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On finite-dimensional motives and Murre's conjecture

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by Uwe Jannsen

To Jacob Murre

§0 Introduction

The conjectures of Bloch, Beilinson, and Murre predict the existence of a certain functorial filtration on the Chow groups (with $\mathbb{Q}$-coefficients) of all smooth projective varieties, whose graded quotients only depends on cycles modulo homological equivalence. This filtration would offer a rather good understanding of these Chow groups, and would allow to prove several other conjectures, like Bloch's conjecture on surfaces of geometric genus 0. In Murre's formulation (cf. 4.1 below) one can check the validity of the conjecture for particular smooth projective varieties, and in fact, a slightly weaker form of the conjecture has been proved for several cases, e.g., for surfaces [Mu1] and several threefolds [GM] (proving parts (A), (B) and (D) of the conjecture, and giving evidence for (C)). But to my knowledge, there are few results for higher-dimensional varieties, and the strongest form of Murre's conjecture (including part (C)) is only known for curves, rational surfaces, and, trivially, for Brauer-Severi varieties.

The first aim of this paper is to exhibit cases, where the full Murre conjecture can be shown. The positive aspect is that we get this for some non-trivial cases of varieties of higher (in fact arbitrarily high) dimension, the negative aspect is that we get this just for some special varieties and special ground fields. In particular, not over some universal domain. As a sample, we get the following:

Theorem 0.1 *Let k be a rational or elliptic function field (in one variable) over a finite field $\mathbb{F}$. Let X_0 be an arbitrary product of rational and elliptic curves over $\mathbb{F}$, and let $X = X_0 \times_{\mathbb{F}} k$. Then Murre's conjecture holds for X.*

One ingredient is the notion of "finite-dimensionality" of motives, as introduced independently by Kimura [Ki] and O'Sullivan [OSu]. Up to now it is only known that the Chow motive of a smooth projective variety is finite-dimensional, if it lies in the tensor category generated by the motives of abelian varieties. But I believe that this notion will be fundamental for further progress on Chow groups, and motivic cohomology in general, in view of the nilpotence properties it implies.

Therefore a second aim of this paper is to investigate finite-dimensionality of motives in several directions. We add some observations to the existing results (cf. [An1] for a survey) which may be interesting in their own right, but also bear on our investigation of Murre's conjecture. In particular, we are interested in Chow endomorphisms and nilpotence results: For a smooth projective variety X of pure dimension d let $CH^d(X \times X)_{\mathbb{Q}}$ be the ring of Chow self-correspondences, i.e., the endomorphism ring $\mathrm{End}(h_{rat}(X))$ of the Chow motive $h_{rat}(X)$ associated to X, and let $J(X) = CH^d(X \times X)_{\mathbb{Q},\, hom}$ be the ideal of homologically trivial correspondences. Then we get the following (a similar result appeared in [DP]):

Theorem 0.2 *Let X be a smooth projective variety over a field k, and let π_+^X be the projector onto the even degree part of the cohomology. Then the following properties are equivalent.*

(i) π_+^X *is algebraic and* $J(X^N)$ *is nilpotent for all* $N > 0$.

(ii) π_+^X *is algebraic and* $J(X^N)$ *is a nil ideal for all* $N > 0$.

(iii) X *is finite-dimensional (i.e., $h(X)$, the motive of X, is finite-dimensional).*

The implication from (iii) to (ii) is Kimura's nilpotence theorem. The other implications give a certain converse. The following again sharpens results of Kimura.

Theorem 0.3 *Let M be a motive modulo rational equivalence over a field k, and assume that M is either oddly or evenly finite-dimensional. For an endomorphism $f \in \mathrm{End}(M)$ let $P(t) = \det(\, t - f \mid H^*(M, \mathbb{Q}_\ell)\,)$ be the characteristic polynomial of f acting on the ℓ-adic cohomology of M ($\ell \neq \mathrm{char}(k)$). Then $P(f) = 0$ in $\mathrm{End}(M)$.*

This fact (which in the even case was also proved by O'Sullivan) is somewhat surprising, because a priori the equality $P(f) = 0$ only holds modulo homological equivalence.

Coming back to Murre's conjecture, it is well-known by now [GP] that part (A) (the Chow-Künneth decomposition) follows from the standard conjecture on algebraicity of the Kuenneth components, and finite-dimensionality. In particular, this applies to abelian varieties over arbitrary ground fields. In this paper, we explore which additional ingredients can give the remaining part of Murre's conjecture.

We start with the case of a finite ground field $\mathbb{F}$. Here it is known by work of Geisser [Gei] and Kahn [Ka] that the conjunction of finite-dimensionality and Tate's conjecture "implies everything". In particular it implies that rational and numerical equivalence agrees (with $\mathbb{Q}$-coefficients). Evidently, the latter also implies Murre's conjecture. However, we are interested in getting some unconditional theorems, and hence we take some pain to single out the minimal conditions to get such results. In particular, we don't want to argue with Tate's conjecture for all varieties, but want to get by with conditions just on the given variety X. (For this, we have to rectify some statements in the literature.) As a sample, we get:

Theorem 0.4 *Let X be a smooth projective variety over the finite field $\mathbb{F}$. Assume that $J(X)$ is a nil ideal (e.g., assume that X is finite-dimensional). Fix an integer $j \geq 0$ and assume that the Tate conjecture holds for $H^{2j}(X \times_{\mathbb{F}} \bar{\mathbb{F}}, \mathbb{Q}_\ell(j))$, and that the Frobenius eigenvalue 1 is semi-simple on $H^{2j}(X \times_{\mathbb{F}} \bar{\mathbb{F}}, \mathbb{Q}_\ell(j))$. Then the cycle map induces an isomorphism*

$$CH^j(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \xrightarrow{\sim} H^{2j}(X_{\mathbb{F}}\overline{\mathbb{F}}, \mathbb{Q}_\ell(j))^{Gal(\overline{\mathbb{F}}/\mathbb{F})},$$

and the motivic cohomology $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$ is of finite exponent for all $i \neq 2j$.

Corollary 0.5 *Let X be a smooth projective variety of pure dimension d over the finite field F, and assume that $h(X)$ is finite-dimensional. Then $H^i_{\mathcal{M}}(X, \mathbb{Z}(d))$ has finite exponent for all $i \neq 2d$.*

This generalizes results of Soulé [So1]. The theorems on function fields over F (like Theorem 0.1) are obtained by considering arbitrary (not necessarily smooth or projective) varieties over F and passing to certain limits of ℓ-adic cohomology, as in [Ja1].

The results in this paper were mainly obtained during a stay at the University of Tokyo during the academic year 2003/2004, and it is my pleasure to thank the department and my host Takeshi Saito for the invitation and the hospitality. I thank the referee for suggesting a more elegant version and proof of Theorem 3.3 (a).

§1 Weil cohomology theories, motives, tensor categories

In this section, we recall some notions and properties needed later. We fix a base field k and consider the category SP_k of smooth projective varieties over k. For X in SP_k, we denote by $Z^j(X)$ the group of algebraic cycles of codimension j on X, *with* $\mathbb{Q}$*-rational coefficients*. The following definition is equivalent to the one in [Kl].

Definition 1.1 Let E be a field of characteristic 0. An E-linear Weil cohomology theory H is a contravariant functor $X \mapsto H(X)$, $(f : X \to Y) \mapsto f^* := H(f) : H(Y) \to H(X)$ from SP_k to the category of graded commutative E-algebras ($a.b = (-1)^{i+j} b.a$ for $a \in H^i(X), b \in H^j(X)$), together with the following data and properties:
(i) $\dim_E H(X) < \infty$, and $H^i(X) = 0$ for $i < 0$ or $i > 2 \cdot \dim X$.
(ii) To each morphism $f : X \to Y$ there is functorially associated an E-linear map $f_* : H(X) \to H(Y)$ which is of degree $\dim Y - \dim X$ if X and Y are irreducible.
(iii) (projection formula) $f_*(f^*y \cdot x) = y \cdot f_*x$ for $f : X \to Y$, $x \in H(X)$ and $y \in H(Y)$.
(iv) (Künneth formula) The association $a \otimes b \mapsto a \times b := pr_X^* a \cdot pr_Y^* b$ gives an isomorphism $H(X) \otimes_E H(Y) \to H(X \times Y)$.
(v) (Poincaré duality) $H(\text{Spec } k) = H^0(\text{Spec } k) \cong E$, and the bilinear pairing $H(X) \times H(X) \xrightarrow{\cdot} H(X) \xrightarrow{f_*} H(\text{Spec } k) \cong E$, $(a, b) \mapsto < a, b > := f_*(a.b)$, is non-degenerate.
(vi) (cycle map) There are cycle class maps $c\ell^j : Z^j(X) \to H^{2j}(X)$ compatible with products, pull-backs f^* and push-forwards f_*, whenever these operations are defined on the algebraic cycles.

Remarks 1.2 (a) It follows that f_* is the transpose of f^* under Poincaré duality.
(b) Let X be a smooth projective variety of pure dimension d, and denote, as usual, by Δ_X the cycle in $Z^d(X \times X)$ corresponding to the diagonal $X \hookrightarrow X \times X$, and also the associated cycles class in $H^{2d}(X \times X)$ for a given Weil cohomology theory H. The Künneth components of the diagonal, $\pi_i \in H^{2d-i}(X) \otimes H^i(X)$ $(i = 1, \ldots, 2d)$ are defined by decomposing $\Delta_X = \sum_{i=0}^{2d} \pi_i$ according to the Künneth isomorphism (1.1)(iv). The standard conjecture $C(X)$ predicts that the π_i are algebraic, i.e., again classes of algebraic cycles.
(c) Let X and $Y \in SP_k$. Any correspondence from X to Y, i.e., any element $\alpha \in Z^*(X \times Y)$, induces an E-linear map, again denoted α, from $H(Y)$ to $H(X)$ by defining $\alpha(x) = (p_Y)_*((p_X)^*(x).c\ell(\alpha))$ for $x \in H(Y)$, where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the projections. The same already holds for any cohomology class $\alpha' \in H^*(X \times Y)$ in place of $c\ell(\alpha)$. Via this interpretation, the element π_i, as cohomological correspondence from X to X, is the identity on $H^i(X)$, and zero on $H^j(X)$ for $j \neq i$.

Examples 1.3 The following are examples of Weil cohomology theories:

(a) If $k = \mathbb{C}$: singular cohomology $H(X) = H^*(X(\mathbb{C}), \mathbb{Q})$ $(E = \mathbb{Q})$.

(b) For arbitrary k: ℓ-adic cohomology $H^*_{et}(X \times_k \overline{k}, \mathbb{Q}_\ell)$, for $\ell \neq \mathrm{char}\, k$ $(E = \mathbb{Q}_\ell)$.

(c) If char $k = 0$: de Rham cohomology $H^*_{dR}(X/k)$ $(E = k)$.

(d) If k is a perfect field of characteristic $p > 0$: crystalline cohomology $H^*_{\mathrm{crys}}(X/B(k)) := H^*_{\mathrm{crys}}(X/W(k)) \otimes_{W(k)} B(k)$ $(E = B(k) := \mathrm{Frac}(W(k))$.

