Change Without Time
Relationalism and Field Quantization

Dissertation
zur Erlangung des Doktorgrades
der Naturwissenschaften (Dr. rer. nat.)
der naturwissenschaftlichen Fakultät II – Physik
der Universität Regensburg

vorgelegt von
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aus Schwandorf

2004

Promotionsgesuch eingereicht am: 19.04.2004

Die Arbeit wurde angeleitet von Prof. Dr. Gustav M. Obermair.

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Tag der mündlichen Prüfung: 23.07.2004

Tempus item per se non est [...]

Titus Lucretius Carus [Car, 459 ff]

Preface

This thesis studies the foundations of time in physics. Its origin lies in the urge to understand the quantum measurement problem. While the emergence of classicality can be well described within algebraic quantum mechanics of infinite systems, this can be achieved only in infinite time. This led me to a study of quantum dynamics of infinite systems, which turned out to be far less unique than in the case of finitely many degrees of freedom. In deciding on the correct time evolution the question appears how time – or rather duration – is being measured. Traditional quantum mechanics lacks a time observable, and for closed systems in an energy eigenstate duration is indeed meaningless. A similar phenomenon shows up in general relativity, where absolute duration (as well as spatial distance) becomes meaningless due to diffeomorphism invariance. However, by relating different parts of a closed system through simultaneity (in quantum mechanics as well as in general relativity), an internal notion of time becomes meaningful.

This similarity between quantum mechanics and general relativity was recognized in the context of quantum gravity by Carlo Rovelli, who proposed a relational concept of quantum time in 1990. He showed in a two-oscillator model that, by using an energy constraint instead of time evolution, the algebra of constants of motion can be quantized and used to relate so-called[1] partial observables. The main problem with the relational concept of time turns out to be the lack of a fixed evolution in the quantum domain, where arbitrary superpositions are allowed for the total system, leading to a possible superposition of different instants of the internal time. At this point a question naturally arises, which to the best of our knowledge has not been asked so far: If the system becomes infinite, can we reconstruct a classical notion of time from algebraic quantum mechanics; and what do inequivalent representations of the algebra of observables mean for time? – When trying to find rules, which guarantee agreement of a general notion of time with the empirical time of classical observers, one is lead to a number of further questions, whose very meaning is not easily clarified: What is time? Why is time a totally ordered set, even a one-dimensional differentiable manifold? Why does time pass by? What determines the direction of time? What is a clock? – During the last decade foundational physics has seen much progress on the subject of time; the central issues are however still unsolved – there is still pretty much to do.

This thesis analyzes the foundation of the relational concept of time in view of various meanings and problems of 'time', and it asks whether a 'relational quantization' of the free quantum field is possible; see the overview on page vi.

[1]The notion of partial observable was introduced only recently [Rov01b], thereby putting the conceptual foundations of relationalism on a solid basis.

It could not accomplish its initial goal, a solution of the age-old quantum measurement problem. Nevertheless we feel confident that a solution of this problem will be possible with a new, physical understanding of time. This will require to free oneself from preconceptions, to doubt the foundations instead of believing in them, an act of pleasant emancipation, which allows to discover the real world behind the shadows we see and too often take for real.

Würzburg, spring 2004

Contents

Overview

Chapter 1

After a brief historical introduction to the phenomenon of time we present mathematical structures often used in talking about time and we discuss the concepts of time as used in Newtonian mechanics, special and general relativity and quantum mechanics. Thereafter we focus on the relational concept of time and reduce time to its essence: a simultaneity relation.

Chapter 2

An introduction to three of the main problems of time is given: 1. The arrow of time in classical and quantum physics from the absolute and relational point of view. 2. The measurement of time with quantum clocks. 3. The meaning of time in quantum gravity (without a fixed background metric).

Chapter 3

We discuss Rovelli's model of two oscillators, which shows the very meaning of the relational concept of time at the quantum level: Via coherence one oscillator acts as a clock for the other one. In the first section we generalize this model on the classical level to the free massless scalar field in one dimension. The second section is concerned with quantization.

Chapter 4

The last chapter briefly discusses the classicality of time and gives an outlook on a general dynamics compatible with quantum measurement collapse.

On arrangement and notation

Each chapter is divided into sections, sections are divided into subsections and, possibly, subsubsections; numbers include the chapter, e.g. /1.2.3.4/ refers to subsubsection 4 of subsection 3 of section 2 of chapter 1, and /1/ refers to the first chapter. Equations are consecutively numbered within each section and contain the number of the chapter and section, e.g. (1.2.3) denotes equation 3 in section 2 of chapter 1.

References are cited in square brackets, e.g. [Rov90]. In cases where the literature is too much to be cited completely, we have tried to include in the references at least recent reviews or original articles.

By "matter" we mean all forms of energy, not only fermionic matter. If not otherwise stated, by finite (infinite) quantum systems we mean quantum systems with a finite (infinite) number of degrees of freedom. We use "quantum mechanics" and "quantum theory" synonymously. With "state" of a quantum system in the traditional Hilbert space formalism we usually mean a state vector (and not a ray), or a density matrix. In /2.1/ "state" does mean a configuration, not a point in phase space. We use the terms "two-oscillator system" and "two oscillators" instead of and synonymously with "double pendulum".

In the context of general relativity, as is customary, we use Einstein's summation convention and the range of greek indices is $0, 1, 2, 3$, while roman indices take the values $1, 2, 3$ corresponding to 3-space.

Throughout the text we use the following symbols and abbreviations:

$\mathbf{T}$	instant
$\mathcal{T}$	set of instants
T	time operator
t	time parameter, or time coordinate
x^0	time coordinate in relativity
$\mathbb{1}_S$	identity mapping on S
$\mathcal{V}$	Poisson algebra
$\mathcal{C}$	Poisson subalgebra of constants of motion
$\mathrm{span}(A)$	vector space spanned by A
$\mathrm{gen}(A)$	algebra generated by A
$\mathrm{Ran}(A)$	range of A
$\mathcal{S}(\mathbb{R}, \mathbb{R})$	real-valued Schwartz functions with domain $\mathbb{R}$
$\mathrm{SYM}(\mathcal{H})$	symmetric operators on the Hilbert space $\mathcal{H}$
$\mathcal{V}$,$\mathcal{V}_2$,$\mathcal{V}_\infty$	Poisson algebras
$\mathcal{C}$,$\mathcal{C}_2$,$\mathcal{C}_\infty$	algebras of constants of motion
$\mathcal{C}_{\infty,g}$	set of generators of $\mathcal{C}_\infty$
a_n^+, a_n	creation and annihilation operators
$a_{ij}^\pm, b_{ij}^\pm$	constants of motion of the free field involving triples of field modes
$\theta(x)$	Heaviside step function ($\theta(x) = 1$, if $x \geq 0$ and $\theta(x) = 0$ otherwise)
$i_X Y$	inner product of the tensors X and Y
$\mathrm{gen}_{\mathrm{Lie}}(A)$	Lie algebra generated by the set A
w.r.t.	with respect to

CHAPTER 1

Concepts of time

In this chapter we study the nature of time in fundamental physical theories. We put emphasis on its relational character.

The prejudice – which has by no means died out in the meantime – consists in the faith that facts in themselves can and should yield scientific knowledge without a free conceptual construction. Such a misconception is possible only because one does not easily become aware of the free choice of such concepts, which, through verification and long usage, appear to be immediately connected with the empirical material.

ALBERT EINSTEIN[1]

Time has for centuries been the subject of scientific investigations and speculations. Only recently has a considerable number of physicists become interested in the fundamental nature of time, when trying to unify the two great theories of the last century, quantum mechanics and general relativity. Since Newtonian mechanics time was steadily "flowing" and was defined up to an overall choice of a unit of time and an origin. According to Newton, *"Absolute, true and mathematical time, of itself, and from its own nature, flows equably, without relation to anything external* [...]*"* [New69]. While special relativity rendered time observer dependent, with the advent of general relativity it became clear that not only time but even spacetime is only defined up to a diffeomorphic change of coordinates (general covariance).

Motivated by a speculation of Carlo Rovelli that time has to be understood first of all in quantum field theory, this work focuses on the relational nature of quantum time. Quantum dynamics predicts the change of expectations for measurement

[1] cited in [Sch82, p. 48]

results during the course of time and therefore involves the measurement of any quantity *as well as time*. This relies on the tacit assumption that the quantum system can act as a carrier of information on time between the (classical) preparation and measurement devices, or, to put it differently, the quantum system always shares the same time with the measurement apparatus, even without any intentional measurement of time being performed. This means, that time is assumed to be a classical observable (i.e. an observable which commutes with all other observables), and moreover always takes the same values for interacting systems. This is not at all clear for quantum systems, where superpositions of states at different times are ruled out by a postulate, not on a physical basis. - What guarantees that a measurement is performed only at a single instant of time? A finite measurement apparatus, being a quantum system, needs an external device to measure time, and so on. Similar to the measurement of any quantum observable we need a von Neumann's chain regarding the generation of instantaneity. We hold the opinion that the possibility of instantaneity in quantum theory has to be explained.

We investigate the intriguing idea that for quantum systems with finitely many degrees of freedom there is no classical time at all (*fundamental timelessness*); we shall reconstruct a classical notion of time on the other hand for a quantum field with its infinitely many degrees of freedom.

If time is no fundamental observable, why should one try to eliminate it from the formalism? (After all, time has proved to be an extremely fruitful concept.) We recall Einstein's thought experiment where it is not observable locally whether the frame is accelerated or a gravitational field is present. This famous equivalence principle stating the unobservability of a quantity[2] was the corner stone of general relativity. Analogously timelessness might give rise to a new fundamental theory, and in fact is believed to play a central role in quantum gravity.

The current chapter introduces the main concepts of time, with special emphasis on the relational concept.

Time serves as a means to structure observations of our own and others in a consistent way and allows us to coordinatize our actions. The notion of a state signifies observations at a specific instant of time. Observations at different instants of time are connected through dynamical laws, causing the predictive power of science.

A simple observation of a dynamical law is that of simultaneous recurrence of events, and in fact was used already five thousand years ago in ancient Egypt, where the position of stars on the night sky provided a seemingly universal, eternal clock time. The problematization of the nature of time began two and a half thousand years ago[3] in presocratic philosophy. Heraclitus compares time

[2]The unobservability of a distinguished inertial frame in special relativity is of the same kind, and in fact – according to his autobiographical notes [ebPAS79] – Einstein was motivated by an analogy with 'impotence' principles of phenomenological thermodynamics to construct perpetual motion machines.

[3]At about the same time in Hinduistic culture one of the Upanisads mentions time. We quote a translation from the Sanskrit original of Maitri Upanisad (VI. 14):

with a river whose water is always changing, while Parmenides holds that not change is real, but only permanence.

Zenon, follower of Parmenides, formulated four paradoxes, the best known of which is the second one, where Achilles cannot win a footrace against a turtle; these paradoxes later entered into the concept of time as a real line. According to Democrit atoms and empty space are permanent, while the structures built out of atoms can change with time. The Pythagoreans, on the contrary, hold that not substance is eternal, but ideas and mathematical laws. For Plato timelessness is ideal while change is not. Aristotle dissents: matter has the potentiality of having a certain form, and motion or change happens when potentiality becomes actuality. Three centuries later Lucretius, follower of Epicurus, writes: "tempus item per se non est, sed rebus ab ipsis consequitur sensus, ..." [Car, *459*]; we quote a translation given by Rovelli [Rov91c, fn. 18]:

> "Time does not exist by itself. Time gets meaning from the objects: from the fact that events are in the past, or that they are here now, or they will follow in the future. It is not possible that anybody may measure time by itself; it may only be measured by looking at the motion of the objects or at their peaceful quiet."

Another five centuries later Augustinus moves backward and contends, contrary to Aristotle, that time is a prerequisite for motion and comes from spirit.

This short journey through the early human history of time (for details see [Mai02, F9̈9,oMa]) shows that there were many concepts of time already before the rise of modern sciences. Contemporary thought has dealt with time a lot again, and we refer the reader the the excessive literature, of which the book [Mac91] collects the most important items from different sciences up to 1991.

Here we are concerned with time in fundamental physical theories. In dealing with such a fundamental notion like time we first of all have to notice that our thinking is very deeply biased towards temporal concepts [Rue82]. Every sentence of our languages encodes time in tense. Everyday experiences incessantly affirm a temporal logic, a clear distinction between "before" and "after". Being performed with various kinds of clocks[4], time measurements are the most popular

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> Because of the subtlety, this is the measure: time is of the thing to be measured. Without a measure, there is no getting hold of the thing to be measured. Moreover, because of its separateness, the thing to be measured becomes the measure for the purpose of making itself known. Someone has said: the one who worships time as Brahman moves on through all the divisions of time that there are, and time moves very far away from him. Someone has said:
> Because of time, beings move on;
> Because of time, they grow up;
> In time they reach their end;
> Time, though unshaped, possesses shapes.

[4]For s short history of timekeeping cf. e.g. [oST95]. Clocks will be discussed in /2.2.2/ and /2.2.4/.

measurements of a physical quantity and make it very difficult for us to imagine a world in which time is not a fundamental concept.

The opinion towards the existence of time at a fundamental level has recently become quite controversial among physicists. Popular physics books published in the last decade proclaimed “Timeless reality” [Ste00], the “End of time” [Bar99a] or “The View from Nowhen” [Pri96]. Julian B. Barbour describes his experience at an international workshop on time asymmetry in 1991 [Bar94]:

> During the workshop, I conducted a very informal straw-poll, putting the following question to each of the 42 participants:
> *Do you believe that time is a truly basic concept that must appear in the foundations of any theory of the world, or is it an effective concept that can be derived from more primitive notions in the same way that a notion of temperature can be recovered in statistical mechanics?*
> The results were as follows: 20 said there was no time at a fundamental level, 12 declared themselves to be undecided or wished to abstain, and 10 believed time did exist at the most basic level. However, among the 12 in the undecided/abstain column, 5 were sympathetic or inclined to the belief that time should not appear at the most basic level of theory.

During the last decades time has become a major topic of research in the (quantum) gravity context, see /2.3/ below, as well as in quantum theory, see /2.2.1/.

This section begins with a brief review of mathematical concepts of time /1.1/. We next locate these concepts in fundamental physical theories, see /1.2/, where time is shown to lose structure with increasing generality of the theories. From this discussion we are led to relationalism /1.3/ as fundamental concept of time that will be explored in the subsequent chapters.

1.1 Modelling time

Different theories use different models of time. In preparation for the next subsection we decompose the *standard notion of time*, the real numbers $\mathbb{R}$, into a hierarchy of substructures, cf. [Rov95, Kro85]:

(A) Let $\mathcal{T}$ be a set with the cardinality of the continuum, $|\mathcal{T}| = |\mathbb{R}|$.
(B) Let $\mathcal{T}$ be equipped with a topology $\mathfrak{T}$.
(C) Let $\mathcal{T}$ be a differentiable manifold with local charts $\varphi_{\mathbf{T}}$ at $\mathbf{T} \in \mathcal{T}$ such that the topology induced by open sets of $\mathbb{R}$ via $\varphi_{\mathbf{T}}^{-1}$ coincides with $\mathfrak{T}$.
(D) Let $\mathfrak{T}$ moreover be isomorphic to the topology of $\mathbb{R}$; this implies that a (global) chart $\varphi : \mathcal{T} \to \mathbb{R}$ (bijective and C^∞) exists.

(E) Let an Euclidean metric d on $\mathcal{T}$ be defined (fixing one global chart φ),

$$d(\mathbf{T}_1, \mathbf{T}_2) := |\varphi(\mathbf{T}_1) - \varphi(\mathbf{T}_2)| \quad (\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{T}).$$

(F) Let a linear order relation $\leq$ on $\mathcal{T}$ be defined by

$$\mathbf{T}_1 \leq \mathbf{T}_2 :\Leftrightarrow \varphi(\mathbf{T}_1) \leq \varphi(\mathbf{T}_2) \quad (\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{T}).$$

(G) Let $\mathcal{T}$ be equipped with the field structure inherited via φ^{-1} from $\mathbb{R}$, i.e. define addition in $\mathcal{T}$ by

$$\mathbf{T}_1 + \mathbf{T}_2 := \varphi^{-1}(\varphi(\mathbf{T}_1) + \varphi(\mathbf{T}_2)) \quad (\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{T}),$$

let the neutral element be $\varphi^{-1}(0)$, and similarly for multiplication:

$$\mathbf{T}_1 \cdot \mathbf{T}_2 := \varphi^{-1}(\varphi(\mathbf{T}_1) \cdot \varphi(\mathbf{T}_2)) \quad (\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{T}),$$

with the neutral element $\varphi^{-1}(1)$.

Conditions (A) to (G) imply that $\mathcal{T}$ is isomorphic to the real numbers $\mathbb{R}$ with Euclidean metric. We call $\mathbf{T} \in \mathcal{T}$ an *instant (of time)* or *moment of time, $\mathcal{T}$ the set of instants,* d the *duration (of time)* or *temporal distance* or *time lapse* and $\leq$ the *time order*. By just *time* we mean $\mathcal{T}$ together with a specification of some of the structures (A) to (G), or possibly others.

ad G: The multiplicative group structure among instants has no physical meaning; neither does addition of instants. What is used in the description of dynamical flows is the sum of durations: Let ϕ_t be a flow with "time t" on a set M, i.e. a mapping $\phi : \mathbb{R} \times M \to M$, $(t, x) \mapsto \phi_t(x)$ with the flow properties $\phi_0 = \mathbb{1}_M$ and $\phi_s \circ \phi_t = \phi_{t+s}$ $(s, t \in \mathbb{R})$. The meaning of s, t and $t + s$ is the following: Given any instant $\mathbf{T}_0 \in \mathcal{T}$ the expression ϕ_t means that the flow ϕ has to be evaluated at that instant $\mathbf{T}_t \in \mathcal{T}$ which is uniquely defined by $d^*(\mathbf{T}_0, \mathbf{T}_t) = t$, where we have introduced a signed duration by $d^*(\mathbf{T}, \mathbf{T}') := \pm d(\mathbf{T}, \mathbf{T}')$, with "+" applying in case $\mathbf{T} \leq \mathbf{T}'$ and "$-$" otherwise. I.e., given any fixed instant $\mathbf{T}_0$, metric and time order provide a bijection between instants and real numbers. The expression $t+s$ signifies that instant $\mathbf{T}_{t+s}$ which is uniquely defined by $d^*(\mathbf{T}_0, \mathbf{T}_{t+s}) = d^*(\mathbf{T}_0, \mathbf{T}_t) + d^*(\mathbf{T}_0, \mathbf{T}_s)$. The flow ϕ can thus be characterized in terms of instants, duration and time order, while the field structure on $\mathcal{T}$ has no physical meaning. Note that the mapping $\mathbf{T}_0 \mapsto \mathbf{T}_t$ $(\mathbf{T}_0 \in \mathcal{T}, t \in \mathbb{R})$ is a time translation; time translations build a group reflecting the additive group of reals[5]. (Similarly, the time that enters into Galilei boosts has the meaning of a duration.)

Note also that independence of the choice of $\mathbf{T}_0$ is implicitly contained in the flow properties. $\mathbf{T}_0$ can be understood as "now" and allows an observer to separate all other instants into two sets, called "past" ($\{\mathbf{T} \in \mathcal{T} : \mathbf{T} \leq \mathbf{T}_0, \mathbf{T} \neq \mathbf{T}_0\}$) and "future" ($\{\mathbf{T} \in \mathcal{T} : \mathbf{T}_0 \leq \mathbf{T}, \mathbf{T} \neq \mathbf{T}_0\}$). Two unsolved problems are connected with this: 1) Is it possible to distinguish the "past" and the "future" on a physical basis? - This is known as the problem of the arrow of time, see /2.1/ below. 2) Can the meaning of $\mathbf{T}_0$ as "now" or "present" be established on a physical basis? - Because of time translation invariance (sometimes also called homogeneity of

[5]The set of reals involved here is not $\mathcal{T}$, but the range of the metric.

time) it is hard to single out exactly one instant (called "now") in an objective way[6]. This might be possible however in a theory describing the subjectiveness of an observer on a physical basis, cf. the discussion in [Kro85, Epilogue]. A combination of both problems amounts to the problem of explaining the *flow of time*[7]: There is a change of the "now" ($\mathbf{T}_0 \mapsto \mathbf{T}'_0$) which causes future instants to become "nows" and past instants later on, "during the flow of time". Related questions are why the past is determined, cannot be influenced and can be remembered, while the future is not determined, can be influenced and not remembered. These are as yet open questions.

If we omit (G), we are left with an affine line.

ad F: Given any two instants $\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{T}$ we have either $\mathbf{T}_1 = \mathbf{T}_2$ ("$\mathbf{T}_1$ and $\mathbf{T}_2$ are *simultaneous*"), or $\mathbf{T}_1 < \mathbf{T}_2$ ($:\Leftrightarrow \mathbf{T}_1 \leq \mathbf{T}_2 \wedge \mathbf{T}_1 \neq \mathbf{T}_2$, "$\mathbf{T}_1$ lies in the conventional past of $\mathbf{T}_2$" or, equivalently, "$\mathbf{T}_2$ lies in the conventional future of $\mathbf{T}_1$"), or $\mathbf{T}_2 < \mathbf{T}_1$ ($:\Leftrightarrow \mathbf{T}_2 \leq \mathbf{T}_1 \wedge \mathbf{T}_1 \neq \mathbf{T}_2$). The question of whether (F) represents some physically observable relation is known as the debate on the arrow of time, see /2.1/. In relativity weaker versions of (F) may appear, see below.

ad E: The topology in (D) is induced by d. As will be seen later, the choice of a specific metric has limited physical meaning. E.g. in relativity duration is dependent on the observer's path in space-time (proper time), see /1.2.4/.

ad D: This allows one to speak of *time intervals*, i.e. sets of instants, which φ maps to intervals of $\mathbb{R}$. And it fixes the one-dimensional character of time and as well its linearity; other possibilities would include a cyclic time ($\mathfrak{T}$ isomorphic to the topology of S^1), a many fingered time ("curves with bifurcations"), a time with endpoints ($\mathfrak{T}$ isomorphic to a closed interval), or even more dimensional times. (See also [Wic03], where a topology generated by half-open intervals is used in order to render time itself asymmetric.)
Time structure (D) together with (F) is known as *topological time* [Mit89,Mit95].

ad C: This condition rules out e.g. a discrete time, which is being considered in quantum gravity /2.3/.

ad B: Some sense of neighborhood seems to be necessary for any notion of continuous time.

ad A: The cardinality of $\mathcal{T}$ could even be lower, cf. our treatment of simultaneity within the relational concept of time in /1.3.3/.

Generalizations of the standard notion of time can be obtained by successively dropping the more specialized items (G,F,E,...), but there is no need for this to be done in the given order. One could e.g. define a metric, linear order and preferred point directly on $\mathcal{T}$ without having a topological or differential structure. (The

[6]While the instant of the big bang in cosmological models, i.e. in certain solutions of Einstein's equations, is certainly a preferred point, it is not with regard to Einstein's equations.

[7]This is not to be confused with the concept of a dynamical flow.

chosen hierarchy will prove useful in the next section, since it is similar to the hierarchy of physical theories.)

For further discussion of mathematical time structures cf. also [Pim95].

1.2 The role of time in physical theories

Historically, time initially played the role of an absolute entity in Newtonian mechanics /1.2.1/ (in contrast to Leibniz' relational concept /1.3.1/). In /1.2.2/ we describe the homogeneous and presymplectic formalisms, which allow to formulate mechanics in a reparametrization invariant way. In Einstein's special theory of relativity time was relativized according to the motion of the observer /1.2.3/, and finally it was understood as being meaningful only in relation to material fields in general relativity /1.2.4/. We will briefly sketch these steps and subsequently ask for the role of time in nonrelativistic /1.2.5/ and relativistic quantum mechanics /1.2.6/. (See also /2.3/ for quantum gravity, where problems with time are most pressing.) For each theory we describe a scheme for time measurement (duration and synchronization), as well as the meaning of simultaneity and causality. (Thermodynamical time will be discussed in the next section in connection with the problem of the direction of time /2.1/.)

We remark that similarly to our itemisation, which largely corresponds to that of Rovelli [Rov95], Bialynicki-Birula [BB94] distinguishes four notions of time: Cosmological time, thermodynamical time, time of the laboratory clock, and the time that enters the definition of the state of the system.

The discussion will show that with increasing generality of the theories' domains of application the corresponding notions of time lose structure.

1.2.1 Newtonian mechanics

In Newtonian mechanics one usually starts from time structure (G) /1.1/ and calls it *absolute (Newtonian) or universal time*[8]. When describing the motion $\mathcal{T} \to \mathbb{R}$, $\mathbf{T} \mapsto \mathbf{x}_i(\mathbf{T})$ of particle i ($i = 1, 2, \cdots$) in 3-space the instants $\mathbf{T} \in \mathcal{T}$ can be identified with the values of a *specific* coordinate t on $\mathcal{T}$, the motion obtaining the form $\mathbf{x}_i(t)$, where t is called *time parameter*.[9] With this choice of t Newton's Second Law takes its usual simple form

$$m_i \frac{\mathrm{d}^2\mathbf{x}_i(t)}{\mathrm{d}t^2} = \mathbf{K}(\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots)\,, \tag{1.2.1}$$

$\mathbf{K}$ being the total force acting on a point particle of mass m_i at position $\mathbf{x}_i$. We can take $\mathbf{K}$ to not explicitly depend on t; otherwise we could incorporate the sources causing the explicit time dependence into the system and treat them as dynamical degrees of freedom.

[8]This refers to observer independence and to independence of position (spatial globality).

[9]Note that t is both coordinate and parameter.

The time parameter t is not unique: Upon linear reparametrization

$$t \mapsto \tilde{t} = \alpha t + \beta \qquad (\alpha, \beta \in \mathbb{R},\ \alpha \neq 0)$$

(1.2.1) remains unchanged. The invariance of (1.2.1) under the symmetry operation $\beta \mapsto \beta'$ is called *time translation invariance.* The arbitrariness of the sign of α for solutions of (1.2.1) is called *time reversal invariance* (cf. /2.1.1/).

Thus Newtonian mechanics effectively requires only time structure (E).

The choice of $|\alpha|$ corresponds to a choice of unit of time, which is required to be the same for all systems in order to make duration (time parameter difference, i.e. Euclidean metric on $\mathbb{R}$) comparable between noninteracting systems. How is duration measured? We describe two methods:

(a) Assume that there is an inertial frame of reference in which a body is not at rest and moving freely, i.e., at different instants it does occupy different places. We define duration in such a way that its velocity is constant, namely the duration between two (instantaneous) configurations of this "clock" system is defined – up to a scalar multiple, the unit of time – as the distance between the positions of the body in 3-space. In order to determine the duration between two configurations of any other system, we determine the duration between the corresponding simultaneous configurations of the clock system. (Here we assume hat both systems do not interact.) The determination of corresponding simultaneous positions is no problem, since Newtonian mechanics allows for action at a distance.
Since there is no universal velocity in Newtonian mechanics, we can and must choose an arbitrary unit of time. Since we can in principle attach our clock (or a copy of it) to *any* system, the definition of duration and unit of time derived from this single clock can be extended to that system and therefore to all of space.

(b) If we do not rely on the presence of this single clock (or any copy of it), we can determine the unit of time from the (static in nature) units of mass, distance and force via (1.2.1): The duration between two configurations of a given system can be measured by attaching to it (with no interaction) a clock system with just one particle of mass m exposed to a nonvanishing constant force, moving (for simplicity) in direction x; one measures the distance Δx covered by the particle during the two simultaneous instants in an inertial frame in which its initial velocity vanishes[10] ($\frac{\mathrm{d}x}{\mathrm{d}t} = 0$, being invariant under $t \mapsto \alpha t$, hence not requiring α to be known). From (1.2.1) duration follows as

$$\Delta t = \sqrt{\frac{2m\Delta x}{K}}\ .$$

[10]This can be achieved e.g. by measuring the distance between the particle and another, but free particle with the same initial velocity.

The synchronization of clocks works the same way, and in principle every position in $\mathbb{R}^3$ can be equipped with the same time[11]. The causal structure is given by the time order of two events, irrespective of their spatial positions.

What about general, not necessarily linear reparametrizations? If we use another time parameter $\tau = f(t)$ instead of t, where $f : \mathbb{R} \to \mathbb{R}$ and $F = f^{-1}$ are at least C^2, then (1.2.1) reads

$$m_i \frac{1}{F'(\tau)^3} \left(F'(\tau) \frac{\mathrm{d}^2 \mathbf{x}_i}{\mathrm{d}\tau^2} - F''(\tau) \frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}\tau} \right) = \mathbf{K} \,, \tag{1.2.2}$$

where we have assumed that $\mathbf{K}$ is not explicitly time-dependent and $F' \neq 0$. Since (1.2.2) looks more complicated than (1.2.1), Poincaré was led to a general postulate of simplicity of equations in physics. As pointed out in [Mit89, ch. 2] however, simplicity has no unique meaning, but instead one can require maximal explanatory power; this favors (1.2.1) against (1.2.2), because it does not contain an unexplained velocity-dependent term corresponding to an apparent force.

1.2.2 Homogeneous and presymplectic formalism

Instead of choosing a certain parametrization one can rephrase classical non-relativistic mechanics (in its Hamiltonian formulation) in a reparametrization invariant way, which is called *homogeneous formalism*, see [Mit95, §7.3 and §8.1] and [Mit89, appendix III]:

Let a Lagrangian system with n degrees of freedom, coordinates q_k, velocities $\dot{q}_k = \frac{\mathrm{d}q_k(t)}{\mathrm{d}t}$ and Lagrangian $L(q_k, \dot{q}_k, t)$ $(k = 1, 2, \ldots, n)$ be given, where t denotes Newton's absolute time. The action is $S\,[q_k(t)] = \int_{t_1}^{t_2} \mathrm{d}t\, L(q_k, \dot{q}_k, t)$. Assume that there are no constraints so that the relation between velocities and momenta, $p_k = \frac{\mathrm{d}L}{\mathrm{d}\dot{q}_k}$, can be inverted; then the Hamiltonian $H = \sum_{k=1}^{n} p_k \dot{q}_k - L$ does not depend on velocities. The Hamiltonian equations read $\dot{q}_k = \frac{\partial H}{\partial p_k}$, $\dot{p}_k = -\frac{\partial H}{\partial q_k}$. We call this the "original system" and all quantities "original" quantities.

We shall now construct from this system a new one, called *parametrized system*: Let an arbitrary diffeomorphic mapping $t \mapsto \tau$ be given. We define the $n+1$ coordinates of the parametrized system,

$$\bar{q}_0(\tau) := t(\tau) \,, \quad \bar{q}_k(\tau) := q_k(t(\tau)) \,,$$

and the new velocities

$$\bar{q}'_\nu(\tau) := \frac{\mathrm{d}\bar{q}_\nu(t(\tau))}{\mathrm{d}\tau} \quad (\nu = 0, 1, 2, \ldots, n) \,.$$

The new action is

$$\bar{S}\,[\bar{q}_\nu(\tau)] := \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, \bar{L}(\bar{q}_\nu, \bar{q}'_\nu) \,,$$

[11] The description of a classical mechanical system by a point in configuration space already makes use of simultaneity in 3-space, by *jointly* fixing values of coordinates of particles located at different positions.

where

$$\bar{L}(\bar{q}_\nu, \bar{q}'_\nu) := \frac{\mathrm{d}\bar{q}_0(\tau)}{\mathrm{d}\tau} L(q_k(t(\tau)), \dot{q}_k(t)|_{t=\bar{q}_0(\tau)}, t)$$
$$= \frac{\mathrm{d}\bar{q}_0(\tau)}{\mathrm{d}\tau} L(\bar{q}_k(\tau), \bar{q}'_k(\tau)\left(\bar{q}'_0(\tau)\right)^{-1}, \bar{q}_0(\tau))$$

is the new Lagrangian having no explicit τ-dependence. Obviously, with $\tau_i := \tau(t_i)$ $(i = 1, 2)$ we have $\bar{S}[\bar{q}_\nu(\tau)] = S[q_k(t)]$ and extremizing $\bar{S}$ w.r.t. $\bar{q}_\nu$ yields an extremum $q_k(t) := \bar{q}_k(\bar{q}_0^{-1}(t))$ of S. The Euler-Lagrange equations

$$\frac{\partial \bar{L}}{\partial \bar{q}_k} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \bar{L}}{\partial \bar{q}'_k}\right) = 0 \qquad (k = 1, 2, \ldots, n)$$

are equivalent to those of the original system.

Now for the Hamiltonian formulation: Let

$$p_k := \frac{\partial \bar{L}}{\partial \bar{q}'_k} = \frac{\partial L}{\partial \dot{q}_k} = p_k \qquad (k = 1, 2, \ldots, n),$$

$$\bar{p}_0 := \frac{\partial \bar{L}}{\partial \bar{q}'_0}$$

and

$$\bar{H}(\bar{q}_\nu, \bar{p}_\nu, \bar{q}'_\nu) := \sum_{\nu=0}^{n} \bar{p}_\nu \bar{q}'_k - \bar{L}(\bar{q}_\nu, \bar{q}'_\nu)$$
$$= \bar{p}_0 \bar{q}'_0 + \sum_{k=1}^{n} p_k \dot{q}_k \frac{\mathrm{d}t}{\mathrm{d}\tau} - L\frac{\mathrm{d}t}{\mathrm{d}\tau} \tag{1.2.3}$$
$$= p_0 \frac{\mathrm{d}t}{\mathrm{d}\tau} + H\frac{\mathrm{d}t}{\mathrm{d}\tau} = (p_0 + H)\frac{\mathrm{d}t}{\mathrm{d}\tau},$$

where we have used the abbreviation $p_0 := \bar{p}_0$. $\bar{H}$ does not depend explicitly on τ, but not all velocities can be eliminated, as we will see shortly.

We first prove that $\bar{L}$ is homogeneous of degree one in the velocities $\bar{q}'_\nu$, i.e.

$$\bar{L} = \sum_{\nu=0}^{n} \bar{q}'_\nu \frac{\partial \bar{L}}{\partial \bar{q}'_\nu}. \tag{1.2.4}$$

For $k = 1, 2, \ldots, n$ we have

$$\bar{q}'_k \frac{\partial \bar{L}}{\partial \bar{q}'_k} = \bar{q}'_k \bar{q}'_0 \left.\frac{\partial L(\bar{q}_l, z, \bar{q}_0)}{\partial z}\right|_{z=\bar{q}'_k(\bar{q}'_0)^{-1}} \left(\bar{q}'_0\right)^{-1}$$
$$= \bar{q}'_k \left.\frac{\partial L(\bar{q}_l, z, \bar{q}_0)}{\partial z}\right|_{z=\bar{q}'_k(\bar{q}'_0)^{-1}}$$

and the remaining $\nu = 0$ term evaluates to

$$\begin{aligned} \bar{q}_0' \frac{\partial \bar{L}}{\partial \bar{q}_0'} &= \bar{q}_0' L + (\bar{q}_0')^2 \sum_{k=1}^{n} \frac{\partial L(\bar{q}_l, z, \bar{q}_0)}{\partial z}\bigg|_{z=\bar{q}_k'(\bar{q}_0')^{-1}} (-1)\bar{q}_k' (\bar{q}_0')^{-2} \\ &= \bar{L} - \sum_{k=1}^{n} \bar{q}_k' \frac{\partial L(\bar{q}_l, z, \bar{q}_0)}{\partial z}\bigg|_{z=\bar{q}_k'(\bar{q}_0')^{-1}} . \end{aligned}$$

Adding both expressions we arrive at (1.2.4), and from this we conclude

$$\frac{\partial \bar{L}}{\partial \bar{q}_\mu'} = \frac{\partial}{\partial \bar{q}_\mu'} \left(\sum_{\nu=0}^{n} \bar{q}_\nu' \frac{\partial \bar{L}}{\partial \bar{q}_\nu'} \right) = \frac{\partial \bar{L}}{\partial \bar{q}_\mu'} + \sum_{\nu=0}^{n} \bar{q}_\nu' \frac{\partial^2 \bar{L}}{\partial \bar{q}_\mu \partial \bar{q}_\nu} \qquad (\mu = 0, 1, 2, \ldots, n) ,$$

hence the matrix $\left(\frac{\partial^2 \bar{L}}{\partial \bar{q}_\mu \partial \bar{q}_\nu} \right)_{\mu,\nu=0,1,2,\ldots,n}$ has eigenvalue 0 and is not invertible (i.e., $\bar{L}$ is singular). Therefore not all velocities $\bar{q}_\nu'$ are expressible in terms of coordinates and momenta. While this is possible by assumption for the velocities $\dot{q}_k$ ($k = 1, 2, \ldots, n$) and thus for $\bar{q}_k'$ ($k = 1, 2, \ldots, n$), we cannot eliminate $\bar{q}_0'$.

