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THE Γ–LIMIT OF THE GINZBURG-LANDAU ENERGY IN AN ELASTIC MEDIUM

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ABSTRACT. The sharp interface limit of a multi-phase Ginzburg-Landau energy is identified in situations where contributions from mechanical interactions are present. In the two phase situation we also pass to the limit in the corresponding Euler-Lagrange equation and recover an elastically modified Gibbs–Thomson law which includes terms involving the Eshelby tensor.

1. INTRODUCTION

In this paper we study solutions of the variational problems

($\mathbf{P}^\varepsilon$) Find a minimizer $(\mathbf{c}, \mathbf{u}) \in X_1 \times X_2$ of

$$E^\varepsilon(\mathbf{c}, \mathbf{u}) := \int_\Omega \left(\varepsilon |\nabla \mathbf{c}|^2 + \frac{1}{\varepsilon}\Psi(\mathbf{c}) + W(\mathbf{c}, \mathcal{E}(\mathbf{u})) \right), \quad \varepsilon > 0,$$

subject to the constraint $\fint_\Omega \mathbf{c} = \mathbf{m}$, with $\mathbf{m} \in \Sigma$.

Solutions of the variational problems are stable stationary solutions to the Cahn–Hilliard system with elasticity studied in [11, 12]. Under some natural assumptions it will turn out that minimizers of the above variational problem are, roughly speaking, of the following form. In most of Ω the quantity $\mathbf{c}$ is close to values that minimize Ψ and the regions where $\mathbf{c}$ is close to minimizers of Ψ are separated by transition layers which are of a thickness proportional to ε. It is the goal of this paper to study the limiting behavior of E^ε and its minimizers as ε tends to zero. The scaling in ε is motivated by former studies for the case when no elastic contributions are present and by formally matched asymptotic expansions by Leo, Lowengrub and Jou [19]. As in the case without elasticity (see [25, 24, 9, 3]) we will use arguments of Γ–convergence theory to identify the asymptotic limit for the functionals E^ε. The variational problem in the Γ–limit turns out to be a partitioning problem with an energy which includes terms stemming from surface energy and bulk contributions taking elastic interactions into account. In this paper we also show for a two-phase situation that we can pass to the limit in the Euler-Lagrange equations for E^ε and we obtain an elastically modified Gibbs–Thomson law in the asymptotic limit in which an energy-momentum tensor appears through the so called Eshelby traction.

The energy E^ε appears in models for phase transitions in multicomponent alloys and we now want to specify its correct form in detail. Assuming that the alloy consists of N components, we denote by c_k $(k = 1, \ldots, N)$ the concentration of component k. Therefore, the vector $\mathbf{c} = (c_1, \ldots, c_N)_{k=1,\ldots,N}$ has to fulfill the constraint $\sum_{k=1}^{M} c_k = 1$, i.e. $\mathbf{c}$ lies in the affine hyperplane

$$\Sigma := \{\mathbf{c}' = (c'_k)_{k=1,\ldots,N} \in \mathbb{R}^N \mid \sum_{k=1}^{N} c'_k = 1\} .$$

The space X_1 appearing in the definition of $(\mathbf{P}^\varepsilon)$ is defined as

$$X_1 := \{\mathbf{c} \in H^1(\Omega, \mathbb{R}^N) \mid \mathbf{c} \in \Sigma \text{ almost everywhere}\}$$

i.e. all H^1-functions fulfilling the constraint for the concentrations. In order to describe mechanical effects we introduce the displacement field $\mathbf{u}(\mathbf{x})$ and the linearized strain tensor

$$\mathcal{E}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^t) .$$

The value of E^ε depends on $\mathbf{u}$ only through $\mathcal{E}(\mathbf{u})$ and hence translations and infinitesimal rotations do not change the value of E^ε. Therefore, in order to obtain coercivity we introduce the space

$$X_2 := \{\mathbf{u} \in H^1(\Omega, \mathbb{R}^n) \mid (\mathbf{u}, \mathbf{v})_{H^1} = 0 \text{ for all } \mathbf{v} \in X_{\mathrm{ird}}\}$$

where

$$X_{\mathrm{ird}} := \{\mathbf{u} \in H^1(\Omega, \mathbb{R}^n) \mid \text{ there exist } b \in \mathbb{R}^n \text{ and a skew symmetric } A \in \mathbb{R}^{n\times n} \text{ such that } \mathbf{u}(x) = b + Ax\}$$

is the space of all infinitesimal rigid displacements. Since translations and rotations do not change the appearing patterns and since we are mainly interested in $\mathbf{c}$, searching for a minimum in $X_1 \times X_2$ leads to a physical meaningful solution.

The term $W(\mathbf{c}, \mathcal{E})$ in $(\mathbf{P}^\varepsilon)$ is the elastic free energy density for which we will make specific assumptions later. We refer to [10, 11, 12] for physically relevant examples of W. We assume that the homogeneous free energy Ψ is such that

$$\Psi \geq 0 \qquad \text{and} \quad \Psi(\mathbf{c}') = 0 \quad \Leftrightarrow \quad \mathbf{c}' \in \{\mathbf{p}_1, ..., \mathbf{p}_M\}, \tag{1}$$

where $\mathbf{p}_1, ..., \mathbf{p}_M \in \Sigma$ are mutually different and $M \geq 2$. We note that the variational problem $(\mathbf{P}^\varepsilon)$ for functions Ψ which do not fulfill the assumption (1) can often be shown to be equivalent to a variational problem for a Ψ that fulfills (1). Let us show how this is possible. Assume that $\tilde{\Psi}$ is a homogeneous free energy that does not necessarily fulfill (1). Let A be any affine function which graph defines a supporting hyperplane for $\tilde{\Psi}$ in exactly $M \geq 2$ points. This means

$$\tilde{\Psi}(\mathbf{c}') \geq A(\mathbf{c}') \qquad \text{for all} \quad \mathbf{c}' \in \Sigma$$

and

$$\tilde{\Psi}(\mathbf{c}') = A(\mathbf{c}') \qquad \text{in exactly} \quad M \quad \text{points}$$

(see Figure 4 for $N = M = 2$). Then we can subtract A from $\tilde{\Psi}$ to obtain a function that fulfills (1). Due to the fact that we impose an integral constraint, the minimization problem $(\mathbf{P}^\varepsilon)$ remains unchanged by this procedure. In particular, all homogeneous free energies that appear in the theory of phase separation can be reduced to fulfill (1) in this way.

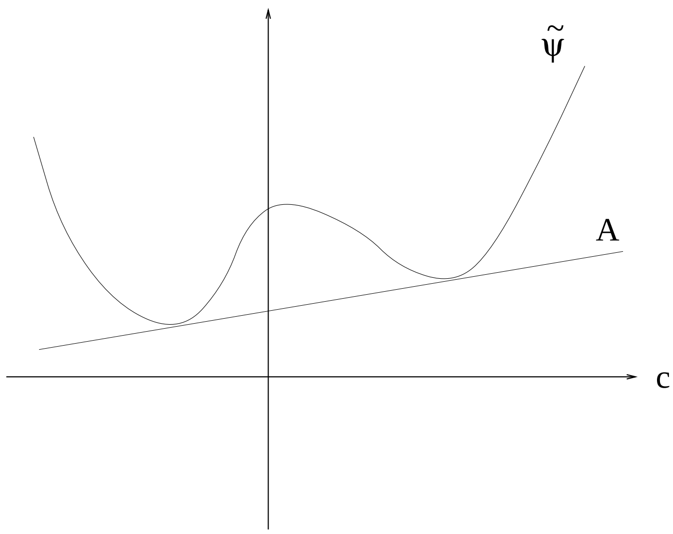

FIGURE 1

Under the assumption (1) it turns out that it is energetically favourable for $\mathbf{c}$ to attain the values $\mathbf{p}_1, ..., \mathbf{p}_M$. And in fact minimizers of E^ε in most of the domain Ω have values close to $\mathbf{p}_1, ..., \mathbf{p}_M$. Our goal is to show that under some natural growth assumptions on Ψ minimizers of E^ε converge to solutions of a partitioning problem. To formulate this partitioning problem we need to introduce some notation. We consider partitions of Ω into measurable sets $\Omega_1,...,\Omega_M$. These sets are assumed to fulfill

$$\sum_{k=1}^{M} \mathcal{X}_{\Omega_k} = 1 \qquad \text{almost everywhere in} \quad \Omega.$$

This means that up to a set of measure zero the sets $\Omega_1,...,\Omega_M$ are a partition of Ω. We want to measure the area of the interface between two sets Ω_k and Ω_l, i.e. we want to measure the area of $\partial\Omega_k \cap \partial\Omega_l \cap \Omega$ in an appropriate generalized sense. Here, it is convenient to use the setting of functions of bounded variation. As general references to functions with bounded variation we refer to the books of Evans and Gariepy [7] and Giusti [15].

Assuming that the sets $\Omega_1, ..., \Omega_M$ are sets with finite perimeter in Ω, i.e. $\mathcal{X}_{\Omega_1}, ..., \mathcal{X}_{\Omega_M}$ lie in $BV(\Omega)$, we can define the interfacial measures as

$$\mu_{kl} = \frac{1}{2}\left(|\nabla \mathcal{X}_{\Omega_k}| + |\nabla \mathcal{X}_{\Omega_l}| - |\nabla(\mathcal{X}_{\Omega_k} + \mathcal{X}_{\Omega_l})|\right).$$

This is a measure theoretic way to define the $(n-1)$–dimensional measure of the interface between the sets Ω_k and Ω_l. Introducing the reduced boundary $\partial^*\Omega_k$ of the sets Ω_k it can be shown (see Bronsard, Garcke and Stoth [4]) that for all open sets $D \subset \Omega$

$$\mu_{kl}(D) = \mathcal{H}^{n-1}\left(\partial^*\Omega_k \cap \partial^*\Omega_l \cap D\right).$$

For $D = \Omega$ and for sets Ω_k and Ω_l with sufficiently smooth boundaries this shows that in fact $\mu_{kl}(\Omega)$ measures the total interfacial area between Ω_k and Ω_l. Let us define the interfaces between the sets Ω_k and Ω_l as

$$\Lambda_{kl} = \partial^*\Omega_k \cap \partial^*\Omega_l \cap \Omega.$$

It will turn out that the Γ–limit E^0 of the functionals $\{E^\varepsilon\}_{\varepsilon>0}$ consists of two summands. One measures the interfacial energy of the individual interfaces Λ_{kl} and the other takes elastic energy contributions into account. The interfacial energy is given as the sum of the area of the interfaces Λ_{kl} weighted by the individual surface tensions σ_{kl}. These surface tensions are defined by means of the homogeneous free energy Ψ. To make this precise we need to introduce the metric d on Σ induced by $\sqrt{\Psi}$. For $\mathbf{c}'_1, \mathbf{c}'_2 \in \Sigma$ we define

$$d(\mathbf{c}'_1, \mathbf{c}'_2) := \inf \Big\{ 2\int_{-1}^{1} \sqrt{\Psi(\gamma(t))}|\gamma'(t)|\, dt \;\;\Big|\;\; \gamma : [-1,1] \to \Sigma \qquad (2)$$
$$\text{is Lipschitz continuous,} \quad \gamma(-1) = \mathbf{c}'_1 \quad \text{and} \quad \gamma(1) = \mathbf{c}'_2 \Big\}.$$

A curve γ for which the infimum in the above expression is attained, is a geodesic with respect to this metric. The curve γ then corresponds to an interfacial layer with minimal energy $\int_{-\infty}^{\infty} \{|\gamma'(t)|^2 + \Psi(\gamma(t))\}\, dt$. In fact, Sternberg [27] proves that a geodesic connecting the points $\mathbf{p}_k$ and $\mathbf{p}_l$ when suitably reparametrised also solves the minimum problem

$$\inf \Big\{ 2 \int_{-\infty}^{\infty} \{|\gamma'(t)|^2 + \Psi(\gamma(t))\}\, dt \mid \gamma : (-\infty, \infty) \to \Sigma \\ \text{is Lipschitz continuous,} \quad \gamma(-\infty) = \mathbf{p}_k \quad \text{and} \quad \gamma(\infty) = \mathbf{p}_l \Big\}$$

which is the variational problem of finding an interfacial layer with minimal energy. Stretching the t–variable by a factor $\frac{1}{\varepsilon}$ then gives that the minimizer realises an interfacial layer which minimizes the first two terms in the energy E^ε when compared to all other one–dimensional interfacial layers. This shows the importance of the metric d and its geodesics when one tries to minimize the functional E^ε. Now the surface tensions σ_{kl} are given as the distance in the metric d between the minimizers $\mathbf{p}_k$ and $\mathbf{p}_l$ of Ψ, i.e.

$$\sigma_{kl} = d(\mathbf{p}_k, \mathbf{p}_l).$$

Our goal is to show that minimizers of E^ε converge (along subsequences) to minimizers of the functional

$$E^0 : L^1(\Omega, \Sigma) \times X_2 \to \mathbb{R} \cup \{\infty\}$$

with

$$E^0(\mathbf{c}, \mathbf{u}) = \begin{cases} \displaystyle\sum_{\substack{k,l=1 \\ k<l}}^{M} \sigma_{kl} \mathcal{H}^{n-1} \left(\partial^* \{\mathbf{c} = \mathbf{p}_k\} \cap \partial^* \{\mathbf{c} = \mathbf{p}_l\}\right) + \int_\Omega W(\mathbf{c}, \mathcal{E}(\mathbf{u})) \\ \qquad\qquad \text{if} \quad \mathbf{c} \in BV(\Omega),\ \Psi(\mathbf{c}) = 0 \text{ a.e. },\ \displaystyle⨍_\Omega \mathbf{c} = \mathbf{m}, \\ \infty \qquad\qquad \text{otherwise.} \end{cases}$$

Here, $\{\mathbf{c} = \mathbf{p}_k\} := \{\mathbf{x} \in \Omega \,|\, \mathbf{c}(\mathbf{x}) = \mathbf{p}_k\}$. We note that $\mathbf{c} \in BV(\Omega)$ implies that the sets $\{\mathbf{c} = \mathbf{p}_k\}$ are sets of finite perimeter in Ω. This follows from the co–area formula applied to the function $\mathbf{x} \mapsto d(\mathbf{c}(\mathbf{x}), \mathbf{p}_k)$ and will become clear from the discussion below.

Finding a minimizer of E^0 can be interpreted as a partitioning problem for the set Ω. Defining for all $\mathcal{E}' \in \mathbb{R}^{n \times n}$

$$W_k(\mathcal{E}') := W(\mathbf{p}_k, \mathcal{E}')$$

as the elastic energy density of phase k, the partitioning problem is as follows.

Find a partition $\Omega = \cup_{k=1}^{M} \Omega_k$ with $\Omega_k \cap \Omega_l = \emptyset$ and

$$\sum_{k=1}^{M} \mathbf{p}_k \frac{|\Omega_k|}{|\Omega|} = \mathbf{m} \tag{3}$$

such that the energy

$$\sum_{\substack{k,l=1 \\ k<l}}^{M} \sigma_{kl} \mathcal{H}^{n-1} \left(\partial^* \Omega_k \cap \partial^* \Omega_l \right) + \sum_{k=1}^{M} \int_{\Omega_k} W_k(\mathcal{E}(\mathbf{u}))$$

becomes minimal. In the case that the convex hull of the vectors $\mathbf{p}_1, ..., \mathbf{p}_M$ is $(M-1)$–dimensional the constraint (3) fixes the volume of the individual phases.

As has been used already, the constraints $\Psi(\mathbf{c}) = 0$ a.e. and $⨍_\Omega \mathbf{c} = \mathbf{m}$ in the definition for E^0 imply (3). This means that the vector of mean values is a convex combination of the minimal points $\mathbf{p}_k$. Therefore we will require for the rest of the section that

$$\mathbf{m} \in conv\{\mathbf{p}_1, ..., \mathbf{p}_M\},$$

where $conv\{\mathbf{p}_1, ..., \mathbf{p}_M\}$ is the convex hull of the points $\mathbf{p}_1, ..., \mathbf{p}_M$. If the mean value does not lie in the convex hull of $\mathbf{p}_1, ..., \mathbf{p}_M$ it is an easy matter to show that the limit of the minimal values of E^ε will be ∞. This means the sequence $\{E^\varepsilon\}_{\varepsilon>0}$ can only have the functional which is identically equal to ∞ as a Γ–limit.