(1.4) Let $\sim$ be an adequate equivalence relation on algebraic cycles, i.e., an equivalence relation on all cycle groups $Z^j(X)$ for all X in SP_k such that product, push-forward and pull-back of cycles is well-defined on the cycle groups $A^j_\sim(X) := Z^j(X)/\sim$ [Ja3]. We recall that we have the adequate equivalence relations rational, algebraic, homological and numerical equivalence with the relationship

$$\alpha \sim_{rat} 0 \quad \Rightarrow \quad \alpha \sim_{alg} 0 \quad \Rightarrow \quad \alpha \sim_{hom} 0 \quad \Rightarrow \quad \alpha \sim_{num} 0$$

The category $\mathcal{M}_\sim(k)$ of ($\mathbb{Q}$-rational) motives modulo $\sim$ over k can be defined as follows. For $X, Y \in SP_k$ the group of correspondences (modulo $\sim$) of degree n from X to Y is defined as $Corr^n_\sim(X,Y) = \oplus_i A^{\dim(X_i)+n}_\sim(X_i \times Y)$, where the X_i are the irreducible components of X. The composition of correspondences $f \in Corr^m_\sim(X,Y)$ and $g \in Corr^m_\sim(Y,Z)$ is defined as $g \circ f = (p_{XZ})_*(p^*_{XY}(f).p^*_{YZ}(g)) \in Corr^{m+n}(X,Z)$, where p_{XZ}, p_{XY} and p_{YZ} are the projections from $X \times Y \times Z$ to $X \times Z$, $X \times Y$ and $Y \times Z$, respectively. Then the objects of $\mathcal{M}_\sim(k)$ can be described as triples (X, p, m), with $X \in SP_k$, $p \in Corr^0_\sim(X,X)$ an idempotent and $m \in \mathbb{Z}$, and one has $Hom((X,p,m),(Y,q,n)) = qCorr^{n-m}(X,Y)p$, with composition given by the above composition of correspondences. The Tate objects are defined by $1(n) = (\mathrm{Spec}k, id, n)$ for $n \in \mathbb{Z}$.

(1.5) The precise definition of a tensor category can be found in [DM]. Let us just recall that it is a category with a bifunctor $(A, B) \mapsto A \otimes B$ together with asssociativity constraints $\psi_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$, commutativity constraints $\phi_{A,B} : A \otimes B \cong B \otimes A$ and an unity constraint $l : 1 \otimes A \cong A$, satisfying certain compatibilities modelled after the situation of the tensor product of vector spaces. A tensor category is called rigid, if it has internal Homs $\underline{\mathrm{Hom}}(A,B)$ (characterized by $\mathrm{Hom}(A \otimes B, C) = \mathrm{Hom}(A, \underline{\mathrm{Hom}}(B,C))$) satisfying some reasonable properties [DM] 1.7. In this case, the dual of an object is defined as $A^\vee = \underline{\mathrm{Hom}}(A, 1)$.

Examples 1.6 (a) In particular, let E be a field. Then the category Vec_E of finite-dimensional E-vector spaces is a rigid E-linear tensor category, with the usual tensor product and the obvious constraints.

(b) The category GrVec_E of finite-dimensional ($\mathbb{Z}$-) graded E-vector spaces V^* is a rigid E-linear tensor category by defining $(V \otimes W)^r = \oplus_{i+j=r} V^i \otimes_E W^j$, taking the associativity constraints from Vec_E, defining $1 = E$ placed in degree 0, and defining $\phi_{A,B}(a \otimes b) = (-1)^{i+j} b \otimes a$ for $a \in A^i$ and $b \in B^j$.

The relationship between the objects introduced in 1.1, 1.3 and 1.4 is as follows.

(1.7) For any adequate equivalence relation $\sim$, the category $\mathcal{M}_\sim(k)$ of motives modulo $\sim$ becomes a rigid $\mathbb{Q}$-rational tensor category by defining $(X, p, m) \otimes (Y, q, n) = (X \times Y, p \times q, m + n)$, and taking the obvious associativity constraint, the unit object $1 =$

$(\mathrm{Spec}(k), id, 0)$, and the commutativity constraints induced by the transpositions $\tau_{X,Y}$: $X \times Y \cong Y \times X$.

Recall that a tensor functor $\Phi : \mathcal{A} \to \mathcal{B}$ between tensor categories is a functor together with functorial isomorphisms $\alpha_{A,B} : \Phi(A) \otimes \Phi(B) \cong \Phi(A \otimes B)$, satisfying some obvious compatibilities with respect to the constraints [DM] 1.8. Then one has:

Lemma 1.8 *Let E be a field. Giving an E-linear Weil cohomology theory H is the same as giving a tensor functor $\Phi : \mathcal{M}_{rat}(k) \longrightarrow \mathrm{GrVec}_E$ with $\Phi(1(-1))$ of degree 2.*

This is well-known, and the proof is straightforward (cf. [An2] 4.1.8.1): Given a Weil cohomology theory H we can extend it to a (covariant) functor on $\mathcal{M}_{rat}(k)$ by defining $H^*(X, p, m) = pH^{*+m}(X)$. Here we have used the fact that correspondences act on the cohomology, cf. 1.2 (ii), and this also gives the functoriality of this association. Conversely, given a tensor functor Φ, we can compose it with the functor $SP_k \to \mathcal{M}_{rat}(k)$ to obtain a Weil cohomology theory. Here, for a morphism $f : X \to Y$, f_* is induced by $\Phi(\Gamma_f)$, where Γ_f is the graph of f, and f^* is induced by $\Phi(\Gamma_f^t)$, where Γ_f^t is the transpose of Γ_f.

§2 Finite-dimensional motives

For any object M in a tensor category $\mathcal{C}$, and every natural number N, the symmetric group S_N acts on the N-fold tensor product $M^{\otimes N}$ as follows. For an elementary transposition $(i, i+1)$, $1 \le i < N$, the induced isomorphism $(i, i+1)_*$ of $M^{\otimes N}$ is induced by applying the comutativity constraint between the i-th and $(i+1)$-st place, i.e., we have $(i, i+1)_* = id_{M^{\otimes(i-1)}} \otimes \phi_{M,M} \otimes id_{M^{\otimes(N-i-1)}}$. One can check that one obtains a well defined action of S_N by decomposing each element σ as a product $\sigma = \prod \tau_\nu$ of such elementary transpositions, and defining $\sigma_* = \prod(\tau_\nu)_*$. Now let $\mathcal{C}$ be a $\mathbb{Q}$-linear pseudo-abelian tensor category. Then, by linearity, the group ring $\mathbb{Q}[S_N]$ acts on $M^{\otimes N}$, and we can define the symmetric product as $\mathrm{Sym}^N M = e_{\mathrm{sym}} M^{\otimes N}$ and the exterior product as $\wedge^N M = e_{\mathrm{alt}} M^{\otimes N}$, where $e_{\mathrm{sym}} = \frac{1}{N!}\sum \sigma$ and $e_{\mathrm{alt}} = \frac{1}{N!}\sum \mathrm{sign}(\sigma)\sigma$, with the sum taken over all $\sigma \in S_N$. Note that these are idempotents in $\mathbb{Q}[S_N]$, and that by the very definition of pseudo-abelian categories, every projector has an image in $\mathcal{C}$.

The following definition goes back to Kimura [Ki] and, independently, to O'Sullivan [OSu] (with a different terminology).

Definition 2.1 An object M in a $\mathbb{Q}$-linear pseudo-abelian tensor category $\mathcal{C}$ is called
(i) evenly finite-dimensional, if there is some $N > 0$ with $\wedge^N M = 0$,
(ii) oddly finite-dimensional, if there is some $N < 0$ with $\mathrm{Sym}^N M = 0$,
(iii) finite-dimensional, if $M = M_+ \oplus M_-$ with M_+ evenly and M_- oddly finite-dimensional, respectively.

For such objects one has the following notion of dimension.

Definition 2.2 If M is finite-dimensional, and if M_+ and M_- are as in definition 2.1 (iii), define the (Kimura-) dimension of M as $\dim M = \dim_+ M + \dim_- M$, where $\dim_+ M := \max\{r \mid \wedge^r M_+ \neq 0\}$ and $\dim_- M := \max\{s \mid \mathrm{Sym}^s M_- \neq 0\}$.

This is well-defined, because M_+ and M_- are unique up to (non unique) isomorphism (loc.cit.).

Examples 2.3 (a) If E is a field of characteristic zero, then any object V in Vec_E is evenly finite-dimensional. In fact, $\wedge^n V$ is the usual alternating power $\wedge_E^n V$, and this is zero for $n > \dim_E V$. One has $\dim V = \dim_E V$.

(b) With E as above, every object in the tensor category GrVec_E (cf. 1.6 (b) for our conventions) is finite-dimensional: For $V = \oplus_{i\in\mathbb{Z}} V^i$ let $V_+ = \oplus_{i\,\mathrm{even}} V^i$ and $V_- = \oplus_{i\,\mathrm{odd}} V^i$. Then one has $\wedge^N_{\mathrm{Gr}} V_+ = \wedge^N_E V_+$ and $\mathrm{Sym}^N_{\mathrm{Gr}} V_- = \wedge^N_E V_-$, by the sign rule for our commutation constraints. Thus $\dim V = \dim_E V$ is the usual dimension of V as an E-vector space (This should not be confused with the rank of V (cf. [DM] p.113) with respect to the structure of GrVec_E as a rigid tensor category, which would be $\mathrm{rank} V = \dim_E V_+ - \dim_E V_-$).

Fix a Weil cohomology theory H^*. For a smooth projective variety X let $\pi_i = \pi_i^X$ be the Künneth components of the diagonal, and let

$$\pi_+ = \pi_+^X = \pi_0 + \pi_2 + \pi_4 + \dots \quad , \quad \pi_- = \pi_-^X = \pi_1 + \pi_3 + \pi_5 + \dots$$

be the projectors onto the even and odd degree part of the cohomology, respectively. Then we have the 'sign conjecture'

Conjecture S(X): The projectors π_+^X and π_-^X are algebraic.

It is implied by standard conjecture $C(X)$ (cf. 1.2 (b)), and hence known for curves, surfaces and abelian varieties, and in general over finite fields. Moreover $S(X)$ and $S(Y)$ imply $S(X \times Y)$ (because $\pi^+_{X\times Y} = \pi^+_X \times \pi^+_Y + \pi^-_Y \times \pi^-_Y$).

If $S(X)$ holds (for the given H), then the motive $h_{\mathrm{hom}}(X)$ modulo homological equivalence (for the given H) is finite-dimensional. In fact, decompose $h_{\mathrm{hom}}(X) = M_+ \oplus M_-$, with $M_\pm = (h_{\mathrm{hom}}(X), \pi_\pm)$, and let

$$b_\pm(X) = \dim H^*(M_\pm) = \dim H^*(X)_\pm \ ,$$

where $H^*(X)_\pm$ is defined as in 2.10 (b). Then $\wedge^{b_+(X)+1} M_+ = 0 = \mathrm{Sym}^{b_-(X)+1} M_-$, because $H^* : \mathcal{M}_{\mathrm{hom}}(k) \to \mathrm{GrVec}_F$ is a faithful tensor functor).

In particular, conjecture $S(X)$ implies that also the motive $h_{\mathrm{num}}(X)$ modulo numerical equivalence is finite-dimensional. However, the following conjecture is much deeper:

Conjecture 2.4 *(Kimura, O'Sullivan) Every motive modulo rational equivalence is finite-dimensional.*

Remark 2.5 If $M = (X, p, m)$ is a motive modulo the equivalence relation $\sim$, then $\mathrm{Sym}^n M = (X^n, e_{\mathrm{sym}} \circ p^{\otimes n}, n \cdot m)$. So this is 0 if and only if $e_{\mathrm{sym}} \circ p^{\otimes n} \sim 0$. Here $p^{\otimes n} = p \times \ldots \times p$ on $(X \times X)^n \cong X^n \times X^n$, and $e_{\mathrm{sym}} \circ p^{\otimes n} = p^{\otimes n} \circ e_{\mathrm{sym}}$, so this is again an idempotent. In fact, for every endomorphism f of $h(X)$ and every $\sigma \in S_n$ obviously $\sigma \circ f^{\otimes n} = f^{\otimes n} \circ \sigma$. Similarly for $\wedge^n$ and e_{alt}.