We have now $\bar{H} = \bar{H}\left(\bar{q}_\nu, \bar{p}_\nu, \bar{q}_0'\right)$ and since we have just reformulated the original system, the variables cannot all be independent. A constraint is obviously already given by (1.2.4), which is equivalent to

$$\bar{H}\left(\bar{q}_\nu, \bar{p}_\nu, \bar{q}_0'\right) = 0 ,$$

or equivalently

$$p_0 + H(q_k, p_k, t) = 0 , \qquad (1.2.5)$$

which fixes the momentum canonically conjugate to $\bar{q}_0 = t$.

Since (1.2.4) means

$$\bar{L} = \sum_{\nu=0}^{n} \bar{p}_\nu \bar{q}_\nu' ,$$

we can write the action in Hamiltonian form as follows:

$$\bar{S}\left[\bar{q}_\nu, \bar{p}_\nu\right] = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \sum_{\nu=0}^{n} \bar{p}_\nu \bar{q}_\nu'$$

This expression is obviously invariant under diffeomorphic transformations of the parameter τ. Not all variables of $\bar{S}$ can be varied independently; we take into account for the constraint (1.2.5) by incorporating it with a Langrangian multiplier $\lambda(\tau)$:

$$\bar{S}\left[\bar{q}_\nu, \bar{p}_\nu, \lambda\right] = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \left(\sum_{\nu=0}^{n} \bar{p}_\nu \bar{q}_\nu' - \lambda\left(\bar{p}_0 + H\right) \right)$$

Variation of this action leads to

$$\begin{aligned} \bar{q}_\nu' &= \lambda \frac{\partial \bar{p}_0}{\partial \bar{p}_\nu} + \lambda \frac{\partial H}{\partial \bar{p}_\nu} , \\ -\bar{p}_\nu' &= \lambda \frac{\partial \bar{p}_0}{\partial \bar{q}_\nu} + \lambda \frac{\partial H}{\partial \bar{q}_\nu} . \end{aligned}$$

For $\nu = 0$ we obtain

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\mathrm{d}\bar{q}_0}{\mathrm{d}\tau} = \lambda \tag{1.2.6}$$

and

$$-\frac{\mathrm{d}p_0}{\mathrm{d}t} = \frac{\partial H}{\partial t} ,$$

while for the indices $k = 1, 2, \dots, n$ we reobtain the Hamiltonian equations of the original system,

$$\dot{q}_k = \frac{\partial H}{\partial p_k} ,$$

$$-\dot{p}_k = \frac{\partial H}{\partial q_k} .$$

Using (1.2.3) the last four equations can also be rewritten:

$$\bar{q}'_\nu = \frac{\partial \bar{H}}{\partial \bar{p}_\nu} , \tag{1.2.7}$$

$$-\bar{p}'_\nu = \frac{\partial \bar{H}}{\partial \bar{q}_\nu} \tag{1.2.8}$$

A Poisson bracket for the parametrized system can also be defined,

$$\overline{\{\bar{f}, \bar{g}\}} := \sum_{\nu=0}^{n} \left(\frac{\partial \bar{f}}{\partial \bar{q}_\nu} \frac{\partial \bar{g}}{\partial \bar{p}_\nu} - \frac{\partial \bar{f}}{\partial \bar{p}_\nu} \frac{\partial \bar{g}}{\partial \bar{q}_\nu} \right) ,$$

where $\bar{f}, \bar{g}$ are functions of $\bar{q}_\nu$ and $\bar{p}_\nu$.

Example. Figure 1.2.1 shows the geometrical meaning of parametrization for the canonical variables considering as example the free particle in one dimension.

Let us now collect, interpret and discuss the results.

(i) We started from a Hamiltonian system with Hamiltonian H and time parameter t and constructed a new Hamiltonian system with one more degree of freedom corresponding to time, with Hamiltonian $\bar{H} = \bar{q}'_0 (\bar{p}_0 + H)$ and a new, arbitrary and physically meaningless parameter τ. The new ("parametrized") system is equivalent to the original one, iff we impose the constraint $\bar{p}_0 + H = 0$.

(ii) An advantage of practical importance for numerical calculations lies in the fact that the Hamiltonian of the parametrized system is not explicitly time-dependent. (This would be especially useful, if the original system had time-dependent constraints.)

(iii) While in the original system time is both a measure of duration and the evolution parameter ("*time* parameter"), these roles are separated in the parametrized system: The time variable $\bar{q}_0 = t$ is on par with the other canonical variables; evolution is controlled by a parameter τ, whose values correspond to instants, but whose metric is physically meaningless.

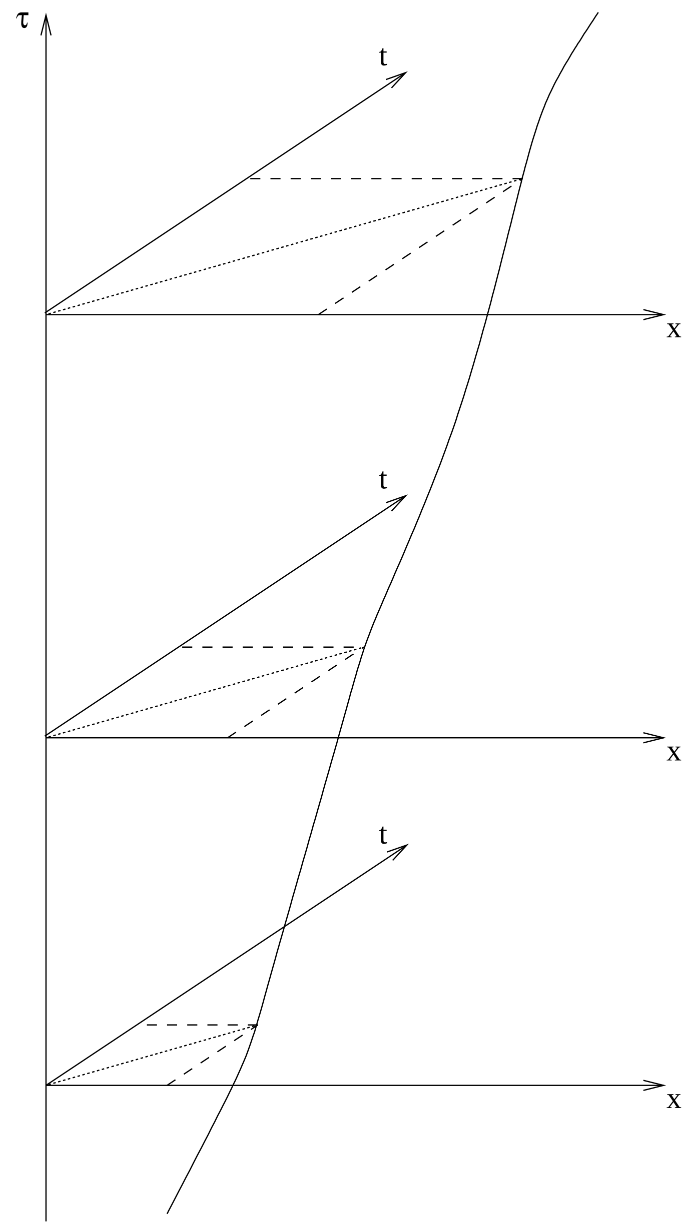

FIGURE 1.2.1. Evolution of a parametrized free particle in one dimension with coordinate $q_1 = x$. The original time evolution $t \mapsto x(t) = vt$ is replaced with the evolution $\tau \mapsto (t(\tau), x(\tau) = vt(\tau))$. The graph may be smoothly deformed along the τ-axis without changing physics.

(iv) In classical mechanics the canonical variable $\bar{q}_0 = t$ is observable indirectly: If we had attached[12] to the original system an (or the unique) ideal clock consisting of a free particle with motion $q_{n+1} =: vt$, then comparison of $\bar{q}_0(\tau)$

[12]Attaching in classical mechanics to a given system another one without interaction does not essentially change the nature of the given system, in contrast with quantum mechanics.

and $\frac{1}{v}\bar{q}_{n+1}(\tau)$ (with the same τ-value, meaning simultaneity) would show that $\bar{q}_0$ has the meaning of absolute time, possibly up to a linear transformation.

(v) If we try to understand the canonical variable $\bar{q}_0$ as a *spatial* coordinate, then we are led back to absolute time: Assume that parametrizing can be understood as attachment of a physical system. The original Hamiltonian was H, the new one is $\bar{p}_0 + H$; thus the attached system has Hamiltonian $H_0 = \bar{p}_0$ and is attached without interaction. The Hamiltonian equations for the attached system read

$$\bar{q}_0' = \frac{\partial H_0}{\partial \bar{p}_0} = 1, \qquad \bar{p}_0' = \frac{\partial H_0}{\partial \bar{q}_0} = 0\,,$$

describing essentially a free particle. The first equation implies $\tau = \bar{q}_0 + \beta'$ ($\beta' \in \mathbb{R}$), so that the evolution is time evolution. Moreover, a free particle is by definition related to absolute time t through $\bar{q}_0 = \alpha t + \beta$, with some constants $\alpha, \beta \in \mathbb{R}$. It follows that τ is linearly dependent on t, i.e. the evolution parameter coincides with absolute time and cannot be chosen arbitrarily.

In other words, we have either a canonical absolute time variable, which is not directly observable (in terms of position measurements), or we are forced to use absolute time as parameter, which is not directly observable either. (The lack of direct observability becomes important in quantum theory /1.2.5/.) In conclusion, the existence of absolute time is postulated in classical mechanics.

In the theory of gauge systems initiated by Dirac [Dir64] the homogeneous formalism is known as "parametrizing" or "rendering a system generally covariant" [HT92, ch. 4]. The latter is according to general relativity, which is invariant under (diffeomorphic) reparametrizations and therefore called generally covariant, see /1.2.4/ below.

Since $\overline{\{\bar{f}, \bar{H}\}} = \frac{\mathrm{d}\bar{f}}{\mathrm{d}\tau}$ is equivalent to $\overline{\{\bar{f}, \bar{p}_0 + H\}} = \frac{1}{\bar{q}_0'}\frac{\mathrm{d}\bar{f}}{\mathrm{d}\tau}$, and with $f\,(q_k, p_k, t) = \bar{f}\,(\bar{q}_\nu, \bar{p}_\nu)$ we have $\frac{\mathrm{d}\bar{f}}{\mathrm{d}\tau} = \bar{q}_0'\frac{\mathrm{d}f}{\mathrm{d}t}$ (assuming $\frac{\partial \bar{f}}{\partial \bar{p}_0} = 0$), it follows $\overline{\{\bar{f}, \bar{p}_0 + H\}} = \frac{\mathrm{d}f}{\mathrm{d}t}$. Hence the gauge transformation generated by the primary first class constraint $\bar{p}_0 + H$ is time evolution. For short one often says "dynamics is (the unfolding of) gauge". – Counting dimensions, we have a $2n + 2$-dimensional phase space of the parametrized system; there is one constraint restricting orbits to a $2n+1$-dimensional hypersurface; after identifying gauge orbits with points in the *physical phase space*, we are left with a $2n$-dimensional manifold, in full agreement with the dimension of the phase space of the original system.

In coordinate-free language the homogeneous formalism can be formulated as follows (see e.g. [Rov90, Rov02a]): The Hamiltonian equations for the original system read

$$i_X\sigma = \mathrm{d}H\,, \tag{1.2.9}$$

where $\sigma = \sum_{k=1}^{n} \mathrm{d}q_k \wedge \mathrm{d}p_k$ is a symplectic (i.e. nondegenerate, closed) [AM78, ch. 3] two-form on phase space Γ. (1.2.9) determines the vector field $X = \frac{\partial}{\partial t}$, whose integral curves $s(t)$ ($s \in \Gamma$) are motions with time parameter t.

The phase space of the parametrized system is $\bar{\Gamma} = \Gamma \times \mathbb{R}^2$ and we define a symplectic form on it as follows:

$$\bar{\sigma} = \sigma + \mathrm{d}\bar{q}_0 \wedge \mathrm{d}\bar{p}_0 \tag{1.2.10}$$

Let $\bar{X}$ be a vector field on $\bar{\Gamma}$ which fulfils

$$i_{\bar{X}}\bar{\sigma} = \mathrm{d}\bar{H}\,, \tag{1.2.11}$$

where as above $\bar{H} = \bar{q}_0'\,(\bar{p}_0 + H)$ and $\bar{q}_0' = \frac{\mathrm{d}\bar{q}_0}{\mathrm{d}\tau}$ with some τ.

With the identification $\bar{X} = \frac{\partial}{\partial\tau}$ this is equivalent to the equations of motion (1.2.7, 1.2.8) of the parametrized system. As we have already shown, on the hypersurface $\Sigma := \left\{p \in \bar{\Gamma} : \bar{p}_0 + H = 0\right\} \subset \bar{\Gamma}$ and with $\bar{q}_0 = t$ these are equivalent to those of the original system. To see this in coordinate-free language, we decompose $\bar{X}$ into components tangent to Γ and $\mathbb{R}^2$:

$$\begin{aligned}
\bar{X} = \frac{\partial}{\partial\tau} &= \sum_{\nu=0}^{n}\left(\frac{\partial\bar{q}_\nu}{\partial\tau}\frac{\partial}{\partial\bar{q}_\nu} + \frac{\partial\bar{p}_\nu}{\partial\tau}\frac{\partial}{\partial\bar{p}_\nu}\right)\\
&= \frac{\partial t}{\partial\tau}\sum_{k=1}^{n}\left(\frac{\partial\bar{q}_k}{\partial t}\frac{\partial}{\partial\bar{q}_k} + \frac{\partial\bar{p}_k}{\partial t}\frac{\partial}{\partial\bar{p}_k}\right) + \frac{\partial\bar{q}_0}{\partial\tau}\frac{\partial}{\partial\bar{q}_0} + \frac{\partial\bar{p}_0}{\partial\tau}\frac{\partial}{\partial\bar{p}_0}\\
&= \frac{\partial t}{\partial\tau}\left.\frac{\partial}{\partial t}\right|_\Gamma + \bar{q}_0'\frac{\partial}{\partial\bar{q}_0} + \bar{p}_0'\frac{\partial}{\partial\bar{p}_0}
\end{aligned}$$

Requiring now $\bar{q}_0 = t$ and restricting $\bar{X}$ to Σ (thereby getting rid of $\frac{\partial}{\partial\bar{p}_0}$) we obtain, using (1.2.10),

$$\begin{aligned}
i_{\bar{X}|_\Sigma}\bar{\sigma}\Big|_\Sigma &= i_{\bar{q}_0' X}\sigma + i_{\bar{q}_0'\frac{\partial}{\partial\bar{q}_0}}\left(\mathrm{d}\bar{q}_0 \wedge \mathrm{d}\bar{p}_0\right)\Big|_\Sigma\\
&= \bar{q}_0'\left(i_X\sigma + \mathrm{d}p_0|_\Sigma\right)\\
&= \bar{q}_0'\left(i_X\sigma - \mathrm{d}H\right)
\end{aligned}$$

on the one hand, and

$$\mathrm{d}\bar{H}\big|_\Sigma = \bar{q}_0'\,\mathrm{d}\left(\bar{p}_0 + H\right)|_\Sigma = 0$$

on the other hand; assuming $\bar{q}_0' \neq 0$ and using the last two formulas, the restriction of (1.2.11) to Σ is immediately seen to be equivalent to (1.2.9). In geometrical terms this means that the integral curves of $\bar{X}\big|_\Sigma$ are the graphs of the integral curves of $\bar{X}$.

The homogeneous formalism is a special case of the *presymplectic formalism*, which requires first a definition:

Definition. A two-form ω on a differentiable manifold M is called *nondegenerate*, if for all $m \in M$

$$\left(\forall y \in T_m(M) \quad \omega\left(x, y\right) = 0\right) \Rightarrow x = 0 \qquad \left(x \in T_m(M)\right),$$

where $T_m(M)$ is the tangent space to M in m.

A symplectic form is a nondegenerate closed two-form. A *presymplectic form* is more general in that the requirement of nondegeneracy is dropped. The two-form $\bar{\sigma}|_\Sigma$ is presymplectic, since $i_{\bar{X}|_\Sigma}\bar{\sigma}\Big|_\Sigma = 0$ while $\bar{X}\big|_\Sigma \neq 0$. (It is true in general, that a closed two-form on an odd-dimensional manifold cannot be nondegenerate.)

In general, for a presymplectic mechanical system with degenerate two-form ω and Hamiltonian $\tilde{H}$ the equation $i_X\omega = \mathrm{d}\tilde{H}$ [13] does not uniquely determine a vector field X, and hence the solutions of the equations of motion, since there may be a nonzero vector field Y with $i_Y\omega = 0$ and thus $i_{X+\alpha Y}\omega = \mathrm{d}\tilde{H}$, where α is an arbitrary scalar. – In our case of the homogeneous formalism we have $\omega = \bar{\sigma}|_\Sigma$, $\mathrm{d}\tilde{H} = \mathrm{d}\bar{H}\big|_\Sigma = 0$, $X = 0$, $Y = \bar{X}\big|_\Sigma$ and the scalar α corresponds to a reparametrization of the solution curves to $\bar{X}\big|_\Sigma$.

While every symplectic system can be cast into a presymplectic one through parametrization, the converse is not true: There are presymplectic systems which do not arise from symplectic ones through parametrization. E.g. the topology might not allow to split off a coordinate $q_0 \in \mathbb{R}$, or the restriction of the presymplectic form to the remaining coordinates might not be symplectic, or the parameter t being uniquely determined by the symplectic evolution on the hypersurface might not coincide with q_0.

For later use we note: Any (not nondegenerate) presymplectic system with constant Hamiltonian is reparametrization invariant.

The generalization of symplectic mechanics, where only a presymplectic form is available, is called *presymplectic mechanics* or *presymplectic formalism*. There is also a theory of canonical transformations of presymplectic systems, see e.g. [CGIR85], and elements of a presymplectic formulation of Lagrangian mechanics can be found in [CR95].

1.2.3 Special relativity

Since there is a maximal signalling velocity (speed of light, c) in special relativity, there is no means to instantaneously compare the readings of clocks at different positions. Consequently the notion of simultaneity becomes problematic. Mittelstaedt's book [Mit89] describes the conceptual foundations in an illuminating way; we will sketch the main points in the light of our discussion of time structures /1.1/ above. (For an axiomatic foundation of special relativity based on free particles and light signals as primitive concepts confer [Sch73]; Mittelstaedt's approach however has the advantage of working out the physically testable assumptions more clearly with the status of axioms, especially the equality of the maximal signalling velocity and the speed of light.)

In principle each point in space can be equipped with a tiny clock, for instance with *Einstein's ideal light clock*: A light signal is reflected back and forth between two parallel mirrors and the number of reflections as counted on one mirror is proportional to duration at this mirror's position. In the limit that the distance between the mirrors approaches zero, a time is being defined at a single position.

[13] For conditions on the solvability of this equation cf. [GNH78].

(In classical mechanics this limit is unproblematic.) To be more precise, we have a set of instants $\mathcal{T}_p$ for any $p \in \mathbb{R}^3$, together with time structures (A) to (F) /1.1/. Especially we have a duration d_p and a time order $\leq_p$ for all $p \in \mathbb{R}^3$, and by analogy with the discussion in /1.1/ we introduce a signed duration d^*_p providing a bijection between instants and the real numbers which are measured by the ideal clock. (With p we denote the points of 3-space, irrespective of what coordinates are assigned to these points by observers.)

Next we want to make instants at different positions comparable, in order to be able to describe motions: Let a particle be located at $p \in \mathbb{R}^3$ at instant $\mathbf{T} \in \mathcal{T}_p$ with local clock reading $t(p) = d^*_p(\mathbf{T}, \mathbf{T}_0) \in \mathbb{R}$, where $\mathbf{T}_0 \in \mathcal{T}_p$ is some fixed instant. Assume we find the particle at position $p' \in \mathbb{R}^3$, where the local clock reads $t(p') = d^*_{p'}(\mathbf{T}', \mathbf{T}'_0) \in \mathbb{R}$ with fixed $\mathbf{T}'_0 \in \mathcal{T}_{p'}$. In order to predict the observable number $t(p')$ (arrival time at p') from $t(p)$ and laws of motion, we must fix a relation between an instant at p and an instant at p' (e.g. $\mathbf{T}_0$ and $\mathbf{T}'_0$), called *simultaneity relation*.

Before continuing with synchronization we must first discuss how to distinguish the points p of the spatial manifold $\mathbb{R}^3$ on a physical basis. For this purpose we need a *reference frame*. By this we mean (compare [Mit89]) a set of reference points ideally filling all of space, where *reference point* means a distinguishable[14], noninteracting material basis located approximately in a point[15] and carrying an infinitesimal Einstein's light clock. The reason to require this is that two points without any distinguishable physical property would have to be considered identical. Empirically we know that any reference frame is topologically isomorphic to $\mathbb{R}^3$ and all points of $\mathbb{R}^3$ are distinguishable.

Given a reference frame we can define coordinates x^k ($k = 1, 2, 3$) on $\mathbb{R}^3$. In general, these coordinates might even depend on local time, i.e. $x^k = x^k(p, t(p))$, but we restrict our considerations to the case of a *coordinate system at rest in the reference frame* (also called a comoving coordinate system), $x^k = x^k(p)$. Empirically we find that the coordinate functions provide an isomorphism between the topology of $\mathbb{R}^3$ and the topology of the reference frame.

A reference frame in which every free body moves along a straight line (or doesn't change its position) in a coordinate system at rest in this frame, is called an *inertial system* or *inertial frame*; if gravitational interaction is negligible, then neutral bodies may serve as free bodies. We call a coordinate system at rest in an inertial frame an *inertial coordinate system*.

Let two free bodies be at rest at points $p_1, p_2 \in \mathbb{R}^3$. If we send light signals from p_1 at constant intervals $\Delta t_1(p_1)$ to p_2, then the signals arrive at p_2 at constant intervals $\Delta t_2(p_2) = \Delta t_1(p_1)$, since the local clocks were constructed in the same way. Moreover, light signals preserve the time order; hence they provide an identification of instants such that we can empirically verify $d^*_{p_1} = d^*_{p_2}$. This relation holds true, if we send the signals backward, thereby identifying different instants. There are even more methods to identify instants: Einstein's method

[14] Note that distinguishability of reference points is required throughout time.

[15] Infinitesimal adjacency, or in other words topology, is assumed to be a primitive notion of the theory, testable directly in experiments.

consists of sending a light signal from p_1 at $\mathbf{T}_1 \in \mathcal{T}_{p_1}$ to p_2, where it arrives at $\mathbf{T}_2 \in \mathcal{T}_{p_2}$ and is immediately reflected back to p_1, where it arrives at $\mathbf{T}'_1 \in \mathcal{T}_{p_1}$; he proposed to identify the instant $\mathbf{T} \in \mathcal{T}_{p_1}$ fulfilling $t_1(\mathbf{T}) = \frac{1}{2}\left(t_1(\mathbf{T}_1) + t_1(\mathbf{T}'_1)\right)$ as simultaneous with $\mathbf{T}_2$; for obvious reasons this synchronization method is often called *radar synchronization*. We cannot however identify any $\mathbf{T} > \mathbf{T}'_1$ or any $\mathbf{T} < \mathbf{T}_1$ with $\mathbf{T}_2$, because this would violate causality (light arriving before being sent)[16]. Altogether, we can define any $\mathbf{T}$ with

$$t_1(\mathbf{T}) = t_1(\mathbf{T}_1) + \varepsilon\left(t_1(\mathbf{T}'_1) - t_1(\mathbf{T}_1)\right) ,$$

where $0 \leq \varepsilon \leq 1$, as *simultaneous* with $\mathbf{T}_2$. The so-called ε*-parameter* shows the conventionality of simultaneity [Jan02]. Since $\mathcal{T}_{p_1}$ and $\mathcal{T}_{p_2}$ carry the same metric, this synchronization procedure (with fixed ε) works for any and for all instants.

There is a long and continuing debate [Ryn01b, Ryn01a, Min02a] on whether ε does have a naturally fixed value. A value $\varepsilon \neq \frac{1}{2}$ would mean that the forward and backward velocity of light (between p_1 and p_2) are different. With the postulate of an isotropic velocity of light Einstein's definition ($\varepsilon = \frac{1}{2}$) is retained. This postulate can however not be verified experimentally. In order to not single out one entity (light) as special (other phenomena might not be isotropic), a more general postulate requires that any synchronization procedure must not single out one preferred spatial direction; from this, again, $\varepsilon = \frac{1}{2}$ follows. A less popular opinion holds that on the contrary space need not be isotropic, and what is measurable is only the two-way velocity of light, never the one-way velocity. In this approach all $\varepsilon \in [0; 1]$ are possible. Recently it was even argued that different forward and backward velocities of light are measurable: By considering circular signal propagation, being observed in a rotating frame, in [GS97] it was claimed – making use of general relativity – that not only can the one-way velocity of light be observed, but also the forward and backward velocities of light can differ for frames with a nonvanishing relative velocity; in [JS03] however the misconceptions therein (mainly the incorrect usage of general relativistic higher order corrections to the special relativistic approximation in the case of circular geometry) and in the recent literature have been clarified, and the Sagnac effect was shown to be in accordance with the principles of special relativity; also the problem of synchronization of clocks on a rotating platform was explored. The assumption of equal forward and backward velocities of light is hence not at odds with experiment.

Let us now return to motion. We pick out a free body which is not at rest in an inertial coordinate system and therefore moves along a straight line. We parametrize this line so that position depends linearly on the parameter, $x^k(t) = v^k t$, and we call the parameter t *time parameter*; it is defined up to linear transformations. Empirically we find that t is proportional to the time $t(x^k)$ measured by the local clocks at x^k, which we now assume to be synchronized by any of the above procedures.

[16]The consistency with causality is a main restriction for the definition of synchronization procedures.

So far our description was based on a fixed inertial frame and a coordinate system at rest in this frame. We have identified the local times in this frame with a synchronization procedure, resulting in a spatially global time coordinate[17]. We thus have a *spacetime* coordinate system, and free bodies move along straight lines in this 4-dimensional coordinate system. – The relativity principle now states that all inertial frames provide equivalent descriptions of a physical system. Let two inertial frames and corresponding spacetime coordinate systems $(\mathbf{x}, t)$ and $(\mathbf{x}', t')$, where the same synchronization procedure has been used, be given. Since free bodies in $(\mathbf{x}, t)$ are also free in $(\mathbf{x}', t')$, straight lines must be mapped into straight lines upon coordinate transformation. In axiomatic relativity it is then shown that together with homogeneity of space and time the linearity of the transformation can be derived. One next requires that the transformations have a group structure and that they preserve the causal structure. The resulting transformation equation contains a free parameter v_∞ (w.r.t.g. $v_\infty \geq 0$) with the meaning of a maximal velocity. Choosing $v_\infty = \infty$ reproduces Galilei transformations, while $v_\infty = c < \infty$ results in the well known Lorentz transformations. The last equation is necessary for $\varepsilon = \frac{1}{2}$ [Mit89, ch. VI], but $\varepsilon = \frac{1}{2}$ is conventional. The true reason for $v_\infty = c$ (conversely implying $\varepsilon = \frac{1}{2}$) lies in the fundamental axiom of *Lorentz invariance*, which means: Given any Lorentz system[18] and any equivalent system (such that all of physics looks the same from within each system), both are connected by a Lorentz transformation:
Given one Lorentz system $L_\mathbf{0}$ with coordinates $(t_\mathbf{0}, \mathbf{x_0}) \in \mathbb{R}^4$, every other Lorentz system $L_\mathbf{v}$ has a relative velocity $\mathbf{v} \in \mathbb{R}^3$ ($|\mathbf{v}| < c$), and its coordinates $(t_\mathbf{v}, \mathbf{x_v}) \in \mathbb{R}^4$ are connected to those of the original system through a Lorentz transformation[19] (up to translations):

$$t_\mathbf{v} = \frac{t_\mathbf{0} - \frac{\mathbf{v}\mathbf{x_0}}{c^2}}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} \ ,$$

$$\mathbf{x_v} = \frac{\mathbf{x_0} - \mathbf{v}t_\mathbf{0}}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}}$$

We thus have no unique time in special relativity, but a 3-parameter family of times of type (F) /1.1/ and bijective (even linear) mappings between them, which however depend on position. ("Time and space are linked together indissolubly.") The metrics of these times are different. (For $\mathbf{v} \neq 0$, $t_\mathbf{v}$ is dilated w.r.t. $t_\mathbf{0}$.) Each time corresponds to an inertial system, and can be measured at an arbitrary point $\mathbf{x} \in \mathbb{R}^3$ (via synchronization). A fixed point in an inertial frame is also called an *inertial observer*, and if we conceive of an inertial observer as a body at rest

[17]Of course, within a finite time only points separated by a finite distance can be synchronized; for an inertial observer the synchronizable region in spacetime is an intersection of a forward and a backward light cone, also called *diamond* or *double cone* in algebraic quantum field theory [Haa96].

[18]A Lorentz system is an inertial coordinate system, which has been extended to a coordinate system on spacetime (using a synchronization procedure), and which has the metric tensor $\eta_{\mu\nu} = \mathrm{diag}(1; -1; -1; -1)$ $(\mu, \nu = 0, 1, 2, 3)$.

[19]The notation is consistent in case $\mathbf{v} = 0$.

in this frame carrying a tiny clock, then its time is by definition the time at its position in this frame.

For an accelerated observer A *proper time* is defined as the time that is measured by a clock carried along with A. In special relativity the metric along worldlines can locally be approximated with the duration of an inertial observer I' moving with the same instantaneous velocity $\mathbf{v}$ as seen from an inertial observer I, i.e.

$$\mathrm{d}\tau_A = \mathrm{d}t_{I'} \,.$$

Integration along the worldline gives the metric of A's proper time,

$$\Delta\tau_A = \int_{t_{I,1}}^{t_{I,2}} \sqrt{1 - \frac{\mathbf{v}^2(t_I)}{c^2}} \mathrm{d}t_I \,.$$

The causal structure, which does not depend on a particular inertial observer's description, depends on both time and space, too: For timelike events it is given by the time order; for two arbitrary spacelike events A, B one can choose inertial frames in which they are arbitrarily time-ordered (A future to B, B future to A or A simultaneous to B), i.e. simultaneity is frame dependent.

Let us finally mention that the presymplectic formalism provides the only possibility for a Lorentz covariant formulation of mechanics, since a distinguished duration (corresponding to a choice of Lorentz frame) is not required in this formalism.

1.2.4 General relativity

As opposed to special relativity /1.2.3/, there is no independent (material) reference frame in general relativity: Since gravity affects all kinds of matter, also reference points interact with the bodies to be referenced. If we assume that a reference point has negligible mass, then it does not affect the motion of other bodies, but its motion is affected by other bodies through the gravitational field they cause at the position of the reference point. Under these circumstances the analogue of inertial motion is free fall: A body is called *freely falling*, if it is not exposed to forces other than gravity. An approximately massless freely falling body is called a reference point. A *reference frame* in general relativity consists of an approximately continuous, but incoherent distribution of matter, which is freely falling. As in special relativity we assume each reference point to be furnished with a tiny and approximately massless Einstein's light clock, so that the presence of the clock does not alter the gravitational field, and the local gravitational field at the position of the clock can be considered homogeneous. For now the reading of this clock does not have the meaning of some duration (which could be compared to other clocks), it just serves to distinguish instants λ along the path of this reference point.

Coordinates
Following Rovelli's proposal [Rov01a] (see also [Rov03, sec. 2.4.6]) we introduce physical coordinates for reference points: Assume that there is an event in the

distant past, where one reference point has 'decayed' into 4 equal reference points ($\mu = 0, 1, 2, 3$), which are all furnished with senders emitting radio signals (in different frequency bands) with a message containing the reading x^μ of their respective Einstein's light clock into all directions of 3-space. Any other reference point can receive these 4 radio signals at any instant λ (provided it is not too far in the past) and thus obtains 4 coordinates $x^\mu = x^\mu(\lambda)$. In general it is not necessary that these 4 real numbers allow to distinguish different reference points and different instants along the path of a reference point, but empirically one finds that this can be ensured in a simple way: One considers a decay where the initial directions of the 4 senders are maximally symmetric: In the rest frame of the 'decaying reference point' they shall move from the center (decay event) to the corners of a tetrahedron. – Here one has to measure angles, but this is possible in special relativity, which valid locally due to

*Einstein's equivalence principl*e [Car97]:

(i) In sufficiently small regions of spacetime, the laws of physics are reduced to those of special relativity.
(ii) It is impossible to detect the existence of a gravitational field.[20]

We thus have coordinates for all reference points and all instants which are not too far in the past, and as long as we neglect spacetime singularities.

Coordinate time
The 'time coordinate' x^0 has no special meaning for the reference point; the choice of the coordinates serves only to label and distinguish the points of spacetime and is a mere matter of convenience. Coordinate time x_0 does not have the meaning of an observable duration for the reference point, it can always be reparametrized.

Proper time
Empirically one finds that for any reference point, with Einstein's light clock reading λ, the dependencies $x^\mu(\lambda)$ obey a second order differential equation:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = C(\lambda) \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \tag{1.2.12}$$

In Newtonian mechanics equation (1.2.2) obtains the simple form (1.2.1) through the choice of a distinguished parametrization (with the meaning of Newtonian time): The term with one first derivative drops out. Except for the term $\Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}$ equation (1.2.12) is equivalent to Newton's Second Law reparametrized, and for vanishing external forces. As in Newtonian mechanics we can get rid of the term with one first derivative [Mit89, ch. VIII]: Define τ by $\frac{\mathrm{d}^2\tau}{\mathrm{d}\lambda^2} = C(\lambda)\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}$ (which determines τ up to linear transformations); then (1.2.12) is easily seen to be

[20] One cannot decide on whether the acceleration of a body is due to the choice of reference frame, or due to actual forces (cf. Einstein's famous elevator thought experiment). One also cannot detect a gravitational field using several bodies, since the proportionality of inertial and gravitational mass is the same for all bodies (weak equivalence principle).

equivalent to

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0 \ . \tag{1.2.13}$$

The value of $C(\lambda)$ can be measured along the path of each reference point, and thus τ is uniquely defined (up to linear transformations). This distinguished parameter τ is called *affine parameter* (of the curve $x^\mu(\tau)$) or *proper time* (of the respective reference point, or along its worldline). In fact locally it agrees with the special relativistic notion of proper time: The connection $\Gamma^\mu_{\alpha\beta}(x)$ is empirically found to arise from a metric $g_{\mu\nu}(x)$ (spacetime is thus a (3+1)-dimensional pseudo-Riemannian manifold); in this case the affine parameter τ is known to be a measure for the length of the curve, $c^2(\mathrm{d}\tau)^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ (see e.g. [Wal84, p. 42f]). On the other hand according to Einstein's relativity principle one can find new coordinates x'^μ in which $\Gamma'^\mu_{\alpha\beta}(x_0) = 0$ and $g'_{\mu\nu}(x_0) = \eta_{\mu\nu} := \mathrm{diag}(1;-1;-1;-1)$ locally around x_0, so that the affine line element $\mathrm{d}\tau$ can be calculated in a (special relativistic) reference frame I_0 in which the reference point is at rest [Mit89, ch. VIII]:

$$c^2(\mathrm{d}\tau)^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \eta_{\mu\nu}\mathrm{d}x'^\mu\mathrm{d}x'^\nu = c^2(\mathrm{d}t'_{I_0})^2$$

Here $\mathrm{d}t'_{I_0}$ is the proper time of special relativity. Furthermore, because of $\Gamma'^\mu_{\alpha\beta}(x_0) = 0$ in the frame I_0 the reference point moves on a straight line and uniformly, $\frac{\mathrm{d}^2x'^\mu}{\mathrm{d}\tau^2} = 0$, so that I_0 is a *local inertial frame.*

Using local inertial frames, also the proper time of bodies falling non-freely along a timelike worldline is physically meaningful. More precisely, it is the duration of proper time; the origin is arbitrary, and the unit of proper time and unit of length are chosen such that the speed of light equals c in a local inertial frame. The duration between two arbitrary timelike events in general depends on the worldline connecting them. While this is already known from the twins paradox in special relativity, here in addition the dynamics of the gravitational field $g_{\mu\nu}$ has to be known.

In conclusion, there is a metric on each worldline called proper time, which depends on the gravitational field $g_{\mu\nu}$ and its canonically conjugate momentum[21] $\pi_{\mu\nu}$. The evolution of $g_{\mu\nu}$ cannot be expressed in dependence on proper time along a single world line; a suitable description of the dynamics of the gravitational field requires more than proper time.

