



**The Principle of the Fermionic  
Projector: An Approach for  
Quantum Gravity?**

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## Abstract

In this short article we introduce the mathematical framework of the principle of the fermionic projector and set up a variational principle in discrete space-time. The underlying physical principles are discussed. We outline the connection to the continuum theory and state recent results. In the last two sections, we speculate on how it might be possible to describe quantum gravity within this framework.

The principle of the fermionic projector [3] provides a new model of space-time together with the mathematical framework for the formulation of physical theories. It was proposed to formulate physics in this framework based on a particular variational principle. Here we explain a few basic ideas of the approach, report on recent results and explain the possible connection to quantum gravity.

It is generally believed that the concept of a space-time continuum (like Minkowski space or a Lorentzian manifold) should be modified for distances as small as the Planck length. We here assume that space-time is discrete on the Planck scale. Our notion of “discrete space-time” differs from other discrete approaches (like for example lattice gauge theories or quantum foam models) in that we do not assume any structures or relations between the space-time points (like for example the nearest-neighbor relation on a space-time lattice). Instead, we set up a variational principle for an ensemble of quantum mechanical wave functions. The idea is that for minimizers of our variational principle, these wave functions should induce relations between the discrete space-time points, which, in a suitable limit, should go over to the topological and causal structure of a Lorentzian manifold. More specifically, in this limit the wave functions should group to a configuration of Dirac seas.

For clarity, we first introduce the mathematical framework (Section 1) and discuss it afterwards, working out the underlying physical principles (Section 2). Then we outline the connection to the continuum theory (Sections 3 and 4) and state some results (Sections 5). Finally, we give an outlook on classical gravity (Section 6) and the field quantization (Section 7).

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# 1 A Variational Principle in Discrete Space-Time

We let  $(H, \langle \cdot | \cdot \rangle)$  be a complex inner product space of signature  $(N, N)$ . Thus  $\langle \cdot | \cdot \rangle$  is linear in its second and antilinear in its first argument, and it is symmetric,

$$\overline{\langle \Psi | \Phi \rangle} = \langle \Phi | \Psi \rangle \quad \text{for all } \Psi, \Phi \in H,$$

and non-degenerate,

$$\langle \Psi | \Phi \rangle = 0 \quad \text{for all } \Phi \in H \quad \implies \quad \Psi = 0.$$

In contrast to a scalar product,  $\langle \cdot | \cdot \rangle$  is *not* positive. Instead, we can choose an orthogonal basis  $(e_i)_{i=1, \dots, 2N}$  of  $H$  such that the inner product  $\langle e_i | e_i \rangle$  equals  $+1$  if  $i = 1, \dots, N$  and equals  $-1$  if  $i = N + 1, \dots, 2N$ .

A *projector*  $A$  in  $H$  is defined just as in Hilbert spaces as a linear operator which is idempotent and self-adjoint,

$$A^2 = A \quad \text{and} \quad \langle A\Psi | \Phi \rangle = \langle \Psi | A\Phi \rangle \quad \text{for all } \Psi, \Phi \in H.$$

Let  $M$  be a finite set. To every point  $x \in M$  we associate a projector  $E_x$ . We assume that these projectors are orthogonal and complete in the sense that

$$E_x E_y = \delta_{xy} E_x \quad \text{and} \quad \sum_{x \in M} E_x = \mathbf{1}. \quad (1)$$

Furthermore, we assume that the images  $E_x(H) \subset H$  of these projectors are non-degenerate subspaces of  $H$ , which all have the same signature  $(n, n)$ . We refer to  $(n, n)$  as the *spin dimension*. The points  $x \in M$  are called *discrete space-time points*, and the corresponding projectors  $E_x$  are the *space-time projectors*. The structure  $(H, \langle \cdot | \cdot \rangle, (E_x)_{x \in M})$  is called *discrete space-time*.

We introduce one more projector  $P$  on  $H$ , the so-called *fermionic projector*, which has the additional property that its image  $P(H)$  is a *negative definite* subspace of  $H$ . We refer to the rank of  $P$  as the *number of particles*  $f := \dim P(H)$ .

A space-time projector  $E_x$  can be used to restrict an operator to the subspace  $E_x(H) \subset H$ . Using a more graphic notion, we also refer to this restriction as the *localization* at the space-time point  $x$ . For example, using the completeness of the space-time projectors (1), we readily see that

$$f = \text{Tr } P = \sum_{x \in M} \text{Tr}(E_x P). \quad (2)$$

The expression  $\text{Tr}(E_x P)$  can be understood as the localization of the trace at the space-time point  $x$ , and summing over all space-time points gives the total trace. When forming more complicated composite expressions, it is convenient to use the short notations

$$P(x, y) = E_x P E_y \quad \text{and} \quad \Psi(x) = E_x \Psi. \quad (3)$$

The operator  $P(x, y)$  maps  $E_y(H) \subset H$  to  $E_x(H)$ , and it is often useful to regard it as a mapping only between these subspaces,

$$P(x, y) : E_y(H) \rightarrow E_x(H).$$

Using (1), we can write the vector  $P\Psi$  as follows,

$$(P\Psi)(x) = E_x P\Psi = \sum_{y \in M} E_x P E_y \Psi = \sum_{y \in M} (E_x P E_y) (E_y \Psi),$$

and thus

$$(P\Psi)(x) = \sum_{y \in M} P(x, y) \Psi(y). \quad (4)$$

This relation resembles the representation of an operator with an integral kernel. Therefore, we call  $P(x, y)$  the *discrete kernel* of the fermionic projector.

