and surface area. In order to model shear flow, we prescribe the inhomogeneous Dirichlet boundary condition  $\vec{g}(\vec{z}) = (z_3, 0, 0)^T$  on the top and bottom boundaries  $\partial_1\Omega = [-2, 2]^2 \times \{\pm 2\}$ . The remaining parameters are given by  $\tilde{\mu} = 1$ ,  $\alpha = 0.05$ , and either  $\tilde{\mu}_\Gamma = 0.05$

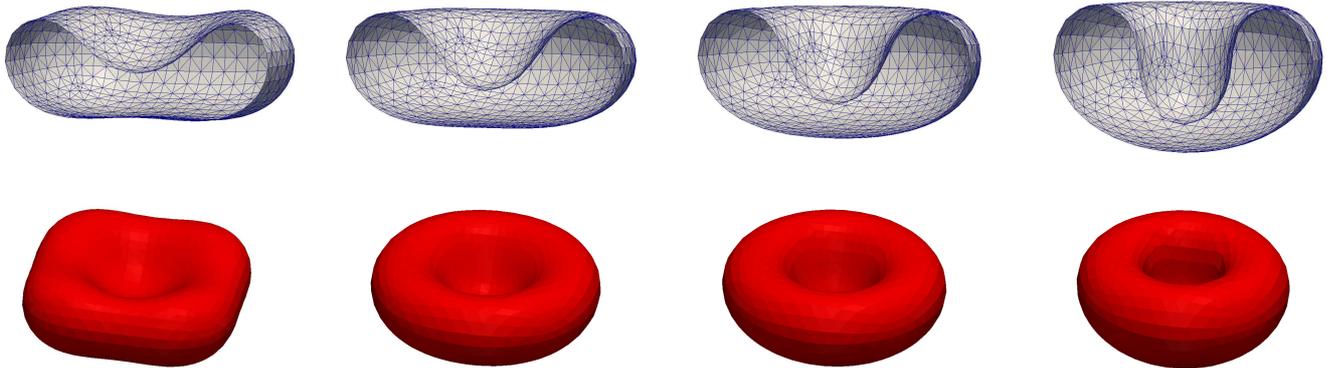


FIG. 1. Flow for a cup-like stomatocyte shape with  $v_r = 0.65$ . The triangulations of  $\Gamma^h$  at times  $t = 0, 5, 10, 20$ . Here  $M_0 = -48.24$  and  $\beta = 0.053$ .

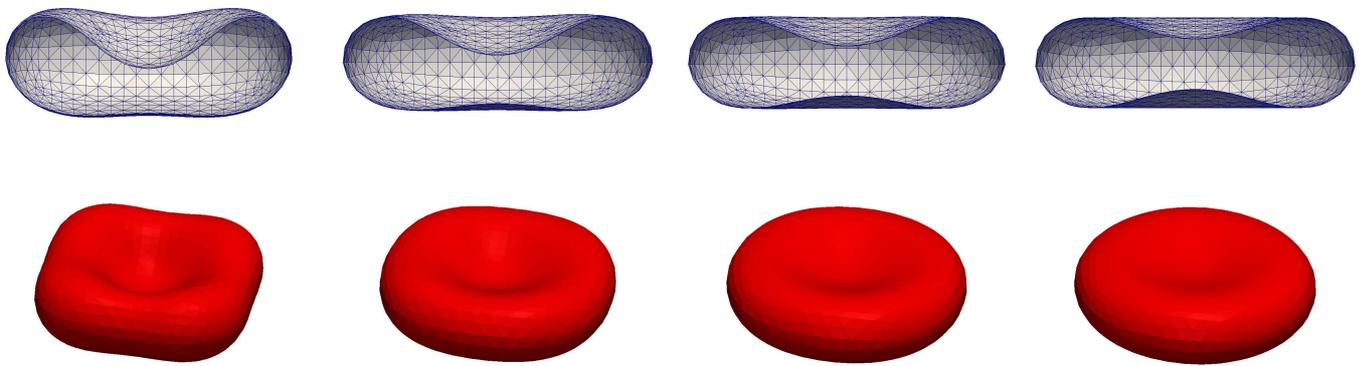


FIG. 2. Same as Figure 1 with  $\beta = 0$ .

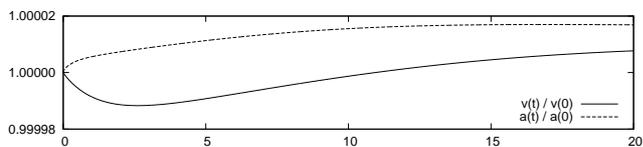


FIG. 3. The evolutions of the relative discrete volume  $v(t)/v(0)$ , and the relative discrete surface area  $a(t)/a(0)$  over time.

or  $\tilde{\mu}_\Gamma = 10$ . See Figures 8 and 9, where we observe tank treading (TT) for  $\tilde{\mu}_\Gamma = 0.05$ , and tumbling (TU) for  $\tilde{\mu}_\Gamma = 10$ . We stress that the tumbling occurs for a viscosity contrast of  $\mu_-/\mu_+ = 1$ , and so is only due to the chosen high surface viscosity  $\tilde{\mu}_\Gamma$ . The transition motion  $\text{TR}_c$  can be seen in Figure 10, where we have chosen  $\tilde{\mu}_\Gamma = 1$ . A plot of the tilt angle  $\theta$  for the simulations in Figures 8–10 can be seen in Figure 11.

In Table I we display the behaviour of the tilt angle  $\theta$  for various values of  $\bar{\varkappa}$  and  $\tilde{\mu}_\Gamma$ . Here the initial shapes of the vesicles, for a reduced volume of  $v_r = 0.83$  and surface area  $a(0) = 6.95$ , were chosen to be numerical approximations of local minimizers for the curvature energy  $\int (\varkappa - \bar{\varkappa})^2 ds$ . These discrete local minimizers were obtained with the help of the gradient flow scheme from [19], and for the choices  $\bar{\varkappa} = \pm 5$  they are displayed in Figure 12. The vesicle is then put in the same shear flow as in Figures 8–10. In fact, these three simulations correspond to the choice  $\bar{\varkappa} = 0$  in Table I. In the table we distinguish between tank treading (TT), the transition motions ( $\text{TR}_c$ ), ( $\text{TR}_{dc}$ ) and tumbling (TU), depending