Synchronization

While in special relativity free bodies at rest in an inertial coordinate system could be used to mark positions, in general relativity freely falling reference points do in general change their distance over (coordinate) time. But there is an analogue of an inertial coordinate system, see [Mit89, ch. VIII], whose presentation we follow closely: One defines *comoving coordinates* as those coordinates, whose spatial values are constant for all freely falling particles and whose time coordinate coincides with the proper time of the respective freely falling particle.

[21] For a Hamiltonian formulation of classical general relativity cf. for instance [Wal84, appendix E].

Given any coordinate system x^λ and the metric tensor $g_{\mu\nu}$ in these coordinates one can find comoving coordinates x'^λ in the following way: Let $u^\mu(x^\lambda)$ be the velocity field of the approximately continuous system of reference points in the given coordinates. One requires the velocity in the comoving coordinates to be

$$u'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} u^\nu = \delta^\mu_0 \ . \tag{1.2.14}$$

This implies $u'_\nu = g'_{\nu 0}$ and $u'_\nu u'^\nu = g'_{00}$. By appropriate normalization of u'^μ one obtains $g'_{00} = 1$, which means that the proper time τ of each reference point coincides with coordinate time x'^0.

Using (1.2.14) and $g'_{00} = 1$ the geodesic equation (1.2.13) in the new coordinates is equivalent to

$$\begin{aligned} 0 &= \frac{\mathrm{d}u'^\mu}{\mathrm{d}\tau} + \Gamma'^\mu_{\alpha\beta} u'^\alpha u'^\beta \\ &= \Gamma'^\mu_{00} u'^0 u'^0 \\ &= \frac{1}{2} g'^{\mu\alpha} \left(g'_{\alpha 0,0} + g'_{0\alpha,0} - g'_{00,\alpha} \right) u'^0 u'^0 \\ &= g'^{\mu\alpha} g'_{\alpha 0,0} u'^0 u'^0 \ , \end{aligned}$$

hence $g'_{\alpha 0,0}$ must vanish. This means in particular that g'_{i0} $(i = 1, 2, 3)$ is time-independent, so that the line element takes the form

$$(\mathrm{d}x'^0)^2 + 2g'_{i0}(x'^k)\mathrm{d}x'^i \mathrm{d}x'^0 + g'_{ij}(x'^\lambda)\mathrm{d}x'^i \mathrm{d}x'^j \ .$$

One may ask whether there is even a coordinate system in which space and time are completely separated, i.e., where also $g'_{i0} = 0$. This would be a comoving and *time-orthogonal* (or *synchronous*) coordinate system. These coordinates are also called *Gaussian normal coordinates* ([Mit89, fn. on p. 162], [Wal84, p. 42f]) and have the following properties: Spacetime is foliated into hypersurfaces $x'^0 =$ const.; geodesics are everywhere orthogonal to the hypersurfaces and have spatial coordinates $x'^i =$ const.; the affine parameter of geodesics (proper time) coincides with x'^0; the hypersurfaces $x'^0 =$ const. consist of simultaneous events for all observers moving along geodesics.[22]

Gaussian normal coordinates do not exist for general spacetimes; as shown in [Mit89, ch. VIII] the normalized velocity field u'^μ has to be curl-free, which can however be experimentally verified for general relativity. The possibility $g'_{i0} = 0$ corresponds to Einstein synchronization when using light signals. (While in special relativity the choice $\varepsilon = \frac{1}{2}$ of the synchronization parameter is completely conventional, general relativity as a theory with stronger self-consistency requirements [Mit89, ch. VIII] puts further conditions on the possibility of radar synchronization with $\varepsilon = \frac{1}{2}$.) With appropriate $g'_{i0} \neq 0$ any two spacelike events

[22] We should perhaps point out the use of a time coordinate common to all freely falling observers: Assume that observer O_1 wants to send a message so that observer O_2 will receive it at some particular time. If O_1 knows the worldlines of himself and O_2, and if he has enough knowledge about the gravitational field along the possible light paths (geodesics) between him and O_2, then he can calculate the time at which to send the message. (In special relativity knowledge about the velocities of O_1 and O_2 would suffice.)

can be made simultaneous as in special relativity. (Recent work [Min02b] shows that the arbitrariness of synchronization can be understood as a choice of gauge in the formalism of gauge theories.)

In conclusion, the general relativistic analogue of the special relativistic inertial frame is a reference frame (an approximately continuous system of reference points, i.e. incoherent, freely falling and nearly massless particles, with clocks attached to them), which has a curl-free velocity field. This latter condition ensures that two reference points leaving both from event A, taking different paths and finally arriving both at event B measure the same time difference between A and B; the twins paradox does not appear.

In cases where one has no reference frame of freely falling particles, but only accelerated particles the introduction of appropriate coordinates is more complicated (see e.g. [PV00, AL03]).

We had assumed that clocks measuring proper time are attached to the reference points. Such clocks are called standard clocks and from the outset we could only assume that they were constructed in the same way (e.g. as tiny Einstein's light clocks). If one assumes that general relativity (or just a Weylian spacetime model) is valid, then one can verify whether a real clock is a standard clock by using elementary procedures based essentially only on freely falling (or even accelerated) bodies and light signals, see [Per94]. On the other hand, the assumption of the existence of clocks in the above reasoning rests on the local validity of special relativity. If one does not assume this characteristic of general relativity from the outset, then one can use the axiomatic approach of Ehlers, Pirani and Schild (EPS) [EPS72] (see also a summary thereof in [LP03, appendix C] as well as [AL94, LÖ1]) for the foundation of spacetime structures, which is based solely on empirical facts about light rays and freely falling particles and does not assume the existence of clocks.

General spacetimes

Proper time and synchronization as sketched above face problems when spacetime singularities are involved, and in general they are meaningful only for bounded regions, not globally, since for general spacetimes one cannot choose Gaussian normal coordinates everywhere. If this were the case, one would have a *cosmic time*, which is used e.g. in Robertson-Walker models of the universe, where the metric is $\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a(t)\,(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2)$. In general there is a hierarchy of causal structures weaker than cosmic time, see. e.g. [HE73, ch. 6], or [Wal84, ch. 8].

At the bottom of this hierarchy lies *temporal orientability* and *chronology*. Both notions are concerned with temporal ordering: The first ensures the existence of past and future locally, and the second excludes the possibility of closed timelike curves [Ear97]. In fact there are homogeneous, but non-isotropic spacetimes (Gödel universes, [HE73, sec. 5.7]), which are temporally orientable, but allow for closed timelike curves and thus for "time travel" [AM00]; these solutions to Einstein's equations are however considered unphysical.

1.2.5 Nonrelativistic quantum mechanics

Nonrelativistic quantum mechanics uses a time parameter t to describe the evolution of the state of a system. This parameter is not an observable of the system itself (see /2.2.1/), but of a classical environment. The evolution of the system is described in such a way that to each value of t corresponds a state of the system, which determines the expectations for instantaneous measurements taking place at an instant, which is determined by the fact that a *classical* clock shows the value t. By this a classical environment and the existence of a Newtonian absolute time is presumed.

For solutions of the time-independent Schrödinger equation with a certain energy E the time evolved state vector is the product of an initial state vector and a complex phase $e^{-\frac{i}{\hbar}Et}$, which is unobservable; the state is therefore called stationary.

The appearance of t-dependent potentials in the time-dependent Schrödinger equation can be understood as an effective influence of a (semi-)classical environment at instant t (as determined by a classical clock). In [BR00] it was shown, that certain t-dependent potentials arise from an interaction of the quantum system with a quantum environment in the limit that the environment can be treated semi-classically; the value of t is here determined by the quantum state of the environment and (at least in principle) observable. This suggests that the description of any t-evolution can be achieved with a quantum environment in an appropriate classical limit, so that a strictly classical notion of time is not required.

Besides the Schrödinger picture the Heisenberg picture is used for the description of time evolution. In the traditional Hilbert space formalism both pictures are equivalent. In field theory the situation is however not so clear. Dirac claimed[23] that the Heisenberg picture is the right picture. There is still much controversy which of the two pictures is the right one, Schrödinger's or Heisenberg's (except for traditional nonrelativistic quantum mechanics, where both are equivalent). E.g., an article published in a respected journal [FFMS02] recently studied a harmonic oscillator coupled to the electromagnetic field and concluded that Schrödinger's picture is wrong, contrary to Heisenberg's, which is right. This argument in favor of Dirac's claim has however shown to be flawed [Nik03]. While Dirac's claim finds support in theories with constraints, in algebraic quantum theory the Schrödinger picture is known to be more general than the (modified) Heisenberg picture [Kad65].

Another alternative is relevant for the time evolution of quantum systems: A system can be modelled as subject to influences of the environment (*open system, exosystem*), or isolated from external influences (*closed system, endosystem*). Both

[23] According to [Rov91c, sec. III.A.] he claimed this in a talk in 1981 (see also [Dir65]; before he had been an adherent of the Schrödinger picture, and even earlier of the Heisenberg picture, cf. different editions of his famous textbook on quantum mechanics), in which he used a single transparency which contained just the following text: $i\hbar\frac{dA}{dt} = [A, H]$*: Heisenberg mechanics is the good mechanics.*

descriptions are quite different, the former having an indeterministic dynamics, the latter a deterministic one.

In this respect we mention that the time evolution of an endosystem is problematic, since it requires some environment to interpret the parameter t in the Schrödinger equation as time. The environment forbids arbitrary reparametrizations $t \mapsto t' = \lambda(t)$ with a diffeomorphism λ, just as the hypothesis of absolute time in classical mechanics. – Without environment there is no measure of duration for the system /1.3.2.3/.

1.2.6 Relativistic quantum mechanics

A relativistic quantum theory is widely believed to be meaningful only as a quantum field theory. (See e.g. [Wei95, p. 169] and the discussions about Malament's theorem [HC01] the Hegerfeldt's paradox [Heg98].)

As in nonrelativistic quantum mechanics the time parameter in quantum field theory corresponds to an external classical time, however in this case the relativistic time of classical special relativity. The arbitrariness of the choice of classical Lorentz frame naturally imposes covariance conditions on the quantum fields.

Due to the quantum nature relativistic field states are not localized at a point, and therefore not at an instant of time [BB94]. Since states (eigenstates of a system of operators) are used to label the state of a system, this state is not instantaneous, too. Thus only smeared in time (and in space) field operators are mathematically meaningful.[24] In a different line of thought is was noticed that the (strong) causality of special relativity may be too strong in the quantum domain: It is only the expectation values which have to be Lorentz covariant; for individual processes this is not necessary, which gives rise to the notion of weak causality [Heg98] (see also [May98]).

Quantum field theory on curved spacetimes [Wal94] with a fixed background metric uses the general relativistic notion of time.

1.2.7 Summary

Let us briefly summarize in which respect the structure of time in Newtonian mechanics has been reduced[25] in increasingly fundamental theories.

In special relativity the temporal ordering of spacelike events, duration and simultaneity become dependent on the choice of inertial frame. Duration transforms linearly between these frames. The choice of synchronization procedure is to some extent arbitrary.

---

[24] Bialynicki-Birula uses a relational argument: "When the state of a system at a given instant becomes ill-defined, time itself becomes diffused. After all, what is the meaning of an instant if there is no element of reality associated with it?"

[25] Healey [Hea02] compares Newtonian mechanics with "a high water mark for the reality of time in physics from which it has been receding ever since".

In general relativity the notion of inertial frame is generalized to a reference frame with curl-free velocity field. If a cosmic time does exist, the events along different worldlines of freely falling observers can be synchronized. The arbitrariness of synchronization remains.

In quantum theory classical notions of time are used, but duration is not observable for isolated systems.

1.3 Relational time

In the previous section the notions of time have been shown to progressively become deconstructed when advancing towards more fundamental theories. What, then, are the bare essentials of every notion of time? - The answer of relationalism is: The nature of time is essentially that it relates observations of physical quantities. Time measurements do not correspond to preferred observables. A time observable could be e.g. the *spatial position* of the hand of a clock. On the other hand not every observable qualifies as a time observable. The relational concept of time reduces the richness of time structures discussed in /1.1/ to merely an equivalence relation having the meaning of simultaneity. It therefore also goes under the term *fundamental timelessness.*

This section begins with a short review of the historical development of relational ideas /1.3.1/, then explains in detail the idea of time as a relation in fundamental theories /1.3.2/ and finally analyzes the underlying notion of simultaneity /1.3.3/.

1.3.1 Relationalism

Most physical theories start with presumed (absolute) structures for space and time, then place objects into spacetime (localization), and describe (or predict) verifiable observations as a result of the interaction of the objects.

An opposing attitude holds that this procedure conceals the nature of space and time. Instead of assuming space and time from the outset, they should be seen as formed from the relationships between objects; objects themselves are formed from relationships between our observations, and only our observations are absolute. This latter attitude is called *relationalism* or *relationism*[26].

Both attitudes have a long history [Bar89], as has the struggle between them[27], which continues until today. As an early proponent of relationalism we have already quoted Lucretius (see above). Participants in this debate include among others Newton, Huygens, Leibniz, Berkeley, Maxwell, Kant, Mach, Poincaré and Einstein [Ear89].

[26] In philosophy the relational idea with regard to time is known as the “relational theory of time” [SO95, ch. 3].

[27] Much of the debate is concerned with rotation (and Newton’s bucket experiment), which is of no importance for our purpose.

Newton's famous scholium on absolute space and time begins with these sentences:

> Hitherto I have laid down the definitions of such words as are less known, and explained the sense in which I would have them to be understood in the following discourse. I do not define time, space, place, and motion, as being well known to all.

This was heavily criticised by Leibniz in his correspondence with Clarke, who defended Newton's view [Sma64, pp. 89-98]:

> ... As for my own opinion, I have said more than once, that I hold space to be something merely relative, as time is; ...
>
> ... the answer is, that his inference would be right, if time was any thing distinct from things existing in time. ...
>
> ... Nothing of time does ever exist, but instants; and an instant is not even itself a part of time. Whoever considers these observations, will easily apprehend that time can only be an ideal thing. ...

Mach published a history of mechanics in 1883 [Mac83]. Therein he argued strongly against Newton's idea of absolute space and absolute time. Newton had argued that inertial motion was relative to absolute space; Mach held that inertial motion was relative to the average of all the masses in the universe. His understanding of time was relational, too:

> It is utterly beyond our power to measure the changes of things by time. Quite the contrary, time is an abstraction, at which we arrive by means of the changes of things.

While Mach had formulated his critique of Newton' absolute space and time rather vaguely, Poincaré clarified the defects in mathematical terms in 1902 and 1905, but he did not create a relational theory of classical mechanics. Einstein on the other hand read Mach's book, and was inspired by a relational philosophy:

> ... time and space are modes by which we think and not conditions in which we live.

But he seemingly did not know about Poincaré's analysis [Bar99b]. He coined the term *Mach's principle*, but missing Poincaré's analysis he interpreted Mach in a "curious way" [Bar99b], thereby causing much confusion. Indeed there are now different versions of Mach's principle; see [Rov03, sec. 2.4.1] for a brief summary and a discussion of their validity in general relativity (some are false). After Einstein's eminent creations of special and general relativity the original Mach-Poincaré line of thought escaped most researchers' attention. There had been publications by Hofmann (1904), Reissner (1915) and Schrödinger (1925), who went into that direction, but it took until the seventies of the last century for the idea of Machian mechanics to be rediscovered by Barbour and Bertotti [BB77] (see /1.3.2.1/).

While Machian mechanics attacks absolutism with relationalism of motion, Einstein's general relativity has another kind of relationalism[28]: It involves only spatiotemporal relations among bodies and events [Ear89]. The famous hole argument /1.3.2.2/ is widely perceived as evidence for the absence of spacetime as a substratum underlying concrete events.

Another kind of relationalism is embodied in the conventionality of simultaneity in special relativity /1.2.3/.

In nonrelativistic physics time can be separated from space and the relational concept of time can be characterized e.g. like that [Coe69]: Time is no mystery, no *a priori*: Time only exists to the extent that it is a property of matter, and it relates the change of any two quantities, whereby the unit of time (duration) cancels out during the evolution of both quantities. Only relative duration is physically meaningful; time can thus be seen in any system as a relation among its observables. (We will make this more precise in /1.3.3/.)

This raises the question for devices to measure relative duration, i.e. for clocks. The position of the sun on the sky and of the stars on the night sky were early clock observables, although they had limited precision. To construct "better" clocks is more than a mere technical problem; let us cite Rovelli [Rov91c]:

> Galileo used his pulse to measure the oscillation period of a pendulum and to discover that it was isochronous. A few years later doctors were using pendulums to measure the periods of people's pulses and to check whether they were isochronous.

Which one is the better clock, the pendulum or the pulse? – A straightforward answer from classical mechanics is that clocks having a less disturbed oscillation mechanism are the better ones. To put it differently, the more isolated the clock system is from other systems, the better is its time. As general relativity tells, gravity can never be switched off, and therefore the metric of time, which essentially is a component of the metric tensor, can always be measured by an isolated clock. We will discuss nonrelativistic (classical and quantum) clocks in /2.2/.

1.3.2 Relational theories

In this subsection we indicate how classical mechanics, general relativity and quantum mechanics can be formulated in a relational way.

1.3.2.1 Classical mechanics

As noted above /1.2.1/, Newtonian mechanics has an absolute, unique time common to all systems: We can in principle adjoin the same ideal clock system to all systems, while the interaction with each system remains negligibly small. Absolute time (duration) is then defined to be – up to a linear transformation – the clock time (duration) measured by the ideal clock.

[28]Earman holds that "there are almost as many versions of relationalism as there are relationalists." [Ear89, p. 12]

However a line of research originating from Mach's critique of Newton (see [Bar03, Bar99a, But01, BB82, Bar74] for contemporary work) exploits a loophole in this reasoning: One cannot attach a physical clock to the whole universe.[29] (In Barbour's words: "The great timekeeper outside the universe does not exist.") Consequently the absence of absolute time is claimed for Newtonian mechanics of the universe, and furthermore the absence of absolute position and rotation, as well as absolute unit of length. All absolute frames are considered equivalent and *shape space* is defined as the set of all configurations of the universe which differ only by changes in absolute frames. I.e., points in shape space correspond to three-dimensional relative configurations of point particles. Within this approach dynamics is then reconstructed from geodesic principles in this shape space, where only solution curves attain physical meaning, not certain parametrizations[30]. Absolute time can be understood as distinguished simplifier by which the expression for the distance between different configurations is made most simple. This relational theory was shown to be equivalent to Newtonian mechanics with vanishing total angular momentum. Hence, this approach tries to base mechanics on a more simple and more restrictive set of axioms, which conforms to Machian principles. To this day there is no experimental evidence consistent with Newtonian mechanics that would falsify Machian classical mechanics.

In this thesis we employ another possibility to make sense of a relational classical mechanics: We drop the assumption of *extensibility* also for parts of the universe, i.e. we consider a closed system and exclude the possibility of attaching additional degrees of freedom, in particular a clock. In such a setting the notion of absolute time becomes meaningless: Assume that any mechanical system S is given and that there is nothing beyond S. Then we have no possibility to measure the duration between any two different instants during the evolution of (the whole of) S, since we cannot attach a noninteracting free classical particle, or any other clock, to S. The evolution of S is then necessarily reparametrization invariant, so that only solution manifolds (without a parametrization) are physically meaningful. – Clearly, giving up extensibility in classical mechanics requires a justification, since empirically it is plainly wrong; it is however a fruitful guiding idea for the quantum regime /1.3.2.3/.

[29]The consideration of all degrees of freedom is essential in this approach. Barbour also argues against the possibility of an internal time; the present author disagrees with this claim [Bar94]:

> "I think it is wrong to attempt to identify certain degrees of freedom as a clock and use them to describe the behaviour of the remainder. Any satisfactory operational definition of time must involve *all* the degrees of freedom of the universe on an equal footing, so any division into clock and residual measured system is misleadingly artificial."

[30]Hence the full symmetry group of Machian mechanics is the *Leibniz group [BB77],* consisting of rotations, time dependent displacements and time reparametrizations, plus spatial scale transformations. See also Earman [Ear89], who discusses a hierarchy of spacetimes (Machian, Leibnizian, Maxwellian, neo-Newtonian, full Newtonian, Aristotelian) with decreasing symmetry groups. Except for the Machian spacetime, all have a fixed time metric.

A reconsideration of the foundations of mechanics, which does not require the notion of external time, has been undertaken by Rovelli [Rov01b,Rov02a,Rov02b]. He argues that there are two kinds of observables: complete observables and partial observables. Partial observables are quantities, which can be measured, while complete observables can be predicted by the theory. E.g. in the case of a single harmonic oscillator in one dimension the function

$$f(q,t;A,\phi) = q - A\sin(\omega t + \phi)\ ,$$

where q is the position and t the time, is a complete observable, since its value is predicted by the theory:

$$f = 0$$

Here A and ϕ are parameters of a motion. The quantities q and t are partial observables, since their values can be measured, but not predicted by the theory. There is no fundamental reason to consider q as being dependent on t, or vice versa; it is only the relation between q and t, given by f, which is meaningful. This idea that only a relation between partial observables makes a complete observable can be generalized and was applied to special relativity, general relativity and quantum theory.

1.3.2.2 General relativity

In analogy to classical mechanics a Machian theory of general relativity has been worked out, see [Bar03] for a recent overview and references. One assumes spacetime to be globally hyperbolic with compact spatial sections, so that it can be foliated into a family of spacelike hypersurfaces (of simultaneous events according to some synchronization) with an induced Riemannian 3-metric on each hypersurface, and the family is parametrized by a 'time' parameter. (In Hamiltonian gravity the initial conditions for time evolution are given by a 3-metric on a hypersurface and a canonically conjugate momentum.) The equivalence classes (3-geometries) of all 3-metrics on a hypersurface under diffeomorphisms on this hypersurface constitute an (instantaneous) physical configuration, i.e. a point in shape space. Starting from a 3-geometry (without time), time evolution results from the Baierlein-Sharp-Wheeler action principle (which is derived from the Einstein-Hilbert action principle). A change of foliation of spacetime then corresponds to a reparametrization of time. Considering conformal transformations of 3-metrics, problems arise in rendering general relativity fully scale-invariant.

Besides modifications towards relationalism, any theory of gravity is already relational in the following sense: Local physics (in particular, proper time for a local observer) depends on the presence of energy-momentum in surrounding regions, since “gravity cannot be switched off”. In general relativity there is no fixed background relative to which a body moves, but the presence of the body has an impact on the gravitational field. Only in the (classically admissible) idealization of infinitesimally small test masses can a time be defined for these test masses, which is independent of them; it would coincide with the reading of an infinitesimal clock at the position of the test mass, and defines proper time. In general, and in particular for real, massive and extended clocks, the evolution of the gravitational field and of the clock have to be considered together.

But there is another reason to call general relativity a relational – as opposed to substantivalist – theory. This is different from the absolute-relational controversy about the existence of absolute position, time, velocity, acceleration or rotation. The point at issue is whether spacetime itself does exist independently of events taking place *on* it (substantivalism), or spacetime is made up from relations among events (relationalism).

During the development of general relativity Einstein was facing this problem. Until 1912 he was searching for a general covariant theory of gravity, but then inventing the *hole argument* he convinced himself that the theory could not be general covariant. In 1915 he finally returned to a general covariant formulation and general relativity was born.

Let us briefly describe the hole argument (see e.g. [Nor99, Ear89, GR99, EN87, Nor93]) and its interpretations. One starts with a general relativistic description $\langle \mathcal{M}, g, T \rangle$ of reality, where $\mathcal{M}$ is the spacetime manifold, and the metric tensor g and the energy-momentum tensor T are related through Einstein's equation. And one assumes that there is a region H ("hole") in which T does vanish. Then one can choose arbitrary diffeomorphisms d on $\mathcal{M}$, which do differ from the identity only inside H. For any such d the descriptions $\langle \mathcal{M}, g, T \rangle$ and $\langle \mathcal{M}, d^* g, d^* T \rangle = \langle \mathcal{M}, d^* g, T \rangle$ are mathematically equivalent, since Einstein's equations are due to general covariance invariant under (active) diffeomorphisms. What does this mean? The answer is that if the mathematical description is correct, then one can change the metric $g \mapsto d^* g$ without affecting physics. Hence, on the one hand g is gauge-dependent, and on the other hand the points inside H cannot be individuated by the values of g, and since $T = 0$ not at all.

The relational interpretation therefore claims that spacetime does not exist a priori, only the gravitational field, from which spacetime may emerge. Substantivalism contrariwise takes spacetime to exist and have metrical properties, while the gravitational field does not exist. As noted in [Rov03], the distinction between both interpretations is reduced to semantics. – A physically meaningful statement about a field ϕ at an event x involves a relation of a value of ϕ with four parameters determining the event x, the latter being either four values of the gravitational field g (in the relational interpretation), or the metric g of spacetime (in the substantivalist interpretation). (A question in both interpretations then is how to individuate spacetime points and what is observable in general relativity [Lus03, LP03, Lus02, PV02, Rov91d, Rov91b, Ber61].)

1.3.2.3 Quantum mechanics

When one adjoins a classical clock C to a quantum mechanical system S in order to measure a duration between observations (instantaneous measurements) of S, one assumes that C and S do not interact. The correlation between observation results and readings of the clock takes place at the classical level, with approximately vanishing interaction. In a genuinely quantum description one cannot neglect mutual influences between (measurements of) S and C, and we have to treat them together as a whole quantum system W. If the state of W separates into a tensor product of states of S and C, then measurements of S and C remain uncorrelated. If W is in an entangled state w.r.t. the decomposition into S and

C, then a measurement of C does project out not only a state on C, but also the corresponding relative state of S; subsequent (or simultaneous) measurements of S are therefore in general correlated with "time" measurements on C, hence the state of S is "time dependent". We say that C provides an *internal* time (within W) for S. In spite of this interpretation there is no fundamental property which qualifies C as a clock and the observable measured on it as time observable. In the next subsection /1.3.3/ we will characterize the conditions such an observable has to fulfil in order to be a reasonable measure for time.

As above (in classical mechanics) we now assume that there is nothing beyond W, i.e. we describe W as an *endosystem*[31]. As a first consequence we have no external time, and evolution becomes reparametrization invariant. Instantaneous states w.r.t. a fictitious external time lose their meaning; only orbits remain as physically meaningful descriptions of the state of W. Within an orbit there is a relation between partial observables (like e.g. that of S and C). – On the other hand, denying extensibility implies that there are no observers and no measurement devices outside W. What then, do observables mean for the endosystem W? The term certainly loses its connection with actual measurements. Primas [Pri94c, AP02] proposes to call the endophysical observables *potential properties.* He shows that in a realistic interpretation of quantum mechanics one has to choose a context in a subjective way. This choice corresponds to the attribution of values to potential properties.

We therefore have two aspects of relationalism in quantum theory [Wei01]:

(i) A relation between simultaneous relative states of subsystems, i.e. a simultaneity relation with respect to value-definiteness of subsystem observables.
(ii) The absence of a relation between the system and anything external to it; this renders any absolute meaning of time unphysical.

For quantum mechanics this relational approach has been worked out for the first time by Rovelli. In his path-breaking article "Quantum mechanics without time: A model" [Rov90] (see also [H9́1, Rov91a, Rov91c, Rov88]) he investigates the conditions under which time emerges from a finite system consisting of two uncoupled quantum oscillators with equal frequencies in one dimension. As this model will be studied in detail below /3/, we sketch it here only briefly[32]:

- First one considers the model at the classical level with an energy constraint for the closed total system. Consequently the system has two independent constants of motion: the relative phase of the oscillators and the energy (or amplitude) of one of them. (The absolute phase is physically vacuous and the energy of the other oscillator follows from the constraint. Classically one could of course adjoin a (ideal) clock and observe the motion of both oscillators in time.) - What remains from a timeless point of view is the

[31]We use the term in the sense of Primas, see [Pri94c] for its comparison with the notion of Rössler and Finkelstein.

[32]Whereas this research was primarily conducted in the context of quantum gravity, we are here concerned with its implications for non-gravitational quantum physics. Thereby we pick up a hypothesis of Rovelli, that the problem of time might have to be solved in the quantum realm.

correlation between the elongations q_1 and q_2 (partial observables) of both oscillators when measured simultaneously. One can then consider $t := q_1$ as time observable which allows (albeit ambiguous) predictions of q_2. There is no duration required for the total system, just a simultaneity relation for both subsystems.

- In the second step the system is quantised in a non-canonical way: The algebra of constants of motion is mapped to an algebra of Hilbert space operators such that the energy constraint is fulfilled for all physical states /3.2.2/. If one requires the total system to be in certain (semiclassical) states, then the expectations of the quantised elongations, $\hat{q}_1$ and $\hat{q}_2$, are correlated and allow to define an approximate time $t := \langle \hat{q}_1 \rangle$. Rovelli shows that (using the semiclassical state) the evolution of $\hat{q}_2$ in the time parameter t is approximately unitary. In the classical limit for $\hat{q}_1$ this agrees with the well known unitary evolution in classical environments with respect to a classical time parameter, showing that the internal quantum time has the correct classical limit.
- The underlying simultaneity relation is not derived in any way, but taken as an empirical valid fundamental structure of possible observations of q_1 and q_2, which are in addition supposed to happen at a single instant. The approach works without an absolute duration, it relies just on instantaneity and simultaneity.

The main problem with this approach is to justify the choice of (a class of) states for the total system which induce a correlation between subsystem observables, so that for macroscopic systems there is a time observable, which can be identified with the empirical classical time /4/.

1.3.3 Relational time from simultaneity

In /1.1/ several time structures were discussed. Here we define a general time structure for the relational approach based on the primitive notions of observation and simultaneity. We thereby reduce time to its core.

By "observation" we mean the event of a measurement and we denote the set of observations by $\mathcal{O}$. We do not assume that such an event is instantaneous in an absolute sense, since we have no notion of an absolute time with respect to which we could define instantaneity. We will however construct a notion of an instant w.r.t. all observations below. Since each measurement has an outcome, we also have a map $r : \mathcal{O} \to R,\ O \mapsto r(O)$, which gives the *result of an observation* in a set R, often the reals $\mathbb{R}$. In nonrelativistic classical mechanics an observation could be that of the x-coordinate of a particle, and the result of the observation would be a real number. R could e.g. also contain values in $\mathbb{R}^3$, or the set $\{0, 1\}$ for binary measurements. In nonrelativistic quantum mechanics we mean by observation not an actual measurement, which would disturb the system, but a potential one, its "result" being the expectation value of the measured observable.

In order to compare observations taking place somewhere in space we assume that we are given a simultaneity relation. In /1.2/ we have discussed for different theories how simultaneity can be empirically established. Simultaneity is a reflexive, symmetric and transitive binary relation in $\mathcal{O}$, i.e. an equivalence relation, which we denote by $\sim_S$.

We begin our discussion in nonrelativistic physics where time and space can be separated and a single simultaneity relation is sufficient.

Let $\mathcal{O}$ be a set of observations, $r : \mathcal{O} \to R$ a result function and $\sim_S$ a simultaneity relation. We furthermore assume that $\mathcal{O}$ is equipped with an *observable structure*, i.e. with a set $\mathcal{A}$ of subsets of $\mathcal{O}$, whose elements $A \in \mathcal{A}$ we call *observables*. This notion of observable coincides with the usual one, where two observations belong to the same observable, if the measurement procedures are essentially the same[33]. Often for $A \in \mathcal{A}$ the set $r(A)$ is a metric space.[34]

From the observable structure and the values of observations we clearly have for any $A \in \mathcal{A}$ and any $O_1, O_2 \in A$ an equivalence relation

$$O_1 \sim_A O_2 \;:\Leftrightarrow\; r(O_1) = r(O_2)\,.$$

This does not mean that the observations O_1 and O_2 contain the same information about the measured system: they need not be simultaneous. If $O_1 \sim_A O_2$ and $O_1 \sim_M O_2$, then O_1 and O_2 are *physically equivalent*. (One could restrict all considerations to classes of physically equivalent observations, but we are doing without.)

Definition 1.3.1. We call a quadruple $(\mathcal{O}, \mathcal{A}, r, \sim_S)$ consisting of a set of observations $\mathcal{O}$, an observable structure $\mathcal{A}$, a result function r and a simultaneity relation $\sim_S$ a *description of a physical system.*

We now ask, whether the values of observations belonging to an observable $A \in \mathcal{A}$ allow to distinguish different equivalence classes of a given equivalence relation $\sim$ in $\mathcal{O}_0 \subset \mathcal{O}$, to be more general. (In a moment we will restrict to simultaneity, $\sim = \sim_S$.) We denote the set of equivalence classes as $\mathcal{X} := \frac{\mathcal{O}_0}{\sim}$ and want to find an observable $A \in \mathcal{A}$ such that

(i) for all $x \in \mathcal{X}$ there exists an observation $O \in \mathcal{O}_0$ with $O \in A$ and $O \in x$ (observations of A parametrize $\mathcal{X}$),
(ii) $\forall O_1, O_2 \in A\ (O_1 \sim O_2 \Rightarrow r(O_1) = r(O_2))$ (the result of observations of A depends only on the equivalence class),
(iii) for all pairs of observations $O_1, O_2 \in \mathcal{O}_0$ with $O_1 \nsim O_2$ there exist corresponding equivalent observations of A with different results, i.e. $\exists O_1', O_2' \in A \cap \mathcal{O}_0 : O_1' \sim O_1 \wedge O_2' \sim O_2 \wedge r(O_1') \neq r(O_2')$ (observed values of A separate the equivalence classes in $\mathcal{X}$).

Example. Consider Newtonian mechanics for massive particles in three dimensions. Let $\mathcal{O}_0$ consist of observations of the x-coordinate of the center of mass

[33] Of course, an observable may be measurable using different measurement procedures.

[34] The set of observables $\mathcal{A}$ has further structure, e.g. in nonrelativistic classical (quantum) mechanics the observables form a Poisson algebra (Jordan-Lie algebra). In particular, $\mathcal{A}$ often has a topological structure; this is however of no importance here.

in a coordinate system and, say, the angular momentum of a single particle. We want to characterize the equilocality relation for all observations by means of an observable. We choose the position observable $A := X$, which consists of all observations of the particle position with real values as results.

This is trivial, since we have a position variable at our hand. Suppose now that we do not. If we know that the electrostatic potential Φ increases monotonously along the x-axis, then we could as well use observations of Φ to decide whether equilocality holds. That is true, because *i)* for each position we can observe Φ, *ii)* there is only one value of Φ for each position, and *iii)* for different positions we observe different values of Φ.

Since there is no obvious natural time observable, we now consider the above question for the simultaneity relation $\sim_S$, i.e. we want to characterise equivalence classes under simultaneity by the values of a physical observable. We can do this in the same way as above (e.g. characterising observations by simultaneous measurements of the position observable in the case of a free particle with a nonvanishing velocity), but we want to remove unphysical structure from our simultaneity relation: It may happen that two equivalence classes have the same values for each observable. Assuming that there is no more physical structure than contained in the set of observations, we can then identify these equivalence classes.

Example. Assume that we have an oscillator in one dimension and only observations of position and momentum. Then we cannot distinguish between different cycles of the motion. We identify one time observable as the combined position-momentum observable; we could as well choose the momentum observable together with the sign of the position observable, but the momentum observable alone would not qualify as a time observable. – If we had only observations of position, then the position observable would of course suffice to distinguish instants. (In the case of the oscillator we know however, that there is more observable structure which we have to "resolve" by a time observable.)

We come now to the definition of a relational time observable:

Definition 1.3.2. Let $(\mathcal{O}, \mathcal{A}, r, \sim_S)$ be a description of a physical system. Given $\mathcal{O}_0 \subset \mathcal{O}$, we define $\mathcal{T} := \frac{\mathcal{O}_0}{\sim_S}$ and call the elements $t \in \mathcal{T}$ *instants (of* $\mathcal{O}_0$*)*. $\mathcal{T}$ is called the *set of instants (of* $\mathcal{O}_0$*)* (cf. /1.1/).

Moreover, an observable $T \in \mathcal{A}$ is called a *(relational) time observable* for the set of observations $\mathcal{O}_0$, if

(i) for all $t \in \mathcal{T}$ there exists an observation $O \in t \cap T \cap \mathcal{O}_0$
(observations of T *parametrise* $\mathcal{T}$*)*
(ii) $\forall O_1, O_2 \in T \; (O_1 \sim_S O_2 \Rightarrow r(O_1) = r(O_2))$
(the result of observations of T *depends only on the instant)*

(iii) if $O_1, O_2 \in \mathcal{O}_0$ with $O_1 \nsim_S O_2$ and

$$\begin{aligned} &\exists A \in \mathcal{A},\, O_1'' \in A,\, O_2'' \in A : \\ &\qquad O_1 \sim_S O_1'' \;\wedge\; O_2 \sim_S O_2'' \;\wedge\; r(O_1'') \neq r(O_2'')\,, \end{aligned} \tag{1.3.1}$$

then there exist corresponding simultaneous observations of T with different results, i.e.

$$\exists O_1', O_2' \in T \cap \mathcal{O}_0 : O_1' \sim_S O_1 \wedge O_2' \sim_S O_2 \wedge r(O_1') \neq r(O_2')$$

(observed values of T separate the equivalence classes in $\mathcal{T}$).