We can now set up our variational principle. We define the *closed chain*  $A_{xy}$  by

$$A_{xy} = P(x, y) P(y, x) = E_x P E_y P E_x; \quad (5)$$

it maps  $E_x(H)$  to itself. Let  $\lambda_1, \dots, \lambda_{2n}$  be the zeros of the characteristic polynomial of  $A_{xy}$ , counted with multiplicities. We define the *spectral weight*  $|A_{xy}|$  by

$$|A_{xy}| = \sum_{j=1}^{2n} |\lambda_j|.$$

Similarly, one can take the spectral weight of powers of  $A_{xy}$ , and by summing over the space-time points we get positive numbers depending only on the form of the fermionic projector relative to the space-time projectors. Our variational principle is to

$$\text{minimize} \quad \sum_{x, y \in M} |A_{xy}^2| \quad (6)$$

by considering variations of the fermionic projector which satisfy the constraint

$$\sum_{x, y \in M} |A_{xy}|^2 = \text{const}. \quad (7)$$

In the variation we also keep the number of particles  $f$  as well as discrete space-time fixed. Using the method of Lagrangian multipliers, for every minimizer  $P$  there is a real parameter  $\mu$  such that  $P$  is a stationary point of the *action*

$$\mathcal{S}_\mu[P] = \sum_{x, y \in M} \mathcal{L}_\mu[A_{xy}] \quad (8)$$

with the *Lagrangian*

$$\mathcal{L}_\mu[A] = |A^2| - \mu |A|^2. \quad (9)$$

This variational principle was first introduced in [3]. In [4] it is analyzed mathematically, and it is shown in particular that minimizers exist:

**Theorem 1.1** *The variational principle (6, 7) attains its minimum.*

## 2 Discussion, the Underlying Physical Principles

We come to the physical discussion. Obviously, our mathematical framework does not refer to an underlying space-time continuum, and our variational principle is set up intrinsically in discrete space-time. In other words, our approach is *background free*. Furthermore, the following physical principles are respected, in a sense we briefly explain.

- The **Pauli Exclusion Principle**: We interpret the vectors in the image of  $P$  as the quantum mechanical states of the particles of our system. Thus, choosing a basis  $\Psi_1, \dots, \Psi_f \in P(H)$ , the  $\Psi_i$  can be thought of as the wave functions of the occupied states of the system. Every vector  $\Psi \in H$  either lies in the image of  $P$  or it does not. Via these two conditions, the fermionic projector encodes for every state  $\Psi$  the occupation numbers 1 and 0, respectively, but it is impossible to describe higher occupation numbers. More technically, we can form the anti-symmetric many-particle wave function

$$\Psi = \Psi_1 \wedge \dots \wedge \Psi_f .$$

Due to the anti-symmetrization, this definition of  $\Psi$  is (up to a phase) independent of the choice of the basis  $\Psi_1, \dots, \Psi_f$ . In this way, we can associate to every fermionic projector a fermionic many-particle wave function which obeys the Pauli Exclusion Principle. For a detailed discussion we refer to [3, §3.2].

- A **local gauge principle**: Exactly as in Hilbert spaces, a linear operator  $U$  in  $H$  is called *unitary* if

$$\langle U\Psi | U\Phi \rangle = \langle \Psi | \Phi \rangle \quad \text{for all } \Psi, \Phi \in H .$$

It is a simple observation that a joint unitary transformation of all projectors,

$$E_x \rightarrow UE_xU^{-1}, \quad P \rightarrow UPU^{-1} \quad \text{with } U \text{ unitary} \quad (10)$$

keeps our action (6) as well as the constraint (7) unchanged, because

$$\begin{aligned} P(x, y) &\rightarrow UP(x, y)U^{-1}, & A_{xy} &\rightarrow UA_{xy}U^{-1} \\ \det(A_{xy} - \lambda\mathbb{1}) &\rightarrow \det(U(A_{xy} - \lambda\mathbb{1})U^{-1}) = \det(A_{xy} - \lambda\mathbb{1}), \end{aligned}$$

and so the  $\lambda_j$  stay the same. Such unitary transformations can be used to vary the fermionic projector. However, since we want to keep discrete space-time fixed, we are only allowed to consider unitary transformations which do not change the space-time projectors,

$$E_x = UE_xU^{-1} \quad \text{for all } x \in M . \quad (11)$$

Then (10) reduces to the transformation of the fermionic projector

$$P \rightarrow UPU^{-1} . \quad (12)$$

The conditions (11) mean that  $U$  maps every subspace  $E_x(H)$  into itself. Hence  $U$  splits into a direct sum of unitary transformations

$$U(x) := UE_x : E_x(H) \rightarrow E_x(H), \quad (13)$$

which act “locally” on the subspaces associated to the individual space-time points.

Unitary transformations of the form (11, 12) can be identified with local gauge transformations. Namely, using the notation (3), such a unitary transformation  $U$  acts on a vector  $\Psi \in H$  as

$$\Psi(x) \longrightarrow U(x) \Psi(x).$$

This formula coincides with the well-known transformation law of wave functions under local gauge transformations (for more details see [3, §1.5 and §3.1]). We refer to the group of all unitary transformations of the form (11, 12) as the *gauge group*. The above argument shows that our variational principle is *gauge invariant*. Localizing the gauge transformations according to (13), we obtain at any space-time point  $x$  the so-called *local gauge group*. The local gauge group is the group of isometries of  $E_x(H)$  and can thus be identified with the group  $U(n, n)$ . Note that in our setting the local gauge group cannot be chosen arbitrarily, but it is completely determined by the spin dimension.

- The **equivalence principle**: At first sight it might seem impossible to speak of the equivalence principle without having the usual space-time continuum. What we mean is the following more general notion. The equivalence principle can be expressed by the invariance of the physical equations under general coordinate transformations. In our setting, it makes no sense to speak of coordinate transformations nor of the diffeomorphism group because we have no topology on the space-time points. But instead, we can take the largest group which can act on the space-time points: the group of all permutations of  $M$ . Our variational principle is obviously *invariant under the permutation group* because permuting the space-time points merely corresponds to reordering the summands in (6, 7). Since on a Lorentzian manifold, every diffeomorphism is bijective and can thus be regarded as a permutation of the space-time points, the invariance of our variational principle under the permutation group can be considered as a generalization of the equivalence principle.

