The Dynamics of the Romer R&D Growth Model with Quality Upgrading

Lutz G. Arnold
and
Wolfgang Kornprobst

University of Regensburg
Department of Economics
Economic Theory
93 040 Regensburg
Germany
Phone: +49-941-943-2705
Fax: +49-941-943-1971
E-mail: lutz.arnold@wiwi.uni-regensburg.de

Abstract

This paper replaces increasing product variety with quality upgrading in the Romer (1990) model. We show that the range of parameters for which a steady state exists can be divided into two subspaces with well-behaved comparative statics and saddle-point dynamics in one subspace, but with "perverse" comparative-statics properties and either equilibrium indeterminacy or instability in the other subspace. In the latter subspace, a parameter change possibly leads to a Hopf bifurcation. Using a theorem in Arnold (in press), these results for the closed economy can also be used to characterize the dynamics of the M-country open-economy version of the model.

JEL classification: F12, O41
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1 Introduction

This paper replaces increasing product variety with quality upgrading à la Grossman and Helpman (1991a, Ch. 4, 1991b) in the Romer (1990) model. As both the Romer model and the Grossman-Helpman model possess steady states with "normal" comparative-statics properties and a unique convergent equilibrium growth path, one might suspect that this turns out to be a non-exciting exercise with little scope for surprising results. However, we show analytically that the range of parameters for which a steady state exists can be partitioned into two subspaces with well-behaved comparative statics and saddle-point dynamics in one subspace, but with "perverse" comparative-statics properties and either equilibrium indeterminacy or instability in the other subspace. In the latter subspace, a parameter change possibly leads to a Hopf bifurcation. Using a theorem in Arnold (in press), we generalize our results to an M-country open economy.

Romer's (1990) original paper is confined to steady-state analysis. The present paper contributes to a strand of the literature that investigates the dynamics of Romer's path-breaking model. Arnold (2000a, 2000b) shows that both the balanced-growth equilibrium of the model and the steady state of the optimal growth path are saddle points. Benhabib, Perli, and Xie (1994) add complementarities between the intermediate goods to the Romer model and highlight the possibility of indeterminacy in equilibrium. Using numerical examples, Asada, Nowak, and Semmler (1998) demonstrate that the Benhabib-Perli-Xie (1994) extension also allows for Hopf bifurcations (as does a variant of the model with physical capital as an input in R&D). Slobodyan (2002) presents some analytical results on the possibility of Hopf bifurcations in this model. Arnold (2006) shows that the steady state is a saddle point if one adds population growth to the model and assumes diminishing returns to knowledge in R&D (i.e., in the Jones 1995 model). Devereux and Lapham (1994) demonstrate that the steady state of the two-country open economy version of the Romer model (first studied by Rivera-Batiz and Romer, 1991) without international knowledge spillovers is unstable. Since quality upgrading is generally recognized as no less important a source of growth than horizontal innovation and since, as the present analysis shows, the respective results differ sharply from the setup proposed by Romer (1990), our paper makes a useful contribution to this strand of the literature.

Section 2 presents the model. Section 3 derives the equations which determine the equilibrium growth path. Sections 4 and 5 are concerned with the steady state and dynamics, respectively. The generalization to an M-country open economy is performed in Section 6. Section 7 concludes.

2 Model

Consider a closed economy inhabited by a continuum of mass one of identical individuals. Each individual supplies L units of labor and maximizes the intertemporal utility function $U(t) \equiv \int_t^\infty [c(\tau)^{1-\sigma} - 1]/(1-\sigma) \cdot e^{-\rho(\tau-t)} d\tau$, where t and τ denote time, $c(t)$ is consumption at time t, ρ (> 0) is the subjective discount rate, and σ (> 0) is the inverse of the intertemporal elasticity of substitution in consumption. There is a single final good, which is used both for consumption and for investment. Depreciation is neglected. Hence, $\dot{K} = Y - c$, where Y is aggregate production and K is the capital stock.[1] Output is produced using labor L_Y and an index of intermediate goods D_Y according to the Cobb-Douglas production function $Y = L_Y^{1-\alpha} D_Y^{\alpha}$ ($0 < \alpha < 1$). In Romer (1990), D_Y is composed of a range of intermediate goods which can be expanded via R&D (cf. also Grossman and Helpman, 1991a, Ch. 3, p. 45). By contrast, we follow Grossman and Helpman (1991a, Ch. 4, p. 87) and let

$$\ln D_Y = \int_0^1 \ln \left[\sum_{\omega=1}^{\Omega(j)} \lambda^{\omega} x_{\omega}(j) \right] dj,$$

where $x_\omega(j)$ is the input of quality ω of intermediate j, $\Omega(j)$ is the highest quality producible in sector j, and λ (> 1) is the size of a quality jump. R&D targeted at intermediate j raises $\Omega(j)$ by one if successful. Let $I(j)\, dt$ denote the probability of a quality jump in industry j in a short time interval dt. We restrict attention to equilibria with $I(j) \equiv I$ uniform for all j and call I the rate of innovation. The number of markets with a quality improvement in dt is $d[\int_0^1 \Omega(j) dj] = I\, dt$. The rate of innovation is a linear function of R&D employment L_A: $I = L_A/a$ ($a > 0$). Each quality of each intermediate good is obtained one-to-one from physical capital. Innovators compete in prices and have a temporary monopoly for their respective innovations. All other agents behave competitively. All markets always clear. Throughout the analysis, we restrict our attention to allocations with $L_A > 0$.

3 Equilibrium

Perfect competition in the final-goods sector implies $wL_Y = (1-\alpha)Y$, where w is the wage rate. Furthermore, for any intermediate j, one unit of quality ω is a perfect substitute for λ units of quality $\omega - 1$. So, letting $p_\omega(j)$ denote the price of quality ω of intermediate j, only the producer $\tilde{\omega}(j)$ with the lowest quality-adjusted price, $p_\omega(j)/\lambda^\omega$, faces a positive demand. This producer's price elasticity of demand is unit elastic, so profits increase as price rises. In equilibrium, the producer of the maximum-quality intermediate, $\Omega(j)$, prices the lower-quality producers out of the market (i.e., $\tilde{\omega}(j) = \Omega(j)$) with the limit price $\lambda\eta r \equiv p$, where r is the interest rate. From the zero-profit condition for the final-goods sector, $px_{\Omega(j)}(j) = Y - wL_Y = \alpha Y$. Letting $z \equiv Y/K$, it follows that $x_{\Omega(j)}(j) = \alpha Y/(\lambda\eta r) \equiv x$, $K = \eta x$, and $r = \alpha Y/(\lambda K) = \alpha z/\lambda$. Turning to the consumers, assuming bounded utility, an optimal consumption profile satisfies the Ramsey rule $\dot{c}/c = (r-\rho)/\sigma = (\alpha z/\lambda - \rho)/\sigma$ and the transversality condition $\lim_{t\to\infty} e^{-\rho t}\mu(t)B(t) = 0$, where B is the consumers' asset holdings and μ is the associated current-value multiplier. Letting $\chi \equiv c/K$, we have $\dot{\chi}/\chi = \dot{c}/c - \dot{K}/K$ or, using $\dot{K} = Y - c$, the definitions of z and χ, and $r = \alpha z/\lambda$,

$$\frac{\dot{\chi}}{\chi} = \chi - \left(1 - \frac{\alpha}{\lambda\sigma}\right) z - \frac{\rho}{\sigma}. \tag{1}$$