If we drop the relational requirement (1.3.1), then we call T an *intrinsic time observable.*

Because of condition (1.3.1) a relational time observable T is more general: It has to have different result values only if there is any observable, whose result values, observed simultaneously with those of T, differ. Time is thus a tool to structure actual observations, while a possibly too fine, unobservable classification through simultaneity is being coarse-grained.

Remark 1.3.3.

1. If T_1 and T_2 are both time observables for the observations $\mathcal{O}_0$, then we have a bijection of observations $T_1 \cap \mathcal{O}_0 \to T_2 \cap \mathcal{O}_0$ which at the same time is a bijection between their result values.
2. Given an observable $A \in \mathcal{A}$, we can associate to any observation $O \in A$ due to (i) an observation $O' \in T$ belonging to the same instant ($\sim_S$- equivalence class). And due to (ii) $r(O')$ does not depend on the choice of $O' \in T$, so that we have a mapping from observations of A to time values $t = r(O')$. This ensures that each observation happens at some “time” t.
3. One usually requires uniqueness of all observations of an observable at an instant, i.e. a generalization of (ii),

$$\forall A \in \mathcal{A}\, \forall O_1, O_2 \in A \; (O_1 \sim_S O_2 \;\Rightarrow\; r(O_1) = r(O_2)) \;. \tag{1.3.2}$$

   This guarantees that the *Heisenberg observables* $A(t) := \{O \in A : O \in t\}$ ($t \in \mathcal{T}$) for all $A \in \mathcal{A}$ have definite values, notwithstanding the possibility of different observations of the same observable at the same instant. Then there exists a injective mapping of the values of A (a value of A being defined as the value of an arbitrary observation in A) to the simultaneous values of T.
4. Many physical theories provide simultaneity and equilocality relations in the set of all observations. In nonrelativistic physics it is however not important where a physical quantity is measured, if it is the same quantity that is measured. E.g. the motion of a pendulum could be broadcasted to a distant observer, who can equally well observe the position of the pendulum.
5. It is straightforward to generalise the definition of a time observable to encompass several of the above defined time observables pasted together, so that they can measure time successively (comprising all instants). (This requires “consecutive pairs” of time observables to have common instants.)

Up to now we have not discussed the structure of the set of values of observations. Even if there were only a discrete set of values or only finitely many observations, then our definition of a time observable would make sense. Often the set of values of the observations of an observable $A \in \mathcal{A}$, which we denote by $R(A)$, has more structure, leading to more structure of the notion of time (cf. /1.1/):

If T is a nonrelativistic time observable and $R(A)$ and $R(T)$ are topological spaces, and if (1.3.2) holds, then the injection $t_T : R(A) \to R(T)$, $r(A(t)) \mapsto r(t)$ $(t \in \mathcal{T})$ may be open, whence $t_T^{-1} : r(t) \mapsto R(A(t))$ is continuous. If these maps are continuous for all A, the system has a *continuous evolution* w.r.t. the time observable T. If $R(A)$ and $R(T)$ are moreover differentiable manifolds, and if t_T^{-1} is a differentiable map, then we can take time derivatives and define e.g. velocities and use differential equations to describe evolution. While the relational time would also make sense for manifolds of higher dimensions, $R(T)$ is usually a one-dimensional manifold.

If $R(T)$ is a metric space with metric d_T, this defines a duration in the set $\mathcal{T}$ of instants:

$$d\,(t_1, t_2) := d_T\,(r(t_1), r(t_2)) \qquad (t_1, t_2 \in \mathcal{T})$$

If there are several time observables, their durations can be different: If $T^{(i)}$ $(i = 1, 2)$ are time observables and $R(T^{(i)})$ are differentiable manifolds and metric spaces such that the metric induces the topology, then the bijection between the instants of both time observables, $t^{(1)} \mapsto t^{(2)}$, induces a diffeomorphism $R(T^{(1)}) \to R(t^{(2)})$, i.e. the change between the two time observables amounts to a reparametrization of the time parameter, $r^{(1)}(t) \mapsto r^{(2)}(t)$.

Given an ordering relation $<$ on $R(T)$ one can also define a time order on $\mathcal{T}$:

$$t < t' \;:\Leftrightarrow\; r(t) < r(t') \qquad (t, t' \in \mathcal{T})$$

This does however not provide us with a substantial direction of time, since one can easily find another time observable T' with the same values and with the opposite ordering (see /2.1/).

We have now analyzed the reconstruction of the standard notion of time /1.1/ as seen from nowhen and from a relational point of view. Physical theories not only tell us how to verify simultaneity and how to measure observables; they also say which observations are possible and which of their values are consistent. This is the subject of a dynamical theory. Deterministic dynamical theories can predict or retrodict the values of all observations, given that the values are known at one instant, by *dynamical laws.* The consistent combinations of the values of all observations for a given physical system are called *solutions* or *orbits.* Orbits are independent of the choice of time observable. (They usually have the structure of 1-dimensional differentiable manifolds.)

Let us conclude this subsection with some remarks on time observables in relativity and quantum mechanics.

In special relativity the description of observations as seen from different Lorentz frames is equivalent. Each Lorentz frame defines a simultaneity relation and we can apply the above reasoning. A change of Lorentz frame also affects the equilocality relation, which like simultaneity is an equivalence relation among observations. Lorentz invariance of a theory can then be seen as the unobservability of certain changes of the simultaneity relation which are accompanied by certain changes of the equilocality relation.

Quantum mechanics puts strong restrictions on possible time observables, since only commuting observables are simultaneously observable. This is e.g. the case if all observations, except for those of the time observable, belong to a subsystem which is disjoint from the subsystem on which the time observable does operate (see /1.3.2.3/). If one wants to have a time observable for all observations, then the time observable must commute with all other observables, i.e. it must be a classical observable. We will return to this idea in chapter /4/.

CHAPTER 2

Problems with time

This chapter gives a brief account of three main problems connected with time: The arrow of time, time measurement, and quantum gravity.

... beyond all day-to-day problems in physics, in the profound issues of principle that confront us today, no difficulties are more central than those associated with the concept of 'time' ...

John Archibald Wheeler [Whe94, p. 7]

In our culture time is experienced as flowing, it has a distinguished present, and it allows for memory of the past as well as expectations for the future. Why is that? This raises, among others, the question for a physical basis of the direction of time, which will be treated and discussed also from the relational point of view in /2.1/.

A second problem is concerned with the measurement of time in quantum physics. There is no time operator although clocks do exist /2.2/. This problem is especially important within the relational approach to time.

A main contemporary motive for considering time problems is rooted in quantum gravity /2.3/. For a generally covariant quantum theory of gravity makes an understanding of the true nature of time indispensable. We sketch the main problems.

Another problem connected with time will be considered later /4.2/: The unitary dynamics of endosystems, which is a main obstacle to a consistent description of the measurement process in quantum mechanics.

2.1 The arrow of time

Everyday experience endorses the passage of time, e.g. a glass can break, but we never see one building up from pieces "spontaneously". Can this be explained on the basis of our microscopic physical theories? At first sight the decay of an atom seems to prove that time has a direction: While relaxation occurs spontaneously, excitation does not. But in fact this rests on the assumption that the boundary conditions of the equations of motion lie in a certain ("time forward") class: We consider the undecayed state as "initial" and require all other states during its dynamics to lie in the "future". Tautologically, then, atoms decay in the future. Substituting the words "initial" and "future" with "final" and "past" would result in a physically equivalent description of the decay process, however with an apparently different unphysical meaning.

In order to avoid wrong intuitions it is wise to use a time symmetric language. In [Alb00] the notion *dynamical conditions* was introduced: Dynamical conditions comprise all information necessary to apply dynamical laws in a predictive way (be it towards the future, i.e. prediction, or towards the past, i.e. retrodiction), e.g. instantaneous positions and momenta in Newtonian mechanics, or instantaneous electric and magnetic fields and charge and current densities in classical electrodynamics, or the state vector in nonrelativistic quantum mechanics.

The dynamical laws of most fundamental theories do not distinguish between prediction and retrodiction and therefore cannot justify everyday experiences, like e.g. breaking glasses. The debate on the arrow (or direction) of time aims at bringing dynamical laws into agreement with everyday experiences. This debate (which has a long history and continues until today [Vaa02]) has suffered from many imprecise statements (e.g. using hidden assumptions, or confusing different concepts), we therefore recast some definitions in a rather mathematical language, following largely [Alb00] and [CLL03b]. After explaining (partial) time-reversal invariance /2.1.1/ we make the important distinction between irreversibility /2.1.2/ and the arrow of time /2.1.3,2.1.4/.

2.1.1 Time-reversal invariance

Intuitively, time-reversal invariance of a physical process amounts to the possibility of watching a movie of this process backward and finding it to be in accordance with all dynamical laws.

E.g. in Newtonian mechanics the instantaneous pictures contain information about the positions of all particles. As pointed out by Albert [Alb00, ch. 1], velocities are not part of an instantaneous configuration, which is per definition a *state*[1]. Velocities always involve two different instants, which can be arbitrarily close. Since velocities are the time derivatives of positions, they change their signs, when the movie is watched backward. They do not enter into Newton's

[1]Albert [Alb00] and Zeh [Zeh99] use the term *state* synonymous with configuration, in contradistinction with a point in phase space.

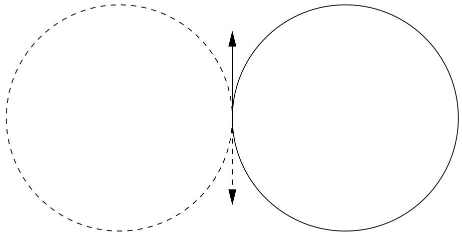

FIGURE 2.1.1. A charged particle moving along a circular path (solid line) with an initial velocity (solid arrow) in a plane perpendicular to a homogeneous magnetic field, and the path of the time reversed motion (dashed line) with an opposite initial velocity (dashed arrow): Under time-reversal the path does not remain invariant.

Second Law (assuming an autonomous system, i.e., an endosystem with absolute time),

$$\frac{\mathrm{d}^2 x_i(t)}{\mathrm{d}t^2} = F_i(x_1(t), \ldots, x_N(t)) ,$$

which only contains second time derivatives. It is equivalent to the following equation:

$$\frac{\mathrm{d}^2 x_i(-t)}{\mathrm{d}t^2} = F_i(x_1(-t), \ldots, x_N(-t))$$

Thus $\mathbf{x}(t)$ fulfills Newton's Second Law, iff $\mathbf{x}(-t)$ fulfills Newton's Second Law[2]; this means that for any sequence $\mathbf{x}_i, \ldots, \mathbf{x}_f$ of instantaneous configurations also the reversed sequence $\mathbf{x}_f, \ldots, \mathbf{x}_i$ can happen. We therefore call Newtonian mechanics *strictly time-reversal invariant.*

Many theories turn out to be not strictly time-reversal invariant: Consider e.g. the Newtonian mechanics of point particles subject to external electric and magnetic fields. The instantaneous states consist of the positions of all particles and the magnitudes and directions of the electric and magnetic fields everywhere in space. Upon time-reversal the magnetic field doesn't change its sign, which in general leads to a different dynamics (cmp. fig. 2.1.1). This theory is not time-reversal invariant.

Another example is Maxwell's electrodynamics. Here the fields are not pregiven, external and instantaneously measurable quantities, but they have physical sources: The electric field is generated by an instantaneously measurable charge distribution and therefore is itself instantaneously measurable. The magnetic field on the other hand is given by the motion of charges and the rate of change of the electric field, and thus is not instantaneously measurable. I.e., the magnetic field is a dynamical condition and not fixed by a choice of an instantaneous state. We are therefore free to choose its transformation behaviour upon time-reversal, so that the equations of motion remain invariant under time-reversal. This is

[2]The choice of the origin of time with respect to which $t \mapsto -t$ is performed is arbitrary, since we assume time-translation invariance.

indeed possible: We just have to change the sign of the magnetic field. This restores Maxwell's equations after a time-reversal and at the same time leaves the instantaneous state invariant. By transforming an instantaneously unobservable quantity we therefore have restored time-reversal invariance.

Albert rightly calls a time-reversal invariance that holds for instantaneous states (and not necessarily for all dynamical conditions) *partial time-reversal invariance*, contrary to many textbooks, which just call it "time-reversal invariance".

The point is that *already partial time-reversal invariance conflicts with everyday experiences* [Alb00]. E.g., given two configurations, one with a glass on the table, and another one with a broken glass on the floor, we can tell which was later: the broken glass; i.e., here we observe only the positions of particles. Furthermore, it is widely believed that quantum measurements basically always involve position measurements. If the classicality of macroscopic phenomena arises from localization processes, then the partial time-reversal invariance of quantum mechanics contradicts everyday experience, too.

The combination of time reversal ($I : t \mapsto -t$) and another transformation J, which restores partial time-reversal invariance, is known as *time-reversal operator* K and acts on solutions $f(t)$, $f(t)$ being a state (i.e. a point in configuration space) for each t, in such a way that its image $f_K(t)$ is also a solution.

For nearly all fundamental theories there are transformations of quantities other than positions, which can restore partial time-reversal invariance after a reversal of time. (See [COE01] for a general definition of the time-reversal operator and examples.) To give another example, in nonrelativistic quantum mechanics the time-dependent Schrödinger equation

$$\mathrm{i}\hbar\partial_t\psi(t) = H(t)\psi(t) \tag{2.1.1}$$

is equivalent to its time reversed and complex conjugated form

$$\mathrm{i}\hbar\partial_t\overline{\psi(-t)} = H^+(-t)\overline{\psi(-t)} \ . \tag{2.1.2}$$

If $H(t) = H^+(-t)$, $\psi(t)$ in (2.1.1) and $\overline{\psi(-t)}$ in (2.1.2) evolve according to the same dynamical law. The time reversal operator can then be defined in the *position* representation as Wigner's anti-unitary operator

$$K : \psi(x,t) \mapsto \overline{\psi(x,-t)} \ .$$

Then ψ and $K\psi$ have the same expectation values for *position* operators, while the expectations for momentum operators differ by a sign[3]. The ambiguity of momentum is no problem, because position and momentum are not simultaneously

[3] If we assume that position eigenstates $|q\rangle$ remain unchanged upon complex conjugation, the latter can also be seen from a complex conjugation of

$$|p\rangle = \int \mathrm{d}q\, e^{\mathrm{i}pq}\, |q\rangle \ ,$$

which gives $|-p\rangle$.

dispersion-free. Note that the state vector ψ is a dynamical condition, since the equation of motion is of first order in time.[4]

Another example is Hamiltonian mechanics, which treats position and momentum on an equal footing (in a "symplectic" way) and therefore – contrary to Newtonian mechanics – is not strictly time-reversal invariant. In order to restore partial time-reversal invariance the sign of all momenta has to be inverted.

For certain decay processes of elementary particles (e.g. Kaons), which violate the combined charge-parity (CP) symmetry, partial time-reversal invariance is expected to be violated, as a consequence of CPT conservation.

Summarizing, we have the following definition:

Definition 2.1.1. A given dynamical law based on the standard notion of time (with parameter t) /1.1/ is *strictly time-reversal invariant*, if it is invariant under the transformation $I : t \mapsto -t$, i.e. if for each solution $f : \mathbb{R} \to \Gamma,\ t \mapsto f(t)$, where Γ is the set of dynamical conditions, $f \circ I : \mathbb{R} \to \Gamma,\ t \mapsto f(-t)$ is also a solution. In other words, the set $\mathbf{S}$ of solutions of the theory fulfils $\mathbf{S} = \mathbf{S} \circ I$.

The dynamical law is *partially time-reversal invariant*, if

(i) there is an operator $J : \Gamma \to \Gamma$, which maps dynamical conditions into dynamical conditions such that the instantaneous state (instantaneously measurable configuration) remains invariant, i.e. $P \circ J = P$, where $P : \Gamma \to X$ maps a dynamical condition to the corresponding instantaneous state.

(ii) for each solution f, $J \circ f \circ I$ is also a solution, i.e. $\mathbf{S} = J \circ \mathbf{S} \circ I$.

The map $K : \mathbf{S} \to \mathbf{S},\ f \mapsto J \circ f \circ I$ is then called *time-reversal operator.*

Remark 2.1.2. In phenomenological thermodynamics a solution $f \in \mathbf{S}$ is given by the time dependence of the state functions, among them the entropy S, which according to the second law does not decrease, $\frac{\mathrm{d}S(t)}{\mathrm{d}t} \geq 0$. Thermodynamics is not strictly time-reversal invariant, since the time-reversed solution $f \circ I$ has the entropy $S'(t) = S(-t)$, which contradicts the second law (except if S is constant): $\frac{\mathrm{d}S'(t)}{\mathrm{d}t} = \frac{\mathrm{d}S(-t)}{\mathrm{d}t} = -\frac{\mathrm{d}S(t')}{\mathrm{d}t'} \leq 0$

To restore partial time-reversal invariance one has to change the sign of S, $KS(t) = -S(-t)$. On the other hand entropy is related to heat transfer δQ and absolute temperature T via $\mathrm{d}S = \frac{\delta Q}{T}$; if δQ does not change its sign, then $KT(t) = -T(-t)$ must hold. If the time-reversal operator is appropriately defined for all other thermodynamical quantities, this non-standard usage of negative absolute temperatures has been shown to be consistent (see [Kro85, p. 128]).

In the abstract relational approach developed in /1.3.3/ each observation is in one and only one equivalence class of the simultaneity relation and therefore is by assumption instantaneous. (Quantities such as momenta in mechanics require two observations at different instants.) Let a description $(\mathcal{O}, \mathcal{A}, r, \sim_S)$ of a physical

[4]According to [Alb00, p. 132] a time-reversal and time-translation invariant theory, whose instantaneous states are also complete dynamical conditions, cannot describe change with its dynamics.

system be given. (This is an abstraction of a solution, albeit reduced to actual observations.) Let furthermore T be a time observable for $\mathcal{O}$ with a time order $<$. We can then reverse the time order without affecting physics; this means that there is a physically equivalent time reversed description of the system: the relational formulation is automatically time-reversal invariant. The reason is that we have no external time against which a time-reversal would be physically meaningful.[5]

2.1.2 Irreversibility

As mentioned in the introduction to the previous chapter, the problem of linearity versus cyclicity of time has long been disputed in human history [F9̈9], often under the terms infinity and finiteness [Obe00]. Modern science is built on the assumption of linear time as represented by the standard notion /1.1/. If – as in the relational approach – time is physically observable within a system, then cyclicity or linearity must be properties of the solutions of the equations of motion of the system:

Definition 2.1.3. A solution of an equation of motion is *reversible* if it is a fixed point or a closed curve in phase space; otherwise it is called *irreversible.*

The *problem of irreversibility* consists in showing that a time-reversal invariant equation of motion (definition 2.1.1) admits an irreversible solution.

Irreversibility is desirable, because a reversible solution does not allow to define a nontrivial time order on all instants. (Note that irreversibility is independent of the time order. It has however often been mixed up [CL03] with the arrow of time /2.1.3,2.1.4/.)

Irreversibility is a problem, because of Poincaré's *recurrence theorem*[6], which states that in an isolated, finite mechanical system, *any* state will be revisited to arbitrary closeness an infinite number of times during the course of time evolution. For a quantum version of Poincaré's theorem cf. [Sch78, Duv02]. In a coarse-grained theory which takes into account our limited precision of measurements, we will therefore always detect recurrence, though possibly after an enormously long duration.

There are several ways to circumvent recurrence:

- *Infinite systems* can have an irreversible solutions, e.g. in scattering of two classical particles. Quantum field theory also allows for irreversible motions [Vit01a]. A well known example are scattering processes, which require a

[5]One can however ask, whether a reversal of the internal time (relative to other observations) leads to another description, which is admissible in the dynamical theory. Of course, this depends on the choice of time observable. If, e.g., the time observable T is given by observations of the position of a free particle, then nature proves to be T-reversal invariant.

[6]The so called reversibility objections (recurrence and microreversibility) against an arrow of time historically appeared for the first time in the discussion of Boltzmann's H-theorem [Dav74, ch. 3].

change of the unitary equivalence class of representations of the underlying field algebra.

- *Open systems* allow for irreversible solutions. An environment can e.g. effect damping of the system. In classical mechanics an external influence can cause a chaotic dynamics, whereby the system loses its memory; in the quantum domain this can be achieved through decoherence.
- An interesting approach [Vit01b, Vit01a] to simulate an "effective" environment of a quantum system doubles the degrees of freedom of the system in order to compensate e.g. the energy flux to/from the hypothetical environment. Thereby a dissipative system can be quantised, while its copy has the time-reversed dynamics.
- Another possibility, which formally works for isolated, finite systems, is to *modify the dynamical theory* in such a way that it becomes irreversible, using dynamical semigroups [Ali02, Lin76]. The dynamics is chosen such that it mimics the influence of an environment.
- A similar, but different method in the quantum realm, which also modifies the dynamics is known as the rigged Hilbert space approach [Boh99, B̈78]. Here the usual Schrödinger equation is considered with asymmetric boundary conditions. Two Gelfand triplets are used, one for the prepared state ("in state") and one for the detected state ("out state"). Time evolution then splits into two semigroups (towards past and future, respectively). In particular, decay processes can be described within this theory. An arrow of time (see below) must however be postulated [Bos00].
- Another modification of quantum dynamics, which introduces a probabilistic element through a random local collapse of the wave function, is known as spontaneous collapse [GRW86].
- Of the several other approaches to embody irreversibility in the quantum domain we just mention one, which relies on K-flows, i.e. on a certain shrinking of the algebra of observables in time. See [Pri97, LM85] for details.

The approaches with a modified dynamics are called intrinsic, while those relying on openness are called extrinsic. Extrinsic irreversibility usually involves ontic as well as epistemic state concepts (see [ABA00], where some of the above enlisted approaches are discussed and compared).

From the relational point of view /1.3/ open system approaches are ruled out. On the other hand the intrinsic approaches do not allow to determine the time of a state from within the system, which is however required in the relational approach. We are then led to the consideration of infinite systems.

In the abstract relational approach /1.3.3/ a description of a physical system with time observable T is irreversible, if the observed T-values can be equipped with a nontrivial time order, which does distinguish different instants.

2.1.3 The arrow of time in classical physics

As pointed out in [CLL03b], in order to explain the apparent passage of time, i.e. why certain processes (like breaking glasses) are observed and their time-reversed images are not, one has to solve the problem of irreversibility /2.1.2/, but this is not enough: One also needs the existence of an arrow or direction of time, which allows to distinguish the two possible time orderings. – Let us make this more precise.

Known dynamical theories only relate observations to conventional time order, but each time order has an "opposite" one: From a notion of time with a (partial) time order $\leq$ one can always define $a \preccurlyeq b \;:\Leftrightarrow\; b \leq a$ and show that $\preccurlyeq$ is a (partial) time order, too. A time order is just a conventional description, in analogy to choosing coordinates on a manifold. In order to select one of the two orderings on a physical basis, i.e. in order to define a physical direction of time, one needs a *physically substantial* argument telling which one of two arbitrary instants a and b is the past/future one.

Definition 2.1.4. Let a physical theory with the standard notion of time be given. Let $\mathcal{T}$ be the set of instants and $M \subset \mathcal{T} \times \mathcal{T}$ a time order, denoted by $\leq$. Let $\mathbf{S}$ be the set of solutions

$$f : \mathcal{T} \to \Gamma$$

of the equations of motion of the theory, where Γ is the set of dynamical conditions (phase space in classical mechanics), and let

$$P : \Gamma \to X$$

map each dynamical condition to the instantaneous state (instantaneously observable configuration), which it represents. Here X is the set of instantaneous states.

We say that the theory has an *arrow of time* (or equivalently, a *direction of time*), if there is a (nonlocal in time) *observable* (correlation)

$$A : X \times X \to \{+1, -1\} \ ,$$

which for all $t_a, t_b \in \mathcal{T}$, $t_a \neq t_b$ and all $f \in \mathbf{S}$ fulfils:

$$A\left(P \circ f(t_a), P \circ f(t_b)\right) = \begin{cases} +1 & \text{if } t_a \leq t_b \\ -1 & \text{if } t_b \leq t_a \end{cases}$$

If $A\left(P \circ f(t_a), P \circ f(t_b)\right) = +1$, we denote the arrow of time as $t_a \overset{A}{\rightarrowtail} t_b$.

If $A\left(P \circ f(t_a), P \circ f(t_b)\right) = -1$, we denote the arrow of time as $t_b \overset{A}{\rightarrowtail} t_a$.

If $t_a \overset{A}{\rightarrowtail} t_b$, we say that t_a lies in the *(substantial) past* of t_b, or that t_b lies in the *(substantial) future* of t_a.

Remark 2.1.5. Clearly, the arrow of time can be used to select one time order on a physical basis. The value of A does only depend on a pair of instants, and not on the choice of solution f. This is a desired property, since for single solutions (or special classes of solutions) an arrow of time could often be defined trivially (see the following example).

Apart from the arrow of time itself only instantaneously observable quantities do enter into our definition.

Example 2.1.6. Consider classical mechanics of a free particle in one dimension and superficially restrict the set of solutions to just those with positive velocity v: The solutions read $x(t) = vt + x_0$, and the dynamical conditions are $f(t) = (x(t), v)$. Since position x is an observable, we can immediately define an arrow of time $A\,(P \circ f(t_a), P \circ f(t_b)) = A\,(x(t_a), x(t_b)) := \text{sign}\,(x(t_b) - x(t_a)) = \text{sign}\,(t_b - t_a)$, which is independent of the chosen solution. (If we allow for solutions with negative velocities, we cannot define an arrow of time, see below.)

A more realistic case is phenomenological thermodynamics, where an arrow of time can be defined from the *observable* entropy S, which due to the Second law increases along one time order $\leq$ only:

$$S(t_1) \leq S(t_2) \;\Leftrightarrow\; t_1 \leq t_2 \tag{2.1.3}$$

Correspondingly, one speaks of the *thermodynamical arrow of time*. If one considers thermodynamics as based on the statistics of a microscopic theory, then one has to decide whether the statistics is the appropriate description of reality, or the microstate. In the latter case there is no arrow of time, since the fundamental theories are partially time-reversal invariant. In the former case there is also no arrow of time, since probabilities are invariant under time reversal and an entropy defined from statistics increases towards the future as it does towards the past.

Remark 2.1.7. If a theory based on the standard notion of time has an observable $S : X \to \mathbb{R}$, whose values increase strictly monotonously along one time order for all solutions $f : \mathcal{T} \to X$, see (2.1.3), then this theory has an arrow of time defined by

$$A(t_a, t_b) := \text{sign}\,(S(f(t_b)) - S(f(t_a))) \ .$$

Observations, which are genuinely nonlocal in time [Atm97, AA98], do not exist in known fundamental theories. The above arrow of time A combines two instantaneous observations of the observable S. It is a common strategy to obtain such an S (whose monotonicity behaviour is isomorphic to a time order) by restricting the set $\mathbf{S}$ of solutions. In particular one imposes boundary conditions guaranteeing the monotonicity.

A well known example is classical electrodynamics, where advanced solutions are discarded. The waves are assumed to propagate from macroscopic coherent sources in the past to the future, and the macroscopic sources in the future are assumed to be incoherent [Pri96, ch. 3]. (This applies to wave theories in general, and is called the *radiation arrow of time*.) – To justify this asymmetry it has been argued that due to the expansion of the universe radiation carries away coherence beyond the event horizon of an observer, so that the increase in (observable) entropy parallels the expansion of the universe [Obe00]. The loss of coherence is based on the assumption that the outgoing (beyond the event horizon) radiation is coherent, while the incoming is not.

Another example is the decay of an unstable state: Essentially a two-level system is interacting with an environment and one assumes the state with an excited two-level system and no excitation in the environment to be the "initial" state, while the state where the excitation has been transferred to the environment is considered to be the "final" one. The time asymmetry is obviously introduced by hand through the choice of asymmetric boundary conditions. With suitable other boundary conditions nothing would prevent a particle from becoming excited in the future. – The same argument applies to the process of decoherence, where a state with localised coherence is considered to be "initial", while a unitarily evolved state with delocalized coherence is dubbed "final".

Lemma 2.1.8. Let a theory have an arrow of time A. Then all solutions of the equations of motion of this theory are irreversible.

PROOF. Assume there is a fixed point solution $f_0 \in \mathbf{S}$. Then $A(t_a, t_b) = A(t_b, t_a)$ and A cannot be an arrow of time. Assume next that there is a closed solution $f_1 \in \mathbf{S}$. Then there are instants $t_a \neq t_b$ with $f_1(t_a) = f_1(t_b)$, and again $A(t_a, t_b) = A(t_b, t_a)$. Hence all solutions must be be irreversible. □

The *problem of the arrow of time* consists in deciding whether an arrow of time does exist in fundamental physics. The crucial point in the definition is that the time order must be observable. The main obstacle is time reversal invariance:

Proposition 2.1.9. If a theory based on the standard notion of time has a partially time-reversal invariant dynamical law, then it does not possess an arrow of time.

PROOF. By assumption for every solution $f : \mathcal{T} \to \Gamma$ also $f_K = J \circ f \circ I$ is a solution. We choose an instant $t \in \mathcal{T}$ with $t > 0$, and as conventional time order the order of the real numbers (such that $-t \leq t$) and assume that there is an arrow of time A. Then we have $A(P \circ f(-t), P \circ f(t)) = +1$, since $-t \neq t$ and $-t \leq t$, and on the other hand $A(P \circ f(-t), P \circ f(t)) = A(P \circ J \circ f(-t), P \circ J \circ f(t)) = A(P \circ f_K(t), P \circ f_K(-t)) = -1$, hence a contradiction. This result does not depend on the choice of conventional time order. (In the case of classical mechanics a similar proposition can be found in [AM78, prop. 4.3.14].)

As in the abstract relational approach /1.3.3/ a theory is automatically time-reversal invariant, it does not possess an arrow of time. (With the relational approach we are therefore making a strong commitment with regard to one of the most fundamental open problems.)

In order to circumvent proposition 2.1.9 one argues that the fundamental physical theories are only idealized theories in that they describe closed systems while all systems (except for the universe) are open. In this line of reasoning it suffices to justify an "arrow of time for the universe". The (classical) evolution of the universe however is governed by the time-reversal invariant Einstein's equations. There are essentially two ways out: The one appeals to a statistics of universes in the tradition of Boltzmann, while the other contents itself with a single solution of Einstein's equations, namely that in which we live.

The first approach was initiated by Gold [Gol62] and is discussed in an illuminating way in [Pri96, ch. 4]. The idea is to relate the entropy of the universe with its expansion. Although it is not yet clear how to define entropy including gravitational interaction, it seems that this is not sufficient to explain the very low entropy of the big bang[7] [Pri96]. Nevertheless, one could introduce the low entropy of the big bang as a postulate, i.e. a restriction to solutions with special boundary conditions. (In the case of a symmetric universe an analogous postulate would be required for the big crunch.) For an extended discussion of the alternatives see the cited literature as well as a number of other books [Dav74, Zeh99, Sch97, Sav95, Ste00, AR97, Pri96, Pri79]. (These also discuss the other arrows of time at length.) Altogether the entropic (or traditional global) approach relies on the questionable statistics of universes.

The second approach [CLL03b, CLL03a] provides a global and local arrow of time in a non-entropic way: It postulates as an additional law that there be only a single solution of Einstein's equation, namely that which we observe as our universe. Together with the input of an asymmetric universe (being currently favoured) the expansion of the universe in cosmic time[8] is a global observable. If the dominant energy condition [CLL03b] is fulfilled, the global arrow of time becomes manifest locally in the timelike local energy flux. This provides a local arrow of time and is possible because just one (asymmetric) solution of Einstein's equations is allowed.

Another way to circumvent proposition 2.1.9 is the assumption that the fundamental dynamical laws of nature are not partially time-reversal invariant.

The case of weak interactions is not clear. (It has been argued that the Kaon decay experiments have not actually been performed in a time-reversed manner, or that the weak interactions might be too weak in order to observe a macroscopic arrow of time [CLL03b, footnote in part V. C.]; apart from short remarks this topic is however not treated in the literature.) Due to lack of new insights we have to follow bad custom and leave weak interactions out of consideration.

If there were an arrow of time, how could it explain our everyday experiences? If we could distinguish past and future instants on a physical basis, why would a glass always be intact in the past and broken in the future, and not vice versa? One strategy is to weaken this question, and to ask why most glasses would do so. That is, one can reduce the question to statistics and try to deduce an increase in entropy towards the future. This is not a trivial task, especially when taking into account all fundamental interactions.

Another twist is added to the problem of the arrow of time by quantum mechanics:

[7] According to Penrose's estimate only 1 in $10^{10^{123}}$ universes will have a big bang compatible with our observations.

[8] The existence of a cosmic time is a precondition for a global arrow of time.

2.1.4 The arrow of time in quantum physics

According to quantum mechanics there are two incompatible time evolutions ("dynamical dualism" [Zeh03]): The unitary evolution, which is partially time-reversal invariant and hence does not allow for an arrow of time /2.1.3/; and the collapse dynamics, or (wave packet) reduction, which is nonunitary and nondeterministic and describes von Neumann measurements.

In order to deal with the latter dynamics we first clarify the notions of determinism and causality[9]:

Definition 2.1.11. A system with time order $\leq$ is said to be *deterministic in the time forward (backward) direction*, if any copy of this system containing exactly the same observations at instant t does so for any $t' \geq t$ ($t' \leq t$).

Of course, no experimenter can produce a copy of the universe, so determinism can be tested only locally in space and time by providing the same boundary conditions locally at a later time, or somewhere else in space.

If there is no arrow of time, then forward and backward determinism have to be equivalent, i.e. if any two copies contain the same observations at an instant, then they do so at any other instant.

Definition 2.1.12. A system with time order $<$ is said to be *causal in the time forward (backward) direction*, if any two copies of this system containing different observations at an instant t do so at any instant $t' < t$ ($t' > t$).

If there is no arrow of time, then forward and backward causality have to be equivalent, i.e. if any two copies contain different observations at an instant, then they do so at any other instant. Thus causality does not require an arrow of time.

From our definitions it is obvious that determinism in the time forward (backward) direction is equivalent to causality in the time forward (backward) direction. The time-symmetric versions are identical, too.

From the definitions we can distinguish four kinds of (in)determinism [DEH01] (compare fig. 2.1.2):

(a) The system is forward deterministic and backward deterministic.
(b) The system is forward deterministic and not backward deterministic.
(c) The system is not forward deterministic and backward deterministic.
(d) The system is not forward deterministic and not backward deterministic.

ad (a):

This case includes classical (non-statistical) theories and unitary quantum dynamics: Determinism does not mean predictability; the former refers to ontic descriptions of systems, the latter to epistemic ones [Pri02, Bis03]. For instance,

[9] Here we are concerned with the question of *fundamental* indeterminacy (or uncertainty), which cannot be blamed to the observer's limited accuracy.

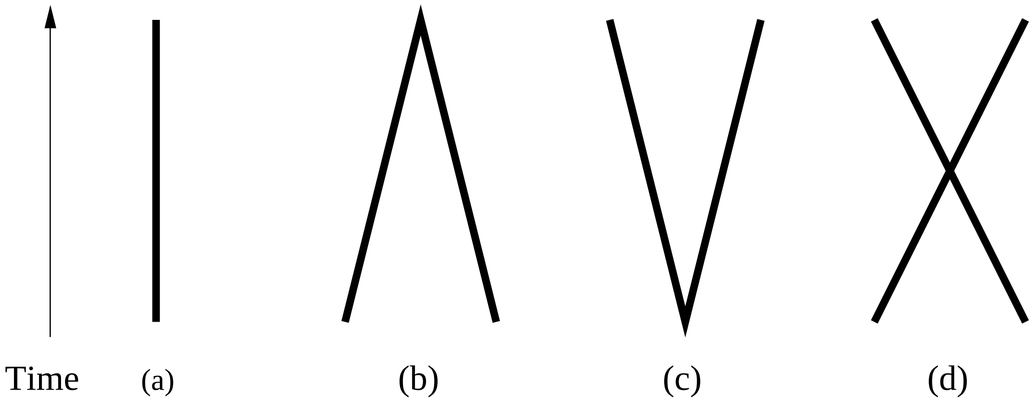

FIGURE 2.1.2. Four types of (in)determinism [DEH01]. (a) One cause has one effect and vice versa. (b) One effect has more than one cause, but these causes have always the same effect. (c) One cause has more than one effect, but these effects have always the same cause. (d) Every effect has more than one cause and each of these causes has more than one effect.

there might be a hidden variable theory according to which the evolution is deterministic, but as long as the hidden variables are not observable the theory lacks predictability.