An immediate objection to the last paragraph is that the symmetry under permutations of the space-time points is not compatible with the topological and causal structure of a Lorentzian manifold, and this leads us to the discussion of the physical principles which are *not* taken into account in our framework. Our definitions involve **no locality** and **no causality**. We do not consider these principles as being fundamental. Instead, our concept is that the causal structure is induced on the space-time points by the minimizer  $P$  of our variational principle. In particular, minimizers should spontaneously break the above permutation symmetry to a smaller symmetry group, which, in a certain limiting case describing the vacuum, should reduce to Poincaré invariance. Explaining in detail how this is supposed to work goes beyond the scope of this short article (for a first step in the mathematical analysis of spontaneous symmetry breaking see [5]). In order to tell the reader right away what we have in mind, we shall first simply assume the causal structure of Minkowski space and consider our action in the setting of relativistic quantum mechanics (Section 3). This naive procedure will *not* work, but it will nevertheless illustrate our variational principle and reveal a basic difficulty. In Section 4 we will then outline the connection to the continuum theory as worked out in [3].

### 3 Naive Correspondence to a Continuum Theory

Let us see what happens if we try to get a connection between the framework of Section 1 and relativistic quantum mechanics in the simplest possible way. To this end, we just replace  $M$  by the space-time continuum  $\mathbb{R}^4$  and the sums over  $M$  by space-time integrals. For a vector  $\Psi \in H$ , the corresponding  $\Psi(x) \in E_x(H)$  as defined by (3) should be a 4-component Dirac wave function, and the scalar product  $\langle \Psi(x) | \Phi(x) \rangle$  on  $E_x(H)$  should correspond to the usual Lorentz invariant scalar product on Dirac spinors  $\bar{\Psi}\Phi$  with  $\bar{\Psi} = \Psi^\dagger \gamma^0$  the adjoint spinor. Since this last scalar product is indefinite of signature  $(2, 2)$ , we are led to choosing  $n = 2$ , so that the spin dimension is  $(2, 2)$ .

In view of (4), the discrete kernel should in the continuum go over to the integral kernel of an operator  $P$  on the Dirac wave functions,

$$(P\Psi)(x) = \int_M P(x, y) \Psi(y) d^4 y.$$

The image of  $P$  should be spanned by the occupied fermionic states. We take Dirac's concept literally that in the vacuum all negative-energy states are occupied by fermions forming the so-called *Dirac sea*. This leads us to describe the vacuum by the integral over the lower mass shell

$$P(x, y) = \int \frac{d^4 k}{(2\pi)^4} (\not{k} + m) \delta(k^2 - m^2) \Theta(-k^0) e^{-ik(x-y)} \quad (14)$$

(we consider for simplicity only one Dirac sea of mass  $m$ ; the factor  $(\not{k} + m)$  is needed in order to satisfy the Dirac equation  $(i\not{\partial}_x - m)P(x, y) = 0$ ).

We now consider our action for the above fermionic projector. Since we do not want to compute the Fourier integral (14) in detail, we simply choose  $x$  and  $y$  for which the integrals in (14) exist (for details see below) and see what we get using only the *Lorentz symmetry* of  $P$ . We can clearly write  $P(x, y)$  as

$$P(x, y) = \alpha (y - x)_j \gamma^j + \beta \mathbf{1}$$

with two complex parameters  $\alpha$  and  $\beta$ . Taking the complex conjugate of (14), we see that

$$P(y, x) = \bar{\alpha} (y - x)_j \gamma^j + \bar{\beta} \mathbf{1}.$$

As a consequence,

$$A_{xy} = P(x, y) P(y, x) = a (y - x)_j \gamma^j + b \mathbf{1} \quad (15)$$

with real parameters  $a$  and  $b$  given by

$$a = \alpha \bar{\beta} + \beta \bar{\alpha}, \quad b = |\alpha|^2 (y - x)^2 + |\beta|^2. \quad (16)$$

Using the formula  $(A_{xy} - b\mathbf{1})^2 = a^2 (y - x)^2$ , one can easily compute the zeros of the characteristic polynomial of  $A_{xy}$ ,

$$\lambda_1 = \lambda_2 = b + \sqrt{a^2 (y - x)^2}, \quad \lambda_3 = \lambda_4 = b - \sqrt{a^2 (y - x)^2}.$$

If the vector  $(y - x)$  is spacelike, we conclude from the inequality  $(y - x)^2 < 0$  that the argument of the above square root is negative. As a consequence, the  $\lambda_j$  appear in *complex conjugate pairs*,

$$\bar{\lambda}_1 = \lambda_3, \quad \bar{\lambda}_2 = \lambda_4.$$

Furthermore, the  $\lambda_j$  all have the same absolute value  $|\lambda_j| =: |\lambda|$ , and thus the action (6) reduces to

$$\mathcal{S}_\mu[A] = |\lambda|^2 (4 - 16\mu).$$

This simplifies further if we choose the Lagrangian multiplier equal to  $\frac{1}{4}$ , because then the action vanishes identically. If conversely  $(y - x)$  is timelike, the  $\lambda_i$  are all real. Using (16), one easily verifies that they are all *positive* and thus  $\mathcal{S}_{\frac{1}{4}}[A] = (\lambda_1 - \lambda_3)^2$ . We conclude that

$$\mathcal{S}_{\frac{1}{4}}[A_{xy}] = \begin{cases} 4a^2 (y - x)^2 & \text{if } (y - x) \text{ is timelike} \\ 0 & \text{if } (y - x) \text{ is spacelike.} \end{cases} \quad (17)$$

This consideration gives a simple *connection to causality*: In the two cases where  $(y - x)$  is timelike or spacelike, the spectral properties of the matrix  $A_{xy}$  are completely different (namely, the  $\lambda_j$  are real or appear in complex conjugate pairs, respectively), and this leads to a completely different form of the action (17). More specifically, if the  $\lambda_j$  are non-real, this property is (by continuity) preserved under small perturbations of  $A_{xy}$ . Thinking of a dynamical situation, this suggests that perturbations of  $P(x, y)$  for spacelike  $(y - x)$  should not effect the action or, in other words, that events at points  $x$  and  $y$  with spacelike separation should not be related to each other by our variational principle. We remark that choosing  $\mu = \frac{1}{4}$  is justified by considering the Euler-Lagrange equations corresponding to our variational principle, and this also makes the connection to causality clearer (see [3, §3.5 and §5]).