Since only maximum-quality intermediates are produced and because of symmetry, we have $\ln D_Y = \int_0^1 \ln[\lambda^{\Omega(j)} x] dj = \ln\Delta + \ln x$, where $\ln\Delta \equiv \ln\lambda \int_0^1 \Omega(j)dj$ is an indicator of the state of technology. Together with $x = K/\eta$, aggregate output becomes $Y = L_Y^{1-\alpha}(\Delta K/\eta)^\alpha$. Using the definition $z \equiv Y/K$, it follows that $K/L_Y = (\Delta\eta)^{\alpha/(1-\alpha)} z^{-\alpha/(1-\alpha)}$ and $Y/L_Y = [(\Delta/\eta)K/L_Y]^\alpha = [\Delta/(\eta z)]^{\alpha/(1-\alpha)}$. Let P_A denote the value of a firm with a maximum-quality intermediate. Free entry into R&D implies $P_A = wa$. Using $wL_Y = (1-\alpha)Y$, we get $P_A = (1-\alpha)aY/L_Y = (1-\alpha)a[\Delta/(\eta z)]^{\alpha/(1-\alpha)}$. Hence, $\dot{P}_A/P_A = \alpha(\dot{\Delta}/\Delta - \dot{z}/z)/(1-\alpha)$. From the definition of Δ, we have $\dot{\Delta}/\Delta = I\ln\lambda$. Hence $\dot{z}/z = I\ln\lambda - [(1-\alpha)/\alpha]\dot{P}_A/P_A$. A quality leader's current stream of profit is $\pi \equiv (p - \eta r)x = (1-1/\lambda)px = (1-1/\lambda)\alpha Y$. The absence of arbitrage opportunities implies $\pi + \dot{P}_A = (r+I)P_A$. Let $\nu \equiv Y/P_A$, so that the free-entry condition becomes $L_Y = a(1-\alpha)\nu$. Substituting $r = \alpha z/\lambda$, $I = (L - L_Y)/a = L/a - (1-\alpha)\nu$, $\pi = (1-1/\lambda)\alpha Y$, and $\nu \equiv Y/P_A$ into the no-arbitrage equation yields $\dot{P}_A/P_A = \alpha z/\lambda + L/a - (1-\alpha/\lambda)\nu$. Inserting this equality and $I = L/a - (1-\alpha)\nu$ into the expression for $\dot{z}/z$ and simplifying terms gives

$$\frac{\dot{z}}{z} = \frac{1-\alpha}{\alpha}\left[\left(\frac{\alpha}{1-\alpha}\ln\lambda - 1\right)\frac{L}{a} + \left(1 - \frac{\alpha}{\lambda} - \alpha\ln\lambda\right)\nu - \frac{\alpha}{\lambda}z\right]. \tag{2}$$

From $\nu = a(1-\alpha)L_Y$, the definition of z, and $Y/L_Y = [\Delta/(\eta z)]^{\alpha/(1-\alpha)}$, we have $\nu = a(1-\alpha)zK[\Delta/(\eta z)]^{-\alpha/(1-\alpha)}$. Differentiating with respect to time and using $\dot{K}/K = z - \chi$, $\dot{\Delta}/\Delta = I\ln\lambda$, $I = L/a - (1-\alpha)\nu$, and (2), we obtain

$$\frac{\dot{\nu}}{\nu} = \left(\ln\lambda - \frac{1}{\alpha}\right)\frac{L}{a} + \left[\frac{1}{\alpha} - \frac{1}{\lambda} - (1-\alpha)\ln\lambda\right]\nu + \left(1 - \frac{1}{\lambda}\right)z - \chi. \tag{3}$$

Given the starting values $K(0)$ and $\Delta(0)$, equations (1)-(3) determine the evolution of the economy through time: (1) determines c $(= \chi K)$, (2) determines Y $(= zK)$, and (3) pins down L_Y $(= a(1-\alpha)\nu)$, L_A $(= L - L_Y)$, and I $(= L_A/a)$. This in turn determines the evolution of K (since $\dot{K} = Y - c$) and Δ (since $\dot{\Delta}/\Delta = I\ln\lambda$). Finally, the prices are determined by $r = \alpha z/\lambda$, $w = (1-\alpha)Y/L_Y$, and $P_A = aw$.

4 Steady state

Let the steady-state value of any variable y be denoted y^*. $(\chi^*, z^*, \nu^*)'$ comprises a steady state of the model with positive growth if $(\chi, z, \nu)' = (\chi^*, z^*, \nu^*)'$ solves (1)-(3), $I^* > 0$, the transversality condition for the consumers' utility-maximization problem is satisfied, utility is bounded, and $(\chi^*, z^*, \nu^*)' > \mathbf{0}$. Let $l \equiv L/a$ and $\phi \equiv \rho/l$. Moreover, define

$$\bar{\sigma} \equiv 1 - \frac{1 - \frac{\alpha}{\lambda}}{\alpha \ln \lambda} \tag{4}$$

and

$$\bar{\phi} \equiv \frac{\alpha(\lambda - 1)}{\lambda(1 - \alpha)} = \frac{\alpha(1 - \bar{\sigma}) \ln \lambda}{1 - \alpha} - 1. \tag{5}$$

Then, from (1)-(3),[2]

$$\chi^* = \frac{l}{\sigma - \bar{\sigma}} \left[\frac{\lambda}{\alpha} \left(\bar{\phi}\sigma - \phi\,\bar{\sigma} \right) + \phi - \bar{\phi} \right] \tag{6}$$

$$z^* = \frac{l}{\sigma - \bar{\sigma}} \frac{\lambda}{\alpha} \left(\bar{\phi}\sigma - \phi\,\bar{\sigma} \right) \tag{7}$$

$$\nu^* = \frac{l}{\sigma - \bar{\sigma}} \frac{1}{\alpha \ln \lambda} \left[\frac{\alpha}{1 - \alpha} (\sigma - 1) \ln \lambda + 1 + \phi \right]. \tag{8}$$

The steady-state rate of innovation is obtained from $I^* = (L - L_Y^*)/a = l - (1 - \alpha)\nu^*$ and (8):

$$I^* = \frac{\alpha \left(1 - \frac{1}{\lambda}\right) l - (1 - \alpha)\rho}{(\sigma - \bar{\sigma})\alpha \ln \lambda}. \tag{9}$$

From (9), the requirement $I^* > 0$ is satisfied in either of two subsets of the parameter space:

$$\begin{array}{rl} \text{region I:} & \sigma < \bar{\sigma},\ \phi > \bar{\phi} \\ \text{region II:} & \sigma > \bar{\sigma},\ \phi < \bar{\phi} \end{array}. \tag{10}$$

Obviously, the comparative statics are "perverse" in region I: since the denominator in (9) is negative (as the numerator also is), increases in L reduce I^*, while increases in a and ρ increase I^*. Interestingly, this can happen neither in the Romer (1990) model nor in the Grossman-Helpman (1991a, Ch. 4, 1991b) model.[3]

Next, we turn to the transversality condition. Constancy of χ, z, and ν in the steady state implies that c, K, and P_A, respectively, grow at the same rate as Y. As $\dot{\mu}/\mu = -\sigma\dot{c}/c$ and the households' financial wealth is $B = K + P_A$, the transversality condition boils down to $\rho > (1 - \sigma)(\dot{Y}/Y)^*$. We note in passing that this condition is also necessary and sufficient for the boundedness of the utility integral. From $Y = L_Y^{1-\alpha}(\Delta K/\eta)^\alpha$, constancy of L_Y^*, and $(\dot{K}/K)^* = (\dot{Y}/Y)^*$, we have $(\dot{Y}/Y)^* = \alpha I^*(\ln \lambda)/(1 - \alpha)$. Let

$$\phi(\sigma) \equiv (1 - \sigma) \frac{\alpha^2(\lambda - 1) \ln \lambda}{(1 - \alpha)(\lambda - \alpha)}. \tag{11}$$

— **Figure 1 here** —

Notice that $\phi(\bar{\sigma}) = \bar{\phi}$. Then, using (9), we find the following conditions for the transversality condition to be satisfied:

$$\begin{array}{ll} \text{in region I:} & \phi < \phi(\sigma) \\ \text{in region II:} & \phi > \phi(\sigma) \end{array}. \tag{12}$$

It remains for us to characterize the range of parameters such that $(\chi^*, z^*, \nu^*)' > \mathbf{0}$. To do so, we notice that

$$\bar{\phi}\sigma - \phi\bar{\sigma} + \phi - \bar{\phi} = \frac{(\lambda - \alpha)[\phi - \phi(\sigma)]}{\alpha\lambda \ln \lambda}. \tag{13}$$

From (10), (12), and (13), we have

$$\begin{array}{ll} \text{in region I:} & 0 < \phi - \bar{\phi} < \phi\bar{\sigma} - \bar{\phi}\sigma \\ \text{in region II:} & 0 > \phi - \bar{\phi} > \phi\bar{\sigma} - \bar{\phi}\sigma \end{array}. \tag{14}$$

Consider the second inequalities in the two lines in (14), respectively. From the definition of the parameter regions in (10), these inequalities also hold true if one adds $\sigma - \bar{\sigma}$ on the left-hand side. Therefore, we have

$$\begin{array}{ll} \text{in region I:} & (1+\phi)(1-\bar{\sigma}) < (1+\bar{\phi})(1-\sigma) \\ \text{in region II:} & (1+\phi)(1-\bar{\sigma}) > (1+\bar{\phi})(1-\sigma) \end{array}. \tag{15}$$