Before proving a result on the limit of the functionals $\{E^\varepsilon\}_{\varepsilon>0}$ let us state the results known on the Γ–limit of the functionals E^ε in the case without elasticity. In the case $N = 2$ due to the constraint $c_1 + c_2 = 1$ the problem can be reduced to a problem defined via a scalar quantity. Modica [24] showed that if a sequence of minimizers $\mathbf{c}^\varepsilon$ to the variational problem with $W = 0$ converges in $L^1(\Omega)$ to a limit $\mathbf{c}$, then the limit defines a partition of Ω with minimal interfacial area. His proof is based on earlier work of him together with Mortola [25]. This result was generalized by Fonseca and Tartar [9] and Sternberg [27] to the vectorial case under the assumption that $M = 2$, i.e. Ψ only has two global minimizer and hence only two phases are present. Both papers also state assumptions on Ψ under which sequences $(\mathbf{c}^\varepsilon)_{\varepsilon>0}$ with $E^\varepsilon(\mathbf{c}^\varepsilon)$ uniformly bounded are compact in L^1. Baldo [3] studied the vector–valued problem with a finite number of global minimizers of Ψ and showed that L^1–limits of minimizers of $(\mathbf{P}^\varepsilon)$ are minimizers of E^0. We also refer to work of Ambrosio [2], who studied the case in which the set of zeros of Ψ can be any compact set in $\mathbb{R}^N$. In particular, he proved compactness in L^1 for sequences $(\mathbf{c}^\varepsilon)_{\varepsilon>0}$ with bounded energy $E^\varepsilon(\mathbf{c}^\varepsilon)$ under very general assumptions.

The paper is organized as follows.

In Section 2 we show that E^0 is in fact the Γ–limit of E^ε. The Euler-Lagrange equation for the sharp interface functional E^0 are derived in Section 3 for the multi-phase situation. The Euler-Lagrange equations lead in particular to an elastically modified Gibbs–Thomson law. In Section 4 we derive this modified Gibbs–Thomson equation in the scalar case as the singular limit of the Euler-Lagrange equations for E^ε. This is possible for absolute minimizers and generalizes results of Luckhaus and Modica [22] to the case with elasticity.

2. The Γ–limit of the elastic Ginzburg–Landau energies

We make the following assumptions.

(A1) $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary.
(A2) The homogeneous free energy $\Psi \in C^1(\mathbb{R}^N, \mathbb{R})$ fulfills (1) and there exist constants $c_4, C_4 > 0$ such that

$$\Psi(\mathbf{c}') \geq c_4 |\mathbf{c}'|^2 - C_4 \qquad \text{for all } \mathbf{c}' \in \Sigma\,.$$

(A3) For the elastic energy density $W \in C^1(\mathbb{R}^N \times \mathbb{R}^{n\times n}, \mathbb{R})$ we assume
(A3.1) $W(\mathbf{c}', \mathcal{E}')$ only depends on the symmetric part of $\mathcal{E}' \in \mathbb{R}^{n\times n}$, i.e. $W(\mathbf{c}', \mathcal{E}') = W(\mathbf{c}', (\mathcal{E}')^t)$ for all $\mathbf{c}' \in \mathbb{R}^N$ and $\mathcal{E}' \in \mathbb{R}^{n\times n}$,
(A3.2) $W_{,\mathcal{E}}(\mathbf{c}', .)$ is strongly monotone uniformly in $\mathbf{c}'$, i.e. there exists a $c_1 > 0$ such that for all symmetric $\mathcal{E}_1', \mathcal{E}_2' \in \mathbb{R}^{n\times n}$

$$(W_{,\mathcal{E}}(\mathbf{c}', \mathcal{E}_2') - W_{,\mathcal{E}}(\mathbf{c}', \mathcal{E}_1')) : (\mathcal{E}_2' - \mathcal{E}_1') \geq c_1 |\mathcal{E}_2' - \mathcal{E}_1'|^2\,,$$

(A3.3) there exists a constant $C_2 > 0$ such that for all $\mathbf{c}' \in \Sigma$ and all symmetric $\mathcal{E}' \in \mathbb{R}^{n\times n}$

$$\begin{aligned} |W(\mathbf{c}', \mathcal{E}')| &\leq C_2(|\mathcal{E}'|^2 + |\mathbf{c}'|^2 + 1)\,,\\ |W_{,\mathbf{c}}(\mathbf{c}', \mathcal{E}')| &\leq C_2(|\mathcal{E}'|^2 + |\mathbf{c}'|^2 + 1)\,,\\ |W_{,\mathcal{E}}(\mathbf{c}', \mathcal{E}')| &\leq C_2(|\mathcal{E}'| + |\mathbf{c}'| + 1)\,. \end{aligned}$$

The growth condition on Ψ will ensure compactness in L^2 for $\{\mathbf{c}^\varepsilon\}_{\varepsilon>0}$ if the sequence $\{(\mathbf{c}^\varepsilon, \mathbf{u}^\varepsilon)\}_{\varepsilon>0}$ has uniformly bounded energy $E^\varepsilon(\mathbf{c}^\varepsilon, \mathbf{u}^\varepsilon)$. The compactness of $\mathbf{c}^\varepsilon$ will be necessary to handle the elastic part of the free energy when we prove the following two theorems.

In the first theorem we state in part a) a compactness result for sequences $\{(\mathbf{c}^\varepsilon, \mathbf{u}^\varepsilon)\}_{\varepsilon>0}$ with bounded energy $E^\varepsilon(\mathbf{c}^\varepsilon, \mathbf{u}^\varepsilon)$ and in part b) and c) the Γ–convergence of E^ε to E^0 is shown.

Theorem 2.1. *Assume (A1)-(A3) and* $\mathbf{m} \in conv\{\mathbf{p}_1, ..., \mathbf{p}_M\}$.

a) Let $\{(\mathbf{c}^\varepsilon, \mathbf{u}^\varepsilon)\}_{\varepsilon>0} \subset H^1(\Omega, \Sigma) \times X_2$ *be such that* $⨍ \mathbf{c}^\varepsilon = \mathbf{m}$ *and*

$$E^\varepsilon(\mathbf{c}^\varepsilon, \mathbf{u}^\varepsilon) \qquad \text{is uniformly bounded.}$$

Then there exists a sequence $\{\varepsilon^\kappa\}_{\kappa\in\mathbb{N}} \subset \mathbb{R}^+$ *with* $\lim_{\kappa\to\infty} \varepsilon^\kappa = 0$ *and* $\mathbf{c} \in L^2(\Omega, \Sigma) \cap BV(\Omega, \Sigma)$, $\mathbf{u} \in X_2$ *such that*

$$\begin{aligned} \mathbf{c}^{\varepsilon^\kappa} &\to \mathbf{c} \qquad \text{in} \qquad L^2(\Omega, \Sigma), \\ \mathbf{u}^{\varepsilon^\kappa} &\to \mathbf{u} \qquad \text{weakly in} \qquad H^1(\Omega, \mathbb{R}^n) \end{aligned}$$

as ε^κ *tends to zero. Furthermore, it holds that* $\mathbf{c} \in \{\mathbf{p}_1, ..., \mathbf{p}_M\}$ *almost everywhere.*

b) For all $\{(\mathbf{c}^{\varepsilon^\kappa}, \mathbf{u}^{\varepsilon^\kappa})\}_{\kappa\in\mathbb{N}} \in H^1(\Omega, \Sigma) \times X_2$ *with* $\mathbf{c}^{\varepsilon^\kappa} \to \mathbf{c}$ *in* $L^1(\Omega, \Sigma)$ *and* $\mathbf{u}^{\varepsilon^\kappa} \to \mathbf{u}$ *in* $L^2(\Omega, \mathbb{R}^N)$ *as* ε^κ *tends to zero, it holds*

$$E^0(\mathbf{c}, \mathbf{u}) \le \liminf_{\kappa\to\infty} E^{\varepsilon^\kappa}(\mathbf{c}^{\varepsilon^\kappa}, \mathbf{u}^{\varepsilon^\kappa}).$$

c) For any $(\mathbf{c}, \mathbf{u}) \in L^1(\Omega, \Sigma) \times X_2$ *and any sequence* $\varepsilon^\kappa \searrow 0$, $\kappa \in \mathbb{N}$, *there exists a sequence* $\{(\mathbf{c}^{\varepsilon^\kappa}, \mathbf{u}^{\varepsilon^\kappa})\}_{\kappa\in\mathbb{N}} \in H^1(\Omega, \Sigma) \times X_2$ *with* $\mathbf{c}^{\varepsilon^\kappa} \to \mathbf{c}$ *in* $L^1(\Omega, \Sigma)$ *and* $\mathbf{u}^{\varepsilon^\kappa} \to \mathbf{u}$ *in* $L^2(\Omega, \mathbb{R}^N)$ *as* $\varepsilon^\kappa \searrow 0$ *such that*

$$E^0(\mathbf{c}, \mathbf{u}) \ge \limsup_{\kappa\to\infty} E^{\varepsilon^\kappa}(\mathbf{c}^{\varepsilon^\kappa}, \mathbf{u}^{\varepsilon^\kappa}).$$

The preceding theorem enables us to show that minimizers of E^ε converge (along subsequences) to minimizers of E^0.

Theorem 2.2. *Under the assumptions of Theorem 2.1 the variational problems* $(\mathbf{P}^\varepsilon)$ *possess minimizers* $(\mathbf{c}^\varepsilon, \mathbf{u}^\varepsilon) \in H^1(\Omega, \Sigma) \times X_2$ *provided that* ε *is small enough. Furthermore, there exists a sequence* $\{\varepsilon^\kappa\}_{\kappa\in\mathbb{N}} \subset \mathbb{R}^+$ *with* $\lim_{\kappa\to\infty} \varepsilon^\kappa = 0$ *and a* $(\mathbf{c}, \mathbf{u}) \in L^2(\Omega, \Sigma) \times H^1(\Omega, \mathbb{R}^n)$ *such that*
i)

$$\begin{aligned} \mathbf{c}^{\varepsilon^\kappa} &\to \mathbf{c} \qquad \text{in} \qquad L^2(\Omega, \Sigma), \\ \mathbf{u}^{\varepsilon^\kappa} &\to \mathbf{u} \qquad \text{in} \qquad H^1(\Omega, \mathbb{R}^n); \end{aligned}$$

ii) $(\mathbf{c}, \mathbf{u})$ *is a global minimizer of* E^0. *In particular*
$\mathbf{c} \in BV(\Omega, \{\mathbf{p}_1, ..., \mathbf{p}_M\})$ *and* $⨍ \mathbf{c} = \mathbf{m}$.

Remark 2.1. *We remark that we are able to prove strong convergence of* $\mathbf{u}^{\varepsilon^\kappa}$ *in* $H^1(\Omega, \mathbb{R}^n)$. *This will be important later when we want to pass to the limit in the Euler–Lagrange equation (see Section 4).*

Proof of Theorem 2.1 a).
From (A3) we derive

$$W(\mathbf{c}', \mathcal{E}') \ge c_3|\mathcal{E}'|^2 - C_3\left(|\mathbf{c}'|^2 + 1\right)$$

and hence, we can use the boundedness of $E^\varepsilon(\mathbf{c}^\varepsilon, \mathbf{u}^\varepsilon)$ and the quadratic growth of Ψ to conclude that

$$\int_\Omega \left(\varepsilon|\nabla \mathbf{c}^\varepsilon|^2 + \frac{1}{\varepsilon}\Psi(\mathbf{c}^\varepsilon) \right) \tag{4}$$

is uniformly bounded if ε is small enough. Our first goal is to show compactness of $(\mathbf{c}^\varepsilon)_{\varepsilon>0}$. Here we use an idea of Ambrosio [2]. We define for all $\mathbf{c}' \in \Sigma$

$$\phi_k(\mathbf{c}') = d(\mathbf{c}', \mathbf{p}_k)$$

which is locally Lipschitz continuous with

$$|D_{\mathbf{c}'}\phi_k(\mathbf{c}')| \leq 2\sqrt{\Psi(\mathbf{c}')} \qquad a.e. \quad \text{in } \Sigma.$$

For a proof see e.g. Fonseca and Tartar [9] and Ambrosio [2]. Defining for $k = 1, ..., M$ and $\varepsilon > 0$

$$f_k^\varepsilon(\mathbf{x}) = \min\left(\phi_k\left(\mathbf{c}^\varepsilon(\mathbf{x})\right), 1\right),$$

one can show via an approximation argument

$$\begin{aligned} |\nabla f_k^\varepsilon(\mathbf{x})| &\leq |D_{\mathbf{c}'}\phi_k(\mathbf{c}^\varepsilon(\mathbf{x}))|\,|\nabla \mathbf{c}^\varepsilon(\mathbf{x})| \\ &\leq 2\sqrt{\Psi(\mathbf{c}^\varepsilon(\mathbf{x}))}\,|\nabla \mathbf{c}^\varepsilon(\mathbf{x})| \\ &\leq \varepsilon|\nabla \mathbf{c}^\varepsilon(\mathbf{x})|^2 + \frac{1}{\varepsilon}\Psi(\mathbf{c}^\varepsilon(\mathbf{x})) \end{aligned}$$

for almost all $\mathbf{x} \in \Omega$. Using (4) it follows

$$f_k^\varepsilon \qquad \text{is uniformly bounded in } BV(\Omega)$$

for all $k \in \{1, ..., M\}$.

Since the embedding of $BV(\Omega)$ into $L^1(\Omega)$ is compact we can conclude that there exists a subsequence $\{\varepsilon^\kappa\}_{\kappa\in\mathbb{N}} \subset \mathbb{R}^+$ tending to zero such that for all $k \in \{1, ..., M\}$

$$f_k^{\varepsilon^\kappa} \to f_k \qquad \text{in } L^1(\Omega) \quad \text{and a.e.} \quad \text{as } \varepsilon^\kappa \quad \text{tends to 0.}$$

The uniform boundedness of the second summand in (4) implies that

$$\Psi(\mathbf{c}^{\varepsilon^\kappa}) \to 0 \qquad \text{in } L^1(\Omega) \quad \text{as } \varepsilon^\kappa \quad \text{tends to 0.}$$

There exists a subsequence (which we again denote by $\{\varepsilon^\kappa\}_{\kappa\in\mathbb{N}}$) with

$$\Psi(\mathbf{c}^{\varepsilon^\kappa}) \to 0 \qquad \text{almost everywhere as } \varepsilon^\kappa \to 0. \tag{5}$$

Defining the increasing function

$$\omega(z) = \inf\{\, \Psi(\mathbf{c}') \,|\, \mathrm{dist}(\mathbf{c}', \{\mathbf{p}_1, ..., \mathbf{p}_M\}) \geq z \,\}$$

we have, due to the assumptions on Ψ (see (1) and (A2)), that

$$\begin{aligned} \omega(z) > 0 \qquad &\text{if} \qquad z > 0, \\ \omega(z) \to 0 \qquad &\text{as} \qquad z \to 0. \end{aligned}$$

Since

$$\Psi(\mathbf{c}') \geq \omega\left(\text{dist}\left(\mathbf{c}', \{\mathbf{p}_1, ..., \mathbf{p}_M\}\right)\right)$$

we get with (5)

$$dist\left(\mathbf{c}^{\varepsilon^\kappa}, \{\mathbf{p}_1, ..., \mathbf{p}_M\}\right) \to 0 \qquad \text{as } \varepsilon \to 0 \tag{6}$$

on $\Omega \backslash \mathcal{N}$, where $\mathcal{N}$ is a set with Lebesgue measure zero. Now we define

$$\Omega_k = \{\, \mathbf{x} \in \Omega \setminus \mathcal{N} \mid \lim_{\kappa \to \infty} d(\mathbf{c}^{\varepsilon^\kappa}(\mathbf{x}), \mathbf{p}_k) = 0 \,\}.$$

Claim 1:

$$\lim_{\kappa \to \infty} \mathbf{c}^{\varepsilon^\kappa}(\mathbf{x}) = \mathbf{p}_k \qquad \text{on} \qquad \Omega_k.$$

Suppose the claim were false. Then, due to the fact that

$$dist\left(\mathbf{c}^{\varepsilon^\kappa}, \{\mathbf{p}_1, ..., \mathbf{p}_M\}\right) \to 0 \qquad \text{as } \varepsilon^\kappa \to 0,$$

we can find a $\mathbf{x} \in \Omega_k$, an $l \in \{1, ..., M\} \setminus \{k\}$ and a subsequence $\{\varepsilon^\alpha\}_{\alpha \in \mathbb{N}} \subset \{\varepsilon^\kappa\}_{\kappa \in \mathbb{N}}$ such that

$$\mathbf{c}^{\varepsilon^\alpha}(\mathbf{x}) \to \mathbf{p}_l \qquad \text{as} \quad \varepsilon^\alpha \quad \text{tends to zero.}$$

Consequently

$$\lim_{\varepsilon^\alpha \to 0} d(\mathbf{c}^{\varepsilon^\alpha}(\mathbf{x}), \mathbf{p}_k) = d(\mathbf{p}_l, \mathbf{p}_k) \neq 0,$$

which is a contradiction and Claim 1 is proved.

