Non-trivial Bounded Harmonic Functions on Cartan-Hadamard Manifolds of Unbounded Curvature

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Introduction

The question under which conditions there exist non-trivial bounded harmonic functions on Riemannian manifolds (M, g) has been of great interest to many mathematicians and still is.

A function $h : M \to \mathbb{R}$ is called *harmonic* if it is a smooth solution of the Laplace equation

$$\Delta_M u = 0, \tag{1}$$

where Δ_M is the Laplace-Beltrami operator on the Riemannian manifold M.

It has been known since 1957, see [Hu], that there are no non-constant bounded harmonic functions on a complete surface, i.e. complete Riemannian manifold of dimension two, with positive curvature. On the other hand, it follows from the Ahlfors-Schwarz Lemma, [Ah], that a simply connected surface with curvature bounded from above by a negative constant is conformally equivalent to the unit disc and consequently possesses non-trivial bounded harmonic functions.

Hence it is a natural question to ask, whether curvature – sectional curvature, to be more precise – is a good criterion in all dimensions to distinguish between Riemannian manifolds which admit non-trivial bounded harmonic functions and so-called *Liouville manifolds*, i.e. Riemannian manifolds where constant functions are the only solutions to equation (1).

As an immediate consequence of the infinitesimal version of the *Harnack inequality* proven by Yau in [Y], Theorem 3", every positive and therefore every bounded harmonic function on a complete Riemannian manifold (of arbitrary dimension) with non-negative (i.e. ≥ 0) curvature is constant.

In case of a *Cartan-Hadamard manifold*, i.e. a complete simply connected Riemannian manifold with non-positive (i.e. ≤ 0) sectional curvature, there is the following conjecture of Greene and Wu, that was (in a slightly relaxed version) also a consideration of Dynkin in [D1]. In the following, $r(x)$ denotes the radial part of $x \in M$:

Conjecture 0.1 (cf. [G-W] and [H-M], p.767).
Let (M, g) be a Cartan-Hadamard manifold with sectional curvatures

$$\mathrm{Sect}_x^M \leq -c r(x)^{-2}$$

for some constant c and all $x \in M$ in the complement of a compact set. Then there are non-constant bounded harmonic functions on M.

Up to now there is no proof known, but there are several affirmative results in this direction. We are going to give a short historical overview of these results and the methods used for their proofs:

For a Cartan-Hadamard manifold M of dimension d there is a natural geometric boundary, the *sphere at infinity* $S_\infty(M)$, such that $M \cup S_\infty(M)$ equipped with the *cone topology* is homeomorphic to the unit ball $B \subset \mathbb{R}^d$ with boundary $\partial B = S^{d-1}$, cf. [E-O´N], [B-O´N] and [Kl]. Using polar coordinates (r, ϑ) for M, a sequence $(r_n, \vartheta_n)_{n\in\mathbb{N}}$ of points in M converges to a point of $S_\infty(M)$ if and only if $r_n \to \infty$ and $\vartheta_n \to: \vartheta$.

Given a continuous function $f : S_\infty(M) \to \mathbb{R}$ the *Dirichlet problem at infinity* is to find a harmonic function $h : M \to \mathbb{R}$ which extends continuously to $S_\infty(M)$ and there coincides with the given function f. The Dirichlet problem at infinity is called *solvable* if this is possible for every such function f. Hence the question whether there exist non-trivial bounded harmonic functions on M is naturally related to the question if the Dirichlet problem at infinity for M is solvable.

In 1983, Anderson proved that the Dirichlet problem at infinity is uniquely solvable for Cartan-Hadamard manifolds with pinched negative curvature, i.e. for complete simply connected Riemannian manifolds M whose sectional curvatures satisfy

$$-a^2 \leq \mathrm{Sect}_x^M \leq -b^2 \text{ for all } x \in M,$$

where $a^2 > b^2 > 0$ are arbitrary constants. See [An], Theorem 3.2. The main idea of the proof was to use *barrier functions* and Perron's method to obtain the desired results. Essentially the same ideas are used by Choi in 1984 to show that in case of a *model manifold* (M, g) the Dirichlet problem at infinity is solvable if the radial curvature is bounded from above by $-A/(r^2 \log(r))$. Hereby a Riemannian manifold (M, g) is called model if it possesses a *pole* $p \in M$ and every linear isometry $\varphi : T_pM \to T_pM$ can be realized as the differential of an isometry $\Phi : M \to M$ with $\Phi(p) = p$, see [C], Theorem 3.6. Choi furthermore provides a criterion, the *convex conic neighbourhood condition*, which yields solvability of the Dirichlet problem at infinity.

Definition 0.2 (cf. [C], Definition 4.6).
Let M be a Cartan-Hadamard manifold. M satisfies the *convex conic neighbourhood condition at* $x \in S_\infty(M)$ if for any $y \in S_\infty(M)$, $y \neq x$, there exist V_x and $V_y \subset M \cup S_\infty(M)$ such that V_x and V_y are disjoint open sets of $M \cup S_\infty(M)$ in terms of the cone topology and $V_x \cap M$ is convex with $\mathscr{C}^2$-boundary. If this condition is satisfied for all $x \in S_\infty(M)$, we say that M satisfies the *convex conic neighbourhood condition.*

Due to [C], Theorem 4.7, the Dirichlet problem at infinity is solvable for a Cartan-Hadamard manifold M with sectional curvature bounded from above by $-c^2$, for $c > 0$, that satisfies the convex conic neighbourhood condition.

Another approach to the Dirichlet problem at infinity is given from probabilistic methods as it is well known that harmonic functions on a Riemannian manifold are characterized by the *mean value property* for geodesic balls, see Theorem 2.9. This property extends under certain conditions to the sphere at infinity, i.e. if the Dirichlet problem at infinity for M is solvable and almost surely $B_\zeta := \lim_{t\to\zeta} B_t$ exists in $S_\infty(M)$, where $(B_t)_{t<\zeta}$ is a Brownian motion on M with lifetime ζ, the unique solution $h : M \to \mathbb{R}$ to the Dirichlet problem at infinity with boundary function f is given as

$$h(x) = \mathbb{E}\left(f \circ B^x_{\zeta^x}\right). \tag{2}$$

Here B^x is a Brownian motion starting in $x \in M$.

On the contrary, considering a Brownian motion on M such that almost surely $\lim_{t\to\zeta^x} B^x_t$ exists in $S_\infty(M)$ for all $x \in M$, one can define the *harmonic measure* μ_x on $S_\infty(M)$, where for a Borel set $U \subset S_\infty(M)$

$$\mu_x(U) := \mathbb{P}\left(B^x_{\zeta^x} \in U\right). \tag{3}$$

For every Borel set $U \subset S_\infty(M)$ the assignment

$$x \mapsto \mu_x(U)$$

defines a bounded harmonic function h_U on M. Using the maximum principle for harmonic functions it follows that h_U is either identically equal to 0 or 1 or takes values in $(0, 1)$. Furthermore, all the harmonic measures μ_x on $S_\infty(M)$ are equivalent. Showing that the harmonic measure class on $S_\infty(M)$ is non-trivial solves the Dirichlet problem at infinity for M as the unique solution for a given continuous boundary function $f : S_\infty(M) \to \mathbb{R}$ is given in the form

$$h(x) = \int_{S_\infty(M)} f(y)\mu_x(dy). \tag{4}$$

This explains why studying the asymptotic behaviour of Brownian motion on M is a convenient method to decide whether the Dirichlet problem for M is solvable or not.

The first results in this direction have been obtained by Prat between 1971 and 1975 (see [P1] and [P2]). He proved that on a Cartan-Hadamard manifold where the sectional curvature is bounded from above by a negative constant $-k^2$, $k > 0$, Brownian motion is transient, i.e. almost surely all paths of the Brownian motion exit from M at the sphere at infinity ([P2], Théorème 1). If in addition the sectional curvatures are bounded from below by a constant $-K^2$, $K > k$, he shows that the angular part $\vartheta(B_t)$ of the Brownian motion almost surely converges when $t \to \zeta$ ([P2], Théorème 2). This is the reason why it makes sense to consider harmonic measures on $S_\infty(M)$ in this situation.

In 1976, Kifer presented a stochastic proof, see [K1], Theorem 2, that on Cartan-Hadamard manifolds with sectional curvature bounded between two negative constants and satisfying a certain additional condition (Condition 1 in [K1]) the Dirichlet problem at infinity can be uniquely solved. However, the proof there was merely given in explicit terms for the two dimensional case. The case of a Cartan-Hadamard manifold (M, g) with pinched curvature without additional conditions and arbitrary dimension was finally treated by Kifer in 1984 in a more accurate version in [K2], Section 3.

Independently of Anderson, in 1983, Sullivan presented a stochastic proof of the fact that on a Cartan-Hadamard manifold with pinched curvature the Dirichlet problem at infinity is uniquely solvable (see [S], Theorem 1). The crucial point has been to prove that the harmonic measure class is non-trivial in this case. He obtains his result as a corollary of the following theorem:

Theorem 0.3 ([S], Theorem 2).
The harmonic measure class on $S_\infty(M) = \partial(M \cup S_\infty(M))$ is positive on each nonvoid open set. In fact, if m_i in M converges to m_∞ in $S_\infty(M)$, then the Poisson hitting measures μ_{m_i} tend weakly to the Dirac mass at m_∞.

In the special case of a Riemannian surface M with negative curvature bounded from above by a negative constant, Kendall gave a stochastic proof that the Dirichlet problem at infinity is uniquely solvable, see [Ke]. He thereby used the fact that every geodesic on the Riemannian surface "joining" two different points on the sphere at infinity divides the surface into two disjoint halfs. Starting in a point x on M, with non-trivial probability Brownian motion will eventually stay inside one of the two halfs up to its lifetime. As this is valid for every geodesic and every starting point x, the non-triviality of the harmonic measure class on $S_\infty(M)$ follows.

Concerning the case of Cartan-Hadamard manifolds of arbitrary dimension several results have been published how the pinched curvature assumption can be relaxed such that still the Dirichlet problem at infinity for M is solvable. To mention just two of them which use probabilistic methods, we refer to [H-M] and [H1]. From there we have the following result which allows that – under certain conditions – one can omit the condition of constant upper or constant lower bound for the sectional curvature:

Theorem 0.4 ([H1], Theorem 1.1 and Theorem 1.2).
Let (M, g) be a Cartan-Hadamard manifold. The Dirichlet problem at infinity for M is solvable if one of the following conditions is satisfied:

i) There exists a positive constant a and a positive and nonincreasing function h with $\int_0^\infty rh(r)dr < \infty$ such that

$$-h(r(x))^2 e^{2ar(x)} \le \mathrm{Ric}_x^M \ \text{ and } \ \mathrm{Sect}_x^M \le -a^2 \text{ for all } x \in M.$$

ii) There exist positive constants r_0, $\alpha > 2$ and $\beta < \alpha - 2$ such that

$$-r(x)^{2\beta} \le \mathrm{Ric}_x^M \ \text{ and } \ \mathrm{Sect}_x^M \le -\frac{\alpha(\alpha-1)}{r(x)^2}$$

for all $x \in M$ with $r(x) \ge r_0$.

It was unknown for quite a long time whether only the existence of a constant negative upper bound for the sectional curvature of a Cartan-Hadamard manifold of dimension $d \ge 2$ could be a sufficient condition for the solvability of the Dirichlet problem at infinity as it was already proven to be in dimension 2. But in 1994, Ancona constructed a Riemannian manifold with sectional curvatures bounded from above by a negative constant such that the Dirichlet problem at infinity for M is not solvable, see [A1] and Chapter 4 for further details. He shows that on his manifold M Brownian motion almost surely exits from M at a single point ∞_M on the sphere at infinity and so evidently the Dirichlet problem at infinity for M is not solvable. Independently of Ancona and using purely analytic methods, there is the work of Borbély who also provides an example of a Riemannian manifold M such that the Dirichlet problem at infinity is not solvable ([B], Theorem 1 and Theorem 2). Unlike Ancona, Borbély shows that his manifold possesses non-trivial bounded harmonic functions.

The present work is now concerned with several questions: Does the manifold of Ancona also possess non-trivial bounded harmonic functions? Does Brownian motion on the manifold of Borbély behave similar to Brownian motion on Ancona's manifold? Then the two examples can be considered to be essentially the "same", at least from the probabilistic point of view. Is there a stochastic representation of the harmonic functions obtained on the manifold of Borbély and are there further harmonic functions besides the ones Borbély already constructed? Furthermore, is there a way to use the ideas of Ancona and Borbély to construct further examples of Riemannian manifolds for which the Dirichlet problem at infinity is not solvable whereas there exist non-trivial bounded harmonic functions? And finally, do the non-trivial bounded harmonic functions, which we construct with probabilistic methods provide full information about the space of all positive (bounded respectively) harmonic functions on M?

We are going to discuss and answer these and related questions in the following chapters. Writing $h(M)$ for the (Banach) space of bounded harmonic functions on a Cartan-Hadamard manifold M, we have the following result of Anderson:

Theorem 0.5 ([An], Theorem 4.3).
Let (M,g) be a Cartan-Hadamard manifold of dimension d, whose sectional curvatures satisfy $-a^2 \le \mathrm{Sect}_x^M \le -b^2$ for all $x \in M$. Then the linear mapping

$$\begin{aligned} P : \mathrm{L}^\infty(S_\infty(M), \mu) &\to h(M), \\ f \mapsto P(f), \quad P(f)(x) &:= \int_{S_\infty(M)} f \, d\mu_x \end{aligned} \tag{5}$$

is a norm-nonincreasing isomorphism onto $h(M)$.

In the situation of Ancona's and – as we are going to show in Chapter 3 – Borbély's manifold Brownian motion almost surely exits from the manifold at a single point of the sphere at infinity independent of the starting point x. Hence all harmonic measures μ_x on M are trivial. On the other hand, we are going to show in Chapter 3, Theorem 3.16 and Theorem 3.27 as well as in Chapter 4, Theorem 4.4, that these manifolds possess non-trivial bounded harmonic functions. From that it is clear that the mapping $P : \mathrm{L}^\infty(S_\infty(M), \mu) \to h(M)$ fails to be surjective as every harmonic function of the form $P(f)$ is necessarily constant.

As for the considered manifolds the Dirichlet problem at infinity is obviously unsolvable we are going to use another "criterion" for the proof that these manifolds possess non-trivial bounded harmonic functions: Due to Dynkin, cf. [D1], Chapter XII, there is a one-to-one correspondence between the space $h(M)$ of all bounded harmonic functions on M and the set of all $\mathscr{A}_{\mathrm{inv}}$-measurable functions up to equivalence. Hereby $\mathscr{A}_{\mathrm{inv}}$ denotes the shift-invariant σ-field on the space $\mathscr{C}(\mathbb{R}_+, \widetilde{M})$ of continuous paths with values in the Alexandroff compactification $\widetilde{M}$ of M.

Given a bounded harmonic function h on M, the function $H : \mathscr{C}(\mathbb{R}_+, \widetilde{M}) \to \mathbb{R}$ with

$$H := \lim_{t\to\zeta}(h \circ \mathrm{pr}_t), \tag{6}$$

where ζ is the lifetime of a Brownian motion B on M, defines an $\mathscr{A}_{\mathrm{inv}}$-measurable function. For H bounded and $\mathscr{A}_{\mathrm{inv}}$-measurable the function $h : M \to \mathbb{R}$ defined by

$$h(x) := \mathbb{E}^x H \tag{7}$$

is harmonic on M. These two mappings are inverse to each other. See Section 2.3, Lemma 2.15, for the proof.

Consequently a Riemannian manifold M does not possess non-trivial bounded harmonic functions if and only if the shift-invariant σ-field $\mathscr{A}_{\mathrm{inv}}$ is trivial, i.e. $\mathbb{P}^x(\mathscr{A}_{\mathrm{inv}}) \subset \{0, 1\}$ for all $x \in M$. This is furthermore equivalent to the fact that the exit sets $U \subset \widetilde{M}$ are all trivial for the Brownian motion on M, i.e. $\mathbb{P}\{B_t^x \in U \text{ eventually}\} \subset \{0, 1\}$ for all $x \in M$. See Theorem 2.18. Therefore the main task is to find non-trivial exit sets for the Brownian motion, i.e. to find (non-trivial) possibilities to distinguish between Brownian paths. The question under which conditions the shift-invariant σ-field for a Riemannian manifold M is trivial and, in case of non-triviality, the question whether one can find $\mathscr{A}_{\mathrm{inv}}$-measurable functions that generate the shift-invariant σ-field, has been considered in many situations. However, in general these questions are not easy to answer. For further information we refer to [Cr1], [Cr2], [F-O], [Cr-O-R].

This work is organized as follows:

Chapter 1 recalls some important definitions and facts concerning differential geometry. In Section 1.3 we give a short idea how the sphere at infinity $S_\infty(M)$ of a Cartan-Hadamard manifold can be obtained and state the Dirichlet problem at infinity. Most of the presented theorems can be found for example in [B-O´N], [E-O´N], [Kl] and [Jo].

Brownian motion on Riemannian manifolds is introduced in Chapter 2, Section 2.1. The Lévy-Characterization and the Strong Markov property for Brownian motion are summarized in Section 2.2, Theorem 2.6 and Theorem 2.7.

Most part of Chapter 2.1 is devoted to the relations between bounded harmonic functions on a Riemannian manifold M and the asymptotic behaviour of Brownian motion on M: in Section 2.3, Theorem 2.9, we recall some well known facts concerning the heat semigroup $(P_t)_{t\in\mathbb{R}_+}$ generated by the Laplacian Δ_M and the mean value property of harmonic functions. Theorem 2.11 is a stochastic criterion for the solvability of the Dirichlet problem at infinity and provides the stochastic representation formula

$$u_f(x) = \mathbb{E}\left(f \circ B_{\zeta^x}^x\right)$$

for the unique solution to the Dirichlet problem at infinity, given a continuous boundary function f. The σ-field of shift-invariant events is defined as well as the set $m\mathscr{C}_{\text{inv}}/_\sim$ of all bounded $\mathscr{A}_{\text{inv}}$-measurable functions up to equivalence. The definition and the proof of the isomorphism $h(M) \to m\mathscr{C}_{\text{inv}}/_\sim$ that we already mentioned above, is contained in Lemma 2.15. From this the *Liouville criterion* for Riemannian manifolds, see Theorem 2.18, is an immediate consequence. We add some facts about transience and recurrence of Brownian motion in Theorem 2.20 and Theorem 2.21. The final section of this chapter introduces the *Martin boundary* and the *minimal Martin boundary* of a Riemannian manifold. The Martin boundary M^* compactifizes the Riemannian manifold M and has the property that every positive harmonic function on M can be obtained as an integral over the Martin boundary M^*. We write down some results under which conditions the Martin boundary is known to be homeomorphic to the sphere at infinity. It turns out that for the manifold we consider in Chapter 3 the Martin boundary has to be at least of dimension 2. We give a short explanation of how this conclusion can be obtained from the results we are going to prove in Chapter 3, Theorem 3.16 and Theorem 3.27.

Chapter 3 is the main part of this work. Herein we first define the Riemannian manifold M we are interested in. The construction of the Riemannian manifold M is the same as Borbély presented in [B]: we consider M as the warped product

$$M := (H \cup L) \times_g S^1,$$

where L is a unit-speed geodesic in the hyperbolic space $\mathbb{H}^2$ of constant sectional curvature -1 and H is one component of $\mathbb{H}^2 \setminus L$. The Riemannian metric γ on M is the warped product metric of the hyperbolic metric on H coupled with the (induced) Euclidean metric on S^1 via the function $g : H \cup L \to \mathbb{R}_+$

$$ds_M^2 = ds_{\mathbb{H}^2}^2 + g \cdot ds_{S^1}^2.$$

By identifying points (ℓ, α_1) and (ℓ, α_2) with $\ell \in L$ and $\alpha_1, \alpha_2 \in S^1$ and choosing the metric "near" L equal to the hyperbolic metric of the three dimensional hyperbolic space $\mathbb{H}^3$ the manifold M becomes complete, simply connected and rotationally symmetric with respect to the axis L. In Section 3.1 we carry out the necessary calculations and list some conditions the function g has to satisfy in order to provide a Riemannian metric on M for which the sectional curvatures are bounded from above by a negative constant. Section 3.2 is dedicated to the description of the sphere at infinity $S_\infty(M)$ of M. In Section 3.3, Lemma 3.1, we write down all the properties the function g has to satisfy such that the Riemannian manifold M becomes an example of a manifold for which the Dirichlet problem at infinity is not solvable. We add some comments that clarify which of the required properties are necessary for the construction of the function g and which of them influence the asymptotic behaviour of the Brownian paths. As the construction of the function g is described in detail in [B] we only sketch the construction in Section 3.4 and refer to Borbély for the detailed proofs.

The probabilistic consideration of the manifold M starts in Section 3.5. We write down the defining stochastic differential equations for the Brownian motion B on M, where we use the component processes R, S and A of B with respect to the global coordinate system $\{(r, s, \alpha) |\, r \in \mathbb{R}_+, s \in \mathbb{R}, \alpha \in [0, 2\pi)\}$ for M.

The non-solvability of the Dirichlet problem at infinity for the Riemannian manifold M is an immediate consequence of Theorem 3.4:

Theorem 3.4. *i) For the Brownian motion B on the Riemannian manifold (M, γ) constructed above the following statement almost surely holds:*

$$\lim_{t \to \zeta} B_t = L(+\infty),$$

independently of the starting point B_0. In particular the Dirichlet problem at infinity for M is not solvable.

ii) There is a submanifold $\overline{S}$ of $S_\infty(M)$ of codimension 1 with the following property: Given a bounded continuous function $f : \overline{S} \to \mathbb{R}$ we can find a non-trivial bounded harmonic function $h : M \to \mathbb{R}$ which has f as limiting boundary function, i.e. $\lim_{p \to \tilde{p}} h(p) = f(\tilde{p})$ where $p \to \tilde{p} \in \overline{S}$. Writing pr_3 for the map $M \to \mathbb{R}$, $(r, s, \alpha) \mapsto \alpha$, almost surely $\lim_{t \to \zeta}(\mathrm{pr}_3 \circ B_t) \equiv \lim_{t \to \zeta} A_t$ exists and takes values in the submanifold $\overline{S}$.

Further, we have for any point $p = (r, s, \alpha) \in M$:

$$h(p) = \mathbb{E}^p \left(f \circ \lim_{t \to \zeta}(\mathrm{pr}_3 \circ B_t) \right) = \mathbb{E}^p \left(f \circ \lim_{t \to \zeta} A_t \right).$$

From part i) it is clear that the asymptotic behaviour of the Brownian motion on M is the same as in the case of the manifold of Ancona. Moreover, it turns out (see Section 3.6, Corollary 3.24, i) of Theorem 3.22) that the component R of the Brownian motion B almost surely goes to infinity when $t \to \zeta$. In combination with Lemma 3.15 and Lemma 3.18 we obtain the proof that the components R, S and A of the Brownian motion on M behave the same way as the corresponding components T, X and Y on the manifold of Ancona, see Theorem 4.4. Moreover, we get a stochastic representation of non-trivial bounded harmonic functions on M. These harmonic functions are different to those Borbély constructs in [B], Theorem 2. Furthermore it is a remarkable fact that in contrary to the manifold of Ancona, where the Brownian motion almost surely has infinite lifetime, we can show that on the manifold M the lifetime ζ of the Brownian motion is almost surely finite, see Corollary 3.24, ii).
The proof of Theorem 3.4 is split up into several lemmata, which in combination yield the statements i) and ii), cf. Lemmata 3.7, 3.12, 3.15, 3.16 and 3.18.

In Section 3.6 we construct further harmonic functions on M. We define a time change of the Brownian motion such that the drift of the component process R of B becomes equal to t, i.e. the time changed component $\widetilde{R}$ behaves similar to the deterministic curve $\mathbb{R}_+ \to \mathbb{R}_+$, $t \mapsto r_0 + t$. We show in Theorem 3.22 that the process

$$Z_t := \widetilde{S}_t - \int_0^{\widetilde{R}_t} q(r)dr$$

almost surely converges in $\mathbb{R}$ when $t \to \widetilde{\zeta}$. We furthermore give a stochastic proof of the fact that

$$u(Z_t) := \max\left\{0, \frac{2}{\pi}\arctan\left(\frac{1}{2}\left(\widetilde{S}_t - \int_0^{\widetilde{R}_t} q(u)du - a\right)\right)\right\}$$

defines a submartingale, cf. Proposition 3.26. Using the submartingale property of $u(Z_t)$, we show in Theorem 3.27 that the random variable

$$Z_{\widetilde{\zeta}} := \lim_{t\to\widetilde{\zeta}} Z_t$$

is a non-trivial shift-invariant random variable, which gives further non-trivial bounded harmonic functions on M. A short sketch of the construction of the function q is given in Section 3.7. This function has been already used by Borbely to prove Theorem 2 of [B].

We finish this chapter with a geometric interpretation how the asymptotic behaviour of the Brownian motion can be "visualized" via a change of coordinates of the manifold M. It turns out that the non-trivial shift-invariant random variable $A_\zeta := \lim_{t\to\zeta} A_t$ can be interpreted as one dimensional angle which indicates from which direction the projection of the Brownian path onto the sphere at infinity attains the point $L(\infty)$. The non-trivial shift-invariant random variable $Z_{\widetilde{\zeta}}$ indicates along which surface of rotation $C_{s_0} \times S^1$ inside of M the Brownian paths finally exit the manifold M. Thereby C_{s_0} is the trajectory starting in $(0, s_0) \in \mathbb{R}_+ \times \mathbb{R}$ of the vector field (3.38), given as

$$V := \frac{\partial}{\partial r} + q(r)\frac{\partial}{\partial s}.$$

This kind of asymptotic behaviour is not known in the "usual" case of a Cartan-Hadamard manifold of pinched negative curvature. There the angular part $\vartheta(B)$ of B carries all information and its limit random variable Θ generates the shift-invariant σ-field of B. Hence all non-trivial information to distinguish between Brownian paths can be obtained by looking at the angular projection of B onto $S_\infty(M)$, which is not sufficient if we consider our manifold.

Chapter 4 introduces the manifold of Ancona published in [A1] which is another example of a Riemannian manifold where the Dirichlet problem at infinity is not solvable. For this manifold the asymptotic behaviour of Brownian motion has been discussed in [A1]. However, Ancona did not deal with the question whether there exist non-trivial bounded harmonic functions on this manifold. In Section 4.1 we give a short summary how the Riemannian manifold M is defined including the conditions which the coupling function $h : \mathbb{R}^2 \to \mathbb{R}_+$ in the Riemannian metric

$$ds_\gamma^2 = dt^2 + e^{2t}dx^2 + h(x,t)^2dy^2$$

for M has to satisfy such that all sectional curvatures of M are bounded from above by a negative constant. Section 4.2 illustrates how the function h is constructed on "stripes" of the form $\mathbb{R} \times [t_i, t_j] \subset \mathbb{R}^2$. The main part of Theorem 4.4 in Section 4.3 is already proven in [A1]. We add in part iii) of this theorem the observation that the shift-invariant random variable $Y_\infty := \lim_{s\to\infty} Y_s$ is non-trivial and therefore provides non-trivial bounded harmonic functions on M. From that it is clear that the manifold M is a non-Liouville manifold. We conclude this chapter with a geometric discussion whether there possibly may exist further non-trivial bounded harmonic functions on M besides the ones we get from Y_∞. In Section 4.4 we extend the given example to arbitrary dimension $d \geq 3$, which was already initiated by Ancona. The most important observation is presented in Theorem 4.7, iii). Due to these results for every continuous bounded function $f : \mathbb{R}^{d-2} \to \mathbb{R}$ we obtain a non-trivial bounded harmonic function $u : M \to \mathbb{R}$ of the form

$$u(m) = \mathbb{E}^m \left[f \circ \left(\lim_{s \to \infty} (Y_s, Z_{1s}, \dots, Z_{ms}) \right) \right],$$

where $Y, Z_1, \dots Z_m$, $m = d - 3$, are components of the Brownian motion on M.

Section 4.5 introduces a second way to define the metric function h, see Lemma 4.10. Using this function instead of the function h of Section 4.2 to define the Riemannian metric, we obtain a three dimensional Riemannian manifold M with the following property: Almost surely the Brownian motion exits from M along the hypersurface $\{x = 0\} \subset M$. However we can still find non-trivial bounded harmonic functions on M, see Theorem 4.12. We illustrate this asymptotic behaviour in Remark 4.13: when looking at the projection of the Brownian motion onto the sphere at infinity one observes that the Brownian motion finally attains the great circle $\{x = 0\}$ on $S_\infty(M)$. The non-trivial information, which yields non-trivial bounded harmonic functions on M, is which point of the circle $\{x = 0\}$ the Brownian paths finally attain.

With this in mind, we modify the function h with the help of a dense sequence $(a_i)_{i \in \mathbb{N}}$ in $\mathbb{R}$ such that the (projections of the) Brownian paths $B_s(\omega)$ are again and again close to the circles $\{x = a_i\}$, $i \in \mathbb{N}$, on the sphere at infinity, i.e. oscillate for $s \to \infty$. In particular, Brownian motion B_s does not converge for $s \to \infty$. However, there are still non-trivial bounded harmonic functions on M. This is proven in Theorem 4.16.

In the last Section 4.6, we collect the obtained results of Section 4.3 and 4.5 to obtain for every dimension $d \geq 3$ a Riemannian manifold M such that the asymptotic behaviour of the Brownian motion can be "predetermined" whereas there exist non-trivial bounded harmonic functions on M.

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Chapter 1

Some Background on Differential Geometry

1.1. Fundamentals and Definitions

We start with some general definitions and facts about Riemannian manifolds, which are used in the following chapters. If not explicitly mentioned, we consider a Riemannian manifold (M, g) of dimension $d \geq 2$ (where the main chapters only treat the case $d = 3$) with Riemannian metric g. The $\mathscr{C}^\infty(M)$-module of all $\mathscr{C}^\infty$-sections $M \to TM$ is denoted by $\Gamma(TM)$. A chart (h, U) for M yields a local basis $\partial_1, \partial_2, \ldots, \partial_d$ for $\Gamma(TM|_U)$, where $\partial_i := (dh)^{-1}(e_i)$, $i = 1, \ldots, d$, and $e_1, e_2, \ldots, e_d$ the standard basis of $\mathbb{R}^d$.

A *linear connection* on TM is a $\mathbb{R}$-linear mapping

$$\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$$

which satisfies the Leibniz rule

$$\nabla(fX) = df \otimes X + f\nabla X, \text{ for all } X \in \Gamma(TM) \text{ and } f \in \mathscr{C}^\infty(M).$$

Using the canonical identification $\Gamma(T^*M \otimes TM) = \mathrm{Hom}_{\mathscr{C}^\infty(M)}(\Gamma(TM), \Gamma(TM))$, a linear connection on M is a $\mathbb{R}$-bilinear mapping

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM), \; (Y, X) \mapsto \nabla_Y X := \nabla(Y, X)$$

which is $\mathscr{C}^\infty(M)$-linear in the first argument and acts as a derivation in the second one. A linear connection on TM is called *metric* or *Riemannian* if it respects the Riemannian metric g on TM in the sense that

$$d(g(X, Y)) = g(\nabla X, Y) + g(X, \nabla Y) \text{ for all } X, Y \in \Gamma(TM).$$

The *Levi-Civita connection* on M is the uniquely determined Riemannian connection on TM which is additionally *torsionfree*, i.e. satisfies

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] = 0 \text{ for all } X, Y \in \Gamma(TM),$$

with the *Lie bracket* $[X, Y] \in \Gamma(TM)$, given as derivation $[X, Y]f := X(Yf) - Y(Xf)$ for $f \in \mathscr{C}^\infty(M)$.

With respect to a chart (h, U) of M the Levi-Civita connection ∇ is uniquely determined by its *Christoffel symbols* $(\Gamma_{ij}^k)_{i,j,k=1,\dots,d} \in \mathscr{C}^\infty(U)$, where $\nabla_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k \partial_k$. In terms of the Riemannian metric g on M one can compute the Christoffel symbols as

$$\Gamma_{ij}^k = \frac{1}{2}\sum_\ell g^{\ell k}[\partial_i g_{j\ell} + \partial_j g_{\ell i} - \partial_\ell g_{ij}],$$

where $g_{ij} := \langle \partial_i, \partial_j \rangle$ and $g^{ij} \in \mathscr{C}^\infty(U)$ with $\sum_j g^{ij} g_{jk} = \delta_{ik}$.

A differentiable curve $\gamma : I \to M$, where $I \subset \mathbb{R}$, is called *geodesic curve* or *geodesic* if $\dot{\gamma} \in \Gamma(\gamma^*(TM))$ is *parallel along* γ with respect to ∇, i.e. if $\nabla_D \dot{\gamma} = 0$ where D is the canonical vector field on I and ∇ the induced connection on $\gamma^*(TM)$. In local coordinates (h, U) a geodesic satisfies the equation

$$\ddot{x}^i(t) + \sum_{jk} \Gamma_{jk}^i(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0, \text{ for } i = 1, \dots, d, \tag{1.1}$$

where $\dot{x}^i(t) := \frac{d}{dt}x^i(\gamma(t))$, etc. and $x^i(\gamma(t))$ is the i-th component of $\gamma(t)$ in the coordinates given by h.

As we are interested in the existence of non-trivial bounded harmonic functions on M as well as the asymptotic behaviour of Brownian motion on M – what is well known to be essentially the same question, see Chapter 2 – we briefly recall the definition of the *Laplace-Beltrami operator* on the Riemannian manifold M: for $f \in \mathscr{C}^\infty(M)$ we have the *Hessian* of f as the section $\nabla df \in \Gamma(T^*M \otimes T^*M)$ with ∇ the induced connection on $\Gamma(T^*M)$. The operator $\Delta : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ given as

$$\Delta f := \operatorname{trace} \nabla df,$$

that means $\Delta f(x) = \sum_i (\nabla df)(e_i, e_i)$ with $e_1, \dots, e_d$ an orthonormal basis for T_xM, is called the *Laplace-Beltrami operator* on M. In local coordinates (h, U) one has the explicit formula

$$\Delta f | U = \sum_{ij} g^{ij}\left(\partial_i\partial_j f - \sum_k \Gamma_{ij}^k \partial_k f\right). \tag{1.2}$$

A function $h \in \mathscr{C}^2(M)$ is called *harmonic*, if $\Delta h \equiv 0$; it is called *subharmonic*, if $\Delta h \geq 0$ and *superharmonic*, if $\Delta h \leq 0$.

In what follows, we restrict ourselves to Riemannian manifolds with strictly negative curvature – the reason for that will be clear in the next chapter. We give a short definition of the concept of *curvature* of a Riemannian manifold: the *curvature tensor* $R \in \Gamma\left(T^*M^{\otimes 3} \otimes TM\right)$ of M is defined as

$$R(X, Y, Z) \equiv R(X, Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

for tangent vectors $X, Y, Z \in \Gamma(TM)$.

Let $x \in M$. For a plane $\operatorname{Lin}\{X, Y\} \subset T_xM$ spanned by two tangent vectors $X, Y \in T_xM$ the sectional curvature is given as

$$\mathrm{Sect}_x^M(\mathrm{Lin}\{X,Y\}) = \frac{\langle R(X,Y)Y,X\rangle}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}.$$

A Riemannian manifold M has *strictly negative sectional curvature* if there is a constant $k \in \mathbb{R}\setminus\{0\}$ such that for all $x \in M$ and all $X, Y \in T_xM$ one has

$$\mathrm{Sect}_x^M(\mathrm{Lin}\{X,Y\}) \leq -k^2.$$

1.2. Cartan-Hadamard Manifolds

We recall some facts about geodesics on Riemannian manifolds before we restrict ourselves to manifolds with negative sectional curvature.

Theorem 1.1. *Let M be a Riemannian manifold, $p \in M$ and $v \in T_pM$. Then there exists an interval $[0,\varepsilon] \subset \mathbb{R}$ and a unique geodesic*

$$\gamma : [0,\varepsilon] \to M$$

with $\gamma(0) = p$ and $\dot\gamma(0) = v$. Furthermore the geodesic γ depends smoothly on p and v.

Proof. See for example [Jo], Theorem 1.4.2. □

For $v \in T_pM$ let γ_v denote the unique geodesic with $\gamma(0) = p$ and $\dot\gamma(0) = v$. As a geodesic curve is locally given as the solution of the differential equation (1.1) it is a consequence of the theory of ordinary differential equations that

$$\mathscr{O}(M) := \{v \in TM : \gamma_v \text{ is defined for } t = 1\}$$

is an open subset of TM with $0 \in \mathscr{O}(M)$, where $0 \in \Gamma(TM)$ denotes the zero section.