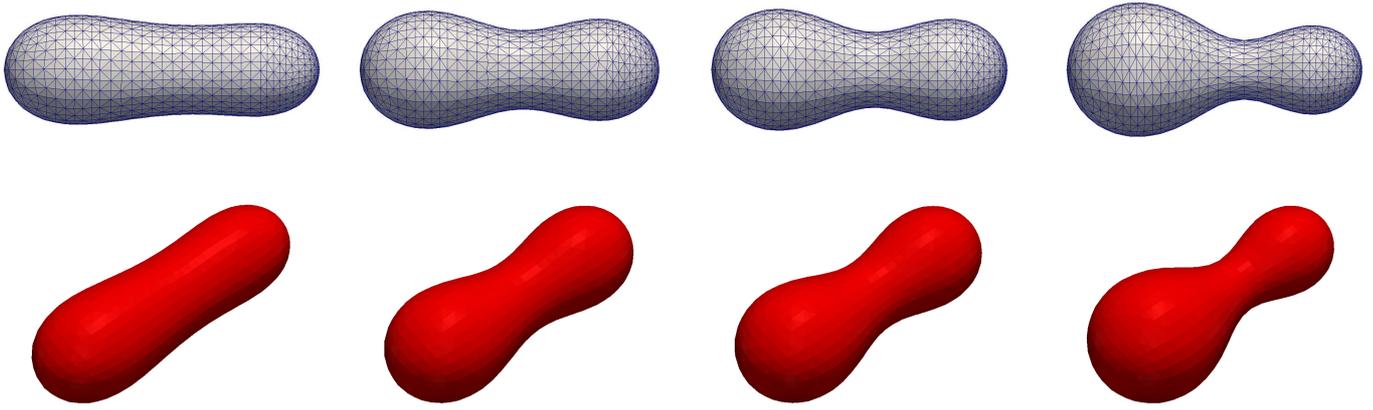


FIG. 4. Flow for a varying-diameter cigar-like shape with  $v_r = 0.75$ . The triangulations of  $\Gamma^h$  at times  $t = 0, 1, 10, 50$ . Here  $M_0 = -33.5$  and  $\beta = 0.46$ .

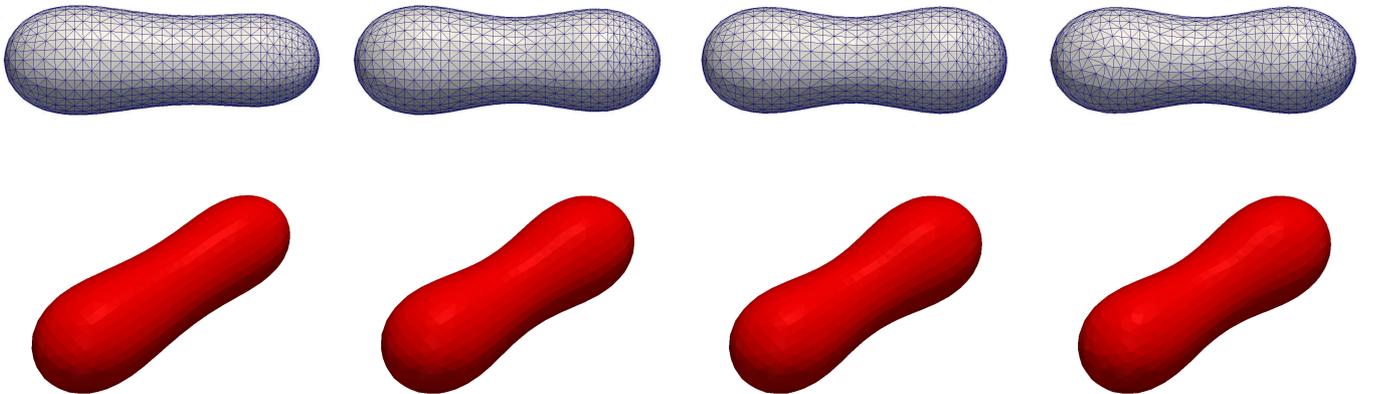


FIG. 5. Same as Figure 4 with  $\beta = 0$ .

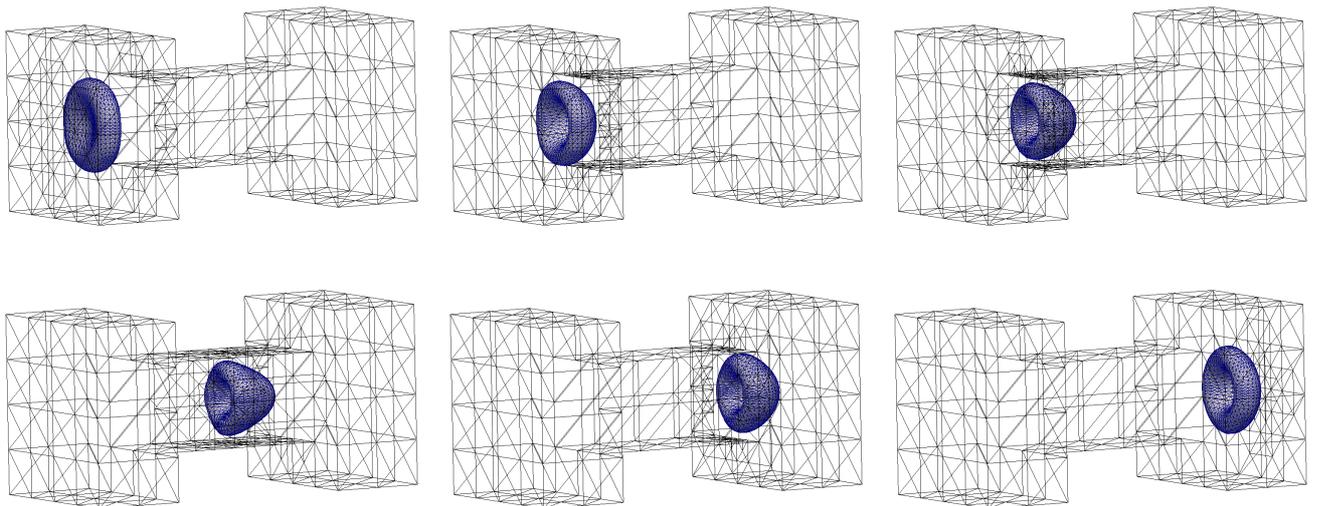


FIG. 6. Flow through a constriction. The plots show the interface  $\Gamma^h$  at times  $t = 0, 0.3, 0.5, 1, 1.4, 1.8$ .