Claim 2:

$$\mathcal{L}^n\left(\Omega \setminus \left(\cup_{k=1}^M \Omega_k\right)\right) = 0.$$

There exists a set $\mathcal{S} \subset \Omega$ of measure zero, such that for all $k \in \{1, ..., M\}$ the sequence $f_k^{\varepsilon^\kappa}$ converges pointwise on $\Omega \setminus \mathcal{S}$ and such that (6) holds on $\Omega \setminus \mathcal{S}$. For each point $\mathbf{x} \in \Omega \setminus \mathcal{S}$ one can find an $l \in \{1, ..., M\}$ and a subsequence $\{\varepsilon^\alpha\}_{\alpha \in \mathbb{N}} \subset \{\varepsilon^\kappa\}_{\kappa \in \mathbb{N}}$ such that $\mathbf{c}^{\varepsilon^\alpha}(\mathbf{x}) \to \mathbf{p}_l$ as ε^α tends to zero. Hence,

$$f_l^{\varepsilon^\alpha}(\mathbf{x}) \to 0 \qquad \text{as} \quad \varepsilon^\alpha \quad \text{tends to} \quad 0.$$

Since the whole sequence $f_l^{\varepsilon^\kappa}$ converges we obtain $\mathbf{x} \in \Omega_l$ and Claim 2 is shown.

Hence we obtain that $\mathbf{c}^{\varepsilon^\kappa}$ converges almost everywhere to a measurable function $\mathbf{c}$ which only attains the values $\{\mathbf{p}_1, ..., \mathbf{p}_M\}$. Assumption (A2) implies

$$|\mathbf{c}^{\varepsilon^\kappa}|^2 \leq \frac{1}{c_4}\left(\Psi(\mathbf{c}^{\varepsilon^\kappa}) + C_4\right)$$

and since $\Psi(\mathbf{c}^{\varepsilon^\kappa}) \to 0$ in $L^1(\Omega)$ we can deduce with the help of the generalized Lebesgue theorem that

$$\mathbf{c}^{\varepsilon^\kappa} \to \mathbf{c} \qquad \text{in } L^2(\Omega).$$

Now we can use the uniform bound on $E^\varepsilon(\mathbf{c}^\varepsilon, \mathbf{u}^\varepsilon)$, the assumptions (A2)-(A3) on Ψ and W respectively and Korn's inequality to conclude

$$\mathbf{u}^{\varepsilon^\kappa} \qquad \text{is uniformly bounded in} \qquad H^1(\Omega, \mathbb{R}^n).$$

This implies the weak convergence of a subsequence and hence a) is shown.

PROOF OF THEOREM 2.1 b).
In case that $\liminf_{\kappa\to\infty} E^{\varepsilon^\kappa}(\mathbf{c}^{\varepsilon^\kappa}, \mathbf{u}^{\varepsilon^\kappa}) = \infty$ the conclusion holds and we can assume without loss of generality that

$$\lim_{\kappa\to\infty} E^{\varepsilon^\kappa}(\mathbf{c}^{\varepsilon^\kappa}, \mathbf{u}^{\varepsilon^\kappa}) < \infty \qquad \text{and} \qquad \mathbf{c}^{\varepsilon^\kappa} \to \mathbf{c} \quad \text{a.e.}\,.$$

Using assumptions (A2)-(A3) and Korn's inequality (see e.g. [32]) we obtain the existence of a constant C such that

$$\int_\Omega \left(\varepsilon^\kappa |\nabla \mathbf{c}^{\varepsilon^\kappa}|^2 + \frac{1}{\varepsilon^\kappa} \Psi(\mathbf{c}^{\varepsilon^\kappa}) \right) + \|\mathbf{u}^{\varepsilon^\kappa}\|_{H^1} \le C < \infty. \tag{7}$$

Since $\lim_{\kappa\to\infty} \int_\Omega \Psi(\mathbf{c}^{\varepsilon^\kappa}) = 0$ we can conclude

$$\Psi(\mathbf{c}^{\varepsilon^\kappa}) \to 0 \qquad \text{a.e.} \tag{8}$$

and hence $\mathbf{c} \in \{\mathbf{p}_1, ...\mathbf{p}_M\}$ almost everywhere. Let

$$\mathcal{R} = 1 + \max\{d(\mathbf{p}_k, \mathbf{p}_l) \,|\, k, l \in \{1, ..., M\}\}$$

and define

$$\varphi_k(\mathbf{c}') = \min\left(d(\mathbf{c}', \mathbf{p}_k), \mathcal{R}\right).$$

Similar as in the proof of part a) of the theorem we can show that

$$\int_\Omega |\nabla \left(\varphi_k(\mathbf{c}^{\varepsilon^\kappa})\right)| \le \int_\Omega \left(\varepsilon^\kappa |\nabla \mathbf{c}^{\varepsilon^\kappa}|^2 + \frac{1}{\varepsilon^\kappa} \Psi(\mathbf{c}^{\varepsilon^\kappa}) \right)$$

and therefore the left hand side is uniformly bounded for $k = 1, ..., M$. The convergence of $\varphi_k(\mathbf{c}^{\varepsilon^\kappa})$ in $L^1(\Omega)$ yields

$$\int_\Omega |\nabla \left(\varphi_k(\mathbf{c})\right)| \le \liminf_{\kappa\to\infty} \int_\Omega |\nabla \left(\varphi_k(\mathbf{c}^{\varepsilon^\kappa})\right)|.$$

Taking the measure theoretic supremum $\bigvee_{k=1}^{M}$ (see Definition 6.1) of the sequence of measures $|\nabla\left(\varphi_k(\mathbf{c}^{\varepsilon^\kappa})\right)|$ then implies

$$\begin{aligned}\bigvee_{k=1}^{M} |\nabla\left(\varphi_k(\mathbf{c})\right)|(\Omega) &\leq \liminf_{\kappa\to\infty} \bigvee_{k=1}^{M} |\nabla\left(\varphi_k(\mathbf{c}^{\varepsilon^\kappa})\right)|(\Omega) \\ &\leq \liminf_{\kappa\to\infty} \int_\Omega \left(\varepsilon^\kappa |\nabla \mathbf{c}^{\varepsilon^\kappa}|^2 + \frac{1}{\varepsilon^\kappa}\Psi(\mathbf{c}^{\varepsilon^\kappa})\right)\end{aligned} \tag{9}$$

(see also Remark 6.1). Then, using the identity

$$\varphi_k(\mathbf{c}) = \sum_{l=1}^{M} d(\mathbf{p}_k, \mathbf{p}_l)\mathcal{X}_{\{\mathbf{c}=\mathbf{p}_l\}}$$

and applying the co–area formula (see [7]) gives

$$\begin{aligned}|\nabla\left(\varphi_k(\mathbf{c})\right)|(\Omega) &= \int_\Omega |\nabla\left(\textstyle\sum_{l=1}^{M} d(\mathbf{p}_k, \mathbf{p}_l)\mathcal{X}_{\{\mathbf{c}=\mathbf{p}_l\}}\right)| \\ &= \int_{-\infty}^{\infty}\int_\Omega |\nabla \mathcal{X}_{\{\left(\sum_{l=1}^{M} d(\mathbf{p}_k,\mathbf{p}_l)\mathcal{X}_{\{\mathbf{c}=\mathbf{p}_l\}}\right)<\lambda\}}|\, d\lambda \\ &\geq \int_{0}^{\sigma_k}\int_\Omega |\nabla \mathcal{X}_{\{\left(\sum_{l=1}^{M} d(\mathbf{p}_k,\mathbf{p}_l)\mathcal{X}_{\{\mathbf{c}=\mathbf{p}_l\}}\right)<\lambda\}}|\, d\lambda \\ &= \sigma_k \int_\Omega |\nabla \mathcal{X}_{\{\mathbf{c}=\mathbf{p}_k\}}|,\end{aligned}$$

where $\sigma_k := \min\{d(\mathbf{p}_k, \mathbf{p}_l)\,|\, l = 1, ..., M\ , l \neq k\}$. This implies that $\Omega_k := \{\mathbf{c} = \mathbf{p}_k\}$ is a set of finite perimeter in Ω. In particular, the reduced boundary $\partial^*\{\mathbf{c} = \mathbf{p}_k\}$ and the interfaces $\Lambda_{kl} = \partial^*\Omega_k \cap \partial^*\Omega_l \cap \Omega$ are defined.

Claim: For all open sets $D \subset \Omega$ it holds

$$\bigvee_{k=1}^{M} |\nabla\left(\varphi_k(\mathbf{c})\right)|(D) = \sum_{\substack{l,m=1\\ l<m}}^{M} d(\mathbf{p}_l, \mathbf{p}_m)\mathcal{H}^{n-1}\left(\Lambda_{lm} \cap D\right). \tag{10}$$

The function $\varphi_k(\mathbf{c})$ jumps on $\partial^*\{\mathbf{c} = \mathbf{p}_l\} \cap \partial^*\{\mathbf{c} = \mathbf{p}_m\}$ by an amount of $|d(\mathbf{p}_k, \mathbf{p}_l) - d(\mathbf{p}_k, \mathbf{p}_m)|$. Hence, with the help of the co–area formula we can show for all open $D \subset \Omega$ (see [30, 3] for details)

$$|\nabla\varphi_k(\mathbf{c})|(D) = \sum_{\substack{l,m=1\\ l<m}}^{M} |d(\mathbf{p}_k, \mathbf{p}_l) - d(\mathbf{p}_k, \mathbf{p}_m)|\mathcal{H}^{n-1}\left(\Lambda_{lm} \cap D\right). \tag{11}$$

Taking the measure theoretic supremum in (11) we obtain

$$\bigvee_{k=1}^{M} |\nabla\varphi_k(\mathbf{c})|(D) \tag{12}$$
$$= \sum_{\substack{l,m=1\\ l<m}}^{M} \max_{k=1,\dots,M} |d(\mathbf{p}_k,\mathbf{p}_l) - d(\mathbf{p}_k,\mathbf{p}_m)|\mathcal{H}^{n-1}\left(\Lambda_{lm}\cap D\right).$$

The claim now follows since a triangle inequality holds for d.

Inequality (9) and the representation (10) give that

$$\sum_{\substack{l,m=1\\ l<m}}^{M} d(\mathbf{p}_l,\mathbf{p}_m)\mathcal{H}^{n-1}\left(\partial^*\{\mathbf{c}=\mathbf{p}_l\}\cap\partial^*\{\mathbf{c}=\mathbf{p}_m\}\cap\Omega\right) \tag{13}$$
$$\leq \liminf_{\kappa\to\infty}\int_\Omega \left(\varepsilon^\kappa|\nabla\mathbf{c}^{\varepsilon^\kappa}|^2 + \frac{1}{\varepsilon^\kappa}\Psi(\mathbf{c}^{\varepsilon^\kappa})\right).$$

It remains to show that the term $\int_\Omega W(\mathbf{c}^{\varepsilon^\kappa},\mathbf{u}^{\varepsilon^\kappa})$ is lower semi-continuous. We can use the growth condition (A2) and the facts (7) and (8) to deduce that

$$\mathbf{c}^{\varepsilon^\kappa}\to\mathbf{c} \quad \text{in} \quad L^2(\Omega,\Sigma) \quad \text{and} \quad \mathbf{u}^{\varepsilon^\kappa}\to\mathbf{u} \quad \text{weakly in} \quad H^1(\Omega,\mathbb{R}^n).$$

The convexity of $W(\mathbf{c}',\mathcal{E}')$ in the variable $\mathcal{E}'$ implies

$$\int_\Omega \left(W(\mathbf{c}^{\varepsilon^\kappa},\mathcal{E}(\mathbf{u}^{\varepsilon^\kappa})) - W(\mathbf{c},\mathcal{E}(\mathbf{u}))\right) =$$
$$= \int_\Omega \left(W(\mathbf{c}^{\varepsilon^\kappa},\mathcal{E}(\mathbf{u}^{\varepsilon^\kappa})) - W(\mathbf{c}^{\varepsilon^\kappa},\mathcal{E}(\mathbf{u}))\right) + \int_\Omega \left(W(\mathbf{c}^{\varepsilon^\kappa},\mathcal{E}(\mathbf{u})) - W(\mathbf{c},\mathcal{E}(\mathbf{u}))\right)$$
$$\geq \int_\Omega W_{,\mathcal{E}}(\mathbf{c}^{\varepsilon^\kappa},\mathcal{E}(\mathbf{u})) : \left(\mathcal{E}(\mathbf{u}^{\varepsilon^\kappa}) - \mathcal{E}(\mathbf{u})\right) + \int_\Omega \left(W(\mathbf{c}^{\varepsilon^\kappa},\mathcal{E}(\mathbf{u})) - W(\mathbf{c},\mathcal{E}(\mathbf{u}))\right).$$

Now the weak convergence of $\mathcal{E}(\mathbf{u}^{\varepsilon^\kappa})$ in $L^2(\Omega,\mathbb{R}^{n\times n})$, the strong convergence of $\mathbf{c}^{\varepsilon^\kappa}$ in $L^2(\Omega,\Sigma)$ and the growth conditions in (A3) give

$$\liminf_{\kappa\to\infty}\left(\int_\Omega \left(W(\mathbf{c}^{\varepsilon^\kappa},\mathcal{E}(\mathbf{u}^{\varepsilon^\kappa})) - W(\mathbf{c},\mathcal{E}(\mathbf{u}))\right)\right) \geq 0.$$

Together with (13) this yields the conclusion of part b) of the theorem.

Proof of Theorem 2.1 c).
In the case that no elastic effects are present, a proof of c) was given by Baldo [3] who used ideas of Modica and Mortola [25, 24]. We generalize the proof to the case that elastic terms are included. Assume $(\mathbf{c},\mathbf{u})\in H^1(\Omega,\Sigma)\times X_2$ with

$$E^0(\mathbf{c},\mathbf{u}) < \infty$$

is given. Then there exist sets $\Omega_1, ..., \Omega_M \subset \Omega$ such that

$$\mathbf{c} = \sum_{k=1}^{M} \mathbf{p}_k \mathcal{X}_{\Omega_k} \qquad a.e.$$

with $\mathcal{X}_{\Omega_k} \in BV(\Omega)$ and $\sum_{k=1}^{M} \mathcal{X}_{\Omega_k} = 1$. In this case Baldo [3] showed for all sequences $\varepsilon^\kappa \searrow 0$ the existence of functions $\mathbf{c}^{\varepsilon^\kappa} \in H^1(\Omega, \Sigma)$ such that

$$\mathbf{c}^{\varepsilon^\kappa} \to \mathbf{c} \quad \text{in} \quad L^1(\Omega, \Sigma)$$

and

$$\sum_{\substack{l,m=1 \\ l<m}}^{M} d(\mathbf{p}_l, \mathbf{p}_m) \mathcal{H}^{n-1} \left(\partial^* \{\mathbf{c} = \mathbf{p}_l\} \cap \partial^* \{\mathbf{c} = \mathbf{p}_m\} \right)$$

$$\geq \limsup_{\kappa \to \infty} \int_\Omega \left(\varepsilon^\kappa |\nabla \mathbf{c}^{\varepsilon^\kappa}|^2 + \frac{1}{\varepsilon^\kappa} \Psi(\mathbf{c}^{\varepsilon^\kappa}) \right).$$

The idea of Baldo's proof is as follows. First he approximates the sets $\Omega_1, ..., \Omega_M$ by partitions consisting of polygonal domains. Then the sharp jumps, separating sets Ω_k and Ω_l, are replaced by smooth transition layers of a thickness proportional to ε. The smooth transition layers are taken to be appropriately scaled geodesics connecting the values $\mathbf{p}_k$ and $\mathbf{p}_l$. This assumes that a geodesic realizing the infimum in the definition of $d(\mathbf{p}_k, \mathbf{p}_l)$ (see (2)) exists. If a minimizer does not exist a curve γ connecting $\mathbf{p}_k$ and $\mathbf{p}_l$ whose energy $2\int_{-1}^{1} \sqrt{\Psi(\gamma(t))}|\gamma'(t)|\,dt$ is close to $d(\mathbf{p}_k, \mathbf{p}_l)$ has to be chosen instead of a geodesic.