Clearly, the time symmetry of this kind of determinism by itself does not imply the impossibility of an arrow of time. Furthermore, if two systems are in an entangled state, then they have the same arrow of time [Pri02].

ad (b), (c):

This asymmetric kind of determinism relies on the assumption that the theory is intrinsically indeterministic. For if hidden variables were able to restore a forward as well as backward deterministic evolution, it would have to be explained why always the same type of determinism (forward, resp. backward) is concealed by the unobservability of hidden variables.

Given an intrinsic indeterminism of type (b) or (c) one can experimentally determine an arrow of time in the following way: One prepares several copies of the system at one instant t_1 of time, where approximately all possible states of the system appear approximately infinitely often. Then one measures the states of all these copies at another instant t_2. If there are copies with the same state at t_2 (t_1), but different states at t_1 (t_2), then the system is not backward (forward) deterministic between t_1 and t_2, otherwise it is. If the same type of (in)determinism holds for all pairs of different instants with the same time order (as between t_1 and t_2), then in cases (b) and (c) one can observe the time order and has an arrow of time.

We thus define a *quantum arrow of time* to be a classical arrow belonging to the classical motion of expectation values of the quantum system, where expectation values are operationally defined as averages over many systems. If there is only one copy of the system (as in the case of the universe), the definition becomes meaningless.

ad (d):

However, as argued in [DEH01], indeterminism in quantum physics with collapse always comes as in (d): Consider e.g. an electron source emitting particles in an x-spin up ($x+$) state and let them pass through a Stern-Gerlach apparatus, which measures their z-spin, the result being indeterminate (with probabilities $\frac{1}{2}$ for $z+$ and $\frac{1}{2}$ for $z-$). The same result would be obtained upon usage of $x-$ states. There is thus more than one cause having the same effects. This kind of indeterminism appearing in von Neumann collapse is symmetric in both directions of time.

Time symmetry of the standard formalism of quantum mechanics with collapse was revealed already in 1964 in the famous paper by Aharonov, Bergman and Lebowitz (ABL) [ABL64][10]. They formulated quantum mechanics including measurement collapse in a time-symmetric way by considering preparation as well as detection. Textbooks usually limit consideration to ensembles constructed in the following way:

> Put all joint events with fixed initial state $\psi^{(i)}_{k_0}$ and one of the possible final states $\psi^{(f)}_l$ ($l = 1, 2, \ldots$) into the same ensemble E_l, and discard all other initial states.

Clearly, this introduces a time asymmetry; time-symmetric ensembles can be constructed in the following way:

> Put all joint events with one of the possible "initial" states $\psi^{(i)}_k$ ($k = 1, 2, \ldots$) and one of the possible "final" states $\psi^{(f)}_l$ ($l = 1, 2, \ldots$) into the same ensemble $E_{k,l}$.

The probability interpretation of quantum mechanics ascribes an approximate relative frequency distribution to the population of the ensembles E_l and $E_{k,l}$ (provided they contain sufficiently enough events). The time-symmetric case is therefore included in the standard interpretation; this remains true for histories, i.e. sequences of measurements.

The reverse does not hold: Given probabilities for ensembles with time-symmetric double-selection one cannot deduce probabilities for ensembles in which the selection is based only on initial (or only on final) observations. As shown by ABL [ABL64], this deduction requires the following additional postulate: Ensembles chosen on the basis of an initial measurement alone possess unambiguous and reproducible probability characteristics.

Adopting the time-symmetric description without this postulate, a system at a time between two noncommuting measurements can be described either by the state after the "initial" observation (for predictive purposes), or by the the state before the "final" observation (for retrodictive purposes).

[10] Note that while their conclusions are correct there is a mistake in the derivation, which cancels itself [Kir02].

One may ask which state between preparation and detection is the correct one in the time-symmetric formalism: the state predicted from preparation, or the state retrodicted from detection? Since however by assumption there is no measurement-like interaction between preparation and measurement, this question is operationally meaningless. (This "inaccessible past" [Pri96] is the characteristic difference with classical mechanics, where any quantity is measurable at any time.)

In [AV91] the concept of a general state was introduced, which comprises all information about a system at a time, including both past and future measurements. The predictive formalism of quantum mechanics, which together with Bayes' theorem allows to calculate retrodictive probabilities, was reformulated in a such a way that it treats both preparation and detection in a symmetric fashion. Recently in [PBJ02] a generalization to biased (preparation as well as) detection procedures was given. (A simpler formalism for retrodiction is desirable in quantum communication and quantum cryptography, for the recipient wants to retrodict the state prepared by the sender.)

Prediction and retrodiction still assume the system to be causal in the time forward direction, and thereby assume the existence of one and only one direction of time, with respect to which inferences can be drawn. If there is no arrow of time, then time forward and backward causal descriptions must be equivalent. The main argument against backward causation is the so-called *bilking argument* [Fay01, Wha98]: Assume that a cause A is later than the effect B. Then appealing to free will one argues that one could intervene after B has occurred (and before A has occurred) in such a way that instead of A a different and to some degree arbitrary cause A' would take place. Now B would have to be caused by A and arbitrary causes A', which cannot be true in general and therefore backward causation is impossible. – While this is true in classical mechanics, an arbitrary intervention without changing A is impossible in quantum mechanics, as pointed out above. Quantum mechanics hence allows for backward causation and in fact some features of quantum mechanics can be interpreted in a new way. E.g. the nonlocal Bell correlations can be reinterpreted in the following way [Wha98, Pri94a, Pri96]: Bell's theorem assumes that the past values of hidden variables cannot be influenced by future measurements. With backward causation this is possible, and measurements on both spatially separated subsystems *cause* the earlier state of the compound system. Hence there is a local propagation of causal effects and the local realistic interpretation of quantum mechanics as desired by Einstein, Podolsky and Rosen can be reestablished. – Of course, such an approach conflicts with common intuition, but there is nothing in quantum mechanics which forbids this interpretation. Also, due to the stochastic nature of quantum mechanics superluminal signalling is impossible in this approach although causal influences are instantaneous.

In [EDZ02] (see also [ED02]) an experiment has been proposed, which suggests backward causation. It consists of the time-reversed generation of an EPR pair. Consider first the time-forward (V-shaped) version of the experiment: A single photon passes through a beam splitter and each of its two "halves" subsequently interacts with a single atom in a cavity along its path. In this way the states of the atoms become entangled (w.r.t. ground and excited states of both). After passing the cavities the photons are not detected. – Now consider the time-backward (Λ-shaped) version of this experiment (called "RPE"): One arranges two distant, independent sources so that they emit a single photon (during a certain time interval) on average. The beams of both sources are directed towards a beam splitter, so that the interference of this single photon with itself can be detected with photon counters behind the beam splitter. (The interference of light coming from uncorrelated sources was discovered by Hanbury-Brown and Twiss; see [Pau86] for a review. It has been experimentally demonstrated at the single photon level.) To complete the time-backward analogue one now puts two cavities, each with a single atom, into the beams. The time-reversed experiment thus works in the following way: A single photon is emitted from two independent sources, its two "halves" pass through the cavities, thereby possibly exciting the atoms inside, and finally are brought to interference at a beam splitter, behind which the single photon is detected. Through the use of the beam splitter the two photon paths become indistinguishable, which entails that the atoms are in an entangled state. On the other hand, if one pulls out the beam splitter and gains knowledge on the photon path, one atom will be excited, and the other not. The astounding fact now is that the experimenter can choose to pull out or not to pull out the beam splitter *only after* the interaction with the atoms has taken place. Thereby the wave function of the atoms (which is experimentally verifiable!) can be chosen ex post! If one claims reality of the wave function at any instant of time, then one has to admit backward causation.

We return to physics without backward causation and consider a modification of quantum mechanics, which explicitly rules out symmetric determinism (a) in favor of symmetric indeterminism (d): In the spontaneous collapse model of Ghirardi, Rimini and Weber (GRW) [GRW86] a local collapse of the wave function in the *position* basis on certain time and distance scales is postulated, thereby slightly modifying the collapse-free unitary dynamics. The probability of spontaneous collapse is taken to be so small that no contradiction with observed measurement results arises. In [DEH01] it is argued, that this kind of indeterminism may be macroscopically time asymmetric: Consider a sufficiently big number of molecules in a box so that collapses happen at almost every instant of time, and assume furthermore that the dynamics without collapse evolves the wave function into a sufficiently delocalized superposition of localised molecule states, so that the collapse dynamics will perform a substantial relocalization. While the delocalization happens in both directions of time, this spontaneous localization is postulated to occur only in one direction of time (towards the "future"). Clearly, this assumption explicitly breaks time-reversal invariance. Nevertheless the indeterminism is symmetric of type (d): Let $|x\rangle$ and $|y\rangle$ be localized states. Then we have an arrow of time intrinsic to the dynamics, where e.g. the following collapse happens:

$$\left.\begin{array}{l}\frac{1}{\sqrt{2}}\left(|x\rangle+|y\rangle\right)\\ \text{or}\\ \frac{1}{\sqrt{2}}\left(|x\rangle-|y\rangle\right)\end{array}\right\}\begin{array}{c}\rightarrowtail\\ \not\leftarrow\end{array}\left\{\begin{array}{l}|x\rangle\\ \text{or}\\ |y\rangle\end{array}\right.$$

The problem then is to justify that the future direction $\rightarrowtail$ is the same as that observed in thermodynamics. Indeed, under the stated assumptions an entropy increase of a nonequilibrium state of the molecules in the container towards the future is very likely; an anti-thermodynamic (entropy decreasing) behaviour is not excluded. The probability of an entropy increase can in principle be calculated from a solution of the equations of motion, which may however turn out to be intractable. The hypothesis of a low entropy of the initial states is not needed here; the increase of entropy is rooted in the intrinsic time asymmetry of an indeterministic dynamics.[11]

Conversely, it has also been argued that a fundamental indeterminism is a necessary condition for the existence of an arrow of time [DEH01, ED99]. As in the GRW case, this condition is however not sufficient; existence of an arrow of time depends on the details of dynamics.

In [ED99] it was argued that in the case of black holes the (indeterministic) evaporation at the event horizon is a sufficient condition for an intrinsic arrow of time, because the increase in entropy due to quantum fluctuations at the event horizon seems to be unrelated to the content of of the black hole; the increase of entropy outside is not connected with an equal decrease inside the event horizon, so that the entropy production (information annihilation) is absolute and defines an arrow of time. – This argument remains however speculative as long as it is not supported by an established theory of quantum gravity. Penrose [HP96] expects that a theory of quantum gravity will reveal an arrow of time in the basic laws of physics. Hawking point of view [HP96] is however not tenable: He argues that causation is time-symmetric, and the arrow of time arises from intrinsic indeterminism (information annihilation) together with some unique initial conditions of the universe's evolution. However, as shown in [ED99], time-symmetric causation is incompatible with intrinsic indeterminism.

The open question therefore is whether a proper theory of quantum gravity does provide an intrinsic indeterminism (which goes beyond indeterminism in non-gravitational quantum physics).

We come to the conclusion that a fundamental indeterminism as in the GRW theory or in black hole evaporation has not been established. Hence quantum physics does not require a fundamental arrow of time, not does it unambiguously provide one. The relational concept of time based on a mere simultaneity relation without an arrow of time is therefore adequate for the description of fundamental physics.

[11] It has been criticised however, that the usage of nonequilibrium states remains unexplained [Cal01].

Let us finally mention a new idea for defining an arrow of time in local quantum field theory: Buchholz considers a class of states of quantum field, which are locally (but not necessarily globally) close to thermal equilibrium. The largest space-time regions in which these "local equilibrium states" can exist, are shown [Buc03] to be timelike cones in a simple model. The singular events at the apices of these cones can be interpreted as "hot bangs" (singularities in temperature); by assuming that they always lie in the past of the temporally subsequent local equilibrium they give rise to a local direction of time for the chosen state of the quantum field without requiring dissipative effects. The model shows how macroscopic time-reversal symmetry is reflected in the microscopic setting.

In the absence of a fundamental arrow of time our impression (or "illusion" [12]) of the flow of time might be explained by a new physics of consciousness, which seems to depend on the possibility of memory. There is some hope that this can be achieved with a proper understanding of quantum mechanics. (See e.g. [Pen89, Pen95, AB02, AAMH99, Vit01b] and [Atm03] for thoughts in this direction, to which the author is inclined. They go beyond the scope of this thesis, however.)

2.2 Time measurement

In this section we assume the existence of an absolute time and investigate how it can be measured with clocks built from classical or quantum free harmonic oscillators. Before we do this, we give a short review of the problem of finding a quantum time observable.

2.2.1 Quantum time observables

In classical mechanics as well as in quantum mechanics there are systems with an oscillatory motion, i.e. whose solutions of the equations of motion are not irreversible /2.1.2/, which do not allow to determine absolute duration by measuring the state of the system. They have no internal time, but may possess a phase operator /2.2.3/. On the other hand, systems with irreversible motion (e.g. free particles) do exist in classical mechanics. In quantum mechanics a footnote in Pauli's famous 1926 Handbuch article [Pau26, p. 60] opened a long discussion on whether a time observable can exist for a quantum system at all. For several decades it was believed that this is not the case; more precisely, Pauli's theorem asserts that there is no self-adjoint operator canonically conjugate to a semi-bounded Hamiltonian. The time operator T should be canonically conjugated to the Hamiltonian H, because $[T, H] = \frac{\mathrm{d}T}{\mathrm{d}t} \stackrel{!}{=} 1$. This is however a too strong statement [Bos01]: 1. T cannot always be defined on the entire phase space. 2. $[T, H] = 1$ needs only be fulfilled on actual trajectories. Moreover, the value of T in a state depends on a reference state (initial conditions), so there is a family

[12] "For us convinced physicists the distinction between past, present and future is an illusion, although a persistent one." (A. Einstein)

of time observables. – But even if one takes the canonically quantized time operator T and the semi-bounded Hamiltonian H to be densely defined, then due to Pauli's theorem $[T, H] = \mathrm{i}\hbar$ [13] cannot be true. However, after more than seven decades and after this statement had entered into numerous textbooks, Pauli's theorem was recognised [Gal02a] to rely on contradictory implicit assumptions. A careful discussion (with regard to the domains of definition of the operators) has shown that for certain semi-bounded Hamiltonians canonically conjugate time operators do exist [Gal02b]. The interpretation as quantised time is however not clear so far. (For instance, there should be a certain probability for time to run backward.)

The hypothetical time observable in Pauli's theorem is assumed to be a selfadjoint operator, or equivalently, due to the spectral theorem, a projection valued (PV) measure[14]. It has however become clear during the last decades that the common textbook wisdom identifying observables with selfadjoint operators is too restricted. More generally, observables have to be represented by positive operator valued (POV) measures[15], which are well established[16] by now [BGL95]. The assumptions of Pauli's theorem are therefore not only self-contradictory, but also too strong. And indeed, POV measures with the meaning of a time observable can be found [BGL94]. E.g., the POV measure for a free particle is explicitly constructed in [Gia97, Bos01]. The idea is to find a POV measure whose expectations coincide with those of the time operator[17] $T = (2p)^{-1}\, q + q\, (2p)^{-1}$ (q and p being position and momentum operators of the particle), and which is also covariant under time translations (rather than a system of imprimitivity as in Pauli's theorem). This POV measure can be evaluated in any state and yields the duration t between the current state and the initially prepared state at $t = 0$. – A general procedure for constructing a POV time observable is still missing.

The time observables discussed so far are clock time observables used to measure the intrinsic time of a system. Such measurements can take place at any instant.

A different question is at which time a measurement event does occur [BF01, Rov98, ORU98, Per86]. An often discussed situation is that of arrival time or time of flight of a particle (see e.g. [ML00, ORU99] and some of the contributions in [MME02]), where the detection of a particle takes place at a single instant

[13] Using $T = t$ and $H = \mathrm{i}\hbar\frac{\partial}{\partial t}$ one obtains $[T, H] = -\mathrm{i}$. The ambiguity of the sign is not understood, but indicates problems with quantum time.

[14] Such observables have been called decision observables, ordinary observables, or sharp observables.

[15] These measures were also called POM (probability operator measures), and the corresponding observables appear also under the names generalized observables, or unsharp observables.

[16] There is however still progress: Recently it has been shown [DHLP02], that there are selfadjoint operators, which have a unique PV measure due to the spectral theorem, and at the same time have a unique POV measure due to certain requirements (such as covariance) with a statistics different from that of the PV measure. If one wants to interpret these operators as observables, then one has to make a choice between the PV measure and POV measure formalisms.

[17] In the PV formalism this operator has been often discussed, see e.g. [Per80, GYS81].

according to von Neumann's measurement collapse. – We point out that in a genuinely quantum description without collapse and without absolute time there is no need that the becoming definite of some observable must happen instantaneously; instantaneity means that a time observable becomes definite simultaneously, which is not necessary in a relational setting.[18] – Von Neumann's measurement theory however assumes instantaneous collapse and thus the question when a measurement event takes place is perfectly justified. There is a certain probability distribution for the occurrence of the event over time, and one wants to find a time observable, which has exactly this distribution for resulting time values. Such observables have been called event time observables (see e.g. [Bus02]).

A general procedure for constructing a POV event time observable measuring the time of occurrence of an effect (more precisely, the duration between the instant of occurrence and the preparation instant) has been given in [BF01]. Let us briefly sketch the idea: Let $A > 0$ be a bounded operator on a Hilbert space $\mathcal{H}$. A is the effect whose occurrence time shall be measured. E.g., A might be a projection operator for a spatial region, in which a particle detector is sensitive. In the Heisenberg picture the expectation $\langle A(t)\rangle_\psi$ of $A(t)$ in a state ψ varies over time. This probability distribution may be such that

$$0 < \int_{\mathbb{R}} \langle A(t)\rangle_\psi \, \mathrm{d}t < \infty \ ,$$

if the particle does enter the detector at all and does not spend an infinitely long time there. For this case it is shown in [BF01], that the total duration between $t = 0$ and the occurrence of the effect, namely

$$B = \int_{\mathbb{R}} A(t) \, \mathrm{d}t \ ,$$

is a positive selfadjoint operator; moreover the POV measure

$$P(I) = B^{-\frac{1}{2}} \int_I A(t) \, \mathrm{d}t \, B^{-\frac{1}{2}}$$

(where I is an interval of the real line) transforms covariantly under time translations,

$$(P(I))\,(t) = P(I + t) \ ,$$

where $(P(I))\,(t)$ is the time evolution of $P(I)$. An event time operator can then be defined as the first moment of this POV measure:

$$T_A = \int_{\mathbb{R}} t P(\mathrm{d}t)$$

This procedure can be used for the construction of event time operators for several characteristic times, e.g. the time delay in scattering theory [CN02], the dwell time in tunnelling [CN02], or the lifetime of an unstable state.

[18] In [Rov98] (which was also criticized in [ORU98]) it is argued that one could attach a second apparatus in order to detect whether the first apparatus has performed a measurement or not; but here again the instantaneity (w.r.t. absolute time) of the second measurement is an assumption, if the overall system is genuinely quantum.

Intimately connected with the search for time observables is the question on the existence and meaning of time-energy uncertainty relations. We do not review this endless debate here, but instead refer to the introductory section of [KAN94]; see [BF02] for a recent rigorous derivation of this uncertainty relation.

2.2.2 Classical oscillator clocks

In /1.2.2/ we used a free classical body as a clock, whose time can be read off (up to linear transformations) by measuring position. The quantum POV time observable for the free particle mentioned in the previous subsection /2.2.1/ has the disadvantage that the measured time value is not classically definite, but probabilistic, so that the clock time sometimes might go backward. To obtain a classical clock time one could use an infinite number of free particles and take the average, or one could take an infinitely massive particle. Both options are not convincing for real clocks; real clocks do usually tick, i.e. they have an oscillation mechanism. Oscillator clocks are no better than free particles regarding quantum stochasticity. They even have the further disadvantage that their motion is not irreversible and thus they only allow to measure a cyclic time. Bosonic quantum fields do however provide an infinite number of oscillators with arbitrary long oscillation periods. They can thus be used to measure a linear time, and we will explore below /4.1/ how a classical notion of time can be obtained from them. In this subsection we study in detail how classical clocks can be constructed from one, two, or infinitely many free oscillators.

One oscillator

The motion of a free harmonic oscillator in one dimension is known to be

$$q(t) = A \cos(\omega t + \phi) \ ,$$

where $q \in \mathbb{R}$ is the position, $A \in \mathbb{R}_0^+$ the amplitude, $\phi \in [0; 2\pi[$ a phase, $\omega \in \mathbb{R}^+$ the frequency and $t \in \mathbb{R}$ a time parameter. The latter coincides with absolute time, since by assumption the oscillator is free. Knowing this motion we can 1) determine A by measuring q at all instants and taking the supremum, and 2) measure q at a single instant and obtain information about t,

$$t = \frac{1}{\omega}\left(2\pi\left(n + \frac{1}{2}\right) - \phi \pm \chi\right) \ , \tag{2.2.1}$$

where $\chi = \pi - \arccos\left(\frac{q}{A}\right)$ and the equation holds for some $n \in \mathbb{Z}$. From q alone we can determine neither n, nor the correct one of the two branches. The positions $q_i = q(t_i)$ $(i = 1, 2)$ are insufficient for measuring the duration $d(t_1, t_2)$ between two instants $t_1, t_2 \in \mathbb{R}$:

$$d(t_1, t_2) := |t_2 - t_1| = \frac{1}{\omega}\left|2\pi(n_2 - n_1) \pm \chi_2 \mp \chi_1\right| \ ,$$

where $\chi_i := \pi - \arccos\left(\frac{q_i}{A}\right)$ and we have to require $A > 0$. While ϕ drops out, we can neither infer the branches uniquely, nor $n_2 - n_1$. The latter would amount to knowing duration in advance, though on a larger scale; if we had an irreversible mechanism at hand, we could use it as an event counter and “outsource” the counting of clock cycles to it, but we do not assume so. Hence we can only measure a *cyclic time*.

In the limit $\omega \to 0$ the oscillator approximates a free particle, and we have $n_i = 0$ (for finite t_i), but the sign ambiguity remains.

With an irreversible mechanism we would be able to distinguish between forward and backward motion, i.e. between positive and negative values of velocity $\dot{q} = -A\omega \sin(\omega t + \phi)$ – independent of the metric of time. This would render the choice of branches unambiguous for the single oscillator clock.

Moreover, two interactionless single oscillator clocks, which in particular do not share their irreversible mechanisms, do not necessarily assign the same time order to a series of instants, if there is no arrow of time /2.1.3/.

Two oscillators

Consider an additional oscillator

$$q'(t) = A' \cos(\omega' t + \phi') \ .$$

We assume that we are able to perform simultaneous measurements of q and q'. For the second oscillator we obtain an equation analogous to (2.2.1),

$$t = \frac{1}{\omega'}\left(2\pi\left(n' + \frac{1}{2}\right) - \phi' \pm \chi'\right) \ , \tag{2.2.2}$$

where $\chi' = \pi - \arccos\left(\frac{q'}{A'}\right)$ and $n' \in \mathbb{Z}$. Equating (2.2.1) and (2.2.2) leads to

$$\omega' n - \omega n' = \Delta_{\pm,\pm}(\omega, \omega') \tag{2.2.3}$$

with $\Delta_{\pm,\pm}(\omega,\omega') \in \mathbb{R}$, the first $\pm$ indicating the two branches of (2.2.1) and the second those in (2.2.2). When is a solution (n, n') of this equation unique?

Lemma 2.2.1. A solution of (2.2.3) is unique, iff $\omega'\omega^{-1}$ is irrational.

Proof. Let (n, n') and $(\tilde{n}, \tilde{n}')$ be solutions to (2.2.3). Then$\omega' n - \omega n' = \omega'\tilde{n} - \omega\tilde{n}'$, whence $\omega'\omega^{-1}(n - \tilde{n}) = n' - \tilde{n}'$. If $\omega'\omega^{-1}$ is irrational, we must have $n = \tilde{n}$ and $n' = \tilde{n}'$. Otherwise we have $r \in \mathbb{Z}$, $s \in \mathbb{N}$ with $\omega'\omega^{-1} = \frac{r}{s}$ and can choose $\tilde{n} := n + s$, $\tilde{n}' := n' + r$, which is a solution different from (n, n'). □

Hence for irrational $\omega'\omega^{-1}$ (ω and ω' being *incommensurable*) position measurements on both oscillators determine time uniquely, up to a possible choice of 4 branches. However, a limited accuracy of position measurements renders the resulting time values non-unique.

Again, if we consider durations (corresponding to 4 measured positions q_1,q_2,q'_1,q'_2), the phases ϕ and ϕ' drop out. And in the limit $\omega, \omega' \to 0$ we also have $n = n' = 0$.

Free field

Next consider the time evolution of a free Klein-Gordon field Φ in one dimension with coordinate x, i.e. a solution of the Klein-Gordon equation

$$\frac{\partial^2}{\partial t^2}\Phi(x,t) - \frac{\partial^2}{\partial x^2}\Phi(t,x) + m^2\Phi(x,t) = 0 \, ,$$

with mass $m \geq 0$. We assume $\Phi \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$ and $\Phi(t, \cdot) =: \Phi_t \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ $(t \in \mathbb{R})$, where $\mathcal{S}(\mathbb{R}, \mathbb{R})$ is the space of real-valued Schwartz functions on $\mathbb{R}$. Denoting the Fourier transformation w.r.t. x by $\hat{\cdot}$, we get

$$\frac{\partial^2}{\partial t^2}\hat{\Phi}_t(k) + \omega(k)^2 \hat{\Phi}_t(k) = 0 \qquad (k \in \mathbb{R}) \,,$$

where $\omega(k) := \sqrt{k^2 + m^2} \in \mathbb{R}_0^+$. The solution reads

$$\hat{\Phi}_t(k) = \hat{\Phi}_0(k) \cos\left(\omega(k)t + \phi(k)\right) \,,$$

where $\phi(k) \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ is an initial phase. Let us fix a k and measure $\hat{\Phi}_t(k)$ for some k at two different instants $t = t_1, t_2$. Since every nontrivial neighborhood of k contains a k' for which $\omega(k')$ is an irrational multiple of $\omega(k)$, we can use the oscillators at k and k' as a clock. According to classical theory this is possible in principle. In order to know the exact values of the Fourier transformed fields, we must know $\Phi(x, t)$ for all $x \in \mathbb{R}$.

An alternative method to measure time makes use of a convergent sequence of field modes $(k_n)_{n\in\mathbb{N}}$. Let $k^* := \lim_{n\to\infty} k_n$. From a measurement of $\hat{\Phi}_t(k^*)$ and $\hat{\Phi}_t(k_n)$ for some n one can infer the value of

$$\left(\omega(k^*) - \omega(k_n)\right) t$$

modulo some recurrence time. In the limit $n \to \infty$ this recurrence time becomes infinite. (Intuitively, this sequence corresponds to a (cascaded) hierarchy of hands of a clock counting seconds, minutes, hours, etc..) In order to resolve k^* and k_n it is however necessary to measure the field at sharply localized values of k, which means that nonlocal observables are unavoidable. Furthermore, this method does work only, if in each neighborhood of k^* there is at least one k_n with $\hat{\Phi}_0(k_n) \neq 0$.

If the field is translation invariant, then the choice of k^* is arbitrary. If translation invariance is broken, then the value $k^* = 0$ is of special interest (cmp. /4/ below); then one can find $k_1, k_2, \ldots$ such that $m\omega(k_m) = m\,|k_m| = \text{const.}$ $(m \in \mathbb{N})$. The greater m, the longer the period of the k_m-oscillator, and the more instants can be distinguished by this oscillator. Not only do the $n_k$again disappear, but also the sign ambiguity: Since $|\chi_{k,1} - \chi_{k,2}|$ becomes arbitrarily small for $k \to 0$ the duration between two configurations of the field is defined unambiguously. This fact makes the field into a clock.

2.2.3 Quantum phase observables

While the measurement of the phase of a single oscillator is classically unproblematic, the existence of a phase observable for a quantum oscillator is one of the oldest and still debated problems of quantum mechanics [Lyn95]. It is similar to the problem of finding a quantum time observable /2.2.1/.

One wants to have a *phase operator* Φ in the traditional Hilbert space setting, i.e. an observable which can be measured on an individual system in any state, and that has eigenstates which have a definite phase in $[0; 2\pi[$ in the classical limit.

The first to define an assumed phase operator was Dirac in 1927. He proposed a polar decomposition [RS80] of the annihilation operator, $a = e^{\mathrm{i}\phi}R$, where ϕ and R are selfadjoint. Then $a^* = Re^{-\mathrm{i}\phi}$, $n := a^*a = R^2$, $e^{\mathrm{i}\phi} = an^{\frac{1}{2}}$, and finally one calculates $\left[e^{\mathrm{i}\phi}, n\right] = e^{\mathrm{i}\phi}$, which can be expanded as a power series in ϕ, whence by comparison of the lowest-order terms one obtains the canonical commutation relation between number and phase, $[n, \phi] = \mathrm{i}$. However, this relation leads to an obvious contradiction already for number states ($l, m \in \mathbb{N}_0$):

$$\mathrm{i}\delta_{lm} = \mathrm{i}\,\langle l|m\rangle = \langle l\,|n\phi - \phi n|\,m\rangle = (l-m)\,\langle l\,|\phi|\,m\rangle$$

Closer inspection shows that $e^{\mathrm{i}\phi}$ is not unitary and thus ϕ is no selfadjoint operator. The problems in finding a phase observable canonically conjugate to number can be traced back to two reasons: 1. The number operator is bounded from below. (This reason is similar to the problem of finding a time observable.) 2. The phase shall be 2π-periodic.

Later Susskind and Glogower defined analogues of $e^{\pm\mathrm{i}\phi}$, namely

$$E = (n+1)^{\frac{1}{2}}\,a = \sum_{n=0}^{\infty} |n\,\rangle\langle\,n+1| \ ,$$

$$E^* = a^*\,(n+1)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} |n+1\,\rangle\langle\,n| \ .$$

However, E is not unitary again, only an isometry: $EE^* = 1$, $E^*E = \mathbb{1} - |0\,\rangle\langle\,0|$ (These operators have the advantage that they are approximately unitary for number states $|n\rangle$with large n.

Most other approaches suffer from similar problems. In fact it has been proved in general, that a Hermitian phase operator cannot be consistently defined within the traditional Hilbert space formalism [Fuj95].

Pegg and Barnett [PB89] were able to recover self-adjoint phase operators by slightly modifying the Hilbert space formalism of quantum mechanics. They defined for $\theta \in \mathbb{R}$ *phase states* by

$$|\theta\rangle = \lim_{s\to\infty}\left(s + \frac{1}{2}\right)^{-\frac{1}{2}} \sum_{n=0}^{s} e^{in\theta}\,|n\rangle \ .$$

Here $|n\rangle$ ($n = 0, \ldots, n$) are number states spanning a finite, $s+1$-dimensional subspace Ψ_s of the infinite dimensional Hilbert space of the harmonic oscillator. Note that only the finite dimensional approximations to $|\theta\rangle$ live in the Hilbert space, but $|\theta\rangle$ itself contains infinitely many excitations.

Time evolution takes a phase state $|\theta\rangle$ into another phase state $|\theta - \omega t\rangle$.

The basic idea of the formalism is to first calculate all expectations using the finite subspaces Ψ_s, and only then to take the limit $s \to \infty$. On each Ψ_s ($s \in \mathbb{N}_0$) one can define a Hermitian phase operator by

$$\Phi_s := \sum_{m=0}^{s} \theta_m\,|\theta_m\rangle\,\langle\theta_m| \ .$$

Here $\theta_m := \frac{2\pi m}{s+1} \in [0; 2\pi[$ so that Φ_s has the correct spectrum. Although Φ_s fails to be Hermitian in the limit $s \to \infty$, its expectations do converge in this limit and the Pegg-Barnett approach asserts that these limit values are observable. This means that one has a phase observable which can be repeatedly measured on phase states without affecting time evolution and at the same time measuring duration (up to multivaluedness).

The Pegg-Barnett formalism has been criticised (see e.g. [VR99b, VPB99, VR99a, Kak02]) and its relevance is still unclear, since no experiments are known, which would allow to discriminate between the traditional and the modified Hilbert space formalism.

A second approach to the phase operator problem argues that the requirement of hermiticity is too restrictive; instead one should identify observables with POV measures. Along these lines of thought *covariant phase observables* have been extensively studied [LP99, LP00, LP01, Pel01a, Pel01b]. Dropping hermiticity one can find infinitely many phase operators [Roy96].

A third class of approaches to the phase operator problem rests upon a change of quantization. We follow this route with the relational quantization scheme below /3.2/. This quantization does actually involve only phase difference observables[19], which is also desirable, since phase measurements are always based on interference.

2.2.4 Quantum oscillator clocks

After having seen how to use classical oscillators as clocks and how to measure the phase of a quantum oscillator, we now want to use a quantum oscillator as a clock.

Consider a single quantum oscillator in the traditional Hilbert space setting. First of all it is necessary to note that not all states are apt for building a clock: The time evolution of number states consists merely in a change of complex phase and therefore does not change expectation values (denoted by $\langle\cdot\rangle$). It is only when a nontrivial superposition of number states is involved, that expectations become time dependent.[20]

Because of their classical limits the coherent states $|z\rangle$ ($z \in \mathbb{C}$) are an important special case of such superpositions. They are defined by

$$|z\rangle = e^{za^+ - \bar{z}a}\,|0\rangle = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\,|n\rangle \;\;,$$

[19]By this we mean an observable, which corresponds to phase differences in the classical limit. Another definition of phase difference observable has been given in [HLP02] in the POVM formalism. There it was shown that one can find phase difference observables, which are not a difference of phase observables.

[20]Actually, for any system with two energy eigenstates $|\psi_{1,2}\rangle$ with energies $E_1 \neq E_2$, the observable $|\psi_1\rangle\langle\psi_2| - |\psi_2\rangle\langle\psi_1|$ has a time dependent expectation value in a nontrivial superposition of $|\psi_2\rangle$ and $|\psi_2\rangle$. See also [Sto03].

where $|n\rangle$ ($n \in \mathbb{N}_0$) are the number states and a^+ and a are creation and annihilation operators. They are related to position q and momentum p through

$$a^+ = \frac{\lambda q - i\lambda^{-1} p}{\sqrt{2\hbar}} \qquad \text{and} \qquad a = \frac{\lambda q + i\lambda^{-1} p}{\sqrt{2\hbar}} \;, \tag{2.2.4}$$

where $\lambda = \sqrt{m\omega}$. Under time evolution with the free Hamiltonian $H = \hbar\omega(N + \frac{1}{2})$, $N = a^+ a$, coherent states remain coherent states:

$$e^{-\frac{\mathrm{i}}{\hbar}Ht} |z\rangle = e^{-\frac{1}{2}\mathrm{i}\omega t} \left|e^{-\mathrm{i}\omega t} z\right\rangle =: e^{-\frac{1}{2}\mathrm{i}\omega t} |z(t)\rangle \tag{2.2.5}$$

While $|z(t)| = \sqrt{\langle N\rangle}$ remains constant, the complex phase of $z(t)$ oscillates and so does the expectation of the position operator q, just as in the classical case:

$$\langle q\rangle = \langle z(t)\,|q|\,z(t)\rangle = \frac{\sqrt{2\hbar}}{\lambda} \left\langle z(t)\,\middle|a + a^+\middle|\,z(t)\right\rangle = \frac{\sqrt{8\hbar\,\langle N\rangle}}{\lambda} \cos(\omega t + \phi)$$

Here we have used $z = |z|\, e^{-\mathrm{i}\phi}$. The uncertainty is calculated as

$$(\Delta q)^2 = \frac{2\hbar}{\lambda^2} \;.$$

With increasing amplitude $|z| \to \infty$, i.e. in the classical limit of an infinite number $\langle N\rangle$, the relative measurement error $\frac{\Delta q}{\langle q\rangle}$ becomes negligible for almost all times.

In order to infer information about time from measurements of position we can simultaneously prepare at an initial instant $t = 0$ an ensemble of oscillators in the states $|z\rangle$, i.e. with particle number $\langle N\rangle$ and initial phase ϕ, and then choose an instant at which we simultaneously measure q among all oscillators. The duration t between both instants is again given by (2.2.1), where now

$$\chi = \pi - \arccos \frac{\lambda\,\langle q\rangle}{\sqrt{8\hbar\,\langle N\rangle}} \;.$$

The problems with the multivaluedness are the same as in the classical case and we must require $\langle N\rangle > 0$.