Now let C be a smooth projective curve over k, and let $x \in C$ be a closed point of degree m. Then we have a decomposition $h_{rat}(C) = 1 \oplus h^1_{rat}(C) \oplus 1(-1)$, with $h^1_{rat}(C) := (C, \Delta_C - \frac{1}{m} x \times C - \frac{1}{m} C \times x, 0)$. As for the notation, note that $\tilde{\pi}_o := \frac{1}{m} x \times C$ and $\tilde{\pi}_2 := \frac{1}{m} C \times x$ are orthogonal idempotents lifting the Künneth components π_0 and π_2, respectively. Hence $\Delta_C - \tilde{\pi}_0 - \tilde{\pi}_2$ is an idempotent lifting the Künneth component $\pi_1 = \Delta_C - \pi_0 - \pi_2$.

Now 1 and $1(r)$, for every $r \in \mathbb{Z}$, are evenly finite-dimensional, since S_2 acts trivially on $1 \otimes 1$ and $1(r) \otimes 1(r)$. The following results thus show that $h(C)$ is finite-dimensional.

Theorem 2.6 *([Ki] 4.2) The motive $h^1_{rat}(C)$ is oddly finite-dimensional. More precisely, one has $Sym^{2g+1} h^1_{rat}(C) = 0$ where g is the genus of C.*

Proposition 2.7 *([Ki]) Let M and N be objects in a $\mathbb{Q}$-linear pseudo-abelian tensor category.*

(a) If M and N are finite-dimensional, then $M \oplus N$ is finite-dimensional, with $\dim M \oplus N \leq \dim M + \dim N$.

(b) If M and N are finite-dimensional, then $M \otimes N$ is finite-dimensional, with $\dim M \otimes N \leq \dim M . \dim N$.

(c) If M is finite-dimensional, then also every direct factor of M.

(d) $M = 0$ if and only if M is finite-dimensional with $\dim M = 0$.

§3 Nilpotence and finite-dimensionality

For each smooth projective variety X over k, let $J(X) \subseteq \mathrm{Corr}^0_{rat}(X, X)$ be the ideal of correspondences which are numerically equivalent to zero. recall the following conjecture.

Conjecture $\mathbf{N(X)}$: $J(X)$ is a nilpotent ideal.

A remarkable consequence of this conjecture would be that there is no phantom motive, i.e., no non-trivial motive which becomes zero after passing to numerical equivalence, and that every idempotent modulo numerical or homological equivalence can be lifted to an idempotent modulo rational equivalence. In fact, for any motive M modulo rational equivalence let $J(M) \subseteq \mathrm{End}(M)$ be the ideal of numerically trivial endomorphisms (so that $J(X) = J(h(X))$ for X in SP_k). Then we have

Lemma 3.1 *Assume that $J(M)$ is a nil ideal. Let M_{num} and M_{hom} be the images of M in $\mathcal{M}_{num}(k)$ and $\mathcal{M}_{hom}(k)$ (with respect to a given Weil cohomology), respectively. Then the following holds.*

(i) If $M_{num} = 0$ (e.g., if $H^*(M) = 0$ for a Weil cohomology theory), then $M = 0$.

(ii) Any idempotent in $\mathrm{End}(M_{num})$ or $\mathrm{End}(M_{hom})$ can be lifted to an idempotent in $\mathrm{End}(M)$, and any two such liftings are conjugate by a unit of $\mathrm{End}(M_{hom})$ lying above the identity of $\mathrm{End}(M_{num})$.

(iii) If the image of $f \in \mathrm{End}(M)$ in $\mathrm{End}(M_{num})$ is invertible, then so is f.

Proof. (i): If id_M maps to zero in $\mathrm{End}(M_{num})$, it is nilpotent, hence zero. (ii) and (iii): These properties holds for any surjection $A \twoheadrightarrow \overline{A} = A/I$ where A is a (not necessarily commutative) ring with unit, and I is a (two-sided) nil ideal. For (iii), it suffices to assume that the element $a \in A$ maps to $1 \in \overline{A}$. But then a is unipotent, hence invertible. As for (ii), if $\overline{e}$ is idempotent in $\overline{A}$ and a is any lift in A, then $(a - a^2)^N = 0$ for some $N > 0$, and it follows easily that $\tilde{e} = (1 - (1 - a)^N)^N$ is an idempotent lifting $\overline{e}$ ([Ki] 7.8). If e and e' are idempotents of A lying above $\overline{e}$, then $u = e'e + (1 - e')(1 - e)$ lies above $1 \in \overline{A}$. Thus u is invertible, and the equality $e'u = e'e = ue$ shows that $e' = ueu^{-1}$.

Conjecture $N(X)$ would follow from the existence of the Bloch-Beilinson filtration [Ja2], or Murre's conjecture [Mu1], or the following conjecture of Voevodsky:

Conjecture 3.2 *([Voe]) If an algebraic cycle z is numerically trivial, it is smash nilpotent, i.e., there is an $n > 0$ such that $z^{\times n} = 0$.*

In fact, as is observed in loc. cit., a smash nilpotent correspondence from X to X is nilpotent; more precisely, $z^{\times n} = 0$ implies $z^n = 0$ in $Corr(X, X)$. The following result gives (in part (b)) another criterion for nilpotence. Here we may consider motives modulo any (fixed) adequate equivalence relation $\sim$. Recall that, for a motive $M = (X, p, m)$ and an endomorphism f of M, the trace of f is defined as $tr(f) =< f.p^t >$, where $< \alpha.\beta >$ is the intersection number of two cycles α and β. This coincides with the trace coming from the rigid tensor category structure of $\mathcal{M}_\sim(k)$.

Theorem 3.3 *Let $f : M \to M$ be an endomorphism of a motive.*
(a) If $\wedge^{d+1} M = 0$ (resp. $\mathrm{Sym}^{d+1} M = 0$), then $\sum_{i=0}^{d} (-1)^{d-i} tr(\wedge^{d-i} f) f^i = 0$ (resp. $\sum_{i=0}^{d} tr(\mathrm{Sym}^{d-i} f) f^i = 0$).
(b) In particular, if M is either evenly finite-dimensional or oddly finite-dimensional, and $d = \dim M$, then there is a monic polynomial $G(t) \in \mathbb{Q}[t]$ of degree d with $G(f) = 0$. If f is numerically equivalent to 0, then f is nilpotent, viz., $f^d = 0$.

This was originally proved by Kimura [Ki] in a slightly weaker form - giving (b) with $d + 1$ instead of d, and not giving the description (a) of $G(t)$. The following corollaries already follow from this original form, except for 3.7.

Corollary 3.4 *If M is a finite-dimensional motive, then the ideal $J_M \subseteq End(M)$ of numerically trivial endomorphisms is nilpotent.*

In fact, by decomposing $M = M_+ \oplus M_-$, it is shown in [Ki] that $J(M)$ is a nil ideal, with degree of nilpotence bounded by $n = (\dim_+ M . \dim_- M + 1).\max(\dim_+ M, \dim_- M) + 1$. By a result of Nagata-Higman (cf. [AK] 7.2.8) it follows that $J(M)$ is in fact a nilpotent ideal, of nilpotence degree $\leq 2^n - 1$ (since we assume $\mathbb{Q}$-coefficients).

Corollary 3.5 *If M is a finite-dimensional motive, and H is any F-rational Weil cohomology theory, then $\dim M = \sum_{i \in \mathbb{Z}} \dim_F H^i(M)$. In particular, the right hand side is independent of H.*

Proof. (cf. [Ki] 3.9 and 7.4) We may assume that M is either evenly or oddly finite-dimensional. Obviously, the dimension decreases under any tensor functor, so $\dim M \geq \dim_F H(M)$. On the other hand, by the nilpotence result (together with 2.7 and 3.1), $\wedge^r M = 0$ if $\wedge^r H(M) = H(\wedge^r M) = 0$, similarly for Sym^r. Thus $\dim M \leq \dim_F H(M)$.

Corollary 3.6 *(compare 2.7) If M and N are finite-dimensional motives, then* $\dim(M \oplus N) = \dim M + \dim N$ *and* $\dim M \otimes N = \dim M \cdot \dim N$.

In fact, this holds for $\sum_{i \in \mathbb{Z}} \dim_F H^i(-)$. For $d = 1$ Theorem 3.3 implies:

Corollary 3.7 *If M is a finite-dimensional motive with* $\dim M = 1$*, then $J(M) = 0$, i.e., on $End(M)$ numerical and rational equivalence coincide. Moreover, $End(M) = \mathbb{Q}$.*

Examples 3.8 (a) If M is an evenly (resp. oddly) finite-dimensional motive of dimension d, then $\wedge^d M$ (resp. $\mathrm{Sym}^d M$) is one-dimensional. In fact, by 3.5 it suffices to show this after applying some Weil cohomology theory, and then it holds, again by 3.5.

(b) If C is a curve of genus g, then $h^1_{rat}(C)$ is oddly finite-dimensional with $\dim h^1_{rat}(C) = 2g$: This follows from 2.6, 3.5, and the fact that for the ℓ-adic cohomology, $\ell \neq$ char k, one has $\dim_{\mathbb{Q}_\ell} H^1(C \times_k \overline{k}, \mathbb{Q}_\ell) = 2g$. Moreover $\mathrm{Sym}^{2g} h^1_{rat}(c) \cong 1(-g)$: First of all, $\mathrm{Sym}^{2g} h^1_{rat}(C)$ is one-dimensional, by (a). Then, by 3.4 and 3.1, we only have to show this isomorphism modulo (some) homological equivalence. But one knows that $h^1_{hom}(C) \cong h^1_{hom}(\mathrm{Jac}(C))$, where $\mathrm{Jac}(C)$ is the Jacobian of C, and that $\wedge^{2g} h^1_{rat}(\mathrm{Jac}(C)) \cong h^{2g}_{rat}(\mathrm{Jac}(C)) \cong 1(-g)$. Here we have used the fact that $\mathrm{Jac}(C)$ is an abelian variety of dimension g, that for an abelian variety A the Künneth components π_i of the diagonal are algebraic, and that for $h^i_{hom}(A) := h_{hom}(A, \pi_i)$ one has a canonical isomorphism $h^i_{hom}(A) \cong \wedge^i h^1_{hom}(A)$.

We can deduce a certain converse of Theorem 3.3. Consider the following, a priori weaker variant of conjecture $N(X)$ (for a smooth projective variety X).

Conjecture $\mathbf{N'(X)}$: $J(X)$ is a nil ideal.

Corollary 3.9 *Let X be a smooth projective variety X, and let H^* be any Weil cohomology theory. Then the following statements are equivalent, where $S(X)$ is meant with respect to H^*:*

(a) $h(X)$ is finite-dimensional.

(b) $S(X)$ holds, and $N(X^n)$ holds for all $n \geq 1$.

(c) $S(X)$ holds, and $N'(X^n)$ holds for all $n \geq 1$.