Apart from the oversimplifications and many special assumptions, the main flaw of this section is that the Fourier integral (14) does not exist for all  $x$  and  $y$ . More precisely,  $P(x, y)$  is a well-defined distribution, which is even a smooth function if  $(y - x)^2 \neq 0$ . But *on the light cone*  $(y - x)^2 = 0$ , this distribution is *singular* (for more details see [3, §2.5]). Thus on the light cone, the pointwise product in (15) is ill-defined and our above arguments fail. The resolution of this problem will be outlined in the next section.

## 4 The Continuum Limit

We now return to the discrete setting of Section 1 and shall explain how to get a rigorous connection to the continuum theory. One approach is to study the minimizers in discrete space-time and to try to recover structures known from the continuum. For example, in view of the spectral properties of  $A_{xy}$  in Minkowski space as discussed in the previous section, it is tempting to introduce in discrete space-time the following notion (this definition is indeed symmetric in  $x$  and  $y$ , see [3, §3.5]).

**Def. 4.1** *Two discrete space-time points  $x, y \in M$  are called **timelike separated** if the zeros  $\lambda_j$  of the characteristic polynomial of  $A_{xy}$  are all real and not all equal. They are said to be **spacelike separated** if the  $\lambda_j$  are all non-real and have the same absolute value.*

The conjecture is that if the number of space-time points and the number of particles both tend to infinity at a certain relative rate, the above “discrete causal structure” should go over to the causal structure of a Lorentzian manifold. Proving this conjecture under suitable assumptions is certainly a challenge. But since we have a precise mathematical framework in discrete space-time, this seems an interesting research program.

Unfortunately, so far not much work has been done on the discrete models, and at present almost nothing is known about the minimizers in discrete space-time. For this reason, there seems no better method at the moment than to impose that the fermionic

projector of the *vacuum* is obtained from a Dirac sea configuration by a suitable regularization process on the Planck scale [3, Chapter 4]. Since we do not know how the physical fermionic projector looks like on the Planck scale, we use the *method of variable regularization* and consider a large class of regularizations [3, §4.1].

When introducing the fermionic projector of the vacuum, we clearly put in the causal structure of Minkowski space as well as the free Dirac equation ad hoc. What makes the method interesting is that we then introduce a *general interaction* by inserting a general (possibly nonlocal) perturbation operator into the Dirac equation. Using methods of hyperbolic PDEs (the so-called *light-cone expansion*), one can describe the fermionic projector with interaction in detail [3, §2.5]. It turns out that the regularization of the fermionic projector with interaction is completely determined by the regularization of the vacuum (see [3, §4.5 and Appendix D]). Due to the regularization, the singularities of the fermionic projector have disappeared, and one can consider the *Euler-Lagrange equations* corresponding to our variational principle (see [3, §4.5 and Appendix F]). Analyzing the dependence on the regularization in detail, we can perform an expansion in powers of the Planck length. This gives differential equations involving Dirac and gauge fields, which involve a small number of so-called *regularization parameters*, which depend on the regularization and which we treat as free parameters (see [3, §4.5 and Appendix E]). This procedure for analyzing the Euler-Lagrange equations in the continuum is called *continuum limit*. We point out that only the singular behavior of  $P(x, y)$  on the light cone enters the continuum limit, and this gives causality.

## 5 Obtained Results

In [3, Chapters 6-8] the continuum limit is analyzed in spin dimension  $(16, 16)$  for a fermionic projector of the vacuum, which is the direct sum of seven identical massive sectors and one massless left-handed sector, each of which is composed of three Dirac seas. Considering general chiral and (pseudo)scalar potentials, we find that the sectors spontaneously form pairs, which are referred to as *blocks*. The resulting *effective interaction* can be described by chiral potentials corresponding to the effective gauge group

$$SU(2) \otimes SU(3) \otimes U(1)^3 .$$

This model has striking similarity to the standard model if the block containing the left-handed sector is identified with the leptons and the three other blocks with the quarks. Namely, the effective gauge fields have the following properties.

- The  $SU(3)$  corresponds to an unbroken gauge symmetry. The  $SU(3)$  gauge fields couple to the quarks exactly as the strong gauge fields in the standard model.
- The  $SU(2)$  potentials are left-handed and couple to the leptons and quarks exactly as the weak gauge potentials in the standard model. Similar to the CKM mixing in the standard model, the off-diagonal components of these potentials must involve a non-trivial mixing of the generations. The  $SU(2)$  gauge symmetry is spontaneously broken.
- The  $U(1)$  of electrodynamics can be identified with an Abelian subgroup of the effective gauge group.

The effective gauge group is larger than the gauge group of the standard model, but this is not inconsistent because a more detailed analysis of our variational principle should give further constraints for the Abelian gauge potentials. Moreover, there are the following differences to the standard model, which we derive mathematically without working out their physical implications.

- The  $SU(2)$  gauge field tensor  $F$  must be simple in the sense that  $F = \Lambda s$  for a real 2-form  $\Lambda$  and an  $su(2)$ -valued function  $s$ .
- In the lepton block, the off-diagonal  $SU(2)$  gauge potentials are associated with a new type of potential, called nil potential, which couples to the right-handed component.