The *exponential map* of (M, g) is the mapping

$$\begin{aligned} \exp : \mathscr{O}(M) &\to M \times M, \\ v &\mapsto (\pi(v), \gamma_v(1)). \end{aligned} \tag{1.3}$$

It is well known that exp maps an open neighbourhood of $0 \in \Gamma(TM)$ diffeomorphically onto an open neighbourhood of the diagonal $\{(p,p) : p \in M\} \subset M \times M$.

A Riemannian manifold M is called *metrically complete* if for every $p \in M$ and every $v \in T_pM$ the unique geodesic γ_v is defined for all $t \in \mathbb{R}$. Obviously metrical completeness is equivalent to $\mathscr{O}(M) = TM$.
Metrical completeness is also equivalent to completeness of M as a metric space. This is part of the theorem of *Hopf-Rinow*:

Theorem 1.2 (Theorem of Hopf-Rinow).
Let M be a connected Riemannian manifold. The following statements are equivalent:

i) M is complete as a metric space, i.e. every Cauchy sequence in M converges.

ii) There exists $p \in M$ for which $\exp_p = \exp|_{T_pM}$ is defined for all $v \in T_pM$.

iii) M is metrically complete, i.e. $\mathscr{O}(M) = TM$.

Furthermore each of the statements i) - iii) implies that any two points $p, q \in M$ can be joined by a geodesic of length $d(p,q)$, i.e. by a minimizing geodesic.

Proof. See for example [Jo], p.26 ff. □

Definition 1.3. A simply connected metrical complete Riemannian manifold M of dimension $d \geq 2$ with $\mathrm{Sect}^M \leq 0$ is called a *Cartan-Hadamard manifold*.

On a Cartan-Hadamard manifold we can introduce a global coordinate system due to the following theorem:

Theorem 1.4 (Cartan-Hadamard).
Every Cartan-Hadamard manifold M of dimension $d \geq 2$ is diffeomorphic to $\mathbb{R}^d$. More precisely: for a Cartan-Hadamard manifold M the mapping $\exp_p : T_pM \to M$ is a diffeomorphism for every $p \in M$.

Proof. See for example [Ha-Th], p.505 f. □

A point $p \in M$ is called a *pole* if $\exp_p : T_pM \to M$ is a diffeomorphism. Hence every point of a Cartan-Hadamard manifold is a pole.

If we fix a pole $0 \in M$ we can identify M with $\mathbb{R}^d \cong \mathbb{R}_+ \times S^{d-1}$ and therefore introduce a system of polar coordinates (r, ϑ) on M, where

$$r(p) := d(0,p) = \| \cdot \| \circ (\exp_0)^{-1}(p)$$

and $\vartheta(p)$ is the unit vector at 0 tangent to the minimizing geodesic that connects 0 and p. We refer to these coordinate representation when later using the expressions *radial part* and *angular part* of the Brownian motion on M.

1.3. The Sphere at Infinity and the Dirichlet Problem at Infinity

Definition 1.5. Let M be a Cartan-Hadamard manifold. Two unit-speed geodesics $\gamma_1, \gamma_2 : \mathbb{R} \to M$ are called *asymptotic* if there is a constant $C > 0$ such that

$$d(\gamma_1(t), \gamma_2(t)) \leq C \text{ for all } t \geq 0.$$

Remark 1.6. (Some facts about asymptotic geodesics, cf. [E-O´N])

i) Orientation-preserving reparametrizations of asymptotic geodesics γ_1 and γ_2 yield again asymptotic geodesics.

ii) Asymptoticity provides an equivalence relation on the set of all geodesics in M. The equivalence classes are called *asymptotic classes of geodesics*. For a geodesic γ let $\gamma(\infty)$ denote the asymptotic class of γ. Denote by $\gamma(-\infty)$ the asymptotic class of γ with the reverse parametrization.

iii) If two asymptotic geodesics γ_1 and γ_2 have a common point they are the same up to parametrization.

iv) For a geodesic γ and a point $p \in M$ there exists (up to parametrization) a unique geodesic γ_1 such that $\gamma_1(0) = p$ and γ_1 is asymptotic to γ.

We can now give the definition of the sphere at infinity of M:

Definition 1.7. Let M be a Cartan-Hadamard manifold. The *boundary at infinity* or the *sphere at infinity* of M is the set

$$S_\infty(M) := \{\gamma(\infty) | \, \gamma : \mathbb{R} \to M \text{ is geodesic}\}. \tag{1.4}$$

Denote $\overline{M} := M \cup S_\infty(M)$.

Let $p, q \in M$ where M is a Cartan-Hadamard manifold. Then it is easy to prove that there is a unique geodesic (up to parametrization) joining p and q. Denote this geodesic by γ_{pq}. For $p \in M$ and $x \in S_\infty(M)$ let $\gamma_{px} : \mathbb{R} \to M$ denote the geodesic through p with $\gamma(\infty) = x$ if it exists and is unique. If finally $x, y \in S_\infty(M)$ we write γ_{xy} for the geodesic with $\gamma_{xy}(-\infty) = x$ and $\gamma_{xy}(\infty) = y$ if it exists and is unique. The following theorem guarantees that under certain conditions for the sectional curvature Sect^M of M we do not have to worry about existence and uniqueness of geodesics as defined above:

Theorem 1.8. *Let M be a Cartan-Hadamard manifold with strictly negative sectional curvature, i.e. $\mathrm{Sect}^M \le -k^2 < 0$ for a constant $k \in \mathbb{R}$, $k \neq 0$. Then for any two points $x, y \in \overline{M}$ there is a unique geodesic γ_{xy} up to parametrization.*

Proof. See [E-O´N] – there a proof is given under much weaker assumptions on the curvature, but for our applications the assumptions made in Theorem 1.8 are sufficient. □

On $\overline{M}$ one can define a topology with respect to which $\overline{M}$ is homeomorphic to the closed ball $\overline{B} \subset \mathbb{R}^n$, and $S_\infty(M)$ is homeomorphic to the boundary sphere $S^{d-1} = \partial B$. This justifies the name sphere at infinity for $S_\infty(M)$:

Fix a pole 0 in the Cartan-Hadamard manifold M. For $v, w \in T_0M$ we denote by $\sphericalangle_0(v, w)$ the angle between v and w in the vector space T_0M. For points $x, y \in \overline{M}$ the *angle* $\sphericalangle_0(x, y)$ is defined as $\sphericalangle_0(\dot\gamma_{0x}(0), \dot\gamma_{0y}(0))$. Let finally $S(T_0M) := \{v \in T_0M : \|v\| = 1\}$.

Definition 1.9 (cf. [C], p.695)). Let M be a Cartan-Hadamard manifold and $0 \in M$ fixed. For every $v \in S(T_0M)$ and $\delta > 0$ the *cone in T_0M of opening angle δ and axis v* is the set:

$$C(v, \delta) := \{x \in \overline{M} | \; \sphericalangle_0(v, \dot\gamma_{0x}(0)) < \delta\}. \tag{1.5}$$

For $r > 0$ we call

$$T(v, \delta, r) := C(v, \delta) \setminus \{p \in M | \, d(0, p) \le r\}$$

the *truncated cone of radius r in $\overline{M}$.*

The proof of the following theorem can again be found in [E-O´N]:

Theorem 1.10. *Let M be a Cartan-Hadamard manifold of dimension d and $0 \in M$ fixed. The set of all $T(v, \delta, r)$ for $v \in S(T_0M), \delta > 0$ and $r > 0$, together with the open balls $B_p(r) = \{q \in M | \, d(p, q) < r\}$ for all $p \in M$ and $r > 0$ defines a local basis of a topology on $\overline{M}$. It is called the cone-topology.*
The cone-topology does not depend on the choice of $0 \in M$. Equipped with this topology $\overline{M}$ is homeomorphic to the closed unit ball $\overline{B} \subset \mathbb{R}^d$, M to the open ball B and $S_\infty(M)$ to the boundary sphere $S^{d-1} = \partial\overline{B}$. Furthermore the topology induced on M by the cone-topology is the original topology of M.

It is useful to remark that a sequence of points $(p_n)_{n\in\mathbb{N}} \in M$ converges to a boundary point $\vartheta_0 \in S_\infty(M)$ if and only if $r(p_n) \to \infty$ and $\vartheta(p_n) \to \vartheta_0$, see [B-O´N] and [H1].

With the sphere at infinity we now have the notion of a boundary of a Cartan-Hadamard manifold M hence we can state the *Dirichlet problem at infinity* for M.

Definition 1.11 (The Dirichlet problem at infinity).
Let M be a Cartan-Hadamard manifold. The *Dirichlet problem at infinity* is to find for a given continuous function $f : S_\infty(M) \to \mathbb{R}$ a continuous function $h : \overline{M} \to \mathbb{R}$ such that h is harmonic in M and $h|_{S_\infty(M)} = f$. Here continuity refers to the cone topology.

Definition 1.12 (Solvability of the Dirichlet problem at infinity).
Let M be a Cartan-Hadamard manifold. We say that *the Dirichlet problem at infinity for M is solvable* if for every continuous function $f : S_\infty(M) \to \mathbb{R}$ there is a function $h : \overline{M} \to \mathbb{R}$ as in the definition above.

Chapter 2

Brownian Motion on Riemannian Manifolds

2.1. Definitions

Let $(\Omega; \mathscr{F}; \mathbb{P}; (\mathscr{F}_t)_{t\in\mathbb{R}_+})$ be a filtered probability space satisfying the usual conditions, i.e. $(\Omega; \mathscr{F}; \mathbb{P})$ is a complete probability space and $(\mathscr{F}_t)$ is an increasing right continuous family of sub-σ-fields of $\mathscr{F}$ with $\bigcup_{t\in\mathbb{R}_+} \mathscr{F}_t = \mathscr{F}$. Let $\widetilde{M} := M \cup c(M)$ be the Alexandroff-compactification of M.

For an adapted stochastic process $(X_t)_{t\in\mathbb{R}_+} : \Omega \to M$ with values in M the *lifetime of* X is a stopping time ζ such that $X_t \in M$ for all $t < \zeta$ and $\mathbb{P}$-a.s. $X_t \to c(M)$ in $\widetilde{M}$ for $t \nearrow \zeta$ on $\{\zeta < \infty\}$. By convention, $X_t := c(M)$ on $\{t \geq \zeta\}$.

Definition 2.1. Let (M, g) be a Riemannian manifold and $(X_t)_{t\in\mathbb{R}_+} : \Omega \to M$ an adapted continuous stochastic process with lifetime ζ.
X is called *semimartingale on* M *with lifetime* ζ if for every $f \in \mathscr{C}^\infty(M)$ the real valued process $f \circ X$ is a continuous real semimartingale with lifetime ζ.

Theorem 2.2 ([Ha-Th], Theorem 7.57).
Let (M, g) *be a Riemannian manifold and let* $(X_t)_{t\in\mathbb{R}_+} : \Omega \to M$ *be an* M*-valued semimartingale. There is a unique linear mapping* $\Gamma(T^*M \otimes T^*M) \to \mathscr{A}, b \mapsto \int b(dX, dX)$ *such that for all* $f, g \in \mathscr{C}^\infty(M)$*:*

i) $df \otimes dg \mapsto [f \circ X, g \circ X]$,

ii) $f \cdot b \mapsto \int (f \circ X) b(dX, dX)$,

where by definition $b(dX, dX) := d(\int b(dX, dX))$*. The expression* $\int b(dX, dX)$ *is called integral of* b *along* X *or* b*-quadratic variation of* X*.*

Definition 2.3. Let (M, g) be a Riemannian manifold and X a semimartingale with values in M.

$$[X, X] := \int g(dX, dX) = \int \langle dX, dX \rangle \in \mathscr{A}$$

is called *Riemannian quadratic variation* of X.

Definition 2.4. Let (M, g) be a Riemannian manifold and $(X_t)_{t\in\mathbb{R}_+}$ an M-valued semimartingale. X is called *martingale* or more precisely ∇*-martingale* if for every $f \in \mathscr{C}^\infty(M)$

$$d(f \circ X) \overset{m}{=} \frac{1}{2}(\nabla df)(dX, dX).$$

Herein $\overset{m}{=}$ means equal up to differentials of local martingales and ∇ is the Levi-Civita connection.

Definition 2.5. Let (M, g) be a Riemannian manifold and $(B_t)_{t\in\mathbb{R}_+} : \Omega \to M$ an adapted continuous stochastic process with lifetime ζ.
B is called *Brownian motion on M with lifetime ζ* if for every $f \in \mathscr{C}^\infty(M)$ the real-valued process

$$f \circ B - \frac{1}{2}\int \Delta f \circ B\, dt$$

is a local martingale with lifetime ζ. We denote by $\mathrm{BM}(M, g)$ the set of all Brownian motions on M and by $\mathrm{BM}_p(M, g)$ all Brownian motions on M starting in p, i.e. with $\mathbb{P} \circ B_0^{-1} = \delta_p$.

2.2. Important Properties of Brownian Motion

We list some facts about Brownian motion on Riemannian manifolds that are used in the following chapters. Most of the proofs can be found for example in [Ha-Th], Chapter 7.

Theorem 2.6 (Lévy-Characterization of M-valued Brownian motions).
Let (M, g) be a Riemannian manifold and $(X_t)_{t\in\mathbb{R}_+} : \Omega \to M$ a semimartingale with values in M. Then the following statements are equivalent:

i) X is a Brownian motion on M.

ii) X is a ∇-martingale with $[f \circ X, f \circ X] = \int \|\operatorname{grad} f \circ X\|^2\, dt$ for every $f \in \mathscr{C}^\infty(M)$.

For the Riemannian quadratic variation $[X, X] = \int g(dX, dX)$ of a Brownian motion X on M it follows that $\int_0^t g(dX, dX) := \left(\int g(dX, dX)\right)_t = (\dim M) \cdot t$.

From the definition of Brownian motion it follows that Brownian motion on M can be considered as an M-valued Markov process with the Laplacian Δ_M as infinitesimal generator (in a suitable way). With this in mind it is not surprising that the Brownian motion on M has the Strong Markov Property:

Theorem 2.7 (Strong Markov Property of Brownian motion).
Let (M, g) be a Riemannian manifold. For $x \in M$ let B^x denote a Brownian motion in M starting in x, i.e. with $\mathbb{P} \circ B_0^{-1} = \delta_x$ and lifetime ζ. Writing $B_t := c(M)$ for $t \geq \zeta$ one can extend B to a process with infinite lifetime and values in $\widetilde{M}$.

Let $H : \mathscr{C}(\mathbb{R}_+, \widetilde{M}) \to \mathbb{R}_+$ be bounded and measurable. Then for every Brownian motion B on M and every stopping time τ the following holds:

$$\mathbb{E}^{\mathscr{F}_\tau}\left(H \circ B_{\tau+\bullet}\right) = \mathbb{E}\left(H \circ B^x_\bullet\right)|_{x=B_\tau} \text{ almost surely on } \{\tau < \infty\}. \tag{2.1}$$

2.3. Brownian Motion and Harmonic Functions

The aim of this work is to use Brownian motion on Cartan-Hadamard manifolds M to study the Dirichlet problem at infinity for M, in particular questions of solvability. Moreover, Brownian motion provides a tool to decide about the existence of non-trivial bounded harmonic functions on Riemannian manifolds even if the Dirichlet problem at infinity for M is not solvable.

To understand the interplay between Brownian motion on M, solvability of the Dirichlet problem at infinity for M and so-called *Liouville properties* of M, we give a short summary of the most important facts:
Let (M,g) be a complete Riemannian manifold and $(P_t)_{t\in\mathbb{R}_+}$ the minimal semigroup generated by the Laplacian $\frac{1}{2}\Delta_M$ on M. That means $(P_t)_{t\in\mathbb{R}_+}$ is a family of linear operators

$$P_t : b(M) \to b(M), \quad t \geq 0$$

on the space $b(M)$ of bounded measurable functions $f : M \to \mathbb{R}$ with the following properties:

i) $P_sP_tf = P_{s+t}f$ for $f \in b(M)$ and $0 \leq s,t$.

ii) $P_tf \geq 0$ for $f \in b(M)$, $f \geq 0$, and $P_t1 \leq 1$.

iii) $(P_tf)(x) - f(x) = \frac{1}{2}\int_0^t (P_s\Delta_M f)(x)ds$ for every test function $f : M \to \mathbb{R}$, i.e. $\mathscr{C}^\infty$-function f with compact support.

iv) $(P_t)_{t\in\mathbb{R}_+}$ is minimal, i.e. for every family $(Q_t)_{t\in\mathbb{R}_+}$ of positive linear operators on $b(M)$ satisfying the properties (i),(ii) and (iii) one has

$$P_tf \leq Q_tf, \quad \text{for } 0 \leq f \in b(M), t \geq 0.$$

One has the following connection between Brownian motion on M and the semigroup $(P_t)_{t\in\mathbb{R}_+}$:

Theorem 2.8 (cf. for example [Ha-Th] Theorem 7.252).
Let (M,g) be a complete Riemannian manifold and $(P_t)_{t\in\mathbb{R}_+}$ the minimal subgroup generated by $\frac{1}{2}\Delta_M$ as above. Then

$$(P_tf)(x) = \mathbb{E}\left((f \circ B_t^x)1_{\{t<\zeta^x\}}\right), \quad f \in b(M), x \in M, \tag{2.2}$$

where $B^x \in \mathrm{BM}_x(M,g)$ is a Brownian motion on M starting in $x \in M$ with lifetime ζ^x.

Theorem 2.9 (cf. [Ha-Th] and [Th]).
Let (M,g) be a Riemannian manifold, B a Brownian motion on M with lifetime ζ and $h : M \to \mathbb{R}$ bounded and measurable. The following statements are equivalent:

i) h is a harmonic function, i.e. $h \in \mathscr{C}^\infty(M)$ and $\Delta_M h \equiv 0$.

ii) $\frac{P_th-h}{t} \to 0$ pointwise for $t \to 0$.

iii) $h(x) = \mathbb{E}(h \circ B^x_\tau)$ *for every* $x \in M$ *and every stopping time* $0 \le \tau < \zeta^x$ *almost surely.*

iv) h *has the mean value property, i.e. for every* $x_0 \in M$ *and every sufficiently small geodesic ball* $B_\varepsilon(x_0) \subset M$ *with radius* ε *centered at* x_0 *one has*

$$h(x) = \mathbb{E}(h \circ B^x_{\tau^x}) \qquad \textit{for } x \in B_\varepsilon(x_0),$$

where $\tau^x := \inf\{t \ge 0 : B^x_t \notin B_\varepsilon(x_0)\}$ *denotes the first exit time of* B^x *from* $B_\varepsilon(x_0)$.

v) $h \circ B$ *is a local martingale for* $B \in \mathrm{BM}(M, g)$.

As an immediate corollary of the theorem above one can derive the maximum principle for harmonic functions:

Corollary 2.10 (Maximum principle for harmonic functions). *Let* (M, g) *be a connected Riemannian manifold and* $h : M \to \mathbb{R}$ *harmonic. Let* $m := \sup_{x\in M} h(x) \in \overline{\mathbb{R}}$. *If there exists* $x_0 \in M$ *with* $h(x_0) = m$, *then* h *is constant.*

Proof. (cf. [Ha-Th] p.534)
From the mean value property it follows that $M_0 := \{x \in M : h(x) = m\}$ is open and obviously closed as h is continuous. Hence $M_0 = M$. □

As we have seen in Section 1.4 a Cartan-Hadamard manifold M of dimension d together with the sphere at infinity $S_\infty(M)$ and equipped with the cone-topology is homeomorphic to the closed unit ball $\overline{B} \subset \mathbb{R}^n$. The theorem above shows that a harmonic function $h : M \to \mathbb{R}$ is determined inside a geodesic ball $B_\varepsilon(x_0)$ by the values of h on the boundary $\partial B_\varepsilon(x_0)$ where the Brownian motion B^x exits the geodesic ball. Under certain conditions this is the same for the "ball" $\overline{M} = M \cup S_\infty(M)$, as shown in the following theorem. From this we get a first idea how Brownian motion can be used to solve the Dirichlet-problem at infinity:

Theorem 2.11. *Let* (M, g) *be a Cartan-Hadamard manifold and* B *a Brownian motion on* M *with lifetime* ζ. *Suppose that for any* $x \in M$ *one has*

$$\mathbb{P}\left\{B^x_{\zeta^x} := \lim_{t\nearrow\zeta^x} B^x_t \textit{ exists}\right\} = 1,$$

where $\lim_{t\nearrow\zeta^x} B^x_t$ *is understood in the topology of* $\overline{M}$, *and that for any* $\theta_0 \in S_\infty(M)$ *and any neighbourhood* U *of* $\theta_0 \in S_\infty(M)$

$$\lim_{x\to\theta_0} \mathbb{P}\left\{B^x_{\zeta^x} \in U\right\} = 1.$$

Then the Dirichlet problem at infinity for M *is solvable.*
More precisely: For any $f \in \mathscr{C}(S_\infty(M))$ *the function*

$$u_f(x) := \mathbb{E}\left(f \circ B^x_{\zeta^x}\right)$$

is the unique solution to the Dirichlet problem at infinity with boundary function f.

Proof. The proof given here can be found for example in [H1], page 3:
From the Strong Markov Property for Brownian motion on M one has for every relatively compact open set D and $x \in D$:

$$u_f(B^x_{\tau_D}) = \mathbb{E}\left(f \circ B^y_\zeta\right)|_{y=B^x_{\tau_D}} = \mathbb{E}^{\mathscr{F}_{\tau_D}}\left(f \circ B^x_{\zeta+\tau_D}\right) = \mathbb{E}^{\mathscr{F}_{\tau_D}}\left(f \circ B^x_\zeta\right),$$

where $\tau_D = \inf\{t \geq 0 : B^x_t \notin D\}$ is the first exit time from D. Hence

$$\mathbb{E}\left(u_f \circ B^x_{\tau_D}\right) = \mathbb{E}\left[\mathbb{E}^{\mathscr{F}_{\tau_D}}\left(f \circ B^x_\zeta\right)\right] = \mathbb{E}\left[f \circ B^x_\zeta\right] = u_f(x),$$

which proves the harmonicity of u_f due to Theorem 2.9.

To prove that u_f has f as boundary function we choose for given $\theta_0 \in S_\infty(M)$ and $\varepsilon > 0$ a neighbourhood U of θ_0 in $S_\infty(M)$ such that $|f(\theta) - f(\theta_0)| < \varepsilon$ for all $\theta \in U$. Then for $x \in M$:

$$|u_f(x) - f(\theta_0)| \leq \mathbb{E}|f(B^x_{\zeta^x}) - f(\theta_0)| \leq \varepsilon \mathbb{P}\{B^x_{\zeta^x} \in U\} + 2\|f\|_\infty \mathbb{P}\{B^x_{\zeta^x} \notin U\}.$$

With $x \to \theta_0$ it follows that $\limsup_{x\to\theta_0} |u_f(x) - f(\theta_0)| < \varepsilon$, so in fact u_f has f as boundary function on $S_\infty(M)$ and therefore is a solution to the Dirichlet problem at infinity for M.

To prove that u_f is the unique solution to the Dirichlet problem assume that u is another solution with boundary function f. Let $(D_n)_{n\in\mathbb{N}}$ be an exhaustion of M consisting of relatively compact sets D_n. Then $\left(u(B_{t\wedge\tau_{D_n}})\right)_{t\in\mathbb{R}_+}$ is a uniformly bounded martingale for every $n \in \mathbb{N}$ and therefore

$$u(x) = \mathbb{E}\left(u(B^x_{t\wedge\tau_{D_n}})\right) \qquad \text{for every } n \in \mathbb{N}.$$

With $t \nearrow \zeta$ and then $n \nearrow \infty$ we get

$$u(x) = \mathbb{E}\left(u(B^x_{\zeta^x})\right) = \mathbb{E}\left(f \circ B^x_{\zeta^x}\right) = u_f(x),$$

which proves the uniqueness. □

Looking at the proof of the theorem above we can easily derive the following corollary concerning *non-solvability of the Dirichlet problem*:

Corollary 2.12. *Let (M,g) be a Cartan-Hadamard manifold with a point $\theta_0 \in S_\infty(M)$ that satisfies the following property: for the Brownian motion B on M with lifetime ζ one has*

$$\mathbb{P}\left\{\lim_{t\to\zeta^x} B^x_t = \theta_0\right\} = 1 \text{ for every } x \in M.$$

Then the Dirichlet problem at infinity for M is not solvable.

Proof. Let $f : S_\infty(M) \to \mathbb{R}$ be a continuous and non-constant function. Suppose there is a continuous solution $h : \overline{M} \to \mathbb{R}$ with h harmonic on M and $h|_{S_\infty(M)} = f$. Then again for an exhaustion $(D_n)_{n\in\mathbb{N}}$ of M as above the process $\left(h(B_{t\wedge\tau_{D_n}})\right)_{t\in\mathbb{R}_+}$ is a uniformly bounded martingale for every $n \in \mathbb{N}$. Letting $t \nearrow \zeta$ and $n \nearrow \infty$, we have for every $x \in M$

$$h(x) = \mathbb{E}\left(h(B^x_{\zeta^x})\right) = \mathbb{E}\left(f(\theta_0)\right) = f(\theta_0).$$

This means that h is necessarily constant and equal to $f(\theta_0)$. For $\theta \in S_\infty(M)$ with $f(\theta) \neq f(\theta_0)$ therefore

$$\lim_{x\to\theta} u(x) = \lim_{x\to\theta} f(\theta_0) = f(\theta_0) \neq f(\theta).$$

Hence u cannot have f as boundary function which is a contradiction. □

As we are going to see in the following Chapters 3 and 4, two of the known examples of Cartan-Hadamard manifolds where the Dirichlet problem at infinity is not solvable have the above property that the Brownian motion almost surely exits the manifold for $t \to$"lifetime" at a single point θ_0 of the sphere at infinity. However we are going to show that in these cases the considered Cartan-Hadamard manifold is not of *Liouville type* but possesses non-trivial bounded harmonic functions. Of course it is not possible to extend these functions continuously to the sphere at infinity.

As we have seen above, Cartan-Hadamard manifolds where the Dirichlet problem at infinity is solvable provide a large family of non-trivial bounded harmonic functions $h : M \to \mathbb{R}$ as for every continuous $f : S_\infty(M) \to \mathbb{R}$ the solution $h : \overline{M} \to \mathbb{R}$ to the Dirichlet problem at infinity with boundary function f is harmonic on M.

It is a more delicate question how to find non-trivial bounded harmonic functions on a Cartan-Hadamard manifold (M, g) if the Dirichlet problem at infinity is not solvable. Using the explicit formula (1.2)

$$\Delta_M = \sum_{ij} g^{ij} \left(\partial_i\partial_j - \sum_k \Gamma^k_{ij}\partial_k\right)$$

for the global chart (r, ϑ) of polar coordinates for M one would have to find a solution $h : M \to \mathbb{R}$ of the second order partial differential equation

$$\sum_{ij} g^{ij} \left(\partial_i\partial_j h - \sum_k \Gamma^k_{ij}\partial_k h\right) = 0.$$

But in general it is not that easy to decide whether there exist solutions and – in case of their existence – to compute the harmonic functions explicitely.

A classical method to prove existence of non-trivial bounded harmonic functions is to use *Perron's principle*: Given a subharmonic function $\varphi : M \to \mathbb{R}$ and a superharmonic function $\psi : M \to \mathbb{R}$ with $\varphi \leq \psi$, then there is a harmonic function $h : M \to \mathbb{R}$ with $\varphi \leq h \leq \psi$. Hence the problem is reduced to finding a pair of a subharmonic function φ and a superharmonic function ψ on M with the above property and such that in addition there is no constant function between φ and ψ. This is the method Borbély uses in [B] to prove the existence of non-trivial bounded harmonic functions on his manifold, cf. [B], page 234.

However, Brownian motion on M provides another approach to the construction of bounded harmonic functions on M:

Let $\mathscr{C} := \mathscr{C}(\mathbb{R}_+; \widetilde{M})$ denote the space of all continuous paths $\alpha : \mathbb{R}_+ \to \widetilde{M}$ on the Alexandroff compactification $\widetilde{M}$ of M, with the property that $\alpha(t) = c(M)$ for all $t \geq \inf\{t \in \mathbb{R}_+ : \alpha(t) = c(M)\}$. For every $t \geq 0$ there is the *canonical projection*

$$\begin{aligned} \mathrm{pr}_t : \mathscr{C} &\to \widetilde{M} \\ \alpha &\mapsto \alpha(t) \end{aligned} \tag{2.3}$$

and the *time shift*

$$\begin{aligned} \vartheta_t : \mathscr{C} &\to \mathscr{C} \\ \alpha &\mapsto \alpha(t + \cdot). \end{aligned} \tag{2.4}$$

The canonical projections generate the σ-field $\mathscr{A} := \sigma\{\mathrm{pr}_t : t \in \mathbb{R}_+\}$ on $\mathscr{C}$ which is filtrated by the sub-σ-fields $\mathscr{A}_t := \sigma\{\mathrm{pr}_s : s \leq t\}$ for $t \in \mathbb{R}_+$.

Definition 2.13. Let $(\mathscr{C}, \mathscr{A})$ be the space of continuous paths $\alpha : \mathbb{R}_+ \to \widetilde{M}$ as above together with the σ-field $\mathscr{A}$ generated by the canonical projections. The *shift-invariant σ-field* or *σ-field of shift-invariant events* is the σ-field

$$\begin{aligned} \mathscr{A}_{\mathrm{inv}} &:= \{A \in \mathscr{A} \mid \vartheta_t A = A \text{ for all } t \geq 0\} \\ &= \sigma\{H : \mathscr{C} \to \mathbb{R} \mid H \text{ is } \mathscr{A}\text{-measurable and } H \circ \vartheta_t = H \text{ for all } t \geq 0\}. \end{aligned} \tag{2.5}$$

The *σ-field of terminal events* is the σ-field

$$\mathscr{A}_\infty := \bigcap_{t>0} \sigma\{\mathrm{pr}_s : s \geq t\}. \tag{2.6}$$

Remark 2.14.

i) $\mathscr{A}_{\mathrm{inv}} \subset \mathscr{A}_\infty$.

ii) A measurable function $H : \mathscr{C} \to \mathbb{R}$ is $\mathscr{A}_\infty$-measurable if and only if there is a family $(g_t)_{t \in \mathbb{R}_+}$ of measurable functions $g_t : \mathscr{C} \to \mathbb{R}$ with $H = g_t \circ \vartheta_t$ for all $t \in \mathbb{R}_+$.

iii) A measurable function $H : \mathscr{C} \to \mathbb{R}$ is $\mathscr{A}_{\mathrm{inv}}$-measurable if and only if there is a measurable function $g : \mathscr{C} \to \mathbb{R}$ with $H = g \circ \vartheta_t$ for all $t \geq 0$.

For $B^x \in \mathrm{BM}_x(M, g)$ a Brownian motion starting in $x \in M$ with lifetime ζ^x let

$$\Phi_x : \Omega \to \mathscr{C},\ \omega \mapsto B^x_\bullet(\omega)$$

be the corresponding path map. From that we get a probability measure $\mathbb{P}^x := \mathbb{P} \circ (\Phi_x)^{-1}$ on $(\mathscr{C}, \mathscr{A})$. By $\mathbb{E}^x$ we denote the corresponding expectation.

Let $m\mathscr{C}$ be the space of all bounded measurable functions $H : \mathscr{C} \to \mathbb{R}$. On $m\mathscr{C}$ define an equivalence relation "$\sim$" as follows:

$$H \sim G :\Longleftrightarrow H = G \quad \mathbb{P}^x\text{-almost surely for every } x \in M.$$

Let $m\mathscr{C}_{\mathrm{inv}}/_\sim$ denote the set of all bounded $\mathscr{A}_{\mathrm{inv}}$-measurable functions $H : \mathscr{C} \to \mathbb{R}$ up to equivalence and $h(M)$ the set of all bounded harmonic functions on M.

Lemma 2.15. *There is a linear isomorphism*

$$\begin{aligned} h(M) &\to m\mathscr{C}_{\mathrm{inv}}/_{\sim} \\ h &\mapsto H := \lim_{t\to\zeta} (h \circ \mathrm{pr}_t)\,. \end{aligned} \tag{2.7}$$

Proof. The mapping above is well-defined because for every $x \in M$:

$$\mathbb{P}^x\{\lim_{t\to\zeta^x} (h\circ \mathrm{pr}_t) \text{ exists}\} = \mathbb{P}\{\lim_{t\to\zeta^x} (h\circ \mathrm{pr}_t \circ \Phi_x) \text{ exists}\} = \mathbb{P}\{\lim_{t\to\zeta^x} (h\circ B_t^x) \text{ exists}\} = 1.$$

The last equation follows from the martingale convergence theorem as $(h(B_t^x))_{t<\zeta^x}$ is a bounded martingale. Obviously the mapping is linear.
Inverse to the above mapping is the mapping:

$$m\mathscr{C}_{\mathrm{inv}}/_{\sim} \to h(M), \qquad H \mapsto h, \text{ where } h(x) := \mathbb{E}^x H.$$

The so defined function h is obviously harmonic: Let $0 \le \tau < \zeta^x$ be a stopping time and $x \in M$. Then using the Strong Markov property and the fact that H is $\mathbb{P}^x$-almost surely $\mathscr{A}_{\mathrm{inv}}$-measurable we get:

$$\mathbb{E}\left(h\circ B_\tau^x\right) = \mathbb{E}\left(\mathbb{E}^{B_\tau^x} H\right) = \mathbb{E}\left(\mathbb{E}\left(H\circ B_\bullet^y\right)|_{y=B_\tau^x}\right) = \mathbb{E}\left(H\circ B_\bullet^x\right) = h(x),$$

which proves harmonicity of h.
For every $x \in M$ we further have $\mathbb{P}^x$-almost surely $\lim_{t\to\zeta^x} (\mathbb{E}^\bullet \circ H)\circ \mathrm{pr}_t = H$. This follows from once again using the Strong Markov property:

$$\begin{aligned} \lim_{t\to\zeta^x} (\mathbb{E}^\bullet H)\circ B_t^x &= \lim_{t\to\zeta^x} \left(\mathbb{E}(H\circ B_\bullet^y)|_{y=B_t^x}\right) = \lim_{t\to\zeta^x} \left(\mathbb{E}^{\mathscr{F}_t}(H\circ B_{t+\bullet}^x)\right) = \\ &= \lim_{t\to\zeta^x} \left(\mathbb{E}^{\mathscr{F}_t}(H\circ B_\bullet^x)\right) = H\circ B_\bullet^x. \end{aligned}$$

The other equation $h = \mathbb{E}^\bullet\left(\lim_{t\to\zeta}(h\circ \mathrm{pr}_t)\right)$ for $h \in h(M)$ harmonic, follows from

$$\mathbb{E}^x\left(\lim_{t\to\zeta}(h\circ \mathrm{pr}_t)\right) = \int \lim_{t\to\zeta}(h\circ B_t^x)\, d\mathbb{P} = \lim_{t\to\zeta}\int h\circ B_t^x\, d\mathbb{P} = \lim_{t\to\zeta}\mathbb{E}\left(h\circ B_t^x\right) = h(x).$$

□

Remark 2.16. Considering the preimage $\Phi^{-1}(\mathscr{A}_{\mathrm{inv}})$ of $\mathscr{A}_{\mathrm{inv}}$ under the path map Φ corresponding to the Brownian motion on M, we have a description of $\Phi^{-1}(\mathscr{A}_{\mathrm{inv}})$ as the union of all *exit sets* of the Brownian motion B.
To be more precise, let $U \subset \widetilde{M}$ be open and define

$$\mathscr{H}_U := \{\alpha \in \mathscr{C}(\mathbb{R}_+; \widetilde{M}) : \alpha(t) \in U \text{ eventually}\}.$$

Then we have up to sets of measure 0:

$$\Phi^{-1}(\mathscr{A}_{\text{inv}}) = \{B_\bullet^{-1}(\mathscr{H}_U) : U \text{ open in } \widetilde{M}\} = \{\,\{B_t \in U \text{ eventually}\} : U \subset \widetilde{M} \text{ open}\}.$$

For the proof note that

$$\{B_t \in U \text{ eventually}\} = \{\omega \in \Omega : \exists\, t(\omega) > 0 \text{ such that } B_t(\omega) \in U \text{ for all } t \geq t(\omega)\}.$$

Obviously $\Phi(\{B_t \in U \text{ eventually}\}) \in \mathscr{A}_{\text{inv}}$. This proves "$\supset$".
For the other inclusion "$\subset$" let $h(x) := \mathbb{E}^x(1_A) = \mathbb{P}^x(A)$ for a shift-invariant set A. The so defined function h is harmonic on M. For a number $a > 0$ consider the open set

$$U := \{x : h(x) > a\} \subset \widetilde{M}.$$

Then $\{B_t \in U \text{ eventually}\} = \{\omega : h(B_t(\omega)) > a \text{ eventually}\}$. As h is a bounded harmonic function, $(h \circ B_t)_{t<\zeta}$ is a bounded martingale. By the Strong Markov property and martingale convergence we obtain:

$$h \circ B_t^x = \mathbb{E}\left(1_A \circ B_\bullet^y\right)|_{y=B_t^x} = \mathbb{E}^{\mathscr{F}_t}\left(1_A \circ B_{t+\bullet}^x\right) = \mathbb{E}^{\mathscr{F}_t}\left(1_{\Phi^{-1}(A)}\right) \to 1_{\Phi^{-1}(A)}.$$

Definition 2.17. Let (M, g) be a connected Riemannian manifold. M is called a *Liouville manifold* if every bounded harmonic function h on M is constant.