Consider first χ^*. In region I,

$$\frac{\lambda}{\alpha}\left(\bar{\phi}\sigma - \phi\,\bar{\sigma}\right) + \phi - \bar{\phi} < \frac{\lambda}{\alpha}\left(\bar{\phi}\sigma - \phi\,\bar{\sigma}\right) + \phi\bar{\sigma} - \bar{\phi}\sigma = \left(\frac{\lambda}{\alpha} - 1\right)\left(\bar{\phi}\sigma - \phi\,\bar{\sigma}\right) < 0. \tag{16}$$

So, from (6) and (10), $\chi^* > 0$. In region II, the reverse inequalities hold true in (16). Again, (6) and (10) yield $\chi^* > 0$. As for z^*, (7), (10), and (14) imply $z^* > 0$. Finally, using (5), the term in square brackets in (8) can be rewritten as

$$\frac{\alpha}{1-\alpha}(\sigma - 1)\ln\lambda + 1 + \phi = \frac{(1+\phi)(1-\bar{\sigma}) - (1+\bar{\phi})(1-\sigma)}{1-\bar{\sigma}}.$$

Evidently, from (8) and (15), $\nu^* > 0$. We can sum up:

Theorem 1: *A steady state exists if, and only if, the parameters are in region I or region II defined by (10) and satisfy (12).*

For "realistic" parameter values, the steady state is located in region II, so that the comparative statics conforms to what one would expect (given (4), $\sigma > 1$ is a simple "realistic" and sufficient condition).

— **Figure 2 here** —

Example 1: Let $\sigma = 2$, $\alpha = 0.4$, and $\lambda = 1.2$. Evidently, $\partial I^*/\partial l > 0 > \partial I^*/\partial \rho$. Further, let $\rho = 0.02$ and $l = 2.005$. Then $I^* = 0.1645$ and $(\dot{Y}/Y)^* = 2\%$.

5 Dynamics

Linearizing (1)-(3) about the steady state yields[4]

$$\begin{pmatrix} \dot{\chi} \\ \dot{z} \\ \dot{\nu} \end{pmatrix} = \begin{pmatrix} \chi^* & -\left(1 - \frac{\alpha}{\lambda\sigma}\right)\chi^* & 0 \\ 0 & -\frac{1-\alpha}{\lambda} z^* & -\bar{\sigma}(1-\alpha)(\ln\lambda) z^* \\ -\nu^* & \left(1 - \frac{1}{\lambda}\right)\nu^* & (\alpha - \bar{\sigma})(\ln\lambda)\nu^* \end{pmatrix} \begin{pmatrix} \chi - \chi^* \\ z - z^* \\ \nu - \nu^* \end{pmatrix} \tag{17}$$

or $\dot{\mathbf{x}} = \mathbf{J}(\mathbf{x} - \mathbf{x}^*)$, where $\mathbf{x} \equiv (\chi, z, \nu)'$ and $\mathbf{J}$ is the Jacobian matrix in (17).

Theorem 2: *A steady state in region I is unstable or indeterminate, a steady state in region II is a saddle point.*

Proof: None of the three variables χ, z, and ν is historically given at $t = 0$. However, as shown below, the definitions of z and ν imply a relation between these two variables, so that the steady state is unstable, a saddle point, or indeterminate, depending on whether the number of negative eigenvalues is zero, one, or two, respectively.

From the Routh-Hurwitz Theorem, the number of negative eigenvalues is equal to three minus the number of variations of sign in the scheme $-1 \,||\, \mathrm{Tr}(\mathbf{J}) \,||\, -\mathrm{B}(\mathbf{J}) + \mathrm{Det}(\mathbf{J})/\mathrm{Tr}(\mathbf{J}) \,||\, \mathrm{Det}(\mathbf{J})$, where $\mathrm{Tr}(\mathbf{J})$ is the trace and $\mathrm{Det}(\mathbf{J})$ is the determinant of the Jacobian in (17) (and $\mathrm{B}(\mathbf{J})$ is another term defined using the elements of the Jacobian). Below, we show that in region I, $\mathrm{Det}(\mathbf{J}) > 0$ and $\mathrm{Tr}(\mathbf{J}) > 0$, so that the Routh-Hurwitz sign scheme is $- \,||\, + \,||\, ? \,||\, +$. Hence, the number of negative eigenvalues is zero or two, and the system is unstable or indeterminate, respectively. In region II, by contrast, $\mathrm{Det}(\mathbf{J}) < 0$ and $\mathrm{Tr}(\mathbf{J}) > 0$. The Routh-Hurwitz sign scheme is $- \,||\, + \,||\, ? \,||\, -$, there is one negative eigenvalue, and the steady state is a saddle point. From (17), the determinant is[5]

$$\mathrm{Det}(\mathbf{J}) = -\frac{\alpha(1-\alpha)}{\lambda}\frac{\sigma - \bar{\sigma}}{\sigma}(\ln\lambda)\chi^* z^* \nu^*. \tag{18}$$

Evidently, $\mathrm{Det}(\mathbf{J}) > 0$ for $\sigma < \bar{\sigma}$ and $\mathrm{Det}(\mathbf{J}) < 0$ for $\sigma > \bar{\sigma}$ (s. Figure 2). This proves that the determinant is positive in region I and negative in region II.

From (17), the trace is given by $\chi^* - (1-\alpha)z^*/\lambda + (\alpha - \bar{\sigma})(\ln\lambda)\nu^*$. Substituting for χ^*, z^*, and ν^* from (6)-(8), using (5), and rearranging terms, this can be rewritten as[6]

$$\mathrm{Tr}(\mathbf{J}) = \frac{l}{\alpha}\frac{f(\sigma,\phi,\alpha,\lambda)}{\sigma - \bar{\sigma}}, \tag{19}$$

where

$$f(\sigma,\phi,\alpha,\lambda) \equiv \left(\alpha + \lambda - 1 + \frac{\alpha - \bar{\sigma}}{1-\bar{\sigma}}\right)[(1-\bar{\sigma})\phi - (1-\sigma)\bar{\phi}] + (\lambda - 1)(\bar{\phi} - \phi) + \frac{\alpha - \bar{\sigma}}{1-\bar{\sigma}}(\sigma - \bar{\sigma}). \tag{20}$$

We have $f(\bar{\sigma},\bar{\phi},\alpha,\lambda) = 0$ and, using (4) and (5),

$$\frac{\partial f(\sigma,\phi,\alpha,\lambda)}{\partial\sigma} = \frac{\alpha^2\left(1 - \frac{1}{\lambda} + \ln\lambda\right) + \alpha\left(\lambda - 1 - \ln\lambda + \frac{1}{\alpha} - 1\right)}{1-\alpha} \equiv f_\sigma(\alpha,\lambda) \tag{21}$$

and

$$\frac{\partial f(\sigma,\phi,\alpha,\lambda)}{\partial\phi} = \frac{\lambda - \alpha}{\alpha\lambda\ln\lambda}(\alpha + \lambda - \alpha\lambda\ln\lambda) \equiv f_\phi(\alpha,\lambda). \tag{22}$$

Since $f_\sigma(\alpha,\lambda)$ and $f_\phi(\alpha,\lambda)$ do not depend on σ or ϕ, $\mathrm{Tr}(\mathbf{J}) = 0$ is a straight line with slope $d\phi/d\sigma|_{f(\sigma,\phi)=0} = -f_\sigma(\alpha,\lambda)/f_\phi(\alpha,\lambda)$ in the (σ,ϕ)-plane. The fact that $\lambda - 1 > \ln\lambda$ implies $f_\sigma(\alpha,\lambda) > 0$ for $0 < \alpha < 1$, $\lambda > 1$. So $f(\sigma,\phi,\alpha,\lambda) < 0$ to the left of the $\mathrm{Tr}(\mathbf{J}) = 0$ line and $f(\sigma,\phi,\alpha,\lambda) > 0$ to the right. Furthermore, from (19), the sign of the trace is the same as the sign of $f(\sigma,\phi,\alpha,\lambda)/(\sigma - \bar{\sigma})$. So for $\sigma < \bar{\sigma}$, it is positive to the left and negative to the right of the $f(\sigma,\phi,\alpha,\lambda) = 0$-locus, while for $\sigma > \bar{\sigma}$, it is positive to the right and negative to the left. Suppose $f_\phi(\alpha,\lambda) < 0$. Then $d\phi/d\sigma|_{f(\sigma,\phi)=0} > 0$, and it follows immediately that the trace is positive both in region I and in region II (see panel (a) in Figure 2). Conversely, suppose $f_\phi(\alpha,\lambda) > 0$, so that the $f(\sigma,\phi,\alpha,\lambda) = 0$-locus is downward-sloping. In this case, $\mathrm{Tr}(\mathbf{J}) > 0$ is satisfied if this locus is steeper than the line $\phi(\sigma)$ defined in (11), i.e., if

$$\frac{\frac{\alpha^2\left(1-\frac{1}{\lambda}+\ln\lambda\right)+\alpha\left(\lambda-1-\ln\lambda+\frac{1}{\alpha}-1\right)}{1-\alpha}}{\frac{\lambda-\alpha}{\alpha\lambda\ln\lambda}(\alpha + \lambda - \alpha\lambda\ln\lambda)} > \frac{\alpha^2(\lambda-1)\ln\lambda}{(\lambda-\alpha)(1-\alpha)}$$