Proof. If $M = h(X) = M_+ \oplus M_-$, where $M_+ = h(X, p_+)$ (resp. $M_- = h(X, p_-)$) is evenly (resp. oddly) finite-dimensional, then $H^*(M_+)$ (resp. $H^*(M_-)$) is the even (resp. odd) degree part of $H^*(M)$), because $H^* : \mathcal{M}_{rat}(k) \to GrVec_E$ is a tensor functor. Thus $p_\pm = \pi^X_\pm$ modulo homological equivalence. Therefore (a) implies $S(X)$, and by 3.4, it also implies $N(X)$. Since (a) also implies finite-dimensionality of $h(X^n) = h(X)^{\otimes n}$, for all $n \geq 1$ (by 2.7), (a) implies (b).
(b) $\Rightarrow$ (c) is trivial.
(c) $\Rightarrow$ (a): If $S(X)$ holds, the $\pi_\pm$ are algebraic projectors modulo homological equivalence, and if $N'(X)$ holds, these lift to orthogonal projectors $\tilde{\pi}_+$ and $\tilde{\pi}_-$ modulo rational equivalence with $\tilde{\pi}_+ + \tilde{\pi}_- = id$ by 3.1 (lift π_+ to a projector $\tilde{\pi}_+$ and let $\tilde{\pi}_- = id - \tilde{\pi}_+$). Let $M_\pm = (X, \tilde{\pi}_\pm, 0)$ modulo rational equivalence. Then $M = M_+ \oplus M_-$, and for $b_\pm = \dim H^*(M_\pm)$ one has $\wedge^{b_+ + 1} M_+ = 0 = \mathrm{Sym}^{b_- + 1} M_-$ modulo homological equivalence. By 3.1 and $N'(X^n)$, for $n = b_+ + 1$ and $n = b_- + 1$, one concludes that this vanishing also holds modulo rational equivalence, i.e., we obtain (a).

Corollary 3.10 *Voevodsky's nilpotence conjecture (cf. 3.2) implies the conjecture of Kimura and O'Sullivan (cf. 2.4).*

Proof. It implies the standard conjecture $D(X)$ (postulating $\sim_{\text{hom}} = \sim_{\text{num}}$), hence $B(X)$, hence $C(X)$, hence $S(X)$. Moreover, it implies $N'(X)$ (cf. the lines after 3.2).

Remark 3.11 O'Sullivan (cf. [An1] Th. 3.33) has proved: Let $\mathcal{C}$ be a rigid tensor subcategory of $\mathcal{M}(k)$. If every motive in $\mathcal{C}$ is finite-dimensional, and if every tensor functor

$$\omega : \mathcal{C} \longrightarrow s\mathrm{Vec}_F$$

(where F is a field of characteristic 0 and sVec_F is the category of finite-dimensional super (i.e., $\mathbb{Z}/2$-graded) vector spaces over K) factors through numerical equivalence, then Voevodsky's conjecture holds for $\mathcal{C}$.

So far we have only applied the nilpotence result in 3.3 (b). Theorem 3.3 (a) gives the following Cayley-Hamilton theorem.

Theorem 3.12 *Let f be an endomorphism of a motive M, and assume that M is either evenly finite-dimensional or oddly dimensional. Let H^* be a Weil cohomology theory, and let $P(t) = \det(t - f \mid H^*(M))$ be the characteristic polynomial of f on $H^*(M)$. Then $P(t)$ is independent of the chosen Weil cohomology theory, and one has $P(g) = 0$.*

Proof. If M is an evenly finite-dimensional motive, its cohomology is even, and by the trace formula [Kl] 1.3.6 c one has $tr(f) = tr(f, H^*(M))$. Therefore one has

$$\sum_{i=0}^{d} (-1)^{d-i} tr(\wedge^{d-i} f) t^i = \sum_{i=0}^{d} (-1)^{d-i} tr(\wedge^{n-i} f \mid \wedge^{n-i} H^*(M)) t^i \,.$$

On the other hand, it is known that the right hand side is the charcteristic polynomial $P(t)$. For an oddly finite-dimensional motive its cohomology is odd and one has $tr(f) = -tr(f, H^*(M))$, so that

$$\sum_{i=0}^{d} tr(\mathrm{Sym}^{d-i} f) t^i = \sum_{i=0}^{d} (-1)^{d-i} tr(\wedge^{n-i} f \mid \wedge^{n-i} H^*(M)) t^i$$

is again equal to $P(t)$. Therefore the claim follows with 3.3 (a).

We now come to the proof of Theorem 3.3. It is straightforward to prove the following two lemmas. (Note that for $n = 3$, Lemma 3.13 is just the definition of composition of correspondences.)

Lemma 3.13 *Let $p_{ij} : X^n \to X \times X$ be the projection onto the i-th and j-th factor $((x_1, \ldots, x_n) \mapsto (x_i, x_j))$. Consider algebraic cycles $f_1, \ldots, f_{n-1}$ on $X \times X$, regarded as correspondence from X to X. Then one has*

$$(p_{1,n})_*(p_{1,2}^* f_1 \cdot p_{2,3}^* f_2 \, \ldots \, p_{n-2,n-1}^* f_{n-2} \cdot p_{n-1,n}^* f_{n-1}) = f_{n-1} \circ f_{n-2} \circ \, \ldots \, \circ f_2 \circ f_1$$

(composition of correspondences on the right hand side).

Lemma 3.14 *Consider morphisms $f : V \to M$, $g : W \to N$ of smooth, projective varieties, and the diagram*

$$\begin{tikzcd}
& V \times W \arrow[dl, "p_V"'] \arrow[dr, "p_W"] \arrow[dd, "f\times g"] & \\
V \arrow[dd, "f"'] & & W \arrow[dd, "g"] \\
& M \times N \arrow[dl, "p_M"'] \arrow[dr, "p_N"] & \\
M & & N
\end{tikzcd}$$

where p_V, p_W, p_M and p_N are the projections. Then, for algebraic cycles α on V and β on W one has

$$(f \times g)_*(p_V^*\alpha \,.\, p_W^*\beta) = p_M^* f_*\alpha \,.\, p_N^* g_*\beta \,,$$

i.e., $(f \times g)_*(\alpha \times \beta) = f_*\alpha \times g_*\beta$ for the exterior products.

Proof of Theorem 3.3 Let us consider the case where M is evenly finite-dimensional, with $\wedge^n M = 0$, where $n = d+1$ (The odd case is similar). If $M = (X, p, m)$, then f is a cycle on $X \times X$ such that $pf = f = fp$. Then, by assumption, we have

$$\sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\ \sigma \circ p \times \ \cdots \ \times p = 0$$

(where we have n factors p) since this endomorphism factors through $\wedge^n M$. This means

$$\sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\ p^*_{1,n+\sigma(1)}p \,.\, p^*_{2,n+\sigma(2)}p \,.\ \cdots\ .\, p^*_{n,n+\sigma(n)}p = 0$$

since $p \times \ldots \times p = p^*_{1,n+1}p \,.\, p^*_{2,n+2}p \,.\ \cdots\ .\, p^*_{n,2n}p$ on X^{2n}, where $p_{ij} : X^{2n} \to X \times X$ is the projection onto the i-th and j-th factor as in 3.13. In particular, we have

$$\sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\ (p_{1,n+1})_*(p^*_{1,n+\sigma(1)}p \,.\ \cdots\ .\, p^*_{n,n+\sigma(n)}p \,.\, p^*_{2,n+2}f^t \,.\, p^*_{3,n+3}f^t \,.\ \cdots\ .\, p^*_{n,2n}f^t) = 0$$

Let $o_1(\sigma) = \{1, \sigma(1), \sigma^2(1), \ldots, \sigma^{s-1}(1)\}$ (with $\sigma^s(1) = 1$) be the orbit of $1 \in \{1, \ldots, n\}$ under $\sigma \in S_n$. Then σ is the product

$$\sigma = (1\, \sigma(1)\, \sigma^2(1)\, \ldots\, \sigma^{s-1}(1)) \cdot \sigma'$$

of the s-cycle $\sigma_1 :\ 1 \mapsto \sigma(1) \mapsto \sigma^2(1) \mapsto \ldots \mapsto \sigma^{s-1}(1) \mapsto 1$ and a product σ' of cycles which are disjoint from σ_1. Thus, in the above sum, the summand corresponding to σ is

$$\operatorname{sign}(\sigma)\, (p_{1,n+1})_*(p^*_{1,n+\sigma(1)}p \,.\, p^*_{n+\sigma(1),\sigma(1)}f \,.\, p_{\sigma(1),n+\sigma^2(1)}p \,.\, p_{n+\sigma^2(1),\sigma^2(1)}f \,.$$

$$\cdots\ .\, p^*_{n+\sigma^{s-1}(1),\sigma^{s-1}(1)}f \,.\, p^*_{\sigma^{s-1}(1),n+1}p \,.\, \beta)$$

where β is the product of the $2(n-s)$ factors $p^*_{i,n+\sigma(i)}p$ and $p^*_{i,n+i}f^t$ with $i \in \overline{o_1(\sigma)} = \{1, \ldots, n\} \smallsetminus o_1(\sigma)$. Writing $X^{2n} = V \times W$, with V being the product of the $2s$ factors at the places i or $n+i$ for $i \in o_1(\sigma)$, and W the product over the $2(n-s)$ factors at the other places, it follows from 3.11 (applied to $p_{1,n+1} : V \to X \times X$ and the structural morphism $W \to \mathrm{Spec}(k)$) and 3.10 that the summand is

$$\mathrm{sign}(\sigma) \cdot (f)^{s-1} \deg(\beta')$$

where β' is a zero cycle on W and $(f)^{s-1}$ is the $(s-1)$-fold self product in $\mathrm{End}(M)$. If σ is an n-cycle, then $\mathrm{sign}(\sigma) = (-1)^{n-1}$, $s = n$, $\beta = \beta' = 1$ and $\deg(\beta') = 1$, so that the summand is $(-1)^{n-1}(f)^{n-1}$. If σ is not an n-cycle, then $s < n$.

This shows that we get a polynomial equation for f with leading term $(-1)^{n-1}(n-1)!\,(f)^{n-1}$. If f is numerically equivalent to zero, then so is β' for $s < n$, so that $\deg(\beta') = 0$ unless σ is an n-cycle. This proves 3.3 (b).

For 3.3 (b) we note that, by choosing a bijection $\rho : \{1, \ldots, n-s\} \to \overline{o_1(\sigma)}$, we may identify W with $X^{n-1} \times X^{n-s}$ and β' with

$$\begin{array}{rcl} \beta'' & = & p^*_{1,n-s+\sigma''(1)}p \cdot \cdots \cdot p^*_{n-s,n-s+\sigma''(n-s)}p \cdot p^*_{1,n-s+1}f^t \cdot \cdots \cdot p^*_{n-s,2(n-s)}f^t \\ & = & (\sigma'' \circ p \times \cdots \times p).(f \times \cdots \times f)^t \,, \end{array}$$

where $\sigma'' = \rho^{-1}\sigma'\rho \in S_{n-s}$. Thus

$$\deg(\beta') = tr(\sigma'' \circ p \times \cdots \times p \circ f \times \cdots \times f) = tr(\sigma'' \circ f \times \cdots \times f)\,.$$

Summing over all $\sigma \in S_n$ with fixed σ_1, and keeping the bijection ρ, we thus get

$$(-1)^{s-1} \sum_{\tau \in S_{n-s}} \mathrm{sign}(\tau) tr(\tau \circ f \times \ldots \times f) = (-1)^{s-1}(n-s)! tr(\wedge^{n-s} f) f^{s-1}\,.$$

After summing over all $\sigma \in S_n$ we then see that the coefficient of f^{s-1} is

$$(-1)^{n-1}(n-1)!\,(-1)^{n-s} tr(\wedge^{n-s} f)\,,$$

because there are $(n-1)!/(n-s)!$ cycles σ_1 containing 1 of length s. This proves 3.3 (b).

§4 Finite fields

Let us recall Murre's conjecture. Let X be a purely d-dimensional smooth projective variety over a field k, fix a Weil cohomology theory, and assume the standard conjecture $C(X)$, i.e., that the Künneth components $\pi_i = \pi_i^X \in H^{2d}(X \times X)$ of the diagonal Δ_X are algebraic.