This construction may violate the mass constraint. Since the error in the mass constraint is only of order ε it is possible to make a small and sufficiently smooth perturbation of the function in the bulk of one of the sets Ω_k, so that the mass constraint is met. Such a perturbation only changes the energy to an amount that vanishes in the limit $\varepsilon^\kappa \searrow 0$. For the details of the proof we refer to Baldo [3].

Since the sequence $\mathbf{c}^{\varepsilon^\kappa}$ is constructed such that

$$\int_\Omega \left(\varepsilon^\kappa |\nabla \mathbf{c}^{\varepsilon^\kappa}|^2 + \frac{1}{\varepsilon^\kappa} \Psi(\mathbf{c}^{\varepsilon^\kappa}) \right) \qquad \text{is bounded,}$$

we can conclude as in the proof of part a) that

$$\mathbf{c}^{\varepsilon^\kappa} \to \mathbf{c} \quad \text{in} \quad L^2(\Omega, \Sigma).$$

Then, the growth condition $|W(\mathbf{c}', \mathcal{E}')| \leq C_2(|\mathcal{E}'|^2 + |\mathbf{c}'|^2 + 1)$ and the generalized convergence theorem of Lebesgue imply

$$\int_\Omega W(\mathbf{c}^{\varepsilon^\kappa}, \mathcal{E}(\mathbf{u})) \to \int_\Omega W(\mathbf{c}, \mathcal{E}(\mathbf{u})) \qquad \text{as} \quad \kappa \to \infty.$$

Hence, we can choose $(\mathbf{c}^{\varepsilon^\kappa}, \mathbf{u}^{\varepsilon^\kappa}) := (\mathbf{c}^{\varepsilon^\kappa}, \mathbf{u})$ to obtain

$$E^0(\mathbf{c}, \mathbf{u}) \geq \limsup_{\kappa\to\infty} E^\varepsilon(\mathbf{c}^{\varepsilon^\kappa}, \mathbf{u}^{\varepsilon^\kappa}),$$

which shows part c) of the theorem. □

Now we show that minimizers of E^ε converge (along subsequences) to minimizers of E^0.

PROOF OF THEOREM 2.2.

The existence of global minimizers $(\mathbf{c}^\varepsilon, \mathbf{u}^\varepsilon)$ of $(\mathbf{P}^\varepsilon)$ for ε small can be shown by the direct method taking into account the assumptions (A2)-(A3) and Korn's inequality.

Since $\mathbf{m} \in conv\{\mathbf{p}_1, ..., \mathbf{p}_M\}$ one can find a partition of Ω into sets $\Omega_1, ..., \Omega_M$ such that $\mathcal{X}_{\Omega_1}, ..., \mathcal{X}_{\Omega_M} \in BV(\Omega)$ and such that the mass constraint (3) is fulfilled. This means that by defining $\mathbf{d} = \sum_{k=1}^M \mathbf{p}_k \mathcal{X}_{\Omega_k}$ we obtain that the integral constraint $\fint \mathbf{d} = \mathbf{m}$ holds. By choosing any $\mathbf{v} \in X_2$ we get a pair $(\mathbf{d}, \mathbf{v})$ such that $E^0(\mathbf{d}, \mathbf{v})$ remains bounded. Part b) of Theorem 2.1 now implies that $E^\varepsilon(\mathbf{c}^\varepsilon, \mathbf{u}^\varepsilon)$ remains bounded as ε tends to zero. Hence by part a) of the previous theorem it follows that there exists a sequence $\{\varepsilon^\kappa\}_{\kappa\in\mathbb{N}} \subset \mathbb{R}^+$ with $\lim_{\kappa\to\infty} \varepsilon^\kappa = 0$ and $\mathbf{c} \in L^2(\Omega, \Sigma)$, $\mathbf{u} \in X_2$ such that

$$\begin{array}{lcl} \mathbf{c}^{\varepsilon^\kappa} \to \mathbf{c} & \text{in} & L^2(\Omega, \Sigma), \\ \mathbf{u}^{\varepsilon^\kappa} \to \mathbf{u} & \text{weakly in} & H^1(\Omega, \mathbb{R}^n) \end{array}$$

as ε^κ tends to zero. In addition it holds that $\mathbf{c} \in \{\mathbf{p}_1, ..., \mathbf{p}_M\}$ almost everywhere. Now by standard arguments in the theory of Γ–convergence it follows that $(\mathbf{c}, \mathbf{u})$ is a minimizer of E^0.

It remains to show the strong convergence of $\mathbf{u}^{\varepsilon^\kappa}$ in $H^1(\Omega, \mathbb{R}^n)$. To show this we use the fact that $\mathbf{u}^{\varepsilon^\kappa}$ minimizes $\hat{\mathbf{u}} \mapsto E^{\varepsilon^\kappa}(\mathbf{c}^{\varepsilon^\kappa}, \hat{\mathbf{u}})$ in the class X_2. Hence, variations in the direction $\boldsymbol{\eta} \in X_2$ give

$$\int_\Omega W_{,\mathcal{E}}(\mathbf{c}^{\varepsilon^\kappa}, \mathcal{E}(\mathbf{u}^{\varepsilon^\kappa}))\nabla\boldsymbol{\eta} = 0.$$

By choosing $\boldsymbol{\eta} = \mathbf{u}^{\varepsilon^\kappa} - \mathbf{u}$ and using the strict monotonicity of W with respect to the variable $\mathcal{E}'$ we obtain:

$$\begin{aligned} c_1 \int_\Omega &|\mathcal{E}(\mathbf{u}^{\varepsilon^\kappa} - \mathbf{u})|^2 \\ &\leq \int_\Omega \left(W_{,\mathcal{E}}(\mathbf{c}^{\varepsilon^\kappa}, \mathcal{E}(\mathbf{u}^{\varepsilon^\kappa})) - W_{,\mathcal{E}}(\mathbf{c}^{\varepsilon^\kappa}, \mathcal{E}(\mathbf{u}))\right) : \left(\mathcal{E}(\mathbf{u}^{\varepsilon^\kappa} - \mathbf{u})\right) \\ &= -\int_\Omega \left(W_{,\mathcal{E}}(\mathbf{c}^{\varepsilon^\kappa}, \mathcal{E}(\mathbf{u}))\right) : \left(\mathcal{E}(\mathbf{u}^{\varepsilon^\kappa} - \mathbf{u})\right). \end{aligned}$$

Since $\mathbf{u}^{\varepsilon^\kappa} \to \mathbf{u}$ weakly in $H^1(\Omega, \mathrm{I\!R}^n)$ and $\mathbf{c}^{\varepsilon^\kappa} \to \mathbf{c}$ strongly in $L^2(\Omega, \Sigma)$ as $\kappa \to \infty$ we can use the assumption (A3) and the inequality above to obtain:

$$\int_\Omega |\mathcal{E}(\mathbf{u}^{\varepsilon^\kappa} - \mathbf{u})|^2 \to 0 \qquad \text{as} \quad \kappa \to \infty.$$

Now Korn's inequality implies strong convergence of $\mathbf{u}^{\varepsilon^\kappa}$ in $H^1(\Omega, \mathrm{I\!R}^n)$ which proves the theorem. □

3. Euler–Lagrange equation for the sharp interface functional

In this section we compute the Euler–Lagrange equation for a minimizer of E^0, i.e. for a minimizer of the elastic partitioning problem. In the following lemma we derive equations by varying the independent variable in such a way that the volume constraint is met for the variations. This way we obtain an identity for divergence free variations.

Lemma 3.1. (Weak formulation of the Euler–Lagrange equation) *Assume Ω is a bounded domain with C^1–boundary. Let $(\mathbf{c}, \mathbf{u}) \in BV(\Omega) \times X_2$ with $⨍ \mathbf{c} = \mathbf{m}$ be a minimizer of E^0.*

Then

$$\int_\Omega \sum_{\substack{k,l=1\\k<l}}^{M} \sigma_{kl} \left(\nabla \cdot \boldsymbol{\xi} - \nu_k \cdot \nabla \boldsymbol{\xi} \nu_k\right) \mu_{kl} \tag{14}$$

$$+ \sum_{k=1}^{M} \int_{\Omega_k} \big(W_k Id - (\nabla \mathbf{u})^t W_{k,\mathcal{E}}\big) : \nabla \boldsymbol{\xi} = 0$$

for all $\boldsymbol{\xi} \in C^\infty(\overline{\Omega}, \mathrm{I\!R}^n)$ with $\nabla \cdot \boldsymbol{\xi} = 0$ and $\boldsymbol{\xi} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Here and below we use the notation $\nu_k = -\frac{\nabla \mathcal{X}_{\Omega_k}}{|\nabla \mathcal{X}_{\Omega_k}|}$.

Remark 3.1. *In Lemma 3.1 we only consider divergence free vector fields $\boldsymbol{\xi}$. Hence, the identity (14) can be rewritten as*

$$-\int_\Omega \sum_{\substack{k,l=1\\k<l}}^{M} \sigma_{kl} \nu_k \cdot \nabla \boldsymbol{\xi} \nu_k \mu_{kl} - \sum_{k=1}^{M} \int_{\Omega_k} \big((\nabla \mathbf{u})^t W_{k,\mathcal{E}}\big) : \nabla \boldsymbol{\xi} = 0. \tag{15}$$

We stated the Euler–Lagrange equations in the above form because we will consider variations along more general vector fields $\boldsymbol{\xi}$ later.

Proof of Lemma 3.1. We consider the family of diffeomorphisms $\Phi(\tau, \cdot)$, $\tau \in [-\tau_0, \tau_0]$ of Ω given by

$$\Phi(0, \mathbf{x}) = \mathbf{x} \qquad \text{and} \qquad \Phi_{,\tau}(\tau, \mathbf{x}) = \boldsymbol{\xi}(\Phi(\tau, \mathbf{x}))$$

for $\mathbf{x} \in \Omega$ and $\tau \in [-\tau_0, \tau_0]$. The mappings $\Phi(\tau, \cdot)$ define a one parametric group of diffeomorphisms and in particular it holds

$$\Phi(\tau, \Phi(-\tau, \mathbf{x})) = \mathbf{x},$$

i.e. $\Phi(\tau, \cdot)$ is the inverse of $\Phi(-\tau, \cdot)$. Via the diffeomorphisms $\Phi(\tau, \cdot)$ we define variations of the independent variable and obtain

$$\begin{aligned}
\mathbf{c}^\tau(\mathbf{x}) &:= \mathbf{c}\left(\Phi(-\tau, \mathbf{x})\right), \\
\mathbf{u}^\tau(\mathbf{x}) &:= \mathbf{u}\left(\Phi(-\tau, \mathbf{x})\right), \\
\Omega_k^\tau(\mathbf{x}) &:= \Phi(\tau, \Omega_k) = \{\Phi(\tau, \mathbf{x}) \,|\, \mathbf{x} \in \Omega_k \}.
\end{aligned}$$

Since

$$\frac{d}{d\tau} det\Phi_{,\mathbf{x}}(\tau, \mathbf{x}) = (\nabla \cdot \boldsymbol{\xi})\left(\Phi(\tau, \mathbf{x})\right) det\Phi_{,x}(\tau, \mathbf{x})$$

we get

$$\begin{aligned}
\frac{d}{d\tau}|\Omega_k^\tau| &= \frac{d}{d\tau}\int_{\Omega_k} |det\Phi_{,\mathbf{x}}(\tau, \mathbf{x})| d\mathbf{x} \\
&= \int_{\Omega_k} (\nabla \cdot \boldsymbol{\xi})\left(\Phi(\tau, \mathbf{x})\right) det\Phi_{,x}(\tau, \mathbf{x})\, d\mathbf{x}
\end{aligned}$$

and since $\boldsymbol{\xi}$ is divergence free, deformations of Ω by $\Phi(\tau, \cdot)$ do not change the volume of the individual phases. Hence the integral constraint

$$⨍_\Omega \mathbf{c}^\tau = \mathbf{m}$$

is fulfilled. Notice that $(\mathbf{c}, \mathbf{u})$ is also a minimizer if we allow $\mathbf{u}$ to vary in the larger class $H^1(\Omega, \mathbb{R}^N)$. Hence, $(\mathbf{c}^\tau, \mathbf{u}^\tau)$ is allowed as a comparison function and we obtain $E^0(\mathbf{c}, \mathbf{u}) \leq E^0(\mathbf{c}^\tau, \mathbf{u}^\tau)$, which implies

$$0 = \frac{d}{d\tau} E^0(\mathbf{c}^\tau, \mathbf{u}^\tau)_{|\tau=0}$$

if the derivative exists. Let us compute the above derivative. Using the identity

$$\mu_{kl} = \frac{1}{2}\left(|\nabla \mathcal{X}_{\Omega_k}| + |\nabla \mathcal{X}_{\Omega_l}| - |\nabla(\mathcal{X}_{\Omega_k} + \mathcal{X}_{\Omega_l})|\right)$$

one can reduce the computation of the derivative of the first term in E^0 to the problem of computing the first variation of area. Determining the first variation of area in the setting of sets of bounded perimeter is standard (see for example Giusti [15]) and we only sketch the arguments. Let $\mathcal{X}^\tau$ be either $\mathcal{X}_{\Omega_k^\tau}$ or $\mathcal{X}_{\Omega_k^\tau \cup \Omega_l^\tau} = \mathcal{X}_{\Omega_k^\tau} + \mathcal{X}_{\Omega_l^\tau}$ and $\mathcal{X} = \mathcal{X}^0$. Going back to the definition of the variation $\int_\Omega |\nabla \mathcal{X}^\tau|$ we obtain with the help of the change of variable formula and an approximation argument

$$\int_\Omega |\nabla \mathcal{X}^\tau| = \int_\Omega |\left(\Phi_{,\mathbf{x}}(\tau,\cdot)\right)^{-1}\nu|\,|\det \Phi_{,\mathbf{x}}(\tau,\cdot)|\,|\nabla\mathcal{X}|, \qquad (16)$$

where $\nu = -\frac{\nabla\mathcal{X}}{|\nabla\mathcal{X}|}$ is the generalized unit normal which is a $|\nabla\mathcal{X}|$–measurable function. The right hand side is differentiable with respect to the parameter τ. Since

$$\frac{d}{d\tau}\left(|\det \Phi_{,\mathbf{x}}(\tau,\mathbf{x})|\right)_{|\tau=0} = \nabla\cdot\boldsymbol{\xi}(\mathbf{x}) \qquad (17)$$

and

$$\frac{d}{d\tau}\left(\left(\Phi_{,\mathbf{x}}(\tau,\mathbf{x})\right)^{-1}\right)_{|\tau=0} = -\nabla\boldsymbol{\xi}(\mathbf{x}) \qquad (18)$$

we obtain

$$\frac{d}{d\tau}\left(\int_\Omega |\nabla\mathcal{X}^\tau|\right)_{|\tau=0} = \int_\Omega \left(\nabla\cdot\boldsymbol{\xi} - \nu\cdot\nabla\boldsymbol{\xi}\nu\right)\,|\nabla\mathcal{X}|.$$

Defining $\nu_k = -\frac{\nabla\mathcal{X}_{\Omega_k}}{|\nabla\mathcal{X}_{\Omega_k}|}$ it holds

$$\nu_k + \nu_l = -\frac{\nabla\mathcal{X}_{\Omega_k}}{|\nabla\mathcal{X}_{\Omega_k}|} - \frac{\nabla\mathcal{X}_{\Omega_l}}{|\nabla\mathcal{X}_{\Omega_l}|} = 0 \qquad \mu_{kl}\text{–almost everywhere}$$

and

$$\frac{\nabla(\mathcal{X}_{\Omega_k}+\mathcal{X}_{\Omega_l})}{|\nabla(\mathcal{X}_{\Omega_k}+\mathcal{X}_{\Omega_l})|} = 0 \qquad \mu_{kl}\text{–almost everywhere}$$