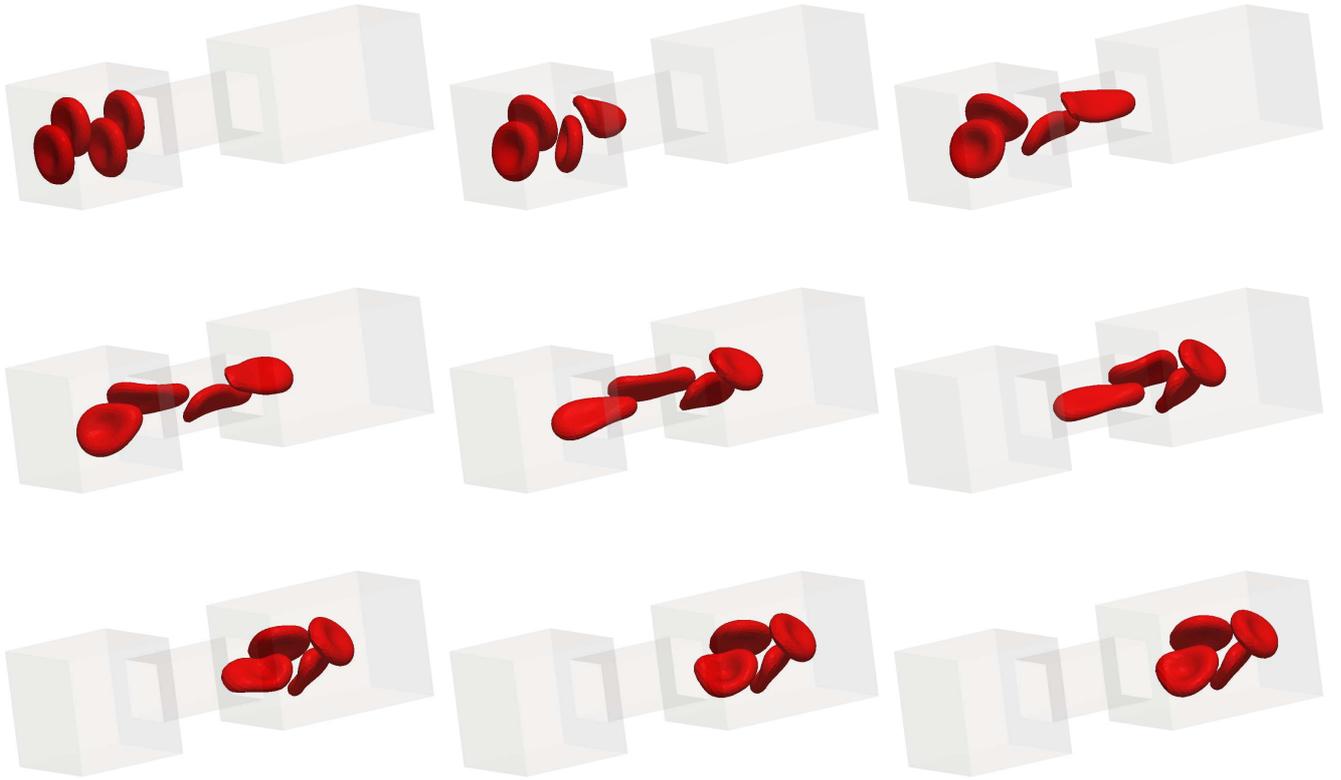


FIG. 7. Flow through a constriction. The plots show the interface  $\Gamma^h$  at times  $t = 0, 0.4, 0.8, 1.2, 1.6, 2.2.4, 2.8$ .

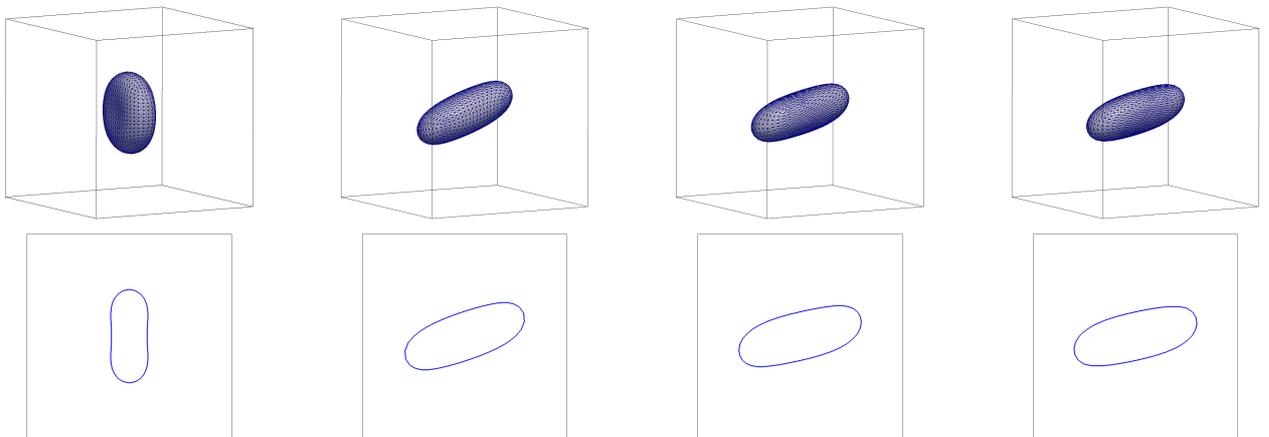


FIG. 8. Shear flow with  $\tilde{\mu} = 1$  and  $\tilde{\mu}_\Gamma = 0.05$  resulting in a tank treading (TT) motion. The plots show the interface  $\Gamma^h$  within  $\bar{\Omega}$ , as well as cuts through the  $x_1$ - $x_3$  plane, at times  $t = 0, 2.5, 5, 7.5$ .

on the behaviour of the tilt angle  $\theta$ .

The results in Table I indicate that the values of the surface viscosity, at which the transitions between TT, TR<sub>c</sub>, TR<sub>dc</sub> and TU take place, strongly depend on the spontaneous curvature. For completeness, we visualize two tumbling motions for  $\bar{\alpha} = \pm 5$  in Figures 13 and 14.

#### D. Shearing for biconcave discocyte shapes

Here we use a scaled variant of the initial interface in Figure 6. In particular,  $v_r = 0.568$  and  $a(0) = 8.95$ . For the remaining parameters we choose the values from §IV C. In particular, we let  $\alpha = 0.05$ , and either  $\tilde{\mu}_\Gamma = 0.05$  or  $\tilde{\mu}_\Gamma = 10$ . The results can be seen in Figures 15 and 16. In both cases we observe a tumbling

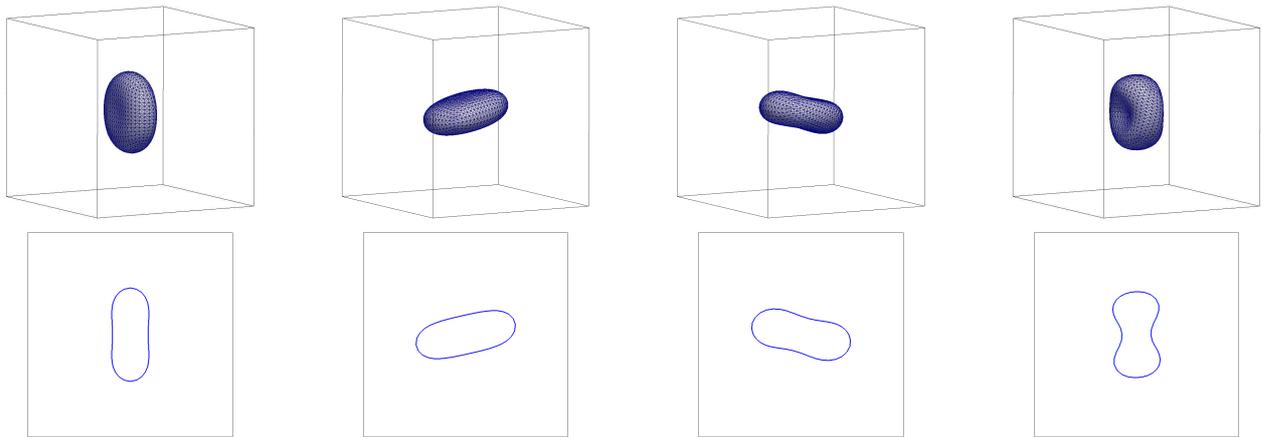


FIG. 9. Shear flow with  $\tilde{\mu} = 1$  and  $\tilde{\mu}_\Gamma = 10$  resulting in a tumbling (TU) motion. The plots show the interface  $\Gamma^h$  within  $\bar{\Omega}$ , as well as cuts through the  $x_1$ - $x_3$  plane, at times  $t = 0, 2.5, 5, 7.5$ . The interface at  $t = 10$  is very close to the plot at  $t = 2.5$ .