Using the lemma above and Remark 2.16 we have the following connection between the Liouville property for a connected Riemannian manifold M, the σ-field $\mathscr{A}_{\text{inv}}$ of shift-invariant events and exit sets of the Brownian motion B on M:

Theorem 2.18 (cf. [Th], p37ff).
Let (M, g) be a connected Riemannian manifold. Then

$$\begin{aligned}(M,g) \textit{ is a Liouville manifold } &\Longleftrightarrow \mathbb{P}^x\left(\mathscr{A}_{\text{inv}}\right) \subset \{0,1\} \textit{ for every } x \in M \\ &\Longleftrightarrow \mathbb{P}\{B_t^x \in U \textit{ eventually}\} \in \{0,1\} \textit{ for every open} \\ &\qquad U \subset \widetilde{M} \textit{ and every } x \in M.\end{aligned} \tag{2.8}$$

Remark 2.19. It remains to note that

$$\mathbb{P}^x\left(\mathscr{A}_{\text{inv}}\right) \subset \{0,1\} \text{ for some } x \in M \Longleftrightarrow \mathbb{P}^x\left(\mathscr{A}_{\text{inv}}\right) \subset \{0,1\} \text{ for every } x \in M.$$

This follows from the fact that for $A \in \mathscr{A}_{\text{inv}}$ the function $h : M \to \mathbb{R}$ with $h(x) := \mathbb{E}^x(1_A) = \mathbb{P}^x(A)$ is harmonic on M. If then $h(x) = 0$ or $h(x) = 1$ for some $x \in M$ we have $h \equiv 0$ or $h \equiv 1$ from the maximum principle.
The same equivalence holds for $\mathbb{P}\{B_t^x \in U \text{ eventually}\}$.

From the theorem above it follows that there exist non-trivial bounded harmonic functions on a Riemannian manifold whenever there are non-trivial exit sets for the Brownian motion, i.e. if there is an open set $U \subset \widetilde{M}$ such that $0 < \mathbb{P}^x\{B_t \in U \text{ eventually}\} < 1$. Considering Brownian paths, this means that whenever there is a non-trivial way to distinguish between Brownian paths $B_t(\omega)$ for $t \to \zeta$ then the Riemannian manifold M fails to have the Liouville property. This is why we look for non-trivial shift invariant random variables and search for non-trivial exit sets for the Brownian motion. With this in mind we can show the existence of non-trivial bounded harmonic functions under suitable conditions even on a Riemannian manifold M for that the Dirichlet problem at infinity is not solvable. We are going to use this fact for example in Chapter 3, Theorem 3.16 and Chapter 4, Theorem 4.4.

To finish this section about the relations between Brownian motion and non-trivial bounded harmonic functions on Riemannian manifolds we just add two little theorems concerning the asymptotic behaviour of Brownian motion:

Theorem 2.20 (cf. [Ha-Th], Theorem 7.260).
Let (M, g) be a Riemannian manifold. Then the Brownian motion B on M is either recurrent or transient, i.e. for every Brownian motion B with lifetime ζ on M we have either

i) $\liminf_{t \to \zeta} d(B_0, B_t) = 0$ *almost surely or*

ii) $\liminf_{t \to \zeta} d(B_0, B_t) = \infty$ *almost surely.*

On Cartan-Hadamard manifolds (M, g), where in addition all sectional curvatures Sect_M are bounded from above by a negative constant $-k^2$, $k > 0$, almost surely all Brownian paths reach the sphere at infinity $S_\infty(M)$ as $t \to \zeta$. This is part of the last theorem:

Theorem 2.21. *Let (M, g) be a Cartan-Hadamard manifold with sectional curvatures bounded from above by a negative constant $-k^2$, $k > 0$. Then Brownian motion B on M is transient. In particular we have for the radial part $r(B)$ of B:*

$$\lim_{t \to \zeta} r(B_t) = \infty.$$

Proof. The proof is an easy consequence of the the fact that Brownian motion is transient on the hyperbolic space of constant curvature $-k^2$. In this case of a so-called *model manifold* transience of Brownian motion can be decided by using a certain finiteness criterion for the coupling function $f(r)$ appearing in the polar coordinate representation of the Riemannian metric. See for example [Ha-Th], Theorem 7.262. Using a standard comparison theorem for Brownian motion on Riemannian manifolds ([Ha-Th], Theorem 7.265) almost surely transience of Brownian motion on M follows. See also [P2], Theorem 1, for a more detailed discussion about the radial part of the Brownian motion. □

2.4. The Martin Boundary

In Chapter 1 we defined the sphere at infinity $S_\infty(M)$ of a Cartan-Hadamard manifold M. We also mentioned that equipped with the cone-topology $\overline{M} = M \cup S_\infty(M)$ is homeomorphic to the closed ball $\overline{B} \subset \mathbb{R}^d$ and $S_\infty(M)$ corresponds to the boundary ∂B of B. However, the boundary $S_\infty(M)$ is a pure "geometric" boundary construction for M as it depends on the asymptotic classes of geodesic rays.

In this section we are going to introduce another boundary for the manifold M which is related to analytic properties of M: the so-called *Martin boundary.* It is constructed in potential theoretic terms and has a natural probabilistic interpretation as is seen in Section 2.3 and the end of this section.

The construction of the Martin boundary relies on the existence of a *Green function* $G(x, y)$ on M defined for all $x, y \in M$ with $x \neq y$ and $\mathscr{C}^2$ outside the diagonal $\{(x,x) : x \in M\}$. A Green function on M can be obtained as a minimal *fundamental solution of the Laplace equation* $\Delta_M u = 0$ *with pole in* $x \in M$. Hereby a $\mathscr{C}^2$-function $F : M \setminus \{x\} \to \mathbb{R}$ is a fundamental solution of $\Delta_M u = 0$ with pole x if $-\Delta_M F = \delta_x$ in the sense of distributions. Denote by τ_U the exit time of a Brownian motion B^x starting in x from a relatively compact open set $U \subset M$, $U \neq M$, containing x and having smooth boundary. Then the measure $\mathscr{G}_U(x, \cdot)$ on U, defined as

$$A \mapsto \mathscr{G}_U(x, A) := \frac{1}{2} \int_0^\infty \mathbb{P}\left(B_t^x \in A; t < \tau_U\right) dt = \frac{1}{2} \mathbb{E} \int_0^{\tau_U} \left(1_A \circ B_t^x\right) dt,$$

is absolutely continuous with respect to the canonical volume measure $\mu_M|_U$ on U with density $G_U(x, \cdot)$. Here $0 \leq G_U(x, \cdot) \in \mathscr{C}^2(U \setminus \{x\}) \cap \mathscr{C}(\overline{U} \setminus \{x\})$ and

$$-\Delta_M G_U(x, \cdot) = \delta_x \text{ on } U \text{ and } G_U(x, \cdot)|_{\partial U} = 0.$$

Choosing a sequence $(U_n)_{n \in \mathbb{N}}$ of relatively compact open sets as above with $U_n \nearrow M$ we have $G_{U_n}(x, \cdot) \nearrow G(x, \cdot)$. Hereby $G(x, \cdot)$, if it is finite, is a positive fundamental solution of the Laplace equation with pole in x for every $x \in M$ and one obtains a symmetric $\mathscr{C}^2$-function $G : (M \times M) \setminus \{(x,x) : x \in M\} \to \mathbb{R}$. Furthermore $G(x, \cdot)$ is minimal in the sense that $0 \leq G(x, \cdot) \leq F$ for every fundamental solution F of $-\Delta_M u = 0$ with pole x.

The existence of such a Green function G on M is guaranteed when the Brownian motion B on M is transient. This is part of the following theorem, wich can be found for example in [Th]:

Theorem 2.22. *Let* (M, g) *be a connected Riemannian manifold. Then the following conditions are equivalent:*

i) Brownian motion B *on* M *is transient.*

ii) M *possesses a Green function.*

iii) $\int_0^\infty \mathbb{P}\left(B_t^x \in K; t < \zeta\right) dt = \mathbb{E}\left[\int_0^\zeta 1_K \circ B_t^x \, dt\right] < \infty$ *for every compact set* $K \subset M$ *and every* $x \in M$.

Remark 2.23. In particular every Cartan-Hadamard manifold M with curvature bounded from above by a negative constant $-k^2$, $k > 0$, possesses a Green function due to Theorem 2.21.

We now follow [K2] with the construction of the Martin boundary of a Riemannian manifold with Green function G.

Theorem 2.24 (Harnack inequality).
Let (M, g) *be a Riemannian manifold. Then for any* $r_2 > r_1 > 0$ *and* $x \in M$ *there exists*

a constant $C(x, r_1, r_2)$ with the following property: if $h : M \to \mathbb{R}$ is harmonic in $B_{r_2}(x)$ and strictly positive then for any $y, z \in B_{r_1}(x)$ holds:

$$h(y) \leq C(x, r_1, r_2) \cdot h(z).$$

Proof. See [K2] and [Mo]. □

Remark 2.25. Let as before $h : M \to \mathbb{R}$ be a positive harmonic function in $B_{r_2}(x)$ for $x \in M$ and $r_2 > 0$ a given constant. Using the infinitesimal version of the Harnack inequality, cf. [Y], Theorem 3", one can estimate the uniform Hölder norm $\|h\|_{r_1}^{\alpha}$ in $B_{r_1}(x)$, for $0 < r_1 < r_2$, $\alpha \in (0, 1]$, cf. [Pi] p.84, as follows:

$$\|h\|_{r_1}^{\alpha} \leq C(x, \alpha, r_1, r_2) \cdot \sup_{y \in B_{r_2}(x)} |h(y)|,$$

where $C(x, \alpha, r_1, r_2)$ is a constant independent of h.

From the above estimate together with the estimate in Theorem 2.24 we obtain the *Harnack Principle* using the Ascoli-Arzela Theorem:

Theorem 2.26 (Harnack Principle).
Let (M, g) be a Riemannian manifold, $r > 0$ and $x \in M$. Let further $(h_n)_{n\in\mathbb{N}} : M \to \mathbb{R}$ be a sequence of positive functions which are harmonic in $B_r(x)$ and such that there exists $y \in B_r(x)$ and $C > 0$ with $h_n(y) \leq C$ for all $n \in \mathbb{N}$. Then the sequence $(h_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence. The limiting function h is harmonic in $B_r(x)$.

For the rest of the chapter let (M, g) be a Cartan-Hadamard manifold of strictly negative curvature. Denote its Green function with G.

Fix a point $0 \in M$ and for $x, y \in M$ define the *normalized Green function at* 0 *with pole* y as the function $k : M \times M \to \mathbb{R}_+$, given as

$$k_y(x) := k(x, y) := \begin{cases} \frac{G(x,y)}{G(0,y)}, & \text{for } y \neq 0, \\ 0, & \text{for } y = 0, x \neq 0, \\ 1, & \text{for } x = y = 0. \end{cases} \tag{2.9}$$

$k_y(x)$ is continuous for $y \in M \setminus \{x\}$.

Let $(y) := (y_i)_{i\in\mathbb{N}}$ a sequence in M with $r(y_i) \to \infty$ and choose an increasing sequence of balls $B_i \nearrow M$ such that $y_j \notin \overline{B_i}$ for all $j \geq i$. Then the corresponding functions $k_{y_j}(x)$ are harmonic in B_i for $j \geq i$. Since $k_{y_j}(0) = 1$ the functions k_{y_j}, $j \geq i$, are uniformly bounded in B_i by the Harnack inequality.

The sequence (y) is called *fundamental* if $(k_{y_i})_{i\in\mathbb{N}}$ converges to a harmonic function $h_{(y)}$ on M. It follows from the Harnack Principle that any sequence $(y_i)_{i\in\mathbb{N}}$ with $r(y_i) \to \infty$ possesses a fundamental subsequence. We call two fundamental sequences in M equivalent if they converge to the same harmonic limit function h.

Definition 2.27 (Martin boundary). The set M^* consisting of all equivalence classes of fundamental sequences in M is called *Martin boundary of* M. Let $\mathscr{M} := M \cup M^*$.

If $y \in M^*$, then $k_y(x) := \lim_{i\to\infty} k_{y_i}(x)$ is a harmonic function on M, where $(y_i)_{i\in\mathbb{N}}$ is a fundamental sequence in the equivalence class determined by y. Hence points $y \in M^*$ correspond uniquely to positive harmonic functions k_y on M.

For any pair $y_1, y_2 \in \mathscr{M}$ define

$$\rho(y_1, y_2) := \sum_{n=1}^{\infty} (m(B_0(n)))^{-2} \int_{B_0(n)} \frac{|k_{y_1}(x) - k_{y_2}(x)|}{1 + |k_{y_1}(x) - k_{y_2}(x)|} \, d\mu_M(x),$$

where $B_0(n)$ is a geodesic ball with center 0 and radius n and μ_M denotes the Riemannian volume on M, cf. [K2], p206.

Theorem 2.28. *The function $\rho : \mathscr{M} \times \mathscr{M} \to \mathbb{R}$ defines a metric on $\mathscr{M}$. Furthermore $\mathscr{M}$ equipped with this metric is a complete compact metric space, whose topology inside of M agrees with the topology of M as a Riemannian manifold and M^* is the topological boundary of $\mathscr{M}$. The structure of $\mathscr{M}$ is independent of the choice of the base point* 0.

Proof. See [Ma], [K2] and [A-S]. □

The Martin boundary M^* of M provides full information about the space of positive (bounded respectively) harmonic functions on M: it turns out that the Riemannian manifold M is a Liouville manifold if and only if the Martin boundary M^* shrinks to a single point. Furthermore every positive harmonic function has a boundary integral representation with respect to the Martin boundary of M as follows:

Theorem 2.29. *Let $h : M \to \mathbb{R}$ be a positive harmonic function. Then there exists a Borel measure μ on M^* such that*

$$h(x) = \int_{M^*} k_y(x)\mu(dy) \text{ for all } x \in M. \tag{2.10}$$

Proof. The proof can be found for example in [Ma], [D2]. □

However, the Borel measure μ on M^* in Theorem 2.29 is not uniquely determined by the harmonic function h. Call a positive harmonic function $u : M \to \mathbb{R}$ *minimal*, if, whenever v is a positive harmonic function on M with $v \le u$, then $v \equiv cu$ for some constant $c \in (0, 1]$. A point $y \in M^*$ is called *minimal boundary point* if the function k_y is minimal. Denote by Δ_0 the set of all minimal boundary points. Δ_0 is called *minimal Martin boundary.* The minimal Martin boundary Δ_0 is a Borel set in the Martin topology. We have the following representation theorem:

Theorem 2.30 (Martin Representation Theorem).
Let $h : M \to \mathbb{R}$ be a positive harmonic function. Then there exists a unique finite measure μ_h supported on the minimal Martin boundary Δ_0 such that

$$h(x) = \int_{\Delta_0} k_y(x)\mu_h(dy).$$

Conversely for every finite measure μ supported on the minimal Martin boundary Δ_0,

$$h(x) := \int_{\Delta_0} k_y(x)\mu(dy) \tag{2.11}$$

is a bounded harmonic function on M.

Proof. For the proof se [Ma]. For further discussion see [Pi], p.285ff. □

Remark 2.31 *(Probabilistic approach to the Martin boundary).*
For B a Brownian motion on M the *h-conditioned Brownian motion* is the (canonical) Markov process B^h on $(\Omega^h, (\mathscr{A}_t^h)_{t\in\mathbb{R}_+}, (\mathbb{P}_x^h)_{x\in M})$, with infinitesimal generator Δ_M^h where

$$\Delta_M^h := [h]^{-1} \circ \Delta_M \circ [h],$$

and $[h] : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$, $f \mapsto fh$ is the multiplication with h. Denote by ζ^h the lifetime of B^h.
The h-conditioned Brownian motion almost surely converges in the Martin topology. This is part of

Theorem 2.32 (cf. [Pi], Theorem 7.2.1 and Theorem 7.2.2).
For every $y \in \Delta_0$, denote with k_y the corresponding harmonic function on M. Then

$$\mathbb{P}_x^{k_y}\left\{\lim_{t\to\zeta^{k_y}} B_x^{k_y} = y\right\} = 1 \text{ for all } x \in M. \tag{2.12}$$

The convergence is understood with respect to the Martin topology.
More general, let $h : M \to \mathbb{R}$ be a positive harmonic function. Then for any $x \in M$

$$\mathbb{P}_x^h\left\{\lim_{t\to\zeta^h} B_t^h \text{ exists}\right\} = 1. \tag{2.13}$$

Furthermore for any measurable $A \subset \Delta_0$

$$\mathbb{P}_x^h\left\{\lim_{t\to\zeta^h} B_t^h \in A\right\} = \frac{1}{h(x)}\int_A k_y(x)\mu_h(dy), \tag{2.14}$$

where μ_h is the measure in the Martin representation for h.

Taking $h \equiv 1$ one obtains for the Brownian motion B on M:

$$\begin{aligned} \mathbb{P}^x\left\{\lim_{t\to\zeta} B_t \text{ exists}\right\} &= 1 \qquad \text{and} \\ \mathbb{P}^x\left\{\lim_{t\to\zeta} B_t \in A\right\} &= \int_A k_y(x)\mu_1(dy). \end{aligned} \tag{2.15}$$

Considering a Cartan-Hadamard manifold M with sectional curvatures bounded between two negative constants $-a^2 < -b^2$, for $a > b > 0$, there is a natural homeomorphism

$$\Phi : M^* \to S_\infty(M) \tag{2.16}$$

from the Martin boundary M^* of M to the sphere at infinity. This has been first proven by Anderson and Schoen in [A-S], Theorem 6.3. Furthermore in this case the Martin boundary and the minimal Martin boundary coincide. More general results in this direction have been obtained for example in [A2].

In the following chapter we are going to present an example of a Riemannian manifold M with unbounded negative curvature. Although the Dirichlet problem at infinity is not solvable for that manifold, we construct non-trivial bounded harmonic functions on M with the help of non-trivial exit sets for the Brownian motion on M. It turns out that in this case the Martin boundary M^* of M has to be at least of dimension 2. This is caused by the following fact: Writing the Brownian motion B on M in a global chart for M as three dimensional diffusion (R_t, S_t, A_t), see Chapter 3, it turns out that the component A_t generates non-trivial bounded harmonic functions on M, see Lemma 3.16 and is – up to a time change – stochastically independent of the components (R, S) of B. On the other hand we obtain non-trivial bounded harmonic functions considering an $\mathscr{A}_{\text{inv}}$-measurable function for the diffusion (R, S), cf. Theorem 3.22 and Theorem 3.27.

Chapter 3

A Non-Liouville Manifold with Degenerate Angular Behaviour of BM

In this chapter we are going to present an example of a Cartan-Hadamard manifold M, where the Dirichlet problem at infinity is not solvable. We first start with the construction of the manifold and some geometrical considerations concerning the Riemannian metric and the sectional curvatures of M. The main part of the chapter is dedicated to the proof that Brownian motion B almost surely exits from M at a single point of the sphere at infinity. However, it turns out that there exist non-trivial bounded harmonic functions on M. We give a stochastic proof of this result and finally conclude with some geometrical interpretations about the asymptotic behaviour of Brownian motion on M. The manifold we are going to present was first constructed by Borbély in [B]. Unlike Borbély who used methods of partial differential equations and differential geometry to prove that his manifold provides an example of a Non-Liouville manifold for that the Dirichlet problem at infinity is not solvable, we are interested in a complete stochastic description of the considered manifold.

Let $L \subset \mathbb{H}^2$ be a fixed unit speed geodesic in the hyperbolic half plane

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

equipped with the hyperbolic metric $ds^2_{\mathbb{H}^2}$ of constant curvature -1. For our purposes one can assume without loss of generality $L := \{(0, y) | y > 0\}$ to be the positive y-axis. Let H denote one component of $\mathbb{H}^2 \setminus L$ and define a Riemannian manifold M as the warped product:

$$M := (H \cup L) \times_g S^1,$$

with Riemannian metric

$$ds^2_M = ds^2_{\mathbb{H}^2} + g \cdot ds^2_{S^1},$$

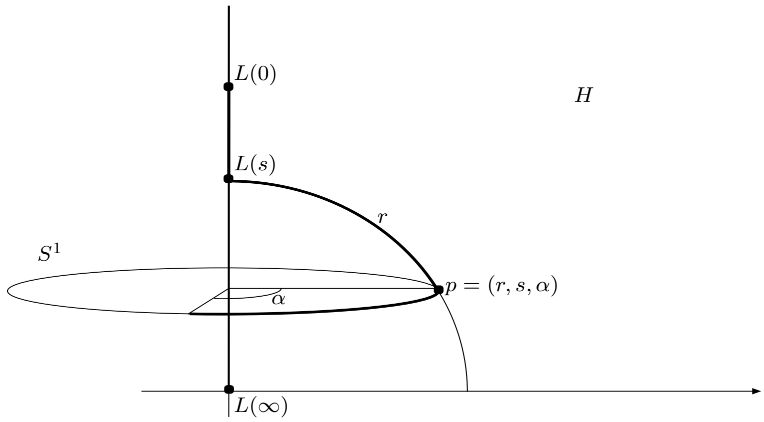
where $g : H \cup L \to \mathbb{R}_+$ is a positive $\mathscr{C}^\infty$-function to be determined later. By identifying points (ℓ, α_1) and (ℓ, α_2) with $\ell \in L$ and $\alpha_1, \alpha_2 \in S^1$, we make M a simply connected space.

We introduce a system of coordinates (r, s, α) on M, where for a point $p \in M$ the coordinate r is the hyperbolic distance between p and the geodesic L – i.e. the hyperbolic length of the perpendicular on L through p. The coordinate s is the parameter on the geodesic $\{L(s) : s \in (-\infty, \infty)\}$ – i.e. the length of the geodesic segment on L joining $L(0)$ and the orthogonal projection $L(s)$ of p onto L. Further $\alpha \in [0, 2\pi)$ is the parameter on S^1 when using the parametrization $e^{i\alpha}$. We sometimes take $\alpha \in \mathbb{R}$, in particular when considering components of the Brownian Motion, thinking of $\mathbb{R}$ as the universal covering of S^1.

In the coordinates (r, s, α) the Riemannian metric on $M \setminus L$ takes the form

$$ds_M^2 = dr^2 + h(r)ds^2 + g(r, s)d\alpha^2,$$

where $h(r) = \cosh^2(r)$.



Coordinates for the Riemannian manifold M

Let $g(r, s) := \sinh^2(r)$ for $r < \frac{1}{10}$ (the complete definition is given below), then the above metric smoothly extends to the whole manifold M, where M is now rotationally symmetric with respect to the axis L and for $r < \frac{1}{10}$ isometric to the three dimensional hyperbolic space $\mathbb{H}^3$ with constant sectional curvature -1, cf. [B]. From that it is clear that the Riemannian manifold (M, g) is complete.

3.1. Computation of the Sectional Curvature

From now on we fix the basis $\partial_1 := \frac{\partial}{\partial r}, \partial_2 := \frac{\partial}{\partial s}, \partial_3 := \frac{\partial}{\partial \alpha}$ for the tangent space T_pM in $p \in M$. Herein the Christoffel symbols of the Levi-Civita connection can be computed as follows – the indices refer to the corresponding tangent vectors of the basis:

$$\Gamma^1_{22} = -\frac{1}{2}h'_r, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{h'_r}{2h}, \qquad \Gamma^3_{13} = \Gamma^3_{31} = \frac{g'_r}{2g},$$
$$\Gamma^1_{33} = -\frac{1}{2}g'_r, \qquad \Gamma^2_{33} = -\frac{g'_s}{2h}, \qquad \Gamma^3_{23} = \Gamma^3_{32} = \frac{g'_s}{2g},$$

all others equal 0. Herein g'_r denotes the partial derivative of the function g with respect to the variable r, g'_s the partial derivative with respect to s, etc.
For the computation of the sectional curvatures Sect^M write $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ for tangent vectors $X, Y \in T_pM$ in terms of the basis $\partial_1, \partial_2, \partial_3$. Then one gets

$$\begin{aligned}\langle R(X,Y)Y,X\rangle = &\underbrace{\left(-\frac{1}{2}h''_{rr} + \frac{1}{4}\frac{(h'_r)^2}{h}\right)}_{=:A}(x_1y_2 - x_2y_1)^2\\ &+ \underbrace{\left(-\frac{1}{2}g''_{rr} + \frac{1}{4}\frac{(g'_r)^2}{g}\right)}_{=:B}(x_1y_3 - x_3y_1)^2\\ &+ \underbrace{\left(-\frac{1}{2}g''_{ss} - \frac{1}{4}g'_rh'_r + \frac{1}{4}\frac{(g'_s)^2}{g}\right)}_{=:C}(x_2y_3 - x_3y_2)^2\\ &+ 2\cdot\underbrace{\left(-\frac{1}{2}g''_{rs} + \frac{1}{4}\frac{g'_sh'_r}{h} + \frac{1}{4}\frac{g'_rg'_s}{g}\right)}_{=:D}(x_1y_3 - x_3y_1)(x_2y_3 - x_3y_2).\end{aligned}$$

as well as

$$\|X \wedge Y\|^2 = h(x_1y_2 - x_2y_1)^2 + g(x_1y_3 - x_3y_1)^2 + gh(x_2y_3 - x_3y_2)^2.$$

For this reason we conclude that the manifold M has strictly negative sectional curvature, i.e. $-k^2 \geq \mathrm{Sect}^M(\mathrm{Lin}\{X,Y\}) = \frac{\langle R(X,Y)Y,X\rangle}{\|X\wedge Y\|^2}$ for a $k > 0$, all $X, Y \in T_pM$ and all $p \in M$, if and only if the following inequalities hold:

$$\frac{1}{2}h''_{rr} - \frac{1}{4}\frac{(h'_r)^2}{h} \geq k^2h, \tag{3.1}$$

$$\frac{1}{2}g''_{rr} - \frac{1}{4}\frac{(g'_r)^2}{g} \geq k^2g, \tag{3.2}$$

$$\frac{1}{2}g''_{ss} + \frac{1}{4}g'_rh'_r - \frac{1}{4}\frac{(g'_s)^2}{g} \geq k^2gh, \tag{3.3}$$

$$\begin{aligned}\frac{1}{g^2h}\left(-\frac{1}{2}g''_{rs} + \frac{1}{4}\frac{g'_sh'_r}{h} + \frac{1}{4}\frac{g'_sg'_r}{g}\right)^2 \leq &\left(\frac{g''_{rr}}{2g} - \frac{(g'_r)^2}{4g^2} - k^2\right)\cdot\\ &\cdot\left(\frac{g''_{ss}}{2gh} + \frac{g'_rh'_r}{4gh} - \frac{(g'_s)^2}{4g^2h} - k^2\right).\end{aligned} \tag{3.4}$$

This can be explained by the fact that the quadratic form

$$q(X,Y,Z) := (A + k^2h)X^2 + (B + k^2g)Y^2 + (C + k^2gh)Z^2 + 2DYZ$$

is non-positive for all $X, Y, Z \in \mathbb{R}$ if and only if

$$-A \geq k^2h \text{ and } -B \geq k^2g \text{ and } -C \geq k^2gh \text{ and } D^2 \leq (B + k^2g)(C + k^2gh).$$

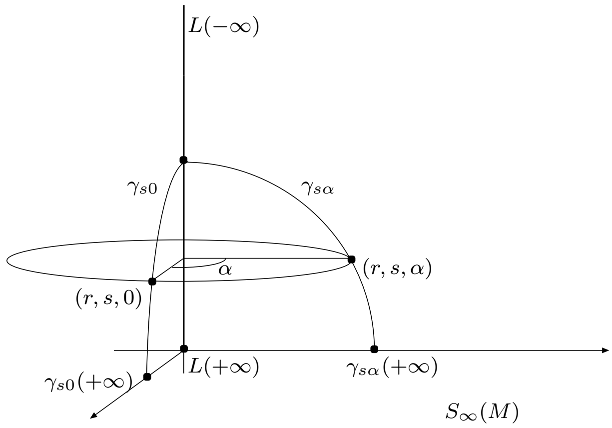
3.2. The Sphere at Infinity $S_\infty(M)$

As it is obvious that, for $(s,\alpha) \in \mathbb{R} \times [0, 2\pi)$ fixed, the curves

$$\gamma_{s\alpha} : \mathbb{R}_+ \to M, \quad r \mapsto (r, s, \alpha)$$

form a foliation of M of geodesic rays we can easily describe the sphere at infinity $S_\infty(M)$ as the union of the "endpoints" (i.e. equivalence classes) $\gamma_{s\alpha}(+\infty)$ of all geodesic rays $\gamma_{s\alpha}$ foliating M together with the equivalence classes $L(+\infty)$ and $L(-\infty)$ determined by the geodesic axis L of M, what in detail means:

$$S_\infty(M) = L(+\infty) \cup \{\gamma_{s\alpha}(+\infty) \,|\, (s,\alpha) \in \mathbb{R} \times [0, 2\pi)\} \cup L(-\infty).$$



Sphere at Infinity $S_\infty(M)$

This explains why it suffices to show that the s-component S_t of the Brownian motion B_t converges to $+\infty$ for $t \to \zeta$, where ζ is the lifetime of the Brownian motion, if we want to show that the Brownian motion B_t on M converges for $t \to \zeta$ to the single point $L(+\infty)$ on the sphere $S_\infty(M)$ at infinity.

3.3. Properties of the Function g

We give a brief description of the properties which the warped product function $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ used in the definition of the Riemannian metric has to satisfy to provide an example of a Riemannian manifold where the Dirichlet problem at infinity is not solvable whereas there exist non-trivial bounded harmonic functions. We do not prove in detail the existence of such a function g but we give a short idea how to construct it and refer to Borbély ([B]) for further details.

Lemma 3.1. *There is a $\mathscr{C}^\infty$-function $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+, (r,s) \mapsto g(r,s)$ satisfying the following properties:*

i) $g'_r \geq 0$ *and* $g'_s \geq 0$.

ii) $g(r,s) = \sinh^2(r)$ *for* $r < \frac{1}{10}$*, and for* $r \geq \frac{1}{10}$ *it holds that:*

$$g'_r \geq h'_r g \quad \text{and} \quad \frac{1}{2}g''_{rr} - \frac{1}{4}\frac{(g'_r)^2}{g} \geq \frac{h'_r g'_r}{8}.$$

iii) Denoting $p(r,s) := \frac{g'_s}{g'_r h}$ *one has* $(ph)'_s \geq 0$ *and for all* $s \in \mathbb{R}$*:*

$$\int_0^\infty p(r,s)dr = \infty.$$

iv) The function $p(r,s)$ *additionally satisfies:*

$$p \leq \frac{1}{1000}, \; p'_s \leq \frac{1}{1000}, \; |p'_r| \leq \frac{5}{1000} \quad \text{and} \quad pp'_r h^2 < \frac{h'_r}{1000}.$$

v) For the construction of p *one further needs that*

$$(ph)'_r \geq 0 \quad \text{and} \quad (ph)''_{rr} \geq 0.$$

vi) As a technical requirement, $p(r,s)$ *has to fulfill:*

$$\int_{r_0}^\infty \frac{1}{p(r,s)h(r)}dr = \infty$$

for every $r_0 > 0$ *and every* $s \in \mathbb{R}$*.*

We mentioned in the lemma above that some of the properties we demand from the function g and the additionally defined function $p(r,s)$ are more or less technical and do not in the first instance influence the behaviour of the Brownian motion on M. The following remark gives a brief discussion about the "importance" of these properties.

Remark 3.2 *(Some comments on the properties listed above).*

i) As already mentioned in the definition of the Riemannian metric on M Property (ii) assures that the Riemannian metric smoothly extends to the geodesic L and that the curvature condition (3.2) is satisfied for a suitable k.

ii) The conditions (iv) stated for $p(r,s)$ assure validity of the curvature condition (3.4) – at least for $k = 1/1000$.

iii) The integral condition (iii) is needed to let the Brownian motion $B = (R,S,A)$ on M converge for $t \to \zeta$ to the single point $L(\infty) \in S_\infty(M)$ which as an immediate consequence implies the non-solvability of the Dirichlet problem at infinity for M. The given condition assures that the drift term in the stochastic differential equation for the component S of the Brownian motion B compared with the drift term appearing in the defining equation for the component R of B grows "fast enough" to force S_t going to ∞ when $t \to$"lifetime". This will be clear in Section 3.5, Lemma 3.18.
In [B], Lemma 2.1, Borbély uses this condition to show that the convex hull of any neighbourhood of $L(\infty)$ is the whole manifold M. This is a natural first step in the construction of a manifold for which the Dirichlet problem at infinity is not solvable (cf. the Introduction and [C], [B]).

iv) Property (vi) is needed for technical reasons. As explained below, one wants to construct the function $g(r,s)$ as solution of the partial differential equation $g'_s(r,s) = p(r,s)h(r)g'_r(r,s)$ with given initial conditions. The above property of p then guarantees the existence of a solution $g(r,s)$ on all of $\mathbb{R}_+ \times \mathbb{R}$ for any initial condition given on a subspace $\Omega \subset \mathbb{R}_+ \times \mathbb{R}$.

3.4. Construction of the Function g

The idea to construct a function g with the properties above is as follows:
As g is given as solution of the partial differential equation

$$g'_s(r,s) = p(r,s)h(r)g'_r(r,s) \tag{3.5}$$

one has to find an appropriate function $p(r,s)$ and the required initial conditions for g to obtain the desired function.

We will see later, that the construction of the metric on M is in some way similar to that given in the example of Ancona, see [A1], as Borbély also defines the function p "stripewise" to control the requirements for p, g respectively, on each region of the form $[r_i, r_j] \times \mathbb{R}$. Yet he is mainly concerned with the definition of the "drift ratio" p, what makes it a little difficult to understand the behaviour of the metric function g and to possibly modify his example for other situations, whereas Ancona gives a more or less direct way to construct the coupling function in the warped product. This as a consequence makes it possible to extend his example to higher dimensions and to adapt it to other desired situations, see for example Section 4.4 and Section 4.5.

We start with a short description how to construct the function

$$p_0(r) := p(r,0),$$

as given in [B]:
p_0 is defined inductively on intervals $[r_n, r_{n+1}]$, where $r_{n+1} - r_n > 3$ and $r_1 > 3$ sufficiently large, see below:
Define $p_0 := 0$ on $[0,2]$ and as a slowly increasing function satisfying conditions (iv) and (v) on $[2,3]$. For $r \in [3, r_1]$ let $p_0(r) := p_0(3)$ be constant, where r_1 is chosen big enough such that $(p_0 h)(r_1) > 1$ and $r_1/h(r_1) < 1/1000$.
On the interval $[r_1, \infty)$ we choose the function p_0 to be decreasing with $\lim_{r\to\infty} p_0(r) = 0$, whereas $p_0 h$ is still increasing and strictly convex. See [B], Lemma 2.3 and Lemma 2.4.