(see Theorem 6.1). Then we obtain using the representation of $|\nabla\mathcal{X}_{\Omega_k}|$ and $|\nabla(\mathcal{X}_{\Omega_k}+\mathcal{X}_{\Omega_l})|$ of Theorem 6.1

$$\begin{aligned}
&\frac{d}{d\tau}\left(\mathcal{H}^{n-1}\left(\partial^*\{\mathbf{c}^\tau = \mathbf{p}_k\}\cap\partial^*\{\mathbf{c}^\tau=\mathbf{p}_l\}\right)\right)_{|\tau=0}\\
&\quad = \frac{d}{d\tau}\left(\mu^\tau_{kl}(\Omega)\right)_{|\tau=0}\\
&\quad = \frac{d}{d\tau}\left(\frac{1}{2}\left(|\nabla\mathcal{X}^\tau_{\Omega_k}| + |\nabla\mathcal{X}^\tau_{\Omega_l}| - |\nabla(\mathcal{X}^\tau_{\Omega_k}+\mathcal{X}^\tau_{\Omega_l})|\right)(\Omega)\right)_{|\tau=0}\\
&\quad = \int_\Omega \left(\nabla\cdot\boldsymbol{\xi} - \nu_k\cdot\nabla\boldsymbol{\xi}\nu_k\right)\mu_{kl}
\end{aligned}$$

which gives the derivative of the first part of E^0. It remains to compute the derivative of $\int_\Omega W(\mathbf{c}^\tau,\mathcal{E}(\mathbf{u}^\tau))$. It holds

$$\int_\Omega W(\mathbf{c}^\tau(\mathbf{y}),\mathcal{E}(\mathbf{u}^\tau)(\mathbf{y}))d\mathbf{y} = \int_\Omega W(\mathbf{c}(\Phi(-\tau,\mathbf{y})),\mathcal{E}(\mathbf{u}(\Phi(-\tau,\mathbf{y})))\,d\mathbf{y}.$$

Setting $\mathbf{x} = \Phi(-\tau, \mathbf{y})$ and using $\Phi_{,\mathbf{x}}(\tau, \Phi(-\tau, \mathbf{y}))\Phi_{,\mathbf{x}}(-\tau, \mathbf{y}) = Id$ we compute

$$\begin{aligned}
\mathcal{E}(\mathbf{u}(\Phi(-\tau,\mathbf{y}))) &= \frac{1}{2}\left(\nabla_{\mathbf{y}}\left(\mathbf{u}(\Phi(-\tau,\mathbf{y}))\right) + \left(\nabla_{\mathbf{y}}\left(\mathbf{u}(\Phi(-\tau,\mathbf{y}))\right)\right)^t\right) \\
&= \frac{1}{2}\left(\nabla\mathbf{u}(\mathbf{x})(\Phi_{,\mathbf{x}}(\tau,\mathbf{x}))^{-1} + \left(\nabla\mathbf{u}(\mathbf{x})(\Phi_{,\mathbf{x}}(\tau,\mathbf{x}))^{-1}\right)^t\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_\Omega W(\mathbf{c}^\tau(\mathbf{y}), \mathcal{E}(\mathbf{u}^\tau)(\mathbf{y}))d\mathbf{y} = \\
&\quad \int_\Omega W\left(\mathbf{c}, \frac{1}{2}\left(\nabla\mathbf{u}(\Phi_{,\mathbf{x}}(\tau,\cdot))^{-1} + \left(\nabla\mathbf{u}(\Phi_{,\mathbf{x}}(\tau,\cdot))^{-1}\right)^t\right)\right) \,|\det \Phi_{,\mathbf{x}}(\tau,\cdot)|
\end{aligned}$$

where the integration in the last line was with respect to the variable $\mathbf{x}$. Using (17), (18) and the growth conditions on W and $W_{,\mathcal{E}}$ (see (A3)) we compute

$$\begin{aligned}
&\frac{d}{d\tau}\left(\int_\Omega W(\mathbf{c}^\tau, \mathcal{E}(\mathbf{u}^\tau))\right)_{|\tau=0} = \\
&= \int_\Omega \left[W(\mathbf{c}, \mathcal{E}(\mathbf{u}))\,\nabla\cdot\boldsymbol{\xi} - W_{,\mathcal{E}}(\mathbf{c}, \mathcal{E}(\mathbf{u})) : \frac{1}{2}\left(\nabla\mathbf{u}\nabla\boldsymbol{\xi} + (\nabla\mathbf{u}\nabla\boldsymbol{\xi})^t\right)\right] \\
&= \int_\Omega \left[W(\mathbf{c}, \mathcal{E}(\mathbf{u}))\,\nabla\cdot\boldsymbol{\xi} - W_{,\mathcal{E}}(\mathbf{c}, \mathcal{E}(\mathbf{u})) : (\nabla\mathbf{u}\nabla\boldsymbol{\xi})\right] \\
&= \int_\Omega \left[W(\mathbf{c}, \mathcal{E}(\mathbf{u}))\,\nabla\cdot\boldsymbol{\xi} - \left((\nabla\mathbf{u})^t W_{,\mathcal{E}}(\mathbf{c}, \mathcal{E}(\mathbf{u}))\right) : \nabla\boldsymbol{\xi}\right]
\end{aligned}$$

where we used the symmetry of $W_{,\mathcal{E}}$ to obtain the last two identities. Since $W(\mathbf{p}_k, \cdot) = W_k(\cdot)$ the lemma is proved. □

It is our goal to derive from the weak form of the Euler–Lagrange equation conditions that hold locally in Ω. We will derive conditions in the bulk, on interfaces and on boundaries of interfaces. We sketch the derivation of these conditions for absolute minimizer that are regular enough.

We assume:

a) the sets Ω_k are Lipschitz,

b) the sets $\Lambda_{kl} = \partial^*\Omega_k \cap \partial^*\Omega_l$ consist of a finite number of oriented C^2–hypersurfaces where the orientation is given by ν_k. If $\partial\Lambda_{kl} \neq \emptyset$ we assume that $\partial\Lambda_{kl}$ consists of a finite number of C^1–$(n-2)$–dimensional surfaces. It is furthermore assumed that these surfaces are either subsets of $\partial\Omega$ or that they meet with the boundary of two or more other interfaces.

c) $\mathbf{u} \in H^2(\Omega_k, \mathrm{I\!R}^n)$,

d) $W \in C^2$.

Assumption c) implies that $\nabla \mathbf{u}$ has traces on $\partial\Omega_k$ that lie in $L^2(\partial^*\Omega_k, \mathbb{R}^n)$. We denote by $\mathrm{div}_{\Lambda_{kl}}$ the tangential divergence with respect to the interface Λ_{kl} and by τ_{kl} the outer unit normal to $\partial\Lambda_{kl}$. Then we compute with the help of the Gauß theorem on manifolds

$$\begin{aligned}\int_\Omega (\nabla \cdot \boldsymbol{\xi} - \nu_k \cdot \nabla \boldsymbol{\xi} \nu_k)\, \mu_{kl} &= \int_{\Lambda_{kl}} \mathrm{div}_{\Lambda_{kl}} \boldsymbol{\xi}\, d\mathcal{H}^{n-1} \\ &= \int_{\Lambda_{kl}} (\mathrm{div}_{\Lambda_{kl}} \nu_k)\, (\boldsymbol{\xi} \cdot \nu_k)\, d\mathcal{H}^{n-1} + \int_{\partial\Lambda_{kl}} (\boldsymbol{\xi} \cdot \tau_{kl})\, d\mathcal{H}^{n-2}.\end{aligned}$$

Since $\nabla \cdot W_{k,\mathcal{E}}(\mathcal{E}(\mathbf{u})) = 0$ almost everywhere in Ω_k and since $W_{k,\mathcal{E}}$ is symmetric we obtain

$$\begin{aligned}&\nabla \cdot \left(W_k Id - (\nabla \mathbf{u})^t\, W_{k,\mathcal{E}}\right) = \\ &= \nabla W_k - (\partial_i \nabla \mathbf{u} : W_{k,\mathcal{E}})_{i=1,\ldots,n} - (\nabla \mathbf{u})^t\, \nabla \cdot W_{k,\mathcal{E}} = 0\end{aligned}$$

which holds almost everywhere in Ω_k. Hence,

$$\begin{aligned}\int_{\Omega_k} \left(W_k Id - (\nabla \mathbf{u})^t\, W_{k,\mathcal{E}}\right) : \nabla \boldsymbol{\xi} &= -\int_{\Omega_k} \nabla \cdot \left(W_k Id - (\nabla \mathbf{u})^t\, W_{k,\mathcal{E}}\right) \cdot \boldsymbol{\xi} \\ &\quad + \int_{\partial\Omega_k} \boldsymbol{\xi} \cdot \left(W_k Id - (\nabla \mathbf{u})^t\, W_{k,\mathcal{E}}\right) \nu_k \\ &= \int_{\partial\Omega_k \cap \Omega} \boldsymbol{\xi} \cdot \left(W_k Id - (\nabla \mathbf{u})^t\, W_{k,\mathcal{E}}\right) \nu_k\end{aligned}$$

where we also used that $\boldsymbol{\xi} \cdot \mathbf{n} = 0$ and $W_{k,\mathcal{E}} \cdot \mathbf{n} = 0$ on $\partial\Omega$ to obtain the last identity. Therefore, we obtain

$$\begin{aligned}&\sum_{\substack{k,l=1\\k<l}}^{M} \sigma_{kl} \left(\int_{\Lambda_{kl}} \mathrm{div}_{\Lambda_{kl}} \nu_k\ (\boldsymbol{\xi} \cdot \nu_k)\ d\mathcal{H}^{n-1} + \int_{\partial\Lambda_{kl}} \boldsymbol{\xi} \cdot \tau_{kl}\, d\mathcal{H}^{n-2} \right) \\ &\qquad + \sum_{k=1}^{M} \int_{\partial\Omega_k \cap \Omega} \boldsymbol{\xi} \cdot \left(W_k Id - (\nabla \mathbf{u})^t\, W_{k,\mathcal{E}}\right) \nu_k d\mathcal{H}^{n-1} = 0\end{aligned} \tag{19}$$

where τ_{kl} is the outward unit normal to $\partial\Lambda_{kl}$. The above identity holds for all $\boldsymbol{\xi} \in C^\infty(\bar{\Omega})$ with $\nabla \cdot \boldsymbol{\xi} = 0$ and $\boldsymbol{\xi} \cdot \mathbf{n} = 0$ on $\partial\Omega$.

Now let Ξ be a smooth $(n-2)$–dimensional subset of $\cup_{\substack{k,l=1\\k<l}}^{M} \partial\Lambda_{kl}$ chosen in such a way that Ξ is either a subset of $\partial\Omega$ or lies in the intersection of three or more interfaces. If $\Xi \subset \partial\Omega$ we obtain by choosing variations in the neighborhood of a point in Ξ that (19) leads to

$$\tau_{kl} = \mathbf{n} \qquad \text{on} \quad \partial\Omega \cap \partial\Lambda_{kl}. \tag{20}$$

This says that the interface intersects the outer boundary orthogonal. Note that we can only choose variations that lead to $\boldsymbol{\xi}$'s with vanishing normal component at the boundary of Ω, which implies that τ_{kl} still can have a normal component. Variations close to points on Ξ have to be done in such a way that the volume of the individual phases remains fixed. If a variation close to a point on Ξ changes volume, the volume constraint still can be met by pushing an interface in the opposite direction a bit further away. Taking into account that the integration with respect to Ξ is $(n-2)$–dimensional whereas the integration with respect to the interfaces is $(n-1)$–dimensional we can obtain a pointwise identity on Ξ (i.e. the identity (20)) by choosing variations with support closer and closer to Ξ. We remark that in constructing these variations we only need to ensure that the volume constraint is met. In particular there is no need to ensure that volume is conserved locally, which means that $\nabla \cdot \boldsymbol{\xi} = 0$ in Ω is not required necessarily.

If Ξ lies in the interior of Ω it has to be the junction of three or more interfaces. We define a set of tuples $\mathcal{I} \subset \{1, ..., M\}^2$ such that

$$\Xi \subset \partial^* \Lambda_{kl} \qquad \text{if and only if} \quad (k,l) \in \mathcal{I}.$$

By choosing variations locally around Ξ we deduce that

$$\sum_{(k,l)\in\mathcal{I}} \sigma_{kl}\tau_{kl} = 0 \qquad \text{on} \quad \Xi. \tag{21}$$

Here again we need to take care of the volume constraint. And again we use the fact that the sets over which we integrate in (19) have different dimension to conclude (21). The identity (21) is the well known Young's law which can be interpreted as a force balance at multiple junctions [31, 16, 4, 13]. Young's law implies conditions for the angles at which interfaces can intersect. For example in the case that three interfaces, which are denoted by $(1,2), (2,3), (1,3)$, meet at a triple junction we obtain

$$\sigma_{12}\tau_{12} + \sigma_{23}\tau_{23} + \sigma_{13}\tau_{13} = 0,$$

which implies that the angles $\theta_1, \theta_2, \theta_3$ between the three surfaces $\Lambda_{12}, \Lambda_{23}, \Lambda_{13}$ fulfill the condition

$$\frac{\sigma_{12}}{\sin\theta_3} = \frac{\sigma_{23}}{\sin\theta_1} = \frac{\sigma_{13}}{\sin\theta_2},$$

where we denote by θ_1 the angle between τ_{12} and τ_{13}, by θ_2 the angle between τ_{12} and τ_{23} and by θ_3 the angle between τ_{13} and τ_{23}. For more discussion on Young's law see [31, 4, 13]. If all surface energies are the same, we obtain that three interfaces meet at $120°$ angles, which are the same as the angles at which surfaces making up soap bubble clusters meet. Bubble clusters realize the least area way to enclose and separate several regions of prescribed volumes (see e.g. Morgan [26], Chapter 13). This would be a solution of the limiting variational problem we considered above in the case that the elastic energy is zero and that all surface tensions are the same.

We have shown that for an absolute minimizer the $(n-2)$–dimensional integral in (19) is zero and we obtain

$$\begin{aligned} T(\boldsymbol{\xi}) \quad := \quad & \sum_{\substack{k,l=1\\k<l}}^{M} \sigma_{kl} \int_{\Lambda_{kl}} \operatorname{div}_{\Lambda_{kl}} \nu_k \; (\boldsymbol{\xi}\cdot\nu_k) \; d\mathcal{H}^{n-1} \qquad (22)\\ & + \sum_{k=1}^{M} \int_{\partial\Omega_k \cap \Omega} \boldsymbol{\xi} \cdot \left(W_k Id - (\nabla \mathbf{u})^t \, W_{k,\varepsilon} \right) \nu_k d\mathcal{H}^{n-1} = 0 \end{aligned}$$

for $\boldsymbol{\xi} \in C^\infty(\overline{\Omega})$ with $\nabla\cdot\boldsymbol{\xi} = 0$ and $\boldsymbol{\xi}\cdot\mathbf{n} = 0$ on $\partial\Omega$. The functional T when defined by (22) is a bounded linear functional on $H^1(\Omega, \mathbb{R}^n)$ with $T(\boldsymbol{\xi}) = 0$ for all divergence free $\boldsymbol{\xi} \in H^1_0(\Omega, \mathbb{R}^n)$. This implies the existence of a function $p \in L^2(\Omega)$ such that

$$T = \nabla p$$

(see Temam [28]), i.e.

$$\begin{aligned} \int_\Omega p \; \nabla\cdot\boldsymbol{\xi} \quad = \quad & \sum_{\substack{k,l=1\\k<l}}^{M} \sigma_{kl} \int_{\Lambda_{kl}} \operatorname{div}_{\Lambda_{kl}} \nu_k \; (\boldsymbol{\xi}\cdot\nu_k) \; d\mathcal{H}^{n-1} \qquad (23)\\ & + \sum_{k=1}^{M} \int_{\partial\Omega_k \cap \Omega} \boldsymbol{\xi} \cdot \left(W_k Id - (\nabla \mathbf{u})^t \, W_{k,\varepsilon} \right) \nu_k d\mathcal{H}^{n-1} \end{aligned}$$

for all $\boldsymbol{\xi} \in H^1_0(\Omega, \mathbb{R}^n)$. Since the left hand side of (23) is zero if $\boldsymbol{\xi}$ is supported in Ω_k, we obtain that p is constant on all connected subsets of Ω_k. The sum of the principal curvatures (i.e. $(n-1)$–times the mean curvature) of the interface Λ_{kl} is given by

$$\kappa_{kl} = \operatorname{div}_{\Lambda_{kl}} \nu_k \qquad \text{on} \quad \Lambda_{kl}.$$

Now we choose a function $\boldsymbol{\xi}$ which fulfills $\boldsymbol{\xi} = V\nu_k$ on Λ_{kl} where V is an arbitrary sufficiently smooth function defined on Λ_{kl} with compact support in the interior of Λ_{kl}. It is furthermore assumed that $\boldsymbol{\xi}$ is chosen such that it is zero on all other interfaces (see for example [15], Chapter 10, for details on the construction of such functions $\boldsymbol{\xi}$). Making use of the fact that V is arbitrary gives

$$\sigma_{kl}\kappa_{kl} + \nu_k \left[W Id - (\nabla \mathbf{u})^t \, W_{,\varepsilon} \right]_l^k \nu_k = [p]_l^k \qquad \text{on} \quad \Lambda_{kl}. \qquad (24)$$

For a quantity q that jumps across an interface Λ_{kl} we define

$$[q]_l^k := q^k - q^l,$$

where q^k and q^l are the traces of q on Λ_{kl} with respect to Ω_k and Ω_l respectively. The quantity $W\,Id - (\nabla \mathbf{u})^t W_{,\mathcal{E}}$ is Eshelby's energy-momentum tensor (see [6]), the term $\nu_k \left[W Id - (\nabla \mathbf{u})^t W_{,\mathcal{E}}\right]_l^k \nu_k$ results from the Eshelby traction and p plays the role of a Lagrange multiplier taking into account that the volume of the phases is prescribed.