(see panel (b) in Figure 2). Simplifying terms yields the equivalent condition[7]

$$1 - \alpha > \alpha\ln\lambda(1 - \alpha\lambda). \tag{23}$$

For $\alpha\lambda < 1$, the term in parentheses is positive, so that, because of $\ln\lambda < \lambda - 1$, (23) is implied by $1-\alpha > \alpha(\lambda-1)(1-\alpha\lambda)$, that is $1 > \alpha\lambda(1+\alpha-\alpha\lambda)$. Maximizing the right-hand side of this latter inequality with respect to $\alpha\lambda$ yields a maximum value of $[(1+\alpha)/2]^2 < 1$ at $\alpha\lambda = (1+\alpha)/2$. This implies the validity of (23). For $\alpha\lambda > 1$, the validity of (23) is obvious.

None of the three variables χ, z, and ν is historically given at $t = 0$. However, by the definitions of z and ν, we have $z = [a(1-\alpha)]^{1-\alpha}(\Delta^{\frac{\alpha}{1-\alpha}}\nu/K)^{1-\alpha}/\eta^{\alpha}$. Evaluating this expression at $t = 0$ and at the steady state, we obtain

$$z(0) - z^* = \frac{[a(1-\alpha)]^{1-\alpha}}{\eta^{\alpha}} \left\{ \left[\frac{\Delta(0)^{\frac{\alpha}{1-\alpha}}}{K(0)} \nu(0) \right]^{1-\alpha} - \left[\left(\frac{\Delta^{\frac{\alpha}{1-\alpha}}}{K} \right)^* \nu^* \right]^{1-\alpha} \right\}. \tag{24}$$

To each eigenvalue q_i $(i = 0, 1, 2)$ corresponds one particular solution $\mathbf{x}(t) - \mathbf{x}^* = \mathbf{b}_i e^{q_i t}$, which satisfies $\dot{\mathbf{x}} = q_i(\mathbf{x} - \mathbf{x}^*)$, where $\mathbf{b}_i = (b_{\chi i}, b_{zi}, b_{\nu i})'$ is the eigenvector associated with q_i. Together with $\dot{\mathbf{x}} = \mathbf{J}(\mathbf{x} - \mathbf{x}^*)$, it follows that $(\mathbf{J} - q_i\mathbf{I})[\mathbf{x}(t) - \mathbf{x}^*] = 0$, where where $\mathbf{I}$ is the 3×3 identity matrix. Inserting $\mathbf{x}(t) - \mathbf{x}^* = \mathbf{b}_i e^{q_i t}$ and setting $t = 0$ gives $(\mathbf{J} - q_i\mathbf{I})\mathbf{b}_i = 0$, which determines the eigenvectors $\mathbf{b}_i$. For instance, for $i = 0$, the second line in this system of equations reads

$$\left(-\frac{1-\alpha}{\lambda} z^* - q_0 \right) b_{z0} - \bar{\sigma}(1-\alpha) \ln \lambda \, z^* b_{\nu 0} = 0. \tag{25}$$

The general solution of $\dot{\mathbf{x}} = \mathbf{J}(\mathbf{x} - \mathbf{x}^*)$ is $\mathbf{x}(t) = \sum_{i=0}^{2} A_i \mathbf{b}_i e^{q_i t}$.

Consider first a steady state in region I. Instability means that $A_i = 0$ for $i = 0, 1, 2$, so that there is no way to reach the steady state if the economy does not happen to be endowed with $\Delta(0)^{\alpha/(1-\alpha)}/K(0) = [\Delta^{\alpha/(1-\alpha)}/K]^*$. In the case of two stable eigenvalues, $A_i = 0$ only for $i = 2$, so $\mathbf{x}(t) - \mathbf{x}^* = \sum_{i=0}^{1} A_i \mathbf{b}_i e^{q_i t}$ and $\mathbf{x}(0) - \mathbf{x}^* = \sum_{i=0}^{1} A_i \mathbf{b}_i$. Given the eigenvectors $\mathbf{b}_0$ and $\mathbf{b}_1$, this gives three equations in the five unknowns $\chi(0)$, $z(0)$, $\nu(0)$, A_0, and A_1. (24) provides one further equation in these unknowns. So we have an underdetermined system of equations with a multiplicity of solutions, i.e. indeterminacy of the equilibrium growth path.

As for a steady state in region II, since $A_i = 0$ for the unstable eigenvalues $(i = 1, 2)$, $\mathbf{x}(t) - \mathbf{x}^* = \mathbf{b}_0 e^{q_0 t}$. Setting $t = 0$ yields $\mathbf{x}(0) - \mathbf{x}^* = \mathbf{b}_0$. So we can replace b_{z0} and $b_{\nu 0}$ in (25) with $z(0) - z^*$ and $\nu(0) - \nu^*$, respectively. Equations (24) and (25) then determine the starting values $z(0)$ and $\nu(0)$. Given $b_{z0} = z(0) - z^*$ and $b_{\nu 0} = \nu(0) - \nu^*$, the first line in $(\mathbf{J} - q_i\mathbf{I})\mathbf{b}_i = 0$ then determines $\chi(0) - \chi^* = b_{\chi_i}$. As argued at the end of Section 3, this determines the evolution of all quantities and prices through time.[8] This completes the proof of Theorem 2. ■

Example 1:[9] In the example introduced at the end of Section 4, the eigenvalues are: $q_0 = -0.0384$, $q_1 = 0.0887$, and $q_2 = 4.7966$.

Remark 1: Theorem 2 provides a complete analytical characterization of the model's dynamics. One may wonder if it is possible to rule out either instability or indeterminacy in region I. But it is easy to construct examples for both kinds of dynamics.

Example 2: Let $\sigma = 0.002$, $\alpha = 0.8$, $\lambda = 2.5$, $l = 0.04$, and $\rho = 0.1$. Then, $q_{0/1} = -0.0460 \pm 0.2188i$ and $q_2 = 0.4046$, which implies indeterminacy.

Example 3: Let the parameters be as in Example 2 except that $\sigma = 0.02$. Then, $q_{0/1} = 0.0069 \pm 0.0647i$ and $q_2 = 0.2656$, so that the steady state is unstable.

Remark 2: Examples 2 and 3 suggest that the model dynamics undergoes a Hopf bifurcation as σ grows from very small to somewhat larger values, so that there is either an unstable limit cycle in the indeterminacy region or a stable limit cycle in the unstable region. In fact, one finds that the real part of the complex eigenvalues $q_{0/1}$ becomes positive as σ increases beyond the critical value $\sigma = 0.0099$.