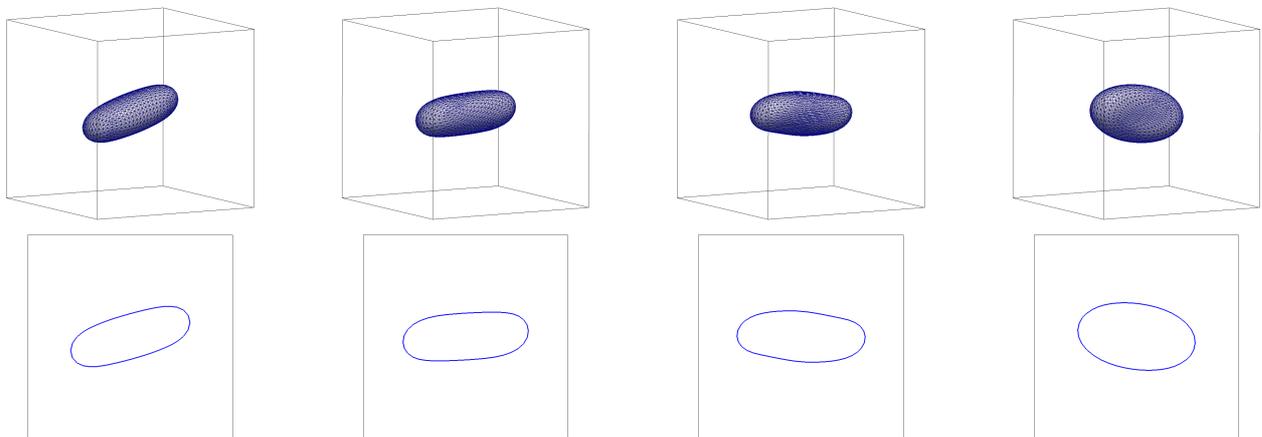


FIG. 10. Shear flow with  $\tilde{\mu} = 1$  and  $\tilde{\mu}_\Gamma = 1$  resulting in a motion in the transition regime with a continuous evolution of the inclination angle ( $\text{TR}_c$ ). The plots show the interface  $\Gamma^h$  within  $\bar{\Omega}$ , as well as cuts through the  $x_1$ - $x_3$  plane, at times  $t = 2.5, 5, 7.5, 10$ .

$\bar{\alpha} = -5$			$\bar{\alpha} = 0$			$\bar{\alpha} = 5$		
$\tilde{\mu}_\Gamma$	$\theta$		$\tilde{\mu}_\Gamma$	$\theta$		$\tilde{\mu}_\Gamma$	$\theta$	
1	[0.05, 0.17]	TT	0.1	[0.15, 0.28]	TT	0.1	[0.16, 0.28]	TT
1.5	[0.02, 0.17]	TT	0.2	[0.11, 0.23]	TT	0.2	[0.12, 0.23]	TT
2	[0.01, 0.14]	TT	0.4	[-0.04, 0.08]	TT	0.4	[-0.02, 0.13]	TT
2.3	[-0.02, 0.14]	$\text{TR}_c$	0.5	[-0.04, 0.11]	$\text{TR}_c$	0.5	[-0.1, 0.03]	$\text{TR}_c$
2.5	[-0.08, 0.16]	$\text{TR}_c$	1	[-0.2, 0.2]	$\text{TR}_c$	1	[-0.25, 0.2]	$\text{TR}_c$
2.7	[-0.08, 0.25]	$\text{TR}_c$	1.6	[-0.4, 0.6]	$\text{TR}_c$	1.3	[-0.4, 0.4]	$\text{TR}_c$
3	[-0.9, 0.9]	$\text{TR}_{dc}$	1.7	[-0.6, 0.6]	$\text{TR}_{dc}$	1.4	[-0.5, 0.4]	$\text{TR}_{dc}$
4	[-0.9, 1]	$\text{TR}_{dc}$	3	[-1, 1]	$\text{TR}_{dc}$	2	[-0.8, 0.7]	$\text{TR}_{dc}$
5	[-1.1, 1.1]	$\text{TR}_{dc}$	4	[-1.2, 1.2]	$\text{TR}_{dc}$	3	[-1.3, 1.2]	$\text{TR}_{dc}$
10	[-1.3, 1.4]	$\text{TR}_{dc}$	8	[-1.4, 1.5]	$\text{TR}_{dc}$	3.3	[-1.4, 1.4]	$\text{TR}_{dc}$
12.5	$(-\frac{\pi}{2}, \frac{\pi}{2}]$	TU	8.5	$(-\frac{\pi}{2}, \frac{\pi}{2}]$	TU	3.4	$(-\frac{\pi}{2}, \frac{\pi}{2}]$	TU
15	$(-\frac{\pi}{2}, \frac{\pi}{2}]$	TU	9	$(-\frac{\pi}{2}, \frac{\pi}{2}]$	TU	4	$(-\frac{\pi}{2}, \frac{\pi}{2}]$	TU
20	$(-\frac{\pi}{2}, \frac{\pi}{2}]$	TU	10	$(-\frac{\pi}{2}, \frac{\pi}{2}]$	TU	5	$(-\frac{\pi}{2}, \frac{\pi}{2}]$	TU

TABLE I. Eventual ranges of  $\theta$  for different values of  $\bar{\alpha}$  and  $\tilde{\mu}_\Gamma$ .

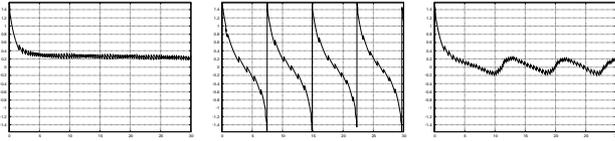


FIG. 11. The tilt angle  $\theta$  for the computations in Figures 8–10. They correspond to the motions TT, TU and  $\text{TR}_c$ , respectively.

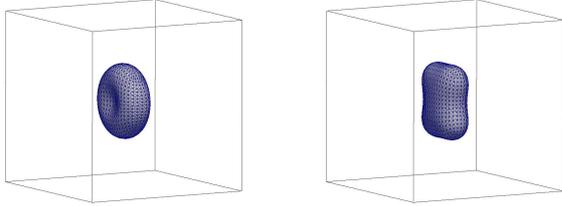


FIG. 12. The vesicles for  $\bar{z} = -5$  (left) and  $\bar{z} = 5$  (right) at time  $t = 0$ .

behaviour. In other situations, typically for small viscosity contrasts  $\mu_-/\mu_+$  and for small membrane viscosities  $\mu_\Gamma$ , a tank treading (TT) evolution occurs, see e.g. §IV C. This does not occur in our numerical simulations for this biconcave shape. We performed computations with the parameters  $(\mu_+, \mu_-, \tilde{\mu}_\Gamma, \alpha) = (1, 10^{-3}, 0, 0.05)$ ,  $(1, 10^{-3}, 0, 10^{-3})$ ,  $(100, 10^{-3}, 0, 0.05)$ ,  $(10^{-2}, 10^{-2}, 0, 0.05)$ ,  $(10^{-3}, 10^{-3}, 0.05, 0.05)$ ,  $(10^{-2}, 10^{-2}, 0.05, 0.05)$  without observing a tank treading (TT) motion.