On the interval $[r_1, r_2]$ for r_2 big enough as given below (and in general on intervals of the form $[r_{2n-1}, r_{2n}]$) one extends $p_0 h$ via a solution of the differential equation

$$y'' = \frac{1}{2y}.$$

Carefully smoothing the function p_0 on the interval $[r_1, r_1 + 1]$ ($[r_{2n-1}, r_{2n-1} + 1]$ respectively) to become $\mathscr{C}^\infty$ does not disturb the properties (iv) and (v) of p_0 and can be done such that $(p_0 h)'' > 1/(4 p_0 h)$ is still valid.

[B], Lemma 2.4, shows that in fact $p_0 = p_0 h/h$ decreases on $[r_{2n-1}, r_{2n}]$ and again with respect to [B], Lemma 2.3, for given r_{2n-1} one can choose the upper interval bound r_{2n} such that

$$\int_{r_{2n-1}}^{r_{2n}} \frac{1}{p_0 h}\, dr > 1 \quad \text{for all } n.$$

This guarantees Property (vi) for p_0.

On the interval $[r_2, r_3]$ for r_3 big enough as given below and in general on the intervals $[r_{2n}, r_{2n+1}]$ let p_0 be constant. As above, smoothing on intervals $[r_{2n}-1, r_{2n}]$ preserves the conditions (iv), (v) and $(p_0 h)'' > \varepsilon/(p_0 h)$, for $0 < \varepsilon < 1/4$ small enough and independent of n but depending on the choice of p_0 on $[2, 3)$.
If we choose for given r_2, r_{2n} respectively, the interval bounds r_3, r_{2n+1} respectively, large enough we get

$$\int_{r_{2n}}^{r_{2n+1}} p_0(r)dr > 1 \quad \text{for all } n,$$

which finally assures Property (iii) for p_0.

Following Borbély in [B], we define $p(r, s)$ via $p_0(r)$ by using a "cut off function" $\chi(r, s)$ as

$$p(r, s) := \chi(r, s)p_0(r).$$

Herein $\chi(r, s)$ is given as

$$\chi(r, s) := \xi(s + \ell(r)),$$

where ξ is smooth and increasing with $\xi(y) := 0$ for $y < 0$, $\xi(y) := 1/2$ for $y > 4$ and $\xi', |\xi''| < 1/2$, $\xi'' + \xi > 0$. The function ℓ, nondecreasing with $\lim_{r\to\infty} \ell(r) = \infty$, is chosen such that $p(r, s)$ satisfies all the required properties listed in Lemma 3.1. For the explicit definition of ℓ and the verification that all the requirements for $p(r, s)$ are fulfilled see [B], p.228ff.

To have an appropriate initial condition for solving the partial differential equation (3.5), define $\widetilde{g_0}(r) := \sinh^2(r)$ on $\left[0, \frac{1}{10}\right]$. On $\left[\frac{1}{10}, \infty\right)$ let $\widetilde{g_0}(r)$ be the solution of the differential equation

$$f' = \frac{1}{\sinh^2(1/10)} h_r' f \quad \text{with initial condition } f\left(\frac{1}{10}\right) = \sinh^2\left(\frac{1}{10}\right).$$

Smoothing $\widetilde{g_0}$ yields a $\mathscr{C}^\infty$-function $g_0(r)$ such that $\widetilde{g_0} = g_0$ on $\left[0, \frac{1}{10} - \delta\right] \cup \left[\frac{1}{10} + \delta, \infty\right)$ for an appropriate δ. The so-defined function g_0 serves as initial condition to solve the partial differential equation

$$g_s'(r, s) = p(r, s)h(r)g_r'(r, s).$$

For a more detailed discussion we refer to [B], as we do not need the explicit construction of g in the ongoing text but only use the properties of the functions g, p_0 and p listed above.

3.5. Brownian Motion on M

Let $(\Omega; \mathscr{F}; \mathbb{P}; (\mathscr{F}_t)_{t\in\mathbb{R}_+})$ be a filtered probability space satisfying the usual conditions and B a Brownian motion on M considered as a diffusion process with generator $\frac{1}{2}\Delta_M$ taking values in the Alexandroff compactification $\widetilde{M} := M \cup \{c(M)\}$ of M as in Section 2.1, Definition 2.5. Further let ζ denote the lifetime of B, i.e. $B_t(\omega) = c(M)$ for $t \geq \zeta(\omega)$, if $\zeta(\omega) < \infty$.

As we have fixed the coordinate system $M = \{(r, s, \alpha) : r \in \mathbb{R}_+, s \in \mathbb{R}, \alpha \in [0, 2\pi)\}$ for our manifold M, we consider the Brownian motion B in the chosen coordinates as well and denote by $(R_t)_{t<\zeta}$, $(S_t)_{t<\zeta}$ and $(A_t)_{t<\zeta}$ the component processes of $(B_t)_{t<\zeta}$.

The generator $\frac{1}{2}\Delta_M$ of B can be written in terms of the basis $\frac{\partial}{\partial r}, \frac{\partial}{\partial s}, \frac{\partial}{\partial \alpha}$ of TM as:

$$\frac{1}{2}\Delta_M = \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{h}\frac{\partial^2}{\partial s^2} + \frac{1}{g}\frac{\partial^2}{\partial \alpha^2}\right) + \left(\frac{h'_r}{4h} + \frac{g'_r}{4g}\right)\frac{\partial}{\partial r} + \frac{g'_s}{4gh}\frac{\partial}{\partial s}. \tag{3.6}$$

Therefore we can write down a system of stochastic differential equations for the components R, S and A of our given Brownian motion:

$$dR_t = \left(\frac{h'_r(R_t)}{4h(R_t)} + \frac{g'_r(R_t, S_t)}{4g(R_t, S_t)}\right)dt + dW^1 \tag{3.7}$$

$$dS_t = \frac{g'_s(R_t, S_t)}{4g(R_t, S_t)h(R_t)} dt + \frac{1}{\sqrt{h(R_t)}} dW^2 \tag{3.8}$$

$$dA_t = \frac{1}{\sqrt{g(R_t, S_t)}} dW^3 \tag{3.9}$$

with a three-dimensional Euclidean Brownian motion $W = (W^1, W^2, W^3)$.

As already mentioned, we are going to read the component A of the Brownian motion with values in the universal covering $\mathbb{R}$ of S^1.

Remark 3.3 *(About the lifetime ζ of the Brownian motion).*
When looking on the defining stochastic differential equations for the components R, S and A of the Brownian motion B on M, it is a remarkable fact that the behaviour of the component A_t does neither influence the behaviour of the components R_t and S_t nor the behaviour of A_t itself as all appearing drift and covariance terms do not depend on the component α of M and so the system of stochastic differential equations does not at all depend on A. It is therefore clear that also the lifetime ζ of B_t does not depend on the component A_t – in particular does not depend on the starting point A_0 of A_t. We are going to use this fact later when we prove the existence of non-trivial bounded harmonic functions on M, see Lemma 3.16.

We are now going to state and prove the main theorem of this chapter which shows that from the stochastic point of view the Riemannian manifold (M, γ) constructed by Borbély in [B] has essentially the same properties as the manifold of Ancona in [A1]. We further give a stochastic construction of non-trivial bounded harmonic functions on M, which is more transparent than the existence proof of Borbély relying on Perron's principle.

Theorem 3.4. *i) For the Brownian motion B on the Riemannian manifold (M,γ) constructed above the following statement almost surely holds:*

$$\lim_{t\to\zeta} B_t = L(+\infty),$$

independently of the starting point B_0. In particular the Dirichlet problem at infinity for M is not solvable.

ii) There is a submanifold $\overline{S}$ of $S_\infty(M)$ of codimension 1 with the following property: Given a bounded continuous function $f : \overline{S} \to \mathbb{R}$ we can find a non-trivial bounded harmonic function $h : M \to \mathbb{R}$ which has f as limiting boundary function, i.e. $\lim_{p\to\tilde{p}} h(p) = f(\tilde{p})$ where $p \to \tilde{p} \in \overline{S}$. Writing pr_3 for the map $M \to \mathbb{R}$, $(r,s,\alpha) \mapsto \alpha$, almost surely $\lim_{t\to\zeta}(\mathrm{pr}_3 \circ B_t) \equiv \lim_{t\to\zeta} A_t$ exists and takes values in the submanifold $\overline{S}$.

Further, we have for any point $p = (r,s,\alpha) \in M$:

$$h(p) = \mathbb{E}^p\left(f \circ \lim_{t\to\zeta}(\mathrm{pr}_3 \circ B_t)\right) = \mathbb{E}^p\left(f \circ \lim_{t\to\zeta} A_t\right).$$

Remark 3.5. We are going to show in Section 3.6 that the harmonic functions we get from Theorem 3.4 are not the only harmonic functions we can find on M; There are further harmonic functions depending on the components R and S of the Brownian motion.

As we have seen at the beginning, to prove the first statement concerning the behaviour of Brownian paths on M, it will be enough to show that $\lim_{t\to\zeta} S_t = \infty$. We are going to split the proof of Theorem 3.4 into several steps that in combination yield both of the statements made in Theorem 3.4.

For the behaviour of the component R of B we need some further information about the geometry of our manifold M:

Proposition 3.6. *Let $a \in \mathbb{R}_+$. Then the region $U_a := \{(r,s,\alpha) \in M | \, r \le a\} \subset M$ is a convex subset of M.*

Proof. We have to show that for two arbitrary points p_0 and p_1 the region U_a also contains the whole geodesic segment joining p_0 and p_1.
Let $\gamma : [0,1] \to M,\ t \mapsto \gamma(t) := (\gamma_r(t), \gamma_s(t), \gamma_\alpha(t))$ with $\gamma(0) = p_0$ and $\gamma(1) = p_1$ denote the geodesic (parametrized by arc length) joining p_0 and p_1. Assuming that $\gamma([0,1])$ is not completely contained in U_a, there exist $t_1 < t_2 \in [0,1]$ such that $\gamma_r(t_1) = \gamma_r(t_2) = a$ and $\gamma((t_1,t_2)) \subset M \setminus U_a$.

Consider the (piecewise smooth) curve

$$\widetilde{\gamma} : [0,1] \to M,\ t \mapsto (\widetilde{\gamma}_r(t), \widetilde{\gamma}_s(t), \widetilde{\gamma}_\alpha(t))$$

defined as follows:

$$\widetilde{\gamma}|_{[0,t_1]} := \gamma|_{[0,t_1]}, \quad \widetilde{\gamma}|_{[t_2,1]} := \gamma|_{[t_2,1]} \quad \text{and } \widetilde{\gamma}(t) := (a, \gamma_s(t), \gamma_\alpha(t)) \text{ for } t \in (t_1, t_2).$$

Then we have

$$\begin{aligned}
\ell(\widetilde{\gamma}) &= \ell(\gamma|_{[0,t_1]}) + \ell(\gamma|_{[t_2,1]}) + \int_{t_1}^{t_2} \|(0, \dot{\gamma}_s(t), \dot{\gamma}_\alpha(t))\|_{(a,\gamma_s(t),\gamma_\alpha(t))} dt \\
&= \ell(\gamma|_{[0,t_1]}) + \ell(\gamma|_{[t_2,1]}) + \int_{t_1}^{t_2} \Big(\underbrace{\cosh^2(a)}_{\leq \cosh^2(\gamma_r(t))} \ \dot{\gamma}_s(t)^2 + \underbrace{g(a, \gamma_s(t))}_{\leq g(\gamma_r(t), \gamma_s(t))} \ \dot{\gamma}_\alpha(t)^2 \Big)^{\frac{1}{2}} dt \\
&< \ell(\gamma|_{[0,t_1]}) + \ell(\gamma|_{[t_2,1]}) + \int_{t_1}^{t_2} \|\dot{\gamma}(t)\|_{\gamma(t)} dt \\
&= \ell(\gamma),
\end{aligned}$$

what contradicts the fact that the geodesic γ has minimal length under all piecewise smooth curves joining p_0 and p_1. Hence necessarily $\gamma([0,1]) \subset U_a$, i.e. the region U_a is convex. □

We are now going to prove three lemmata about the behaviour of the component processes R, S and A of the Brownian motion when $t \to \zeta$. As for the moment we do not have any knowledge about the lifetime ζ of B. That means we consider the case of infinite lifetime as well as the case of finite lifetime. It is a direct consequence of what we are going to show in the following section (see Section 3.6, Remark 3.23) that indeed the lifetime ζ of the Brownian motion is almost surely finite and that the component R_t converges almost surely to ∞ when $t \to \zeta$.

Lemma 3.7. *The component R of the Brownian motion B converges almost surely for $t \to \zeta$ and*

$$\lim_{t \to \zeta} R_t > 2 \qquad \textit{almost surely.}$$

Proof. Define $u : M \to \mathbb{R}_+,\ (r, s, \alpha) \mapsto 1 - \tanh(r/2)$. Then we have

$$\begin{aligned}
\frac{\partial}{\partial r} u &= -\frac{1}{2} \frac{1}{\cosh^2(r/2)}, \\
\frac{\partial^2}{\partial r^2} u &= \frac{1}{2} \tanh(r/2) \frac{1}{\cosh^2(r/2)} .
\end{aligned}$$

and therefore

$$\begin{aligned}
\Delta_M u &= \frac{1}{2} \tanh(r/2) \frac{1}{\cosh^2(r/2)} - \left(\frac{h'_r}{2h} + \frac{g'_r}{2g} \right) \frac{1}{2} \frac{1}{\cosh^2(r/2)} \\
&= \frac{1}{2} \frac{1}{\cosh^2(r/2)} (\tanh(r/2) - \tanh(r)) - \frac{1}{4} \frac{g'_r}{g} \frac{1}{\cosh^2(r/2)} \\
&\leq \frac{1}{2} \frac{1}{\cosh^2(r/2)} (\tanh(r/2) - \tanh(r)) \\
&= \frac{1}{2} \frac{1}{\cosh^2(r/2)} \left(\frac{\sinh(r)}{\cosh(r) + 1} - \frac{\sinh(r)}{\cosh(r)} \right) \leq 0,
\end{aligned} \tag{3.10}$$

where we have used $h(r) = \cosh^2(r)$ and $g'_r \geq 0$.

Hence u is a non-negative bounded Δ_M-superharmonic function. Consequently $(u(B_t))_{t<\zeta}$ is a non-negative (right-)continuous supermartingale and therefore almost surely has a limit in $[0,1]$ for $t \to \zeta$, which implies that almost surely R_t converges for $t \to \zeta$.

Moreover, as $p(r,s) = 0$ for $r \leq 2$, we know that the function $g(r,s)$ does not depend on the variable s for $r \leq 2$. This implies that the sectional curvatures Sect^M in the convex region $\{(r,s,\alpha) | \, r \leq 2\} \subset M$ are bounded. Consequently the Brownian motion cannot explode inside this region within finite time. We are going to explain this fact below (see Remark 3.10). On the other hand, the defining stochastic differential equation (3.7)

$$dR_t = \left(\frac{h'_r(R_t)}{4h(R_t)} + \frac{g'_r(R_t, S_t)}{4g(R_t, S_t)} \right) dt \; + dW^1,$$

with non-negative drift term $\frac{h'_r(R_t)}{4h(R_t)} + \frac{g'_r(R_t,S_t)}{4g(R_t,S_t)}$, implies that $R_t \geq Y_t$ almost surely for $t \leq \zeta$, where Y is the solution of the stochastic differential equation

$$dY = dW^1$$

with $Y_0 = R_0$. This is an application of the "comparison theorem of Ikeda-Watanabe", which can be found for example in [Ha-Th], p.341. As R_t dominates a Euclidean Brownian motion it is clear that every converging path of B with infinite lifetime has necessarily limit $+\infty$, in particular it exits from the region $\{(r,s,\alpha) | \, r \leq 2\}$. Hence $\lim_{t \to \zeta} R_t > 2$ which finishes the proof. □

Remark 3.8 *(The superharmonic function u used in the proof above).*
The choice of the superharmonic function u is motivated by the following proposition that can be found in [A1]:

Proposition 3.9 (cf. [A1], Prop.6.1).
Let M be a complete simply connected Riemannian manifold whose sectional curvatures are bounded from above by $-k^2$, for a real constant $k > 0$, and let C be a closed convex subset of M, $C \neq \emptyset$.
Set $u(m) := 1 - \tanh(\mathrm{dist}(m, C)/2)$, $m \in M$. Then u is a superharmonic function on M.

In our situation we take the geodesic axis L as a convex subset of M and the coordinate r of a point $p = (r,s,\alpha) \in M$ is the Riemannian distance $\mathrm{dist}(p, L)$ of p from the convex set L. Therefore we could have used the function $u = 1 - \tanh(r/2)$ as a superharmonic function due to the proposition above without giving a separate proof of its superharmonicity.

As mentioned in the foregoing proof we want to give an explanation of the fact that the Brownian motion B of M cannot explode inside the convex set $\{(r,s,\alpha) | \, r \leq 2\}$ in finite time. This is caused by bounds for the sectional curvature in the considered region wich have the following effect on the *radial part* of the Brownian motion:

Remark 3.10 *(The effect of curvature bounds on the behaviour of the Brownian motion).*
It is well known that on a complete Riemannian manifold M of dimension d with strictly negative sectional curvature one can introduce a system of polar coordinates (r, ϑ) where

for a point $p \in M$ the coordinate r denotes the Riemannian distance $\mathrm{d}(0,p)$ of p from a fixed point 0 of M (the *pole*) and $\vartheta \in S^{d-1}$ the unit vector at 0 tangent to the minimizing geodesic that connects 0 to p. Considering a Brownian motion B on M the *radial part* $r(B_t)$ of B satisfies – according to geometric Itô formula, see [Ha-Th], Theorem 7.145 – the stochastic differential equation

$$dr(B_s) = dW_s + \frac{1}{2}\Delta_M r(B_s)ds \quad \text{for } s \leq \zeta, \tag{3.11}$$

with W a Euclidean real Brownian motion, cf. [Ha-Th] for details.

Assume now that in a convex region $U \subset M$, where $0 \in U$, the sectional curvatures Sect^M of M are bounded, in particular $\mathrm{Sect}_p^M \geq -c^2$ for a constant $c \in \mathbb{R}$ and all $p \in U$. Then one can immediately deduce that the Ricci curvature in radial direction satisfies the following inequality

$$\mathrm{Ric}_p(\partial^M, \partial^M) \geq -(\dim(M) - 1)c^2 = -(d-1)c^2$$

for all $p \in U$.

Comparing M with the model $\mathbb{M} = \{x \in \mathbb{R}^d : \|x\| < 1/c\}$ equipped with the Riemannian metric

$$g_x(u,v) := \frac{4\langle u, v\rangle}{(1 - \|x\|^2 c^2)^2}$$

the comparison theorem for the Laplacian (cf. [Ha-Th], Theorem 7.243) yields:

$$\Delta_M r(p) \leq (d-1)\; c \cdot \coth(c \cdot r(p))$$

for all $p \in U$. Herein the right-hand-side of the inequality equals $\Delta_{\mathbb{M}} r(\tilde{p})$ where $\tilde{p} \in \mathbb{M}$ with $r(\tilde{p}) = r(p)$.
For the computation of $\Delta_{\mathbb{M}} r(\tilde{p})$ one uses the fact that on a model

$$\Delta_{\mathbb{M}} r(\tilde{p}) = (\dim(\mathbb{M}) - 1) \cdot \frac{f'(r(\tilde{p}))}{f(r(\tilde{p}))}$$

where f is the coupling function in the polar coordinate representation

$$ds^2 = dr^2 + f(r)^2 d\vartheta^2$$

of the Riemannian metric on $\mathbb{M}$. Since $\mathbb{M}$ is a model, f does not depend on ϑ. For the model used here, $f(r) = 1/c \cdot \sinh(c \cdot r)$, which provides the given expression.

Returning to the radial part of our Brownian motion we can make use of the upper inequality by applying a comparison theorem for stochastic differential equations (cf. for example [Ha-Th], Theorem 6.49): As the drift term of the defining differential equation (3.11) of $r(B_s)$ can be estimated from above, one concludes that $r(B_s) \leq Y_s$ for $s \leq \tau$ where τ is the first exit time of B from U and Y is the solution of the stochastic differential equation

$$dY_s = dW_s + 1/2 \cdot (\dim(M) - 1)\, c \cdot \coth(c Y_s) ds$$

starting in $Y_0 = 0$. It is an application of the theory of one dimensional Itô stochastic differential equations (cf. [Ha-Th], Theorem 6.50) to show that Y_s almost surely has infinite lifetime. It is therefore a matter of fact, that also the radial part $r(B)$ of B cannot go to infinity within finite time inside the region U, in other words: $r(B_t) \to \infty$ in finite time is only possible if B finally exits from U.

To proceed with the proof of Theorem 3.4 we need some technical remarks about Δ_M-superharmonic functions:

Proposition 3.11. *Let $u : M \to \mathbb{R}, (r, s, \alpha) \mapsto u(r, s)$, be nonincreasing in s and r and further superharmonic with respect to*

$$L := \frac{\partial^2}{\partial r^2} + \frac{1}{h}\frac{\partial^2}{\partial s^2} + \left(\frac{h'_r}{2h} + \frac{1}{2}h'_r\right)\frac{\partial}{\partial r}.$$

Then u is also Δ_M-superharmonic on M.

Proof. We have $u'_r \le 0$, $u'_s \le 0$ and $Lu \le 0$. Hence

$$\begin{aligned} \Delta_M u &= \frac{\partial^2}{\partial r^2}u + \frac{1}{h}\frac{\partial^2}{\partial s^2}u + \left(\frac{h'_r}{2h} + \frac{g'_r}{2g}\right)\frac{\partial}{\partial r}u + \frac{g'_s}{2gh}\frac{\partial}{\partial s}u \\ &\le \frac{\partial^2}{\partial r^2}u + \frac{1}{h}\frac{\partial^2}{\partial s^2}u + \left(\frac{h'_r}{2h} + \frac{1}{2}h'_r\right)\frac{\partial}{\partial r}u = Lu \le 0. \end{aligned} \tag{3.12}$$

Here we used $g'_s \ge 0$ and

$$\frac{g'_r}{g} \ge h'_r \text{ for all } (r, s) \in \mathbb{R}_+ \times \mathbb{R},$$

because for $r \le 1/10$ the function $g(r, s)$ is just $\sinh^2(r)$ and so obviously $g'_r \ge h'_r g$ whereas $\frac{g'_r}{g} \ge h'_r$ for $r \ge 1/10$ is due to Property (ii) of the function g. □

Lemma 3.12. *The component S_t of the Brownian motion B_t converges almost surely for $t \to \zeta$ and*

$$\lim_{t\to\zeta} S_t \in (-\infty, \infty] \qquad \textit{almost surely.}$$

Proof. For an arbitrary $s_0 \in \mathbb{R}$ let $u_{s_0} : M \to \mathbb{R}$, $(r, s, \alpha) \mapsto u_{s_0}(r, s)$, be given as:

$$u_{s_0}(r, s) := \begin{cases} 1 & \text{if } s \le s_0, \\ 1 - \frac{2}{\pi}\arctan\left((s - s_0)\sinh(r)\right) & \text{if } s > s_0. \end{cases} \tag{3.13}$$

Then

$$\begin{aligned} \frac{\partial}{\partial r}u_{s_0} &= -\frac{2}{\pi}\,\frac{(s - s_0)\cosh(r)}{1 + (s - s_0)^2\sinh^2(r)}, \\ \frac{\partial^2}{\partial r^2}u_{s_0} &= -\frac{2}{\pi}\,\frac{(s - s_0)\sinh(r) + (s - s_0)^3\sinh^3(r) - 2(s - s_0)^3\sinh(r)\cosh^2(r)}{(1 + (s - s_0)^2\sinh^2(r))^2}, \end{aligned}$$

$$\frac{\partial}{\partial s} u_{s_0} = -\frac{2}{\pi} \frac{\sinh(r)}{1 + (s - s_0)^2 \sinh^2(r)},$$
$$\frac{\partial^2}{\partial s^2} u_{s_0} = -\frac{2}{\pi} \frac{-2(s - s_0) \sinh^3(r)}{(1 + (s - s_0)^2 \sinh^2(r))^2}.$$

As u is obviously decreasing in s and r according to Proposition 3.11 it is enough to show that $Lu \leq 0$ (at least on a suitable subset of M), for the partial differential operator L defined as in the proposition. Then u_{s_0} is also a Δ_M-superharmonic function.

With the negative function $n : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $n(r, s) := -\frac{\pi}{2} \cdot h(r) \cdot \left(1 + (s - s_0)^2 \sinh^2(r)\right)^2$ we have for $s \geq s_0$

$$\begin{aligned} n(r,s) \cdot Lu_{s_0} &= -2(s - s_0) \sinh^3(r) + h(s - s_0) \sinh(r) \\ &\quad + h(s - s_0)^3 \sinh^3(r) - 2h(s - s_0)^3 \sinh(r) \cosh^2(r) \\ &\quad + \frac{1}{2} \cdot h'_r(s - s_0) \cosh(r) \left(1 + (s - s_0)^2 \sinh^2(r)\right) \\ &\quad + \frac{1}{2} \cdot hh'_r(s - s_0) \cosh(r) \left(1 + (s - s_0)^2 \sinh^2(r)\right). \end{aligned} \tag{3.14}$$

Using $h(r) = \cosh^2(r)$ we can proceed:

$$\begin{aligned} n(r,s) \cdot Lu_{s_0} =& -2(s - s_0) \sinh^3(r) + (s - s_0) \sinh(r) \cosh^2(r) \\ &+ (s - s_0)^3 \sinh^3(r) \cosh^2(r) - 2(s - s_0)^3 \sinh(r) \cosh^4(r) \\ &+ (s - s_0) \sinh(r) \cosh^2(r) + (s - s_0)^3 \sinh^3(r) \cosh^2(r) \\ &+ (s - s_0) \cosh^4(r) \sinh(r) + (s - s_0)^3 \cosh^4(r) \sinh^3(r) \\ =& (s - s_0) \sinh(r) \cosh^2(r) \left(-2 \tanh^2(r) + 2 + \cosh^2(r)\right) \\ &+ (s - s_0)^3 \sinh(r) \cosh^4(r) \left(2 \tanh^2(r) - 2 + \sinh^2(r)\right). \end{aligned}$$

Obviously $-2 \tanh^2(r) + 2 + \cosh^2(r)$ is strictly positive for all $r \in \mathbb{R}_+$ and $2 \tanh^2(r) - 2 + \sinh^2(r)$ is positive at least for $r \geq 1$. Hence we have

$$n(r, s) \cdot Lu_{s_0} \geq 0 \text{ for } r \geq 1.$$

As $n(r, s)$ is negative for all $(r, s) \in \mathbb{R}_+ \times \mathbb{R}$, the function u_{s_0} has to be superharmonic with respect to L in the region $\{(r, s, \alpha) | r \geq 1\} \subset M$, which implies that u_{s_0} is also Δ_M-superharmonic on the region $U := \{(r, s, \alpha) | r \geq 1\} \subset M$.

The set U is an *absorbing region* for B, i.e. the Brownian motion finally reaches U before leaving the manifold M. Consequently we know that the process $u_{s_0}(B_t)$ is a positive supermartingale as soon as B reaches U and therefore $u_{s_0}(B_t)$ almost surely admits a limit in $[0, 1]$ for $t \to \zeta$, cf. [D-M] and [A1].

In particular $u_{s_0}(B_t)$ almost surely converges for all $s_0 \in \mathbb{Q}$. For that reason there is a set N of $\mathbb{P}$-measure 0, such that

$$\Omega \setminus N \subset \bigcap_{s_0 \in \mathbb{Q}} \left\{ u_{s_0}(B_t) \text{ converges for } t \to \zeta \text{ and } \lim_{t \to \zeta} R_t \geq 1 \right\}$$

We want to show that $S_t(\omega)$ almost surely converges for $t \to \zeta$ and $\omega \in \Omega \setminus N$. Assuming that $S_t(\omega)$ does not possess a limit for $t \to \zeta$, there is a $s_0 \in \mathbb{Q}$ with $S_t(\omega) \geq s_0 + \delta$ and $S_t(\omega) \leq s_0$ again and again for a $\delta > 0$. Then again and again $u_{s_0}(B_t(\omega)) = 1$ and $u_{s_0}(B_t(\omega)) = 1 - \frac{2}{\pi} \arctan((S_t(\omega) - s_0) \sinh(R_t(\omega)))$. Because $S_t(\omega) - s_0 \geq \delta$ and $R_t(\omega) \to: r \geq 2$ for $t \to \zeta$ one has $(S_t(\omega) - s_0) \sinh(R_t(\omega)) \to: C \geq \delta \sinh(2) > 0$, what implies $1 - \frac{2}{\pi} \arctan\left((S_t(\omega) - s_0) \sinh(R_t(\omega))\right) \to: \tilde{c} < 1$. This is a contradiction to the fact that $u_{s_0}(B_t(\omega))$ converges for $t \to \zeta$.

Hence we have shown that $S_t(\omega)$ converges in $[-\infty, \infty]$ for $t \to \zeta$ and all $\omega \in \Omega \setminus N$ which shows that S_t has a limit for $t \to \zeta$ almost surely.

To finish the proof of the lemma we have to exclude $-\infty$ as a possible value for $\lim_{t \to \zeta} S_t$. As in [A1] we use a supermartingale argument (see the remark after the proof) to obtain

$$u_{s_0}(p) \geq \mathbb{P}^p \left\{ \lim_{t \to \zeta} S_t = -\infty \right\} \qquad \text{for all } s_0 \in \mathbb{Q},$$

where $p = (r, s, \alpha) \in M$ is the starting point of the Brownian motion B.

Assume now that $\mathbb{P}^p \{\lim_{t \to \zeta} S_t = -\infty\} = \delta > 0$ for a point $p = (r, s, \alpha) \in M$. As $u_{s_0}(p) \to 0$ for $s_0 \to -\infty$ and $p \in M$ fixed, one can choose $s_0 < s, s_0 \in \mathbb{Q}$, such that

$$u_{s_0}(p) = 1 - \frac{2}{\pi} \arctan((s - s_0) \sinh(r)) < \delta,$$

which yields the desired contradiction. Hence $\lim_{t \to \zeta} S_t \in (-\infty, \infty]$ almost surely. □

Remark 3.13. To derive the results concerning the component S_t of our Brownian motion we made use of the functions u_{s_0} that are proven to be Δ_M-superharmonic on the region $\{(r, s, \alpha) | \, r \geq 1\} \subset M$. As we know from Lemma 3.7, the Brownian motion B will eventually enter this region and stay inside up to the lifetime ζ. However, there is no (stopping) time, from that on the Brownian motion stays inside $\{(r, s, \alpha) | r \geq 1\}$ for all time. Hence we cannot think of $u_{s_0}(B_t)$ being a supermartingale on a stochastic interval $[\tau, \zeta[$.

To nevertheless use convergence of bounded supermartingales and supermartingale inequalities as done in the proof above we construct a supermartingale $(W_t)_{t<\zeta}$ depending on the given superharmonic functions u_{s_0} and the component S_t of B_t as follows:

Let $\sigma_n \nearrow \zeta, \sigma_n < \sigma_{n+1}$, a sequence of stopping times with $R_{\sigma_n} \geq 3/2$. Here we use from Lemma 3.7 that $\lim_{t \to \zeta} R_t \geq 2$ almost surely. Define

$$\tau_n' := \inf\{t > \sigma_n | R_t = 1\} \text{ and } \tau_n := \tau_n' \wedge \sigma_{n+1}, \text{ where by convention } \inf \emptyset = \infty.$$

As $B_t(\omega) \in \{(r, s, \alpha) | r \geq 1\}$ eventually, there is, for any ω outside a set of measure 0, an $N(\omega)$ such that $\tau_n(\omega) = \sigma_{n+1}(\omega)$ for all $n \geq N(\omega)$.

Without loss of generality let $p = (r, s, \alpha) \notin \{(r, s, \alpha)|r \geq 1\}$ denote the starting point of the Brownian motion B and $s_0 < s$, $s_0 \in \mathbb{Q}$. The case where B_t starts inside $\{(r, s, \alpha)|r \geq 1\}$ can be treated analogously, mainly by omitting the first term in the following expression.

We then define an adapted process $(W_t)_{t\leq\zeta}$ given as:

$$W_t := 1_{[0,\sigma_1]}(t) \cdot u_{s_0}(p) + \sum_{n=1}^{\infty} 1_{]\sigma_n,\tau_n]}(t) \cdot u_{s_n}(B_t) + \sum_{n=1}^{\infty} 1_{]\tau_n,\sigma_{n+1}]}(t) \cdot u_{s_n}(B_{\tau_n}),$$

where by convention $S_{\tau_0} := s$ and for every $n \in \mathbb{N}$ the function $s_n : \Omega \to \mathbb{R}$ is given as:

$$s_n(\omega) := \sum_{i=1}^{n} \left(S_{\sigma_i(\omega)}(\omega) - S_{\tau_{i-1}(\omega)}(\omega)\right) + s_0.$$

The process $(W_t)_{t\leq\zeta}$ then has the following properties:

i) $W|_{]\tau_n,\sigma_{n+1}]}$, $W|_{[0,\sigma_1]}$ respectively, is a supermartingale as a pathwise constant process.
ii) $W|_{]\sigma_n,\tau_n]}$ is a supermartingale because $B|_{]\sigma_n,\tau_n]} \in \{(r, s, \alpha)|r \geq 1\}$, where u_{s_0} is Δ_M-superharmonic for every $s_0 \in \mathbb{R}$.
iii) $W_{\tau_n(\omega)}(\omega) = W_{\tau_n(\omega)+}(\omega)$ and

$$W_{\sigma_n(\omega)+}(\omega) = u_{s_n(\omega)}\left(B_{\sigma_n(\omega)}(\omega)\right) \leq u_{s_{n-1}(\omega)}\left(B_{\tau_{n-1}(\omega)}(\omega)\right) = W_{\sigma_n(\omega)}(\omega).$$

From that we conclude that $(W_t)_{t<\zeta}$ is a bounded supermartingale, which therefore almost surely has a finite limit $\lim_{t\to\zeta} W_t$ and the supermartingale inequality yields:

$$\begin{aligned}
u_{s_0}(p) &\geq \mathbb{E}^p\left(\lim_{t\to\zeta} W_t\right) = \int_{\{\lim_{t\to\zeta} W_t=1\}} \lim_{t\to\zeta} W_t \, d\mathbb{P}^p + \int_{\{\lim_{t\to\zeta} W_t\neq 1\}} \lim_{t\to\zeta} W_t \, d\mathbb{P}^p \\
&\geq \mathbb{P}^p\left\{\lim_{t\to\zeta} W_t = 1\right\} = \mathbb{P}^p\left\{\lim_{t\to\zeta} 1_{]\sigma_{N(\cdot)},\zeta]}(t) \cdot u_{s_{N(\cdot)}}(B_t) = 1\right\} \\
&\geq \mathbb{P}^p\left\{\lim_{t\to\zeta} S_t = -\infty\right\}.
\end{aligned}$$

This is the inequality we used in the proof above.

Before we are going to show that also the component A_t of our Brownian motion B_t converges to an almost surely finite random variable A_ζ, we will state another little proposition that will be used in the proof of Lemma 3.15:

Proposition 3.14. *There is a constant $C \in \mathbb{R}_+$ such that a given function $u : M \to \mathbb{R}$, $(r, s, \alpha) \mapsto u(r, \alpha)$ that is nonincreasing in r, convex in α and L_C-superharmonic for*

$$L_C := \frac{\partial^2}{\partial r^2} + \frac{1}{Ce^r}\frac{\partial^2}{\partial \alpha^2} + \frac{1}{2}\frac{\partial}{\partial r},$$

is also Δ_M-superharmonic on $\{(r, s, \alpha) \in M|r \geq 0,5\}$.

Proof. We know from Property (ii) of the coupling function g that

$$g'_r \geq h'_r g \quad \text{and} \quad \frac{1}{2}g''_{rr} - \frac{1}{4}\frac{(g'_r)^2}{g} \geq \frac{h'_r g'_r}{8} \qquad \text{for } r \geq \frac{1}{10}.$$

Hence we get

$$\frac{g''_{rr}}{g} \geq \frac{h'_r g'_r}{4g} + \frac{(g'_r)^2}{2g^2} \geq \frac{(h'_r)^2}{4} + \frac{(h'_r)^2}{2} = 3\cosh^2(r)\sinh^2(r) \geq 1, \qquad \text{for } r \geq 0,5.$$

Using a Gronwall-type argument (cf. [A1]) we conclude that $g(r,s) \geq C \cdot e^r$ for all $r \geq 0,5$, all $s \in \mathbb{R}$ and a constant $C \in \mathbb{R}_+$.