6 M-Country Open Economy

Arnold (in press) analyzes a class of growth models with the "Dixit-Norman property" (cf. Dixit and Norman, 1980, Chapter 4), that a world economy made up of several countries replicates the equilibrium of a hypothetical integrated economy without restrictions on factor movements under certain conditions. The Romer model with quality upgrading considered here belongs to this class of growth models. So we can immediately infer from the analysis in Arnold (Theorem 1, in press) the conditions under which a world economy made up of several countries of the type described in Section 2 (with labor immobility) behaves just like the hypothetical integrated world economy (with complete labor mobility):

Theorem 3: *Suppose the world economy is made up of M (≥ 2) countries with identical tastes and technologies in each country. Suppose further that there is free trade in the final good and the intermediates, financial capital is perfectly mobile, and knowledge spillovers are international in scope. Then, the M-country world economy replicates the equilibrium of the hypothetical integrated world economy if, and only if, physical capital is mobile internationally and/or multinational firms or international patent licensing are allowed for.*

Proof: The proof can be sketched as follows. The replication of the M-country world economy equilibrium is possible if, and only if, the world production levels can be split across countries in such a way that activity levels are non-negative in each country. There are three productive activities: final goods production, intermediate goods production, and R&D. Final goods production and R&D are internationally mobile, in that nothing pins down their location. The production of intermediates is tied to the country where they have been invented if, and only if, multinational firms and international patent licensing are ruled out. There are two primary factors of production: labor and physical capital. While labor is immobile by assumption, ("old") physical capital, once installed, can be assumed to be mobile or not (whereas, due to the assumption of free trade in the final good, "new" physical capital can be transported abroad without any frictions). We use lower-case letters with a superscript m to denote the country-m ($m = 1, \ldots, M$) levels of variables denoted by upper-case letters so far. Let k^m denote physical capital *owned by* country-m residents and k'^m capital *used in* country m. Similarly, let a^m denote the number of intermediates whose highest quality has been *invented in* country m and a'^m the number of leading-edge intermediates *produced in* country m. Since intermediate goods production is the only use of physical capital,

$$k'^m = a'^m x, \quad m = 1, \ldots, M, \tag{26}$$

where x is the uniform quantity produced of each intermediate. If physical capital is immobile (i.e., $k'^m = k^m$) and intermediates have to be produced where they have been invented ($a'^m = a^m$), then (26) is not satisfied generally. With physical capital mobility, k'^m is free to adjust so that (26) is satisfied. With multinationals or patent licensing, a'^m adjusts so that (26) holds. In both cases, replication is feasible, and the allocation of labor l^m to final goods production or R&D is indeterminate. If both physical capital and intermediate goods production are internationally mobile, there is another degree of freedom. ■

Theorem 3 says that if either physical capital is mobile internationally or innovative products can be manufactured abroad (within multinational firms or due to international patent licensing) or both, then the analysis of the closed economy in Sections 3-5 carries over to a world economy made up of M identical countries (except for size) with international knowledge spillovers and free trade. Even if the parameters are in region II, so that the integrated equilibrium is determinate, the division of the integrated-equilibrium input vectors for R&D and final goods across countries is indeterminate. Indeterminacy of the equilibrium adds another degree of indeterminacy. Obviously, the presence of several countries does not help avoid the possible instability of the integrated equilibrium.

7 Conclusions

The integration of quality upgrading à la Grossman and Helpman (1991a, Ch. 4, 1991b) into the Romer (1990) model gives rise to unusual comparative-statics effects and interesting dynamics, which cannot occur in either of the two models. Moreover, if physical capital is mobile internationally and/or multinational firms or international patent licensing are allowed for, then these results carry over to a M-country world economy with international knowledge spillovers. These findings contribute to our understanding of the dynamics of one of the most prominent endogenous growth models.

Notes

[1]The time argument is suppressed unless this might cause confusion.

[2]See Appendix A.

[3]Grossman and Helpman (1991a, Ch. 4, 1991b) assume logarithmic utility. The results that their model produces well-behaved comparative statics and that there is a unique convergent growth path do not hinge on this assumption. As for the steady-state rate of innovation, $I^* = [(\lambda - 1)l - \rho] \,/\, [(\sigma - 1)\ln\lambda + \lambda]$, this follows from the fact that the denominator is positive (as $\lambda > 1 + \ln\lambda$, it is greater than $\sigma\ln\lambda + 1 > 0$).

[4]See Appendix A.

[5]See Appendix A.

[6]See Appendix A.

[7]See Appendix A.

[8]See Appendix A.

[9]Outputs from computations with Maple for this and the following examples can be found in Appendix B.

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Appendix A: Referee's Appendix

Derivation of χ^*, z^* and ν^*

In steady state, $\dot{\chi}/\chi = \dot{z}/z = \dot{\nu}/nu = 0$. Hence, (1)-(3) become

$$\chi^* = z^* \left(1 - \frac{\alpha}{\lambda\,\sigma}\right) + \frac{\rho}{\sigma} \tag{A.1}$$

$$z^* = \frac{\lambda}{\alpha}\left[l\left(\frac{\alpha}{1-\alpha}\ln\lambda - 1\right) - \nu^*\alpha\,\bar{\sigma}\ln\lambda\right] \tag{A.2}$$

$$\nu^* = \left[\chi^* - z^*\left(1 - \frac{1}{\lambda}\right) - l\left(\ln\lambda - \frac{1}{\alpha}\right)\right]\frac{1}{(\alpha - \bar{\sigma})\ln\lambda}, \tag{A.3}$$

respectively. Substituting χ^* from (A.1) into (A.3) gives

$$\nu^* = \frac{z^*\frac{1}{\lambda}\left(1 - \frac{\alpha}{\sigma}\right) + \frac{\rho}{\sigma} - l\left(\ln\lambda - \frac{1}{\alpha}\right)}{(\alpha - \bar{\sigma})\ln\lambda}.$$

Eliminating z^* using (A.2), we obtain

$$\begin{aligned}
\nu^* &= \left\{\frac{1}{\alpha}\left[l\left(\frac{\alpha}{1-\alpha}\ln\lambda - 1\right) - \nu^*\alpha\,\bar{\sigma}\ln\lambda\right]\left(1 - \frac{\alpha}{\sigma}\right)\right. \\
&\quad \left. + \frac{\rho}{\sigma} - l\left(\ln\lambda - \frac{1}{\alpha}\right)\right\}\frac{1}{(\alpha - \bar{\sigma})\ln\lambda} \\
\nu^* + \nu^*\bar{\sigma}\left(1 - \frac{\alpha}{\sigma}\right)\ln\lambda\frac{1}{(\alpha - \bar{\sigma})\ln\lambda} &= \left[\frac{1}{\alpha}l\left(\frac{\alpha}{1-\alpha}\ln\lambda - 1\right)\left(1 - \frac{\alpha}{\sigma}\right) + \frac{\rho}{\sigma}\right. \\
&\quad \left. - l\left(\ln\lambda - \frac{1}{\alpha}\right)\right]\frac{1}{(\alpha - \bar{\sigma})\ln\lambda} \\
\nu^*\alpha\left(1 - \frac{\bar{\sigma}}{\sigma}\right)\frac{\ln\lambda}{(\alpha - \bar{\sigma})\ln\lambda} &= \left\{l\left[\frac{\alpha}{1-\alpha}\left(1 - \frac{1}{\sigma}\right)\ln\lambda + \frac{1}{\sigma}\right] + \frac{\rho}{\sigma}\right\}\frac{1}{(\alpha - \bar{\sigma})\ln\lambda}.
\end{aligned}$$

Rearranging terms yields (8).

In order to obtain z^*, plug (8) into (A.2), factor out $1/(\sigma - \bar{\sigma})$, simplify, use definition (5), and factor out l:

$$\begin{aligned}
z^* &= \frac{\lambda}{\alpha}\left(\left(\frac{\alpha}{1-\alpha}\ln\lambda - 1\right)l - \frac{\bar{\sigma}}{\sigma - \bar{\sigma}}\left\{\left[\frac{\alpha}{1-\alpha}(\sigma - 1)\ln\lambda + 1\right]l + \rho\right\}\right) \\
&= \frac{\lambda}{\alpha}\frac{1}{\sigma - \bar{\sigma}}\left[\frac{\alpha(1-\bar{\sigma})\ln\lambda}{1-\alpha}\sigma l - \sigma l - \bar{\sigma}\rho\right] \\
&= \frac{\lambda}{\alpha}\frac{l}{\sigma - \bar{\sigma}}(\bar{\phi}\sigma - \phi\bar{\sigma}).
\end{aligned}$$

Finally, the formula for χ^* follows from inserting (7) into (A.1) and rearranging terms:

$$\begin{aligned}
\chi^* &= \frac{l}{\sigma - \bar{\sigma}}\cdot\frac{\lambda}{\alpha}(\bar{\phi}\sigma - \phi\,\bar{\sigma})\left(1 - \frac{\alpha}{\lambda\sigma}\right) + \frac{\rho}{\sigma} \\
&= \frac{l}{\sigma - \bar{\sigma}}\left[\frac{\lambda}{\alpha}(\bar{\phi}\sigma - \phi\,\bar{\sigma})\frac{\lambda\sigma - \alpha}{\lambda\sigma} + \frac{\phi\,(\sigma - \bar{\sigma})}{\sigma}\right] \\
&= \frac{l}{\sigma - \bar{\sigma}}\left[\frac{\lambda}{\alpha}\left(\bar{\phi}\sigma - \phi\,\bar{\sigma}\right) + \phi - \bar{\phi}\right].
\end{aligned}$$