### E. Shearing for budded shape (two arms)

We start a scaled variant of the final shape from Figure 4 in a shear flow experiment. In particular, the initial shape is axisymmetric, with reduced volume  $v_r = 0.75$  and  $a(0) = 5.43$ . We set  $\tilde{\mu} = \tilde{\mu}_\Gamma = 1$ ,  $\alpha = 0.05$  and use the same domain and boundary conditions as in §IV C. See Figure 17 for a run with  $\beta = 0.1$  and  $M_0 = -33.5$ . We observe that the shape of the vesicle changes drastically, with part of the surface growing inwards. This is similar to the shapes observed in Figure 1, where the presence of a lower reduced volume led to cup-like stomatocyte shapes. We repeat the same experiment for  $\beta = 0$  in Figure 18. Now the budding shape loses its strong non-convexity completely, as can be clearly seen in the plots of the two-dimensional cuts in Figure 18. Plots of the bending energy  $\alpha E(\Gamma^h)$  are shown in Figure 19, where we recall that the energy inequality in (4) does not hold for the inhomogeneous boundary conditions employed in the present simulations.

### F. Shearing for a seven-arm starfish

We consider simulations for a scaled version of the final shape from Barrett *et al.* [19, Fig. 23] with reduced volume  $v_r = 0.38$  and  $a(0) = 10.54$  inside the domain  $\bar{\Omega} = [-2, 2]^3$ . We set  $\tilde{\mu} = \tilde{\mu}_\Gamma = \alpha = 1$ . In order to maintain the seven-arm shape during the evolution we set  $\beta = 0.05$  and  $M_0 = 180$ . The first experiment is for no-slip boundary conditions on  $\partial\Omega$  and shows that the seven arms grow slightly, see Figure 20. If we use the shear flow boundary conditions from §IV C, on the other hand, we observe the behaviour in Figure 21, where we have changed the value of  $\alpha$  to 0.05. The vesicle can be seen tumbling, with the seven arms remaining intact. Repeating the same experiment with  $\beta = 0$  yields the simulation in Figure 22. Not surprisingly, some of the arms of the vesicle are disappearing.

### G. Shearing for a torus

Here we use as the initial shape a Clifford torus with reduced volume  $v_r = 0.71$  and  $a(0) = 13.88$ . We let  $\tilde{\mu} = \tilde{\mu}_\Gamma = 1$ ,  $\alpha = 0.05$  and use the same domain and boundary conditions as in §IV C. See Figures 23, where the torus appears to tumble whilst undergoing strong deformations.

## V. CONCLUSIONS

We have introduced a parametric finite element method for the evolution of bilayer membranes by coupling a general curvature elasticity model for the membrane to (Navier–)Stokes systems in the two bulk phases and to a surface (Navier–)Stokes system. The model is based on work by Arroyo and DeSimone [43] and our main purpose was to study the influence of the area difference elasticity (ADE) and of the spontaneous curvature on the evolution of the membrane. In contrast to most other works, we discretized the full bulk (Navier–)Stokes systems coupled to the surface (Navier–)Stokes system and for the first time coupled this to a bending energy involving ADE and spontaneous curvature.

The numerical simulations led to the following findings.

- The proposed numerical method conserves the volume enclosed by the membrane and the surface area of the membrane to a high precision.
- For a flow through a constriction membranes deformed drastically, which was resolved by the proposed parametric approach with no problems.
- The biconcave discocyte shape in our numerical simulations always led to a tumbling (TU) type behaviour. In particular, also for very low viscosity contrasts and very low surface viscosities no tank treading (TT) motion was observed.

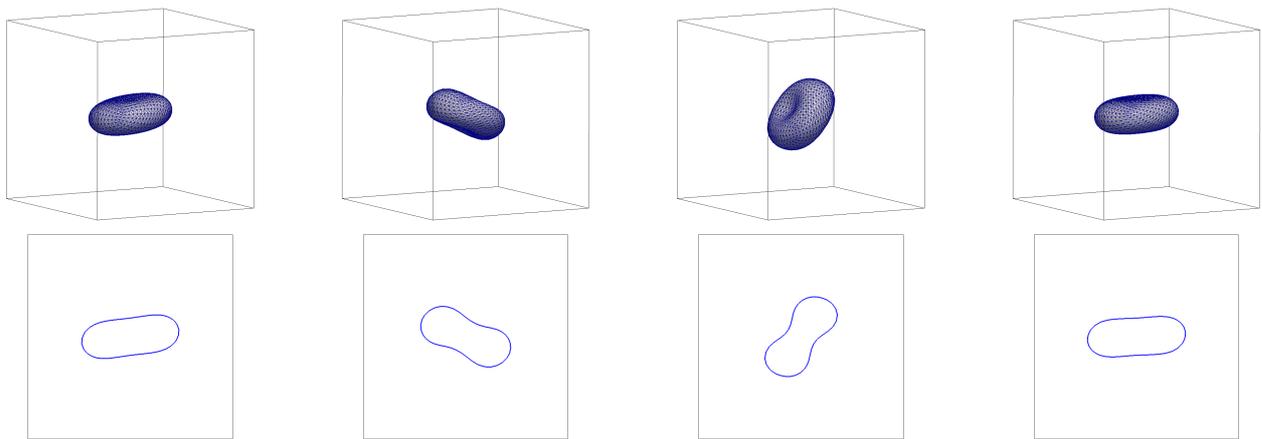


FIG. 13. Shear flow with  $\tilde{\mu} = 1$  and  $\tilde{\mu}_T = 15$  resulting in a tumbling (TU) motion. Here  $\tilde{\alpha} = -5$ . The plots show the interface  $\Gamma^h$  within  $\bar{\Omega}$ , as well as cuts through the  $x_1$ - $x_3$  plane, at times  $t = 10, 12.5, 15, 17.5$ . The interface at  $t = 20$  is very close to the plot at  $t = 12.5$ .

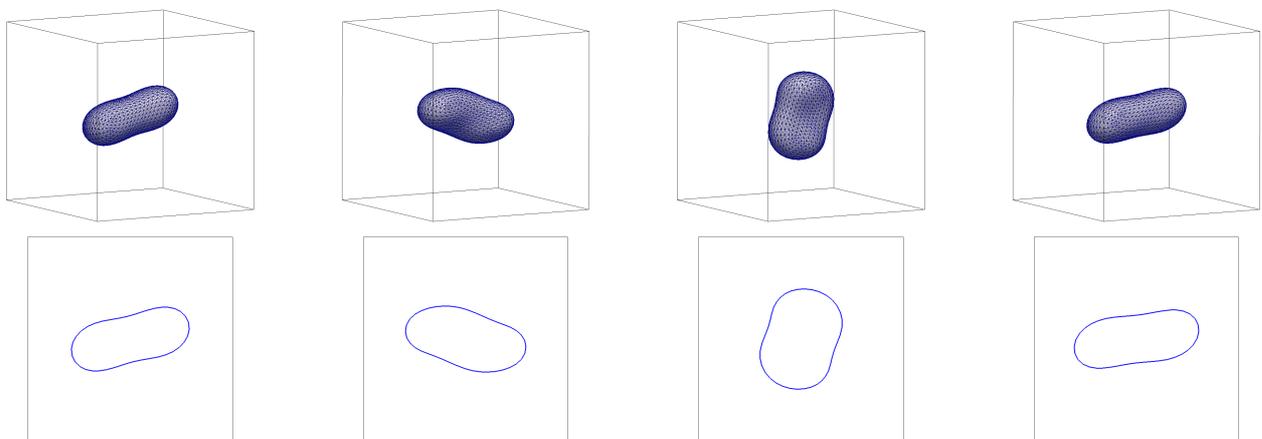


FIG. 14. Shear flow with  $\tilde{\mu} = 1$  and  $\tilde{\mu}_T = 5$  resulting in a tumbling (TU) motion. Here  $\tilde{\alpha} = 5$ . The plots show the interface  $\Gamma^h$  within  $\bar{\Omega}$ , as well as cuts through the  $x_1$ - $x_3$  plane, at times  $t = 10, 12.5, 15, 17.5$ . The interface at  $t = 20$  is very close to the plot at  $t = 12.5$ .