Conjecture 4.1 *(Murre, [Mu1]) (A) X has a Chow-Künneth decomposition, i.e., the π_i^X lift to an orthogonal set of idempotents $\{\tilde{\pi}_i\}$ with $\sum_i \tilde{\pi}_i = \Delta_X$ in $CH^d(X \times X)$.*
(B) The correspondences $\tilde{\pi}_{2j+1}, \ldots, \tilde{\pi}_{2d}$ act as zero on $CH^j(X)$.
(C) Let $F^\nu CH^j(X) = \mathrm{Ker}\tilde{\pi}_{2j} \cap \mathrm{Ker}\tilde{\pi}_{2j-1} \cap \ldots \cap \mathrm{Ker}\tilde{\pi}_{2j-\nu+1} \subseteq CH^j(X)$. Then the descending filtration $F^\cdot$ is independent of the choice of the $\tilde{\pi}_i$.
(D) $F^1CH^j(X) = CH^j(X)_{hom} := \{z \in CH^j(X) | z \sim_{hom} 0\}$.

It is known [Ja2] that this conjecture, taken for all smooth projective varieties, is equivalent to the conjecture of Bloch-Beilinson on a certain functorial filtration on Chow groups, and that this Bloch-Beilinson filtration would be equal to the filtration $F^{\cdot}$ defined above. The advantage of Murre's conjecture is that it can be formulated and proved for specific varieties, and that will be used below.

Remarks 4.2 (a) The condition in 4.1(A) is called the Chow-Künneth decomposition, because it amounts to saying that the Künneth decomposition $h_{hom}(X) = \oplus_{i=0}^{2d} h^i(X)$ in $\mathcal{M}_{hom}(k)$, with $h^i(X) = (X, \pi_i^X)$, can be lifted to a decomposition $h_{rat}(X) = \oplus_{i=0}^{2d} \tilde{h}^i(X)$ in the category $\mathcal{M}_{rat}(k)$ of Chow motives, via $\tilde{h}^i(X) = (X, \tilde{\pi}_i)$.
(b) Conjecture 4.1 (A) would follow from the nilpotence of $J(X)$, i.e., from conjecture $N(X)$ (cf. 3.1), and hence from finite-dimensionality of $h_{rat}(X)$. On the other hand it is known that the Bloch-Beilinson-Murre conjecture would imply the conjecture (2.4) of Kimura-O'Sullivan ([AK], [An1]). Let us note here that, more precisely, Murre's conjecture for X, $X \times X$ and $X \times X \times X$ implies $N(X)$ ([Ja2] pp. 294, 295), so that Murre's conjecture for all sufficient high powers X^N implies finite-dimensionality of X (3.9; note that we have assumed $C(X)$, hence $S(X)$).

Now let k be a finite field. Then the standard conjecture $C(X)$ holds for every smooth projective variety X over k [KM]. It is furthermore known (cf. [Ja3] 4.17) that the conjecture of Bloch-Beilinson-Murre over k is equivalent to the equality $\sim_{rat} = \sim_{hom}$ where homological equivalence is taken with respect to any Weil cohomology theory H satisfying weak Lefschetz (cf. [Kl] p. 368 or [KM] p. 74) (e.g., the ℓ-adic cohomology (1.3 (b)) for any fixed $\ell \neq \mathrm{char}(k)$). Again we want to make this more precise.

Theorem 4.3 *Let k be a finite field. The equality $\sim_{rat} = \sim_{hom}$ on $X \times X$ implies Murre's conjecture for X. Conversely, Murre's conjecture for X and $X \times X$ implies the equality $\sim_{rat} = \sim_{hom}$ for X.*

Proof. The first claim is trivial. For the second claim we use a result of Soulé:

Proposition 4.4 *([So1] Prop. 2) Let X be smooth projective over k. The k-linear Frobenius $F : X \to X$ acts on $CH^j(X)$ as the multiplication by q = cardinality of k.*

Given this, assume Murre's conjecture for X and $X \times X$. We may assume that X is irreducible of dimension d. Let $\tilde{\pi}_0, \ldots, \tilde{\pi}_{2d}$ be orthogonal idempotents lifting the Künneth components $\pi_0^X, \ldots, \pi_{2d}^X$ of the diagonal, and define $\tilde{h}^i(X) = (X, \tilde{\pi}_i)$ in the category $\mathcal{M}_{rat}(k)$ of Chow motives. By Murre's conjecture for $X \times X$,

$$CH^d(X \times X)_{hom} = \bigoplus_{r<2d} \widetilde{\pi_r^{X \times X}}\, CH^d(X \times X)$$

where $\widetilde{\pi_r^{X \times X}} = \sum_{\mu+\nu=r} (\tilde{\pi}_{2d-\mu})^t \times \tilde{\pi}_\nu$ lifts the Künneth component $\pi_r^{X\times X} = \sum_{\mu+\nu=r} \pi_\mu^X \times \pi_\nu^X$ of $X \times X$ (note that $(\pi_{2d-\mu}^X)^t = \pi_\mu^X$). But

$$((\tilde{\pi}_{2d-\mu})^t \times \tilde{\pi}_\nu)\, CH^d(X \times X) \quad = \quad \tilde{\pi}_\nu) \circ CH^d(X \times X) \circ \tilde{\pi}_{2d-\mu}\,,$$

and for $\alpha \in CH^d(X \times X)$ we have

$$\tilde{\pi}_\nu \alpha \tilde{\pi}_{2d-\mu} \tilde{\pi}_i CH^j(X) = 0 \ \text{ for } \ i \neq 2d - \mu\,.$$

On the other hand, for $i = 2d - \mu$ and $\mu + \nu < 2d$ we have

$$\tilde{\pi}_\nu \alpha \tilde{\pi}_{2d-\mu} \tilde{\pi}_i CH^j(X) \subseteq \tilde{\pi}_\nu CH^j(X)$$

with $\nu < i$. This shows that $CH^d(X \times X)_{hom}$ acts trivially on $Gr^i_F CH^j(X)$ for Murre's filtration, because

$$F^i CH^j(X) = \sum_{m \leq 2j-i} \tilde{\pi}_m CH^j(X)$$

by 4.1 (B) and (C). In other words, the correspondences in $CH^d(X \times X)$ act on $Gr^i_F CH^j(X)$ modulo homological equivalence, and then this quotient just depends on the motive modulo homological equivalence $h^{2j-i}(X) = (X, \pi^X_{2j-i})$. Let $P_i(t) = \det(\, t - F^* \mid H^*(X)\,)$ be the characteristic polynomial of the k-linear Frobenius $F : X \to X$ acting on the cohomology. It is known from [KM] that

$$P_i(t) = \det(\, t - F^* \mid H^i(X \times_k \overline{k},\, \mathbb{Q}_\ell))$$

for any $\ell \neq \mathrm{char}(k)$ and hence, by Deligne's proof of the Weil conjectures, that $P_i(t)$ is in $\mathbb{Z}[t]$, and has zeros with complex absolute values $q^{i/2}$.

By the Cayley-Hamilton theorem, $P_i(F)$ acts as zero on $H^i(X)$, hence $P_{2j-\nu}(F)$ acts as zero on $Gr^\nu_F CH^j(X)$. Since $F = q^j$ on $CH^j(X)$ by Soulé's result, and $P_{2j-\nu}(q^i) \neq$ for $\nu \neq 0$, we deduce $Gr^\nu_F CH^j(X) = 0$ for $\nu \geq 1$. q.e.d.

One can prove part of Murre's conjecture from finite-dimensionality, by applying ideas of Soulé [So1], Geisser [Gei], and Kahn [Ka].

Theorem 4.5 *Let k be a finite field, and let X/k be a smooth projective variety such that $J(X)$ is a nil ideal (e.g., assume that $h_{rat}(X)$ is finite-dimensional). Then there is a unique Chow-Künneth decomposition $h_{rat}(X) = \oplus_{i=0}^{2d} \widetilde{h}^i(X)$, and one has*

$$CH^j(\widetilde{h}^i(X)) = 0 \text{ for } i \neq 2j\,.$$

In particular, parts (A), (B) and (C) of Murre's conjecture hold for X and, moreover, $\tilde{\pi}_i$ acts as zero on $CH^j(X)$ for all $i \neq 2j$, so that $F^\nu = 0$ for all $\nu \geq 1$.

Proof. The existence of the Chow-Künneth decomposition was noted in 4.2. Let $P_i(t) = \det(t - F \mid H^i(X))$ be as above. By Cayley-Hamilton we have $P_i(F) = 0$ in $\mathrm{End}(h^i_{hom}(X))$, so that $P_i(F)^r = 0$ in $\mathrm{End}(\widetilde{h}^i(X))$ for some $r \geq 1$, by assumption. Therefore

$$0 = P_i(F)^r \cdot CH^j(\widetilde{h}^i(X)) = P_i(q^j)^r \cdot CH^j(\widetilde{h}^i(X))\,,$$

but $P_i(q^j) \neq 0$ for $j \neq 2i$, by Deligne's proof of the Weil conjecture. The claimed consequences for Murre's conjecture are now immediate. Finally, the uniqueness of the Chow-Künneth decomposition is seen as follows. Let $P(t) = \prod P_i(t)$. Then $P(F)$ is homologically trivial, so that $P(F)^r = 0$ for some $r \geq 0$ in $\mathrm{End}(h_{rat}(X))$. Again by Deligne, the polynomials $P_\nu(t)$ are also pairwise coprime, so that, for each $i \in \{0, \ldots, 2d\}$ there are polynomials $a_i(t)$ and $b_i(t)$ in $\mathbb{Q}(t)$ with

$$a_i(t) P_i(t)^r + b_i(t) Q_i(t)^r = 1\,,$$

where $Q_i(t) = \prod_{j \neq i} P_j(t)$. Then the elements $\tilde{\pi}_i = b_i(F)Q_i(F)^r$ are pairwise orthogonal idempotents in $\mathrm{End}(h_{rat}(X))$ summing up to 1, and $\tilde{\pi}_i$ is a lift of the i-th Künneth projector π_i^X, as follows from Cayley-Hamilton. Finally, by 3.1 every other idempotent lifting π_i^X is of the form $(1+a)\tilde{\pi}_i(1+a)^{-1}$ with $a \in J(X)$, cf. [Ja2] 5.4. But every endomorphism of $h_{rat}(X)$ commutes with F (cf. [So1] Prop. 2 ii)), hence with $\tilde{\pi}_i$, so that we obtain $\tilde{\pi}_i$ again. This shows the uniqueness of the Chow-Künneth decomposition.

Remarks 4.6 (a) The above proof, together with the fact that the Frobenius F commutes with all morphisms of Chow motives ([So1] Prop. 2 ii)), shows that the full (tensor) subcategory $\mathcal{M}_{rat}^{fin}(k) \subset \mathcal{M}_{rat}(k)$ consisting of the finite-dimensional motives, possesses a unique weight grading in the sense of [Ja3] 4.11, i.e., a grading lifting the weight grading of $\mathcal{M}_{hom}(k)$: Every motive M has a unique decomposition $M = \oplus_i M^i$ with $H^i(M) = H^i(M^i)$ and $H^j(M^i) = 0$ for $j \neq i$, and this filtration is respected by any morphism.
(b) By 2.6 and 2.7, the category $\mathcal{M}_{rat}^{fin}(k)$ contains $\mathcal{M}_{rat}^{av}(k)$, the full rigid pseudo-abelian tensor subcategory of $\mathcal{M}_{rat}(k)$ generated by the motives of abelian varieties. Hence the assumptions of Theorem 4.5 hold for (products of) curves, abelian varieties, Fermat hypersurfaces of degree m invertible in k, and Kummer or Enriques surfaces.