For $u : M \to \mathbb{R}$, $u(r,s,\alpha) \equiv u(r,\alpha)$ nonincreasing in r, convex in α and superharmonic with respect to L_C, one has:

$$\begin{aligned}\Delta_M u =& \frac{1}{h}\frac{\partial^2}{\partial s^2}u + \frac{\partial^2}{\partial r^2}u + \frac{1}{g}\frac{\partial^2}{\partial \alpha^2}u + \frac{g'_s}{2gh}\frac{\partial}{\partial s}u + \left(\frac{h'_r}{2h} + \frac{g'_r}{2g}\right)\frac{\partial}{\partial r}u \\ =& \frac{\partial^2}{\partial r^2}u + \frac{1}{g}\frac{\partial^2}{\partial \alpha^2}u + \left(\frac{h'_r}{2h} + \frac{g'_r}{2g}\right)\frac{\partial}{\partial r}u \leq L_C u \leq 0, \qquad \text{for } r \geq 0,5.\end{aligned}$$

This relies on the fact that $g^{-1} \leq (Ce^r)^{-1}$ and

$$\left(\frac{h'_r}{2h} + \frac{g'_r}{2g}\right)\frac{\partial}{\partial r}u \leq \left(\frac{h'_r}{2h} + \frac{h'_r}{2}\right)\frac{\partial}{\partial r}u \leq \frac{1}{2}\frac{\partial}{\partial r}u$$

for r such that $h'_r/(2h) + 1/2 \cdot h'_r = \tanh(r) + \cosh(r)\sinh(r) \geq 1/2$, which is true for $r \geq 1/2$. □

Lemma 3.15. *The component A_t of the Brownian motion B_t converges almost surely for $t \to \zeta$ and*

$$\lim_{t\to\zeta} A_t \in (-\infty, \infty) \quad \textit{almost surely.}$$

Proof. Let $C \in \mathbb{R}$ and L_C be given as in the proposition above.
For an arbitrary $\alpha_0 \in \mathbb{R}$ define $u_{\alpha_0} : M \to \mathbb{R}, (r,s,\alpha) \mapsto u_{\alpha_0}(r,\alpha)$ as:

$$u_{\alpha_0}(r,\alpha) := \begin{cases} 1 & \text{if } \alpha \leq \alpha_0, \\ 1 - \frac{2}{\pi}\arctan\left(\frac{1}{2}(\alpha-\alpha_0)\sqrt{C}\cdot e^{r/2}\right) & \text{if } \alpha > \alpha_0. \end{cases} \tag{3.15}$$

Then

$$\begin{aligned}\frac{\partial}{\partial r}u_{\alpha_0} &= -\frac{2}{\pi}\cdot\frac{\frac{1}{4}(\alpha-\alpha_0)\sqrt{C}e^{r/2}}{1+\frac{1}{4}(\alpha-\alpha_0)^2Ce^r}, \\ \frac{\partial^2}{\partial r^2}u_{\alpha_0} &= -\frac{2}{\pi}\cdot\frac{\frac{1}{8}(\alpha-\alpha_0)\sqrt{C}e^{r/2} - \frac{1}{32}(\alpha-\alpha_0)^3C^{3/2}e^{3/2r}}{\left(1+\frac{1}{4}(\alpha-\alpha_0)^2Ce^r\right)^2}, \\ \frac{\partial^2}{\partial \alpha^2}u_{\alpha_0} &= -\frac{2}{\pi}\cdot\frac{-\frac{1}{4}(\alpha-\alpha_0)C^{3/2}e^{3/2r}}{\left(1+\frac{1}{4}(\alpha-\alpha_0)^2Ce^r\right)^2}.\end{aligned}$$

Obviously u_{α_0} is nondecreasing in r and convex in α. Moreover:

$$\begin{aligned} L_C\, u =& \frac{\partial^2}{\partial r^2}\, u_{\alpha_0} + \frac{1}{Ce^r}\frac{\partial^2}{\partial \alpha^2}\, u_{\alpha_0} + \frac{1}{2}\frac{\partial}{\partial r}\, u_{\alpha_0} \\ =& -\frac{2}{\pi}\left[\frac{\frac{1}{8}(\alpha-\alpha_0)\sqrt{C}e^{r/2} - \frac{1}{32}(\alpha-\alpha_0)^3C^{3/2}e^{3r/2} - \frac{1}{4}(\alpha-\alpha_0)\sqrt{C}e^{r/2}}{\left(1+\frac{1}{4}(\alpha-\alpha_0)^2Ce^r\right)^2}\right. \\ &+ \left.\frac{\frac{1}{8}(\alpha-\alpha_0)\sqrt{C}e^{r/2} + \frac{1}{32}(\alpha-\alpha_0)^3C^{3/2}e^{3r/2}}{\left(1+\frac{1}{4}(\alpha-\alpha_0)^2Ce^r\right)^2}\right] = 0. \end{aligned}$$

Using the proposition above u_{α_0} is Δ_M-superharmonic for every $\alpha_0 \in \mathbb{R}$ on the region $\{(r,s,\alpha) \in M | r \geq 1/2\}$. Now the assertion that the component A_t of the Brownian motion B_t converges almost surely for $t \to \zeta$ follows analogously to the proof of Lemma 3.12 as well as the fact that $\lim_{t\to\zeta} A_t \neq -\infty$ almost surely.

For the remaining statement that also $\lim_{t\to\zeta} A_t \neq \infty$, i.e. that A_t converges to a finite random variable A_ζ, define new functions $u_{\alpha_0} : M \to \mathbb{R}$, for $\alpha_0 \in \mathbb{R}$, as

$$u_{\alpha_0}(r,\alpha) := \begin{cases} 1 - \frac{2}{\pi}\arctan\left(\frac{1}{2}(\alpha_0-\alpha)\sqrt{C}\cdot e^{r/2}\right) & \text{if } \alpha \leq \alpha_0 \\ 1 & \text{if } \alpha > \alpha_0. \end{cases} \tag{3.16}$$

It turns out that these functions are also superharmonic on the absorbing region $\{(r,s,\alpha) \in M | r \geq 1/2\}$ of M and yield the desired (supermartingale) inequality to conclude that $\mathbb{P}^p\{\lim_{t\to\zeta} A_t = \infty\} = 0$. □

We are now in the situation to prove statement b) of our main Theorem 3.4. Therefore it obviously suffices to show that the random variable $A_\zeta := \lim_{t\to\zeta} A_t$ is not almost surely constant. For a given continuous function $f : S^1 \to \mathbb{R}$ the function

$$h : M \to \mathbb{R},\ h(p) := \mathbb{E}^p\left(f \circ A_\zeta\right) = \mathbb{E}^p\left(f \circ \lim_{t\to\zeta}\left(\mathrm{pr}_3 \circ B_t\right)\right)$$

is then Δ_M-harmonic on M and if $p := (r,s,\alpha)$ tends to $L(\infty) \in S_\infty(M)$ such that $s \to \infty$ and $\alpha \to: \alpha_\infty \in \mathbb{R}$, one has $h(p) \to f(\alpha_\infty)$.

Lemma 3.16. *The component A_t of the Brownian motion B_t converges almost surely to a non-trivial (i.e. almost surely non-constant) random variable $A_\zeta : \Omega \to \mathbb{R}$.*

Proof. According to (3.7) the component A_t of B_t is given by means of the stochastic differential equation

$$dA_t = \frac{1}{\sqrt{g(R_t,S_t)}}dW_t$$

with a one dimensional real Brownian motion W.
One has therefore

$$A_t = A_0 + \int_0^t \frac{1}{\sqrt{g(R_u,S_u)}}dW_u,$$

where the Brownian integral

$$\int_0^t \frac{1}{\sqrt{g(R_u, S_u)}} dW_u,$$

obviously does not depend on A and therefore in particular does not depend on the initial value A_0 of A; we mentioned this before. Moreover, the lifetime ζ of the Brownian motion is also independent of A_0. If we now consider the almost surely finite random variable $\lim_{t\to\zeta} A_t$, we have

$$A_\zeta := \lim_{t\to\zeta} A_t = A_0 + \int_0^\zeta \frac{1}{\sqrt{g(R_u, S_u)}} dW_u,$$

where the integral-term does not depend on the starting point A_0 from A. Therefore it is clear, that A_ζ cannot be almost sure equal to a constant, independent of the starting point (A_0, R_0, S_0) of the Brownian motion. This proves that A_ζ is a non-trivial shift-invariant random variable, which finishes the proof.

□

We are now going to state and prove the final Lemma, which together with the foregoing Lemmata 3.7, 3.12 and 3.15 yields the fact that the Brownian motion will almost surely exit the manifold M at the single point $L(\infty) \in S_\infty(M)$.
As we already mentioned in Section 3.2, for this result it suffices to show that almost surely $\lim_{t\to\zeta} S_t = \infty$.

The following proposition explains how the special choice of the function g influences the behaviour of the components S_t and R_t of the Brownian motion. In fact it will be shown that in certain regions of M the drift vector of the defining stochastic differential equation will urge the component S_t to grow "faster" than the component R_t, which in conclusion implies that $\lim_{t\to\zeta} S_t = \infty$. As it is obvious that the effect of the drift vector in the stochastic differential equation for the component S is mainly influenced by the explicit definition of the function $g(r, s)$ we will briefly recall the properties of g that are used below to formulate the proposition:

As we have seen in Section 3.4 the function $g(r, s)$ is given as the solution of the partial differential equation

$$g'_s(r, s) = p(r, s)h(r)g'_r(r, s),$$

where the function $p(r, s)$ is defined as $p(r, s) = \chi(r, s)p_0(r) \equiv \xi(s+\ell(r))p_0(r)$. See Section 3.4 for the explicit definition of the functions ξ, ℓ and p_0.
As $\lim_{r\to\infty} \ell(r) = \infty$ and ℓ is increasing, for given s one can choose r_s big enough such that $s + \ell(r) > 4$ for all $r \geq r_s$, what implies that $\xi(s + \ell(r)) = \frac{1}{2}$.
We also recall the fact that there is a sequence $r_1 < r_2 < \cdots < r_{2n-1} < r_{2n} < r_{2n+1} \nearrow \infty$ such that $p_0(r)$ is constant for $r \in [r_{2n}, r_{2n+1}]$, $n \in \mathbb{N}$. It is the main part of the following proposition to show that we can (if necessary) slightly modify the "stripes", where p_0 is constant, such that within these stripes preferably the absolute value $|S_t|$ of the s-component of the Brownian motion B_t grows. As it is easy to see, enlarging the intervals $[r_{2n}, r_{2n+1}]$ does not influence the eventual behaviour of the components R_t, S_t and A_t of

the Brownian motion which we proved in the preceding lemmata. With this in mind we can now state the following technical proposition.

Proposition 3.17. *Let $a, b \in \mathbb{R}$, $a < b$ and $\ell : \mathbb{R} \to \mathbb{R}$ the function used in the definition of the coupling function g, (c.f. Section 3.4). Let further $\eta > 0$.*
For given $r_{2n} \geq 1$ with $\ell(r_{2n})+a > 4$ one can find $r_{2n+1} > r_{2n}$ such that for every Brownian motion $B_t = (R_t, S_t, A_t)$ starting in $p = (r, s, \alpha)$, where $a < s < b$ and $r = \frac{1}{2}(r_{2n} + r_{2n+1})$, one has

$$\mathbb{P}^p\{R_\tau = r_{2n} \text{ or } R_\tau = r_{2n+1}\} \leq \eta.$$

Herein τ is the first exit time of B from the set

$$U_{ab} := \{(r, s, \alpha) \in M | \, a < s < b, r_{2n} < r < r_{2n+1}\}.$$

Proof. To find r_{2n+1} with the desired properties we assume for the moment that $p_0(r) = c$ for a constant $c \in \mathbb{R}_+$ for all $r \in [r_{2n}, \infty]$.

It is

$$g_s'(r, s) = p(r, s)h(r)g_r'(r, s) \quad \text{with } p(r, s) = \xi(s + l(r))p_0(r).$$

As ℓ is increasing and for all $a < s < b$, $r \in [r_{2n}, \infty[$ we have by assumption $s + \ell(r) > a + \ell(r_{2n}) > 4$, i.e. $\xi(s + \ell(r)) = \frac{1}{2}$ for all $r \geq r_{2n}$ and all $a < s < b$. This means that we can use $p(r, s) = \frac{1}{2} \cdot c$ for all $a < s < b$ and $r \in [r_{2n}, \infty]$.
Let $\sigma(r, s) := \exp(\varepsilon r - \delta s)$, where $\varepsilon, \delta > 0$. Then for $a < s < b$ and $r \geq r_{2n}$:

$$\begin{aligned}
\sigma^{-1}\Delta_M\sigma =& \frac{\delta^2}{h} + \varepsilon^2 - \delta \cdot \frac{g_s'}{2gh} + \varepsilon\left(\frac{h_r'}{2h} + \frac{g_r'}{2g}\right) \\
&\leq \frac{\delta^2}{\cosh^2(r_{2n})} + \varepsilon^2 - \delta \cdot p(r, s) \cdot \frac{g_r'}{2g} + \varepsilon\left(1 + \frac{g_r'}{2g}\right) \\
&= \frac{\delta^2}{\cosh^2(r_{2n})} + \varepsilon^2 - \delta \cdot \frac{1}{2}c \cdot \frac{g_r'}{2g} + \varepsilon\left(1 + \frac{g_r'}{2g}\right) \\
&= \frac{\delta^2}{\cosh^2(r_{2n})} - \frac{1}{4}c \cdot \delta \cdot \frac{g_r'}{2g} + \varepsilon^2 + \varepsilon + \frac{g_r'}{2g}\left(\varepsilon - \frac{1}{4}c \cdot \delta\right) \\
&\leq \frac{\delta^2}{\cosh^2(r_{2n})} - \frac{1}{4}c \cdot \delta \cdot \cosh(r_{2n})\sinh(r_{2n}) + \varepsilon^2 + \varepsilon + \frac{g_r'}{2g}\left(\varepsilon - \frac{1}{4}c \cdot \delta\right) \\
&= \delta \cdot \left(\frac{\delta}{\cosh^2(r_{2n})} - \frac{1}{4}c \cdot \cosh(r_{2n})\sinh(r_{2n})\right) + \varepsilon^2 + \varepsilon + \frac{g_r'}{2g}\left(\varepsilon - \frac{1}{4}c \cdot \delta\right).
\end{aligned}$$

Choose $\delta < \frac{1}{8}c \cdot \cosh^3(r_{2n})\sinh(r_{2n})$, then

$$\frac{\delta}{\cosh^2(r_{2n})} - \frac{1}{4}c \cdot \cosh(r_{2n})\sinh(r_{2n}) < -\delta \cdot \frac{1}{8}c \cdot \cosh(r_{2n})\sinh(r_{2n}).$$

For $\varepsilon < \frac{1}{8}c \cdot \delta$ one has:

$$\frac{g_r'}{2g}\left(\varepsilon - \frac{1}{4}c \cdot \delta\right) < -\frac{1}{8}c \cdot \frac{g_r'}{2g} \cdot \delta.$$

If we combine these two inequalities with the preceding estimate for $\sigma^{-1}\Delta_M \sigma$ we get

$$\begin{aligned}
\sigma^{-1}\Delta_M \sigma &< -\frac{1}{8}c \cdot \delta \cdot \cosh(r_{2n})\sinh(r_{2n}) + \varepsilon^2 + \varepsilon - \frac{1}{8}c \cdot \frac{g_r'}{2g} \cdot \delta \\
&\leq -\frac{1}{8}c \cdot \delta \cdot \cosh(r_{2n})\sinh(r_{2n}) + \varepsilon^2 + \varepsilon - \frac{1}{8}c \cdot \delta \cdot \cosh(r_{2n})\sinh(r_{2n}) \\
&\leq 0 \quad \text{for } \varepsilon \text{ small enough.}
\end{aligned}$$

With δ and ε chosen as above, $\sigma(r,s) := \exp(\varepsilon r - \delta s)$ is then Δ_M-superharmonic on $\{(r,s,\alpha) | a < s < b, r_{2n} \leq r\}$.

Let further $\varphi(r) := \exp(-r)$. Then

$$\Delta_M \varphi = e^{-r} - \left(\frac{h_r'}{2h} + \frac{g_r'}{2g}\right) e^{-r} \leq e^{-r}(1 - \tanh(r) - \cosh(r)\sinh(r)) \leq 0 \quad \text{for } r \geq 1.$$

Hence $\varphi(r) = \exp(-r)$ is Δ_M-superharmonic on $\{(r,s,\alpha) | a < s < b, r_{2n} \leq r\}$ as well.

We then choose r_{2n+1} so large that at the same time

$$\begin{aligned}
\exp(\delta(b-a)) \cdot \exp\left(-\frac{1}{2}\varepsilon(r_{2n+1} - r_{2n})\right) &\leq \frac{1}{2}\eta \ \text{ and} \\
\exp\left(-\frac{1}{2}(r_{2n+1} - r_{2n})\right) &\leq \frac{1}{2}\eta.
\end{aligned}$$

Then σ and φ are Δ_M-superharmonic functions on the set

$$U_{ab} := \{(r,s,\alpha) \in M | \, a < s < b, r_{2n} < r < r_{2n+1}\}$$

and for a Brownian motion B_t with starting point $p = (r,s,\alpha)$, where $a < s < b$ and $r = 1/2 \cdot (r_{2n+1} + r_{2n})$, one has:

$$\begin{aligned}
\sigma(p) \geq \mathbb{E}^p(\sigma(B_\tau)) = \int_\Omega \sigma(B_\tau)\, d\mathbb{P}^p &\geq \int_{\{R_\tau = r_{2n+1}\}} \exp(\varepsilon \cdot r_{2n+1} - \delta \cdot s)\, d\mathbb{P}^p \\
\geq \exp(\varepsilon \cdot r_{2n+1}) \int_{\{R_\tau = r_{2n+1}\}} &\exp(-\delta \cdot b)\, d\mathbb{P}^p \\
= \exp(\varepsilon \cdot r_{2n+1}) \cdot \exp(-\delta \cdot b)\, &\mathbb{P}^p\{R_\tau = r_{2n+1}\}.
\end{aligned}$$

This yields

$$\begin{aligned}\mathbb{P}^p\{R_\tau = r_{2n+1}\} &\le \exp(\delta\cdot b)\cdot\exp(-\varepsilon\cdot r_{2n+1})\cdot\exp\left(\frac{1}{2}\varepsilon\cdot(r_{2n+1}+r_{2n})-\delta\cdot s\right)\\ &\le \exp(\delta\cdot b)\cdot\exp(-\varepsilon\cdot r_{2n+1})\cdot\exp\left(\frac{1}{2}\varepsilon\cdot(r_{2n+1}+r_{2n})\right)\cdot\exp(-\delta\cdot a)\\ &= \exp(\delta\cdot(b-a))\cdot\exp\left(-\frac{1}{2}\varepsilon\cdot(r_{2n+1}-r_{2n})\right)\le\frac{1}{2}\eta.\end{aligned}$$

On the other hand we have due to the superharmonicity of φ:

$$\begin{aligned}\varphi(p) \ge \mathbb{E}^p(\varphi(B_\tau)) &\ge \int_{\{R_\tau=r_{2n}\}}\exp(-R_\tau)\,d\mathbb{P}^p\\ &= \int_{\{R_\tau=r_{2n}\}}\exp(-r_{2n})\,d\mathbb{P}^p = \exp(-r_{2n})\,\mathbb{P}^p\{R_\tau=r_{2n}\},\end{aligned}$$

which implies

$$\begin{aligned}\mathbb{P}^p\{R_\tau = r_{2n}\} \le \exp(r_{2n})\exp\left(-\frac{1}{2}(r_{2n+1}+r_{2n})\right) &=\\ = \exp\left(-\frac{1}{2}(r_{2n+1}-r_{2n})\right)&\le\frac{1}{2}\eta.\end{aligned}$$

Putting this together we arrive at:

$$\mathbb{P}^p\{R_\tau = r_{2n} \text{ or } R_\tau = r_{2n+1}\}\le\eta,$$

which finishes the proof. □

We are now able to prove the last remaining Lemma:

Lemma 3.18. *Almost surely*

$$\lim_{t\to\zeta} S_t = \infty.$$

Proof. Since we proved in Lemma 3.12 that almost surely $\lim_{t\to\zeta} S_t \in (-\infty,\infty]$, it suffices to show that $|S_t|\to\infty$ almost surely for $t\to\zeta$. As the Brownian motion B on M is transient, we know that the absolute value of at least one of the components R_t and S_t of B_t has to go to ∞ as $t\to\zeta$. Thus without loss of generality we may assume that $\lim_{t\to\zeta} R_t=\infty$ almost surely because we already know that $S_t(\omega)\to\infty$ whenever $\lim_{t\to\zeta} R_t(\omega)$ is finite.

Using the foregoing proposition we can further assume that the sequence $r_1 < r_2 < \dots < r_{2n} < r_{2n+1} \nearrow \infty$ is chosen such that

$$\mathbb{P}^{p_n}\{R_{\tau_n} = r_{2n} \text{ or } R_{\tau_n} = r_{2n+1}\}\le 2^{-n}$$

where τ_n is the first exit time from the set

$$U_n := \{(r,s,\alpha)\in M\,|\, r_{2n} < r < r_{2n+1}, -n < s < n\}$$

of a Brownian motion started in $p_n = \big(\frac{1}{2}(r_{2n+1}+r_{2n}), s, \alpha\big)$ for $-n < s < n$.

Let B_t be a Brownian motion on M with starting point $p_0 = (r_0, s_0, \alpha_0)$. By assumption one has stopping times $t_n \nearrow \infty$ with $R_{t_n(\omega)}(\omega) = \frac{1}{2}(r_{2n+1}+r_{2n})$. Define $p_n(\omega) := B_{t_n(\omega)}(\omega)$.

We further define $\tau_n := \inf\{t > 0 |\, B_t^{p_n} \in M \setminus U_n\}$ and $\sigma_n := \inf\{t > t_n |\, B_t \in M \setminus U_n\}$. Then $\sigma_n = t_n + \tau_n$ and we have

$$\begin{aligned}
\mathbb{P}^{p_0}\{|S_{\sigma_n}| \geq n\} &= \mathbb{E}^{p_0}\left(1_{\{|pr_2|\geq n\}} \circ B_{\sigma_n}\right) \\
&= \int \mathbb{E}\left(1_{\{|pr_2|\geq n\}} \circ B_{\tau_n}^{p_n}\right)|_{p_n = B_{t_n}} d\mathbb{P}^{p_0} \\
&= \int \mathbb{P}^{p_n(\omega)}\{|S_{\tau_n}| \geq n\} d\mathbb{P}^{p_0}(\omega) \\
&\geq (1-2^{-n})\mathbb{P}^{p_0}(\Omega) = 1 - 2^{-n}.
\end{aligned}$$

Hence $\mathbb{P}^{p_0}\{|S_{\sigma_n}| < n\} = 1 - \mathbb{P}^{p_0}\{|S_{\sigma_n}| \geq n\} \leq 2^{-n}$ for all $n \in \mathbb{N}$ and therefore:

$$\sum_{n=1}^{\infty} \mathbb{P}^{p_0}\{|S_{\sigma_n}| < n\} \leq \sum_{n=1}^{\infty} 2^{-n} \leq \frac{1}{1-1/2} < \infty.$$

Borel-Cantelli provides in this situation:

$$\mathbb{P}^{p_0}\left\{\limsup\{|S_{\sigma_n}| < n\}\right\} = 0,$$

hence $\mathbb{P}^{p_0}\{\liminf\{|S_{\sigma_n}| \geq n\}\} = 1$, which means that almost surely $|S_{\sigma_n}| \geq n$ eventually, and therefore almost surely $S_{\sigma_n} \to \infty$ (as $\lim_{t\to\zeta} S_t \in (-\infty, \infty]$). For that reason S_t almost surely possesses a subsequence $S_{\sigma_n} \to \infty$, which implies that $S_t \to \infty$. □

3.6. Non-Trivial Shift-Invariant Events for B

As we have seen in Chapter 2 there is a one-to-one correspondence between the σ-field $\mathscr{A}_{\text{inv}}$ of shift-invariant events for B up to equivalence and the set of bounded harmonic functions on M. We are going to use this fact and give a probabilistic proof for the existence of non-trivial shift-invariant random variables, which as an immediate consequence yields the existence of non-trivial bounded harmonic functions on M. Furthermore we get a stochastic representation of the constructed harmonic functions that can be interpreted as "solutions of a modified Dirichlet problem at infinity". For this the considered boundary functions are not of "pure angular type" as in the usual Dirichlet problem at infinity, where the boundary function is a function $f : S_\infty(M) \cong S^{d-1}(M) \to \mathbb{R}$ and the stochastic representation of the solution h only depends on the angular part $\vartheta(B) \in S^{d-1}(M)$ of the Brownian motion.

In our situation it turns out (as seen in Section 3.5, Lemma 3.16) that the shift-invariant random variable A_ζ is non-trivial and can be interpreted as "1-dimensional angle" on the sphere $S_\infty(M)$ at infinity. As we will see in Chapter 4 this has the same meaning as that of the random variable Y_∞ in Ancona's example: A_ζ gives the direction on the sphere at infinity, wherefrom the Brownian motion B converges to the single point $L(\infty)$. See also the geometrical interpretation below. As the limit $B_\zeta \in S_\infty(M)$ is trivial this is obviously the only (non-trivial) information for Brownian paths to differ when just looking at their projection onto $S_\infty(M)$. This is why one could believe that the random variable A_ζ itself already generates the shift-invariant σ-field $\mathscr{A}_{\text{inv}}$, what as a consequence would imply that every bounded harmonic function h on M has a stochastic representation

$$h(x) = \mathbb{E}^x[f \circ A_\zeta] = \mathbb{E}^x\left[\lim_{t\to\zeta} f \circ (\text{pr}_3(B_t))\right]$$

with $f : S^1 \to \mathbb{R}$ continuous. However, in his paper Borbély gives a way to construct a family of rotationally invariant, i.e. independent of α, harmonic functions $\psi(r,s)$ that obviously cannot be of the above form. As he uses "Perron's principle" for the construction he does not really get a representation of the obtained harmonic functions. We have learnt from this that there must be a further way to obtain non-trivial shift-invariant events when just using the components S_t and R_t of the Brownian motion or a "combination" of them. Indeed, it turns out that the random variable

$$\lim_{t\to\zeta}\left(\widetilde{S}_t - \int_0^{\widetilde{R}_t} q(r)dr\right)$$

yields the desired non-trivial shift-invariant random variable. Herein $\widetilde{S}$ and $\widetilde{R}$ are time-changed versions of S and R and $q : \mathbb{R}_+ \to \mathbb{R}$ is a function already constructed by Borbely. From that we get additional (to that depending on the component α) harmonic functions via the stochastic representation

$$h(x) = \mathbb{E}^x\left[g \circ \left(\lim_{t\to\zeta}\left(\widetilde{S}_t - \int_0^{\widetilde{R}_t} q(r)dr\right)\right)\right],$$

where $g : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function.

In the following we are going to prove the results mentioned above with probabilistic methods. As we use the same function $q(r)$ as Borbély we will just write down its properties that are essential for the proof and refer to the following Section 3.7 for a short sketch of Borbély's construction. We conclude this chapter with a geometric interpretation of the results obtained in Section 3.8.

Fix $a \in \mathbb{R}$ and let $T_0 \in \mathbb{R}$ such that $p_0(r)h(r) > 240$ and $\sqrt{h(r)} = \cosh(r) > 80$ for $r \geq T_0$. Let further $T_1 > T_0$ such that $\chi(r,s) = 1/2$ for $r \geq T_1$ and $s \geq a-1$. Then we have $p(r,s) = 1/2 \cdot p_0(r)$ for $r \geq T_1$ and $s \geq a-1$.

Lemma 3.19. *There is a $\mathscr{C}^\infty$-function $q : \mathbb{R}_+ \to \mathbb{R}$ with the following properties:*

i)

$$q(r) = -\frac{\sinh(r)}{\cosh^2(r)} = \left(\frac{1}{\sqrt{h}}\right)' \quad \textit{for } r \leq T_1.$$

ii) For $r > T_1$ the function q satisfies the inequalities

$$-3|q| < q' < \frac{1}{\cosh(r)}, \qquad \left(\frac{1}{\sqrt{h}}\right)' \leq q \leq \frac{p_0}{2} - \frac{40}{h}.$$

iii) There is a $T_2 > T_1$ such that

$$q(r) = \frac{p_0(r)}{2} - \frac{40}{h(r)} \text{ for } r \geq T_2.$$

Proof. See the following Section 3.7 and [B]. □

As we will see in Section 3.8 (and have already seen in Proposition 3.17 and Lemma 3.18) the "drift ratio" $p(r,s) = \frac{g_s'}{g_r' h}$ influences the interplay of the components S_t and R_t of the Brownian motion and because of that determines the behaviour of the Brownian paths. For this reason it is more convenient to work with a time changed version $\widetilde{B}_t$ of our Brownian motion, where the drift of the component $\widetilde{R}_t$ is just t and the drift of $\widetilde{S}_t$ is essentially given by p. This can be realized with a time change $\langle\tau\rangle$ defined as follows:
Let

$$T(t) := \int_0^t \left(\frac{h_r'}{4h} + \frac{g_r'}{4g}\right)(S_u, R_u)\, du$$

and $\tau_t := T^{-1}(t) \equiv \inf\{s \in \mathbb{R}_+ : T(s) \geq t\}$ for $t \leq T(\zeta)$. The components $\widetilde{R}_t, \widetilde{S}_t, \widetilde{A}_t$ of the time changed Brownian motion $\widetilde{B}_t := B_{\tau_t}$ are given for $t \leq \widetilde{\zeta} := T(\zeta)$ by the following system of stochastic differential equations:

$$d\widetilde{R}_t = dt + \frac{1}{\sqrt{\frac{h_r'}{4h}(\widetilde{R}_t) + \frac{g_r'}{4g}(\widetilde{R}_t, \widetilde{S}_t)}}\, dW^1 \tag{3.17}$$

$$d\widetilde{S}_t = \frac{g_s'(\widetilde{R}_t, \widetilde{S}_t)}{(gh_r' + g_r'h)(\widetilde{R}_t, \widetilde{S}_t)}\, dt + \frac{1}{\sqrt{h(\widetilde{R}_t)\left(\frac{h_r'}{4h}(\widetilde{R}_t) + \frac{g_r'}{4g}(\widetilde{R}_t, \widetilde{S}_t)\right)}}\, dW^2 \tag{3.18}$$

$$d\widetilde{A}_t = \frac{1}{\sqrt{g(\widetilde{R}_t, \widetilde{S}_t)\left(\frac{h_r'}{4h}(\widetilde{R}_t) + \frac{g_r'}{4g}(\widetilde{R}_t, \widetilde{S}_t)\right)}}\, dW^3. \tag{3.19}$$

As $\widetilde{B}$ is the time changed Brownian motion B, one has from Section 3.5:

i) $\lim_{t\to\widetilde{\zeta}} \widetilde{R}_t$ exists almost surely and $\lim_{t\to\widetilde{\zeta}} \widetilde{R}_t > 2$ almost surely,

ii) $\lim_{t\to\widetilde{\zeta}} \widetilde{S}_t = \infty$ almost surely,

iii) $\lim_{t\to\widetilde{\zeta}} \widetilde{A}_t$ exists almost surely and is almost surely finite.

We need two technical results before proving the main theorem of this section:

Remark 3.20. For every $(r,s) \in \mathbb{R}_+ \times \mathbb{R}$:

$$\frac{h'_r}{4h} + \frac{g'_r}{4g} \geq \frac{1}{10}.$$

Proof. For $r \leq 1/10$ one has $g(r,s) = \sinh^2(r)$ and from that

$$\frac{h'_r}{4h} + \frac{g'_r}{4g} = \frac{\sinh(r)}{2\cosh(r)} + \frac{\cosh(r)}{2\sinh(r)} \geq 5.$$

For $r > 1/10$ one has from Lemma 3.1, ii) that $g'_r \geq g h'_r$. Hence

$$\frac{h'_r}{4h} + \frac{g'_r}{4g} \geq \frac{\sinh(r)}{2\cosh(r)} + \frac{1}{2}\sinh(r)\cosh(r) \geq \frac{1}{10},$$

which yields the desired inequality. □

Proposition 3.21.

$$\int_0^{\widetilde{\zeta}} \frac{1}{\cosh(\widetilde{R}_t)}\, dt < \infty \quad \textit{and} \quad \int_0^{\widetilde{\zeta}} \frac{1}{\cosh^2(\widetilde{R}_t)}\, dt < \infty \quad \textit{almost surely.}$$

Proof. One has

$$\cosh(r) = \frac{1}{2}\left(e^r + e^{-r}\right) \geq \frac{1}{2}e^r,$$
$$\cosh^2(r) = \frac{1}{4}\left(e^{2r} + 2 + e^{-2r}\right) \geq \frac{1}{4}e^{2r}.$$

As $r \geq 0$ and therefore $e^{2r} \geq e^r$ it is sufficient to show that $\int_0^{\widetilde{\zeta}} e^{-\widetilde{R}_t}\, dt < \infty$ almost surely. We show that

$$\mathbb{E}\left[\int_0^{\widetilde{\zeta}} \frac{1}{e^{\widetilde{R}_t/6}}\, dt\right] < \infty.$$

This yields the desired claim.

Let

$$M_t := \frac{1}{6}\int_0^t \frac{1}{\sqrt{\frac{h'_r}{4h}(\widetilde{R}_s) + \frac{g'_r}{4g}(\widetilde{R}_s, \widetilde{S}_s)}}\, dW^1.$$

Then due to Remark 3.20 we can estimate the term appearing in the denominator by the constant $1/10$. We therefore obtain the estimate:

$$\frac{1}{2}[M]_t = \frac{1}{72}\int_0^t \frac{1}{\frac{h'_r}{4h}(\widetilde{R}_s) + \frac{g'_r}{4g}(\widetilde{R}_s, \widetilde{S}_s)}\, ds \leq \frac{1}{72}\int_0^t 10\, ds = \frac{5}{36}\, t.$$

From the Novikov criterion it follows that $\mathscr{E}(-M) := e^{-M - \frac{1}{2}[M]}$ is a martingale with

$$\mathbb{E}\left(e^{-M_t - \frac{1}{2}[M]_t}\right) = 1 \text{ for all } t \in \mathbb{R}_+.$$

Using this, the upper bound for $\frac{1}{2}[M]_t$ and the explicit representation $\widetilde{R}_t = t + 6M_t$, we get:

$$\mathbb{E}\left[\frac{1}{e^{\widetilde{R}_t/6}}\right] = \mathbb{E}\left[\frac{e^{-M_t - \frac{1}{2}[M]_t}}{e^{\frac{1}{6}t - \frac{1}{2}[M]_t}}\right] \leq \mathbb{E}\left[\frac{e^{-M_t - \frac{1}{2}[M]_t}}{e^{\frac{1}{36}t}}\right] = e^{-\frac{1}{36}t}. \tag{3.20}$$

It follows that

$$\mathbb{E}\left[\int_0^{\widetilde{\zeta}} \frac{1}{e^{\widetilde{R}_t/6}}\, dt\right] \leq \int_0^\infty \mathbb{E}\left[\frac{1}{e^{\widetilde{R}_t/6}}\right] dt \leq \int_0^\infty e^{-\frac{1}{36}t}\, dt < \infty. \tag{3.21}$$

□

We can now state and prove the main theorem of this chapter:

Theorem 3.22. *Let $q : \mathbb{R}_+ \to \mathbb{R}$ be as in Lemma 3.19 and*

$$Z_t := \widetilde{S}_t - \int_0^{\widetilde{R}_t} q(r)dr.$$

Then $\lim_{t\to\widetilde{\zeta}} Z_t$ exists almost surely and is almost surely finite.