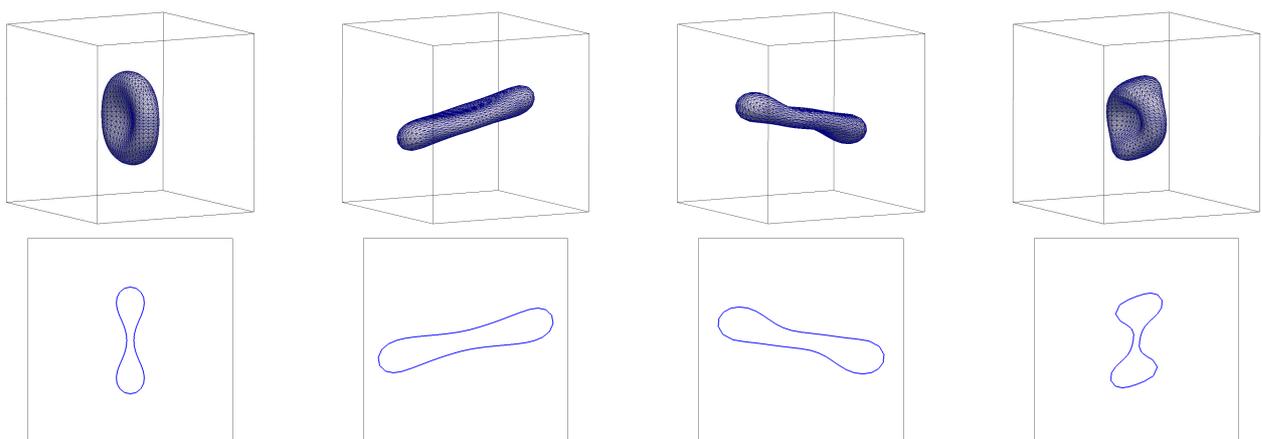


FIG. 15. Shear flow with  $\tilde{\mu} = 1$  and  $\tilde{\mu}_T = 0.05$  resulting in a tumbling (TU) motion. The plots show the interface  $\Gamma^h$  within  $\bar{\Omega}$ , as well as cuts through the  $x_1$ - $x_3$  plane, at times  $t = 0, 2.5, 5, 7.5$ . The interface at  $t = 10$  is very close to the plot at  $t = 2.5$ .

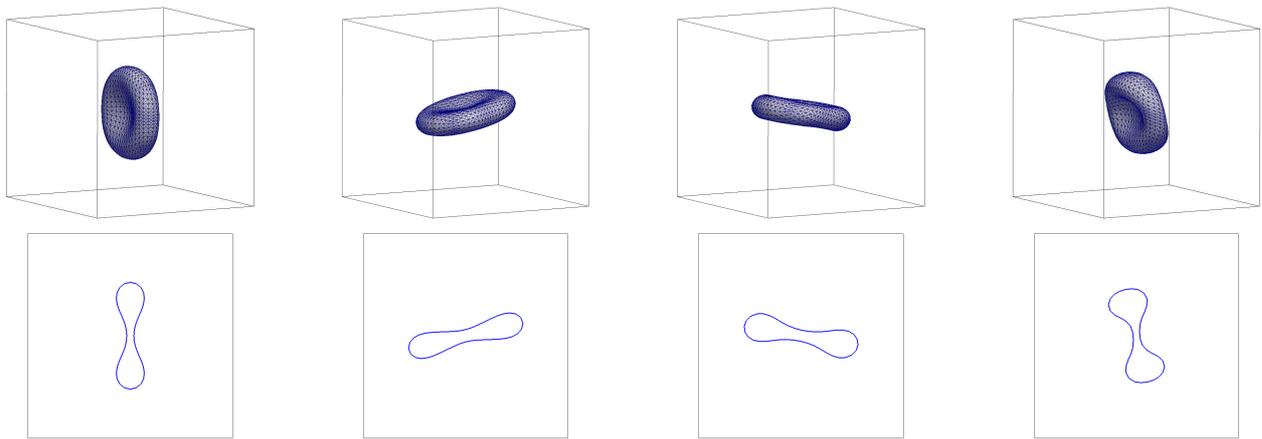


FIG. 16. Shear flow with  $\tilde{\mu} = 1$  and  $\tilde{\mu}_\Gamma = 10$  resulting in a tumbling (TU) motion. The plots show the interface  $\Gamma^h$  within  $\overline{\Omega}$ , as well as cuts through the  $x_1$ - $x_3$  plane, at times  $t = 0, 2.5, 5, 7.5$ . The interface at  $t = 10$  is very close to the plot at  $t = 2.5$ .