Derivation of the linearized dynamic system

The local dynamics of the system described by (1)-(3) can be analyzed by linearizing the system. As for $\dot{\chi}$, we get

$$\begin{aligned}\dot{\chi} &\approx \underbrace{\dot{\chi}(\chi^*, z^*, \nu^*)}_{=0} + \left.\frac{\partial \dot{\chi}}{\partial \chi}\right|_{\chi^*, z^*, \nu^*} (\chi - \chi^*) + \left.\frac{\partial \dot{\chi}}{\partial z}\right|_{\chi^*, z^*, \nu^*} (z - z^*) + \left.\frac{\partial \dot{\chi}}{\partial \nu}\right|_{\chi^*, z^*, \nu^*} (\nu - \nu^*) \\ &= \chi^*(\chi - \chi^*) - \chi^* \left(1 - \frac{\alpha}{\lambda\,\sigma}\right)(z - z^*).\end{aligned}$$

Analogously,

$$\dot{z} \approx -\frac{1-\alpha}{\lambda} z^*(z - z^*) + \left[\left(1 - \frac{\alpha}{\lambda} - \alpha \ln \lambda\right)\frac{1-\alpha}{\alpha} z^*\right](\nu - \nu^*)$$

$$\dot{\nu} \approx \nu^* \left(\frac{1}{\alpha} - \frac{1}{\lambda} - (1-\alpha)\ln \lambda\right)(\nu - \nu^*) + \left(1 - \frac{1}{\lambda}\right)\nu^*(z - z^*) - \nu^*(\chi - \chi^*).$$

Replacing the terms $1 - \alpha/\lambda - \alpha \ln \lambda$ and $1/\alpha - 1/\lambda - (1-\alpha)\ln\lambda$ with $-\bar{\sigma}\alpha \ln \lambda$ and $(\alpha - \bar{\sigma})\ln \lambda$, respectively, yields the system in (17).

Derivation of Det(J)

The determinant of $\mathbf{J}$ is:

$$\begin{aligned}\mathrm{Det}(\mathbf{J}) &\equiv -\chi^* \left(\frac{1-\alpha}{\lambda}\right) z^* \nu^* (\alpha - \bar{\sigma}) \ln \lambda - \left(1 - \frac{\alpha}{\lambda\sigma}\right)\chi^* z^* \alpha \bar{\sigma} \frac{1-\alpha}{\alpha} \nu^* \ln \lambda \\ &\quad + \chi^* z^* \alpha \bar{\sigma} \frac{1-\alpha}{\alpha}\left(1 - \frac{1}{\lambda}\right)\nu^* \ln \lambda \\ &= \chi^* z^* \nu^* (1-\alpha) \ln \lambda \left[-\frac{\alpha - \bar{\sigma}}{\lambda} - \left(1 - \frac{\alpha}{\lambda\sigma}\right)\bar{\sigma} + \left(1 - \frac{1}{\lambda}\right)\bar{\sigma}\right] \\ &= -\frac{\alpha(1-\alpha)}{\lambda}\frac{\sigma - \bar{\sigma}}{\sigma}(\ln \lambda)\chi^* z^* \nu^*.\end{aligned}$$

Derivation of Tr(J)

The trace of $\mathbf{J}$ is

$$\mathrm{Tr}(\mathbf{J}) = \chi^* - \frac{1-\alpha}{\lambda} z^* + \nu^* \left(1 - \frac{\bar{\sigma}}{\alpha}\right)\alpha \ln \lambda.$$

Inserting the expressions for χ^*, z^*, and ν^*, we get

$$\begin{aligned}\mathrm{Tr}(\mathbf{J}) &= \frac{l}{\sigma - \bar{\sigma}}\left[(\bar{\phi}\sigma - \phi\,\bar{\sigma})\frac{\lambda - 1 + \alpha}{\alpha} + \phi - \bar{\phi} + \frac{\alpha - \bar{\sigma}}{1-\alpha}(\sigma - 1)\ln \lambda + (1+\phi)\frac{\alpha - \bar{\sigma}}{\alpha}\right] \\ &= \frac{l}{\sigma - \bar{\sigma}}\left\{(\bar{\phi}\sigma - \phi\,\bar{\sigma})\frac{\lambda - 1 + \alpha}{\alpha} + \phi - \bar{\phi} + \left(1 - \frac{\bar{\sigma}}{\alpha}\right)\left[\frac{\alpha}{1-\alpha}(\sigma - 1)\ln \lambda + (1+\phi)\right]\right\}\end{aligned}$$

or, using definition (5),

$$
\begin{aligned}
\mathrm{Tr}(\mathbf{J}) &= \frac{l}{\sigma-\bar{\sigma}}\left\{(\bar{\phi}\sigma-\phi\,\bar{\sigma})\frac{\lambda-1+\alpha}{\alpha}+\phi-\bar{\phi}+\left(1-\frac{\bar{\sigma}}{\alpha}\right)\left[1+\phi-\frac{1-\sigma}{1-\bar{\sigma}}(1+\bar{\phi})\right]\right\} \\
&= \frac{l}{\alpha(\sigma-\bar{\sigma})}\Big\{\alpha(\bar{\phi}\sigma-\phi\,\bar{\sigma}+\phi-\bar{\phi})+(\lambda-1)(\bar{\phi}\sigma-\phi\,\bar{\sigma})+ \\
&\quad+\frac{\alpha-\bar{\sigma}}{1-\bar{\sigma}}\left[(1-\bar{\sigma})(1+\phi)-(1-\sigma)(1+\bar{\phi})\right]\Big\} \\
&= \frac{l}{\alpha(\sigma-\bar{\sigma})}\Big\{\alpha\left[(1-\bar{\sigma})\phi-(1-\sigma)\bar{\phi}\right]+(\lambda-1)\left[(1-\bar{\sigma})\phi-(1-\sigma)\bar{\phi}+\bar{\phi}-\phi\right]+ \\
&\quad+\frac{\alpha-\bar{\sigma}}{1-\bar{\sigma}}\left[(1-\bar{\sigma})\phi-(1-\sigma)\bar{\phi}+\sigma-\bar{\sigma}\right]\Big\} \\
&= \frac{l}{\alpha(\sigma-\bar{\sigma})}\Big\{\left(\alpha+\lambda-1+\frac{\alpha-\bar{\sigma}}{1-\bar{\sigma}}\right)\left[(1-\bar{\sigma})\phi-(1-\sigma)\bar{\phi}\right]+(\lambda-1)\left(\bar{\phi}-\phi\right)+ \\
&\quad+\frac{\alpha-\bar{\sigma}}{1-\bar{\sigma}}\left(\sigma-\bar{\sigma}\right)\Big\}.
\end{aligned}
$$

The expression in braces is the function f defined in (20), so (19) follows.

The derivatives of $f(\sigma,\phi,\alpha,\lambda)$

The derivative of $f(\sigma,\phi,\alpha,\lambda)$ with respect to σ is

$$
\begin{aligned}
f_\sigma(\alpha,\lambda) &= \left(\alpha+\lambda-1+\frac{\alpha-\bar{\sigma}}{1-\bar{\sigma}}\right)\bar{\phi}+\frac{\alpha-\bar{\sigma}}{1-\bar{\sigma}} \\
&= (\alpha+\lambda)\bar{\phi}+(1+\bar{\phi})\frac{\alpha-\bar{\sigma}}{1-\bar{\sigma}}-\bar{\phi}.
\end{aligned}
$$

Eliminating $\bar{\phi}$ using (5) yields

$$
f_\sigma(\alpha,\lambda)=\frac{(\alpha+\lambda)\alpha(\lambda-1)}{\lambda(1-\alpha)}+1-\alpha\ln\lambda.
$$

Factoring out $1/(1-\alpha)$ and rearranging terms, we get (21).