A minimizer of the energy E^0 fulfills

$$\nabla \cdot W_{,\mathcal{E}}(\mathbf{c}, \mathbf{u}) = 0$$

in the sense of distributions. This implies that the normal component of $W_{,\mathcal{E}}$ does not jump across the boundary of Ω_k. Hence, it holds

$$[W_{,\mathcal{E}}]_l^k \, \nu_k = 0 \qquad \text{on} \quad \Lambda_{kl}.$$

This implies that (24) can be rewritten as

$$\sigma_{kl}\kappa_{kl} + [W]_l^k - [\nabla \mathbf{u}\, \nu_k]_l^k \cdot (W_{,\mathcal{E}}\nu_k) = [p]_l^k \qquad \text{on} \quad \Lambda_{kl} \tag{25}$$

where $W_{,\mathcal{E}}\nu_k$ can be determined by taking the trace of $W_{,\mathcal{E}}$ on Λ_{kl} either with respect to Ω_k or with respect to Ω_l. The term $W_{,\mathcal{E}}\nu_k$ is the traction at the interface (see [10]). The identity (25) is a stress modified version of the Gibbs–Thomson equation (see e.g. [5, 18, 20, 17, 10]).

Below we show that the function p cannot attain different values in parts of Ω_k that are not connected. To show this we still suppose that $(\mathbf{c}, \mathbf{u})$ is an absolute minimizer of the functional E^0. Hence we can also allow for variations that change the volume of the connected components, if one guarantees that the overall volume of the phases remains the same. Again we consider a situation in which the minimizer $(\mathbf{c}, \mathbf{u})$ fulfills the regularity conditions a)–d). Let us sketch the idea of the argument which shows that p is constant on Ω_k.

Assume there are two maximally connected subsets Ω_1' and Ω_2' of Ω_{k_0}, $k_0 \in \{1, ..., M\}$. Then there exists a chain of sets

$$\Omega_1' = D_1, D_2, ..., D_{m-1}, D_m = \Omega_2'$$

whose interiors are mutually disjoint and which are such that each of the sets $D_1, D_2, ..., D_{m-1}, D_m$ is a maximally connected subset of one the sets $\Omega_1, ..., \Omega_M$ and such that

$$\mathcal{H}^{n-1}(\partial^* D_\alpha \cap \partial^* D_{\alpha+1}) > 0 \qquad \text{for} \quad \alpha = 1, ..., m-1.$$

If such a chain would not exist, we could deduce that a partition of Ω into two sets with non-vanishing volume and zero interfacial area exists. But this is a contradiction to the isoperimetric inequality for partitions of Ω.

The idea now is to push a certain amount of volume from Ω_1' through the sets $D_2, ..., D_{m-1}$ into the set Ω_2' which by assumption belongs to the same phase as the set Ω_1'. Then we want to exploit the fact that this variation cannot increase the energy and hence the derivative of the energy along this variation has to be zero. We choose families of diffeomorphisms $\Phi_\alpha(\tau, \cdot)$, $\tau \in [-\tau_0, \tau_0]$, $\alpha \in \{1, ..., m-1\}$, such that

$$\Phi_\alpha(\tau, \cdot) = Id \qquad \text{on all interfaces besides } \partial^* D_\alpha \cap \partial^* D_{\alpha+1},$$

$$\Phi_{\alpha,\tau}(0, \mathbf{x}) = V_\alpha(\mathbf{x}) \nu_\alpha(\mathbf{x})$$

where V_α is a sufficiently smooth function on $\partial^* D_\alpha \cap \partial^* D_{\alpha+1}$ with compact support and ν_α is the unit normal on $\partial^* D_\alpha \cap \partial^* D_{\alpha+1}$ pointing from D_α into $D_{\alpha+1}$. We choose the functions V_α such that

$$\int_{\partial^* D_\alpha \cap \partial^* D_{\alpha+1}} V_\alpha \, d\mathcal{H}^{n-1} = 1.$$

Choosing diffeomorphisms $\Phi_\alpha(\tau, \cdot)$ with normal velocities V_α on $D_\alpha \cap \partial^* D_{\alpha+1}$ and such that Φ_α is constant on the other interfaces we obtain that the volume of the sets $D_\alpha(\tau)$ changes in the following way

$$\begin{aligned} \frac{d}{d\tau}|D_1(\tau)| &= 1, \\ \frac{d}{d\tau}|D_\alpha(\tau)| &= 0, \qquad \alpha = 2, ..., m-1, \\ \frac{d}{d\tau}|D_m(\tau)| &= -1. \end{aligned}$$

This can be done by an appropriate normalization of the typical construction of normal variations on an interface (see e.g. Giusti [15]). The equations above guarantee that the composition Φ of the $(m-1)$–diffeomorphisms $\Phi_1, ..., \Phi_{m-1}$ preserves the total volume of the individual phases. Computing the first variation $\frac{d}{d\tau} E^0(\mathbf{c}^\tau, \mathbf{u}^\tau)_{|\tau=0}$ using the smoothness assumptions from above yields

$$0 = \sum_{\alpha=1}^{m-1} \int_{\partial^* D_\alpha \cap \partial^* D_{\alpha+1}} V_\alpha \left\{ \sigma_{k_\alpha l_\alpha} \kappa_\alpha + \nu_\alpha \left[W Id - (\nabla \mathbf{u})^t W_{,\varepsilon} \right]_\alpha^{\alpha+1} \nu_\alpha \right\} d\mathcal{H}^{n-1}.$$

By κ_α we denote $(n-1)$ times the mean curvature of $\partial^* D_\alpha \cap \partial^* D_{\alpha+1}$ and $\sigma_{k_\alpha l_\alpha}$ is the surface tension related to $\partial^* D_\alpha \cap \partial^* D_{\alpha+1}$, i.e. $k_\alpha, l_\alpha \in \{1, ..., M\}$ are chosen such that $\partial^* D_\alpha \cap \partial^* D_{\alpha+1}$ belongs to $\Lambda_{k_\alpha l_\alpha}$. The term in the brackets $\{\cdot\}$ is equal to the jump of p across the interface (see (24)). Denoting by p_α the value of p on D_α we obtain

$$\begin{aligned} 0 &= \sum_{\alpha=1}^{m-1} (p_{\alpha+1} - p_\alpha) \int_{\partial^* D_\alpha \cap \partial^* D_{\alpha+1}} V_\alpha = \sum_{\alpha=1}^{m-1} (p_{\alpha+1} - p_\alpha) \\ &= p_m - p_1 \end{aligned}$$

which proves that p_α is constant in each of the sets $\Omega_1, ..., \Omega_M$.

Let us summarize the conditions a regular absolute minimizer of E^0 has to fulfill.

Strong formulation of the Euler–Lagrange equations
Assume an absolute minimizer of E^0 fulfills the regularity assumptions a)–d).
Then there exists a Lagrange multiplier $\boldsymbol{p} = (p_1, ..., p_M)$ with $\sum_{k=1}^M p_k = 0$ such that

In the sets Ω_k it holds:

1.)

$$\nabla \cdot [W_{k,\mathcal{E}}(\mathcal{E}(\mathbf{u}))] = 0.$$

On the boundary of Ω it holds:
2.)

$$W_{k,\mathcal{E}}\, \mathbf{n} = 0.$$

On the interfaces Λ_{kl} it holds
3.)

$$\sigma_{kl}\kappa_{kl} + \nu_k \left[W Id - (\nabla \mathbf{u})^t\, W_{,\mathcal{E}} \right]_l^k \nu_k = p_k - p_l,$$

4.)

$$[W_{,\mathcal{E}}\, \nu_k]_l^k = 0,$$

5.)

$$[\mathbf{u}]_l^k = 0.$$

For co-dimension–2 junctions Ξ of three or more interfaces we define a set of tuples $\mathcal{I} \subset \{1, ..., M\}^2$ such that

$$\Xi \subset \partial^* \Lambda_{kl} \qquad \text{if and only if} \quad (k,l) \in \mathcal{I}.$$

6.) On these junctions *Young's law* has to hold, i.e.

$$\sum_{(k,l) \in \mathcal{I}} \sigma_{kl} \tau_{kl} = 0 \qquad \text{on} \quad \Xi.$$

7.) Interfaces that intersect the outer boundary do these with a *right angle.*

Remark 3.2. *1.) Condition 5.) means that the interface is coherent; i.e. two phases neither separate nor slip at the interface. This implies that* $[\nabla \mathbf{u}]_l^k \tau = 0$ *for all* τ *that are tangent to* Λ_{kl} *and hence, the tangential part of the gradient does not jump. This together with (24) conditions 3.) and 4.) gives*

$$\sigma_{kl}\kappa_{kl}\nu_k + \left[W Id - (\nabla \mathbf{u})^t W_{,\mathcal{E}}\right]_l^k \nu_k = (p_k - p_l)\nu_k \qquad on \quad \Lambda_{kl},$$

2.) The argument in 1.) together with the discussion leading to the strong formulation of the Euler–Lagrange equations shows that any function fulfilling the regularity requirements a)–d) and the conditions 1.)–7.) above also fulfills

$$\int_\Omega (\nabla \cdot \boldsymbol{\xi})\, p \;=\; \sum_{\substack{k,l=1\\k<l}}^{M} \sigma_{kl} \int_{\Lambda_{kl}} (\nabla \cdot \boldsymbol{\xi} - \nu_k \cdot \nabla \boldsymbol{\xi} \nu_k)\, \mu_{kl} + $$
$$+ \sum_{k=1}^{M} \int_{\Omega_k} \left(W_k Id - (\nabla \mathbf{u})^t W_{k,\mathcal{E}}\right) : \nabla \boldsymbol{\xi}$$

for all for all $\boldsymbol{\xi} \in C^\infty(\overline{\Omega}, \mathrm{I\!R}^n)$ *with* $\boldsymbol{\xi} \cdot \mathbf{n} = 0$ *on* $\partial\Omega$*. Here,* p *is given by* $p \equiv p_k$ *on* Ω_k*.*

4. The Gibbs–Thomson equation as a singular limit in the scalar case

In this section we study the asymptotic limit of the diffuse interface model in the binary case. Due to the constraint $c_1 + c_2 = 1$ we can express the dependence on the concentration vector by the scalar quantity

$$c = c_1 - c_2$$

which is the difference of the two concentrations.

The free energy then has the form (after normalizing constants)

$$E^\varepsilon(c, \mathbf{u}) := \int_\Omega \left(\varepsilon|\nabla c|^2 + \frac{1}{\varepsilon}\Psi(c) + W(c, \mathcal{E}(\mathbf{u}))\right), \quad \varepsilon > 0,$$

where $\Psi : \mathrm{I\!R} \to \mathrm{I\!R}$ is assumed to have two global minimizer with height zero. For simplicity of the presentation we rescale such that Ψ attains the global minimum zero at ± 1, i.e.

$$\Psi \geq 0 \qquad \text{and} \qquad \Psi(c') = 0 \qquad \Leftrightarrow \qquad c' = \pm 1. \tag{26}$$

We want to show that we can pass to the limit in the Euler–Lagrange equation for a minimizer of E^ε to obtain the modified Gibbs–Thomson equation studied in the previous section. The result we obtain generalizes a result of Luckhaus and Modica [22] to the case that elastic energy contributions are taken into account.

In the whole of this section we assume:

(B1) Ω is a bounded domain with C^1–boundary,

(B2) there exist constants $c_4, C_4 > 0$ such that

$$\Psi(c') \geq c_4 |c'|^2 - C_4 \qquad \text{for all} \qquad c' \in \mathbb{R}.$$

Furthermore it is assumed that $\Psi = \Psi^1 + \Psi^2$ where Ψ^1 is non–negative and convex and $\Psi^2_{,c}$ has sub-linear growth. In addition we assume that the growth of $\Psi^1_{,c}$ is bounded by Ψ^1 in the sense that for all $\delta > 0$ there exists a constant C_δ such that

$$|\Psi^1_{,c}(c')| \leq \delta \Psi^1(c') + C_\delta,$$

(B3) the assumptions (A3) for the elastic energy density W hold.

Under these assumptions one can show analogously to the vector case the existence of an absolute minimizer of the following scalar minimum problem.

(P^ε) Find a minimizer of

$$E^\varepsilon(c, \mathbf{u}) := \int_\Omega \left(\varepsilon |\nabla c|^2 + \frac{1}{\varepsilon} \Psi(c) + W(c, \mathcal{E}(\mathbf{u})) \right), \quad \varepsilon > 0$$

in the class of all functions $(c, \mathbf{u}) \in H^1(\Omega) \times X_2$ that fulfill the constraint $⨍_\Omega c = m$, where $m \in (-1, 1)$ is a given constant.

In the previous section we showed that in the limit $\varepsilon \searrow 0$ these variational problems lead to

(P^0) Find a minimizer of

$$E^0(c, \mathbf{u}) := \frac{\sigma}{2} \int_\Omega |\nabla c| + \int_\Omega W(c, \mathcal{E}(\mathbf{u})),$$

in the class of all functions $(c, \mathbf{u}) \in BV(\Omega) \times X_2$ that fulfill the constraints $c \in \{-1, 1\}$ a.e. and $⨍_\Omega c = m$, where $m \in (-1, 1)$ is a given constant.

In the following theorem we state the first variation of problem (P^ε) with respect to the independent variable. Our goal later is to pass to the limit $\varepsilon \searrow 0$ in the first variation formula that we obtain. This is appropriate since for problem (P^0) variations with respect to the dependent variable c are not possible. This is due to the fact that for the limiting problem c is constrained to attain the values ± 1.

Theorem 4.1. *Under the assumptions (B1)–(B3) a pair* $(c^\varepsilon, \mathbf{u}^\varepsilon) \in H^1(\Omega) \cap X_2$ *that is a solution to the variational problem* $(\mathbf{P}^\varepsilon)$ *fulfills*

$$\int_\Omega \left\{ \left(\varepsilon |\nabla c^\varepsilon|^2 + \frac{1}{\varepsilon}\Psi(c^\varepsilon) \right) \nabla \cdot \boldsymbol{\xi} - 2\varepsilon \nabla c^\varepsilon \cdot \nabla \boldsymbol{\xi} \nabla c^\varepsilon + \right. \tag{27}$$

$$\left. + \left(W(c^\varepsilon, \mathcal{E}(\mathbf{u}^\varepsilon)) Id - (\nabla \mathbf{u})^t W_{,\mathcal{E}}(c^\varepsilon, \mathcal{E}(\mathbf{u}^\varepsilon)) \right) : \nabla \boldsymbol{\xi} \right\} = \int_\Omega \lambda^\varepsilon c^\varepsilon \nabla \cdot \boldsymbol{\xi}$$

for all $\boldsymbol{\xi} \in C^\infty(\overline{\Omega}, \mathbb{R}^n)$ *with* $\boldsymbol{\xi} \cdot \mathbf{n} = 0$ *on* $\partial\Omega$. *Here,*

$$\lambda^\varepsilon = -\!\!\!\!\!\!\int_\Omega \left\{ \frac{1}{\varepsilon} \Psi_{,c}(c^\varepsilon) + W_{,c}(c^\varepsilon, \mathcal{E}(\mathbf{u}^\varepsilon)) \right\} \tag{28}$$

is a Lagrange multiplier.