## 6 Outlook: The Classical Gravitational Field

The permutation symmetry of our variational principle as discussed in Section 2 guarantees that the equations obtained in the continuum limit are invariant under diffeomorphisms. This gives us the hope that classical gravity might already be taken into account, and that even quantum gravity might be incorporated in our framework if our variational principle is studied beyond the continuum limit. Unfortunately, so far these questions have hardly been investigated. Therefore, at this point we leave rigorous mathematics and must enter the realm of what a critical scientist might call pure speculation. Nevertheless, the following discussion might be helpful to give an idea of what our approach is about, and it might also give inspiration for future work in this area.

The only calculations for gravitational fields carried out so far are the calculations for linearized gravity [7, Appendix B]. The following discussion of classical gravity is based on these calculations. For the metric, we consider a linear perturbation  $h_{jk}$  of the Minkowski metric  $\eta_{jk} = \text{diag}(1, -1, -1, -1)$ ,

$$g_{jk}(x) = \eta_{jk} + h_{jk}(x).$$

In linearized gravity, the diffeomorphism invariance corresponds to a large freedom to transform the  $h_{jk}$  without changing the space-time geometry (this freedom is usually referred to as “gauge freedom”, but we point out for clarity that it is not related to the “local gauge freedom” as discussed in Section 2). This freedom can be used to arrange that (see e.g. [8])

$$\partial^k h_{jk} = \frac{1}{2} \partial_j h_{kl} \eta^{kl}.$$

Computing the corresponding Dirac operator and performing the light-cone expansion, the first-order perturbation of the fermionic projector takes the form

$$\Delta P(x, y) = \mathcal{O}(\not{x} z^{-1}) + \mathcal{O}(\xi^{[k} \gamma^{l]} z^{-1}) + \mathcal{O}(m) + \mathcal{O}((h_{ij})^2) \quad (18)$$

$$\left. \begin{aligned} & + \frac{1}{2} \left( \int_x^y h_j^k \right) \xi^j \frac{\partial}{\partial y^k} P(x, y) \\ & - \frac{i}{16\pi^3} \left( \int_x^y (2\alpha - 1) \gamma^i \xi^j \xi^k (h_{jk,i} - h_{ik,j}) \right) d\alpha z^{-2} \\ & - \frac{1}{32\pi^3} \left( \int_x^y \varepsilon^{ijklm} (h_{jk,i} - h_{ik,j}) \xi^k \xi_l \rho \gamma_m \right) d\alpha z^{-2} \end{aligned} \right\} \quad (19)$$

$$+ \frac{i}{32\pi^3} \left( \int_x^y (\alpha^2 - \alpha) \xi^j \gamma^k G_{jk} \right) d\alpha z^{-1}, \quad (20)$$

where we set  $\xi \equiv y - x$ , the integrals go along straight lines joining the points  $x$  and  $y$ ,

$$\int_x^y f d\alpha = \int_0^1 f(\alpha y + (1 - \alpha)x) d\alpha,$$

and  $z^{-1}, z^{-2}$  are distributions which are singular on the light cone,

$$z^{-1} = \frac{\text{PP}}{\xi^2} + i\pi\delta(\xi^2)\epsilon(\xi^0), \quad z^{-2} = \frac{\text{PP}}{\xi^4} - i\pi\delta'(\xi^2)\epsilon(\xi^0)$$

(where PP denotes the principal part, and  $\epsilon$  is the step function  $\epsilon(x) = 1$  if  $x > 0$  and  $\epsilon(x) = -1$  otherwise). In this formula we only considered the most singular contributions on the light cone and did not take into account the higher orders in the rest mass  $m$  of the Dirac particles (for details see [7]). Nevertheless, the above formula gives us some general information on how the fermionic projector depends on a classical gravitational field. The contribution (19) describes an ‘‘infinitesimal deformation’’ of the light cone corresponding to the fact that the gravitational field affects the causal structure. Since it involves at most first derivatives of the metric, the curvature does not enter, and thus (19) can be compensated by a gauge and an infinitesimal coordinate transformation. The diffeomorphism invariance of the equations of the continuum limit ensures that the contribution (19) drops out of these equations (and this can also be verified by a direct computation of the closed chain). We conclude that the equations of the continuum limit will be governed by the contribution (20). It is remarkable that the *Einstein tensor*  $G_{jk}$  appears. Thus, provided that the equations of the continuum limit give sufficiently strong constraints, we obtain the vacuum Einstein equations.

The situation becomes even more interesting if fermionic matter is involved. In this case, the wave function  $\Psi$  of a particle (or similarly anti-particle) will lead to a perturbation of the fermionic projector of the form

$$\Delta P(x, y) = -\Psi(x)\overline{\Psi(y)}. \quad (21)$$

Performing a multi-pole expansion around  $x$ , the zeroth moment  $-\Psi(x)\overline{\Psi(x)}$  corresponds to the electromagnetic current and should be taken care of by the Maxwell equations. The first moment

$$-(y-x)^j \Psi(x)\partial_j\overline{\Psi(x)}$$

is proportional to the energy-momentum tensor of the Dirac wave function. Imposing that this contribution should be compensated by the first moment of (20), we obtain a relation of the form

$$\frac{i}{32\pi^3} \frac{1}{6} \xi^j G_{jk} z^{-1} = \xi^j T_{jk}[\Psi]. \quad (22)$$