As in loc. cit., one can extend the above result to higher algebraic K-theory, or rather motivic cohomology with rational coefficients. Recall that, for any smooth variety V over a field L its $\mathbb{Q}$-rational motivic cohomology can be defined by $H^i_{\mathcal{M}}(V, \mathbb{Q}(j)) = K_{2j-i}(V)^{(j)}$, where

$$K_m(V)^{(j)} := \{\, x \in K_m(V)_{\mathbb{Q}} \mid \psi_n(x) = n^j x \text{ for all } n \geq 1 \,\}\,.$$

Here ψ_n is the n-th Adams operator, acting on the algebraic K-group $K_m(V)$. One has $H^{2j}_{\mathcal{M}}(V, \mathbb{Q}(j)) = K_0(V)^{(j)} \cong CH^j(V)$ by the Riemann-Roch theorem for Chow groups. It is known (and follows from the fact just recalled) that algebraic correspondences modulo rational coefficients act on motivic cohomology, so that motivic cohomology extends to a covariant functor on $\mathcal{M}_{rat}(k)$ by defining

$$H^i_{\mathcal{M}}(M, \mathbb{Q}(j)) = pH^{i+2n}_{\mathcal{M}}(X, \mathbb{Q}(j+n))$$

for $M = (X, p, n)$.

Theorem 4.7 *Under the assumptions of theorem 4.5, one has*

$$H^i_{\mathcal{M}}(\tilde{h}^\nu(X), \mathbb{Q}(j)) = 0 \text{ for } \nu \neq 2j\,.$$

In particular, $H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) = 0$ *for* $j > d = \dim(X)$.

Proof. This follows as above, by using that F acts on $H^i(X, \mathbb{Q}(j))$ as q^j, because $F = \psi_q$ on $K_m(X)$ [So2] 6.1, while $P_\nu(q^j) \neq 0$ for $\nu \neq 2j$.

It remains to investigate part (D) of Murre's conjecture, i.e., the equality $F^1CH^j(X) = CH^j(X)_{hom}$. Recall that the Tate conjecture for $H^{2j}(\overline{X}, \mathbb{Q}_\ell)$ states the surjectivity of the cycle map

$$CH^j(X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \longrightarrow H^{2j}(\overline{X}, \mathbb{Q}_\ell(j))^\Gamma\,,$$

where $\Gamma = Gal(k^{sep}/k)$ is the absolute Galois group of k.

Theorem 4.8 *Let X be smooth, projective of pure dimension d. Assume that*
(i) $J(X)$ is a nil ideal (e.g., assume that X is finite-dimensional),
(ii) the Tate conjecture holds for $H^{2j}(\overline{X}, \mathbb{Q}_\ell)$ and $H^{2(d-j)}(\overline{X}, \mathbb{Q}_\ell)$, and
(iii) the eigenvalue 1 of F is semi-simple on $H^{2j}(\overline{X}, \mathbb{Q}_\ell(j))$.
Then the following holds.
(a) $\sim_{rat} = \sim_{num}$ on $CH^\nu(X)$ (i.e., $CH^\nu(X)_{num} = 0$), for $\nu = j,\ d-j$.
(b) $H^i_{\mathcal{M}}(X, \mathbb{Q}(\nu)) = 0$ for all $i \neq 2\nu$, for $\nu = j,\ d-j$.

Proof. This follows from results of Geisser [Gei] and Kahn [Ka]. Let us give a brief argument, for avoiding a little problem with the arguments given in [Ka], and for getting a statement used below.

By Poincaré duality, (iii) also holds for $d-j$, so it suffices to consider $\nu = j$. Then, by theorems 4.5 and 4.7, it suffices to consider $\tilde{h}^{2j}(X)$ instead of X in the statements. Now it is well-known (cf. [Ta] (2.9)) that the assumptions on the Tate conjecture and the semi-simplicity of F imply that $\sim_{\rm num} = \sim_{\rm hom}$ on $CH^\nu(X)$ for $\nu = j$ and $d-j$, and that

$$A^\nu_{\rm num}(X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} H^{2j}(\overline{X}, \mathbb{Q}_\ell(j))^\Gamma \text{ for } \nu = j,\ d-j$$

via the cycle map. Let $P_{2j}(t) = \det(t - F \mid H^{2j}(X))$ where H is the Weil cohomology theory given by ℓ-adic cohomology, $\ell \neq p = \mathrm{char}(k)$. Write $P_{2j}(t) = Q(t)(t-q^j)^\rho$ with $(t-q^j) \nmid Q(t)$ and some $\rho \geq 0$. By assumption, there is an integer $r > 0$ with $P_{2j}(F)^r = 0$ in $\mathrm{End}(\tilde{h}^{2j}(X))$. Now we have $1 = q(t)Q(t)^r + r(t)(t-q^j)^{\rho r}$ with polynomials $q(t)$ and $r(t)$ in $\mathbb{Q}$. This shows that $Q'(F) = q(F)Q(F)^r$ and $P'(F) = r(F)P(F)^r$, with $P(t) = (t-q^j)^\rho$, are orthogonal idempotents in $\mathrm{End}(\tilde{h}^{2j}(X))$ with $Q'(F)+P'(F) = 1$. Let $M_1 = P'(F)\tilde{h}^{2j}(X)$ and $M_2 = Q'(F)\tilde{h}^{2j}(X)$. Then $M = M_1 \oplus M_2$ and $CH^j(M_1) = 0$ as in the proof of 4.5, because $Q(F)^r M_1 = 0$ and $Q(q^j) \neq 0$, and similarly $H^i_{\mathcal{M}}(M_1, \mathbb{Q}(j)) = 0$.

We now claim that $M_2 \cong 1(-j)^\rho$. Then the claims follow, because it is clear that $\sim_{\rm rat} = \sim_{\rm hom}$ on $1(-j)$, and well-known (by work of Quillen) that $H^i_{\mathcal{M}}(1(-j), \mathbb{Q}(j)) = H^{i-2j}(\mathrm{Spec}k, \mathbb{Q}(0)) = 0$ for $i \neq 2j$ if k is a finite field.

The characteristic polynomial of F on $P'(F)H^{2j}(\overline{X}, \mathbb{Q}_\ell(j))$ is $(t-q^j)^\rho$. Hence

$$H(M_2(j)) = Q'(F)H^{2j}(\overline{X}, \mathbb{Q}_\ell(j)) \cong H^{2j}(\overline{X}, \mathbb{Q}_\ell(j))^{F=1} \cong \mathbb{Q}_\ell^\rho$$

as a Galois module, by semi-simplicity (iii). By Tate's conjecture (ii), this cohomology has a basis given by algebraic cycles. Using the equality $A^j_{hom}(X) = \mathrm{Hom}(1, h_{hom}(X)(j)) = \mathrm{Hom}(1, (M_2)_{hom}(j))$, and the identification of the composition map

$$\mathrm{Hom}(1, h_{hom}(X)(j)) \times \mathrm{Hom}(h_{hom}(X)(j), 1) \to Hom(1,1) = \mathbb{Q}$$

sending (α, β) to $\beta \circ \alpha$ with the intersection number pairing

$$A^j_{hom}(X) \times A^{d-j}_{hom}(X) \to \mathbb{Q}$$

sending (α, β) to $< \alpha \,.\, \beta >$ we now get two maps $1^\rho \xrightarrow{\varphi} M_2(-j) \xrightarrow{\psi} 1^\rho$ whose composition is the identity. (Note that the above intersection number pairing is non-degenerate, because $A^\nu_{hom}(X) = A^\nu_{num}(X)$ for $\nu = j$ and $d-j$, as remarked above.) Therefore 1^ρ becomes a direct factor of $(M_2)_{hom}$, and we conclude that $\varphi : 1^\rho \cong (M_2)_{hom}$ is an isomorphism with inverse ψ, because $H(M_2(j)) \cong \mathbb{Q}_\ell^\rho$ as was shown above. But this implies that one also has an isomorphism $1^\rho \cong M_2(j)$ in the category of Chow motives, because $J(M_2)$ is a nil ideal, and $J(1) = 0$.

Corollary 4.9 *If $J(X)$ is a nil ideal (e.g., if X is finite-dimensional), then $H^i_{\mathcal{M}}(X, \mathbb{Q}(d)) = 0$ for $i \neq 2d$.*

Proof. In fact, it is clear that the Tate conjecture holds in degrees 0 and $2d$, and that the corresponding cohomology groups are semi-simple Galois representations. We remark that the bijectivity of $c\ell^d : CH^d(X) \otimes \mathbb{Q}_\ell \to H^{2d}(\overline{X}, \mathbb{Q}_\ell(d))$ is known without the assumption on X, by higher class field theory (note that we have $\mathbb{Q}$-coefficients).

Corollary 4.10 *Assume that $J(X)$ is a nil ideal, the Tate conjecture holds for X (i.e., for all cohomology groups of X), and the eigenvalue 1 is semi-simple on all groups $H^{2j}(\overline{X}, \mathbb{Q}_\ell(j)$. Then*
(a) $CH^j(X)_{num} = 0$, and $CH^j(X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} H^{2j}(\overline{X}, \mathbb{Q}_\ell(j)^\Gamma$ via the cycle map (strong Tate conjecture) for all $j \geq 0$,
(b) $K_m(X) \otimes \mathbb{Q} = 0$ for all $m \neq 0$ (Parshin conjecture).

Remarks 4.11 (a) The problems with the arguments in [Ka] concern the meaning of the statement that rational and numerical equivalence agree on X. In this paper, the meaning is that $\sim_{rat} = \sim_{num}$ on $CH^j(X)$ for all j, and this would also fit with the assumptions in [Ka]. It does not imply that one can identify $h_{rat}(X)$ and $h_{num}(X)$ as written in the parenthesis following loc.cit. Cor. 2.2, because that would rather mean that rational and numerical equivalence agree on $X \times X$. Similarly, the reference in [Ka] 2.2 to [Gei] th. 3.3 has to be completed, because in the latter reference the argument is by assuming $\sim_{rat} = \sim_{num}$ for all varieties, and deducing an action of $\mathrm{End}(h_{num}(X))$ on $K_a(X)^{(j)}$, which again requires $\sim_{rat} = \sim_{num}$ on $X \times X$. Finally, in the proof of [Ka] Théorem 1.10, the reference to [Mi] th. 2.6 has to be taken with similar care, because again, in that reference the (strong) Tate conjecture is assumed for all varieties, and in principle used for a product of two varieties when deducing semi-simplicity of the category $\mathcal{M}_{hom}(k)$ and considering the question of isomorphy of two motives. The final conclusion is that the stated results in [Ka] remain correct, while the proofs have to be modified - basically by noting that in the considered cases it suffices to consider morphisms between Tate objects $1(j)$ and $h(X)$ instead of endomorphisms of $h(X)$.
(b) In principle, the proof given in [An1] 4.2 is correct, but the short formulation might disguise the fact that, to my knowledge, it does not suffice to assume the Tate conjecture and 1-semi-simplicity just for $H^{2j}(\overline{X}, \mathbb{Q}_\ell(j))$ if one wants to get the results for $CH^j(X)$.
(c) In view of 4.6 (b), the assumptions of Corollary 4.10 hold, e.g., for arbitrary products of elliptic curves [Sp], for abelian varieties of dimension ≤ 3 [So1] Th. 4, Fermat surfaces of degree m invertible in k and dimension ≤ 3 (loc. cit.), for rational, Enriques or Kummer surfaces, and for many abelian varieties. In particular, for the sub tensor category of $\mathcal{M}_{rat}(k)$ generated by elliptic curves one gets $\sim_{\mathrm{rat}} = \sim_{\mathrm{hom}}$, and hence the validity of Murre's conjecture.