If instead of measuring q on all oscillators at a single instant, we measure q on a single oscillator at different instants, the time resulting from a substitution of $\langle q\rangle$ with the measured values of q may occasionally run backwards, compared to the classical time.

A phase operator apt for coherent states would e.g. be

$$\Phi := \int_{\mathbb{R}^2} d^2 z\; (\ln z - \ln \bar{z})\, |z\rangle\,\langle z| \;,$$

or (in order to avoid problems with the multivaluedness of the ln-terms)

$$\Phi := \int_{\mathbb{R}^2} d^2 z\, \frac{z}{|z|}\, |z\rangle\,\langle z| \;,$$

which corresponds to $e^{-\mathrm{i}\phi}$ in the classical limit. This approach to the phase operator problem is known as Paul formalism. A problem with Paul's Φ is – as with most phase operators in the traditional Hilbert space setting – its lack of unitarity.

If we want to have a phase operator which can be used in any state of the oscillator system, then as explained above /2.2.3/ we have to leave traditional quantum mechanics.

2.3 Quantum gravity

A quantum theory of gravity is still missing [Rov00], there are two main candidates[21]: loop quantum gravity (LQG) and string theory. Smolin [Smo00] believes, that both could be correct and complementary, describing nature on different length scales. LQG is assumed to be valid below the Planck length ($\sim 10^{-35}$m) and string theory at intermediate scales, between the Planck length and the smallest length scales of particle physics. The main conceptual difference between both theories is that string theory assumes a spacetime with a fixed background metric (sometimes called a 'stage' on which strings act), while LQG tries to take diffeomorphism invariance of general relativity seriously. LQG [Rov87,Thi01,GR99,Thi02] is in the tradition of canonical quantum gravity (CQG), to which we restrict the following considerations. It is the more fundamental of both approaches, and it requires a new understanding of space and time.

In CQG classical general relativity is formulated as a Hamiltonian field theory with fields $g_{\mu\nu}$ and canonically conjugate momenta $\pi_{\mu\nu}$. Spacetime is foliated into a 1-parameter family of 3-dimensional spacelike hypersurfaces with constant time coordinate, and the Hamiltonian effects the evolution within this family of hypersurfaces along coordinate time. Classical general relativity is a gauge theory with the group of active diffeomorphisms of spacetime as gauge group /1.2.4/. In particular it is invariant under reparametrizations of (coordinate) time[22]. This implies that the Hamiltonian H is a Hamiltonian constraint.

The second step in CQG is a canonical quantization of the Hamiltonian theory. One has two options [BI99]: 1) Solve the constraint classically and quantize the reduced theory, or 2) quantize the gauge theory and solve the quantum constraint.

Choosing option 1 requires to find some internal time (i.e. an observable corresponding basically to one of the degrees of freedom of the theory, to which the motion of the other degrees of freedom can then be related; e.g. local proper time of some observer, or cosmic time of a simple cosmological model) w.r.t. which evolution becomes physically meaningful. This has not been achieved over decades and constitutes a first aspect of the problem of time in CQG.

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[21] Cf. [Rov03, appendix B] for an extended discussion of the history (see also [Pul02]) and [Smo03] for a recent review.

[22] As a generalization of Hamiltonian systems the notion of a *reparametrization invariant system* has been introduced [H97́]; each Hamiltonian system then corresponds to a reparametrization invariant system with a particular choice of auxiliary rest frame.

Choosing option 2 leads to four equations constraining the quantum state ψ in a suitable Hilbert space: three momentum constraints which ensure diffeomorphism invariance within a spacelike hypersurface, and one Hamiltonian constraint (*Wheeler-DeWitt equation*)

$$\hat{H}\psi = 0 \ , \tag{2.3.1}$$

where $\hat{H}$ and ψ depend on the metric of the hypersurface only. This equation differs from the Schrödinger equation in that no $\frac{\partial\psi}{\partial t}$ term is present, which makes it difficult to interpret it as an equation describing time evolution: in quantum theory a time parameter is required for evolution.[23] It is a second aspect of the problem of time in CQG that no time has been found at the quantum level, which would allow to understand the Wheeler-DeWitt equation as an evolution equation.

Even if an internal time could be found, there would be several problems [Kuc99, Kuc92,Kuc91,Ish92,Hea02] (problem of global time, problem of spacetime fitting, multiple-choice problem, problem of many-fingered time-evolution).

These problems gave rise to the hypothesis that CQG has no time at all. Of course, in an appropriate semiclassical approximation a classical notion of time must reemerge, see e.g. [Kie94]. – LQG uses loop variables for the description of the gravitational field and thereby simplifies the treatment of diffeomorphism invariance; it allows for a canonical quantization of general relativity, but it suffers from the same problems with time.

A more radical hypothesis states that there is no time in quantum mechanics either. On the one hand this would diminish the conflict with diffeomorphism invariance of gravity, but if there is no fundamental time, at a first glance our experience of change would be an illusion. The idea of quantum mechanics without time is the subject of the next chapter.

[23] There are more arguments for the incompatibility of diffeomorphism invariance and quantum mechanics:

Diffeomorphisms may change the causal structure, a timelike curve may become spacelike. But the standard interpretation of quantum mechanics requires an observer who performs measurements, and the simplest model for an observer is a timelike curve. Hence the observer ceases to exist after the diffeomorphic transformation. – Above /1.3.2.2/ we mentioned already the long debate on observability even in classical general relativity.

As our discussion in /1.2.4/ has shown, infinitesimal adjacency is a primitive concept in general relativity. Quantum mechanics renders exactly this notion meaningless due to localization problems. Consequently, there are problems of defining a reference frame in quantum gravity.

CHAPTER 3

Relational field quantization

In this chapter we introduce a new quantization of the free massless scalar field, by generalizing a model of Rovelli, which is based on the relational concept of time.

At a fundamental level, we should, simply, forget time.

CARLO ROVELLI [Rov01c]

This chapter studies the mechanics of simple systems, basically free harmonic oscillators, whose role as clocks was discussed already in /2.2/; by contrast, here we use the relational concept of time introduced in /1.3/. In the first section /3.1/ we review the classical part of a model of Rovelli and generalize it to a massless Klein-Gordon field in one dimension. In a second step /3.2/ we explain Rovelli's quantization method (roughly outlined already in /1.3/) and generalize it to the free massless scalar quantum field in one dimension.

3.1 Relational classical mechanics

This section introduces and discusses the classical part of Rovelli's model [Rov90], which is based on the relational concept of time /1.3/. Next we generalize this model to a field in one dimension. We stress once more that there is no need for an elimination of time in classical mechanics; this analysis merely serves as a preparation for quantization, leading to a quantum theory that does not suffer from an unphysical notion of (parameter) time in section /3.2/.

3.1.1 Two oscillators

Rovelli's model is concerned with one of the simplest systems having more than one degree of freedom: two noninteracting harmonic oscillators (double pendulum) in one dimension. After introducing the notation of the traditional treatment of this system in classical mechanics /3.1.1.1,3.1.1.2/ we switch to the presymplectic formulation without external time due to Rovelli /3.1.1.3/. Next we exhibit the physical algebra of this reparametrization invariant system /3.1.1.4/ and derive relations between partial observables /3.1.1.5/. We discuss the limitations of the model /3.1.1.6/ and finally sketch the general theory of evolving constants of motion /3.1.1.7/.

3.1.1.1 Hamiltonian formulation

Let two noninteracting harmonic oscillators with canonical coordinates q_i ($i = 1, 2$), conjugate momenta p_i ($i = 1, 2$) and with equal masses $m_1 = m_2 = 1$ and frequencies $\omega_i > 0$ be given. The free Hamiltonian reads

$$H = \frac{1}{2}\left(p_1^2 + \omega_1^2 q_1^2 + p_2^2 + \omega_2^2 q_2^2\right) .$$

The solutions of the Hamiltonian equations of motion are ($i = 1, 2$)

$$\begin{aligned} q_i &= a_i \sin\left(\omega_i t + \phi_i\right) \ , \\ p_i &= a_i \omega_i \cos\left(\omega_i t + \phi_i\right) \ , \end{aligned}$$

where $a_1, a_2 \geq 0$ are amplitudes and $\phi_i \in [0; 2\pi[$ are initial phases, with "initial" meaning the point $t = 0$ on the trajectories, t being the observable time parameter of classical mechanics. In geometric language, on the 4-dimensional phase space Γ (which is the cotangent bundle, $\Gamma = T^*Q$, of the two dimensional configuration space $Q = \mathbb{R} \times \mathbb{R}$) the symplectic form $\sigma = \mathrm{d}q_1 \wedge \mathrm{d}p_1 + \mathrm{d}q_2 \wedge \mathrm{d}p_2$ (which is the exterior derivative of the canonical 1-form on Γ) together with the Hamiltonian function H determine the Hamiltonian vector field X_H via $i_{X_H}\sigma = \mathrm{d}H$. Moreover, $X_H = \partial_t$ is complete, i.e. its flow exists for all values ('times') of the flow parameter $t \in \mathbb{R}$.

For the time evolution of both oscillators to be comparable we require

$$\omega_1 = \omega_2 =: \omega \ .$$

(The Hamiltonian could also contain an interaction term proportional to $q_1 q_2$, provided that after diagonalisation the frequencies of both oscillators were equal.) Following Rovelli [Rov90], we write the solutions as

$$\begin{aligned} q_1(\tau) &= \sqrt{2A}\sin\tau \\ q_2(\tau) &= \sqrt{2E - 2A}\sin(\tau + \phi) \end{aligned} \tag{3.1.1}$$

where $\phi = \phi_2 - \phi_1$ is the relative phase, $A = \frac{1}{2}a_1^2\omega^2$ is the energy of the first oscillator, $E = \frac{1}{2}\omega^2(a_1^2 + a_2^2)$ is the total energy and $\tau = \omega t + \phi_1$ is a rescaled and shifted time parameter. In traditional classical mechanics /1.2.1/ both the elongations q_1, q_2, and differences in the time parameter τ are observable, hence the dependencies $q_i(\tau)$ can be verified experimentally, except for shifts in τ.

3.1.1.2 Constants of motion

Before we impose an energy constraint we define the algebra of observables of classical mechanics, which is a Poisson algebra [Lan98]:

Definition 3.1.1. A real vector space $\mathcal{V}$ equipped with two bilinear maps,

$$\begin{aligned} \circ : \mathcal{V} \times \mathcal{V} &\to \mathcal{V} \qquad \text{(anticommutator, Jordan product)} \\ [\cdot,\cdot] : \mathcal{V} \times \mathcal{V} &\to \mathcal{V} \qquad \text{(commutator)} \end{aligned}$$

is called a *Jordan-Lie algebra*, if

(i) $[\cdot,\cdot]$ is a Lie bracket, i.e. antisymmetric and fulfilling the Jacobi identity

$$[[f,g],h] + [[h,f],g] + [[g,h],f] = 0$$

(Hence, $(\mathcal{V},[\cdot,\cdot])$ is a Lie algebra.)

(ii) $\circ$ is symmetric and respects the associator identity

$$(f \circ g) \circ h - f \circ (g \circ h) = k\,[[f,h],g]$$

with some $k \in \mathbb{R}$,

(iii) both together fulfil the Leibniz property

$$[f, g \circ h] = [f,g] \circ h + g \circ [f,h] \ .$$

If there is an element $1 \in \mathcal{V}$ with $f \circ 1 = f$ $(f \in \mathcal{V})$, it is called *identity*.

If $\circ$ is associative (i.e. $k = 0$), $\mathcal{V}$ is a *Poisson algebra*.

An immediate consequence of the Leibniz property is $[f,1] = 0$ for all $f \in \mathcal{V}$.

The algebra of observables of classical mechanics is the Poisson algebra of real (measurable) functions on a symplectic manifold (Γ,σ). Often one chooses $\mathcal{V} = C^\infty(\Gamma,\mathbb{R})$, $\mathcal{V} = C_0(\Gamma,\mathbb{R})$, $\mathcal{V} = C_b(\Gamma,\mathbb{R})$, $\mathcal{V} = L^\infty(\Gamma,\mathbb{R})$, or especially for $\Gamma = \mathbb{R}^{2n}$ $\mathcal{V} = P(2n)$, the Poisson algebra of all polynomials of the Cartesian coordinate functions q_i and p_i. Since the energy is an observable, one requires $H \in \mathcal{V}$. The anticommutator is defined as pointwise multiplication, $f \circ g = fg$, and the commutator is the Poisson bracket[1], $[f,g] = \{f,g\}$.

Definition 3.1.2. An observable $A \in \mathcal{V}$, which has a vanishing commutator with the Hamiltonian $H \in \mathcal{V}$, i.e. $[A,H] = 0$, is called a *constant of motion*.

Lemma 3.1.3. In classical mechanics the constants of motion form a Poisson subalgebra $\mathcal{C}$ of the algebra of observables $\mathcal{V}$.

PROOF. Given $f, g, H \in \mathcal{V}$ with $[H,f] = 0$ and $[H,g] = 0$, the anticommutator $f \circ g$ is also a constant of motion, since from the Leibniz property

$$[H, f \circ g] = [H,f] \circ g + f \circ [H,g] = 0 \ .$$

Because of Jacobi's identity

$$[H,[f,g]] = -[g,[H,f]] - [f,[g,H]] = 0 \ .$$

[1] In geometric terms the Poisson bracket of two functions f and g on the symplectic manifold is defined as $\{f,g\} := \sigma(X_f, X_g)$, where X_f and X_g are defined by $i_{X_f}\sigma = \mathrm{d}f$ and $i_{X_g}\sigma = \mathrm{d}g$.

also the commutator is in $\mathcal{V}_c$. □

3.1.1.3 Presymplectic formulation

Arbitrary time reparametrizations are not allowed in classical mechanics: Time has to be identical with the unique absolute time up to a linear transformation. As we have seen in /1.2.2/, classical mechanics can however be formulated in such a way that the evolution parameter is separated from the time coordinate. This allows for a reparametrization invariant evolution, and a measurement of time still requires an extension of the system.

We now drop the assumption of extensibility /1.3.2/, i.e. we switch to an endophysical description and assume that there is nothing beyond the two-oscillator system. In particular, the evolution parameter τ in (3.1.1) does no more need to be linearly related to absolute time t. The value of τ (more precisely, differences in τ) ceases to be observable; *we have no notion of duration for the endosystem comprising both oscillators.* The dependencies $q_i(\tau)$ now cannot be verified experimentally according to this description, which for this reason is also called *timeless.* (To counter the obvious objection that in order to have physically meaningful properties a physical system must not be strictly closed we refer to [Pri94c].)

To avoid confusion we warn the reader that in the remaining part of this chapter we will frequently have to switch between the exophysical description (with time) and the endophysical description (without time) to illuminate the difference.

Let us explore the endophysical description of the two-oscillator system corresponding to the exophysical one introduced above /3.1.1.1/: We assume that the energy $E \in \mathbb{R}$ is strictly conserved[2],

$$C := H - E = 0 \ . \tag{3.1.2}$$

This constraint[3,4] defines a 3-dimensional compact hypersurface $\Sigma \subset \Gamma$. This is the starting point of Rovelli's pioneering work [Rov90, Rov91c, H9́1, Rov91a]; it was motivated by the analogy of (3.1.2) with the Hamiltonian constraint in canonical quantum gravity.

In the theory of constrained systems one calls the phase space Γ of the original system the *extended phase space* and the hypersurface Σ the *constraint surface.* The constraints (which in the general case could be more than one) generate a gauge group via the Poisson brackets (infinitesimally, $F \mapsto F + \varepsilon \{F, C\}$, with infinitesimal ε; the orbits of the gauge group in phase space are obtained by choosing $F = q, p$). Hájíček [H9́5] pointed out that there are two types of constrained systems, which are quite different with respect to time evolution: *Gauge systems* and *parametrized systems.* In the former, the solution curves (of the equations

[2]For closed systems the boundedness from below of the Hamiltonian is not necessary; it is required for open systems in order to ensure the existence of stable states.

[3]In Dirac's formalism for quantization with constraints [Dir64] one writes $H - E \sim 0$, where $\sim$ means "weakly vanishing".

[4]Strictly speaking, the function $C : \Gamma \to \mathbb{R}$, $x \mapsto C(x)$ is the constraint function, and $C = 0$ is the constraint condition.

of motion) in phase space are transversal to the orbits of the gauge group, while in the latter each solution curve lies within a gauge orbit. The latter is trivially the case for the constrained two-oscillator system, since (the generators of) time evolution and gauge are the same, $\{H, \cdot\} = \{H - E, \cdot\} = \{C, \cdot\}$. (In the terminology of Dirac [Dir64], C is a first-class primary constraint, and no further constraints (secondary, or second-class) are required.) Since $\mathrm{d}H = \mathrm{d}C = 0$, this system is reparametrization invariant /1.2.2/.[5]

In /1.2.2/ we parametrized a system and found that a subsequent restriction to the hypersurface $H + p_0 = 0$ together with the requirement $q_0 = t$ is equivalent to the original system. Here we consider the two-oscillator system as already parametrized and proceed in the opposite direction: we render $q_0 = t$ endophysically meaningless (since there is not absolute time), and we reduce the phase space. The two-oscillator system does however not arise from a symplectic system through parametrization: Assume that a symplectic system (with $4 - 2 = 2$-dimensional phase space) has the coordinates q' and p' and Hamiltonian $H'(q', p')$, and is extended by the canonical variable $q_0 = t$ and its conjugate momentum p_0 as in /1.2.2/, so that the resulting system is the two-oscillator system. The restriction $H'(q', p') + p_0 = 0$ yields a 3-dimensional manifold, but since q_0 is unbounded, this cannot be topologically equivalent to the compact constraint surface Σ. We thus have a genuine presymplectic system $(\Sigma, \bar{\sigma})$ with manifold Σ, where the presymplectic form $\bar{\sigma}$ is defined as the pull-back of σ to Σ.

Since we can change the parametrization of any trajectory diffeomorphically ($\tau \mapsto \tau'$) without altering physics, the mathematical concept representing a *physical solution* is a 1-dimensional manifold with a differentiable structure, but without a fixed parametrization. We allow also $\tau \mapsto -\tau$, since we have no preferred direction of time. Hence, here we use time structure (D), see /1.1/.

The reduced phase space $\Gamma_{\text{phys}} := \Sigma/\sim$ is a partition of the constraint manifold Σ into gauge equivalence classes, where $x \sim x'$ means that x and x' are related by a gauge transformation. Since in our case gauge transformations are time evolutions, Γ_{phys} consists of the solutions of the system, where by a solution we mean a solution curve modulo parametrization. The points in Γ_{phys} are in 1-1 correspondence with the initial conditions at some arbitrary, but fixed initial instant. For the two-oscillator system the points in Γ_{phys} are coordinatized by the parameters A and ϕ. Γ_{phys} is a 2-dimensional manifold and it can be easily seen to have the topology of a 2-sphere S^2: The ranges of the parameters are $0 \leq A \leq E$ and $0 \leq \phi < 2\pi$. For $A = 0$ and $A = E$, respectively, there is a unique physical solution, irrespective of ϕ, corresponding to the poles of the sphere.

3.1.1.4 Complete observables

The physical observables in a theory with constraints are the elements of the Poisson algebra, which commute with the constraints. In our case they have to commute with $H - E$; hence the physical observables are just the constants of

[5] Confer [H9́7] for a general definition of a reparametrization invariant system in geometrical terms.

motion. In contradistinction with the partial observables to be introduced in the /3.1.1.5/ we call them *complete observables*.

Expressing both A and ϕ as phase space functions,

$$\begin{aligned} 4A &= 2E + \omega^{-2}\left(p_1^2 - p_2^2\right) + q_1^2 - q_2^2 \ , \\ \tan\phi &= \frac{p_1 q_2 - p_2 q_1}{\omega^{-1} p_1 p_2 + \omega q_2 q_1} \ , \end{aligned}$$

we find that they have vanishing Poisson brackets with H [6]:

$$\begin{aligned} \{A, H\} &= 0 \ , \\ \{\phi, H\} &= 0 \end{aligned}$$

We choose as Poisson algebra $\mathcal{V} = \mathcal{V}_2$ the algebra of all polynomials in q_i, p_i $(i = 1, 2)$ including the identity $1 : \Gamma \to \mathbb{R}$, $(q, p) \mapsto 1$. This ensures $H \in \mathcal{V}_2$ and $A \in \mathcal{V}_2$, but we have $\tan\phi \notin \mathcal{V}_2$. The following functions are however in $\mathcal{V}_2$:

$$\begin{aligned} L_x &= \sqrt{A\,(E-A)}\cos\phi = \frac{1}{2}\left(\omega^{-1} p_1 p_2 + \omega q_2 q_1\right) \\ L_y &= \sqrt{A\,(E-A)}\sin\phi = \frac{1}{2}\left(p_2 q_1 - p_1 q_2\right) \\ L_z &= A - \frac{E}{2} \qquad\qquad\quad = \frac{1}{4}\left(\omega^{-1} p_1^2 - \omega^{-1} p_2^2 + \omega q_1^2 - \omega q_2^2\right) \end{aligned}$$

It can be easily seen that they generate a Poisson subalgebra

$$\mathcal{C}_2 := \mathrm{gen}\left(\{L_x, L_y, L_z, 1\}\right)$$

of $\mathcal{V}_2$ and $\{H, \mathcal{C}_2\} = 0$. We call C_2 the *algebra of constants of motion* or *algebra of complete observables* of the two-oscillator system.

From $\{q_i, p_j\} = \delta_{ij}$ one moreover calculates

$$\begin{aligned} \{L_x, L_y\} &= L_z \ , \\ \{L_y, L_z\} &= L_x \ , \\ \{L_z, L_x\} &= L_y \ . \end{aligned}$$

A simple calculation also yields

$$4\omega^2 \mathbf{L}^2 := 4\omega^2\left(L_x^2 + L_y^2 + L_z^2\right) = H^2 = (C+E)^2 \ , \tag{3.1.3}$$

so that $(C+E)^2 \in \mathcal{C}_2$. For quantization it is desirable that the constraint C is in $\mathcal{C}_2$, which is not the case here, but for $H \geq 0$ and $E > 0$ the constraint C is equivalent to $C\,(H+E) = C\,(C+2E) = (C+E)^2 - E^2$, which lies in $\mathcal{C}_2$.

[6]Since A and ϕ are two functionally independent constants of motion and our system has two degrees of freedom, we have here a maximally superintegrable system [TTW01].

3.1.1.5 Partial observables

In /1.3.2.1/ we have mentioned the example of a harmonic oscillator and distinguished complete observables (motion) and partial observables (position and time). In the endophysical approach to the two-oscillator system time is not observable, hence not even a partial observable. We assume that the positions of both oscillators are partial observables. This requires that a primitive notion of instantaneity is empirically meaningful, so that we can make instantaneous observations of q_1 and q_2. Moreover, Rovelli's approach to complete and partial observables takes – without mentioning it explicitly – an empirically meaningful notion of simultaneity $\sim$ (cf. our discussion in /1.3.3/) to be given.

The combination of simultaneous observations of the partial observables gives a complete observable. While in classical nonrelativistic theories the absolute Newtonian time is always an independent partial observable, and all other partial observables are time dependent, there is no distinguished independent observable in the timeless two-oscillator model. All we can do is: relate simultaneous observations of the partial observables[7] q_1 and q_2 and infer the values of one of them, e.g. q_1, from the values obtained when measuring another one, e.g. q_2, simultaneously. Given the values of the constants of motion A and ϕ we can eliminate the unphysical parameter τ and obtain from (3.1.1) the complete observable

$$q_2(q_1) = \sqrt{\frac{E}{A} - 1}\left(q_1 \cos\phi \pm \sqrt{2A - q_1^2}\sin\phi\right) .$$

This is to be interpreted as follows: Upon a measurement of q_1 leading to a value $q_1 \in \mathbb{R}$ a prediction of the outcome of a measurement of q_2 is possible: Its value will be one of the two values $q_2(q_1)$. Thus q_1 is an oscillator clock time as discussed in /2.2.2/ and its value plays the role of an (internal) time parameter for oscillator 2. Moreover, q_1 is not a time observable /1.3.3/ for the two-oscillator system, because due to the sign ambiguity the value of q_1 does not uniquely determine the value of q_2.

The complete observable $q_2(q_1)$ is built from constants of motion,

$$q_2(t) = \left(L_z + \frac{E}{2}\right)^{-1}\left(L_y t \pm L_x\sqrt{2L_z + E - t^2}\right) \quad (t \in [-\sqrt{2A}, \sqrt{2A}]) . \tag{3.1.4}$$

When seen as a Taylor series in t all coefficients of this expression can be approximated by elements of $\mathcal{C}_2$.

Using a complete observable it is thus possible to describe change without absolute time.

[7]Obviously, without time one oscillator would have only one instantaneously measurable partial observable, namely position; to make meaningful statements about a relation between partial observables one needs at least two degrees of freedom.

3.1.1.6 Limitations of the model

Let us now examine in what respect the two-oscillator model is too restrictive and can be generalized:

1. Its dynamics is solvable analytically, so that the algebra of constants of motion is known explicitly. This is required for quantization, but in general an exceptionally rare case. In particular for realistic models with interaction no general method is known to overcome this limitation.

2. The model has only two degrees of freedom. In /3.1.2/ we generalize to a field, albeit without interaction.

3. The kinematical group of the model is the Galilei group. If instead we use the Poincaré group and consider a Lorentz boost with velocity v parallel to the line connecting the two oscillators, then the time is dilated by a factor $\sqrt{1-\left(\frac{v}{c}\right)^2}$ in the moving frame. If the oscillation does not happen in the spatial domain (e.g., in electric field strength), or perpendicular to the boost, then the elongations of the oscillators are unaffected by the boost. If in (3.1.1) the phase difference ϕ were also dilated by $\sqrt{1-\left(\frac{v}{c}\right)^2}$, we would obtain exactly the same motion in the moving frame and in the rest frame. This means that absolute time does not show up, just a transformation of constants of motion is required to realize a Lorentz boost.

Consider now the general case (where a Lorentz boost may also affect the elongations) of a system with a Lorentz covariant equation of motion: A solution of the equation of motion in the rest frame can be Lorentz transformed and becomes a solution of the equation of motion in a moving frame. Since a choice of solution corresponds to a choice of the constants of motion, this induces a transformation of the constants of motion, disregarding absolute time.

4. Even in the case of free oscillators the equality of their frequencies is a main limitation. In /2.2.2/ we have discussed broadly if and how one can infer the position of one oscillator from that of the other in the general case.

3.1.1.7 Evolving constants of motion

In a theory which deals with solutions of the equations of motion, disregarding the global time parameter, the question naturally arises how evolution is possible at all, if only constants of motion are physical. In the case of the two-oscillator system we have already answered this question in that the family of observables (3.1.4) represents evolution w.r.t. the internal time $t = q_1$. The general case of a dynamical system with N degrees of freedom has been investigated by Rovelli [Rov91c]. Here we sketch the classical part of this theory.

Let H be a Hamiltonian constraint on a presymplectic space with coordinates q_n, p_n ($n = 1, \ldots, N$). Let us choose $t := q_1$ to be the clock time. Then for $i = 2, \ldots, N$ t-dependent families of observables $Q_i(t) = Q_i(t; q_n, p_n)$ are wanted, which measure the value of the coordinate q_i at the time $t = q_1$. On the one hand due to reparametrization invariance w.r.t. absolute time they have to fulfil

$$\{Q_i(t; q_n, p_n), H(q_n, p_n)\} = 0 \qquad (t \in \mathrm{Ran}(q_1)) \ .$$

On the other hand the requirement of constancy along each trajectory does not fix the constant value of the observable; this is done by the equation

$$Q_i(t; t, q_2, \ldots p_n) = q_i \qquad (t \in \mathrm{Ran}(q_1)) \ ,$$

which forces the single, instantaneous observable $Q_i(t)$ (t fixed!) to have the desired value q_i at that instant t, which fulfils $t = q_1$. In other words, the value of the observable is determined by the value of the variable q_i at that point along the trajectory, where the trajectory intersects the simultaneity surface $q_1 = t$. The t-dependent families $Q_i(t)$ are called *evolving constants of motion*. E.g., in the two-oscillator system the family (3.1.4) of complete observables is an evolving constant of motion.

The two conditions defining an evolving constant of motion can be generalized to the case where the internal time is not given directly as a phase space coordinate, but where it is an arbitrary function on phase space, $q_T = q_T(q_n, p_n)$. Moreover, an observable does not need to be restricted to yield one of the phase space coordinates; instead any phase space function $q = q(q_n, p_n)$ defines an evolving constant of motion Q: The two conditions then become

$$\{Q(T), H\} = 0 \qquad (T \in \mathrm{Ran}(q_T)) \ ,$$
$$Q(q_T) = q \ .$$

We finally mention a method to propagate $Q(T)$ in T [Rov91c, H$\acute{9}$1, Rov91a], at least at the points where $T = q_T$:

$$\left.\frac{\partial Q(T)}{\partial T}\right|_{T=q_T} \{q_T, H\} = \left.\frac{\partial Q(T)}{\partial T}\right|_{T=q_T} \frac{\mathrm{d}q_T}{\mathrm{d}t} = \frac{\mathrm{d}Q(q_T)}{\mathrm{d}t} = \frac{\mathrm{d}q}{\mathrm{d}t} = \{q, H\}$$

Here the parameter t can be chosen arbitrarily. The Poisson brackets on the left and right hand can be calculated and thus one obtains $\left.\frac{\partial Q(T)}{\partial T}\right|_{T=q_T}$.

3.1.2 Free field

We now generalize the foregoing results to a scalar field in one dimension, proceeding similarly: After introducing the Hamiltonian formulation (with absolute time) /3.1.2.1/ we exhibit the algebra of constants of motion /3.1.2.2/. In order to obtain them explicitly we have to confine ourselves to the noninteracting case; for the same reason we also specialize to a massless field and to one dimension only. The basic idea is that we can identify simple constants of motion involving triples of field modes, two of whose frequencies add up to the third one. After switching to the presymplectic formulation /3.1.2.3/ we discuss the meaning of the constants of motion /3.1.2.4/ and consider particular partial observables /3.1.1.5/.

3.1.2.1 Hamiltonian formulation

We consider a free Klein-Gordon field ξ with mass m in one dimension. (Later on we will specialize to $m = 0$.) The phase space is $\Gamma = \mathcal{S}(\mathbb{R}, \mathbb{R}) \times \mathcal{S}(\mathbb{R}, \mathbb{R})$, where $(\xi, \eta) \in \Gamma$ consists of the initial field $\xi(x)$ and its initial momentum $\eta(x)$.

On Schwartz space $\mathcal{S}(\mathbb{R}, \mathbb{R})$ we have a scalar product $\langle f, g\rangle := \int_{\mathbb{R}} f(x)g(x)\mathrm{d}x$, which allows us to write the Hamiltonian as a spatial integral:

$$H(\xi, \eta) = \frac{1}{2}\langle \eta, \eta\rangle + \frac{1}{2}\left\langle \left(m^2 - \frac{\partial^2}{\partial x^2}\right)\xi, \xi\right\rangle$$

Performing a Fourier transform (symbolized by $\hat{\cdot}$) we have

$$\begin{aligned} H(\hat{\xi}, \hat{\eta}) &= \frac{1}{2}\langle \hat{\eta}, \hat{\eta}\rangle + \frac{1}{2}\left\langle \omega^2\hat{\xi}, \hat{\xi}\right\rangle \\ &= \int_{\mathbb{R}} H(k)\mathrm{d}k \ , \end{aligned}$$

where $\omega(k) = \sqrt{k^2 + m^2}$, and $\hat{\xi}, \hat{\eta} \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ are the Fourier transforms of ξ, η, and

$$H(k) = \frac{1}{2}\hat{\eta}(k)^2 + \frac{1}{2}\omega(k)^2\hat{\xi}(k)^2$$

is the Hamiltonian density.

Canonically, one has position observables $q(f) = \left\langle \hat{\xi}, f\right\rangle$ and momentum observables $p(f) = \langle \hat{\eta}, f\rangle$, which are continuous linear functionals of $\hat{\xi}$ and $\hat{\eta}$, respectively. Here $f \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ is a test function. – Instead of smearing out the fields, we can also use point limits and define a continuous family of observables by $q_{k_0} := \left\langle \hat{\xi}(k), \delta(k - k_0)\right\rangle = \hat{\xi}(k_0)$ and $p_{k_0} := \langle \hat{\eta}(k), \delta(k - k_0)\rangle = \hat{\eta}(k_0)$.

Since q_k and p_k are continuous in k and the rational numbers are dense in the reals, one can restrict to observables with rational values of k. One assumes that these, in turn, are approximated in the limit $L \to \infty$ by the Fourier transforms q_{k_n}, p_{k_n} of position and momentum observables in a box of length L with periodic boundary conditions. Here we have $k_n = nk^{(L)}$, $n \in \mathbb{Z}\setminus\{0\}$ and $k^{(L)} = \frac{2\pi}{L}$. We exclude $n = 0$, because we will consider the the infrared singularity in the massless case later on.

To simplify notation we write just n instead of the discretized index k_n, e.g. q_n instead of q_{k_n}. It is assumed that in the limit $L \to \infty$ the Hamiltonian H can be approximated by "box" Hamiltonians

$$H^{(L)} = \sum_{n\in\mathbb{Z}\setminus\{0\}} H_n \ ,$$

where

$$H_n = \frac{1}{2}\left(p_n^2 + \omega_n^2 q_n^2\right) \ .$$

(We omit the superscript (L) for notational convenience.)

The symplectic form reads

$$\sigma = \sum_{n\in\mathbb{Z}\setminus\{0\}} \mathrm{d}q_n \wedge \mathrm{d}p_n \ ,$$

and the basic Poisson brackets are

$$\{q_n, p_m\} = \delta_{nm} \ .$$

The equations of motion are those of independent harmonic oscillators with the well known solutions:

$$q_n(t) = A_n \sin(\omega_n t + \phi_n) \ , \tag{3.1.5}$$

$$p_n(t) = A_n \omega_n \cos(\omega_n t + \phi_n) \ , \tag{3.1.6}$$

where $A_n \in \mathbb{R}_0^+$ are amplitudes and $\phi_n \in [0; 2\pi[$ initial phases.

We take the Poisson algebra generated by the q_n and p_n with the pointwise product as commutator and the Poisson bracket as anti-commutator to be the *algebra of observables* $\mathcal{V}_\infty^{(L)}$ *of the free field.* (Again, we may omit the superscript (L) and write $\mathcal{V}_\infty$.) As already seen in lemma 3.1.3 the *algebra of constants of motion of the free field*, which we denote by $\mathcal{C}_\infty$, is a Poisson subalgebra of the algebra of observables.

3.1.2.2 Constants of motion

In order to study the explicit form of the constants of motion we introduce complexified observables

$$z_n = \sqrt{\frac{\omega_n}{2}} q_n + \frac{\mathrm{i}}{\sqrt{2\omega_n}} p_n \ ,$$

with the inverse relations

$$q_n = \frac{z_n + \overline{z}_n}{\sqrt{2\omega_n}} \ , \tag{3.1.7}$$

$$p_n = \sqrt{\omega_n} \frac{z_n - \overline{z}_n}{\mathrm{i}\sqrt{2}} \ . \tag{3.1.8}$$

From $\{q_n, p_m\} = \delta_{nm}$ we have $\{\overline{z}_n, z_m\} = \mathrm{i}\delta_{nm}$ and in the following calculations $\overline{z}$ and z behave similarly to creation and annihilation operators, disregarding of course operator ordering. Furthermore, $H_n = \omega_n \overline{z}_n z_n$, $\{H, z_m\} = \{H_n, z_m\} = i\omega_n z_n \delta_{nm}$ and $\{H, \overline{z}_m\} = \{H_n, \overline{z}_m\} = -i\omega_n \overline{z}_n \delta_{nm}$. From these equations one easily shows:

Lemma 3.1.4. Let $f = \prod_{i=1}^{M} \overline{z}_{m_i} \prod_{j=1}^{N} z_{n_j}$ with $m_i, n_j \in \mathbb{N}$ and $M, N \in \mathbb{N}_0$. Then

$$\{H, f\} = -\mathrm{i} \left(\sum_{i=1}^{M} \omega_{m_i} - \sum_{j=1}^{N} \omega_{n_j} \right) f \ .$$

In the case of a massless field (with dispersion relation $\omega(k) = |k|$) f is a constant of motion, iff

$$\sum_{i=i}^{M} |m_i| - \sum_{j=1}^{N} |n_j| = 0 \ . \tag{3.1.9}$$

From now on we consider a massless field.