This calculation was too naive, because the left side of the equation involves the singular distribution  $z^{-1}$ , whereas the right side is smooth. This is also the reason why the method of the continuum limit as developed in [3] cannot be applied directly to the gravitational field. On the other hand, this is not to be expected, because the formalism of the continuum limit only gives dimensionless constants, whereas the gravitational constant has the dimension of length squared. These extra length dimensions enter (22) by the factor  $z^{-1}$ . The simplest method to convert (22) into a reasonable differential equation is to argue that the concept of the space-time continuum should be valid only down to the Planck

scale, where the discreteness of space-time should lead to some kind of “ultraviolet regularization.” Thus it seems natural to replace the singular factor  $z^{-1}$  by the value of this factor on the Planck scale. This leads to an equation of the form

$$G_{jk} \frac{1}{l_P^2} \sim T_{jk}, \quad (23)$$

where  $l_P$  denotes the Planck length. These are the precisely the Einstein equations. We point out that the above argument is not rigorous, in particular because the transition from (22) to (23) would require a methods which go beyond the formalism of the continuum limit. Nevertheless, our consideration seems to explain why the Planck length enters the Einstein equations and in particular why the coupling of matter to the gravitational field is so extremely weak. Also, we get some understanding for how the Einstein equations could be derived from our variational principle.

## 7 Outlook: The Field Quantization

We hope that in our approach, the field quantization is taken into account as soon as one goes beyond the continuum limit and takes into account the discreteness of space-time. Since the basic mechanism should be the same for the gravitational field as for any other bosonic field, we can just as well consider here for simplicity an electromagnetic field. The basic ideas are quite old and were one of the motivations for thinking of the principle of the fermionic projector [6]. Nevertheless, the details have not been worked out in the meantime, simply because it seemed more important to first get a rigorous connection to the continuum theory by analyzing the continuum limit. Thus the following considerations are still on the same speculative level as nine years ago. In order to convey the reader some of the spontaneity of the early text, we here simply give a slightly revised English translation of [6, Section 1.4].

In preparation of the discussion of field quantization, we want to work out why quantized bosonic fields are needed, i.e. what the essence of a “quantization” of these fields is. To this aim, we shall analyze to which extent we can get a connection to quantum field theory by considering *classical* gauge fields. For simplicity, we restrict attention to one type of particles and an electromagnetic interaction, but the considerations apply just as well to a general interaction including gravitational fields. Suppose that when describing the interacting system of fermions in the continuum limit we get the system of coupled differential equations

$$(i\partial + eA - m) \Psi = 0, \quad F^{ij} = e \bar{\Psi} \gamma^i \Psi. \quad (24)$$

These equations are no longer valid at energies as high as the Planck energy, because the approximations used in the formalism of the continuum limit are no longer valid. Our variational principle in discrete space-time should then still describe our system, but at the moment we do not know how the corresponding interaction looks like. For simplicity, we will assume in what follows that the fermions do not interact at such high energies. In this way, we get in the classical Maxwell equations a natural cutoff for very large momenta.

When describing (24) perturbatively, one gets Feynman diagrams. To this end we can proceed just as in [1]: We expand  $\Psi$  and  $A$  in powers of  $e$ ,

$$\Psi = \sum_{j=0}^{\infty} e^j \Psi^{(j)}, \quad A = \sum_{j=0}^{\infty} e^j A^{(j)}$$

and substitute these expansions in the differential equations (24). In these equations, the contributions to every order in  $e$  must vanish, and thus one solves for the highest appearing index  $(j)$ . In the Lorentz gauge, we thus obtain the formal relations

$$\Psi^{(j)} = - \sum_{k+l=j-1} (i\partial - m)^{-1} \left( A^{(k)} \Psi^{(l)} \right), \quad A_i^{(j)} = - \sum_{k+l=j-1} \square^{-1} \left( \overline{\Psi}^{(k)} \gamma_i \Psi^{(l)} \right), \quad (25)$$

which by iterative substitutions can be brought into a more explicit form. Taking into account the pair creation, we obtain additional diagrams which contain closed fermion lines, due to the Pauli Exclusion Principle with the correct relative signs. In this way we get all Feynman diagrams.

We come to the renormalization. Since we obtain all the Feynman diagrams of quantum field theory, the only difference of our approach to standard quantum field theory is the natural cutoff for large momenta. In this way all ultraviolet divergences disappear, and the difference between naked and effective coupling constants becomes finite. One can (at least in principle) express the effective coupling constants in terms of the naked coupling constants by adding up all the contributions by the self-interaction. Computations using the renormalization group show that the effective masses and coupling constants depend on the energy. The effective constants at the Planck scale should be considered as our naked coupling constants.

The fact that the theory can be renormalized is important for us, because this ensures that the self-interaction can be described merely by a change of the masses and coupling constants. But renormalizability is not absolutely necessary for a meaningful theory. For example, the renormalizability of diagrams is irrelevant for classes of diagrams which (with our cutoff) are so small that they are negligible. Furthermore, one should be aware that the introduction of a cutoff is an approximation which has no justification. In order to understand the self-interaction at high energies one would have to analyze our variational principle without using the formalism of the continuum limit.

We explained the connection to the Feynman diagrams and the renormalization in order to point out that perturbative quantum field theory is obtained already with *classical* bosonic fields if one studies the coupled interaction between the classical field and the fermions. With second quantization of the gauge fields one can obtain the Feynman diagrams using Wick's theorem in a more concise way, but at this point it is unnecessary both from the mathematical and physical point of view to go over from classical to quantized bosonic fields. In particular, one should be aware of the fact that all the high precision tests of quantum field theory (like the Lamb shift and the anomalous  $g$  factor) are actually no test of the field quantization. One does not need to think of a photon line as an "exchange of a virtual photon"; the photon propagator can just as well be considered simply as the operator  $\square^{-1}$  in (25), which appears in the perturbation expansion of the coupled differential equations (24). Also the equation  $E = \hbar\omega$ , which in a graphic language tells us about the "energy of one photon," does not make a statement on the field quantization. This can be seen as follows: In physics, the energy appears in two different contexts. In classical field theory, the energy is a conserved quantity following from the time translation invariance of the Lagrangian. In quantum theory, on the other hand, the sum of the frequencies of the wave functions and potentials is conserved in any interaction, simply because in the perturbation expansion plane waves of different frequencies are orthogonal. These "classical" and "quantum mechanical" energies are related to each other via the equation  $E = \hbar\omega$ . Planck's constant can be determined without referring to the electromagnetic field (for example via the Compton wavelength of the electron).