Proof. In the proof we omit the index ε. Let $\boldsymbol{\xi} \in C^\infty(\bar{\Omega}, \mathbb{R}^n)$ be such that $\boldsymbol{\xi} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Then we choose a one parameter family of diffeomorphisms $\Phi(\tau, \cdot)$, $\tau \in [\tau_0, \tau_0]$ of Ω defined via the solutions of the following initial value problems

$$\Phi(0, \mathbf{x}) = \mathbf{x} \qquad \text{and} \qquad \Phi_{,\tau}(\tau, \mathbf{x}) = \boldsymbol{\xi}(\Phi(\tau, \mathbf{x}))$$

where $\mathbf{x} \in \Omega$ and $\tau \in [-\tau_0, \tau_0]$. We define

$$\begin{aligned} c^\tau(\mathbf{x}) &= c(\Phi(-\tau, \mathbf{x})) - -\!\!\!\!\!\!\int_\Omega c(\Phi(-\tau, \mathbf{y})) d\mathbf{y} + m, \\ \mathbf{u}^\tau(\mathbf{x}) &= \mathbf{u}(\Phi(\tau, \mathbf{x})) \end{aligned} \tag{29}$$

which is allowed as a comparison function in the minimization problem (P^ε). Note also that $c^0 = c$. Now we want to compute

$$\frac{d}{d\tau} \left\{ \int_\Omega \left(\varepsilon |\nabla c^\tau|^2 + \frac{1}{\varepsilon} \Psi(c^\tau) + W(c^\tau, \mathcal{E}(\mathbf{u}^\tau)) \right) \right\}_{|\tau=0}.$$

This term is zero because c is the minimum in (P^ε). First we compute

$$\int_\Omega |\nabla c^\tau(\mathbf{y})|^2 d\mathbf{y} = \int_\Omega |\left(\Phi_{,\mathbf{x}}(\tau, \mathbf{x})\right)^{-t} \nabla c(\mathbf{x})|^2 \, |\det \Phi_{,\mathbf{x}}(\tau, \mathbf{x})| \, d\mathbf{x}$$

and obtain

$$\frac{d}{d\tau} \left(\int_\Omega |\nabla c^\tau|^2 \right)_{|\tau=0} = \int_\Omega \left\{ |\nabla c|^2 \nabla \cdot \boldsymbol{\xi} - 2\nabla c \cdot \nabla \boldsymbol{\xi} \nabla c \right\}.$$

When we compute the derivative of the Ψ–term the mass correction in (29) will give a term which is a summand in the formula for the Lagrange multiplier. We get

$$\begin{aligned}
&\frac{d}{d\tau}\left(\int_\Omega \Psi(c^\tau(\mathbf{y}))d\mathbf{y}\right)_{|\tau=0}\\
&\quad=\frac{d}{d\tau}\left(\int_\Omega \Psi\left(c(\mathbf{x})-\fint_\Omega c(\Phi(-\tau,\mathbf{y}))d\mathbf{y}+m\right)\,|\det\Phi_{,\mathbf{x}}(\tau,\mathbf{x})|\,d\mathbf{x}\right)_{|\tau=0}\\
&\quad=\int_\Omega \Psi(c)\nabla\cdot\boldsymbol{\xi}-\int_\Omega\left(\Psi_{,c}(c)\left(\fint_\Omega c\nabla\cdot\boldsymbol{\xi}\right)\right)\\
&\quad=\int_\Omega \Psi(c)\nabla\cdot\boldsymbol{\xi}-\int_\Omega\left(c\nabla\cdot\boldsymbol{\xi}\left(\fint_\Omega \Psi_{,c}(c)\right)\right).
\end{aligned}$$

For the elastic term we compute similar as above

$$\begin{aligned}
&\int_\Omega W(c^\tau(\mathbf{y}),\mathcal{E}(\mathbf{u}^\tau(\mathbf{y}))d\mathbf{y}=\\
&=\int_\Omega W\left(c-\fint_\Omega c^\tau+m,\frac{1}{2}\left(\nabla\mathbf{u}(\Phi_{,\mathbf{x}})^{-1}+\left(\nabla\mathbf{u}(\Phi_{,\mathbf{x}})^{-1}\right)^t\right)\right)|\det\Phi_{,\mathbf{x}}|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{d}{d\tau}\left(\int_\Omega W(c^\tau(\mathbf{y}),\mathcal{E}(\mathbf{u}^\tau(\mathbf{y}))d\mathbf{y}\right)_{|\tau=0}=\\
&=\int_\Omega\left(W\,\nabla\cdot\boldsymbol{\xi}-((\nabla\mathbf{u})^tW_{,\mathcal{E}})):\nabla\boldsymbol{\xi}-W_{,c}\left(\fint_\Omega c\nabla\cdot\boldsymbol{\xi}\right)\right)
\end{aligned}$$

where W, $W_{,\mathcal{E}}$ and $W_{,c}$ in the last line are evaluated at $(c,\mathcal{E}(\mathbf{u}))$. This completes the proof of the theorem. □

Remark 4.1. *Let us point out that the Lagrange multiplier λ^ε also fulfills the identity*

$$\int_\Omega\left(2\varepsilon\nabla c^\varepsilon\nabla\zeta+\frac{1}{\varepsilon}\Psi_{,c}(c^\varepsilon)\zeta+W_{,c}(c^\varepsilon,\mathcal{E}(\mathbf{u}^\varepsilon))\zeta\right)=\int_\Omega\lambda^\varepsilon\zeta \tag{30}$$

for all $\zeta\in L^\infty(\Omega)\cap H^1(\Omega)$ which are the Euler–Lagrange equations one obtains by variations with respect to the dependent variable (take $\zeta\equiv 1$ in (30) to obtain (28)). Luckhaus and Modica [22] *who considered the case without elasticity started with these equations and set $\zeta=c^\varepsilon\nabla\cdot\boldsymbol{\xi}$ and then derived the identity (27). Formally it is also possible to derive equation (27) from (30) in the case with elastic interactions. We did not follow this strategy because in the case with elasticity there is not enough regularity known to make the formal calculations rigorous.*

In the next theorem we state that the Lagrange multipliers λ^ε converge (along subsequences as ε tends to zero) to a Lagrange multiplier of the limiting partitioning problem (P^0). This shows that the equation for the chemical potential leads to the Gibbs Thomson law in the limit as ε tends to zero.

Theorem 4.2. *Suppose (B1)–(B3) are fulfilled and assume $(c^\varepsilon, \mathbf{u}^\varepsilon)_{\varepsilon>0}$ are solutions of the variational problems $(\mathbf{P}^\varepsilon)$. Then for each subsequence $(\varepsilon^\kappa)_{\kappa\in\mathbb{N}} \searrow 0$ such that*

$$c^{\varepsilon^\kappa} \to c \quad in \quad L^1(\Omega), \tag{31}$$

$$\mathbf{u}^{\varepsilon^\kappa} \to \mathbf{u} \quad in \quad L^2(\Omega, \mathbb{R}^n) \tag{32}$$

it holds

$$\lambda^{\varepsilon^\kappa} \to \lambda,$$

where λ is a Lagrange multiplier for the minimum problem $(\mathbf{P}^\varepsilon)$, i.e.

$$\int_\Omega \sigma\left(\nabla\cdot\boldsymbol{\xi} - \nu\cdot\nabla\boldsymbol{\xi}\nu\right)|\nabla\mathcal{X}_{\{c=-1\}}| + \int_\Omega \left(W Id - (\nabla\mathbf{u})^t W_{,\mathcal{E}}\right) : \nabla\boldsymbol{\xi} = \int_\Omega \lambda c \nabla\cdot\boldsymbol{\xi} \tag{33}$$

for all $\boldsymbol{\xi} \in C^\infty(\overline{\Omega}, \mathbb{R}^n)$ with $\boldsymbol{\xi}\cdot\mathbf{n} = 0$ on $\partial\Omega$. Here, $\nu = -\frac{\nabla\mathcal{X}_{\{c=-1\}}}{|\nabla\mathcal{X}_{\{c=-1\}}|}$ is the generalized outer unit normal to $\{c=-1\}$ which is a $|\nabla\mathcal{X}_{\{c=-1\}}|$–measurable function.

Remark 4.2. *1.) Theorem 2.1 c) yields the existence of a subsequence such that the converge properties (31), (32) hold.*
2.) Since E^0 is the Γ–limit of E^ε we conclude that $(c, \mathbf{u})$ is an absolute minimizer of E^0.
3.) Notice that the term λc appearing in the term on the right hand side of (33) is constant in the sets $\{c = -1\}$ and $\{c = 1\}$. Setting $p = \lambda c$ we obtain the formulation of the Euler–Lagrange equation used in the previous section.

PROOF OF THEOREM 4.2. As in the proof of Theorem 2.2 we can conclude that the convergence is even stronger than (31), (32). Precisely, we obtain

$$\begin{aligned} c^{\varepsilon^\kappa} &\to c \ \text{ in } \ L^2(\Omega, \Sigma), \\ \mathbf{u}^{\varepsilon^\kappa} &\to \mathbf{u} \ \text{ in } \ H^1(\Omega, \mathbb{R}^n), \\ E^{\varepsilon^\kappa}(c^{\varepsilon^\kappa}, \mathbf{u}^{\varepsilon^\kappa}) &\to E^0(c, \mathbf{u}). \end{aligned}$$

We also derive that $(c, \mathbf{u})$ is a global minimizer of E^0 and

$$c \in \{-1, 1\} \qquad \text{almost everywhere.}$$

The above convergence properties together with the growth condition on W imply

$$\int_\Omega W(c^{\varepsilon^\kappa}, \mathbf{u}^{\varepsilon^\kappa}) \to \int_\Omega W(c, \mathbf{u}),$$

which yields that also

$$\int_\Omega \left(\varepsilon^\kappa |\nabla c^{\varepsilon^\kappa}|^2 + \frac{1}{\varepsilon^\kappa} \Psi(c^{\varepsilon^\kappa}) \right) \to \sigma \int_\Omega |\nabla \mathcal{X}_{\{c=-1\}}| \tag{34}$$

where $\sigma = d(-1,1) = 2\int_{-1}^1 \sqrt{\Psi(z)}\, dz$. Here we used the fact that the geodesic realizing the infimum in (2) is always realized by a straight connection of two points on the real line. Hence, we compute

$$d(c_1', c_2') = 2\int_{c_1'}^{c_2'} \sqrt{\Psi(z)}\, dz \qquad \text{for all} \qquad c_1' < c_2'.$$

Our goal is to pass to the limit in the equation

$$\int_\Omega \Big\{ \left(\varepsilon^\kappa |\nabla c^{\varepsilon^\kappa}|^2 + \frac{1}{\varepsilon^\kappa} \Psi(c^{\varepsilon^\kappa}) \right) \nabla \cdot \boldsymbol{\xi} - 2\varepsilon^\kappa \nabla c^{\varepsilon^\kappa} \cdot \nabla \boldsymbol{\xi} \nabla c^{\varepsilon^\kappa} + \tag{35}$$

$$\big(W(c^{\varepsilon^\kappa}, \mathcal{E}(\mathbf{u}^{\varepsilon^\kappa})) Id - (\nabla \mathbf{u}^{\varepsilon^\kappa})^t W_{,\mathcal{E}}(c^{\varepsilon^\kappa}, \mathcal{E}(\mathbf{u}^{\varepsilon^\kappa})) \big) : \nabla \boldsymbol{\xi} \Big\} = \int_\Omega \lambda^{\varepsilon^\kappa} c^{\varepsilon^\kappa} \nabla \cdot \boldsymbol{\xi}$$

to obtain the weak formulation of the Gibbs–Thomson equation. Here it comes into play that we were able to show strong convergence of $\nabla \mathbf{u}^{\varepsilon^\kappa}$ in $L^2(\Omega, \mathbb{R}^n)$. Taking the growth condition on W and $W_{,\mathcal{E}}$ into account we can pass to the limit in the terms involving W and $W_{,\mathcal{E}}$ to obtain

$$\int_\Omega \big(W(c^{\varepsilon^\kappa}, \mathcal{E}(\mathbf{u}^{\varepsilon^\kappa})) Id - (\nabla \mathbf{u}^{\varepsilon^\kappa})^t W_{,\mathcal{E}}(c^{\varepsilon^\kappa}, \mathcal{E}(\mathbf{u}^{\varepsilon^\kappa})) \big) : \nabla \boldsymbol{\xi} \to$$

$$\int_\Omega \big(W(c, \mathcal{E}(\mathbf{u})) Id - (\nabla \mathbf{u})^t W_{,\mathcal{E}}(c, \mathcal{E}(\mathbf{u})) \big) : \nabla \boldsymbol{\xi}$$

as ε^κ tends to zero. It remains to pass to the limit in the terms involving $\nabla c^{\varepsilon^\kappa}$ and Ψ. To obtain the limit we can use the ideas of Luckhaus and Modica [22] who studied the case without elastic energy contributions. For completeness we present the details.

Similarly as in Section 5 we define

$$\phi(c') = d(c', -1) = 2\int_{-1}^{c'} \sqrt{\Psi(z)}\, dz$$

and obtain

$$\phi(c^{\varepsilon^\kappa}) \to \begin{cases} \sigma = \phi(1) & \text{if} \quad z \in \{c = 1\}, \\ 0 & \text{if} \quad z \in \{c = -1\} \end{cases} \tag{36}$$

almost everywhere. In addition, we have for almost all $\mathbf{x}$

$$\begin{aligned} |\nabla\phi(c^{\varepsilon^\kappa}(\mathbf{x}))| &= \phi_{,c}(c^{\varepsilon^\kappa}(\mathbf{x}))|\nabla c^{\varepsilon^\kappa}(\mathbf{x})| \\ &= 2\sqrt{\Psi(c^{\varepsilon^\kappa}(\mathbf{x}))}|\nabla c^{\varepsilon^\kappa}(\mathbf{x})| \\ &\leq \frac{1}{\varepsilon^\kappa}\Psi(c^{\varepsilon^\kappa}(\mathbf{x})) + \varepsilon^\kappa|\nabla c^{\varepsilon^\kappa}(\mathbf{x})|^2. \end{aligned} \tag{37}$$

This implies that $\nabla\phi(c^{\varepsilon^\kappa})$ is uniformly bounded in $L^1(\Omega)$. A sequence of functions converging almost everywhere and whose weak derivatives are uniformly bounded in $L^1(\Omega)$ also converges in $L^1(\Omega)$. Hence we use (36) to conclude that $\phi(c^{\varepsilon^\kappa}) \to \phi(c)$ in $L^1(\Omega)$. Using the lower semi-continuity property of BV–functions we conclude

$$\begin{aligned} \int_\Omega |\nabla(\phi(c))| &\leq \liminf_{\kappa\to\infty} \int_\Omega 2\sqrt{\Psi(c^{\varepsilon^\kappa}(\mathbf{x}))}|\nabla c^{\varepsilon^\kappa}(\mathbf{x})| \\ &\leq \liminf_{\kappa\to\infty} \int_\Omega \left(\frac{1}{\varepsilon^\kappa}\Psi(c^{\varepsilon^\kappa}(\mathbf{x})) + \varepsilon^\kappa|\nabla c^{\varepsilon^\kappa}(\mathbf{x})|^2\right). \end{aligned}$$

On the other hand with the same arguments as in the proofs of the Theorems 2.1 c) and 2.2 we obtain (see (34))

$$\int_\Omega |\nabla(\phi(c))| = \lim_{\kappa\to\infty} \int_\Omega \left(\frac{1}{\varepsilon^\kappa}\Psi(c^{\varepsilon^\kappa}(\mathbf{x})) + \varepsilon^\kappa|\nabla c^{\varepsilon^\kappa}(\mathbf{x})|^2\right)$$

and

$$\begin{aligned} \int_\Omega |\nabla(\phi(c))| &= \lim_{\kappa\to\infty} \int_\Omega |\nabla(\phi(c^{\varepsilon^\kappa}(\mathbf{x}))| \\ &= \lim_{\kappa\to\infty} \int_\Omega 2\sqrt{\Psi(c^{\varepsilon^\kappa}(\mathbf{x}))}|\nabla c^{\varepsilon^\kappa}(\mathbf{x})| \\ &= \lim_{\kappa\to\infty} \int_\Omega \left(\frac{1}{\varepsilon^\kappa}\Psi(c^{\varepsilon^\kappa}(\mathbf{x})) + \varepsilon^\kappa|\nabla c^{\varepsilon^\kappa}(\mathbf{x})|^2\right). \end{aligned} \tag{38}$$