As for (22), differentiating $f(\sigma,\phi,\alpha,\lambda)$ with respect to ϕ and simplifying terms gives:

$$
\begin{aligned}
f_\phi(\alpha,\lambda) &= \left(\alpha+\lambda-1+\frac{\alpha-\bar{\sigma}}{1-\bar{\sigma}}\right)(1-\bar{\sigma})-(\lambda-1) \\
&= \alpha(1-\bar{\sigma})+(\lambda-1)(1-\bar{\sigma})+\alpha-\bar{\sigma}-(\lambda-1) \\
&= \alpha(1-\bar{\sigma})-\lambda\bar{\sigma}+\alpha
\end{aligned}
$$

Expand $-\lambda\bar{\sigma}+\alpha$ to $\lambda(1-\bar{\sigma})-(\lambda-\alpha)$, collect the $(1-\bar{\sigma})$-terms, and plug in (4) to obtain

$$
\begin{aligned}
f_\phi(\alpha,\lambda) &= \alpha(1-\bar{\sigma})+\lambda(1-\bar{\sigma})-(\lambda-\alpha) \\
&= (\alpha+\lambda)\frac{\lambda-\alpha}{\alpha\lambda\ln\lambda}-(\lambda-\alpha) \\
&= \frac{\lambda-\alpha}{\alpha\lambda\ln\lambda}\left(\alpha+\lambda-\alpha\lambda\ln\lambda\right).
\end{aligned}
$$

Derivation of (23)

Starting from

$$\frac{\frac{\alpha^2\left(1-\frac{1}{\lambda}+\ln\lambda\right)+\alpha\left(\lambda-1-\ln\lambda+\frac{1}{\alpha}-1\right)}{1-\alpha}}{\frac{\lambda-\alpha}{\alpha\lambda\ln\lambda}(\alpha+\lambda-\alpha\lambda\ln\lambda)} > \frac{\alpha^2(\lambda-1)\ln\lambda}{(\lambda-\alpha)(1-\alpha)},$$

cancel the terms $1-\alpha$, $\lambda-\alpha$, and $\alpha\ln\lambda$ and multiply through to get

$$\alpha^2(\lambda-1+\lambda\ln\lambda)+\alpha\left(\lambda^2-\lambda-\lambda\ln\lambda+\frac{\lambda}{\alpha}-\lambda\right) > \alpha(\lambda-1)(\alpha+\lambda-\alpha\lambda\ln\lambda).$$

Simplifying gives (23).

Derivation of (24)

$$z=\frac{Y}{K}=\frac{\Delta^\alpha}{\eta^\alpha}\frac{L_Y^{1-\alpha}}{K^{1-\alpha}}=\frac{\Delta^\alpha}{\eta^\alpha}\frac{[a(1-\alpha)\nu]^{1-\alpha}}{K^{1-\alpha}}=\frac{[a(1-\alpha)]^{1-\alpha}}{\eta^\alpha}\left(\frac{\Delta^{\frac{\alpha}{1-\alpha}}}{K}\,\nu\right)^{1-\alpha}.$$

Determination of initial values in region II

Equation (25) with $b_{z0}=z(0)-z^*$ and $b_{\nu0}=\nu(0)-\nu^*$ reads

$$\left(-\frac{1-\alpha}{\lambda}\,z^*-q_0\right)[z(0)-z^*]-\bar{\sigma}(1-\alpha)\ln\lambda\,z^*[\nu(0)-\nu^*]=0.$$

This equation and (24) determine the initial values of $\nu(0)$ and $z(0)$. Furthermore, $\nu(0)$ directly determines the amount of labor employed in the production sector in $t=0$ via $L_Y(0)=a(1-\alpha)\nu(0)$. The first line in $(\mathbf{J}-q_i\mathbf{I})\mathbf{b}_i=0$ then determines $\chi(0)$, which in turn pins down the starting value of consumption through the definition of χ: $c(0)=K(0)\chi(0)$.

Appendix B: Examples

Notation

This appendix shows the Maple outputs associated with Examples 1-3 in the main text. "sigma" and "sigma_bar" correspond to σ and $\bar{\sigma}$, and analogously for all other Greek variables. "v" corresponds to ν.

Maple output for example 1

```
> restart;
> with(LinearAlgebra):
> lambda:=1.2;l:=2.005;alpha:=.4;sigma:=2.;rho:=.02;
```

$$\lambda := 1.2$$

$$l := 2.005$$

$$\alpha := 0.4$$

$$\sigma := 2.$$

$$\rho := 0.02$$

```
> phi_bar:=alpha*(1-1/lambda)/(1-alpha);
```

$$phi\_bar := 0.1111111111$$

```
> phi_(_sigma):=alpha^2*(1-sigma)*log(lambda)*(1-1/lambda)/((1-alpha/la
> mbda)*(1-alpha));
```

$$\mathrm{phi\_}(\_\sigma) := -0.01215477046$$

```
> phi:=rho/l;
```

$$\phi := 0.009975062344$$

```
> sigma_bar:=1-(1-alpha/lambda)/(alpha*log(lambda));
```

$$sigma\_bar := -8.141358248$$

```
> X_star:=l*(lambda*(phi_bar*sigma-phi*sigma_bar)/alpha+phi-phi_bar)/(s
> igma-sigma_bar);
```

$$X\_star := 0.1599756506$$

```
> z_star:=l*lambda*(phi_bar*sigma-phi*sigma_bar)/(alpha*(sigma-sigma_ba
> r));
```

$$z\_star := 0.1799707807$$

```
> v_star:=(alpha/(1-alpha)*log(lambda)*(sigma-1)+1+phi)*l/(alpha*log(la
> mbda)*(sigma-sigma_bar));
```

$$v\_star := 3.067492692$$

```
> M:=Matrix([[X_star,-X_star*(1-alpha/(lambda*sigma)),0],[0,-z_star*(1-
> alpha)/lambda,z_star*(1-alpha)*(1-alpha/lambda-alpha*log(lambda))/alph
> a],[-v_star,v_star*(1-1/lambda),v_star*(1/alpha-1/lambda-(1-alpha)*log
> (lambda))]]);
```

$$M := \begin{bmatrix} 0.1599756506 & -0.1333130422 & 0 \\ 0 & -0.08998539033 & 0.1602832489 \\ -3.067492692 & 0.5112487821 & 4.776925795 \end{bmatrix}$$

```
> Eigenvalues(M);
```

$$\begin{bmatrix} 4.79658807353882376 + 0.\, I \\ 0.0887063455944296226 + 0.\, I \\ -0.0383783638632547858 + 0.\, I \end{bmatrix}$$

Maple output for example 2

```
> restart;
```

```
> with(LinearAlgebra):
```

```
> lambda:=2.5;l:=0.04;alpha:=.8;sigma:=.02;rho:=.1;
```

$$\lambda := 2.5$$

$$l := 0.04$$

$$\alpha := 0.8$$

$$\sigma := 0.02$$

$$\rho := 0.1$$

```
> phi_bar:=alpha*(1-1/lambda)/(1-alpha);
```

$$phi\_bar := 2.400000000$$

```
> phi_(_sigma):=alpha^2*(1-sigma)*log(lambda)*(1-1/lambda)/((1-alpha/la
> mbda)*(1-alpha));
```

$$\mathrm{phi\_}(\_\sigma) := 2.535430354$$

```
> phi:=rho/l;
```

$$\phi := 2.500000000$$

```
> sigma_bar:=1-(1-alpha/lambda)/(alpha*log(lambda));
```

$$sigma\_bar := 0.0723468323$$

```
> X_star:=l*(lambda*(phi_bar*sigma-phi*sigma_bar)/alpha+phi-phi_bar)/(s
> igma-sigma_bar);
```

$$X\_star := 0.2408624279$$

```
> z_star:=l*lambda*(phi_bar*sigma-phi*sigma_bar)/(alpha*(sigma-sigma_ba
> r));
```

$$z\_star := 0.3172758383$$

```
> v_star:=(alpha/(1-alpha)*log(lambda)*(sigma-1)+1+phi)*l/(alpha*log(la
> mbda)*(sigma-sigma_bar));
```

$$v\_star := 0.09575714315$$

```
> M:=Matrix([[X_star,-X_star*(1-alpha/(lambda*sigma)),0],[0,-z_star*(1-
> alpha)/lambda,z_star*(1-alpha)*(1-alpha/lambda-alpha*log(lambda))/alph
> a],[-v_star,v_star*(1-1/lambda),v_star*(1/alpha-1/lambda-(1-alpha)*log
> (lambda))]]);
```

$$M := \begin{bmatrix} 0.2408624279 & 3.612936418 & 0 \\ 0 & -0.02538206706 & -0.004206489505 \\ -0.09575714315 & 0.05745428589 & 0.06384529512 \end{bmatrix}$$

```
> Eigenvalues(M);
```

$$\begin{bmatrix} 0.265558499569270356 + 0.\,I \\ 0.00688357819536491480 + 0.0646694111662209776\,I \\ 0.00688357819536491480 - 0.0646694111662209776\,I \end{bmatrix}$$