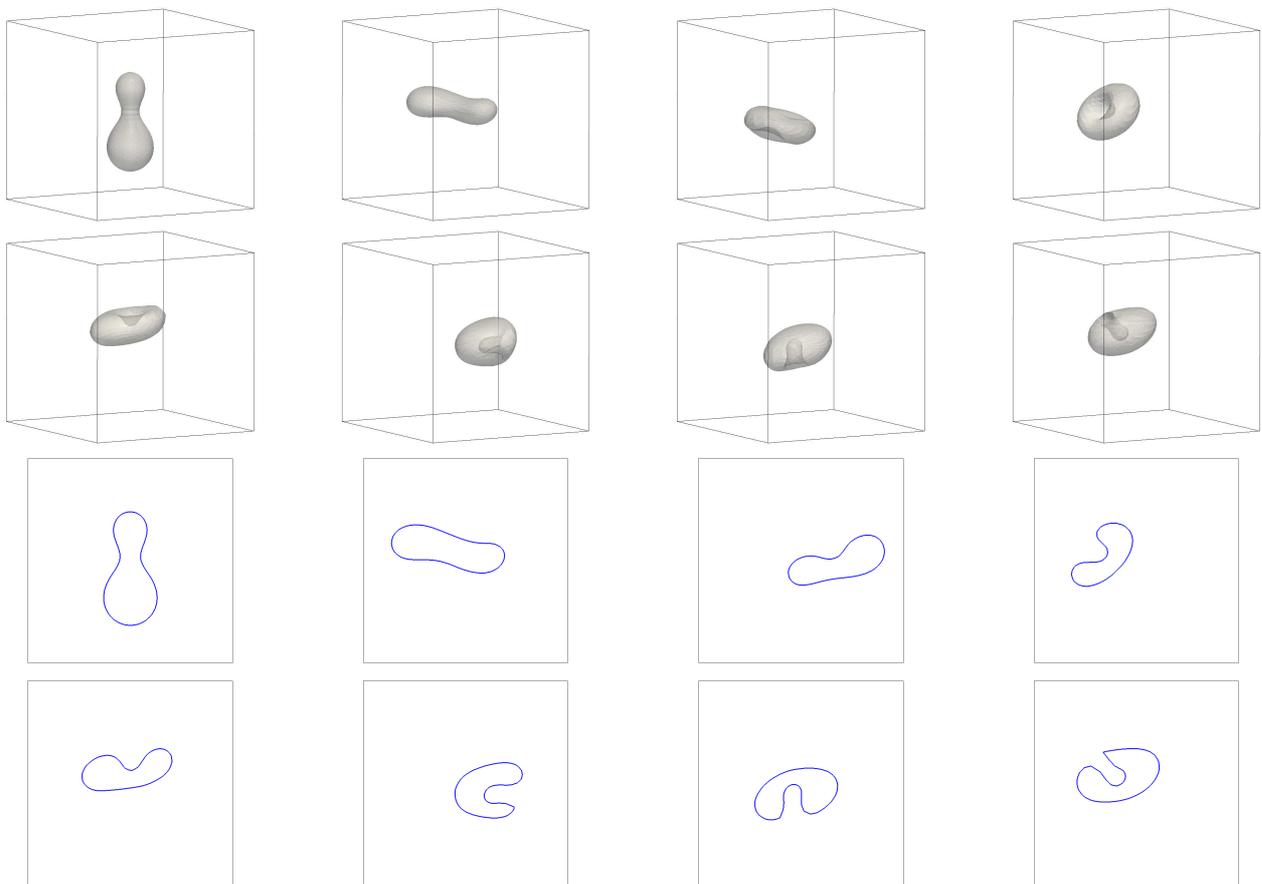


FIG. 17. Shear flow for a budding shape with  $\tilde{\mu} = \tilde{\mu}_\Gamma = 1$ . Here  $\beta = 0.1$  and  $M_0 = -33.5$ . The plots show the interface  $\Gamma^h$  within  $\overline{\Omega}$ , as well as cuts through the  $x_1$ - $x_3$  plane, at times  $t = 0, 5, 15, 17.5, 20, 25, 27.5, 32.5$ .

- The transition from a tank treading (TT) motion to a transition motion (TR) and to a tumbling (TU) motion depended strongly on the surface viscosity. We observed that the surface viscosity alone with no viscosity contrast between inner and outer fluid can lead to a transition from tank treading to a

TR-motion and to tumbling.

- The surface viscosity at which a transition between the different motions TT, TR and TU occur, strongly depends on the spontaneous curvature. In particular, we observed that for negative

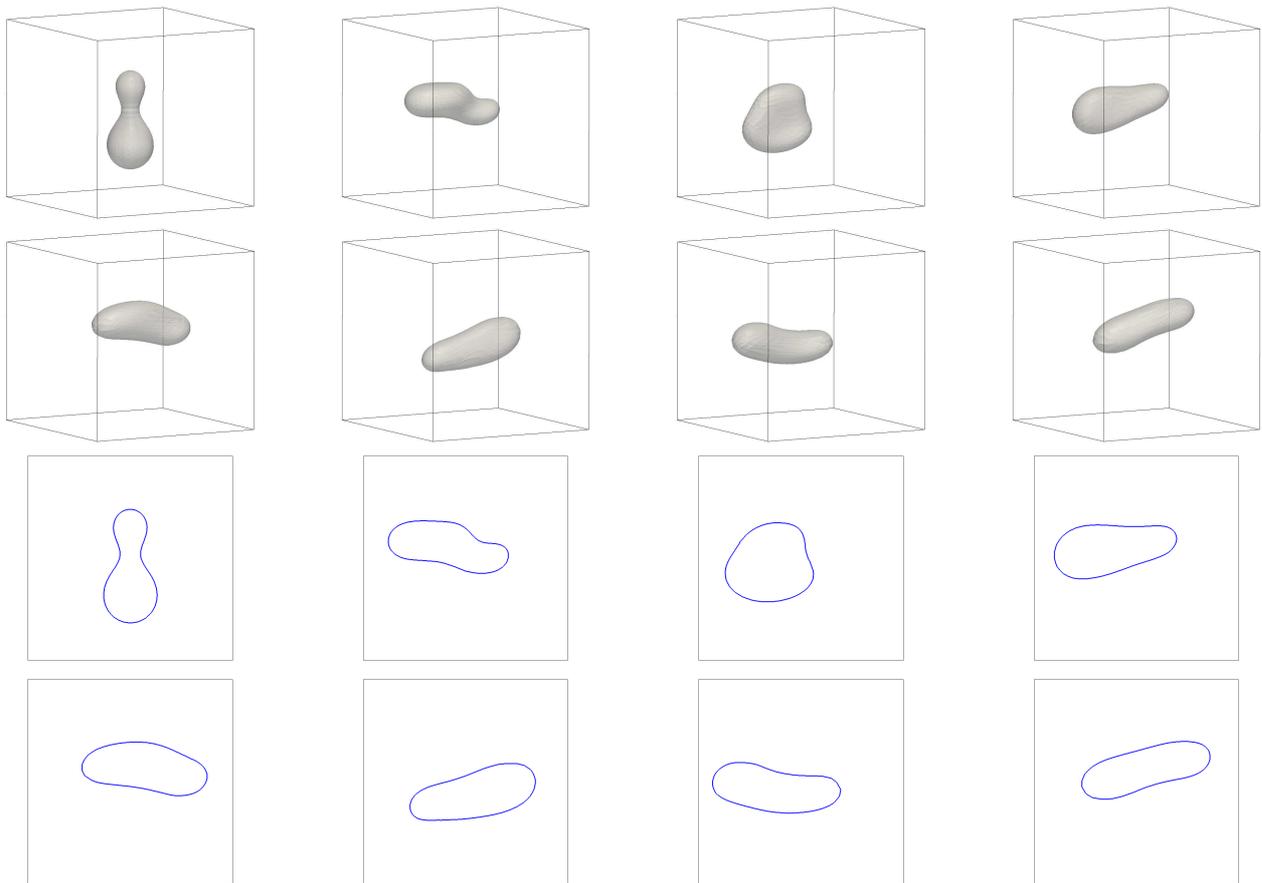


FIG. 18. Same as Figure 17 but with  $\beta = 0$ .

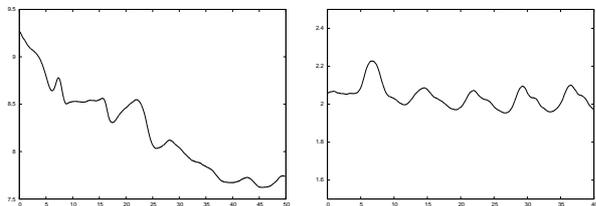


FIG. 19. The bending energy  $\alpha E(\Gamma^h)$  for the computations in Figures 17 and 18.

spontaneous curvature all transitions occurred for larger values of the surface viscosity. For positive spontaneous curvature we observed that tumbling occurred already for much smaller values of the surface viscosity. Here we recall that our sign convention for curvature means that spheres have negative mean curvature.

- In some cases, shear flow can lead to drastic shape changes, in particular for the ADE-model. For example, we observed the transition of a budded pear-like shape to a cup-like stomatocyte shape in shear flow if an ADE-model was used for the curvature elasticity.