Hence,

$$\begin{aligned} \lim_{\kappa\to\infty} \int_\Omega \left|\frac{1}{\varepsilon^\kappa}\Psi(c^{\varepsilon^\kappa}) + \varepsilon^\kappa|\nabla c^{\varepsilon^\kappa}|^2 - |\nabla\phi(c^{\varepsilon^\kappa})|\right| &= \\ \lim_{\kappa\to\infty} \int_\Omega \left(\frac{1}{\varepsilon^\kappa}\Psi(c^{\varepsilon^\kappa}) + \varepsilon^\kappa|\nabla c^{\varepsilon^\kappa}|^2 - 2\sqrt{\Psi(c^{\varepsilon^\kappa})}|\nabla c^{\varepsilon^\kappa}|\right) &= 0. \end{aligned}$$

Therefore,

$$\lim_{\kappa\to\infty} \int_\Omega \left(\varepsilon^\kappa|\nabla c^{\varepsilon^\kappa}|^2 + \frac{1}{\varepsilon^\kappa}\Psi(c^{\varepsilon^\kappa})\right) \nabla\cdot\boldsymbol{\xi} = \lim_{\kappa\to\infty} \int_\Omega \nabla\cdot\boldsymbol{\xi}|\nabla\phi(c^{\varepsilon^\kappa})|.$$

We now show that the limit on the right hand side is equal to

$$\int_\Omega \nabla\cdot\boldsymbol{\xi}|\nabla\phi(c)|.$$

In order to conclude this, we need to show

$$|\nabla\phi(c^{\varepsilon^\kappa})| \to |\nabla\phi(c)| \qquad \text{in the sense of Radon measures.} \tag{39}$$

This is a consequence of (38) and can be deduced as follows. For all compact sets $K \subset \Omega$ the lower semi-continuity of $|\nabla\phi(c^{\varepsilon^\kappa})|$ on open sets implies

$$\begin{aligned}\limsup_{\kappa\to\infty} \int_K |\nabla\phi(c^{\varepsilon^\kappa})| &= \limsup_{\kappa\to\infty} \left(\int_\Omega |\nabla\phi(c^{\varepsilon^\kappa})| - \int_{\Omega\setminus K} |\nabla\phi(c^{\varepsilon^\kappa})| \right) \\ &\leq \int_\Omega |\nabla\phi(c)| - \int_{\Omega\setminus K} |\nabla\phi(c)| = \int_K |\nabla\phi(c)|.\end{aligned}$$

Since a sequence of Radon measures converges weak-* if and only if it is lower semi-continuous on open sets and upper semi-continuous on compact sets (see Evans and Gariepy [7]) the convergence (39) follows.

Using that

$$\begin{aligned} 0 &= \lim_{\kappa\to\infty} \int_\Omega \left(\frac{1}{\varepsilon^\kappa}\Psi(c^{\varepsilon^\kappa}) + \varepsilon^\kappa|\nabla c^{\varepsilon^\kappa}|^2 - 2\sqrt{\Psi(c^{\varepsilon^\kappa})}|\nabla c^{\varepsilon^\kappa}| \right) \\ &= \lim_{\kappa\to\infty} \int_\Omega \left(\sqrt{\frac{1}{\varepsilon^\kappa}\Psi(c^{\varepsilon^\kappa})} - \sqrt{\varepsilon^\kappa}|\nabla c^{\varepsilon^\kappa}| \right)^2 \end{aligned} \tag{40}$$

and that $\int_\Omega \left(\varepsilon^\kappa|\nabla c^{\varepsilon^\kappa}|^2 + \frac{1}{\varepsilon^\kappa}\Psi(c^{\varepsilon^\kappa})\right)$ is uniformly bounded, we obtain

$$\begin{aligned} &\lim_{\kappa\to\infty} \int_\Omega \left| \frac{1}{\varepsilon^\kappa}\Psi(c^{\varepsilon^\kappa}) - \varepsilon^\kappa|\nabla c^{\varepsilon^\kappa}|^2 \right| = \\ &= \lim_{\kappa\to\infty} \int_\Omega \left| \left(\sqrt{\frac{1}{\varepsilon^\kappa}\Psi(c^{\varepsilon^\kappa})} - \sqrt{\varepsilon^\kappa}|\nabla c^{\varepsilon^\kappa}| \right) \left(\sqrt{\frac{1}{\varepsilon^\kappa}\Psi(c^{\varepsilon^\kappa})} + \sqrt{\varepsilon^\kappa}|\nabla c^{\varepsilon^\kappa}| \right) \right| \\ &= 0 \end{aligned}$$

which can be interpreted as equipartition of energy. This terminology is chosen because the two summands making up the energy term $\int_\Omega \left(|\varepsilon^\kappa \nabla c^{\varepsilon^\kappa}|^2 + \frac{1}{\varepsilon^\kappa}\Psi(c^{\varepsilon^\kappa})\right)$ are approximately equal for small ε^κ. Identity (40) and the fact that

$$\nabla\phi(c^{\varepsilon^\kappa}) = \phi_{,c}(c^{\varepsilon^\kappa})\nabla c^{\varepsilon^\kappa} = 2\sqrt{\Psi(c^{\varepsilon^\kappa})}\nabla c^{\varepsilon^\kappa} \qquad \text{a.e.}$$

gives

$$\lim_{\kappa\to\infty}\int_\Omega 2\varepsilon^\kappa \nabla c^{\varepsilon^\kappa}\cdot\nabla\boldsymbol{\xi}\nabla c^{\varepsilon^\kappa}$$

$$= \lim_{\kappa\to\infty}\int_\Omega 2\varepsilon^\kappa \frac{\nabla c^{\varepsilon^\kappa}}{|\nabla c^{\varepsilon^\kappa}|}\cdot\nabla\boldsymbol{\xi}\frac{\nabla c^{\varepsilon^\kappa}}{|\nabla c^{\varepsilon^\kappa}|}|\nabla c^{\varepsilon^\kappa}|^2$$

$$= \lim_{\kappa\to\infty}\int_\Omega 2 \frac{\nabla c^{\varepsilon^\kappa}}{|\nabla c^{\varepsilon^\kappa}|}\cdot\nabla\boldsymbol{\xi}\frac{\nabla c^{\varepsilon^\kappa}}{|\nabla c^{\varepsilon^\kappa}|}\sqrt{\Psi(c^{\varepsilon^\kappa})}|\nabla c^{\varepsilon^\kappa}|$$

$$= \lim_{\kappa\to\infty}\int_\Omega \frac{\nabla \phi(c^{\varepsilon^\kappa})}{\nabla\phi(c^{\varepsilon^\kappa})|}\cdot\nabla\boldsymbol{\xi}\frac{\nabla \phi(c^{\varepsilon^\kappa})}{|\nabla\phi(c^{\varepsilon^\kappa})|}|\nabla\phi(c^{\varepsilon^\kappa})|.$$

It remains to pass to the limit in the last identity. Defining the abbreviations $\nu^{\varepsilon^\kappa} := -\frac{\nabla\phi(c^{\varepsilon^\kappa})}{|\nabla\phi(c^{\varepsilon^\kappa})|}$ and $\nu := -\frac{\nabla\phi(c)}{|\nabla\phi(c)|}$ we need to show

$$\lim_{\kappa\to\infty}\int_\Omega \nu^{\varepsilon^\kappa}\cdot\nabla\boldsymbol{\xi}\nu^{\varepsilon^\kappa}|\nabla\phi(c^{\varepsilon^\kappa})| = \int_\Omega \nu\cdot\nabla\boldsymbol{\xi}\nu|\nabla\phi(c)|. \tag{41}$$

To conclude we construct a smooth approximation of the normals ν and show that these approximations are also good approximations for ν^{ε^κ} (see also [23, 14, 4]). In fact since $\phi(c)\in BV(\Omega)$ we obtain the existence of approximative normals $g^\delta\in C_0^\infty(\Omega,\mathbb{R}^n)$ with $|g^\delta|\le 1$ such that

$$\int_\Omega \left(1-g^\delta\cdot\nu\right)|\nabla\phi(c)|\le\delta.$$

Since

$$\nabla\phi(c^{\varepsilon^\kappa})\to\nabla\phi(c) \qquad \text{in the sense of measures}$$

and since (38) holds, we obtain

$$\begin{aligned}\lim_{\kappa\to\infty}\int_\Omega(1-g^\delta\cdot\nu^{\varepsilon^\kappa})|\nabla\phi(c^{\varepsilon^\kappa})| &= \lim_{\kappa\to\infty}\int_\Omega\left(|\nabla\phi(c^{\varepsilon^\kappa})|+g^\delta\cdot\nabla\phi(c^{\varepsilon^\kappa})\right)\\ &= \int_\Omega|\nabla\phi(c)|+\int_\Omega g^\delta\cdot\nabla\phi(c)\\ &= \int_\Omega\left(1-g^\delta\cdot\nu\right)|\nabla\phi(c)|\le\delta.\end{aligned}$$

Using that g^δ and ν^{ε^κ} have norm less or equal to one we compute

$$\left|\nu^{\varepsilon^\kappa}-g^\delta\right|^2 = |\nu^{\varepsilon^\kappa}|^2-2\,g^\delta\nu^{\varepsilon^\kappa}+|g^\delta|^2\le 2\,(1-g^\delta\cdot\nu^{\varepsilon^\kappa}).$$

The last two computations give

$$\lim_{\kappa\to\infty}\int_\Omega\left|\nu^{\varepsilon^\kappa}-g^\delta\right|^2|\nabla\phi(c^{\varepsilon^\kappa})|\le 2\delta.$$

Therefore, we can conclude

$$\limsup_{\kappa\to\infty}\left(\int_\Omega \nu^{\varepsilon^\kappa}\cdot\nabla\boldsymbol{\xi}\nu^{\varepsilon^\kappa}|\nabla\phi(c^{\varepsilon^\kappa})| - \int_\Omega \nu\cdot\nabla\boldsymbol{\xi}\nu|\nabla\phi(c)|\right) =$$
$$\limsup_{\kappa\to\infty}\left(\int_\Omega g^\delta\cdot\nabla\boldsymbol{\xi}g^\delta|\nabla\phi(c^{\varepsilon^\kappa})| - g^\delta\cdot\nabla\boldsymbol{\xi}g^\delta|\nabla\phi(c)|\right) + \mathcal{O}(\delta)$$

which shows (41) because $|\nabla\phi(c^{\varepsilon^\kappa})| \to |\nabla\phi(c)|$ in the sense of measures and since δ can be chosen arbitrarily small.

It remains to prove convergence of the Lagrange multipliers $\lambda^{\varepsilon^\kappa}$. Here we choose a $\boldsymbol{\xi} \in C_0^\infty(\Omega, \mathbb{R}^n)$ with the property that

$$\int_\Omega \nabla\cdot\boldsymbol{\xi}\, c > 0. \tag{42}$$

This is possible since $c \in \{-1, 1\}$ almost everywhere and since $⨍ c \in (-1, 1)$ which implies that $\sigma\int_\Omega|\nabla c| = \int_\Omega|\nabla\phi(c)| \neq 0$. Choosing such a $\boldsymbol{\xi}$ in (35) we can conclude convergence of the Lagrange multipliers from the convergence of the left hand side and the fact that $\lim_{\varepsilon^\kappa\to 0}\int_\Omega(\nabla\cdot\boldsymbol{\xi})\, c^{\varepsilon^\kappa} = \int_\Omega(\nabla\cdot\boldsymbol{\xi})\, c > 0$.

□

5. Discussion

In this paper we were able to show that the minimizers of the Ginzburg–Landau free energy converge to minimizers of a sharp interface partitioning problem when the interfacial thickness tends to zero. Passing to the limit in the corresponding Euler–Lagrange equation was possible in the case of binary systems. We recovered an elastically modified Gibbs–Thomson law in the asymptotic limit. To generalize this result to more components would require a generalization of the result of Luckhaus and Modica [22] to the vector valued situation which is not known yet.

Moreover, we studied convergence of minimizers of the Ginzburg–Landau functional, i.e. convergence of globally stable *stationary* solutions to the Cahn–Hilliard system. To identify the asymptotic limit in the vector valued *evolution* problem with elasticity one would have to combine the matched formal asymptotic expansions of Leo, Lowengrub and Jou [19] (who studied the binary case with elasticity) and Bronsard, Garcke and Stoth [4] (who studied the multi–component case without elasticity). It should be possible to use the methods of Luckhaus and Sturzenhecker [23] and Bronsard, Garcke and Stoth [4] to obtain a conditional existence result for the vector valued sharp interface model with elastic contributions.

It is desirable to have a regularity theory for minimizers of the sharp interface functional. In the two phase case the main question is whether it is possible to show that minimizers are almost area minimizing in the sense of Almgren [1]. Then the general regularity theory for minimal surfaces is applicable [8]. In this context we refer to work of Lin [21], who developed a regularity theory in a case where bulk terms are included by a scalar field. In our case we search a vector valued displacement field which solves an elliptic system with in general discontinuous coefficients. Therefore, it is not clear how to conclude that minimizers of the sharp interface functional are almost area minimizing.

6. Appendix

Definition 6.1. (Measure theoretic supremum) *Let $\mu_1, ..., \mu_M$ be Radon measures defined on Ω. Then we define the measure theoretic supremum $\bigvee_{k=1}^M \mu_k$ on all open sets $D \subset \Omega$ by*

$$\left(\bigvee_{k=1}^M \mu_k\right)(D) := \sup\left\{\sum_{k=1}^N \mu_k(B_k) \mid B_k \subset D, \ \textit{open, pairwise disjoint}\right\}.$$

Remark 6.1. *i) The measure theoretic supremum $\bigvee_{k=1}^M \mu_k$ is the smallest measure that dominates each of the measures μ_k on all Borel sets. ii) Assume the M sequences of measures $\{\mu_k^l\}_{l\in\mathbb{N}}$, $k = 1, ..., M$, fulfill $\mu_k(D) \leq \liminf_{l\to\infty} \mu_k^l(D)$ for all open $D \subset \Omega$. Then it holds*

$$\left(\bigvee_{k=1}^M \mu_k\right)(D) \leq \liminf_{l\to\infty} \left(\bigvee_{k=1}^M \mu_k^l\right)(D)$$

for all open $D \subset \Omega$.

Theorem 6.1. *Let $\mathcal{X}_k \in BV(\Omega)$, $k = 1, ..., M$, define a partition of the open and bounded set Ω, i.e. we require $\mathcal{X}_k \in \{0,1\}$ and $\sum_{k=1}^M \mathcal{X}_k = 1$ a.e.*

Then it holds for all open sets $\Omega' \subset \Omega$

$$|\nabla \mathcal{X}_k|(\Omega') = \sum_{m=1, m\neq k}^M \mathcal{H}^{n-1}\left(\partial^*\Omega_k \cap \partial^*\Omega_m \cap \Omega'\right),$$

$$|\nabla(\mathcal{X}_k + \mathcal{X}_l)|(\Omega')$$
$$= \sum_{m=1, m\neq k,l}^N \left[\mathcal{H}^{n-1}\left(\partial^*\Omega_k \cap \partial^*\Omega_m \cap \Omega'\right) + \mathcal{H}^{n-1}\left(\partial^*\Omega_l \cap \partial^*\Omega_m \cap \Omega'\right)\right],$$

and for $\mu_{kl} := \frac{1}{2}(|\nabla \mathcal{X}_k| + |\nabla \mathcal{X}_l| - |\nabla(\mathcal{X}_k + \mathcal{X}_l)|)$ *it holds*

$$\mu_{kl}(\Omega') = \mathcal{H}^{n-1}\left(\partial^*\Omega_k \cap \partial^*\Omega_l \cap \Omega'\right).$$

In addition, it holds

$$\frac{\nabla \mathcal{X}_k}{|\nabla \mathcal{X}_k|} + \frac{\nabla \mathcal{X}_l}{|\nabla \mathcal{X}_l|} = 0$$

and

$$\frac{\nabla(\mathcal{X}_k + \mathcal{X}_l)}{|\nabla(\mathcal{X}_k + \mathcal{X}_l)|} = 0$$

μ_{kl}*–almost everywhere.*

For a proof see Vol'pert [29].

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