Proof. From Itô's formula we have

$$\begin{aligned}
dZ_t &= d\widetilde{S}_t - q(\widetilde{R}_t)\, d\widetilde{R}_t - \frac{1}{2}q'(\widetilde{R}_t)\, d\widetilde{R}_t d\widetilde{R}_t \\
&= \frac{g_s'}{gh_r' + g_r'h}\, dt + \frac{1}{\sqrt{h\left(\frac{h_r'}{4h} + \frac{g_r'}{4g}\right)}}\, dW^2 - q\, dt - q \cdot \frac{1}{\sqrt{\frac{h_r'}{4h} + \frac{g_r'}{4g}}}\, dW^1 \\
&\quad - \frac{1}{2}q' \cdot \frac{1}{\frac{h_r'}{4h} + \frac{g_r'}{4g}}\, dt \\
&= \underbrace{\left(\frac{g_s'}{gh_r' + g_r'h} - p\right) dt}_{(4)} + \underbrace{\frac{1}{\sqrt{h\left(\frac{h_r'}{4h} + \frac{g_r'}{4g}\right)}}\, dW^2}_{(2)} + \underbrace{(p-q)\, dt}_{(5)} - \underbrace{q \cdot \frac{1}{\sqrt{\frac{h_r'}{4h} + \frac{g_r'}{4g}}}\, dW^1}_{(1)} \\
&\quad \underbrace{- \frac{1}{2}q' \cdot \frac{1}{\frac{h_r'}{4h} + \frac{g_r'}{4g}}\, dt}_{(3)}.
\end{aligned} \tag{3.22}$$

To prove the pathwise convergence of Z_t for $t \to \widetilde{\zeta}$ we show that $\int_0^{\widetilde{\zeta}}$ over each of the five terms (1) – (5) in the above formula converges pathwise.

We split the proof in five steps, where each one proves the convergence of the corresponding term of (3.22).

Choose $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ and such that $\widetilde{R}_t(\omega)$, $\widetilde{S}_t(\omega)$ and $\widetilde{A}_t(\omega)$ is a convergent path of $B_t(\omega)$ for every $\omega \in \Omega'$.

(1) Let

$$M_r^1 := \int_0^r q(\widetilde{R}_t) \frac{1}{\sqrt{\frac{h_r'}{4h}(\widetilde{R}_t) + \frac{g_r'}{4g}(\widetilde{R}_t, \widetilde{S}_t)}} \, dW_t^1. \tag{3.23}$$

For real-valued continuous local martingales X we have

$$\{[X]_\infty < \infty\} = \{\lim_{t\to\infty} X_t \text{ exists in } \mathbb{R}\} \quad \text{except for a set of measure 0.}$$

As every local martingale with finite lifetime can be transformed by a change of time into a local martingale with almost surely infinite lifetime, cf. for example [Ha-Th] p.236, it suffices to show that $\lim_{t\to\widetilde{\zeta}}[M^1]_t$ exists almost surely in $\mathbb{R}$.

From Lemma 3.19 we have that $|q(r)| < C_1$ for a constant $C_1 > 0$ and all $r \in \mathbb{R}_+$.
In i) we have seen that $\lim_{t\to\widetilde{\zeta}} \widetilde{R}_t > 2$ almost surely. Hence for every path $\widetilde{R}_t(\omega)$, $\omega \in \Omega'$, there exists $N(\omega) < \widetilde{\zeta}(\omega)$ such that $\widetilde{R}_t(\omega) > 1/10$ for all $t \geq N(\omega)$.
Then for all $t \geq N(\omega)$

$$\frac{h_r'}{4h}(\widetilde{R}_t(\omega)) + \frac{g_r'}{4g}(\widetilde{R}_t(\omega), \widetilde{S}_t(\omega)) \geq \frac{h_r'}{4h}(\widetilde{R}_t(\omega)) + \frac{1}{4}h_r'(\widetilde{R}_t)(\omega) \geq C_2 \cosh(\widetilde{R}_t(\omega))$$

for a constant $C_2 > 0$. For $\omega \in \Omega'$ therefore holds

$$\begin{aligned}
[M^1]_{\widetilde{\zeta}} &= \int_0^{\widetilde{\zeta}} q^2(\widetilde{R}_t) \frac{1}{\frac{h_r'}{4h}(\widetilde{R}_t) + \frac{g_r'}{4g}(\widetilde{R}_t, \widetilde{S}_t)} \, dt \leq \int_0^{\widetilde{\zeta}} \frac{C_1^2}{\frac{h_r'}{4h}(\widetilde{R}_t) + \frac{g_r'}{4g}(\widetilde{R}_t, \widetilde{S}_t)} \, dt \\
&\leq C_1^2 \underbrace{\int_0^{N(\omega)} \frac{1}{\frac{h_r'}{4h}(\widetilde{R}_t) + \frac{g_r'}{4g}(\widetilde{R}_t, \widetilde{S}_t)} \, dt}_{<\infty} + \underbrace{\frac{C_1^2}{C_2} \int_{N(\omega)}^{\widetilde{\zeta}} \frac{1}{\cosh(\widetilde{R}_t)} \, dt}_{<\infty, \text{ due to Prop. 3.21}} < \infty.
\end{aligned} \tag{3.24}$$

(2) Let further

$$M_r^2 := \int_0^r \frac{1}{\sqrt{h(\widetilde{R}_t)\left(\frac{h_r'}{4h}(\widetilde{R}_t) + \frac{g_r'}{4g}(\widetilde{R}_t, \widetilde{S}_t)\right)}} \, dW_t^2. \tag{3.25}$$

Then with Remark 3.20 one has for every $\omega \in \Omega'$

$$[M^2]_{\widetilde{\zeta}} = \int_0^{\widetilde{\zeta}} \frac{1}{h(\widetilde{R}_t)\left(\frac{h_r'}{4h}(\widetilde{R}_t) + \frac{g_r'}{4g}(\widetilde{R}_t, \widetilde{S}_t)\right)} \, dt \leq \int_0^{\widetilde{\zeta}} \frac{10}{\cosh^2(\widetilde{R}_t)} dt < \infty. \tag{3.26}$$

The last inequality follows again from Proposition 3.21. Consequently $\lim_{t\to\widetilde{\zeta}} M_t^2$ exists almost surely as well.

We now focus on the convergence of the drift terms of Z_t. To keep the following formulas readable we omit the dependence of $\omega \in \Omega'$, yet the expressions are to be understood pathwise.

(3) Again from Lemma 3.19 we have that $|q'(r)| < C_3$ for a constant $C_3 > 0$ and all $r \in \mathbb{R}_+$. So as in **(1)**

$$\left| \int_0^{\widetilde{\zeta}} \frac{1}{2} q'(\widetilde{R}_t) \frac{1}{\frac{h'_r}{4h}(\widetilde{R}_t) + \frac{g'_r}{4g}(\widetilde{R}_t, \widetilde{S}_t)} \, dt \right| \leq C_3 \int_0^{\widetilde{\zeta}} \frac{1}{\frac{h'_r}{4h}(\widetilde{R}_t) + \frac{g'_r}{4g}(\widetilde{R}_t, \widetilde{S}_t)} \, dt < \infty. \tag{3.27}$$

(4) With the definition $p(r, s) = g'_s/(hg'_r)$ of p we have the following estimate for the first drift term of Z_t:

$$\left| \frac{g'_s}{gh'_r + hg'_r} - p \right| = \left| p \cdot \frac{gh'_r}{gh'_r + hg'_r} \right| = \left| p \cdot \frac{1}{1 + \frac{hg'_r}{gh'_r}} \right| \leq \left| p \cdot \frac{1}{1+h} \right| \tag{3.28}$$

Using this and the fact that $p(r, s) \leq C_4$ for all $(r, s) \in \mathbb{R}_+ \times \mathbb{R}$ and a constant $C_4 > 0$ (see Lemma 3.1) we get

$$\begin{aligned} \left| \int_0^{\widetilde{\zeta}} \left(\frac{g'_s(\widetilde{R}_t, \widetilde{S}_t)}{gh'_r(\widetilde{R}_t, \widetilde{S}_t) + hg'_r(\widetilde{R}_t, \widetilde{S}_t)} - p(\widetilde{R}_t, \widetilde{S}_t) \right) dt \right| &\leq \int_0^{\widetilde{\zeta}} p(\widetilde{R}_t, \widetilde{S}_t) \cdot \frac{1}{1 + h(\widetilde{R}_t)} \, dt \\ &\leq C_4 \int_0^{\widetilde{\zeta}} \frac{1}{\cosh^2(\widetilde{R}_t)} \, dt < \infty. \end{aligned} \tag{3.29}$$

(5) For the remaining drift term of Z_t we again recall Lemma 3.19: according to that there is a $T_2 \in \mathbb{R}_+$ such that $q(r) = \frac{1}{2} p_0(r) - 40/h(r)$ for all $r \geq T_2$.

For $\omega \in \Omega'$ we fix $N(\omega)$ such that $\widetilde{R}_t(\omega) \geq T_2$ for all $t \geq N(\omega)$. Presently we only know $\lim_{t \to \widetilde{\zeta}} R_t > 2$. Thus we just state here that it is possible to find such a $N(\omega)$ and refer to the following Remark 3.23 for the verification. Using this we get:

$$\begin{aligned} \left| \int_0^{\widetilde{\zeta}} \left(p(\widetilde{R}_t(\omega), \widetilde{S}_t(\omega)) - q(\widetilde{R}_t(\omega)) \right) dt \right| &\leq \underbrace{\left| \int_0^{N(\omega)} \left(p(\widetilde{R}_t(\omega), \widetilde{S}_t(\omega)) - q(\widetilde{R}_t(\omega)) \right) dt \right|}_{<\infty} + \\ &+ \left| \int_{N(\omega)}^{\widetilde{\zeta}} \left(p(\widetilde{R}_t(\omega), \widetilde{S}_t(\omega)) - \frac{1}{2} p_0(\widetilde{R}_t(\omega)) + \frac{40}{h(\widetilde{R}_t(\omega))} \right) dt \right|. \end{aligned} \tag{3.30}$$

As

$$\int_0^{\widetilde{\zeta}} \frac{40}{h(\widetilde{R}_t)} \, dt = \int_0^{\widetilde{\zeta}} \frac{40}{\cosh^2(\widetilde{R}_t)} \, dt < \infty \text{ almost surely,}$$

it is sufficient to show that

$$\int_{N(\omega)}^{\widetilde{\zeta}} \left(p(\widetilde{R}_t, \widetilde{S}_t) - \frac{1}{2} p_0(\widetilde{R}_t) \right) dt < \infty \text{ almost surely.}$$

We have seen in Section 3.4 that $p(r,s) = \chi(r,s)p_0(r)$ where $\chi(r,s) = \xi(s+\ell(r))$ with $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ and $\xi(y) = 1/2$ for $y \geq 4$. If we choose $N'(\omega)$ ($N'(\omega) > N(\omega)$ say) such that $\widetilde{S}_t(\omega) > 4$ for all $t \geq N'(\omega)$ – this is possible as we know that $\lim_{t\to\widetilde{\zeta}} \widetilde{S}_t = \infty$ almost surely – we have

$$p(\widetilde{R}_t(\omega), \widetilde{S}_t(\omega)) = \frac{1}{2}p_0(\widetilde{R}_t(\omega))$$

for all $t \geq N'(\omega)$ and therefore

$$\int_{N(\omega)}^{\widetilde{\zeta}} \left(p(\widetilde{R}_t, \widetilde{S}_t) - \frac{1}{2}p_0(\widetilde{R}_t) \right) dt = \int_{N(\omega)}^{N'(\omega)} \left(p(\widetilde{R}_t, \widetilde{S}_t) - \frac{1}{2}p_0(\widetilde{R}_t) \right) dt < \infty.$$

This finally proves the existence of $\lim_{t\to\widetilde{\zeta}} Z_t$ in $\mathbb{R}$. □

Remark 3.23. In the foregoing proof we used the fact that not only $\lim_{t\to\widetilde{\zeta}} \widetilde{R}_t > 2$ but that for every $\omega \in \Omega'$ there exists a number $N(\omega)$ such that $\widetilde{R}_t(\omega) > T_2$ for all $t \geq N(\omega)$. This relies on the following observation:

Let

$$Z'_t := \widetilde{S}_t - \int_0^{\widetilde{R}_t} \frac{1}{2}p_0(u)\, du.$$

Then using again Itô's formula we obtain the same representation of dZ'_t as in (3.22) when replacing the drift term (3) with

$$-\frac{1}{4}p'_0(\widetilde{R}_t)\frac{1}{\frac{h'_r}{4h} + \frac{g'_r}{4g}}\, dt$$

and the drift term (5) with

$$\left(p - \frac{1}{2}p_0\right) dt.$$

As also p'_0 is bounded

$$\Big| \int_0^{\widetilde{\zeta}} \frac{1}{4}p'_0(\widetilde{R}_t) \left(\frac{h'_r}{4h} + \frac{g'_r}{4g}\right)^{-1} dt \Big| < \infty \text{ almost surely}$$

with exactly the same argument as in **(3)**. Further we have already shown in **(5)** that $\int_0^{\widetilde{\zeta}} \left(p(\widetilde{R}_t, \widetilde{S}_t) - \frac{1}{2}p_0(\widetilde{R}_t)\right) dt < \infty$. The last argument relies on the fact that $\lim_{t\to\widetilde{\zeta}} \widetilde{S}_t = \infty$ almost surely.

It is then an immediate consequence that $\lim_{t\to\widetilde{\zeta}} Z'_t$ almost surely exists in $\mathbb{R}$. From this it follows at once that $\lim_{t\to\widetilde{\zeta}} \widetilde{R}_t = \infty$ almost surely. Otherwise $\lim_{t\to\widetilde{\zeta}} \int_0^{\widetilde{R}_t} \frac{1}{2}p_0(u)du$ would be finite as well, what in combination with the fact that $\lim_{t\to\widetilde{\zeta}} \widetilde{S}_t = \infty$ almost surely leads to a contradiction. Using $\lim_{t\to\widetilde{\zeta}} \widetilde{R}_t = \infty$ it follows at once that the lifetime $\widetilde{\zeta}$ of the time changed Brownian motion $\widetilde{B}$ is almost surely infinite.

We formulate a corollary concerning the "original" – i.e. non time changed – Brownian motion B. Together with Theorem 3.4 we then have a complete description of the asymptotic behaviour of the Brownian motion B on M.

Corollary 3.24. *Let $B_t = (R_t, S_t, A_t)$ the Brownian motion on M with lifetime ζ. Then*

i) $\lim_{t\to\zeta} R_t = \infty$ *almost surely.*

ii) ζ *is almost surely finite.*

Proof.
i) is an easy consequence of Remark 3.23 above where we showed that $\lim_{t\to\widetilde{\zeta}} \widetilde{R}_t = \infty$ almost surely. The paths of R_t are the same as the paths of the time changed component $\widetilde{R}_t$ just with different parametrization.
ii) Furthermore we showed in the proof of Theorem 3.22, **(1)**, that $\lim_{t\to\widetilde{\zeta}} M_t^1$ almost surely exists. But M_t^1 was just the "noise" part of $\widetilde{R}_t$. If we reverse the time-change for B, the "noise" part $\int_0^t dW_s^1$ of R_t has then to converge as well. As W^1 is a real Brownian motion $\int_0^\zeta dW_t^1$ cannot converge unless ζ is almost surely finite. □

Remark 3.25. It is important to remark that the lifetime ζ of the Brownian motion on M is almost surely finite. This is the only difference in the asymptotic behaviour of the Brownian motion on the manifold of Borbély and the manifold of Ancona which we are going to discuss in the following chapter.

What we have proven up to now is the pathwise convergence of the process

$$Z_t = \widetilde{S}_t - \int_0^{\widetilde{R}_t} q(r)dr \text{ for } t \to \widetilde{\zeta}.$$

Obviously this limit $Z_{\widetilde{\zeta}}$ is a shift-invariant random variable for the Brownian motion B. However to get non-trivial shift-invariant events (and from there non-trivial bounded harmonic functions) we have to rule out the case that the limit $Z_{\widetilde{\zeta}}$ is a constant independent of the starting point (r, s, α) of the Brownian motion B. This can be done by using a submartingale argument. We first prove a proposition to obtain a bounded (and therefore uniformly integrable) submartingale "containing" the process Z_t to which we can apply the submartingale inequality. We could also use the fact that for $a \in \mathbb{R}$ as chosen before Lemma 3.19 the function

$$u_a(r,s) := \max\left\{0, \frac{2}{\pi}\arctan\left(\delta\left(s - \int_0^r q(u)du - a\right)\right)\right\}$$

is subharmonic for a suitable δ that does not depend on a, which is proven in [B]. But we give here a direct stochastic proof that $u_a(\widetilde{R}_t, \widetilde{S}_t)$ is a submartingale for $a \in \mathbb{R}$ as above and $\delta = \frac{1}{2}$.

Proposition 3.26. *Let $a \in \mathbb{R}$ be as in Lemma 3.19. Then the process $u(Z_t) \equiv u(\widetilde{R}_t, \widetilde{S}_t)$ given as*

$$u(Z_t) := \begin{cases} 0 & \text{for } Z_t < a, \\ \frac{2}{\pi} \arctan\left(\frac{1}{2}\left(\widetilde{S}_t - \int_0^{\widetilde{R}_t} q(u)du - a\right)\right) & \text{for } Z_t \geq a, \end{cases}$$

is a submartingale.

Proof. To show the submartingale property we consider the drift terms of $u(Z_t)$. Thereby we can obviously restrict ourselves to the case $Z_t \geq a$, i.e. $\widetilde{S}_t - \int_0^{\widetilde{R}_t} q(u)du \geq a$. For $Z_t < a$ the term $u(Z_t)$ is equal to 0. Writing down Itô's formula for $u(Z_t)$ we obtain by omitting the local martingale parts (which is indicated by $\overset{m}{=}$):

$$\begin{aligned} \frac{\pi}{2} du(\widetilde{S}_t, \widetilde{R}_t) &\overset{m}{=} \frac{g_s'}{gh_r' + g_r'h} \cdot \frac{\frac{1}{2}}{1 + \frac{1}{4}(Z_t - a)^2}\, dt + \frac{-\frac{1}{2}q(\widetilde{R}_t)}{1 + \frac{1}{4}(Z_t - a)^2}\, dt \\ &+ \frac{1}{2} \cdot \frac{\frac{1}{4}(Z_t - a)}{\left(1 + \frac{1}{4}(Z_t - a)^2\right)^2} \cdot \frac{1}{h\left(\frac{h_r'}{4h} + \frac{g_r'}{4g}\right)}\, dt + \frac{1}{2} \cdot \frac{-\frac{1}{2}q'(\widetilde{R}_t)}{1 + \frac{1}{4}(Z_t - a)^2} \cdot \frac{1}{\frac{h_r'}{4h} + \frac{g_r'}{4g}}\, dt \\ &+ \frac{1}{2} \cdot \frac{-\frac{1}{8}q^2(\widetilde{R}_t)(Z_t - a)}{\left(1 + \frac{1}{4}(Z_t - a)^2\right)^2} \cdot \frac{1}{\frac{h_r'}{4h} + \frac{g_r'}{4g}}\, dt \end{aligned} \tag{3.31}$$

If we can show that this drift term is positive, then the submartingale property of $u(Z_t)$ follows at once.

The factor $\frac{1}{2} \cdot \frac{1}{1+1/4(Z_t-a)^2}$ appearing in all terms of the above sum is strictly positive so we do not have to worry about it. It therefore remains to show that

$$\frac{g_s'}{gh_r' + hg_r'} - q - \frac{2q'}{\frac{h_r'}{h} + \frac{g_r'}{g}} - \frac{1}{2} \cdot \frac{\frac{1}{4}(Z_t - a)}{1 + \frac{1}{4}(Z_t - a)^2} \cdot \left(\frac{2 + q^2 h}{h\left(\frac{h_r'}{4h} + \frac{g_r'}{4g}\right)}\right) \tag{3.32}$$

is positive if $Z_t \geq a$.

Observe that for $x \geq 0$ the function $\frac{1/4\,x}{1+1/4\,x^2}$ is bounded from above by $1/4$. Hence we get with $g_s' = pg_r'h$

$$(3.32) \geq \frac{pg_r'h}{gh_r' + hg_r'} - q - \frac{2q'}{\frac{h_r'}{h} + \frac{g_r'}{g}} - \frac{1}{8} \cdot \frac{8 + 4q^2h}{h\left(\frac{h_r'}{h} + \frac{g_r'}{g}\right)} \tag{3.33}$$

As the denominator $gh_r' + hg_r'$ is positive it remains to show that

$$pg_r'h - qgh_r' - qhg_r' - 2q'gh - g\left(1 + \frac{1}{2}q^2h\right) \tag{3.34}$$

is positive where $Z_t \geq a$.

For the proof we consider four different intervals for r, which correspond to the definition of p and q. Remind that $h(r) = \cosh^2(r)$ for all $r \in \mathbb{R}_+$.

(1) $\widetilde{R}_t \leq \frac{1}{10}$: We know that for $r \leq \frac{1}{10}$ the function $g(r,s)$ equals $\sinh^2(r)$, i.e. $p = 0$, and that $q(r) = -\frac{\sinh(r)}{\cosh^2(r)}$.

Hence

$$(3.34) = \frac{2\sinh^2(\widetilde{R}_t)\left(2\cosh^2(\widetilde{R}_t) - \sinh^2(\widetilde{R}_t)\right)}{\cosh(\widetilde{R}_t)} - \sinh^2(\widetilde{R}_t)\left(1 + \frac{\sinh^2(\widetilde{R}_t)}{2\cosh^2(\widetilde{R}_t)}\right) =$$

$$= \frac{2\sinh^2(\widetilde{R}_t)(\cosh^2(\widetilde{R}_t) + 1)}{\cosh(\widetilde{R}_t)} - \sinh^2(\widetilde{R}_t)\left(1 + \frac{\sinh^2(\widetilde{R}_t)}{2\cosh^2(\widetilde{R}_t)}\right) \geq 0.$$

(2) $\frac{1}{10} \leq \widetilde{R}_t \leq T_1$: For $\frac{1}{10} \leq r \leq T_1$ we know that $g'_r \geq gh'_r$ and $q = -\frac{\sinh(r)}{\cosh^2(r)}$.

Hence

$$(3.34) = pg'_r h + \frac{2\sinh^2(\widetilde{R}_t)}{\cosh(\widetilde{R}_t)} g + \sinh(\widetilde{R}_t) g'_r - \frac{4\sinh^2(\widetilde{R}_t) - 2\cosh^2(\widetilde{R}_t)}{\cosh(\widetilde{R}_t)} g$$

$$- g\left(1 + \frac{\sinh^2(\widetilde{R}_t)}{2\cosh^2(\widetilde{R}_t)}\right) \qquad \text{(as } g'_r \geq gh'_r \text{ and } p \geq 0\text{)}$$

$$\geq g\left[\frac{2\sinh^2(\widetilde{R}_t)}{\cosh(\widetilde{R}_t)} + 2\sinh^2(\widetilde{R}_t)\cosh(\widetilde{R}_t) - \frac{4\sinh^2(\widetilde{R}_t) - \cosh(\widetilde{R}_t)}{\cosh(\widetilde{R}_t)} - \left(1 + \frac{\sinh^2(\widetilde{R}_t)}{2\cosh^2(\widetilde{R}_t)}\right)\right]$$

$$= g\left[\frac{2\sinh^2(\widetilde{R}_t)\cosh^2(\widetilde{R}_t) + 2}{\cosh(\widetilde{R}_t)} - \left(1 + \frac{\sinh^2(\widetilde{R}_t)}{2\cosh^2(\widetilde{R}_t)}\right)\right] \geq 0.$$

Before treating the remaining cases where $\widetilde{R}_t \geq T_1$ we remark that T_1 is chosen such that

$$p(r,s) = \frac{1}{2}p_0(r) \text{ for } r \geq T_1 \text{ and } s \geq a - 1.$$

If we have $\widetilde{R}_t \geq T_1$ and additionally assume $Z_t = \widetilde{S}_t - \int_0^{\widetilde{R}_t} q(r)dr \geq a$ then

$$\widetilde{S}_t \geq a + \int_0^{\widetilde{R}_t} q(r)dr.$$

This yields automatically $\widetilde{S}_t \geq a - 1$ because $q(r) \geq 0$ for $r \geq T'_1$, where $T'_1 \geq T_1$, and for $r \leq T'_1$ we have $q(r) \geq -\frac{\sinh(r)}{\cosh^2(r)}$, what means

$$\int_0^{\widetilde{R}_t} q(r)dr \geq \int_0^{T'_1} -\frac{\sinh(r)}{\cosh^2(r)}dr \geq \int_0^{\infty} -\frac{\sinh(r)}{\cosh^2(r)}dr = -1.$$

This is why we can replace p with $\frac{1}{2}p_0$ in the two remaining cases.

(3) $\underline{T_1 \le \widetilde{R}_t \le T_2}$: For $T_1 \le r \le T_2$ we have

$$\frac{\sinh(r)}{\cosh^2(r)} \ge -q(r) \ge \frac{40}{h(r)} - \frac{1}{2}p_0(r) \quad \text{and} \quad 3|q(r)| \ge -q'(r) \ge -\frac{1}{\cosh(r)}.$$

It follows that

$$\begin{aligned}
(3.34) &\ge \frac{40}{h} g h_r' - \frac{1}{2} p_0 g h_r' + 40 g_r' - 2g\cosh(\widetilde{R}_t) - g\left(1 + \frac{1}{2}q^2 h\right) && (\text{as } g_r' \ge g h_r') \\
&\ge g\left(\frac{40}{h}h_r' + 40h_r' - \frac{1}{2}p_0 h_r' - 2\cosh(\widetilde{R}_t) - \left(1 + \frac{1}{2}q^2 h\right)\right) && (\text{as } p_0 < 1) \\
&\ge g\left(\frac{80\sinh(\widetilde{R}_t)}{\cosh(\widetilde{R}_t)} + 79\cosh(\widetilde{R}_t)\sinh(\widetilde{R}_t) - 2\cosh(\widetilde{R}_t) - \left(1 + \frac{1}{2}q^2 h\right)\right) \\
&\ge g\left(\frac{80\sinh(\widetilde{R}_t)}{\cosh(\widetilde{R}_t)} + 79\cosh(\widetilde{R}_t)\sinh(\widetilde{R}_t) - 2\cosh(\widetilde{R}_t) - \left(1 + \frac{1}{2}\cosh^2(\widetilde{R}_t)\right)\right) \\
&\ge 0,
\end{aligned}$$

where we have to remark that T_0 is chosen such that $\sqrt{h(r)} = \cosh(r) > 80$ for $r \ge T_0$, see Lemma 3.19, what implies that $T_0 \ge 5$ and therefore $T_1 \ge 5$.

(4) $\underline{\widetilde{R}_t \ge T_2}$: We now have $q(r) = \frac{1}{2}p_0(r) - \frac{40}{h}$.

Hence

$$\begin{aligned}
(3.34) &= -\frac{1}{2}p_0 g h_r' + \frac{40}{h} g h_r' + 40 g_r' - 2p_0' g h - \frac{80 h_r'}{h} g - g\left(1 + \frac{1}{8}p_0^2 h - 20p_0 + \frac{800}{h}\right) \\
&\ge g\left(-\frac{1}{2}p_0 h_r' - \frac{40h_r'}{h} + 40h_r' - 2p_0' h - 1 - \frac{1}{8}p_0^2 h + 20p_0 - \frac{800}{h}\right).
\end{aligned}$$

As $p_0 < 1$ and $p_0' < 1$ we have

$$\begin{aligned}
(3.34) &\ge g\left(-\frac{1}{2}h_r' - 80\frac{\sinh(\widetilde{R}_t)}{\cosh(\widetilde{R}_t)} + 40h_r' - \frac{17}{8}h - 1 + 20p_0 - \frac{800}{h}\right) \\
&\ge g\left(79\cosh(\widetilde{R}_t)\sinh(\widetilde{R}_t) - 81 - \frac{17}{8}\cosh^2(\widetilde{R}_t) - \frac{800}{\cosh^2(\widetilde{R}_t)}\right) \ge 0.
\end{aligned}$$

□

We are now in the situation to prove the second main theorem of this chapter. With that we get a purely stochastic proof that there exist non-trivial bounded harmonic functions h on M depending only on the variables r and s. Furthermore we have an explicit stochastic representation of these harmonic functions via the formula

$$h(r, s, \alpha) \equiv h(r, s) = \mathbb{E}^{(r,s,\alpha)}\left[g \circ \left(\lim_{t \to \widetilde{\zeta}} Z_t\right)\right]$$

with $g : \mathbb{R} \to \mathbb{R}$ a bounded continuous function.

Theorem 3.27. *Consider $Z_t = \widetilde{S}_t - \int_0^{\widetilde{R}_t} q(u)du$ as before. Then the limit random variable $\lim_{t\to\widetilde{\zeta}} Z_t$ is non-trivial, i.e. is not almost surely a constant independent of the starting point of the Brownian motion B.*

Proof. Assume that $\lim_{t\to\widetilde{\zeta}} Z_t$ almost surely equals a constant C, where C is independent of the starting point (r, s, α) of the Brownian motion B.

From Proposition 3.26 we know that

$$u(Z_t) := \begin{cases} 0, & \text{for } Z_t < a \\ \frac{2}{\pi}\arctan\left(\frac{1}{2}(Z_t - a)\right), & \text{for } Z_t \geq a \end{cases}$$

is a bounded submartingale.
When starting B in $(0, a+1, 0)$ we get from the submartingale inequality:

$$0 < \frac{2}{\pi}\arctan\left(\frac{1}{2}\right) = \frac{2}{\pi}\arctan\left(\frac{1}{2}\left(\widetilde{S}_0 - \int_0^{\widetilde{R}_0} q(r)dr - a\right)\right) \leq \mathbb{E}^{(0,a+1,0)}\left[\lim_{t\to\widetilde{\zeta}} u(Z_t)\right].$$

In particular $\mathbb{P}\{\lim_{t\to\widetilde{\zeta}} u(Z_t) \neq 0\} > 0$. This implies that $C > a$, because otherwise we would have $\lim_{t\to\widetilde{\zeta}} u(Z_t) = 0$ almost surely in contradiction to the inequality above.

When now starting B in the point $(0, C+1, 0)$ we again have the supermartingale inequality to derive

$$\begin{aligned}\frac{2}{\pi}\arctan\left(\frac{1}{2}(C+1-a)\right) &= \frac{2}{\pi}\arctan\left(\frac{1}{2}\left(\widetilde{S}_0 - \int_0^{\widetilde{R}_0} q(r)dr - a\right)\right) \\ &\leq \mathbb{E}^{(0,C+1,0)}\left[\lim_{t\to\widetilde{\zeta}} u(Z_t)\right] = \mathbb{E}^{(0,C+1,0)}\left[\frac{2}{\pi}\arctan\left(\frac{1}{2}(C-a)\right)\right] \\ &= \frac{2}{\pi}\arctan\left(\frac{1}{2}(C-a)\right) < \frac{2}{\pi}\arctan\left(\frac{1}{2}(C+1-a)\right),\end{aligned}$$

which is a contradiction. Consequently $\lim_{t\to\widetilde{\zeta}} Z_t$ has to be a non-trivial random variable, what finishes the proof. □

3.7. Construction of the Function q

In Lemma 3.19 we listed the properties the function q has to satisfy. As the explicit construction of q is already done in [B] we just give a short description (following Borbély) how to get a function q with the required properties:

Let $a \in \mathbb{R}_+$ and T_0 such that $p_0(r)h(r) > 240$ and $\sqrt{h(r)} > 80$ for $r > T_0$. Let further $T_1 > T_0$ such that $p(r,s) = \frac{1}{2}p_0(r)$ for $r \geq T_1$ and $s \geq a-1$.

For $r \leq T_1$ the function q is defined as

$$q(r) := \left(\frac{1}{\sqrt{h}}\right)' = -\frac{\sinh(r)}{\cosh^2(r)}.$$

For $r \geq T_1$ choose a strictly increasing $\mathscr{C}^\infty$-extension of $-\frac{\sinh(r)}{\cosh^2(r)}$ such that $q(r) > 0$ for r large enough and

$$\left(\frac{1}{\sqrt{h}}\right)'' < q' < \frac{1}{\sqrt{h}}.$$

As $\frac{1}{2}p_0(r) - \frac{40}{h(r)} > 0$ with

$$\lim_{r\to\infty}\left(\frac{1}{2}p_0(r) - \frac{40}{h(r)}\right) = 0$$

(due to the construction of p_0, see Section 3.4) there is an $r > T_1$ with $q(r) = \frac{1}{2}p_0(r) - \frac{40}{h(r)}$. Let

$$T_2 := \inf\left\{r > T_1 : q(r) = \frac{1}{2}p_0(r) - \frac{40}{h(r)}\right\}.$$

For $r \geq T_2$ set

$$q(r) := \frac{1}{2}p_0(r) - \frac{40}{h(r)}.$$

The desired function q is then a smoothened version of the function defined above.

Borbély shows in [B], p.232, that indeed the function q obtained this way has the additionally required properties

$$-3|q| \leq q' \leq \frac{1}{\sqrt{h}} \qquad \text{and} \qquad \left(\frac{1}{\sqrt{h}}\right) \leq q \leq \frac{1}{2}p_0 - \frac{40}{h}.$$

3.8. Geometric Interpretation of the Asymptotic Behaviour of Brownian Motion

We conclude this chapter with some observations to explain how the behaviour of the Brownian paths on M can be interpreted geometrically.

We have seen in Section 3.5, Lemma 3.16 and Section 3.6, Theorem 3.22 that the random variables $\lim_{t\to\zeta} A_t$ and $\lim_{t\to\widetilde{\zeta}}\left[\widetilde{S}_t - \int_0^{\widetilde{R}_t} q(r)dr\right]$ serve as non-trivial shift-invariant random variables for B and hence yield non-trivial shift-invariant events for B.
As we indicated in Section 3.6 the random variable $\lim_{t\to\zeta} A_t$ can be interpreted as a one dimensional angle that gives the direction on the sphere $S_\infty(M)$ at infinity, from where the Brownian path attains the point $L(\infty) \in S_\infty(M)$. This is in some way related to the behaviour of Brownian motion on a three dimensional Riemannian manifold, where one can solve the Dirichlet problem at infinity: Here, we have the converging angular part $\vartheta(B)$ of the Brownian motion as a two-dimensional angle, the limit of which gives the

direction on $S_\infty(M)$ where the Brownian motion exits the manifold. In our case the exit set of B is the point $L(\infty) \in S_\infty(M)$ but we can still see from where the Brownian paths arrive at this point.

More striking than the random variable $\lim_{t\to\zeta} A_t$ is the meaning of the non-trivial shift-invariant random variable we get from the components S_t and R_t, $\widetilde{S}_t$, $\widetilde{R}_t$ respectively, of B. This obviously does not occur in the "usual case" where the Dirichlet problem at infinity is solvable.

To give a possible geometric interpretation of the non-trivial shift-invariant random variable $\lim_{t\to\widetilde{\zeta}} \left[\widetilde{S}_t - \int_0^{\widetilde{R}_t} q(r)dr\right]$ we again have a look at the stochastic differential equations for $\widetilde{S}_t$ and $\widetilde{R}_t$:

$$\begin{aligned} d\widetilde{R}_t &= \qquad dt + \frac{1}{\sqrt{\frac{h_r'}{4h} + \frac{g_r'}{4g}}}\, dW_t^1 \\ d\widetilde{S}_t &= \frac{g_s'}{gh_r' + g_r'h}\, dt + \frac{1}{\sqrt{h\left(\frac{h_r'}{4h} + \frac{g_r'}{4g}\right)}}\, dW_t^2. \end{aligned}$$

We have seen in Section 3.6 that the local martingale parts

$$M_t^1 = \int_0^t \left(\frac{h_r'}{4h} + \frac{g_r'}{4g}\right)^{-1/2} dW^1 \text{ and } M_t^2 = \int_0^t \left(h\left(\frac{h_r'}{4h} + \frac{g_r'}{4g}\right)\right)^{-1/2} dW^2$$

of $\widetilde{R}_t$ and $\widetilde{S}_t$ converge almost surely for $t \to \widetilde{\zeta}$.
That is the reason why the component $\widetilde{R}_t$, when observed at times t near $\widetilde{\zeta}$ (or when starting B near $L(\infty)$), should behave similar to the solution $r(t) := r_0 + t$ of the deterministic differential equation

$$\dot{r} = 1.$$

From the stochastic differential equation above we have

$$\widetilde{S}_t = S_0 + \int_0^t \frac{g_s'(\widetilde{R}_t, \widetilde{S}_t)}{g(\widetilde{R}_t, \widetilde{S}_t)h_r'(\widetilde{R}_t) + h(\widetilde{R}_t)g_r'(\widetilde{R}_t, \widetilde{S}_t)} ds + M_t^2$$

where the local martingale M_t^2 converges for $t \to \widetilde{\zeta}$ and $\widetilde{R}_t$ is expected to behave like $r_0 + t$, when the starting point (r_0, s_0, α_0) of B is chosen near to $L(\infty)$. One could therefore expect $\widetilde{S}_t$ to behave (for t near to $\widetilde{\zeta}$) like the solution $s(t)$ starting in s_0 of the deterministic differential equation

$$\dot{s} = \frac{g_s'(r(t), s)}{g(r(t), s)h_r'(r(t)) + h(r(t))g_r'(r(t), s)}. \tag{3.35}$$

The problem is to find a rigorous way to describe what "should behave like" really means.