Proposition 3.1.5. The only polynomials up to second order in z and $\overline{z}$, which are constants of motion, are linear combinations of

$$h_n^{\pm} := \overline{z}_n z_{\pm n}$$

($n \in \mathbb{Z}\setminus\{0\}$) and a multiple of 1.

The only polynomials of third order in z, $\overline{z}$, which are constants of motion, are linear combinations of ($m, n \in \mathbb{Z}\setminus\{0\}$)

$$a_{mn}^{\pm} := z_m z_n \overline{z}_{\pm(|m|+|n|)}$$

and

$$b_{mn}^{\pm} := \overline{z}_m \overline{z}_n z_{\pm(|m|+|n|)} \ .$$

Proof. From (3.1.9) it is clear that there must be both a $\overline{z}$- and a z-term and only the asserted combinations of indices are possible. ☐

The following relations are obvious[8]:

$$\begin{aligned} h_n^{\pm} &= \overline{h_{\pm n}^{\pm}} \\ b_{mn}^{\pm} &= \overline{a_{mn}^{\pm}} \\ a_{mn}^{\pm} &= a_{nm}^{\pm} \\ b_{mn}^{\pm} &= b_{nm}^{\pm} \end{aligned}$$

For later use we define

$$\mathcal{C}_{\infty,g} := \left\{ h_n^{\pm}, a_{mn}^{\pm}, b_{mn}^{\pm} : \ m, n \in \mathbb{Z}\setminus\{0\} \right\} \cup \{1\} \ .$$

One can also easily calculate the following commutators, which we will frequently use below. We distinguish the two independent sign alternatives by a prime and abbreviate with $\pm \cdot \pm'$ the sign of $(\pm 1)\cdot(\pm' 1)$:

$$\left\{ h_n^{\pm}, h_m^{\pm'} \right\} = -\mathrm{i}\delta_{\pm n,m} h_n^{\pm\cdot\pm'} + \mathrm{i}\delta_{n,\pm' m} h_m^{\pm\cdot\pm'}$$

$$\left\{ a_{ij}^{\pm}, h_n^{\pm'} \right\} = -\mathrm{i}\delta_{in} a_{j,\pm' i}^{\pm} - \mathrm{i}\delta_{jn} a_{i,\pm' j}^{\pm} + \mathrm{i}\delta_{\pm(|i|+|j|),\pm' n} a_{ij}^{\pm\cdot\pm'}$$

$$\left\{ b_{ij}^{\pm}, h_n^{\pm'} \right\} = +\mathrm{i}\delta_{i,\pm' n} b_{ij}^{\pm} + \mathrm{i}\delta_{j,\pm' n} b_{ij}^{\pm} - \mathrm{i}\delta_{\pm(|i|+|j|),n} b_{ij}^{\pm\cdot\pm'}$$

[8] Setting formally $z_n = \bar{z}_n = 1$ one could furthermore extend the allowed range of the indices of $a_{mn}^{\pm}$ and $b_{mn}^{\pm}$ to $m, n \in \mathbb{Z}$ (including 0) and regard $h_n^{\pm}$ as a special third order constant of motion: $h_n^{\pm} = b_{n,0}^{\pm} = a_{\pm n,0}^{\mathrm{sign}(n)}$.

$$\begin{aligned}\left\{a^{\pm}_{ij}, a^{\pm'}_{mn}\right\} = &-\mathrm{i}\delta_{i,\pm'(|m|+|n|)} z_m z_n z_j \overline{z}_{\pm(|m|+|n|+|j|)}\\ &- \mathrm{i}\delta_{j,\pm'(|m|+|n|)} z_m z_n z_i \overline{z}_{\pm(|m|+|n|+|i|)}\\ &+ \mathrm{i}\delta_{n,\pm(|i|+|j|)} z_m z_i z_j \overline{z}_{\pm'(|m|+|j|+|j|)}\\ &+ \mathrm{i}\delta_{m,\pm(|i|+|j|)} z_n z_i z_j \overline{z}_{\pm'(|n|+|j|+|j|)}\end{aligned}$$

$$\begin{aligned}\left\{b^{\pm}_{ij}, b^{\pm'}_{mn}\right\} = &+\mathrm{i}\delta_{i,\pm'(|m|+|n|)} \overline{z}_m \overline{z}_n \overline{z}_j z_{\pm(|m|+|n|+|j|)}\\ &+ \mathrm{i}\delta_{j,\pm'(|m|+|n|)} \overline{z}_m \overline{z}_n \overline{z}_i z_{\pm(|m|+|n|+|i|)}\\ &- \mathrm{i}\delta_{n,\pm(|i|+|j|)} \overline{z}_m \overline{z}_i \overline{z}_j z_{\pm'(|m|+|j|+|j|)}\\ &- \mathrm{i}\delta_{m,\pm(|i|+|j|)} \overline{z}_n \overline{z}_i \overline{z}_j z_{\pm'(|n|+|j|+|j|)}\end{aligned}$$

$$\begin{aligned}\left\{a^{\pm}_{ij}, b^{\pm'}_{mn}\right\} = &-\mathrm{i}\delta_{im} z_j \overline{z}_{\pm(|i|+|j|)} \overline{z}_n z_{\pm'(|m|+|n|)}\\ &- \mathrm{i}\delta_{in} z_j \overline{z}_{\pm(|i|+|j|)} \overline{z}_m z_{\pm'(|m|+|n|)}\\ &- \mathrm{i}\delta_{jm} z_i \overline{z}_{\pm(|i|+|j|)} \overline{z}_n z_{\pm'(|m|+|n|)}\\ &- \mathrm{i}\delta_{jn} z_i \overline{z}_{\pm(|i|+|j|)} \overline{z}_m z_{\pm'(|m|+|n|)}\\ &+ \mathrm{i}\delta_{\pm(|i|+|j|),\pm'(|m|+|n|)} z_i z_j \overline{z}_m \overline{z}_n\end{aligned} \qquad (3.1.10)$$

While taking the commutator with $h^{\pm}_n$ leaves span$(\mathcal{C}_{\infty,g})$ invariant, the other commutators may lead to higher order polynomials. In general, taking the commutator of two polynomials f, g in $z_\cdot, \overline{z}_\cdot$ gives a polynomial $\{f, g\}$, which is (at least) two degrees smaller than their product, and in which each $z_\cdot$ term (each $\overline{z}_\cdot$ term) in f has canceled a $\overline{z}_\cdot$ term (a $z_\cdot$ term) in g. Taking the commutator of a polynomial f with $a^\cdot_\cdot$, which has the structure $zz\overline{z}$ (second degree in z, first degree in $\overline{z}$), increases the degree of f in z by one. Analogously, the commutator with $b^\cdot_\cdot$ increases the degree in $\overline{z}$ by one.

Theorem 3.1.6. *The Poisson algebra $\mathcal{C}_\infty$ of constants of motion polynomial in q_n and p_n of the free massless scalar field in 1+1 dimensions is generated by $\mathcal{C}_{\infty,g}$,*

$$gen\,(\mathcal{C}_{\infty,g}) = \mathcal{C}_\infty \ .$$

Proof. The inclusion gen$(\mathcal{C}_{\infty,g}) \subset \mathcal{C}_\infty$ is obvious. In order to prove the converse inclusion let $f \in \mathcal{C}_\infty$. Using (3.1.7,3.1.8) we rewrite f as a polynomial in z and $\overline{z}$. For $\{H, f\}$ to vanish all summands of f must vanish due to linear independence; up to a scalar multiple they are of the form considered in lemma 3.1.4. It suffices to show that each such term can be obtained from $\mathcal{C}_{\infty,g}$ by taking commutators and anti-commutators: Let f be as in lemma 3.1.4. We proceed by complete induction over the degree of f. If $M + N \leq 3$, then according to proposition 3.1.5 we immediately have $f \in \mathcal{C}_{\infty,g}$ (up to a scalar multiple).

Now for the case of a constant of motion f of degree $M + N > 3$. Our induction hypothesis states that all constants of motion up to degree $M + N - 1$ are in gen$(\mathcal{C}_{\infty,g})$. We show that f can be obtained from such polynomials. We first exclude a simple case: If there are i_0, j_0 ($1 \leq i_0 \leq M$, $1 \leq j_0 \leq N$) with

$m_{i_0} = \pm n_{j_0}$, then

$$f = \left(\prod_{i=1, i\neq i_0}^{M} \overline{z}_{m_i} \prod_{j=1, j\neq j_0}^{N} z_{n_j} \right) h^{\pm}_{m_{i_0}} ,$$

where the expression in the parentheses is a constant of motion, since

$$\sum_{i=1, i\neq i_0}^{M} |m_i| - \sum_{j=1, j\neq j_0}^{N} |n_j| = 0 .$$

Moreover it is of degree $M + N - 2$ and therefore by hypothesis in gen$(\mathcal{C}_{\infty,g})$. Since also $\mathrm{h}^{\pm}_{\mathrm{m}_{\mathrm{i}_0}} \in \mathcal{C}_{\infty,g}$, we have $f \in$ gen$(\mathcal{C}_{\infty,g})$.

Because of $M+N > 3$ we have either $M \geq 2$, or $N \geq 2$. In case $M \geq 2$ we define

$$R = \prod_{i=3}^{M} \overline{z}_{m_i} \prod_{j=1}^{N} z_{n_j}$$

so that $f = R\overline{z}_{m_1}\overline{z}_{m_2}$. The expression $R\overline{z}_{|m_1|+|m_2|}$ is a constant of motion of degree $M + N - 1$ and therefore in gen$(\mathcal{C}_{\infty,g})$. We decompose $R = R'\overline{z}^l_{|m_1|+|m_2|}$, where R' does not contain the factor $\overline{z}_{|m_1|+|m_2|}$ and $0 \leq l < M + N - 2$. Finally we calculate the commutator of R with $b^+_{m_1 m_2} \in \mathcal{C}_0$:

$$\begin{aligned} \left\{R\overline{z}_{|m_1|+|m_2|}, b^+_{m_1 m_2}\right\} &= \left\{R\overline{z}_{|m_1|+|m_2|}, \overline{z}_{m_1}\overline{z}_{m_2} z_{|m_1|+|m_2|}\right\} \\ &= \mathrm{i}R\overline{z}_{m_1}\overline{z}_{m_2} + \overline{z}_{|m_1|+|m_2|} \left\{R, \overline{z}_{m_1}\overline{z}_{m_2} z_{|m_1|+|m_2|}\right\} \\ &= \mathrm{i}R\overline{z}_{m_1}\overline{z}_{m_2} + \mathrm{i}lR'\overline{z}_{m_1}\overline{z}_{m_2}\overline{z}^l_{|m_1|+|m_2|} \\ &= \mathrm{i}\,(l+1)\,f \end{aligned}$$

The second last equation holds, because above we excluded the possibility that f (and thus R) contains z_{m_1} or z_{m_2}, so that $\overline{z}_{m_1}$ and $\overline{z}_{m_2}$ commute with R. We conclude that f is the commutator of constants of motion of degree lower than $M + N$. The second case ($N \geq 2$) can be treated analogously, if one interchanges the roles of $\overline{z}$ and z and takes the commutator with $a^+_{..}$ instead of $b^+_{..}$. □

3.1.2.3 Presymplectic formulation

As in /3.1.1.3/ we switch to an endophysical description. We drop the assumption of extensibility and employ the constraint

$$C := H - E = \sum_{n\in\mathbb{Z}\setminus\{0\}} H_n - E = 0$$

with $E \in \mathbb{R}$. For the solutions (3.1.5), (3.1.6) it takes the form

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} A_n^2 \omega_n^2 = E .$$

The topology is nontrivial, since the field is an infinite dimensional Hamiltonian system [CM74]. But even if one imposes a cutoff confining to N field modes, it remains nontrivial: The ranges of the parameters are $0 \leq A_n^2\omega_n^2 \leq E$ and $0 \leq \phi_n < 2\pi$. The A_n alone form an N-sphere S^N, while the ϕ_n form an N-torus T^N. The whole structure is however not simply the direct product, since as in the case of two oscillators at points where $A_n = 0$ for some n different values of ϕ_n are indistinguishable.

3.1.2.4 Complete observables

To unravel the physical meaning of the elements of $\mathcal{C}_{\infty,g}$ we express them through q_n and p_n, and via (3.1.5) and (3.1.6) also through A_n and ϕ_n:

$$\begin{aligned} h_n^+ &= \frac{1}{2\omega_n}p_n^2 + \frac{\omega_n}{2}q_n^2 = \omega_n^{-1}H_n &&= A_n^2\omega_n \\ h_n^- + \overline{h_n^-} &= \frac{1}{\omega_n}p_n p_{-n} + \omega_n q_n q_{-n} &&= \omega_n A_n A_{-n}\cos\left(\phi_n - \phi_{-n}\right) \\ h_n^- - \overline{h_n^-} &= \mathrm{i}\left(q_n p_{-n} - q_{-n}p_n\right) &&= \omega_n A_n A_{-n}\sin\left(\phi_n - \phi_{-n}\right) \end{aligned}$$

In the following expressions the same choice is made on both sign alternatives:

$$\begin{aligned} a_{mn}^{\pm} + b_{mn}^{\pm} = {} & \frac{1}{\sqrt{2\omega_m\omega_n\omega_{\pm(|m|+|n|)}}}\big(\omega_m\omega_n\omega_{|m|+|n|}q_m q_n q_{\pm(|m|+|n|)} \\ & + \omega_m q_m p_n p_{\pm(|m|+|n|)} \\ & + \omega_n p_m q_n p_{\pm(|m|+|n|)} \\ & - \omega_{|m|+|n|}p_m p_n q_{\pm(|m|+|n|)}\big) \\ = {} & \sqrt{\frac{\omega_m\omega_n\omega_{|m|+|n|}}{2}}A_m A_n A_{\pm(|m|+|n|)}\sin\big(\phi_m + \phi_n - \phi_{\pm(|m|+|n|)}\big) \end{aligned}$$

In the last equation we have made use of $\omega_{|m|+|n|} = \omega_m + \omega_n$, whereby t-dependent terms cancel. A similar calculation gives

$$\begin{aligned} (-\mathrm{i})\left(a_{mn}^{\pm} - b_{mn}^{\pm}\right) = {} & \frac{1}{\sqrt{2\omega_m\omega_n\omega_{\pm(|m|+|n|)}}}\big(p_m p_n p_{\pm(|m|+|n|)} \\ & + \omega_m\omega_{|m|+|n|}q_m p_n q_{\pm(|m|+|n|)} \\ & + \omega_n\omega_{|m|+|n|}p_m q_n q_{\pm(|m|+|n|)} \\ & - \omega_m\omega_n q_m q_n p_{\pm(|m|+|n|)}\big) \\ = {} & \sqrt{\frac{\omega_m\omega_n\omega_{|m|+|n|}}{2}}A_m A_n A_{\pm(|m|+|n|)}\cos\big(\phi_m + \phi_n - \phi_{\pm(|m|+|n|)}\big) \end{aligned}$$

Combining the last two equations leads to:

$$2\mathrm{i}b_{mn}^{\pm} = \sqrt{\frac{\omega_m\omega_n\omega_{|m|+|n|}}{2}}A_m A_n A_{\pm(|m|+|n|)}e^{\mathrm{i}(\phi_m + \phi_n - \phi_{\pm(|m|+|n|)})} \qquad (3.1.11)$$

The (in)dependence of observables can be analyzed when considering only finitely many field modes $n = -N, \ldots, -1, 1, \ldots, N$ ($N \in \mathbb{N}$ being an ultraviolet cutoff):

- The $2N$ amplitudes A_n are determined by h_n^+.

- Given the amplitudes, N further relations between the initial phases are given by h_n^-:

$$\tan\left(\phi_n - \phi_{-n}\right) = \frac{h_n^- - \overline{h_n^-}}{h_n^- + \overline{h_n^-}} \qquad (n = 1, \ldots, N) \tag{3.1.12}$$

- Given the amplitudes, the expressions $a_{mn}^{\pm}$ and $b_{mn}^{\pm}$ together determine the expressions $\phi_m + \phi_n - \phi_{\pm(|m|+|n|)}$ involving three initial phases up to a multiple of 2π. (The frequencies of two of these initial phases add up to the frequency of the third one.) Taking into account h_n^- one can constrict to positive m, n and $+\left(|m| + |n|\right)$, and one sees that there are only $N-1$ linearly independent ones among the expressions $\phi_m + \phi_n - \phi_{m+n}$, since with $m = 1$ for $n = 1, \ldots, N-1$ the initial phase ϕ_{n+1} can be obtained from $\phi_1 + \phi_n - \phi_{n+1}$ and ϕ_1 recursively; ϕ_1 itself cannot be determined, since this would amount to measuring an absolute phase.
- Altogether there are $2N + N + N - 1 = 4N - 1$ independent constants of motion, just one less than in the exophysical situation with the $4N$ parameters A_n and ϕ_n. The Hamiltonian constraint finally reduces the number of independent variables to $4N - 2$. (Compare the two-oscillator case, where $N = 1$.)

In the limit where $L, N \to \infty$ such that $Nk^{(L)} = Nk_1 = \text{const.}$, the field modes become dense, and because of the continuity[9] of ϕ_k in k the choice of one branch of $\arctan\left(\phi_1 + \phi_{k_i} - \phi_{k_i+k_1}\right)$ for some i determines the branches for all i; the 2π-ambiguity thus disappears.

We have thus shown that in this limit already the constants of motion

$$\begin{aligned} &h_n^+ : \; n = -N, \ldots, -1, 1, \ldots, N \;, \\ &h_n^- : \; n = 1, \ldots, N \;, \\ &a_{1,n}^+, b_{1,n}^+ : \; n = 1, \ldots, N-1 \;, \end{aligned}$$

which are in $\mathcal{C}_{\infty,g}$, uniquely determine a solution (a point in reduced phase space).

3.1.2.5 Partial observables

The elements of $\text{gen}(\mathcal{C}_{\infty,g})$ are complete observables, since their values – once they are known at one instant – can be predicted (and retrodicted): they are constant.

The q_n and p_n are partial observables, since their values cannot predicted without resorting to results of related simultaneous measurements. One can use e.g. $a_{mn}^{\pm} + b_{mn}^{\pm}$ to relate the partial observables q_n, q_m and $q_{\pm(|m|+|n|)}$, where the momenta p_l $\left(l = m, n, \pm\left(|m| + |n|\right)\right)$ can be obtained from q_l and h_l^+:

$$p_l = \pm\sqrt{2\omega_l h_l^+ - \omega_l^2 q_l^2}$$

Predictions of one partial observable (e.g. $q_{\pm(|m|+|n|)}$) in general require two other partial observables (e.g. q_n and q_m). (An exception is e.g. the case $m = n$.) This

[9] This derives via (3.1.5) and (3.1.6) from the continuity of q_k and p_k, which in turn stems from the fact that $\hat{\xi}, \hat{\eta} \in \mathcal{S}\left(\mathbb{R}, \mathbb{R}\right)$.

prediction is in general not unique, since an instantaneous configuration of the $q_{.}$ contains no information about the sign of the $p_{.}$ (cf. /2.1/).

We are interested in relating q_n ($n > 1$) with q_1, since in the limit $L \to \infty$ the oscillation period of the q_1-oscillator becomes infinitely long, effectively approximating a free particle, and thus providing an internal clock time with range $\mathbb{R}$. We have from (3.1.7) and (3.1.8)

$$\begin{aligned} q_n(q_1) &= A_n \sin\left(n \arcsin \frac{q_1}{A_1} + \phi_n - n\phi_1\right) \\ &= A_n T_s \cos(\phi_n - n\phi_1) + A_n T_c \sin(\phi_n - n\phi_1) \ , \end{aligned} \tag{3.1.13}$$

where $T_s = \sin\left(n \arcsin \frac{q_1}{A_1}\right)$ and $T_c = \cos\left(n \arcsin \frac{q_1}{A_1}\right)$ are expressions involving powers of $\frac{q_1}{A_1}$ and arcsin has 2 branches. While $A_1 = \omega_1^{-\frac{1}{2}}(h_1^+)^{\frac{1}{2}}$ and $A_n = \omega_n^{-\frac{1}{2}}(h_n^+)^{\frac{1}{2}}$ are constants of motion, q_1 is a partial observable; the terms $\cos(\phi_n - n\phi_1)$ and $\sin(\phi_n - n\phi_1)$ are constants of motion, since they can be obtained as the real and imaginary part, respectively, of

$$\begin{aligned} e^{\mathrm{i}(\phi_n - n\phi_1)} &= e^{\mathrm{i}(\phi_n - \phi_{n-1} - \phi_1)} e^{\mathrm{i}(\phi_{n-1} - \phi_{n-2} - \phi_1)} \cdots e^{\mathrm{i}(\phi_2 - \phi_1 - \phi_1)} \\ &= \prod_{\nu=1}^{n-1} 2\mathrm{i} b_{\nu,1}^+ \left(\sqrt{\frac{\omega_\nu \omega_1 \omega_{\nu+1}}{2}} A_\nu A_1 A_{\nu+1}\right)^{-1} \\ &= \prod_{\nu=1}^{n-1} 2\sqrt{2}\mathrm{i} b_{\nu,1}^+ \left(h_\nu^+ h_1^+ h_{\nu+1}^+\right)^{-\frac{1}{2}} \ , \end{aligned} \tag{3.1.14}$$

where we have used equation (3.1.11). Strictly speaking, $(h_\nu^+)^{-1} \notin \mathcal{C}_\infty$, but it can be approximated through constants of motion using the formula $x^{-1} = \sum_{\nu=0}^{\infty}(1-x)^\nu$ $(x \neq 0)$. Equation (3.1.13) thus relates the partial observables q_n and q_1 via constants of motion.

A similar relation between q_n and q_{-1} holds in the case $n < -1$. The initial phases ϕ_1 and ϕ_{-1} are also connected via a constant of motion, see (3.1.12) for $n = 1$, so that all q_n with $n \in \mathbb{Z}\setminus\{0\}$ can be related via constants of motion to q_1.

In the limit of infinite box length ($L, n \to \infty$ such that $k_n = nk_1 = nk^{(L)} = n\frac{2\pi}{L} =$ const.) the relation $q_{k_n}(q_{k_1})$ can be interpreted as a time dependence $q_k(t)$, where the time parameter t *depends on the infrared behaviour of the state of the field* and allows to predict q_k up to a twofold ambiguity.

3.2 Relational quantization

In this section we discuss the quantization of the algebra of constants of motion. After introducing canonical quantization and quantization in general /3.2.1/ we describe Rovelli's [Rov90] quantization of the two-oscillator system /3.2.2/. In /3.2.3/ we treat the free field.

3.2.1 On quantization

Quantization is a procedure for obtaining a quantum theory whose classical limit coincides with a given classical theory and whose prediction in the quantum domain cannot be falsified. Although there are numerous quantization methods (main classes are Hilbert space based methods, algebraic methods and path integral methods) and plenty of literature, no unique method of quantization could be singled out as the physically correct one; after all there may be inequivalent quantum theories having the same classical limit and different but untested, or untestable predictions.

The best known quantization method is traditional *canonical quantization*. It starts from the Poisson algebra $\mathcal{V}$ /3.1.1.2/ of observables of a classical mechanical system with n degrees of freedom and the goal is to translate the observables into symmetric operators on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$. A well known theorem due to Groenewold and van Hove (see [Giu03] for a review and references and also an introduction to canonical quantization) states that a full quantization of the Poisson algebra $\mathcal{V} = \mathcal{V}_{\text{poly}} := P(2n)$ of all polynomials of canonical positions q_i and momenta p_i $(i = 1, \ldots, n)$ does not exist. The best one can do is to quantize a Lie subalgebra $\mathcal{V}_{\text{quant}}$ of $\mathcal{V}_{\text{poly}}$ (i.e., a subalgebra w.r.t. the Poisson brackets only), e.g. the algebra of polynomials of at most first order,

$$\mathcal{V}_{\text{quant}} = \mathcal{V}_{\text{poly,1}} := \text{span}\left(\{q_i, p_i : i = 1, \ldots, n\} \cup \{1\}\right) .$$

(Here 1 is the observable which maps all of phase space to the value $1 \in \mathbb{R}$.) $\mathcal{V}_{\text{poly,1}}$ is a Lie subalgebra, because $\{q_i, p_j\} = \delta_{ij} \cdot 1 \in \mathcal{V}_{\text{poly,1}}$ and $\{q_i, q_j\} = \{p_i, p_j\} = 0 \in \mathcal{V}_{\text{poly,1}}$.

Le us make these statements more precise. In general there are two rules how to choose a pair of subalgebras $(\mathcal{V}_{\text{irr}}, \mathcal{V}_{\text{quant}})$ of $\mathcal{V}_{\text{poly}}$ for quantization, compare [Giu03]:

Canonical quantization rules:

(a) $\mathcal{V}_{\text{irr}} \subset \mathcal{V}_{\text{poly}}$ must contain 'basic observables': With the help of these basic observables one must be able to coordinatize phase space. $\mathcal{V}_{\text{irr}}$ shall be minimal in this respect.
Clearly, for $\mathcal{V}_{\text{irr}} = \mathcal{V}_{\text{poly,1}}$ the basic observables $\{q_i, p_i : i = 1, \ldots, n\}$ provide such a coordinatization and there is no proper Lie subalgebra of $\mathcal{V}_{\text{poly,1}}$ with this property.

(b) $\mathcal{V}_{\text{quant}}$ shall be the maximal Lie subalgebra of $\mathcal{V}_{\text{poly}}$ containing $\mathcal{V}_{\text{irr}}$, for which a Lie homomorphism (*quantization map*)

$$Q : \mathcal{V}_{\text{quant}} \to \text{SYM}(\mathcal{H}),\ f \mapsto Q(f)$$

(where $\mathrm{SYM}(\mathcal{H})$ is the set of symmetric operators[10] on $\mathcal{H}$) can be found, i.e. a linear map which intertwines the Lie structures[11] $\{\cdot,\cdot\}$ and $\frac{1}{\mathrm{i}\hbar}[\cdot,\cdot]$ on $\mathcal{V}_{\mathrm{poly}}$ and $\mathrm{SYM}(\mathcal{H})$, respectively,

$$Q\left(\{f,g\}\right) = \frac{1}{\mathrm{i}\hbar}\left[Q(f),Q(g)\right] \qquad (f,g\in\mathcal{V}_{\mathrm{quant}})\ , \tag{3.2.1}$$

so that $Q(\mathcal{V}_{\mathrm{irr}})$ acts almost irreducibly, i.e. up to finite multiplicity on $\mathcal{H}$.

For $\mathcal{V}_{\mathrm{irr}} = \mathcal{V}_{\mathrm{poly},1} = \mathcal{V}_{\mathrm{quant}}$ the canonical quantization map

$$\begin{aligned} Q_{\mathrm{can},1} : \mathcal{V}_{\mathrm{poly},1} &\to \mathrm{SYM}(\mathcal{H}),\\ 1 &\mapsto \mathbb{1}\ ,\\ q_i &\mapsto Q(q_i)\ ,\\ p_i &\mapsto Q(p_i)\ , \end{aligned}$$

where $Q(q_i)\psi(q) = q_i\psi(q)$ and $Q(p_i)\psi(q) = -\mathrm{i}\hbar\partial_{q_i}\psi(q)$ are defined on the common invariant dense domain $\mathcal{D} = \mathcal{S}\left(\mathbb{R}^n,\mathbb{C}\right)$, fulfils both quantization rules, except for maximality of $\mathcal{V}_{\mathrm{quant}}$:

As an extension of $Q_{\mathrm{can},1}$ one can consider

$$\mathcal{V}_{\mathrm{quant}} = \mathcal{V}_{\mathrm{poly},2} := \mathrm{span}\left(\left\{q_i,p_i,q_i^2,p_i^2,q_ip_i : i=1,\ldots,n\right\}\cup\{1\}\right)$$

and one finds that

$$\begin{aligned} Q_{\mathrm{can},2} : \mathcal{V}_{\mathrm{poly},2} &\to \mathrm{SYM}(\mathcal{H}),\\ 1 &\mapsto \mathbb{1}\ ,\\ q_i &\mapsto Q(q_i)\ ,\\ p_i &\mapsto Q(p_i)\ ,\\ q_i^2 &\mapsto \left(Q(q_i)\right)^2\ ;\\ p_i^2 &\mapsto \left(Q(p_i)\right)^2\ ,\\ q_ip_i &\mapsto \frac{1}{2}\left(Q(q_i)Q(p_i)+Q(p_i)Q(q_i)\right)\ . \end{aligned}$$

fulfils both quantization rules [Giu03].

Note that $Q_{\mathrm{can},2}$ maps products of classical observables, i.e. more of the associative structure than $Q_{\mathrm{can},1}$, into quantum operators. If one wants to carry over more of the associative structure of the Poisson algebra into the quantum realm (e.g. finding quantum operators for q_i^3 and p_i^3) then contradictions will arise. The statement of maximality is a consequence of a theorem of Groenewold and van Hove (see also [Got98a, Got98b], where also classical phase spaces with other topologies are discussed).

[10] Since already the canonical operators $Q(q_i)$ and $Q(p_i)$ are not defined everywhere on $\mathcal{H} = L^2(\mathbb{R}^n)$, a more precise notation is $\mathrm{SYM}(\mathcal{H},\mathcal{D})$, where $\mathcal{D}$ is a common invariant domain on which all operators in $\mathrm{SYM}(\mathcal{H},\mathcal{D})$ are defined and symmetric, and which lies dense in $\mathcal{H}$; $\mathcal{D}$ could for instance be the space of Schwartz functions on which both position and momentum operators are usually defined.

[11] There is no rule saying how the associative structure of the Poisson algebra has to be mapped; the so-called 'symmetrization rule' can be implemented only partially, see below.

Moreover it can be proved that $\mathcal{V}_{\text{poly},2}$ and $\mathcal{V}_{\text{poly}(\infty,1)}$ (the set of polynomials of at most first order in the $p_.$ whose coefficients are polynomials of arbitrary order in the $q_.$) are the only subalgebras of $\mathcal{V}_{\text{poly}}$ containing $\mathcal{V}_{\text{irr}} = \mathcal{V}_{\text{poly},1}$, which fulfil both quantization rules.

Besides these rules there is no known algorithm for choosing the pair $(\mathcal{V}_{\text{irr}}, \mathcal{V}_{\text{quant}})$. – This way one may obtain empirically inequivalent quantum theories [Wal01], the correct one of which can be determined only by experiments; in particular it has to reproduce classical mechanics in the classical limit. There are however several ways to obtain a classical limit [Wer95]. One frequently considers a family of quantum theories with different values of $\hbar$ in the limit $\hbar \to 0$. The canonical quantization procedure emphasizes the Lie algebraic structure: In the limit $\hbar \to 0$ the commutator of the classical limits of quantum observables equals the Poisson bracket (Dirac's condition). – There are other quantization procedures emphasizing different aspects, e.g. the connection between products of classical observables and the quantum anti-commutator [Lan98, ch. II]. Another often employed criterion for the correct classical limit states that in the spectrum of a quantum observable must coincide with the values that the corresponding classical observable may take (correspondence limit).

For constrained classical systems there is another possibility of quantization: Instead of applying the constraint on the classical level and quantizing then, one can also quantize the classical system on the unconstrained phase space and apply the constraint on the quantum level. An main advantage of the second possibility is that one does not have to solve the constraint explicitly. In the relational approach this is however necessary and the first alternative is chosen.

3.2.2 Two oscillators

We now explain and discuss Rovelli's (relational) quantization [Rov90] of the two oscillator system /3.1.1/, which can be based either on the angular momentum algebra, or equivalently on oscillator algebras.

3.2.2.1 Quantization based on the angular momentum algebra

In /3.1.1.4/ the algebra $\mathcal{C}_2$ of constants of motion of the two-oscillator system was seen to be the angular momentum algebra. Its quantization with $\mathcal{V}_{\text{irr}} = \mathcal{V}_{\text{quant}} = \mathcal{C}_2$ is straightforward, and the constraint can be easily realized, because the eigenvalue of $\hat{\mathbf{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ is a good quantum number and $C = 0$ (compare (3.1.3)) becomes after quantization equivalent to

$$\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = \frac{E^2}{4\omega^2} .$$

The quantum operators $\hat{L}_i = Q(L_i)$ and $\hat{L}_i^2 = Q(L_i^2)$ $(i = x, y, z)$[12] are defined on the Hilbert space $\mathcal{H}$ of spherical harmonics $\psi_{\ell m} = |\ell, m\rangle$ in the usual way (compare any textbook on quantum mechanics) such that $\psi_{\ell m}$ are joint eigenstates of $\hat{\mathbf{L}}^2$ and $\hat{L}_z$ with the respective eigenvalues $\hbar^2\ell(\ell+1)$ (ℓ being integer, or

[12] More precisely one should write $\widehat{L_i^2}$, but one has $Q(L_i^2) = Q(L_i)Q(L_i)$.

half-integer) and $m\hbar$ ($m = -\ell, \ldots, \ell$). The constraint confines the allowed states to a subspace of fixed angular momentum ℓ:

$$\hbar^2 \ell(\ell+1) = \frac{E^2}{4\omega^2} \tag{3.2.2}$$

Upon quantization the complete observable (3.1.4) becomes a parametrized family of self-adjoint operators on $\mathcal{H}$ built from constants of motion:

$$\hat{q}_2(t) = f(\hat{L}_z)\left(\hat{L}_y t \pm g(\hat{L}_z, t)\hat{L}_x g(\hat{L}_z, t)\right) f(\hat{L}_z)$$

Here the classical expression was symmetrized, and the functions f and g are (for self-adjoint arguments) defined by

$$f(x) = \left(x + \frac{E}{2}\right)^{-\frac{1}{2}} ,$$
$$g(x,t) = \left(2x + E - t^2\right)^{\frac{1}{4}} .$$

Classically, the allowed range of the parameter t is given by the condition

$$|t| \leq \sqrt{2A} = \sqrt{2L_z + E} .$$

Put differently, for a given value of t, the complete observable $q_2(t)$ is defined only on that region $\Gamma(t)$ of phase space, where

$$L_z \geq \frac{1}{2}\left(t^2 - E\right) .$$

Outside this region g becomes imaginary.

Hájiček [H91] has shown that $q_2(t)$ is not normal (i.e., not diagonalizable) by evaluating transition matrix elements of $[q_2(t), (q_2(t))^+]$ between two particular states with real and imaginary g, respectively. Rovelli replied [Rov91a] that it is not $q_2(t)$ which has to be quantized; instead one has to multiply $q_2(t)$ with a projection

$$P(t) = \theta\left(\frac{1}{2}(E - t^2) + L_z\right) ,$$

obtaining the classical observable

$$\tilde{q}_2(t) = q_2(t)P(t) = \begin{cases} q_2(t) & \text{on } \Gamma(t) , \\ 0 & \text{otherwise} . \end{cases}$$

Quantum mechanically, this leads to the symmetric operator

$$\hat{\tilde{q}}_2(t) = \hat{P}(t)\hat{q}_2(t)\hat{P}(t) .$$

Here $\hat{P}(t)$ is not the straightforward quantization of $P(t)$, since this would not remedy the problems with normality at the boundary of $\Gamma(t)$. Rovelli's solution uses a slightly modified projection, which includes an additional $\hbar$-term (vanishing in the classical limit):

$$\hat{P}(t) := \theta\left(\frac{1}{2}(E - t^2) + \hat{L}_z - \hbar\right) = \sum_{m > \frac{t^2 - E}{2\hbar} + 1} |\ell, m\rangle \langle \ell, m|$$

With this definition $\hat{\hat{q}}_2(t)$ can be shown to be normal.

3.2.2.2 Quantization based on oscillator algebras

An alternative way of quantization is obtained by using (2.2.4) to express the components of angular momentum through annihilation and creation operators for both oscillators $i = 1, 2$, with $\left[a_i, a_i^+\right] = 1$. The quantization map then becomes

$$\begin{aligned} Q(L_x) &= \frac{\hbar}{2}\left(a_1^+ a_2 + a_1 a_2^+\right) \ , \\ Q(L_y) &= \frac{\mathrm{i}\hbar}{2}\left(a_1^+ a_2 - a_1 a_2^+\right) \ , \\ Q(L_z) &= \frac{\hbar}{2}\left(a_2^+ a_2 - a_1^+ a_1\right) \ , \\ Q(1) &= \mathbb{1} \ . \end{aligned}$$

One easily verifies:

$$Q\left(\{L_i, L_j\}\right) = \frac{1}{\mathrm{i}\hbar}\left[Q(L_i), Q(L_j)\right] \qquad (i, j = x, y, z)$$

(And since $\mathrm{gen}\left(\{L_x, L_y, L_z, 1\}\right) = \mathrm{span}\left(L_x, L_y, L_z, 1\right)$, the quantum algebra also is finite dimensional.)