Since the classical and quantum mechanical energies are both conserved, it is clear that the relation  $E = \hbar\omega$  must hold in general. (Thus the energy transmitted by a photon line of frequency  $\omega$  really is  $\hbar\omega$ .)

After these considerations there remain only a few effects which are real tests of the field quantization. More precisely, these are the following observations,

1. Planck's radiation law
2. the Casimir effect
3. the wave-particle duality of the electromagnetic field, thus for example the double-slit experiment

For the derivation of Planck's radiation law, one uses that the energy of an electromagnetic radiation mode cannot take continuous values, but that its energy is quantized in steps of  $\hbar\omega$ . The Casimir effect measures the zero point energy of the radiation mode. In order to understand field quantization, one needs to find a convincing explanation for the above observations. However, the formalism of quantum field theory does not immediately follow from the above observations. For example, when performing canonical quantization one assumes that each radiation mode can be described by a quantum mechanical oscillator. This is a possible explanation, but it is not a compelling consequence of the discreteness of the energy levels.

We shall now explain how the above observations could be explained in the framework of the principle of the fermionic projector. In order to work out the difference between the continuum limit and the situation in discrete space-time, we will discuss several examples. It will always be sufficient to work also in discrete space-time with the classical notions. For example, by an electromagnetic wave in discrete space-time we mean a variation of the fermionic projector which in the continuum limit can be described via a perturbation of the Dirac operator by a classical electromagnetic field.

We begin with a simple model in discrete space-time, namely a completely filled Dirac sea and an electromagnetic field in the form of a radiation mode. We want to analyze the effect of a variation of the amplitude of the electromagnetic wave. In the continuum limit, we can choose the amplitude arbitrarily, because the Maxwell equations will in any case be satisfied. However, the situation is more difficult in discrete space-time. Then the variation of the amplitude corresponds to a variation of the fermionic projector. However, when performing the perturbation expansion for  $P$  in discrete space-time, we need to take into account several contributions which could be left out in the continuum limit. These additional contributions do not drop out of the Euler-Lagrange equations corresponding to our variational principle. If these equations are satisfied for a given fermionic projector  $P$ , we cannot expect that they will still hold after changing the amplitude of the electromagnetic wave. More generally, in discrete space-time there seems to be no continuous family  $P(\tau)$  of solutions of the Euler-Lagrange equations. This means in particular that the amplitude of the electromagnetic wave can take only discrete values.

Alternatively, the difference between the continuum limit and the description in discrete space-time can be understood as follows: In discrete space-time, the number  $f$  of particles is an integer. If for different values of  $f$  we construct a fermionic projector of the above form, the amplitude of the corresponding electromagnetic wave will in general be different. Let us assume for simplicity that for each  $f$  (in a reasonable range) there is exactly one such projector  $P_f$  with corresponding amplitude  $A_f$ . Since  $f$  is not known,

we can choose  $f$  arbitrarily. Thus the amplitude of the wave can take values in the discrete set  $\{A_f\}$ . In the continuum limit, however, the fermionic projector is an operator of infinite rank. Thus it is clear that now we do not get a restriction for the amplitude of the electromagnetic wave, and the amplitude can be varied continuously.

We conclude that in discrete space-time a natural “quantization” of the amplitude of the electromagnetic wave should appear. Before we can get a connection to the Planck radiation and the Casimir effect, we need to refine our consideration. Namely, it seems unrealistic to consider an electromagnetic wave which is spread over the whole of space-time. Thus we now consider a wave in a four-dimensional box (for example with fixed boundary values). Let us assume that the box has length  $L$  in the spatial directions and  $T$  in the time direction. In this case, again only discrete values for the amplitude of the wave should be admissible. But now the quantization levels should depend on the size of the box, in particular on the parameter  $T$ . Qualitatively, one can expect that for smaller  $T$  the amplitude of the wave must be larger in order to perturb the fermionic projector in a comparable way. This means that the quantization levels become finer if  $T$  becomes larger. Via the classical energy density of the electromagnetic field, the admissible amplitudes  $\{A_j\}$  can be translated into field energies of the wave. Physically speaking, we create a wave at time  $t$  and annihilate it a later time  $t + T$ . Since, according to our above consideration, the relation  $E = \hbar\omega$  should hold in any interacting system, we find that the field energy must be “quantized” in steps of  $\hbar\omega$ . On the other hand, we just saw that the quantization levels depend on  $T$ . In order to avoid inconsistencies, we must choose  $T$  such that the quantization steps for the field energy are just  $\hbar\omega$ .

In this way we obtain a condition which at first sight seems very strange: If we generate an electromagnetic wave at some time  $t$ , we must annihilate it at some later time  $t + T$ . Such an additional condition which has no correspondence in the continuum limit, is called a *non-local quantum condition*. We derived it under the assumption of a “quantization” of the amplitude from the equations of the continuum limit (classical field equations, description of the interaction by Feynman diagrams). Since the Euler-Lagrange equations of discrete space-time should in the continuum limit go over to the classical equations, a solution in discrete space-time should automatically satisfy the non-local quantum condition.