Considering the solutions $r(t)$, $s(t)$ of the deterministic differential equations above, one has to remark that $\Gamma_{s_0} : \mathbb{R}_+ \to \mathbb{R}_+ \times \mathbb{R}$ given as $\Gamma_{s_0}(t) := (t, s(t))$ with $\Gamma_{s_0}(0) = (0, s_0)$ is the trajectory of the "drift" vector field

$$V_d = \frac{\partial}{\partial r} + \frac{g'_s}{gh'_r + hg'_r}\frac{\partial}{\partial s} \tag{3.36}$$

starting in $(0, s_0) = L(s_0)$.
As we are going to see below (see Remark 3.28), the "endpoint" $\Gamma_{s_0}(+\infty) \equiv \lim_{t\to\infty} \Gamma_{s_0}(t)$ of all the trajectories is just $L(\infty) \in S_\infty(M)$. Furthermore, for every point $(r, s) \in \mathbb{R}_+ \times \mathbb{R}$ there is exactly one trajectory Γ_{s_0} of V_d with $\Gamma_{s_0}(r) = (r, s)$, i.e. the union

$$\bigcup_{s_0 \in \mathbb{R}} \Gamma_{s_0}$$

defines a foliation of H. Recall that H is one component of $\mathbb{H} \setminus L$ and $M = (H \cup L) \times_g S^1$. If we define a coordinate transformation

$$\begin{aligned} \Phi : \mathbb{R}_+ \times \mathbb{R} &\to \mathbb{R}_+ \times \mathbb{R} \\ (r, s) &\mapsto (r, s_0) \equiv (\Phi_r(r, s), \Phi_s(r, s)), \end{aligned} \tag{3.37}$$

where s_0 is the starting point of the unique trajectory Γ_{s_0} with $\Gamma_{s_0}(r) = (r, s)$ we obtain coordinates for $\mathbb{R}_+ \times \mathbb{R}$ where the trajectories Γ_{s_0} of V_d are just horizontal lines.

If we also apply the coordinate transformation Φ to the components $\widetilde{R}_t$ and $\widetilde{S}_t$ of the Brownian motion we can interpret the behaviour of the components $\widetilde{R}_t$ and $\widetilde{S}_t$ with respect to the trajectories Γ_{s_0} of V_d, that means with respect to the deterministic solutions $r(t)$ and $s(t)$, when looking at the new components $\Phi(\widetilde{R}_t, \widetilde{S}_t) = (\Phi_r(\widetilde{R}_t, \widetilde{S}_t), \Phi_s(\widetilde{R}_t, \widetilde{S}_t))$. The component $\Phi_r(\widetilde{R}_t, \widetilde{S}_t)$ obviously equals $\widetilde{R}_t$. Yet, if one knew that for $t \to \widetilde{\zeta}$ the new component $\Phi_s(\widetilde{R}_t, \widetilde{S}_t)$ possesses a non-trivial limit, that would mean that the Brownian paths (their projection onto $(H \cup L)$, to be precise) finally attain the point $L(\infty) \in S_\infty(M)$ from the direction of one (limiting) trajectory Γ_{s_0}, where $s_0 = \lim_{t\to\widetilde{\zeta}} \Phi_s(\widetilde{R}_t, \widetilde{S}_t)$. Hence another non-trivial information about the asymptotic behaviour would be along which trajectory (or more precisely: along which surface of rotation $\Gamma_{s_0} \times S^1$) the Brownian path finally exits the manifold M.

The remaining problem is to verify that the so-defined new component $\Phi_s(\widetilde{R}_t, \widetilde{S}_t)$ converges to a non-trivial random variable for $t \to \widetilde{\zeta}$. As we have seen, $\Phi_s(\widetilde{R}_t, \widetilde{S}_t)$ is defined to be the starting point of the deterministic curve $s(t)$, satisfying the differential equation (3.35) with $s(\widetilde{R}_t) = \widetilde{S}_t$. But the solution $s(t)$ is of the form

$$s(t) = s_0 + \int_0^t f(r(u), s(u))du$$

with $f = g'_s/(gh'_r + hg'_r)$. In particular, $s(t)$ explicitly depends on $s(u)$ for $u \leq t$. That is the reason why, when applying Itô's formula to $\Phi_s(\widetilde{R}_t, \widetilde{S}_t)$, there appear first order derivatives of the flow

$$\Psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+ \times \mathbb{R}, \qquad (r, s) \mapsto \Gamma_s(r)$$

with respect to the variable s. Estimating these terms does not seem to be trivial and does not provide good estimates to prove convergence of $\Phi_s(\widetilde{R}_t, \widetilde{S}_t)$ when $t \to \widetilde{\zeta}$.

One possibility to circumvent this problem is to find a vector field V on $T(\mathbb{R}_+ \times \mathbb{R})$ of the form $\partial/\partial r + f(r)\partial/\partial s$ whose trajectories also foliate H and are not "far away" from the trajectories Γ_{s_0} of V_d – in particular the trajectories of V have to exit M through the point $L(\infty) \in S_\infty(M)$ as well.

As we have seen in Section 3.6, Theorem 3.22,

$$\left| \frac{g_s'}{gh_r' + hg_r'} - p \right| \le \left| p \cdot \frac{1}{1+h} \right| .$$

Further for $r \ge T_2$ the function $q(r)$ is defined as $\frac{1}{2}p_0 - \frac{40}{h}$, in particular $q(r)$ does not differ much from the function $p(r,s)$ which equals $\frac{1}{2}p_0$ for r and s large. Hence $q(r)$ is a good approximation for $g_s/(gh_r' + hg_r')$ for r large, and does not depend on the variable s.

We therefore consider the vector field

$$V := \frac{\partial}{\partial r} + q(r)\frac{\partial}{\partial s}. \tag{3.38}$$

Starting in $(0, s_0) \in \mathbb{R}_+ \times \mathbb{R}$ the trajectories C_{s_0} of V have the form

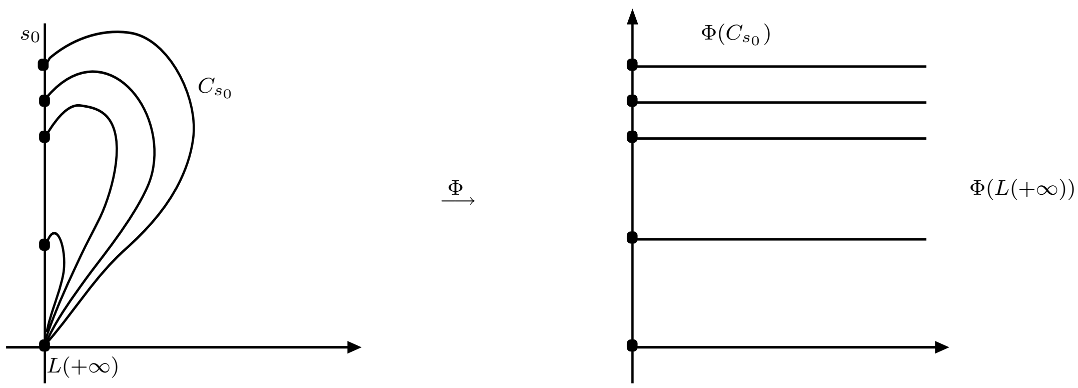
$$C_{s_0}(t) = \left(t, s_0 + \int_0^t q(u)du \right) .$$

As we are going to see below, we also have $\lim_{t\to\infty} C_{s_0}(t) = L(\infty)$, see Remark 3.28, and the union

$$\bigcup_{s_0 \in \mathbb{R}} C_{s_0}$$

forms a foliation of H.

For $(r,s) \in \mathbb{R}_+ \times \mathbb{R}$ there is exactly one trajectory C_{s_0} of V with $C_{s_0}(r) = s$. Its starting point s_0 can be computed as $s_0 = s - \int_0^r q(u)du$. We can therefore define a coordinate transformation



Effect of the coordinate transformation Φ

$$\begin{aligned} \Phi : \mathbb{R}_+ \times \mathbb{R} &\to \mathbb{R}_+ \times \mathbb{R}, \\ (r,s) &\mapsto \left(r, s - \int_0^r q(u)du \right). \end{aligned} \tag{3.39}$$

As seen in the picture above, the trajectories C_{s_0} of V are horizontal lines in the new coordinate system.

In the changed coordinate system the components $\widetilde{R}_t$ and $\widetilde{S}_t$ of B then look like

$$\Phi(\widetilde{R}_t, \widetilde{S}_t) = \left(\widetilde{R}_t, \widetilde{S}_t - \int_0^{\widetilde{R}_t} q(u)du \right) \equiv \left(\Phi_r(\widetilde{R}_t, \widetilde{S}_t), \Phi_s(\widetilde{R}_t, \widetilde{S}_t) \right). \tag{3.40}$$

As we have proven in Section 3.6, Theorem 3.22,

$$\lim_{t \to \widetilde{\zeta}} \left[\widetilde{S}_t - \int_0^{\widetilde{R}_t} q(u)du \right] \equiv \lim_{t \to \widetilde{\zeta}} \Phi_s(\widetilde{R}_t, \widetilde{S}_t)$$

exists and is a non-trivial shift-invariant random variable. Therefore the non-triviality of $\lim_{t \to \widetilde{\zeta}} \Phi_s(\widetilde{R}_t, \widetilde{S}_t)$ provides the possibility to differentiate between Brownian paths when examining along which of the trajectories C_{s_0} of V, i.e. more precisely along which surface of rotation $C_{s_0} \times S^1$, the path finally exits the manifold M. Hence the trajectories C_{s_0} of the vector field V provide a set of "directions" to distinguish between Brownian paths. This is the geometric meaning of $\lim_{t \to \widetilde{\zeta}} \left[\widetilde{S}_t - \int_0^{\widetilde{R}_t} q(u)du \right]$.

It finally remains to complete the section with the proof that the trajectories of the vector field V as well as the trajectories of the vector field V_d exit the manifold M in the point $L(\infty)$. This is done in the final remark:

Remark 3.28.

$$\lim_{t \to \infty} C_{s_0}(t) = L(\infty) \quad \text{and} \quad \lim_{t \to \infty} \Gamma_{s_0}(t) = L(\infty) \quad \text{for every } s_0 \in \mathbb{R}.$$

Proof. It is enough to show that the "s-component" of each trajectory C_{s_0}, Γ_{s_0} resp., converges to ∞ with $t \to \infty$. The s-component of C_{s_0} is

$$s(t) = s_0 + \int_0^t q(r)dr,$$

the s-component of Γ_{s_0}

$$s(t) = s_0 + \int_0^t \frac{g_s'(r, s(r))}{g(r, s(r))h_r'(r) + h(r)g_r'(r, s(r))} dr.$$

Since for $t \geq T_2$ we have

$$q(r) = \frac{1}{2} p_0(r) - \frac{40}{h},$$

it follows immediately that $\lim_{t \to \infty} \int_0^t q(r)dr = \infty$ because $\int_0^\infty \frac{40}{h(r)} dr < \infty$ and

$$\lim_{t \to \infty} \int_0^t p_0(r)dr = \infty$$

due to Section 3.3, Lemma 3.1, Property (iii).

For the second term we notice that $s_0 + \int_0^t f(r, s(r))dr$ with $f = g'_s/(gh'_r + hg'_r)$ is nondecreasing as the integrand is positive. Moreover we have seen above and in the foregoing sections that

$$\left| \frac{g'_s}{gh'_r + hg'_r} - p \right| \leq \left| p \cdot \frac{1}{1+h} \right| .$$

As

$$\int_0^\infty p(r,s) \frac{1}{1+h(r)} dr \leq \int_0^\infty \frac{1}{1+h(r)} dr < \infty$$

it suffices to show that $\lim_{t\to\infty} \int_0^t p(s(r), r)dr = \infty$. This is true as $s(r) \geq s_0$ for all $r \leq t$ and therefore for r large enough we have $p(s(r), r) = \frac{1}{2} p_0(r)$. Then the claimed result follows exactly as above. $\square$

Chapter 4

Further Constructions of Non-Liouville Manifolds of Unbounded Curvature

In the foregoing chapter we presented the example of Borbély and gave a stochastic proof that the Dirichlet problem at infinity for the constructed manifold M is not solvable. We furthermore showed that there exist non-trivial bounded harmonic functions on M obtained from the non-trivial shift-invariant random variables $A_\zeta := \lim_{t\to\zeta} A_t$ and $Z_{\tilde\zeta} := \lim_{t\to\tilde\zeta} Z_t$. As already mentioned, the example given in [B] was not the first to provide such a manifold. In 1994 Ancona gave an example of a Riemannian manifold for which the Dirichlet problem at infinity is not solvable (cf. [A1]). As one possibility to prove the non-solvability of the Dirichlet problem at infinity he used Brownian motion on M and showed that all Brownian paths exit the manifold M almost surely at a single point ∞_M of the sphere at infinity. We adopted some of his ideas in the foregoing chapter to derive the same result for the manifold of Borbély. However, Ancona did not deal with the existence of non-trivial bounded harmonic functions on his manifold.

As it turns out – considering probabilistic properties – the two manifolds of Ancona and Borbély are essentially the same. Hence it is quite obvious that there also exist non-trivial bounded harmonic functions on the manifold of Ancona, to be obtained with the help of non-trivial shift-invariant random variables we derive as $\mathbb{P}$-a.s. limits of Brownian functionals. As we are going to show below (see Section 4.1) it is quite easy and the same proof as in Section 3, Lemma 3.16 to show that $\lim_{s\to\infty} Y_s$, where Y is one component of the Brownian motion, at once yields a non-trivial shift-invariant random variable and therefore the existence of non-trivial bounded harmonic functions on M. Though it may be obvious from the construction of the manifold and from geometrical considerations that there is a second non-trivial shift-invariant random variable corresponding to the random variable $\lim_{t\to\tilde\zeta} Z_t$ of Section 3.6, it does not seem obvious to prove its existence. We give an explanation of that below.

However, compared with the example of Borbély it is an advantage of Ancona's manifold that the metric is constructed more or less explicitly – and not with the help of the drift ratio $p(s,r)$ as in [B]. This makes it possible to extend the example on the to higher dimensions and to slightly modify the construction to obtain additional examples of manifolds, where the Dirichlet problem at infinity is not solvable: considering [A1], Theorem B, there is a Riemannian manifold (M, g) where for every point $\vartheta_0 \in S_\infty(M)$ and every neighbourhood U_{ϑ_0} of ϑ_0 with probability 1 Brownian motion B on M hits U_{ϑ_0} infinitely many times. In particular, the angular part $\vartheta(B_s)$ of B_s "oscillates" for $s \to \infty$, i.e. does not converge at all. We slightly modify the construction to obtain a Riemannian manifold (M, γ) such that the angular part $\vartheta(B_s)$ of the Brownian motion on M does not converge on $S_\infty(M)$ but we can show that there exist non-trivial bounded harmonic functions on M, see Theorem 4.16.

In the following sections we give a short description of the manifold constructed by Ancona – in particular of the construction of the Riemannian metric on M. For explicit details we refer to [A1], and as many of the methods are analogous to the ones presented in Section 3 we omit most of the proofs. We again give a geometrical interpretation and then extend the given examples to higher dimensions.

4.1. The Manifold of Ancona

The following notations are the same as in [A1]:

Let (M, γ) be a Riemannian manifold of dimension 3 defined as

$$M := \mathbb{R}^3 = \{(x, y, t) : x, y, t \in \mathbb{R}\}$$

with Riemannian metric γ given in the global coordinates x, y and t as

$$ds_\gamma^2 = dt^2 + e^{2t}dx^2 + h(x,t)^2 dy^2,$$

where $h : \mathbb{R}^2 \to \mathbb{R}_+$ is a smooth positive function, nondecreasing in t and fulfilling $h(x, t) = e^t$ for $t \leq 0$.

M is complete and has strictly negative sectional curvature $\text{Sect}_M \leq -\alpha^2$ for $\alpha \in [-1; 1] \setminus \{0\}$ if and only if

$$\frac{h''_{tt}}{h} \geq \alpha^2, \tag{4.1}$$

$$\left(\frac{h'_t}{h} + \frac{h''_{xx}}{e^{2t}h}\right) \geq \alpha^2, \tag{4.2}$$

$$\left(\frac{h''_{xt}}{e^t h} - \frac{h'_x}{e^t h}\right)^2 \leq \left(\frac{h''_{tt}}{h} - \alpha^2\right)\left(\frac{h'_t}{h} + \frac{h''_{xx}}{e^{2t}h} - \alpha^2\right). \tag{4.3}$$

For the idea how such a function $h : \mathbb{R}^2 \to \mathbb{R}$ can be constructed see the remark after Lemma 4.1.
The sphere at infinity $S_\infty(M)$ is given as

$$S_\infty(M) = \{\zeta_{(x,y)} : (x, y) \in \mathbb{R}^2\} \cup \{\infty_M\}$$

where $\zeta_{(x,y)} := \tau_{(x,y)}(\infty)$ denotes the equivalence class determined by the geodesic

$$\tau_{(x,y)} : \mathbb{R} \to M, \quad t \mapsto (x, y, t)$$

for $(x, y) \in \mathbb{R}^2$ fixed and $t \to \infty$. The single point ∞_M is the common equivalence class $\tau_{(x,y)}(-\infty)$ determined by all geodesic rays $\tau_{(x,y)}$ as $t \to -\infty$. The proof that $\tau_{(x_1,y_1)}(-\infty) = \tau_{(x_2,y_2)}(-\infty)$ for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ is easily done.

We furthermore have the Laplace Beltrami operator on M given by:

$$\Delta_M = \frac{\partial^2}{\partial t^2} + \frac{1}{e^{2t}}\frac{\partial^2}{\partial x^2} + \frac{1}{h^2}\frac{\partial^2}{\partial y^2} + \left(1 + \frac{h'_t}{h}\right)\frac{\partial}{\partial t} + \frac{h'_x}{e^{2t}h}\frac{\partial}{\partial x}.$$

It is obvious that the constructions of Ancona and Borbély follow the same principles: both manifolds are constructed as warped products of complete Riemannian manifolds or subspaces of them. The coefficients of the Riemannian metric do not depend on the coordinate y, α respectively, and hence the Laplace-Beltrami operator does not depend in first order on the variable y, α respectively. As we are going to see below this has the effect that also the component Y_t of the Brownian motion B on Ancona's manifold M as well as the component A_t in Chapter 3 are local martingales that possesses a non-trivial limit Y_ζ, A_ζ respectively, for $t \to \zeta$. Here ζ denotes again the lifetime of the Brownian motion on M. This is the same result we used in the foregoing chapter to prove the existence of non-trivial bounded harmonic functions on the manifold of Borbély.

Before we are going to state the main theorem of this chapter concerning the Brownian motion on the constructed manifold, we give a short list of the properties the warped product function $h(x, t)$ has to satisfy

4.2. Properties and Construction of the Function h

In the following text $(B_s)_{s<\zeta} = (X_s, Y_s, T_s)_{s<\zeta}$ denotes the Brownian motion on M with lifetime ζ.

For $j \in \mathbb{N}$ let $[t_{4j+1}, t_{4j+2}]$ be an interval in $\mathbb{R}$ with $t_{4j+2} \geq t_{4j+1} + 1$. Denote with Φ_j the function on $[t_{4j+1}, t_{4j+2}]$ defined as follows:

$$\Phi_j(t) := 1 + \int_{t_{4j+1}}^{t} \varphi_j(s)e^{-2s}ds \quad \text{for } t \in [t_{4j+1}, t_{4j+2}], \tag{4.4}$$

where φ_j is a smooth increasing function with $\varphi_j(t) = 0$ for $t \in \left[t_{4j+1}, t_{4j+1} + \frac{1}{2}\right)$ and $\varphi_j(t) = \varphi_j$ on $\left(t_{4j+1} + \frac{3}{4}, \infty\right)$ for a positive constant $\varphi_j \in \mathbb{R}_+$. See Remark 4.2 for the choice of the φ_j.

Let further $\beta_j : [-j, j] \to \mathbb{R}_+$ be given as

$$\beta_j(x) := \underbrace{\sqrt{e^{j+1} + e^{2j+2}}}_{=:b_j} + e^x. \tag{4.5}$$

The existence of a suitable warped product function $h : \mathbb{R}^2 \to \mathbb{R}_+$ such that the corresponding Riemannian manifold has strictly negative sectional curvatures and provides an example where the Brownian motion B almost surely converges to the single point ∞_M of $S_\infty(M)$ is part of the following lemma:

Lemma 4.1. *There is a smooth function $h : \mathbb{R}^2 \to \mathbb{R}_+$ and a sequence $(t_i)_{i\in\mathbb{N}}$ of real numbers with $t_0 \geq 0$ and $t_{i+1} \geq t_i + 1$ for all $i \in \mathbb{N}$, satisfying the following properties:*

i) $h(x,t) = e^t$ *for* $t \leq t_0$ *and* $t_{4j+3} < t < t_{4(j+1)}$ *for all* $j \in \mathbb{N}$.

ii) $h(x,t) = e^t \cdot \exp(\Phi_j(t)\beta_j(x))$ *on* $U_j := \{(x,y,t) : |x| < j, t_{4j+1} \leq t \leq t_{4j+2}\}$ *and one has*

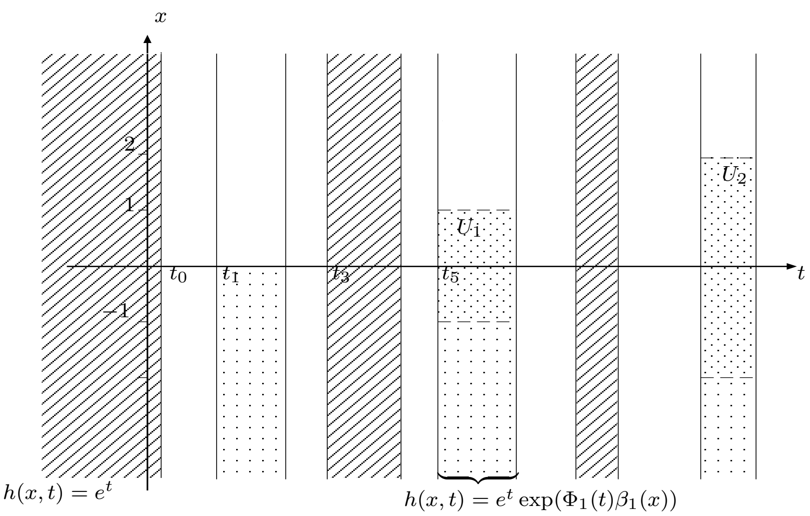
$$\mathbb{P}^m \{T_\tau = t_{4j+1} \text{ or } T_\tau = t_{4j+2}\} \leq 2^{-j},$$

if $\tau := \inf\{t \geq 0 : B_t^m \notin U_j\}$ *denotes the first exit time of the Brownian motion* $B = (X,Y,T)$, *starting in* $m = (x,y,t)$ *with* $|x| < j$ *and* $t = (t_{4j+1} + t_{4j+2})/2$, *from the set* U_j.

iii) h *is increasing in* x *and in* t *and depends only on the variable* t *for "large"* x.

iv) h *satisfies the curvature conditions (4.1),(4.2) and (4.3).*

Proof. The proof can be found in [A1], Propositions 2.1 and 3.3, Lemmata 3.4, 3.9, 4.1, 4.2 and 5.2. □



Definition of the function $h : \mathbb{R}^2 \to \mathbb{R}_+$

Remark 4.2.

i) The condition $h(x,t) = e^t$ for $t \leq t_0$ and $t_{4j+3} \leq t \leq t_{4(j+1)}$ implies that on the corresponding "stripes" $\mathbb{R}^2 \times [t_{4j+3}, t_{4(j+1)}]$ the Brownian motion B on M behaves like a time changed Euclidean Brownian motion with an additional drift term t in the T-component (see (4.7) ff. where we compute the defining stochastic differential equation for B). Hence the exit time τ_j from $\mathbb{R}^2 \times [t_{4j+3}, t_{4j+3} + 1]$ of a Brownian motion started in (x,y,t) with $t=(2t_{4j+3}+1)/2$ is independent of j. As an immediate consequence, Borel-Cantelli yields that B has almost surely infinite lifetime if $\lim_{s\to\zeta} T_s=\infty$.

ii) Condition (ii) immediately implies that the x-component of the Brownian motion grows in some sense "faster" than the t-component of B – at least within the sets U_j. From that it follows again with a simple Borel-Cantelli argument and using the Strong Markov Property that as an (immediate) consequence of $\lim_{s\to\zeta} T_s=\infty$ we can deduce that $\lim_{s\to\zeta} X_s=\infty$ almost surely if we know in addition that $\lim_{s\to\zeta} X_s$ almost surely exists in $(-\infty,\infty]$.

iii) We requested in Chapter 3, Lemma 3.1, property (iii) that the "drift ratio" $p(r,s)$ has to fulfill the condition $\int_0^\infty p(r,s)dr=\infty$ for every $s\in\mathbb{R}$. In this context we define the "drift ratio" as $h'_x/(e^{2t}h'_t)$, see below. Then we have from [A1], Lemma 4.2, that

$$\int_0^\infty \delta_K(t)dt=\infty \tag{4.6}$$

for every compact set $K\subset\mathbb{R}$ and $\delta_K(t):=\inf\left\{h'_x/(e^{2t}h'_t):x\in K\right\}$. For the proof it is necessary to choose on the intervals $[t_{4j+1},t_{4j+2}]$ the constant φ_j and then the upper interval bound t_{4j+2} large enough such that

$$\int_{t_{4j+1}}^{t_{4j+2}} \delta_j(t)dt\geq 1.$$

Herein $\delta_j(t):=\inf\left\{h'_x/(e^{2t}h'_t):|x|\leq j\right\}$ for $t_{4j+1}\leq t\leq t_{4j+2}$. (cf. [A1], Proposition 2.1). This corresponds to the condition for the drift ratio of Chapter 3.

iv) Condition (iii) can be understood as follows: for every interval $[t_{4j},t_{4j+3}]$ there is a constant K_j such that $h(x,t)$ is independent of the variable x for $|x|>K_j$.

v) The construction of the function $h(x,t)$ is clear on the region $t\leq 0$, on the "stripes" $[t_{4j+1},t_{4(j+1)}]\times\mathbb{R}$ and on the sets U_j for $j\in\mathbb{N}$, as the map h is explicitely given for these parts of M.
Moreover one can require $h(x,t)$ to be given in the form $h(x,t)=e^t\cdot\exp\left(\Phi_j(t)\beta_j(x)\right)$ on the set

$$J_j:=\{(x,t):x\leq x_0-a_0e^{-t_{4j}},\ t_{4j+1}\leq t\leq t_{4j+2}\}$$

for a constant a_0 (see [A1], p.202) and every $j\in\mathbb{N}$. This is indicated in the picture above and outlined in [A1], Remark 3.7. The proof that the so defined "pieces" of h can be smoothly glued together such that the curvature conditions for h are still satisfied is given in Proposition 3.3 and Lemma 3.9 of [A1].

4.3. Theorem A of Ancona

The Riemannian manifold (M, γ) constructed above serves as an example to prove the following theorem which is one of the main theorems of [A1]:

Theorem 4.3 (cf. [A1], Theorem A).
There is a complete simply connected Riemannian manifold M of dimension 3 with sectional curvatures bounded from above by -1, and a point $\zeta_0 \in S_\infty(M)$ such that

i) the Brownian motion B_s on M almost surely has infinite lifetime,

ii) with probability 1, the Brownian motion B_s exits from M at ζ_0.

Consider the Brownian motion $(B_s)_{s<\zeta} = (X_s, Y_s, T_s)_{s<\zeta}$ with lifetime ζ on the Riemannian manifold (M, γ). Then the components X, Y and T of B are given as solutions of the following system of stochastic differential equations:

$$dT_s = \frac{1}{2}\left(1 + \frac{h_t'(X_s, T_s)}{h(X_s, T_s)}\right) ds + dW_s^3, \tag{4.7}$$

$$dX_s = \frac{h_x'(X_s, T_s)}{2h(X_s, T_s) \cdot e^{2T_s}} ds + \frac{1}{e^{2T_s}} dW_s^1, \tag{4.8}$$

$$dY_s = \frac{1}{h(X_s, T_s)} dW_s^2, \tag{4.9}$$

with a three dimensional Euclidean Brownian motion (W^1, W^2, W^3).

The proof of Theorem A is an immediate consequence of the following theorem:

Theorem 4.4. *Let M be the Riemannian manifold constructed above and let $(B_s)_{s<\zeta} = (X_s, Y_s, T_s)_{s<\zeta}$ be the Brownian motion on M with lifetime ζ. Then the following statements hold true for the components of B:*

i) $\lim_{s\to\zeta} T_s = \infty$ almost surely.

ii) $\lim_{s\to\zeta} X_s = \infty$ almost surely.

*iii) $\lim_{s\to\zeta} Y_s$ exists almost surely and is almost surely finite.
Furthermore $\lim_{s\to\zeta} Y_s$ is almost surely a non-trivial shift invariant random variable. Hence for every bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ there is a non-trivial bounded harmonic function $u : M \to \mathbb{R}$, $m \mapsto u(m)$, given as*

$$u(m) := \mathbb{E}^m \left[f \circ \left(\lim_{s\to\zeta} Y_s \right) \right].$$

In particular, M possesses non-trivial bounded harmonic functions.

iv) $\zeta = \infty$ almost surely.

Proof. It follows from [A1], Lemma 5.1, that almost surely $\lim_{t\to\zeta} T_s = \infty$, $\lim_{s\to\zeta} Y_s$ exists and is finite and $\lim_{s\to\zeta} X_s$ exists in $(-\infty, \infty]$. As remarked above, $\lim_{s\to\infty} X_s = \infty$ follows from Property (ii) of the function $h(x,t)$ and $\lim_{s\to\zeta} T_s = \infty$. A proof of this can be found in [A1], Proposition 5.4. Last, the almost surely infinite lifetime of B_s is a consequence of Property (i) of $h(x,t)$ and is due to [A1], Lemma 5.3 and Proposition 5.4.

For the last claim that $\lim_{s\to\zeta} Y_s$ is almost surely a non-trivial shift-invariant random variable recall from (4.9) that the component Y_s of B_s is a local martingale given by

$$Y_s = Y_0 + \int_0^s \frac{1}{h(X_r, T_r)}\, dW_r^2,$$

where the integrand is independent of Y_s and in particular independent of the starting point Y_0 of Y_s. The proof is then completely analogous to the proof of 3, Lemma 3.16. $\square$

Remark 4.5 *(Some geometrical aspects).*
As indicated at the beginning of this chapter, we will finish this section with some geometrical considerations: We have defined the drift ratio $h'_x/(e^{2t}h'_t)$ for (M,γ) which is essentially the drift we have for the x-component of the time changed Brownian motion $\widetilde{B}_s := B_{\tau_s}$ where $\tau_s := T^{-1}(s)$ and

$$T(s) = \int_0^s \frac{1}{2}\left(1 + \frac{h'_t}{h}\right) dr.$$

Consider now the drift vector field on M

$$V_d := \frac{\partial}{\partial t} + \frac{h'_x}{e^{2t}h'_t}\frac{\partial}{\partial x}.$$

Let $\Gamma_{x_0} : \mathbb{R} \to \mathbb{R}^2, s \mapsto \Gamma_{x_0}(s)$, be the trajectory of V_d starting in the point $(x_0, 0) \in \mathbb{R}^2$. Then it is an easy consequence of (4.6) that for $s \to \infty$ all the trajectories Γ_{x_0} of V_d exit from M at the point ∞_M. Moreover as $h'_x = 0$ for $t \le 0$, the trajectories Γ_{x_0} are horizontal lines for $s \le 0$ and there coincide with the geodesics $\tau_{(x_0,0)}$ defined at the beginning of this chapter. From that it is clear that for $s \to -\infty$ all the trajectories Γ_{x_0} have their "origin" in the point ∞_M. Furthermore the trajectories Γ_{x_0} form a foliation of $\mathbb{R}^2$ and so give rise to a coordinate transformation for $\mathbb{R}^2$.

With respect to that it arises as a natural question if there are – in analogy to the example of Borbély – further non-trivial shift invariant events for B (and therefore further non-trivial bounded harmonic functions) besides the ones we get from the non-trivial shift invariant random variable $\lim_{s\to\infty} Y_s$. One can think of the possibility to distinguish between different paths of B in looking along which trajectory Γ_{x_0} the Brownian path exits from the manifold M. If we define a coordinate transformation $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ assigning to each point (x,t) the point (x_0, t), where x_0 is the starting point of the unique trajectory Γ_{x_0} with $\Gamma_{x_0}(t) = (x,t)$ analogously to Section 3.8, and consider the diffusion $\Phi(X_s, T_s)$, there remains the problem to explicitly compute the new x-component. We already pointed this out in Section 3.8. In contrast to the example of Borbély it is not that easy to determine a function $q(t)$ which is a good approximation for the drift ratio $h'_x/(e^{2t}h'_t)$ for large t – maybe the function $\delta_K(t)$ could do that part for a certain compact set K. However, as mentioned before, the function h does not depend on the variable x for large x, which means that the drift ratio vanishes for x large. Thus we cannot find any lower bound $\neq 0$ for $h'_x/(e^{2t}h'_t)$ depending only on the variable t. We did not succeed yet in finding a suitable upper bound for the drift ratio. However, it seems to be quite likely that one can find non-trivial bounded harmonic functions using the drift ratio and the drift vector field V_d. We leave this as an open question.

4.4. Theorem A Extended to Higher Dimensions

In the following section we work out an idea that was already given in Ancona; the extension of Theorem A to higher dimensions. We are going to state this in the following theorem.

Theorem 4.6 (cf. [A1], p.217).
For every $d \geq 3$ there is a complete simply connected Riemannian manifold (M, γ) of dimension d with sectional curvatures bounded from above by $-1/4$, and a point $\zeta_0 \in S_\infty(M)$ such that:

i) almost surely Brownian motion B on M has infinite lifetime,

ii) with probability 1, Brownian motion B exits from M at ζ_0.

As in the foregoing chapter we briefly sketch the construction of a Riemannian manifold of dimension d with sectional curvatures bounded from above by $-1/4$ following Ancona and then consider the behaviour of the components of the Brownian motion B_s on M with lifetime ζ for $t \to \infty$. As we are interested in the existence of non-trivial bounded harmonic functions on the manifold, we additionally give a way to construct, given a bounded continuous function $f : \mathbb{R}^{d-2} \to \mathbb{R}$, a non-trivial bounded harmonic function $u : M \to \mathbb{R}$ which possesses f as "boundary function" in a suitable sense.

We start with the construction of the Riemannian manifold (M, γ) using again the notations of Ancona:

Let $d := m+3$ for an integer $m \geq 1$ and let $M := \mathbb{R}^d = \{(x, y, z_1, \ldots, z_m, t) : x, y, t, z_j \in \mathbb{R}\}$ be equipped with the Riemannian metric

$$ds_\gamma^2 = dt^2 + e^{2t}dx^2 + h(x,t)^2 dy^2 + e^{2t} \sum_{j=1}^{m} dz_j^2.$$

The function $h(x,t)$ is constructed as in the foregoing section satisfying the properties of Lemma 4.1. However, to guarantee Property (ii) for the Brownian motion $B = (X, Y, Z_1, \ldots, Z_m, T)$ on M, starting in the point $m = (x, y, z_1, \ldots, z_m, t)$ with $|x| < j$ and $t = (t_{4j+1} + t_{4j+2})/2$, one has to slightly modify the choice of the sequence $(t_i)_{i \in \mathbb{N}}$.

Remark that, as in the case of dimension 3, for $(x, y, z_1, \ldots, z_m) \in \mathbb{R}^{d-1}$ fixed one has the geodesic

$$\tau_{(x,y,z_1,\ldots,z_m)} : \mathbb{R} \to M,\ t \mapsto (x, y, z_1, \ldots, z_m, t)$$

and as before all the geodesics $\tau_{(x,y,z_1,\dots,z_m)}$ emanate from the same point ∞_M of the sphere at infinity. Hence we can describe the sphere at infinity $S_\infty(M)$ as the set

$$S_\infty(M) = \{\zeta_{(x,y,z_1,\dots,z_m)} : (x,y,z_1,\dots,z_m) \in \mathbb{R}^{d-1}\} \cup \{\infty_M\},$$

where $\zeta_{(x,y,z_1,\dots,z_m)} := \tau_{(x,y,z_1,\dots,z_m)}(\infty)$ denotes the equivalence class determined by the geodesic $\tau_{(x,y,z_1,\dots,z_m)}$ for $t \to \infty$.