Theorem 4.12 *Under the assumptions of Corollary 4.10, the regulator map*

$$H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} H^i(\overline{X}, \mathbb{Q}_\ell(j))^\Gamma$$

is an isomorphism for all $i, j \in \mathbb{Z}$, where $\Gamma = Gal(\overline{k}/k)$ and $\ell \neq char\ k$.

Proof. This is clear from 4.10 and the fact that $H^i(\overline{X}, \mathbb{Q}_\ell(j))^\Gamma = 0$ for $i \neq 2j$ by Deligne's proof of the Weil conjectures. For the definition and properties of these regulator maps we refer to [Ja1] ch. 8, where they are deduced from Chern characters on higher algebraic K-theory constructed by Gillet. They exist for any smooth variety U over k instead of X, and coincide with the cycle maps for $i = 2j$, via the isomorphisms $K_0(U)^{(j)} \cong CH^j(U)$.

The following application will be used in the next section.

Corollary 4.13 *If C is an elliptic curve or a rational curve, and X is a product of elliptic curves, then, for every open $U \subseteq C$, and all $i, j \in \mathbb{Z}$, the regulator map*

$$H^i_{\mathcal{M}}(X \times U,\, \mathbb{Q}(j)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to H^i(\overline{X \times U},\, \mathbb{Q}_\ell(j))^\Gamma$$

is an isomorphism, and the eigenvalue 1 of F on $H^i(\overline{X \times U},\, \mathbb{Q}_\ell(j))$ is semi-simple.

This is a special case of the following conjecture (in which k is still a finite field).

Conjecture 4.14 *([Ja1] 12.4) For any separated scheme of finite type Z over k, and all $a, b \in \mathbb{Z}$, the regulator map*

$$H^M_a(Z, \mathbb{Q}(b)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to H^{ét}_a(\overline{Z}, \mathbb{Q}_\ell(b))^\Gamma$$

is an isomorphism, and the eigenvalue 1 of Frobenius on $H^{ét}_a(\overline{Z}, \mathbb{Q}_\ell(b))$is semi-simple.

Again we refer to [Ja1] ch. 8 for the definition and properties of these homological versions of the regulator maps. Corollary 4.13 now follows from the following lemma, because the assertion of 4.12 holds for $X \times C$ and for $X \times \mathrm{Spec}(k(x))$, for any closed point $x \in C \setminus U$.

Lemma 4.15 *(a) ([Ja1] 8.4) If Z is smooth and of pure dimension d, then the regulator map in 4.14 coincides with the regulator map*

$$H^{2d-a}_{\mathcal{M}}(Z, \mathbb{Q}(d-b)) \to H^{2d-a}_{ét}(\overline{Z}, \mathbb{Q}_\ell(d-b))^\Gamma\,.$$

(b) ([Ja1] Th. 12.7 b)) If $Z' \subseteq Z$ is closed, $U = Z \smallsetminus Z'$, and Conjecture 4.14 holds for two of the three schemes Z, Z', U, then it also holds for the third one.

Although in this paper, we always used Chow groups and motivic cohomology groups with $\mathbb{Q}$-coefficients, we note that we can also get consequence for groups with $\mathbb{Z}$-coefficients, as in Soulé's paper [So1]. Recall that, for a smooth variety X over a field L one has motivic cohomology with $\mathbb{Z}$-coefficients, which can for example be defined as $H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) = CH^j(X, 2j-i)$, where the latter groups are the higher Chow groups as defined by Bloch [Bl]. By definition, these groups vanish for $j < 0$ or $i > 2j$ or $i > d + j$, where $d = \dim(X)$, and it is known that $CH^a(X, b) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K_b(X)^{(a)}$ so that $H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) \otimes_{\mathbb{Z}} \mathbb{Q} = H^i_{\mathcal{M}}(X, \mathbb{Q}(j))$. Moreover $H^{2j}(X, \mathbb{Z}(j)) = CH^j(X, 0) = CH^j(X)_{\mathbb{Z}}$, the usual Chow groups with integral coefficients. Finally, for an irreducible smooth projective variety X of dimension d, the group $CH^d(X \times X, \mathbb{Z})$ of integral correspondences is a ring and acts on the motivic cohomology $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$. The additive category of integral motives (modulo rational equivalence) is defined by the same formalism as recalled in section 1. We denote the objects as $(X, p, m)_{\mathbb{Z}}$ where p is now an integral idempotent correspondence, and define $h(X)_{\mathbb{Z}} = (X, id, 0)_{\mathbb{Z}}$, the integral motive corresponding to X and $1(j)_{\mathbb{Z}} = (\mathrm{Spec}(k), id, j)_{\mathbb{Z}}$, the j-fold Tate twist of the trivial motive 1.

Corollary 4.16 *Under the assumptions of Theorem 4.8, the groups $H^i_{\mathcal{M}}(X,\mathbb{Z}(j))$ have finite exponent for $i \neq 2j$. For $H^{2j}_{\mathcal{M}}(X,\mathbb{Z}(j)) = CH^j(X)_{\mathbb{Z}}$, the subgroup $CH^j(X)_{\mathbb{Z},num}$ of numerically trivial cycles has finite exponent, and the quotient group $A^j(X)_{\mathbb{Z},num} = CH^j(X)_{\mathbb{Z}}/CH^j(X)_{\mathbb{Z},num}$ is isomorphic to $\mathbb{Z}^\rho$, where $\rho = \dim_{\mathbb{Q}_\ell} H^{2j}(X_{\overline{k}}, \mathbb{Q}_\ell(j))^{Gal(\overline{F}/F)}$.*

Proof. In the proof of Theorem 4.8 it was shown that there is an isomorphism of $\mathbb{Q}$-linear motives

$$h(X) \cong M_1 \oplus 1(-j)^\rho\,,$$

with $Q(F)M_1 = 0$ for a polynomial $Q(t) \in \mathbb{Z}[t]$ with $Q(q^j) \neq 0$. Then there are morphisms

$$h(X) \xrightarrow{\alpha} 1(-j)^\rho \xrightarrow{\beta} h(X)$$

with $\alpha\beta = id$, and for the idempotent $\beta\alpha$ one has $Q(F)(id - \beta\alpha) = 0$ ($id - \beta\alpha$ is the idempotent corresponding to M_1). Thus there exist integers $N_1, N_2, N_3 > 0$ such that $N_1\alpha$ and $N_2\beta$ lift to integral correspondences $\tilde{\alpha}$ and $\tilde{\beta}$, and $N_3Q(F)(N_1N_2 - \tilde{\beta}\tilde{\alpha}) = 0$ in $CH^d(X\times X)_{\mathbb{Z}}$. Now the argument of Soulé for Chow groups ([So1] Prop. 2 i)) immediately extends to higher Chow groups to show that F acts on $H^i_{\mathcal{M}}(X,\mathbb{Z}(j))$ as multiplication by q^j. We conclude $N_3\tilde{\beta}\tilde{\alpha}H^i_{\mathcal{M}}(X,\mathbb{Z}(j)) = N_1N_2N_3Q(F)H^i_{\mathcal{M}}(X,\mathbb{Z}(j)) = NH^i_{\mathcal{M}}(X,\mathbb{Z}(j))$ with the non-zero integer $N = N_1N_2N_3Q(q^j)$. But the the composition

$$H^i_{\mathcal{M}}(X,\mathbb{Z}(j)) \xrightarrow{\tilde{\alpha}_*} H^i_{\mathcal{M}}(1(-j)^\rho,\mathbb{Z}(j)) \xrightarrow{\tilde{\beta}_*} H^i_{\mathcal{M}}(X,\mathbb{Z}(j))$$

is zero for $i \neq 2j$, because $H^i_{\mathcal{M}}(1(-j),\mathbb{Z}(j)) = H^{i-2j}_{\mathcal{M}}(\mathrm{Spec}(k),\mathbb{Z})$, which is known to be zero for $i \neq 2j$ if k is a finite field. For $i = 2j$ we have $H^{i-2j}_{\mathcal{M}}(\mathrm{Spec}(k),\mathbb{Z}) = \mathbb{Z}$, and thus $NH^{2j}_{\mathcal{M}}(X,\mathbb{Z}(j))$ is isomorphic to $\mathbb{Z}^\rho$ (note that $\tilde{\alpha}_*\tilde{\beta}_* = N_1N_2$, which can be checked after tensoring with $\mathbb{Q}$, where it holds by definition). We deduce that the torsion group of $CH^j(X)_{\mathbb{Z}}$ is killed by N, and coincides with $CH^j(X)_{\mathbb{Z},num}$: it is contained in the latter group, and the quotient embeds into the group $CH^j(X)_{\mathbb{Q}} = A_{num}(X)_{\mathbb{Q}}$.

Corollary 4.17 *Let X be a smooth projective variety over the finite field k such that the associated motive (with $\mathbb{Q}$-coefficients) is finite-dimensional (or that $J(X)$ is a nil ideal). Then the group $H^i_{\mathcal{M}}(X,\mathbb{Z}(j))$ has finite exponent for $j > d = \dim(X)$, and $H^i_{\mathcal{M}}(X,\mathbb{Z}(d))$ has finite exponent for $i < 2d$.*

Proof. The last statement follows from 4.16, because the Tate conjecture and the semi-simplicity hold for H^0 and H^{2d}. Formally, the first statement follows as well, since the condition on the Tate conjecture is empty here, but we give a simpler direct proof: Let the integral polynomial $P(t) = \prod_{i=0}^{2d} P_i(t)$ be as in the proof of Theorem 4.5. Then the assumption implies that $P(F)^r = 0$ in $CH^d(X \times X,\mathbb{Q})$, for some integer $r \geq 1$. Therefore $NP(F)^r = 0$ in $CH^d(X \times X,\mathbb{Z})$ for some integer $N \geq 1$. Because F acts as q^j on $H^i_{\mathcal{M}}(X,\mathbb{Z}(j))$, the integer $NP(q^j)$ annihilates this group, but one has $P(q^j) \neq 0$ for $j \notin \{0,\ldots,2d\}$ by Deligne's proof of the Weil conjectures.

§5 Global function fields

Using the last section, we will deduce some results for global function fields. The most complete results are obtained for certain isotrivial varieties. Let F be a finite field, let C be a smooth projective geometrically irreducible curve over F, and let $k = F(C)$ be its function field.

Theorem 5.1 *Let W be a smooth projective variety over k, and assume that, after possibly passing to a finite extension F'/F, W is isomorphic to $Y \times_F k$, where Y/F is a smooth projective variety such that the assumptions of Corollary 4.10 hold for $X = Y \times_F C$. Then the strong Tate conjecture holds for W, i.e., the cycle maps induce isomorphisms*

$$A^j_{hom}(W) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} H^{2j}_{ét}(W_{\overline{k}}, \mathbb{Q}_\ell(j))^{Gal(\overline{k}/k)},$$

for all $j \geq 0$, and the Abel-Jacobi map

$$CH^j(W)_{hom} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} H^1_{cont}(G_k, H^{2j-1}_{ét}(W_{\overline{k}}, \mathbb{Q}_\ell(j)))$$

is an isomorphism for all $j \geq 0$. Furthermore Murre's conjecture holds for W, with the filtration $F^1CH^j(W) = CH^j(W)_{hom}$ and $F^2CH^j(W) = 0$, and numerical and homological equivalence agree on W (i.e., on all Chow groups of W). Finally one has $H^i_{\mathcal{M}}(W, \mathbb{Q}(j)) = 0$ for $2j - i \neq 0, 1$, i.e., $K_m(W)_{\mathbb{Q}} = 0$ for $m \geq 2$.

Remarks 5.2 The assumptions of the theorem hold, if C is a rational or elliptic curve and X is a product of rational or elliptic curves (or the motive of X is contained in the rigid tensor subcategory generated by elliptc curves and Artin motives).