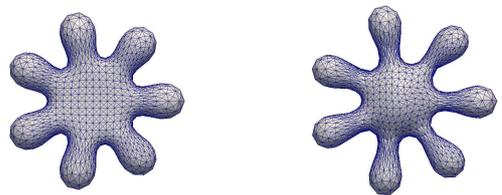


FIG. 20. Flow for a seven-arm figure with  $v_r = 0.38$ . Here  $\beta = 0.05$  and  $M_0 = 180$ . The triangulations  $\Gamma^h$  at times  $t = 0$  and 5.

- The ADE-model can also lead to starfish-type shapes with several arms, see e.g. [1, 15]. In computations for a seven-arm starfish for a model involving an ADE type energy, we observed that in shear flow the overall structure seems to be quite robust. In particular, the seven arms deformed but remained present even in a tumbling motion. However, arms tend to disappear if the area difference elasticity term is neglected.

Thus we have shown that the proposed numerical method is a robust tool to simulate bilayer membranes for quite general models which in particular take the full hydrody-

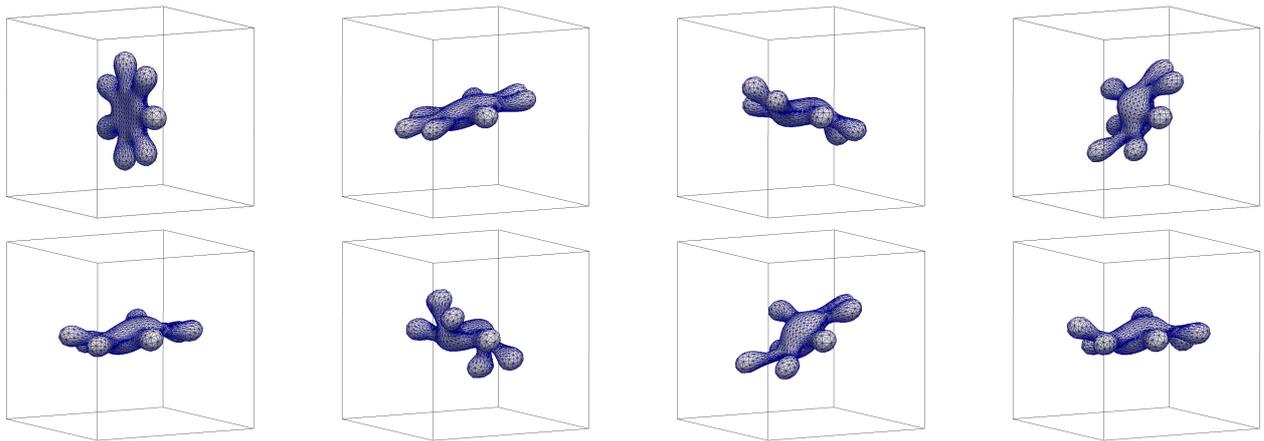


FIG. 21. Shear flow for a budding shape with  $\tilde{\mu} = \tilde{\mu}_\Gamma = 1$ . Here  $\beta = 0.05$  and  $M_0 = 180$ . The plots show the interface  $\Gamma^h$  within  $\bar{\Omega}$  at times  $t = 0, 2.5, 5, 7.5, 10, 12.5, 15, 17.5$ .

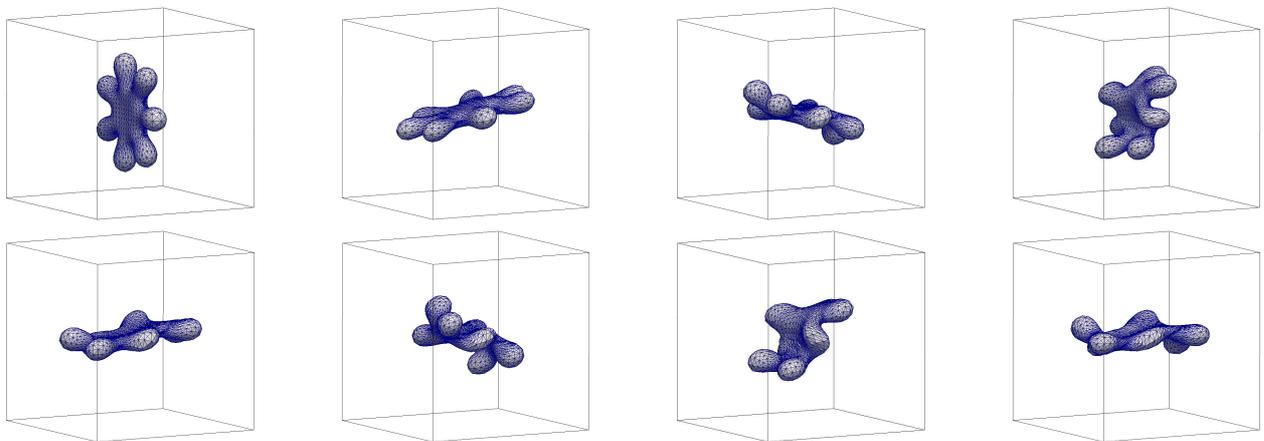


FIG. 22. Same as Figure 21 with  $\beta = 0$ .

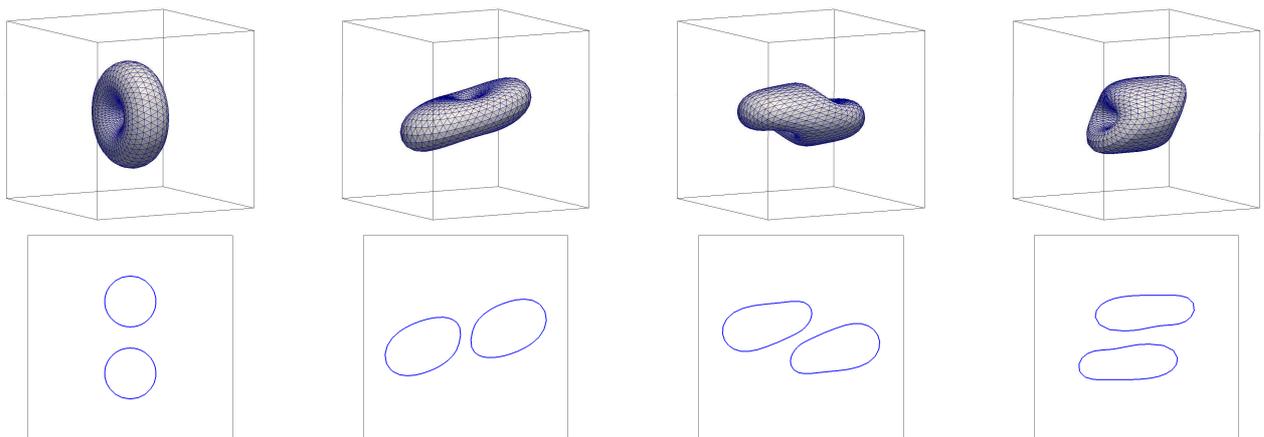


FIG. 23. Shear flow for a torus with  $\tilde{\mu} = \tilde{\mu}_\Gamma = 1$ . The plots show the interface  $\Gamma^h$  within  $\bar{\Omega}$ , as well as cuts through the  $x_1$ - $x_3$  plane, at times  $t = 0, 2.5, 5, 7.5$ . The interface at  $t = 10$  is very close to the plot at  $t = 2.5$ .

namics and a curvature model involving area difference elasticity and spontaneous curvature into account.

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