On M we have the Laplace-Beltrami operator

$$\Delta_M = \frac{\partial^2}{\partial t^2} + \frac{1}{e^{2t}}\frac{\partial^2}{\partial x^2} + \frac{1}{h^2}\frac{\partial^2}{\partial y^2} + \frac{1}{e^{2t}}\sum_{j=1}^{m}\frac{\partial^2}{\partial z_j^2} + \left(1 + m + \frac{h'_t}{h}\right)\frac{\partial}{\partial t} + \frac{h'_x}{e^{2t}h}\frac{\partial}{\partial x}.$$

We can therefore interpret the Brownian motion $B = (X, Y, Z_1, \dots, Z_m, T)$ with lifetime ζ on M as solution of the following system of stochastic differential equations:

$$\begin{aligned}
dT_s &= \frac{1}{2}\left(1 + m + \frac{h'_t(X_s,T_s)}{h(X_s,T_s)}\right) ds + dW_s^d \\
dX_s &= \frac{h'_x(X_s,T_s)}{2h(X_s,T_s)\cdot e^{2T_s}}\, ds + \frac{1}{e^{2T_s}}\, dW_s^1 \\
dY_s &= \frac{1}{h(X_s,T_s)}\, dW_s^2 \\
dZ_{1s} &= \frac{1}{e^{T_s}}\, dW_s^3 \\
&\vdots \\
dZ_{ms} &= \frac{1}{e^{T_s}}\, dW_s^{d-1},
\end{aligned}$$

with a Euclidean Brownian motion $(W^1, W^2, \dots, W^d)$ on $\mathbb{R}^d$.

Theorem 4.6 now is an immediate consequence of the following theorem:

Theorem 4.7. *For $d \geq 3$ let (M, γ) denote the Riemannian manifold of dimension d constructed above and let $(B_s)_{s<\zeta} = (X_s, Y_s, Z_{1s}, \dots, Z_{ms}, T_s)_{s<\zeta}$ be the Brownian motion on M with lifetime ζ. Then the following statements hold true:*

i) $\lim_{s\to\zeta} T_s = \infty$ *almost surely.*

ii) $\lim_{s\to\zeta} X_s = \infty$ *almost surely.*

iii) The limits $\lim_{s\to\zeta} Y_s$, $\lim_{s\to\zeta} Z_{1s}$, $\dots$, $\lim_{s\to\zeta} Z_{ms}$ *almost surely exist and are almost surely finite.*
Furthermore $\lim_{s\to\zeta} Y_s$, $\lim_{s\to\zeta} Z_{1s}$, $\dots$, $\lim_{s\to\zeta} Z_{ms}$ *are almost surely non-trivial shift-invariant random variables. Consequently for every bounded continuous function $f : \mathbb{R}^{d-2} \to \mathbb{R}$ there is a non-trivial bounded harmonic function $u : M \to \mathbb{R}$, $m \mapsto u(m)$ given as*

$$u(m) := \mathbb{E}^m \left[f \circ \left(\lim_{s \to \zeta} (Y_s, Z_{1s}, \dots, Z_{ms}) \right) \right].$$

In particular, M possesses non-trivial bounded harmonic functions.

iv) $\zeta = \infty$ *almost surely.*

Proof. We give a short sketch of the proof as most part of it can be adopted directly from Ancona and the proof of Theorem 4.4.

Statement i) is a consequence of the fact that the real valued function on M given by $(x, y, z_1, \dots, z_m, t) \mapsto e^{-t}$ is Δ_M-superharmonic and that the sectional curvatures of M are bounded on each region of the form $\{t \le a\}$ for $a \in \mathbb{R}$, see [A1], Lemma 5.1.

Part iv) immediately follows from the construction of the function h and $\lim_{s\to\zeta} T_s = \infty$, cf. [A1], Lemma 5.3.

The claimed result $\lim_{s\to\zeta} X_s \in (-\infty, \infty]$ follows as in [A1], Lemma 5.1 because as $m \geq 1$ a smooth function $u(x, y, z_1, \dots, z_m, t) \equiv u(x, t)$ on M is Δ_M-superharmonic if it is nonincreasing in the variables x and t and in addition superharmonic with respect to

$$L := \frac{\partial^2}{\partial t^2} + \frac{1}{e^{2t}} \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}.$$

By a Borel-Cantelli argument, using Property (ii) of the metric function h and $\lim_{s\to\zeta} T_s = \infty$, we obtain that in fact $\lim_{s\to\zeta} X_s = \infty$, cf. [A1], Lemma 5.4.

To prove the almost sure existence and finiteness of $\lim_{s\to\zeta} Y_s$ and $\lim_{s\to\zeta} Z_{is}$ for $i = 1, \dots, m$, we note the following fact:

A function $u : M \to \mathbb{R}$ with $u(x, y, z_1, \dots, z_m, t) \equiv u(y, t)$, $u(z_i, t)$ respectively, is Δ_M-superharmonic on the absorbing region $\{t > 0\} \subset M$ if it is nonincreasing in t, convex in y, z_i respectively, and superharmonic with respect to

$$\begin{aligned} L &= \frac{\partial^2}{\partial t^2} + \frac{1}{e^t} \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial t}, \\ L_i &= \frac{\partial^2}{\partial t^2} + \frac{1}{e^t} \frac{\partial^2}{\partial z_i^2} + \frac{\partial}{\partial t} \quad \text{respectively.} \end{aligned}$$

Using this, it follows that the functions

$$u(y,t) = \begin{cases} 1 & \text{for } y \le y_0 \quad (y \ge y_0 \text{ resp.}) \\ 1 - \frac{2}{\pi} \arctan\left(\frac{1}{2} |y - y_0| e^{\frac{t}{2}} \right) & \text{for } y > y_0 \quad (y < y_0 \text{ resp.}) \end{cases}$$

$$u(z_i,t) = \begin{cases} 1 & \text{for } z_i \le z_{i0} \quad (z \ge z_{i0} \text{ resp.}) \\ 1 - \frac{2}{\pi} \arctan\left(\frac{1}{2} |z_i - z_{i0}| e^{\frac{t}{2}} \right) & \text{for } z_i > z_{i0} \quad (z_i < z_{i0} \text{ resp.}) \end{cases}$$

are Δ_M-superharmonic on $\{t > 0\}$ for every $y_0, z_{i0} \in \mathbb{R}$.

Now it follows as in [A1], Lemma 5.1, that almost surely $Y_\zeta := \lim_{s\to\zeta} Y_s$, $Z_{i\zeta} := \lim_{s\to\infty} Z_{is}$ for all $i = 1\dots, m$ exist and are almost surely finite shift-invariant random variables.

The proof that the random variables Y_ζ and $Z_{i\zeta}$ are almost surely non-trivial is obtained as in Section 4.3, Theorem 4.4, from the fact that the component processes $(Y_s)_{s<\zeta}$ and $(Z_{is})_{s<\zeta}$ are local martingales and that within their integral representations

$$Y_s = Y_0 + \int_0^s h(X_r, T_r)^{-1}\, dW_r^2 \text{ and } Z_{is} = Z_{i0} + \int_0^s e^{-T_r}\, dW_r^{2+i}$$

the integrand is independent of the component process Y and Z itself and in particular of its starting point.
The existence of harmonic functions $u : M \to \mathbb{R}$ as claimed is now an immediate consequence. □

4.5. Further Constructions and Considerations

Let again (M, γ) denote the Riemannian manifold of dimension 3 with $M := \{(x, y, t) : x, y, t \in \mathbb{R}\}$ and Riemannian metric

$$ds_\gamma^2 = dt^2 + e^{2t}dx^2 + h^2(x, t)dy^2$$

as considered in Section 4.1.
As we have seen in Section 4.3, the behaviour of Brownian motion on M depends on the choice of the function $h : \mathbb{R}^2 \to \mathbb{R}$. Property (ii) of Lemma 4.1 is the main reason why the x-component of the Brownian motion B_s on M is forced to converge to ∞ for $s \to \zeta$. To obtain further examples of Riemannian manifolds where Brownian motion B shows interesting asymptotic properties, it is therefore obvious that one has to modify the definition of the function h. In [A1], p.215, there is pointed out a second possibility to define the function h such that the Riemannian manifold M possesses strictly negative sectional curvatures. Ancona uses this function in the first step of the proof of the following theorem:

Theorem 4.8 (cf. [A1], Theorem B).
There exists a complete simply connected Riemannian manifold M of dimension 3, with sectional curvatures bounded from above by -1 and such that

- *i) the Brownian motion B_s on M almost surely has infinite lifetime,*
- *ii) with probability 1, every point on the sphere at infinity $S_\infty(M)$ is a cluster point of B_s (when $s \to \infty$).*

Proof. For the proof see [A1], p.215. □

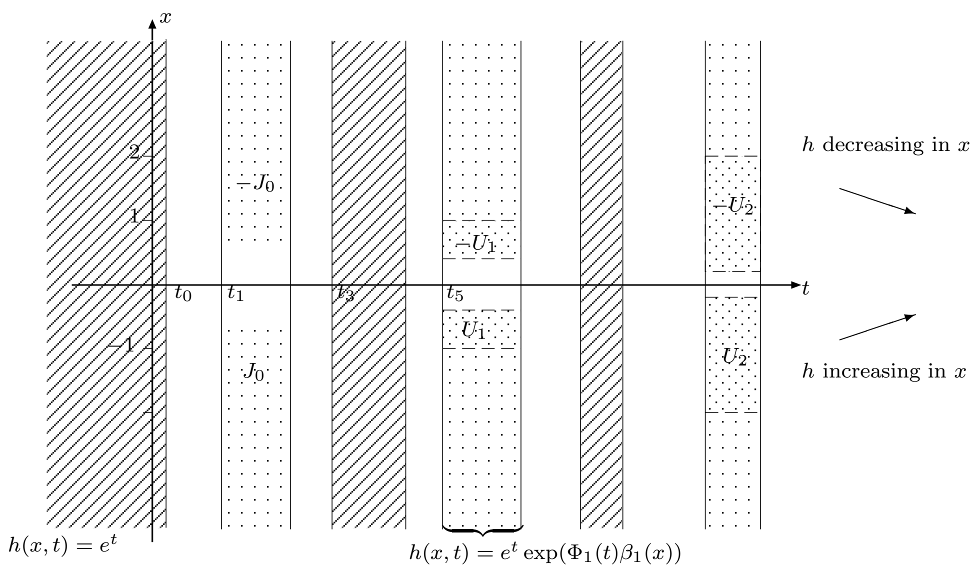
Remark 4.9. Up to now we did not succeed in deciding whether there exist non-trivial bounded harmonic functions on the manifold that Ancona constructed to prove Theorem B. This is caused by the fact that to prove the theorem above one has to make the metric function h dependent on the variable y to be able to control the asymptotic behaviour of the component Y of the Brownian motion as well. See [A1], p. 213f, *first step.* If h additionally depends on y, we have to change the Laplace-Beltrami operator on M to:

$$\Delta_M = \frac{\partial^2}{\partial t^2} + \frac{1}{e^{2t}}\frac{\partial^2}{\partial x^2} + \frac{1}{h^2}\frac{\partial^2}{\partial y^2} + \left(1 + \frac{h'_t}{h}\right)\frac{\partial}{\partial t} + \frac{h'_x}{e^{2t}h}\frac{\partial}{\partial x} + \frac{h'_y}{h^3}\frac{\partial}{\partial y}.$$

As Δ_M now depends in first order on the variable y the component Y of the Brownian motion fails to be a local martingale and becomes dependent on the behaviour of the path $Y(\omega)$. As a consequence, the method used before to obtain non-trivial shift-invariant random variables does not work here.

However, we are going to use the ideas of Ancona to give an example of a Riemannian manifold (M, γ) of dimension 3 such that the Brownian motion B_s on M does not possess a limit for $s \to \infty$ on the sphere at infinity $S_\infty(M)$, whereas there still exist non-trivial bounded harmonic functions on M.

We start with the definition of the function $h : \mathbb{R}^2 \to \mathbb{R}_+$ (remember the definition of the functions Φ_j and β_j at the beginning of Section 4.2):



Definition of the function $h : \mathbb{R}^2 \to \mathbb{R}_+$

Lemma 4.10. *There is a smooth function $h : \mathbb{R}^2 \to \mathbb{R}_+$ and a sequence $(t_i)_{i\in\mathbb{N}}$ of real numbers with $t_0 \geq 0$ and $t_{i+1} \geq t_i + 1$ for all $i \in \mathbb{N}$ satisfying the following properties:*

i) $h(x,t) = e^t$ *for* $t \leq t_0$ *and* $t_{4j+3} \leq t \leq t_{4(j+1)}$ *for all* $j \in \mathbb{N}$.

ii) $h(x,t)$ *is an even function of* x *which is increasing in* x *for* $x \leq 0$ *and decreasing in* x *for* $x \geq 0$. *The function* h *increases in the variable* t.

iii) $h(x,t) = e^t \cdot \exp(\Phi_j(t)\beta_j(x))$ *on* $J_j := \{(x,y,t) : x < -2a_0e^{-t_{4j}}, t_{4j+1} \leq t_{4j+2}\}$, *where* $a_0 > 0$ *is a constant (see [A1], p.203).*

iv) The sequence $(t_i)_{i\in\mathbb{N}}$ *can be chosen such that*

$$\mathbb{P}^m \{T_\tau = t_{4j+1} \text{ or } T_\tau = t_{4j+2}\} \leq 2^{-j},$$

if $\tau := \inf\{t \geq 0 : B_t^m \notin U_j\}$ *denotes the first exit time of the Brownian motion from the set* $U_j := \{(x,y,t) : -j < x < -a_0 e^{-t_{4j}}, t_{4j+1} \leq t_{4j+2}\}$*, where* B^m *is the Brownian motion on* M *starting in* $m = (x,y,t)$ *with* $-j \leq x \leq -2a_0 e^{-t_{4j}}$ *and* $t = (t_{4j+1} + t_{4j+2})/2$.

Remark 4.11. It is obvious that due to the symmetry of the dependence of the function h on the variable x we also have from (iv) that

$$\mathbb{P}^{m'} \{T_{\tau'} = t_{4j+1} \text{ or } T_{\tau'} = t_{4j+2}\} \leq 2^{-j},$$

if $\tau' := \inf\{t \geq 0 : B_t^m \notin -U_j\}$ denotes the first exit time of the Brownian motion from the set $-U_j := \{(x,y,t) : a_0 e^{-t_{4j}} < x < j, t_{4j+1} \leq t_{4j+2}\}$, where $B^{m'}$ is the Brownian motion on M starting in $m' = (x,y,t)$ with $2a_0 e^{-t_{4j}} < x < j$ and $t = (t_{4j+1} + t_{4j+2})/2$.

Proof. The existence of a function $h : \mathbb{R}^2 \to \mathbb{R}$ with the properties above can be derived from [A1], Proposition 3.3, Remark 3.8 and Lemmata 4.1, 4.2, 5.2. □

We have the following theorem about the asymptotic behaviour of B_s for $s \to \zeta$:

Theorem 4.12. *Let* (M, γ) *be the Riemannian manifold with Riemannian metric* γ *obtained from Lemma 4.10 above and let* $(B_s)_{s<\zeta} = (X_s, Y_s, T_s)_{s<\zeta}$ *be the Brownian motion on* M *with lifetime* ζ*. Then the following statements hold true:*

i) $\lim_{s\to\zeta} T_s = \infty$ *almost surely.*

ii) $\lim_{s\to\zeta} X_s = 0$ *almost surely.*

iii) $\lim_{s\to\zeta} Y_s$ *almost surely exists and is almost surely finite.*
Furthermore $\lim_{s\to\zeta} Y_s$ *is almost surely a non-trivial shift-invariant random variable. Hence for every bounded continuous function* $f : \mathbb{R} \to \mathbb{R}$ *there is a non-trivial bounded harmonic function* $u : M \to \mathbb{R}$, $m \mapsto u(m)$, *given as*

$$u(m) := \mathbb{E}^m \left[f \circ \left(\lim_{s\to\zeta} Y_s \right) \right].$$

In particular, M *possesses non-trivial bounded harmonic functions.*

iv) Almost surely $\zeta = \infty$.

Proof. i) and iv) follow exactly as in [A1], Lemma 5.1.

For the proof that $\lim_{s\to\zeta} Y_s$ almost surely exists and is finite, observe that for each $y_0 \in \mathbb{R}$ the functions $u_{y_0} : M \to \mathbb{R}$, $(x,y,t) \mapsto u_{y_0}(x,y,t)$, given as

$$u_{y_0}(x,y,t) := \begin{cases} 1 & \text{for } y \leq y_0 \quad (y \geq y_0 \text{ resp.}), \\ 1 - \frac{2}{\pi} \arctan\left(\frac{1}{2}|y - y_0| e^{t/2}\right) & \text{for } y > y_0 \quad (y < y_0 \text{ resp.}) \end{cases}$$

are superharmonic on $\{t \geq 0\}$ with respect to

$$\Delta_M = \frac{\partial^2}{\partial t^2} + \frac{1}{e^{2t}}\frac{\partial^2}{\partial x^2} + \frac{1}{h^2}\frac{\partial^2}{\partial y^2} + \left(1 + \frac{h_t'}{h}\right)\frac{\partial}{\partial t} + \frac{h_x'}{e^{2t}h}\frac{\partial}{\partial x},$$

where h is as in Lemma 4.10. Clearly h_x' does not have influence on the sign of $\Delta_M u_{y_0}$. Using this assertion, iii) follows exactly as in [A1], Lemma 5.1 and Theorem 4.4.

To prove that $\lim_{s\to\zeta} X_s$ almost surely exists and is finite, we recall (following [A1], Lemma 5.5) that a function $u : M \to \mathbb{R}$, $(x, y, t) \mapsto u(x, t)$, is Δ_M-superharmonic on $\{t \geq 0\}$ if u is convex in x, decreasing in the variable t, decreasing in x as long as $x \leq 0$, increasing in x for $x > 0$ and superharmonic with respect to

$$L := \frac{\partial^2}{\partial t^2} + \frac{1}{e^t}\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}.$$

For every positive $a \in \mathbb{R}$ the functions $u_a : M \to \mathbb{R}$, $(x, y, t) \mapsto u_a(x, t)$, with

$$u_a(x,t) := \begin{cases} 1 & \text{for } |x| \geq \frac{\pi}{2a}, \\ 1 - \frac{2}{\pi}\arctan\left(\frac{\cos(a\cdot x/2)}{\sinh\left(ae^{-t/2}\right)}\right) & \text{for } |x| < \frac{\pi}{2a} \end{cases}$$

fulfill the requirements above and serve as Δ_M-superharmonic functions on the absorbing region $\{t \geq 0\}$ of M. Using these superharmonic functions and the already proven fact that $\lim_{s\to\zeta} T_s = \infty$, it follows that $\lim_{s\to\zeta} X_s$ almost surely exists.
We can now use the supermartingale inequality (see also Section 3, Remark after Lemma 3.12) to obtain for every $m \in M$ and every $a > 0$:

$$u_a(m) \geq \mathbb{E}^m\left(\lim_{s\to\zeta} u_a(B_s)\right) \geq \mathbb{P}^m\left\{\lim_{s\to\zeta} X_s = -\infty\right\} + \mathbb{P}^m\left\{\lim_{s\to\zeta} X_s = +\infty\right\}.$$

From that it follows that $\lim_{s\to\infty} X_s$ is almost surely finite.

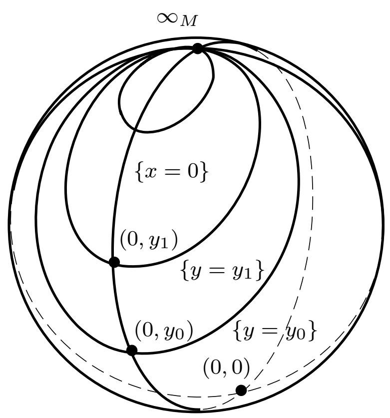
The fact that $\lim_{s\to\zeta} X_s = 0$ almost surely follows again from the Strong Markov Property of the Brownian motion together with a Borel-Cantelli argument; we hereby make use of Property (iv) of Lemma 4.10 and the already proven fact that $\lim_{s\to\infty} X_s$ is almost surely finite. Hence preferably the Brownian paths eventually exit the sets U_j at the upper boundary $x = -a_0 e^{-t_{4j}}$ and the sets $-U_j$ at the lower boundary $x = a_0 e^{-t_{4j}}$. The choice of the sequence $(t_i)_{i\in N}$ with $t_i \nearrow \infty$ yields that $\left|a_0 e^{-t_{4j}}\right| \to 0$. From this it follows that $\lim_{s\to\zeta} X_s = 0$ almost surely. ☐

Remark 4.13. We have just constructed an example of a Riemannian manifold (M, γ) with the property that the Brownian motion B almost surely exits from M along the hypersurface $\{x = 0\} \subset M$, whereas the y-component still possesses a non-trivial limit. In terms of the sphere at infinity this has the following meaning: On $S_\infty(M)$ we can observe that the projection of B onto $S_\infty(M)$ approaches the circle

$$\{x = 0\} := \{\zeta_{(x,y)} \in S_\infty(M) : x = 0\}$$

determined by the coordinate $x = 0$, but one can distinguish between Brownian paths by looking to which point of the circle the Brownian path converges.

In the picture below there is the circle $\{x = 0\}$ together with some possible limit points $\zeta_{(0,y_0)} := (0, y_0)$, $\zeta_{(0,y_1)} := (0, y_1)$ for Brownian paths, determined by the intersections of the "circles" where y is constant with $x = 0$. Note that in this picture $0 < y_0 < y_1$:



$S_\infty(M)$ with the circle $\{x = 0\}$

We are now going to present the changes that have to be made in the definition of the Riemannian manifold above to obtain an example of a Riemannian manifold such that the Brownian motion B_s does not converge for $s \to \infty$ whereas there exist non-trivial bounded harmonic functions.

Looking at the example above, we managed to force the Brownian paths to exit from the manifold M at the circle $\{x = 0\}$ of the sphere at infinity $S_\infty(M)$, whereas there is still "freedom" to choose at which point $\zeta_{(0,y)} \in \{x = 0\}$ the Brownian path will exit from M. This degree of freedom – i.e. the non-triviality of the random variable $Y_\infty := \lim_{s\to\zeta} Y_s$ – provides non-trivial bounded harmonic functions for the Riemannian manifold M.

It is therefore obvious that in order to guarantee the existence of non-trivial bounded harmonic functions on the Riemannian manifold M we should preserve the behaviour of the y-component of the Brownian motion B on M. In particular, this can be achieved in changing the metric function h as long as we do not make it dependent of the variable y. The idea is now to change the metric function h by shifting the sets U_j with the help of a dense sequence $(a_j)_{j\in\mathbb{N}}$ in $\mathbb{R}$ such that the x-component of the Brownian motion is eventually close to the a_j and so obviously cannot converge.
When considering the projections of the Brownian paths on the sphere at infinity $S_\infty(M)$ this means that we want to make the Brownian path "oscillate" between the different circles $\{x = a_j\} \subset S_\infty(M)$ for $s \to \zeta$, whereas the path is eventually close to the circle $\{y = y_0\}$, when $y_0 = \lim_{s\to\zeta} Y_s(\omega)$.

Remark 4.14. For the proof of Theorem B, Ancona uses a dense sequence $(a_j, b_j)_{j\in\mathbb{N}}$ in $\mathbb{R}^2$ and modifies the metric function h such that for j odd he makes h dependent of the variable y by just replacing x with y.
Doing this with the function h of Lemma 4.10, it follows that almost surely $\lim_{s\to\zeta} X_s = \lim_{s\to\zeta} Y_s = 0$, where still $\lim_{s\to\zeta} T_s = \infty$ holds for the component T of the Brownian

motion B ([A1], Lemma 5.5). This means that all Brownian paths exit from M almost surely at the single point $(0,0) := \zeta_{(0,0)} \in S_\infty(M)$, see the picture above. With the help of the dense sequence $(a_j, b_j)_{j\in\mathbb{N}}$ one can now make the Brownian path oscillating between the different circles $\{x = a_j\} \subset S_\infty(M)$ and at the same time between the circles $\{y = b_j\} \subset S_\infty(M)$. This proves that every point of $S_\infty(M)$ is a cluster point for the Brownian motion when $s \to \zeta$.

Fix now a dense sequence $(a_j)_{j\in\mathbb{N}}$ in $\mathbb{R}$. Let $(t_i)_{i\in\mathbb{N}}$ and $h : \mathbb{R}^2 \to \mathbb{R}_+$ as in Lemma 4.10. We then define the function $\widetilde{h} : \mathbb{R}^2 \to \mathbb{R}_+$ as follows:

$$\widetilde{h}(x,t) := \begin{cases} h(x,t) & \text{for} \quad t_{4j+3} + \frac{1}{2} \le t \le t_{4(j+1)} - \frac{1}{2},\ x \in \mathbb{R}, \\ h(x - a_j, t) & \text{for} \quad t_{4j} - \frac{1}{2} < t < t_{4j+3} + \frac{1}{2},\ x \in \mathbb{R}. \end{cases} \tag{4.10}$$

As $h(x,t) = e^t$ for $t \le t_0$ and $t_{4j+3} \le t \le t_{4(j+1)}$ this definition leads to a well defined smooth function $\widetilde{h}$ such that the obtained Riemannian manifold $(M, \widetilde{\gamma})$ still has the same description of the sphere at infinity $S_\infty(M)$ as above and still has sectional curvatures bounded from above by $-1/4$. In order to verify that in fact $\mathrm{Sect}_M \le -1/4$, it may be necessary to replace the constant $b_j := \sqrt{e^{j+1} + e^{2j+2}}$ in the definition of the function β_j (see before Lemma 4.1) with a new constant $\widetilde{b}_j$ depending on a_j. However, this does not influence at all the properties of the function $\widetilde{h}$.

Clearly the following lemma holds:

Lemma 4.15. *Let $(M, \widetilde{\gamma})$ be the Riemannian manifold obtained when using the function $\widetilde{h} : \mathbb{R}^2 \to \mathbb{R}_+$ as above to define the Riemannian metric $\widetilde{\gamma}$. Let*

$$\begin{aligned} U_j &:= \{(x,y,t) : t_{4j+1} < t < t_{4j+2},\ -j + a_j < x < -2a_0 e^{-4t_j} + a_j\} \quad \textit{and} \\ -U_j &:= \{(x,y,t) : t_{4j+1} < t < t_{4j+2},\ 2a_0 e^{-4t_j} + a_j < x < j + a_j\}. \end{aligned}$$

We then have:

$$\begin{aligned} \mathbb{P}^m\{T_\tau = t_{4j+1} \text{ or } T_\tau = t_{4j+2}\} &\le 2^{-j} \quad \textit{as well as} \\ \mathbb{P}^{m'}\{T_{\tau'} = t_{4j+1} \text{ or } T_{\tau'} = t_{4j+2}\} &\le 2^{-j}, \end{aligned} \tag{4.11}$$

where $\tau := \inf\{t \ge 0 : B_t^m \notin U_j\}$ and $\tau' := \inf\{t \ge 0 : B^{m'} \notin -U_j\}$ denote the exit times of the Brownian motion B^m from the set U_j, when started in the point $m = (x,y,t)$ with $t = (t_{4j+1} + t_{4j+2})/2$ and $-j + a_j < x < -2a_0 e^{-4t_j} + a_j$, as well as of the Brownian motion $B^{m'}$ started in the point $m' = (x,y,t)$ with $t = (t_{4j+1} + t_{4j+2})/2$ and $2a_0 e^{-4t_j} + a_j < x < j + a_j$ from the set $-U_j$.

We can now easily derive the following theorem:

Theorem 4.16. *Let $(M, \widetilde{\gamma})$ be the Riemannian manifold defined above. Let $B = (X, Y, T)$ be the Brownian motion on M with lifetime ζ. Then the following statements hold:*

i) Almost surely $\lim_{s\to\zeta} T_s = \infty$.

ii) Almost surely $\lim_{s\to\zeta} Y_s$ *exists and is finite.*
Furthermore $Y_\zeta := \lim_{s\to\zeta} Y_s$ *is an almost surely non-trivial shift-invariant random variable. Hence for every bounded continuous function* $f : \mathbb{R} \to \mathbb{R}$ *there is a non-trivial bounded harmonic function* $u : M \to \mathbb{R}$, $m \mapsto u(m)$, *with*

$$u(m) := \mathbb{E}^m \left[f \circ \left(\lim_{s\to\zeta} Y_s \right) \right].$$

iii) Almost surely $\zeta = \infty$.

iv) With probability 1*, every real number is a cluster point for the component* X *of the Brownian motion* B *as* $s \to \zeta$*. In particular, the Brownian motion does not converge on the sphere at infinity when* $s \to \infty$*.*

Proof. The only assertion to show is iv). Using Lemma 4.15 above, it follows from the Strong Markov Property that with probability $\geq 1 - 2^{-j}$ the Brownian motion started in $m \in M$ hits the set

$$\left\{ (x, y, t) : \, |x - a_j| \leq 2a_0 e^{-t_{4j}}, t_{4j+1} < t < t_{4j+2} \right\}.$$

For a point $a \in \mathbb{R}$ choose a subsequence $(a_{j_k})_{k\in\mathbb{N}}$ of $(a_j)_{j\in\mathbb{N}}$ with $a_{j_k} \to a$. Then from $t_i \nearrow \infty$ and Borel-Cantelli it follows that with probability 1 the component X_s of the Brownian motion is eventually arbitrarily close to a. This proves iv). □

We are going to finish this chapter with some ideas how the Theorems 4.4, 4.7 and 4.16 presented in the foregoing chapters can be used to obtain a large variety of Riemannian manifolds of dimension $d \geq 3$, where Brownian motion shows different asymptotic behaviour.

4.6. Some Concluding Remarks

In the foregoing chapters we presented two different possibilities to influence the asymptotic behaviour of the components of the Brownian motion: On the one hand there is the possibility to enforce certain limit values for the components; for example we showed in Theorem 4.12 how to achieve $\lim_{s\to\zeta} X_s = 0$. It is not difficult to see how to obtain $\lim_{s\to\zeta} X_s = a$ for an arbitrary $a \in \mathbb{R}$. Furthermore it is not difficult to achieve "oscillating" components, as described in Theorem 4.16. On the other hand we can obtain non-trivial shift-invariant random variables as limits of component processes as long as we assure that the metric function h is independent of the respective variable, see Theorem 4.7.

Putting all this together we can prove the following theorem:

Theorem 4.17. *Let* $n \geq 2$ *and* $n_1, n_2, n_3 \in \mathbb{N}$ *with* $n_1 + n_2 + n_3 = n$*. Then for each* $(a_1, \ldots, a_{n_1}) \in (\mathbb{R} \cup \{-\infty, \infty\})^{n_1}$ *there is a complete simply connected Riemannian manifold* $(M; \gamma)$*, where*

$$M := \{(x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}, z_1, \ldots, z_{n_3}, t) : x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}, z_1, \ldots, z_{n_3} \in \mathbb{R}\},$$

with sectional curvatures bounded from above by $-1/4$ *such that for the Brownian motion*

$$(B_s)_{s<\zeta} = ((X_{is})_{i\le n_1}, (Y_{is})_{i\le n_2}, (Z_{is})_{i\le n_3}, T_s)_{s<\zeta}$$

with lifetime ζ *on* M *holds:*

i) Almost surely $\lim_{s\to\zeta} T_s = \infty$.

ii) Almost surely $\lim_{s\to\zeta} X_{is} = a_i$ *for every* $i = 1, \ldots, n_1$.

iii) With probability 1*, every point in* $\mathbb{R}^{n_2}$ *is a cluster point for* $(Y_{1s}, \ldots, Y_{n_2 s})$ *as* $s \to \zeta$.

iv) Almost surely $\lim_{s\to\zeta} Z_{is}$ *exists for every* $i = 1, \ldots, n_3$ *and is finite. Furthermore* $\lim_{s\to\zeta} Z_{is}$ *is an almost surely non-trivial shift-invariant random variable. Hence for every bounded continuous function* $f : \mathbb{R}^{n_3} \to \mathbb{R}$ *there is a non-trivial bounded harmonic function* $u : M \to \mathbb{R}$, $m \mapsto u(m)$*, with*

$$u(m) := \mathbb{E}^m\left[f \circ \left(\lim_{s\to\zeta}(Z_{1s}, \ldots, Z_{n_3 s})\right)\right].$$

In particular, M *possesses non-trivial bounded harmonic functions if* $n_3 \neq 0$.

v) Almost surely $\zeta = \infty$.

Proof. From the foregoing chapters and the considerations above, it is clear that it suffices to find a suitable Riemannian metric γ on M, such that all the required properties can be fulfilled.

To simplify notations we assume without loss of generality that $n_1 = n_2 = n_3 = 1$, i.e. we give the explicit metric for a Riemannian manifold

$$M = \{(x, y, z, t) : x, y, z, t \in \mathbb{R}\}$$

and an arbitrary point $a_1 \in \mathbb{R} \cup \{-\infty, \infty\}$. We denote by $h : \mathbb{R}^2 \to \mathbb{R}$ the metric function constructed in Lemma 4.10 if $a_1 \notin \{-\infty, \infty\}$ or constructed in Lemma 4.1 if $|a_1| = \infty$. We furthermore fix a dense sequence $(b_i)_{i\in\mathbb{N}}$ in $\mathbb{R}$ and denote by $\widetilde{h} : \mathbb{R}^2 \to \mathbb{R}$ the metric function with the properties of Lemma 4.15.

Let $(t_i)_{i\in\mathbb{N}}$ the sequence of real numbers mentioned in the construction of the metric function h. We then define the Riemannian metric γ on M in the global coordinates x, y, z, t:

On $t \le t_0$:

$$ds_\gamma^2 = dt^2 + e^{2t}\,dx^2 + e^{2t}\,dy^2 + e^{2t}\,dz^2,$$

on $t_{4(2j)} < t < t_{4(2j+1)}$, for $j \in \mathbb{N}$:

$$ds_\gamma^2 = dt^2 + e^{2t}\,dx^2 + h_{a_1}^2(x, t)\,dy^2 + e^{2t}\,dz^2,$$

and on $t_{4(2j+1)} < t < t_{4(2j+2)}$, for $j \in \mathbb{N}$:

$$ds_\gamma^2 = dt^2 + \widetilde{h}^2(y,t)\,dx^2 + e^{2t}\,dy^2 + e^{2t}\,dz^2.$$

Herein the function $h_{a_1}(x,t)$ is defined as follows:
If $|a_1| = \infty$, then $h_{a_1}(x,t) := h(\text{sgn}(a_1)x, t)$ with the function h as in Lemma 4.1.
If $|a_1| < \infty$, then $h_{a_1}(x,t) := h(x - a_1, t)$ with the function h as in Lemma 4.10.

As on the stripes $\mathbb{R} \times [t_{4j+3}, t_{4(j+1)}]$ the functions h, $\widetilde{h}$ respectively, equals e^t the metric defined above is a Riemannian metric, and the manifold (M,γ) has (obviously) sectional curvatures bounded from above by $-1/4$.

It is now an immediate result that for the components X, Y, Z and T of the Brownian motion B with lifetime ζ holds:

i) $\lim_{s\to\zeta} T_s = \infty$ almost surely.

ii) $\lim_{s\to\zeta} X_s = a_1$ almost surely.

iii) With probability 1, every point of $\mathbb{R}$ is a cluster point for Y_s as $s \to \zeta$.

iv) $\lim_{s\to\zeta} Z_s$ almost surely exists and is finite. $\lim_{s\to\zeta} Z_s$ is an almost surely non-trivial shift-invariant random variable.

v) Almost surely $\zeta = \infty$.

This proves the theorem in the special case $n_1 = n_2 = n_3 = 1$. From the definition of γ above, it is clear how to extend the definition of γ in adding further cases for the variable t and defining the metric γ on the corresponding stripe with the help of suitable functions h_{a_i}, $\widetilde{h}$ respectively, as in Lemma 4.1, 4.10 or 4.15. $\square$

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