We will prove a somewhat more general result. For any smooth variety W over $k = F(C)$ define the *arithmetic étale cohomology* as

$$H^i_{ar}(W, \mathbb{Q}_\ell(j)) := \varinjlim H^i(X_{U'} \times_F \overline{F}, \mathbb{Q}_\ell(j))^{Gal(\overline{F}/F)},$$

where $U \subseteq C$ is some non-empty open, $X \to U$ is a smooth model for W ($W \cong X \times_U k$), and the limit is over all non-empty open subschemes $U' \subseteq U$, with $X_{U'} = X \times_U U'$. By standard limit theorems this cohomology does not depend on the choice of U and X. Moreover, this cohomology is functorial in W, receives a cycle class and allows an action of Chow correspondences if W is smooth and proper. In fact, there are regulator maps

$$H^i_{\mathcal{M}}(W, \mathbb{Q}(j)) \longrightarrow H^i_{ar}(W, \mathbb{Q}_\ell(j))$$

by taking the limit of the regulator maps

$$H^i_{\mathcal{M}}(X'_U, \mathbb{Q}(j)) \longrightarrow H^i(X'_U \times_F \overline{F}, \mathbb{Q}_\ell(j))^{Gal(\overline{F}/F)}$$

discussed at the end of the previous section and noting that motivic cohomology commutes with filtered inductive limits, so that the limit on the left hand side is $H^i_{\mathcal{M}}(W, \mathbb{Q}(j))$.

Let V be an ℓ-adic representation of G_k (i.e., a finite-dimensional $\mathbb{Q}_\ell$-vector space with continuous action of G_k). Call V *arithmetic*, if it comes from a representation of the fundamental group $\pi_1(U, \overline{\eta})$, where $U \subseteq C$ is a non-empty open and $\overline{\eta} = \mathrm{Spec}(\overline{k})$ the geometric generic point of U. Then we define the *arithmetic Galois cohomology* of V as

$$H^i_{ar}(G_k, V) := \varinjlim H^1(\pi_1(U'_{\overline{F}}, \overline{\eta}), V)^{Gal(\overline{F}/F)}\,,$$

where the limit is again over the non-empty open $U' \subseteq U$. (This is $H^i_{ar}(\mathrm{Spec}(k), \mathcal{F})$ for the ℓ-adic sheaf $\mathcal{F}$ on $\mathrm{Spec}(k)$ corresponding to V, if one defines arithmetic étale cohomology more generally for an *arithmetic* ℓ-adic sheaf $\mathcal{G}$ on W, i.e., one that extends to some model X over some $U \subseteq C$ as above, cf. [Ja1] 11.7, 12.15.) This definition is functorial in V. With these notations we have the following.

Theorem 5.3 *Let W be a smooth projective variety over k. Assume the condition*
$(*)$ There is a scheme X of finite type over C with generic fiber $W = X \times_C k$ such that for some non-empty open $U \subseteq C$, Conjecture 4.14 holds for $X_U = X \times_C U$ and the fibres $X_t = X \times_C t$ for all closed points $t \in U$.
Then the regulator maps induce isomorphisms

$$H^i_{\mathcal{M}}(W, \mathbb{Q}(j)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} H^i_{ar}(W, \mathbb{Q}_\ell(j))\,, \tag{5}$$

for all $i, j \in \mathbb{Z}$. Moreover, $H^i_{\mathcal{M}}(W, \mathbb{Q}(j)) = 0$ for $i - 2j \neq 0, 1$, i.e., $K_m(W)_{\mathbb{Q}} = 0$ for $m > 1$. For $i - 2j = 1$ one has isomorphisms $H^i_{ar}(W, \mathbb{Q}_\ell(j)) \cong H^1_{ar}(G_k, H^i(W_{\overline{k}}, \mathbb{Q}_\ell(j)))$. For $i - 2j = 0$, the cycle maps induce isonorphisms for all $j \geq 0$

$$A^j_{hom}(W) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} H^{2j}_{\acute{e}t}(W_{\overline{k}}, \mathbb{Q}_\ell(j))^{Gal(\overline{k}/k)} \quad \textit{(strong Tate conjecture)} \tag{6}$$

$$CH^j(W)_{hom} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} H^1_{cont}(G_k, H^{2j-1}_{\acute{e}t}(W_{\overline{k}}, \mathbb{Q}_\ell(j))) \quad \textit{(Abel-Jacobi map).} \tag{7}$$

If W has a Chow-Künneth decomposition (e.g., if standard conjecture $C(W)$ holds and condition $(*)$ also hold for $W \times_k W$), then Murre's conjecture holds for W, with $F^1CH^j(W) = CH^j(W)_{hom}$ and $F^2CH^j(W) = 0$.

Proof. The first isomorphism is clear from the above, since the maps

$$H^i_{\mathcal{M}}(X_{U'}, \mathbb{Q}(j)) \longrightarrow H^i(X_{U'} \times_F \overline{F}, \mathbb{Q}_\ell(j))^{Gal(\overline{F}/F)}$$

are isomorphisms for all sufficiently small $U' \subseteq U$, by Lemma 4.15 (a) and (b). The next three claims follow from [Ja1] Thm. 12.16, diagram (12.16.3) and Rem. 12.17 b).

Now assume that W is of pure dimension d and has a Chow-Künneth decomposition, i.e., the Künneth projectors π_i are algebraic, and lift to an orthogonal set of idempotents $\tilde{\pi}_i$ in $CH^d(W \times W)$. Since the cycle maps (6) and (7) are functorial with respect to correspondences, it follows that, for the filtration $F^\nu CH^j(W)$ defined in the theorem, the action of correspondences on $Gr^\nu_F CH^j(W)$ factors through homological equivalence, and that $\pi_i = \delta_{i,2j-\nu} id$ (Kronecker symbol) on $Gr^\nu_F CH^j(W)$. From this the remaining parts of Murre's conjecture follow easily, including the given description of the filtration.

Finally assume that the Künneth components π_i are algebraic and that condition $(*)$ holds for $W \times W$. For any smooth projective variety V over k let F^ν_ℓ be the descending filtration on the continuous étale cohomology $H^i_{cont}(V, \mathbb{Q}_\ell(j))$ coming from the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p_{cont}(G_k, H^q(V_{\overline{k}}, \mathbb{Q}_\ell(j))) \Rightarrow H^{p+q}_{cont}(V, \mathbb{Q}_\ell(j)) .$$

Then, by (7) for $W \times W$, the cycle map $CH^d(W \times W) \to H^{2d}((W \times W)_{\overline{k}}, \mathbb{Q}_\ell(d))$ (which is compatible with the cycle map (5) for $W \times W$ and $(i, j) = (2d, d)$) induces an injection

$$CH^d(W \times W)_{hom} \longrightarrow Gr^1_{F_\ell} H^{2d}_{cont}(W \times W, \mathbb{Q}_\ell(d)) \cong H^1(G_k, H^{2d-1}((W \times W)_{\overline{k}}, \mathbb{Q}_\ell(d))) .$$

On the other hand, the filtration F^ν_ℓ is respected under the action of correspondences, and $F^\nu.F^\mu \subseteq F^{\nu+\mu}$ under cup product. This shows that $J(W) = CH^d(W \times W)$ is an ideal of square zero. Hence W has a Chow-Künneth decomposition.

Proof of Theorem 5.1: We may assume that Y is of pure dimension d. Next we observe that a finite constant field extension does not matter, because we have Galois descent for étale cohomology with $\mathbb{Q}_\ell$-coefficients and motivic cohomology with $\mathbb{Q}$-coefficients. Thus we may assume that $W = Y \times_F k$. Then it is clear that Theorem 5.1 follows from 5.3, except possibly for the statement on Murre's conjecture. But, in the situation of 5.1, the pull-back via the morphism $W = Y_k \to Y$ induces isomorphism $H^i(Y_{\overline{F}}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^i(W_{\overline{k}}, \mathbb{Q}_\ell)$ by proper and smooth base change. This shows that the projectors $\tilde{\pi}^Y_i$ of a Chow-Künneth decomposition for Y (which exist by the assumptions on Y) map to idempotents lifting the Künneth components of W under the pull back $CH^d(X \times X) \to CH^d(W \times W)$. Therefore W has a Chow-Künneth decomposition, and we can apply 5.3.

While the emphasis of this paper was to investigate conjectures, results and conditions for fixed varieties, we conclude with statements on all varieties over a given field. From Theorem 5.3 we get:

Corollary 5.4 *If conjecture 4.14 holds for all (smooth) varieties over F, then the results of theorem 5.3 hold for all smooth projective varieties W over function fields k in one variable over F. In particular, the strong Tate conjecture and the Murre's conjecture hold over such k.*

The reduction to the smooth case is done by lemma 4.15 and induction on dimension. Moreover, we note:

Proposition 5.5 *Conjecture 4.14 holds for all varieties over F if and only if the following holds for all smooth projective varieties X over F:*
(i) Tate's conjecture (surjectivity of (5)),
(ii) the eigenvalue 1 is semi-simple on $H^{2j}(X_{\overline{F}}, \mathbb{Q}_\ell(j))$ for all j,
(iii) the Chow motive $h_{rat}(X)$ is finite-dimensional.

Proof. First we note that the properties (i) - (iii) for all $X \in SP_F$ are equivalent to conjecture 4.14 for all $X \in SP_F$. This follows from theorems 4.12 and 3.9, and the fact that $S(X)$ holds for all $X \in SP_F$. Secondly, conjecture 4.14 holds for all varieties if it holds for smooth projective varieties. The proof goes like in [Ja1] 12.7, but instead of assuming resolution of singularities, one may use de Jong's version: Let Z be any reduced separated algebraic F-scheme. By [dJ] there is a smooth projective variety X and a morphism $f : X \to Z$ which is generically étale. Choose a dense smooth open $U \subseteq Z$ such that the restriction $g : V = f^{-1}(U) \to U$ is finite étale. By induction on dimension and lemma 4.15 (b) it suffices to prove conjecture 4.14 for U, and we may assume that it holds for V. But g induces degree-respecting pull-backs g^* and push-forwards g_* in motivic and étale cohomology making the diagrams

$$\begin{array}{ccc} H^i_{\mathcal{M}}(V, \mathbb{Q}(j)) \otimes \mathbb{Q}_\ell & \longrightarrow & H^i_{ét}(V_{\overline{F}}, \mathbb{Q}_\ell(j))^{Gal(\overline{F}/F)} \\ \downarrow g_* & & \downarrow g_* \\ H^i_{\mathcal{M}}(U, \mathbb{Q}(j)) \otimes \mathbb{Q}_\ell & \longrightarrow & H^i_{ét}(U_{\overline{F}}, \mathbb{Q}_\ell(j))^{Gal(\overline{F}/F)} \end{array}$$

commutative; similarly with g^*. On the other hand, one has $g_*g^* = m$ on both sides, where m is the degree of g. This implies that the bottom line is a retract of the top line, and hence that conjecture 4.14 for V implies conjecture 4.14 for U.

Finally we indicate that the above results can easily be generalized to function fields k of arbitrary transcendence degree over F, by replacing C by any variety over F and using the same definitions of arithmetic étale and Galois cohomology, and a corresponding Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p_{ar}(G_k, H^q(W_{\overline{k}}, \mathbb{Q}_\ell(j))) \Rightarrow H^{p+q}_{ar}(W, \mathbb{Q}_\ell(j)).$$

This gives the following result.

Proposition 5.6 *If conjecture 4.14 holds for all (smooth projective) varieties over $\mathbb{F}_p$, then the strong Tate conjecture, the equality of numerical and homological equivalence and the conjecture of Bloch-Beilinson-Murre hold over all fields of characteristic p.*

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