Of course, the just-derived condition makes no physical sense. But our system of one radiation mode is also oversimplified. Thus before drawing further conclusions, let us consider the situation in more realistic situations: In a system with several radiation modes, we cannot (in contrast to the situation with canonical quantization) treat the different modes as being independent, because the variation of the amplitude of one mode will influence the quantization levels of all the other radiation modes. This mutual influence is non-local. Thus an electromagnetic wave also changes the energy levels of waves which are in large spacelike distance. The situation becomes even more complicated if fermions are brought into the system, because then the corresponding Dirac currents will also affect the energy levels of the radiation modes. The complexity of this situation has two consequences: First, we can make practically no statement on the energy levels, we only know that the quantization steps are  $\hbar\omega$ . Thus we can describe the energy of the lowest level only statistically. It seems reasonable to assume that they are evenly distributed in the interval  $[0, \hbar\omega)$ . Then we obtain for the possible energy levels of each radiation mode on average the values  $(n + \frac{1}{2})\hbar\omega$ . Secondly, the non-local quantum conditions are now so complicated that we can no longer specify them. But it seems well possible that they can be satisfied in a realistic physical situation. We have the conception that such non-local quantum conditions determine all what in the usual statistical interpretation of quantum

mechanics is said to be “undetermined” or “happens by chance”. We will soon come back to this point when discussing the wave-particle dualism.

After these consideration we can explain the above observations 1. and 2.: Since the energy of each radiation mode is quantized in steps of  $\hbar\omega$ , we obtain Planck’s radiation law, whereas the average energy of  $\frac{1}{2}\hbar\omega$  of the ground state energy explains the Casimir effect. We conclude that under the assumption of a “quantization” of the amplitude of the electromagnetic wave we come to the same conclusions as with canonical quantization. The reason is that with the Feynman diagrams and the equation  $E = \hbar\omega$  we had all the formulas for the quantitative description at our disposal, and therefore it was sufficient to work with a very general “discreteness” of the energy levels.

We come to the wave-particle dualism. Since this is a basic effect in quantum mechanics, which appears similarly for bosons and fermions, we want to discuss this point in detail. First we want to compare our concept of bosons and fermions. Obviously, we describe bosons and fermions in a very different way: the wave functions of the fermions span the image of the projector  $P$ , whereas the bosons correspond (as described above) to the discrete excitation levels of the classical bosonic fields. In our description, the Fock space or an equivalent formalism does not appear. It might not seem satisfying that in this way the analogy in the usual description of bosons and fermions, namely the mere replacements of commutators by anti-commutators, gets lost. However, we point out that the *elementary* bosons and fermions differ not only by their statistics but also in the following important point. For the fermions (leptons, quarks) we have a conservation law (lepton number, baryon number), not so for the gauge bosons. This difference is taken into account in our formalism: Every fermion corresponds to a vector in  $P(H)$ . We can transform fermions into each other and can create/annihilate them in pairs. But we cannot change the total number  $f$  of particles. In particular, we cannot annihilate a single fermion. In contrast, since the gauge bosons merely correspond to discrete values of the bosonic fields. They can be generated or annihilated arbitrarily in the interaction, provided that the conservation law for energy and momentum is satisfied.

In order to clarify the connection to the Fock space, we briefly mention how we describe composite particles (for example mesons or baryons). They are all composed of the elementary fermions. Thus a particle composed of  $p$  components corresponds to a vector of  $(P(H))^p$ . This representation is not suitable for practical purposes. It is more convenient to use for the elementary fermions the Fock space formalism. Then the creation/annihilation operators for the composite particle are a product of  $p$  fermionic creation/annihilation operators. If  $p$  is even (or odd), we can generate with these creation/annihilation operators the whole bosonic (or fermionic) Fock space. In this way, we obtain for composite particles the usual formalism. However, we point out that in our description this formalism has no fundamental significance.

Due to the different treatment of elementary fermions and bosons, we need to find an explanation for the wave-particle dualism which is independent of the particular description of these particles. For a fermion, this is a vector  $\Psi \in P(H)$ , for a boson the gauge field. Thus in any case, the physical object is not the pointlike particle, but the wave. At first sight this does not seem reasonable, because we have not at all taken into account the particle character. Our concept is that the particle character is a consequence of a “discreteness” of the interaction described by our variational principle. In order to specify what we mean by “discreteness” of the interaction, we consider the double slit experiment. We work with an electron, but the consideration applies just as well to a photon, if the wave function of the electron is replaced by the electromagnetic field. When it hits

the photographic material on the screen, the electron interacts with the silver atoms, and the film is exposed. In the continuum limit we obtain the same situation as in wave mechanics: the waves originating at the two slits are superposed and generate on the screen an interference pattern. Similar to our discussion of the electromagnetic radiation mode, the continuum limit should describe the physical situation only approximately. But when considering the variational principle in discrete space-time, the situation becomes much more complicated. Let us assume that the interaction in discrete space-time is “discrete” in the sense that the electron prefers to interact with only one atom of the screen. This assumption is already plausible in the continuum limit. Namely, if the electron interacts with a silver atom, one electron from the atom must be excited. Since this requires a certain minimal energy, the kinetic energy of the electron hitting the screen can excite only a small number of atoms. Thus the interaction between electron and the screen can take place only at individual silver atoms; the electron cannot pass its energy continuously onto the screen.

Under this assumption we get on the screen an exposed dot, and thus we get the impression of a pointlike particle. At which point of the screen the interaction takes place is determined by the detailed form of the fermionic projector  $P$  in discrete space-time. With the notion introduced above, we can also say that which silver atom is exposed is determined by non-local quantum conditions. At this point, the non-locality and non-causality of our variational principle in discrete space-time becomes important. Since the non-local quantum conditions are so complicated, we cannot predict at which point of the screen the interaction will take place. Even if we repeat the same experiment under seemingly identical conditions, the global situation will be different. As a consequence, we can only make statistical statements on the measurements. From the known continuum limit we know that the probabilities for the measurements are given by the usual quantum mechanical expectation values.

At this point we want to close the discussion. We conclude that the principle of the fermionic projector raises quite fundamental questions on the structure of space-time, the nature of field quantization and the interpretation of quantum mechanics. Besides working out the continuum limit in more detail, it will be a major goal of future work to give specific answers to these questions.

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