Maple output for example 3

```
> restart;
> with(LinearAlgebra):
> lambda:=2.5;l:=0.04;alpha:=.8;sigma:=.002;rho:=.1;
```

$$\lambda := 2.5$$

$$l := 0.04$$

$$\alpha := 0.8$$

$$\sigma := 0.002$$

$$\rho := 0.1$$

```
> phi_bar:=alpha*(1-1/lambda)/(1-alpha);
```

$$phi\_bar := 2.400000000$$

```
> phi_(_sigma):=alpha^2*(1-sigma)*log(lambda)*(1-1/lambda)/((1-alpha/la
> mbda)*(1-alpha));
```

$$\mathrm{phi\_}(\_\sigma) := 2.581999484$$

```
> phi:=rho/l;
```

$$\phi := 2.500000000$$

```
> sigma_bar:=1-(1-alpha/lambda)/(alpha*log(lambda));
```

$$sigma\_bar := 0.0723468323$$

```
> X_star:=l*(lambda*(phi_bar*sigma-phi*sigma_bar)/alpha+phi-phi_bar)/(s
> igma-sigma_bar);
```

$$X\_star := 0.2559942578$$

```
> z_star:=l*lambda*(phi_bar*sigma-phi*sigma_bar)/(alpha*(sigma-sigma_ba
> r));
```

$$z\_star := 0.3128553821$$

```
> v_star:=(alpha/(1-alpha)*log(lambda)*(sigma-1)+1+phi)*l/(alpha*log(la
> mbda)*(sigma-sigma_bar));
```

$$v\_star := 0.1224302907$$

```
> M:=Matrix([[X_star,-X_star*(1-alpha/(lambda*sigma)),0],[0,-z_star*(1-
> alpha)/lambda,z_star*(1-alpha)*(1-alpha/lambda-alpha*log(lambda))/alph
> a],[-v_star,v_star*(1-1/lambda),v_star*(1/alpha-1/lambda-(1-alpha)*log
> (lambda))]]);
```

$$M := \begin{bmatrix} 0.2559942578 & 40.70308699 & 0 \\ 0 & -0.02502843057 & -0.004147882450 \\ -0.1224302907 & 0.07345817442 & 0.08162939896 \end{bmatrix}$$

```
> Eigenvalues(M);
```

$$\begin{bmatrix} 0.404618673073697422 + 0.\,I \\ -0.0460117234418486731 + 0.218788686402117656\,I \\ -0.0460117234418486731 - 0.218788686402117656\,I \end{bmatrix}$$

Maple output for the Hopf bifurcation

```
> restart;
> with(LinearAlgebra):
> lambda:=2.5;l:=.04;alpha:=.8;sigma:=.0099240639294999999;rho:=.1;
```

$$\lambda := 2.5$$

$$l := 0.04$$

$$\alpha := 0.8$$

$$\sigma := 0.0099240639294999999$$

$$\rho := 0.1$$

```
> phi_bar:=alpha*(1-1/lambda)/(1-alpha);
```

$$phi\_bar := 2.400000000$$

```
> phi_(_sigma):=alpha^2*(1-sigma)*log(lambda)*(1-1/lambda)/((1-alpha/la
> mbda)*(1-alpha));
```

$$\text{phi\_}(\_\sigma) := 2.561498553$$

```
> phi:=rho/l;
```

$$\phi := 2.500000000$$

```
> sigma_bar:=1-(1-alpha/lambda)/(alpha*log(lambda));
```

$$sigma\_bar := 0.0723468323$$

```
> X_star:=l*(lambda*(phi_bar*sigma-phi*sigma_bar)/alpha+phi-phi_bar)/(s
> igma-sigma_bar);
```

$$X\_star := 0.2504080855$$

```
> z_star:=l*lambda*(phi_bar*sigma-phi*sigma_bar)/(alpha*(sigma-sigma_ba
> r));
```

$$z\_star := 0.3144872685$$

```
> v_star:=(alpha/(1-alpha)*log(lambda)*(sigma-1)+1+phi)*l/(alpha*log(la
> mbda)*(sigma-sigma_bar));
```

$$v\_star := 0.1125834459$$

```
> M:=Matrix([[X_star,-X_star*(1-alpha/(lambda*sigma)),0],[0,-z_star*(1-
> alpha)/lambda,z_star*(1-alpha)*(1-alpha/lambda-alpha*log(lambda))/alph
> a],[-v_star,v_star*(1-1/lambda),v_star*(1/alpha-1/lambda-(1-alpha)*log
> (lambda))]]);
```

$$M := \begin{bmatrix} 0.2504080855 & 7.823964262 & 0 \\ 0 & -0.02515898148 & -0.004169518239 \\ -0.1125834459 & 0.06755006754 & 0.07506409540 \end{bmatrix}$$

```
> Eigenvalues(M);
```

$$\begin{bmatrix} 0.300313199426305910 + 0.\,I \\ -0.315236817769251610\,10^{-11} + 0.104354008490003313\,I \\ -0.315236817769251610\,10^{-11} - 0.104354008490003313\,I \end{bmatrix}$$

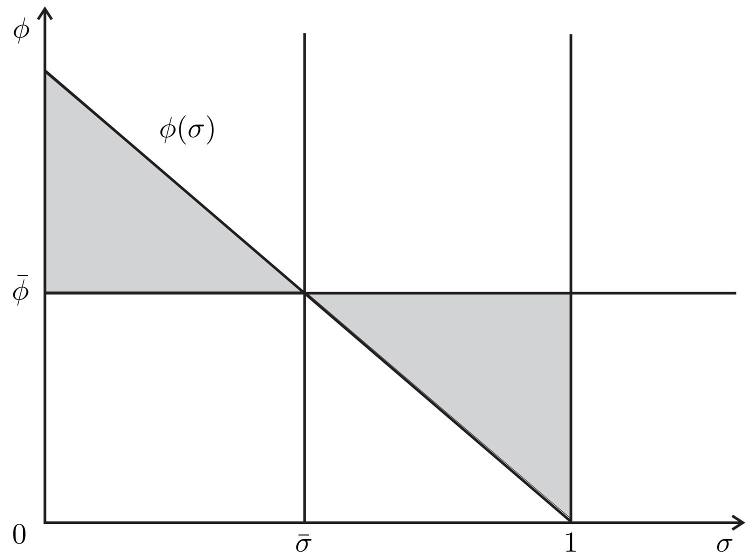

Figure 1: Admissible and non-admissible parameter values

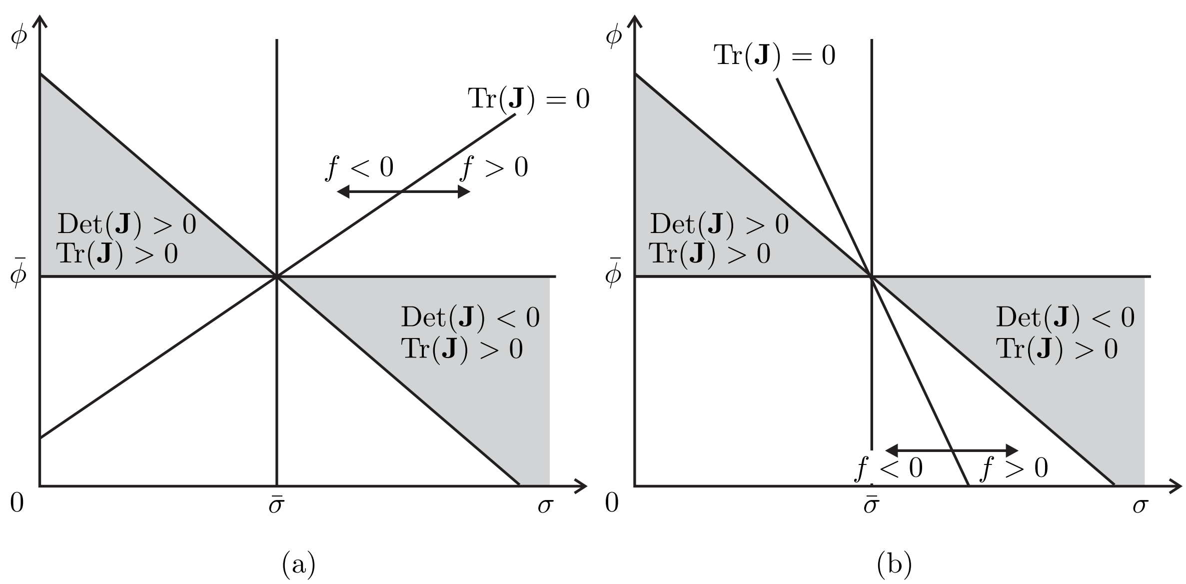

Figure 2: Signs of trace and determinant