The constraint $C = 0$ takes the form

$$Q(L_x)^2 + Q(L_y)^2 + Q(L_z)^2 = \hbar^2\left(\frac{N_1 + N_2}{2}\right)^2 + \hbar^2\frac{N_1 + N_2}{2} = \frac{E^2}{4\omega^2} \ ,$$

where $N_i = a_i^+ a_i$ $(i = 1, 2)$. The constraint can be solved explicitly in a basis of number states $|n, m\rangle$ for both oscillators with the ansatz

$$(N_1 + N_2)\,|n, m\rangle = 2j\,|n, m\rangle \tag{3.2.3}$$

and takes the form

$$j(j + 1) = \frac{E^2}{4\hbar^2\omega^2} \ . \tag{3.2.4}$$

The ansatz requires $n + m = 2j$. Thus all states

$$|m\rangle\rangle := |j - m, j + m\rangle \qquad (m = -j, \ldots, j)$$

fulfil the constraint independently.

This quantization is equivalent to the one based on the angular momentum algebra /3.2.2.1/.

3.2.2.3 Discussion

The quantized constraint condition (3.2.2) or (3.2.4) of Rovelli arises from a quantization of the *squared* Hamiltonian $H^2 = 4\omega^2\mathbf{L}^2$ and the requirement that this observable must take the value E^2. Alternatively, in the quantization scheme based on oscillator algebras /3.2.2.2/, one can directly quantize the constraint $H - E = 0$ (being *linear* in H), which leads to

$$Q(H) = \hbar\omega\left(N_1 + N_2 + 1\right) = E$$

when taking the zero point energy of both oscillators into account. The quantum states fulfilling this condition are the same as above (direct product of number states with constant total occupation number), but the subspace spanned by these states has a different dependence on E, namely using (3.2.3)

$$2j + 1 = n + m + 1 = \frac{E}{\hbar\omega}$$

instead of (3.2.4). (The difference corresponds to a shift $\hbar^2\omega^2$ in E^2.)

This discrepancy between both quantizations of the constraint is however irrelevant, since in our endophysical description the numerical value of E cannot be compared to that of another system: E depends on the unit of time which is meaningless for the endosystem. (This does not preclude the possibility of comparing the energy of one subsystem to the energy of another subsystem.) In order to avoid a contradiction it is therefore sufficient that both quantizations have the same state spaces after imposition of the constraint, regardless with which value of E they are labeled.

The quantization of H requires that $H \in \mathcal{C}_2$, which is not the case with the above definition of $\mathcal{C}_2$; there is however no problem with choosing instead

$$\mathcal{C}_2 = \mathrm{gen}_{\mathrm{Lie}}\left(\{L_x, L_y, L_z, 1, H\}\right) = \mathrm{span}\left(L_x, L_y, L_z, 1, H\right)$$

and extending the quantization map with $Q(H) = \hbar\omega\,(N_1 + N_2 + 1)$. In the next subsection we also have the constraint among the complete observables.

3.2.2.4 Directions of research

The idea of relational time and quantization was picked up by many authors since 1990. We give a short chronological list of the directions of research based on, or similar to Rovelli's two-oscillator model:

- A number of similar models with constraints was discussed in [Tat92] using algebraic quantization.
- In [LE96] a reparametrization invariant system consisting of an oscillator and a free particle serving as a clock was considered.
- In [Ash98b, Ash98a] the coherent state quantization of Rovelli's model was introduced and an explicit form for coherent states on reduced phase space was given and discussed.
- In [Mon01] the relational evolution of generally covariant systems with vanishing Hamiltonian and finitely many degrees of freedom was studied in general and for two concrete systems.
- In [GP01] four models of relational time evolution in generally covariant quantum systems were studied. In particular, the choice of admissible clock variables was shown to be not completely arbitrary.
- In [Ohk00] a two-oscillator system with a constraint consisting of the difference (rather than the sum) of the energies of the oscillators was discussed.
- In [ES02, Elz03a, Elz03b] a model with two compactified extra-dimensions is studied, in which a discrete cyclic time is accessible only 'stroboscopically' through quasi-local measurements.

The concept of evolving constants of motion was discussed in [And95]. There is a number of other publications aimed mainly at understanding the problem of time in quantum gravity; we only give the most recent reference: [GPP04]

3.2.3 Free field

In this subsection we generalize the quantization of the two-oscillator system to infinitely many oscillators, i.e. to a free scalar field in one dimension, whose Poisson algebra $\mathcal{C}_\infty$ of constants of motion was examined in /3.1.2/.

3.2.3.1 Quantization map

The idea of our quantization is similar to canonical quantization: We map $z_n \mapsto a_n$ and $\bar{z}_n \mapsto a_n^+$, where a_n and a_n^+ are annihilation and creation operators on the Hilbert space of oscillator n. We are thus interested in a quantization map Q, which takes (complex) observables into quantum operators:

$$\begin{aligned} Q(1) &= \mathbb{1} \\ Q(h_n^+) &= a_n^+ a_n \\ Q(h_n^-) &= a_n^+ a_{-n} \\ Q(a_{mn}^\pm) &= a_m a_n a^+_{\pm(|m|+|n|)} \\ Q(b_{mn}^\pm) &= a_m^+ a_n^+ a_{\pm(|m|+|n|)} \end{aligned} \tag{3.2.5}$$

One easily sees that Q maps the real observables $\frac{1}{2}(h_n^- + h_{-n}^-)$, $\frac{1}{2\mathrm{i}}(h_n^- - h_{-n}^-)$, $\frac{1}{2}(a_{mn}^\pm + b_{mn}^{\pm -})$ and $\frac{1}{2\mathrm{i}}(a_{mn}^\pm - b_{mn}^{\pm -})$ into symmetric operators. Because of $m \neq \pm(|m| + |n|) \neq n$ $(m, n \in \mathbb{Z} \setminus \{0\})$ the operator ordering is no issue, except for $Q(h_n^+)$, whose ordering is as in the canonical case.

If we consider only the two field modes n and $-n$, then identifying n with 1 and $-n$ with 2 we re-obtain the algebra of the two-oscillator system from $h_{\pm n}^{\pm'}$:

$$\begin{aligned} Q(L_x) &= \frac{\hbar}{2}\left(a_1^+ a_2 + a_1 a_2^+\right) \simeq \frac{\hbar}{2} Q(h_n^- + h_{-n}^-) \ , \\ Q(L_y) &= \frac{\mathrm{i}\hbar}{2}\left(a_1^+ a_2 - a_1 a_2^+\right) \simeq \frac{\hbar}{2} Q(h_n^- - h_{-n}^-) \ , \\ Q(L_z) &= \frac{\hbar}{2}\left(a_2^+ a_2 - a_1^+ a_1\right) \simeq \frac{\hbar}{2} Q(h_{-n}^+ - h_n^+) \ , \\ Q(1) &= \mathbb{1} \simeq Q(1) \end{aligned}$$

This identification rest on the fact that $\omega_n = \omega_{-n}$.

So far we have defined only the quantum image of the generating set $\mathcal{C}_{\infty,g}$. The observables in $\mathcal{C}_{\infty,g}$ provide a coordinatization of the classical phase space: Each solution curve of the underlying Hamiltonian system is fixed by the values of all constants of motion, and because of $\mathcal{C}_\infty = \mathrm{gen}(\mathcal{C}_{\infty,g})$ these are in turn fixed by the values of the generating observables $\mathcal{C}_{\infty,g}$. We thus choose

$$\mathcal{V}_{\mathrm{irr}} = \mathcal{C}_{\infty,g}$$

for our canonical quantization of the field and furthermore

$$\mathcal{V}_{\text{quant}} = \text{gen}_{\text{Lie}}(\mathcal{C}_{\infty,g}) \ ,$$

the Lie algebra generated by $\mathcal{C}_{\infty,g}$, which is a Lie subalgebra of $\mathcal{C}_\infty$ and trivially contains $\mathcal{V}_{\text{irr}} = \mathcal{C}_{\infty,g}$ as required. Note that nothing of the associative structure of $\mathcal{C}_\infty$ is contained in $\mathcal{V}_{\text{quant}}$. (We will prove a Groenewold-van Hove-like theorem below in the contrary case.) One can also see, that even when considering the bare sets, one has $\mathcal{V}_{\text{quant}} \neq \mathcal{C}_\infty$: E.g. $\bar{z}_n^2 z_n^2 = h_n^+ \cdot h_n^+ \in \mathcal{C}_\infty$, but $\bar{z}_n^2 z_n^2$ cannot be obtained by taking Poisson brackets of elements of $\mathcal{C}_{\infty,g}$: Only the bracket (3.1.10) leads to expressions of second order in $\bar{z}_.$ and second order in $z_.$, but no combination of the indices yields $\bar{z}_n^2 z_n^2$.

In the canonical cases $Q_{\text{can},1}$ and $Q_{\text{can},2}$ the sets generating $\mathcal{V}_{\text{quant}}$ do even *linearly span* the classical algebra $\mathcal{V}_{\text{quant}}$, which is then quantized. Here we have $\text{gen}(\mathcal{C}_{\infty,g}) = \mathcal{V}_{\text{quant}}$, but $\mathcal{V}_{\text{quant}} \neq \text{span}(\mathcal{C}_{\infty,g})$, because Poisson brackets of $a^\pm_{..}$ and $b^\pm_{..}$ lead to polynomials of arbitrary high order in $z_.$ and $\bar{z}_.$. It is thus not sufficient to define the image of $\mathcal{C}_{\infty,g}$ under Q; we have to map all of $\mathcal{V}_{\text{quant}}$.

A naive rule for mapping all of $\mathcal{V}_{\text{quant}}$ would state: Given any polynomial $f \in \mathcal{V}_{\text{quant}}$, replace each $\bar{z}_n$ by a_n^+ and each z_n by a_n. This rule does however not work, because of operator ordering: For any polynomial containing $\bar{z}_n$ and z_n one could order $\bar{z}_n$ and z_n arbitrarily, but the quantum image would in depend on the ordering.

A proper definition of the quantization map Q certainly has to fulfil (3.2.1). This way we can define the quantum image of $\{f, g\}$, i.e. for polynomials of arbitrary order, but uniqueness of this definition is not assured: There could possibly be $f' \neq f$ and $g' \neq g$ with $\{f, g\} = \{f', g'\}$ classically, but $[Q(f), Q(g)] \neq [Q(f'), Q(g')]$ due to operator ordering. – Using the computer algebra system *maple* we have calculated Poisson brackets and commutators of several low order constants of motion, and we always found equation (3.2.1) to not lead to contradictions. We were however not able to prove the following

Conjecture 3.2.1. The map $Q : \mathcal{C}_{\infty,g} \to \text{SYM}(\mathcal{H})$ given by (3.2.5) can be uniquely extended to $\mathcal{V}_{\text{quant}}$ using rule (3.2.1).

Let us sketch the difficulties in proving the above conjecture. The lowest order polynomials containing a $z_l \bar{z}_l$-term (for which operator ordering becomes relevant after quantization) are the 4th order expressions ($m, n \in \mathbb{Z} \setminus \{0\}$)

$$\begin{aligned}\left\{a^\pm_{mn}, b^{\pm'}_{mn}\right\} &= \left\{z_m z_n \bar{z}_{\pm(|m|+|n|)}, \bar{z}_m \bar{z}_n z_{\pm'(|m|+|n|)}\right\} \\ &= \left(\bar{z}_n z_n + z_m \bar{z}_m\right) z_{\pm'(|m|+|n|)} \bar{z}_{\pm(|m|+|n|)} =: P_4 \ ,\end{aligned}$$

where the ordering of the factors in both summands of P_4 is arbitrary. The quantization of P_4 is given by rule (3.2.1):

$$\begin{aligned} Q(P_4) &= \frac{1}{\mathrm{i}\hbar}\left[Q(a^{\pm}_{mn}), Q(b^{\pm'}_{mn})\right] \\ &= \frac{1}{\mathrm{i}\hbar}\left[a_m a_n a^{+}_{\pm(|m|+|n|)}, a^{+}_m a^{+}_n a_{\pm'(|m|+|n|)}\right] \\ &= \frac{1}{\mathrm{i}\hbar}\left(a^{+}_n a_n + a_m a^{+}_m\right) a_{\pm'(|m|+|n|)} a^{+}_{\pm(|m|+|n|)} \end{aligned}$$

Hence an operator ordering is defined through (3.2.1); it is also unique for P_4, since there is no other way for obtaining the 4th order expression P_4 from (a linear combination of) Poisson brackets other than from a Poisson bracket of two third oder polynomials, which are easily seen to be fixed uniquely. Note also that – as required by the symmetry of $a^{\pm}_{mn}$ as well as $b^{\pm'}_{mn}$ – $Q(P_4)$ is symmetric in m and n, since

$$a^{+}_n a_n + a_m a^{+}_m = a^{+}_n a_n + 1 + a^{+}_m a_m = a^{+}_m a_m + a_n a^{+}_n \ .$$

In our view the main idea for a proof of the above conjecture is complete induction over the degree of the polynomials: Assuming that the quantization of all polynomials up to order n (obtained from building linear combinations of Poisson brackets of elements of $\mathcal{C}_{\infty,g}$) is unique, one would have to show that the quantization of all polynomials of order $n+1$ is unique (regarding operator ordering), too. We did not succeed in proving this. We considered a polynomial P_{n+1} with $n+1$ factors ($z_{\cdot}$ or $\bar{z}_{\cdot}$), which does emerge as a Poisson bracket $\{P_{n-\nu}, P_{\nu+3}\}$ ($\nu = 0, \ldots, n-3$) from polynomials with $n-\nu$ and $\nu+3$ factors (since always two factors cancel out when evaluating the Poisson bracket). Using this mechanism of emergence also for lower order polynomials, it can be shown that P_{n+1} is a linear combination of 'telescopic Poisson brackets'

$$\{\{\cdots\{\{*,*\},*\}\cdots\},*\}$$

involving only $a^{\pm}_{ij}$ and $b^{\pm}_{ij}$-terms (abbreviated with $*$); the number of $*$-term is $N-1$. The proof is based essentially on the Jacobi identity. (Poisson brackets with $h^{\pm}_n$ can be neglected, since $\{*, h^{\pm}_n\} = *$ up to a scalar, which allows to recursively eliminate $h^{\pm}_n$ using the Jacobi identity.)

This decomposition of P_{n+1} into a linear combination of 'telescopic Poisson brackets' is however not unique, e.g.

$$\begin{aligned} \mathrm{i} z_1 z_2 z_5 \bar{z}_8 &= \{z_1 z_2 \bar{z}_3, z_3 z_5 \bar{z}_8\} = \left\{a^{+}_{1,2}, a^{+}_{3,5}\right\} \\ &= \{z_1 z_5 \bar{z}_6, z_6 z_2 \bar{z}_8\} = \left\{a^{+}_{1,5}, a^{+}_{6,2}\right\} \ . \end{aligned}$$

Although the constant of motion $z_1 z_2 z_5 \bar{z}_8$ arises as Poisson bracket of different expressions, it has a unique quantization, since it does not contain a factor $z_l \bar{z}_l$ and thus operator ordering is arbitrary. (It can be easily seen that operator ordering can become relevant only if the telescopic Poisson bracket expression contains both a $a^{\pm}_{ij}$ and a $b^{\pm'}_{i'j'}$-term.) The appearance of a factor $z_l \bar{z}_l$ is always accompanied by the presence of another summand, identical except for possibly the coefficient and the replacement of $z_l \bar{z}_l$ with $z_k \bar{z}_k$ for some $k \neq l$. This possible appearance of several summands in a telescopic Poisson bracket expression makes the situation complicated; we could not prove that if a telescopic Poisson bracket expression can be written as a linear combination of other such expressions, then the quantization of both must be the same. This would prove the conjecture.

3.2.3.2 Groenewold-van Hove theorem

We have already seen that $(h_n^+)^2 \notin \mathcal{V}_{\text{quant}}$. What happens, if we add $h_m^+ h_n^+$ $(m, n \in \mathbb{Z}\backslash\{0\})$ to the generating set and try to find a quantization map? – We will show that such a quantization does not exist.

Let now $\mathcal{V}_{\text{quant}}$ be arbitrary, but contain $\text{gen}_{\text{Lie}}(\mathcal{C}_{\infty,g} \cup \{h_m^+ h_n^+ : m, n \in \mathbb{Z}\backslash \{0\}\})$. We assume that a quantization map Q fulfilling (3.2.5) does exist. We do not need to make any assumption on the image of $h_m^+ h_n^+$ under Q, but we assume that $h_m^+ h_n^+$ is in the domain of Q. This will suffice to prove a contradiction. (The proof of Groenewold and van Hove, which assumes that for instance q^3 is in the domain of the quantization map, is similar in that respect.) In the following we use $\hbar = 1$.

Lemma 3.2.2. For $\nu \in \mathbb{N}$, if $Q((h_n^-)^\nu)$ is defined, then

$$Q((h_n^-)^\nu) = \alpha_\nu + \gamma_\nu \left(Q(h_n^-)\right)^\nu$$

with some constants $\alpha_\nu, \gamma_\nu \in \mathbb{C}$.

Proof. We consider $Q((h_n^-)^\nu)$ as a polynomial in $a_n^+, a_n, a_{-n}^+, a_{-n}$. From $\{h_n^+, (h_n^-)^\nu\} = -2\mathrm{i}(h_n^-)^\nu$ (cf. /3.1.2.2/) it follows that

$$\left[Q(h_n^+), Q((h_n^-)^\nu)\right] = \mathrm{i}Q\left(\left\{h_n^+, (h_n^-)^\nu\right\}\right) = \mathrm{i}Q\left(-\nu\mathrm{i}(h_n^-)^\nu\right) = \nu Q((h_n^-)^\nu) \ .$$

Since $Q(h_n^+)$ is the number operator for mode n, $Q((h_n^-)^\nu)$ can contain only terms of ν-th power in a_n^+ or a_n. Analogously, $\left\{h_{-n}^+, (h_n^-)^\nu\right\} = \nu\mathrm{i}(h_n^-)^\nu$ and

$$\left[Q(h_{-n}^+), Q((h_n^-)^\nu)\right] = \mathrm{i}Q\left(\left\{h_{-n}^+, (h_n^-)^\nu\right\}\right) = \mathrm{i}Q\left(2\mathrm{i}(h_n^-)^\nu\right) = -\nu Q((h_n^-)^\nu)\,,$$

so that $Q((h_n^-)^\nu)$ can contain only terms of ν-th power in a_{-n}^+ or a_{-n}. Apart from a constant term $Q((h_n^-)^\nu)$ is thus a linear combination of terms of the form $\left(a_n^\#\right)^\nu \left(a_{-n}^\#\right)^\nu$, where $a^\#$ means a or a^+. Since on the other hand

$$\left[Q(h_n^-), Q((h_n^-)^\nu)\right] = \mathrm{i}Q\left(\left\{h_n^-, (h_n^-)^\nu\right\}\right) = \mathrm{i}Q\left(0\right) = 0 \ ,$$

the possibilities $\left(a_n^+\right)^\nu \left(a_{-n}^+\right)^\nu$, $\left(a_n\right)^\nu \left(a_{-n}\right)^\nu$ and $\left(a_n\right)^\nu \left(a_{-n}^+\right)^\nu$ are ruled out, as a simple calculation of commutators shows. We thus have $Q((h_n^-)^\nu) = \alpha_\nu + \gamma_\nu \left(a_n^+\right)^\nu \left(a_{-n}\right)^\nu = \alpha_\nu + \gamma_\nu \left(Q(h_n^-)\right)^\nu$, where of course the operator ordering is arbitrary.

As for the coefficient γ_ν, it can be shown that it is a scalar multiple of $\mathbb{1}$: Assume that after writing γ_ν as a sum of normal ordered terms it contains a term $(a_m^+)^\mu a_m^\lambda$ (with $m \neq n$) and either $\mu > 0$ or $\lambda > 0$. Then in case $\mu \neq \lambda$ we arrive at the contradiction:

$$Q(\left\{h_m^+, (h_n^-)^\nu\right\}) = \frac{1}{\mathrm{i}}\left[Q(h_m^+), Q((h_n^-)^\nu)\right] = \frac{1}{\mathrm{i}}\left[a_m^+ a_m, \gamma_\nu\right]\left(a_n^+\right)^\nu \left(a_{-n}\right)^\nu \neq 0$$

In the case $\mu = \lambda > 0$ we write $\gamma = (a_m^+)^\mu a_m^\mu \tilde{\gamma}$ (with $\tilde{\gamma} \neq 0$ containing neither a_m^+ nor a_m) and also obtain a contradiction:

$$Q(\{h_m^-, (h_n^-)^\nu\}) = \frac{1}{\mathrm{i}} \left[a_m^+, (a_m^+)^\mu a_m^\mu\right] a_{-m}\tilde{\gamma} \left(a_n^+\right)^\nu (a_{-n})^\nu \neq 0$$

□

Now we use this lemma in the case $\nu = 2$; the following proof shows that $Q((h_n^-)^2)$ is defined:

Lemma 3.2.3. $Q((h_n^+)^2)$ is at most of second order in a_n and at most of second order in a_n^+.

Proof. We write $Q((h_n^+)^2)$ as a polynomial in a_n^+ and a_n with normal ordered summands. We consider a summand proportional to $(a_n^+)^\mu a_n^\nu$. Then we have $[[(a_n^+)^\mu a_n^\nu, Q(h_n^-)], Q(h_n^-)] = [[(a_n^+)^\mu a_n^\nu, a_n^+], a_n^+] a_{-n}^2 = \nu(\nu-1)(a_n^+)^\mu a_n^{\nu-2} a_{-n}^2$. On the other hand, the classical brackets are $\{\{(h_n^+)^2, h_n^-\}, h_n^-\} = -2(h_n^-)^2$, implying

$$\left[\left[Q((h_n^+)^2), Q(h_n^-)\right], Q(h_n^-)\right] = Q\left(2(h_n^-)^2\right) = 2\alpha_2 + 2\gamma_2 \left(a_n^+\right)^2 (a_{-n})^2 \ ,$$

where the last equation is due to the preceding lemma. We conclude that $Q((h_n^+)^2)$ can contain only terms $(a_n^+)^\mu a_n^\nu$ with either $\nu = 2$ and $\mu = 2$, or with $\nu < 2$ and arbitrary μ. By analogously taking the commutators with $Q(h_{-n}^-)$ and Poisson brackets with h_{-n}^- we can also show $\mu \leq 2$. □

A generalization of this result is also possible:

Lemma 3.2.4. $Q((h_n^+)^\lambda)$ is at most of λ-th order in a_n and most of λ-th order in a_n^+.

Proof. The proof is analogous to the preceding lemma, but one has to take the λ-fold commutator with $Q(h_n^-)$. □

We are now able to constrain the explicit form of $Q((h_n^+)^2)$:

Lemma 3.2.5. $Q((h_n^+)^2) = \vartheta_n + \zeta_n a_n^+ a_n + \gamma_n (a_n^+)^2 a_n^2$ with $\vartheta_n, \zeta_n, \gamma_n \in \mathbb{C}$ for all $n \in \mathbb{Z}\setminus\{0\}$.

Proof. From the classical bracket $\{(h_n^+)^2, h_n^+\} = 0$ we have the commutator $[Q((h_n^+)^2), a_n^+ a_n] = 0$, which can be easily seen to imply that $Q((h_n^+)^2)$ can contain only terms proportional to $(a_n^+)^\mu a_n^\nu$ with $\mu = \nu$. $Q((h_n^+)^2)$ thus has the asserted form, and, since all three contributions to $Q((h_n^+)^2)$ are linearly independent, a reasoning similar to that of lemma 3.2.2 can be applied for the coefficients $\vartheta_n, \zeta_n, \gamma_n$ separately, implying $\vartheta_n, \zeta_n, \gamma_n \in \mathbb{C}$. □

A generalization, which is not needed in the sequel, can be proved similarly:

Lemma 3.2.6. $Q\left((h_n^+)^\lambda\right)$ is a polynomial in $Q(h_n^+)$ of at most λ-th degree.

Now we can prove the explicit form of $Q((h_n^+)^2)$:

Proposition 3.2.7. For $n \in \mathbb{Z}\setminus\{0\}$ we have $\gamma_n = 1$ and $\zeta_n = 2$, or equivalently

$$Q((h_n^+)^2) = \vartheta_n + a_n^+ a_n a_n a_n^+ \ .$$

PROOF. A straightforward calculation shows (for $i, j \in \mathbb{Z}\setminus\{0\}$)

$$\left\{\left\{a_{ij}^+, b_{ij}^+\right\}, a_{i,|i|+|j|}^+\right\} = \left(h_i^+ + h_{|i|+|j|}^+\right) a_{i,|i|+|j|}^+ \ . \tag{3.2.6}$$

The quantum version of this equation reads

$$Q\left(\left\{\left\{a_{ij}^+, b_{ij}^+\right\}, a_{i,|i|+|j|}^+\right\}\right) = Q\left(h_i^+ a_{i,|i|+|j|}^+\right) + Q\left(h_{|i|+|j|}^+ a_{i,|i|+|j|}^+\right) \ .$$

The left hand side evaluates to:

$$\begin{aligned} Q\left(\left\{\left\{a_{ij}^+, b_{ij}^+\right\}, a_{i,|i|+|j|}^+\right\}\right) &= \frac{1}{\mathrm{i}}\left[\frac{1}{\mathrm{i}}\left[Q\left(a_{ij}^+\right), Q\left(b_{ij}^+\right)\right], Q\left(a_{i,|i|+|j|}^+\right)\right] \\ &= \left(a_i^+ a_i + 2 + a_{|i|+|j|}^+ a_{|i|+|j|}\right) a_{2|i|+|j|}^+ a_{|i|+|j|} a_i \end{aligned}$$

And the right hand side is connected with $Q((h_n^+)^2)$:

$$\begin{aligned} Q\left(h_i^+ a_{i,|i|+|j|}^+\right) &= Q\left(\frac{1}{2\mathrm{i}}\left\{(h_i^+)^2, a_{i,|i|+|j|}^+\right\}\right) \\ &= -\frac{1}{2}\left[Q\left((h_i^+)^2\right), Q\left(a_{i,|i|+|j|}^+\right)\right] \\ &= -\frac{1}{2}\left[\vartheta_i + \zeta_i a_i^+ a_i + \gamma_i (a_i^+)^2 a_i^2, a_{2|i|+|j|}^+ a_i a_{|i|+|j|}\right] \\ &= \left(\frac{1}{2}\zeta_i + \gamma_i a_i^+ a_i\right) a_{2|i|+|j|}^+ a_i a_{|i|+|j|} \ , \end{aligned}$$

and analogously

$$Q\left(h_{|i|+|j|}^+ a_{i,|i|+|j|}^+\right) = \left(\frac{1}{2}\zeta_{|i|+|j|} + \gamma_{|i|+|j|} a_{|i|+|j|}^+ a_{|i|+|j|}\right) a_{2|i|+|j|}^+ a_i a_{|i|+|j|} \ .$$

Comparing the left and right hand sides we obtain $\gamma_i = 1$, $\gamma_{|i|+|j|} = 1$ and $\zeta_i + \zeta_{|i|+|j|} = 4$. Especially we have $\zeta_1 + \zeta_3 = 4$, $\zeta_1 + \zeta_4 = 4$, $\zeta_3 + \zeta_4 = 4$, whence $2\zeta_1 = 4$ and $\zeta_1 = 2$. For $j = 1, 2, \ldots$ we obtain from $\zeta_1 + \zeta_{1+|j|} = 4$ also $\zeta_j = 2$. This result implies $\zeta_i = 2$ also for $i < 0$, since (setting $j = 1$) $\zeta_i + \zeta_{|i|+1} = \zeta_i + 2 = 4$. □

Clearly, as far as only commutators are evaluated the value of ϑ_n is unimportant. Moreover, one easily sees that $Q((h_n^+)^2)$ is symmetric.

Lemma 3.2.8. $Q(h_i^+ h_j^+)$ is a polynomial of at most first degree in $Q(h_i^+)$ and $Q(h_j^+)$.

PROOF. We can assume $Q(h_i^+ h_j^+)$ to be normal ordered. From $\left\{h_i^+ h_j^+, h_i^+\right\} = 0$ we have $\left[Q(h_i^+ h_j^+), a_i^+ a_i\right] = 0$, hence as above we conclude that $Q(h_i^+ h_j^+)$ must be a polynomial in $Q(h_i^+) = a_i^+ a_i$. The reasoning for $Q(h_j^+)$ is analogous.

To prove that $Q(h_i^+ h_j^+)$ is at most of first oder in $Q(h_i^+)$ one writes $Q(h_i^+ h_j^+)$ as a polynomial in $Q(h_i^+)$ and considers a summand of order $\nu \geq 2$:

$$\begin{aligned}\left[\left[\left(Q(h_i^+)\right)^\nu, Q(h_i^-)\right], Q(h_i^-)\right] &= \left[\left[(a_i^+ a_i)^\nu, a_i^+\right], a_i^+\right] a_{-i}^2 \\ &= \nu(\nu-1)(a_i^+ a_i)^{\nu-2} a_{-i}^2 \neq 0\end{aligned}$$

On the other hand we have $\left\{\left\{h_i^+ h_j^+, h_i^-\right\}, h_i^-\right\} = -\mathrm{i}\left\{h_i^- h_j^+, h_i^-\right\} = 0$, so that $\left[\left[Q(h_i^+ h_j^+), Q(h_i^-)\right], Q(h_i^-)\right] = 0$. Again, the reasoning for $Q(h_j^+)$ is analogous. □

Proposition 3.2.9. $Q(h_i^+ h_j^+)$ cannot be consistently defined.

PROOF. Using (3.2.6) a simple calculation yields

$$\left\{\left\{a_{ij}^+, b_{ij}^+\right\}, \left\{\left\{a_{ij}^+, b_{ij}^+\right\}, a_{i,|i|+|j|}^+\right\}\right\} = \left(h_i^+ + h_{|i|+|j|}^+\right)^2 a_{i,|i|+|j|}^+ . \tag{3.2.7}$$

On the other hand we have:

$$\begin{aligned}&\left\{h_i^+ h_{|i|+|j|}^+, -\frac{1}{2\mathrm{i}}\left\{a_{i,|i|+|j|}^+, (h_i^+)^2 + (h_{|i|+|j|}^+)^2\right\}\right\} = \\ &\qquad \left\{h_i^+ h_{|i|+|j|}^+, \left(h_i^+ + h_{|i|+|j|}^+\right) a_{i,|i|+|j|}^+\right\} = \left(h_i^+ + h_{|i|+|j|}^+\right)^2 a_{i,|i|+|j|}^+\end{aligned} \tag{3.2.8}$$

A straightforward calculation (using computer algebra) yields for the quantized left hand side of (3.2.7):

$$\begin{aligned}&2a_i^+ a_i^2 a_{|i|+|j|}^+ a_{|i|+|j|}^2 a_{2|i|+|j|}^+ \\ &+ a_i (a_{|i|+|j|}^+)^2 a_{|i|+|j|}^3 a_{2|i|+|j|}^+ \\ &+ 5a_i a_{|i|+|j|}^+ a_{|i|+|j|}^2 a_{2|i|+|j|}^+ \\ &+ 5a_i^+ a_i^2 a_{|i|+|j|} a_{2|i|+|j|}^+ \\ &+ (a_i^+)^2 a_i^3 a_{|i|+|j|} a_{2|i|+|j|}^+ \\ &+ 4a_i a_{|i|+|j|} a_{2|i|+|j|}^+\end{aligned}$$

For the evaluation of the left hand side of (3.2.8) we use due to the preceding lemma

$$Q(h_i^+ h_{|i|+|j|}^+) = \alpha_3 a_i^+ a_i a_{|i|+|j|}^+ a_{|i|+|j|} + \alpha_2 a_i^+ a_i + \alpha_1 a_{|i|+|j|}^+ a_{|i|+|j|} + \alpha_0$$

and obtain:

$$\begin{aligned}&2\alpha_3 a_i^+ a_i^2 a_{|i|+|j|}^+ a_{|i|+|j|}^2 a_{2|i|+|j|}^+ \\ &+ \alpha_3 a_i (a_{|i|+|j|}^+)^2 a_{|i|+|j|}^3 a_{2|i|+|j|}^+ \\ &+ (3\alpha_3 + \alpha_2 + \alpha_1) a_i a_{|i|+|j|}^+ a_{|i|+|j|}^2 a_{2|i|+|j|}^+ \\ &+ (3\alpha_3 + \alpha_2 + \alpha_1) a_i^+ a_i^2 a_{|i|+|j|} a_{2|i|+|j|}^+ \\ &+ \alpha_3 (a_i^+)^2 a_i^3 a_{|i|+|j|} a_{2|i|+|j|}^+ \\ &+ (\alpha_3 + \alpha_2 + \alpha_1) a_i a_{|i|+|j|} a_{2|i|+|j|}^+\end{aligned}$$

Comparing the coefficients of both expressions we obtain from the first five lines, respectively, $\alpha_3 = 1$ and $\alpha_2 + \alpha_1 = 2$. This latter equation cannot satisfy however the remaining equality of the sixth line; there is a discrepancy $a_i a_{|i|+|j|} a^+_{2|i|+|j|}$, which stems from operator ordering. (For this discrepancy to show up it we found it necessary to consider the commutator of two expressions, in both of which operator ordering is relevant.) □

In conclusion, we haven proved the following no-go theorem similar to that of Groenewold and van Hove:

THEOREM 3.2.10. *A canonical quantization of the Poisson algebra of constants of motion of a free massless scalar field in one dimension, based on a Lie algebra* $\mathcal{V}_{quant}$ *containing* $gen_{Lie}(\mathcal{C}_{\infty,g} \cup \{h^+_m h^+_n : m, n \in \mathbb{Z}\setminus\{0\}\})$ *does not exist.*

3.2.3.3 Constraint

The constraint takes the simple form

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \hbar\omega_n h^+_n = E \ ,$$

where we have subtracted the zero point energy. The subspace of states fulfilling this constraint is spanned by all tensor products $\otimes_{i\in\mathbb{Z}\setminus\{0\}} |n_i\rangle$ of number states of single field modes with

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} |k|\, n_k = \frac{E}{\hbar\omega_1} \ .$$

(Clearly, the constraint entails ultraviolet-finiteness.)
The operators $Q(a^{\pm}_{mn}) = a_m a_n a^+_{\pm(|m|+|n|)}$ and $Q(b^{\pm}_{mn}) = a^+_m a^+_n a_{\pm(|m|+|n|)}$ preserve the constraint by just redistributing the occupation numbers between three field modes.

3.2.3.4 Complete and partial observables

The quantization of the complete observable $q_n(q_1)$ (3.1.13) is straightforward; it requires lots of symmetrization for the products (3.1.14).

In the limit of infinite box length the expression (3.1.14) contains a factor

$$\left(\frac{a^+_1}{\sqrt{a^+_1 a_1}}\right)^{n-1} \ .$$

This operator is not defined for $n_1 = 0$, but can – as well as its adjoint operator – be regularized; see [LE96] for a discussion of the free particle limit in a relational model of two oscillators.

As already stated above /2.2.4/, $n_1 = 0$ does not allow to use field mode 1 as a clock; this would allow only for a single instant. In the Fock representation all but finitely many field modes are in their vacuum state. Understanding time as infrared behaviour of the field, all Fock states thus have the same time. In the next chapter we shall argue that a proper understanding of time as a physical quantity requires inequivalent representations.

CHAPTER 4

Change without time

The classicality of time is discussed and an outlook on general dynamics in view of quantum measurement is given.

A hard-boiled positivist may have difficulties to appreciate such a quantum endophysics since it refers by definition to some kind of Platonic universe, and not to empirical facts. [...] *Every operationally meaningful description we can give is observer-dependent or contextual. That is, for a hard-boiled positivist, universally valid natural laws do not exist.*

Hans Primas [Pri94c]

4.1 Classicality of time

Since a non-cyclic physical time always requires some infinity, we consider the possibility that time essentially is the infrared behaviour of a field /4.1.1/. The main problem with the relational concept of time is rooted in quantum mechanics, where arbitrary superpositions are possible. As a partial solution to this problem we propose to understand time as an observable at infinity of a quantum system with infinitely many degrees of freedom, allowing for instantaneity /4.1.2/.

4.1.1 Physical time

Taking the endophysical point of view and assuming that time is nothing more than what is measurable by actual clocks we find that in classical mechanics there are essentially only two types of clocks: A free particle and systems with an oscillation mechanism. In order to measure a linear (non-cyclic) time the latter systems must have oscillators with arbitrary long oscillation periods; in case of free oscillators the limit of infinite oscillation period corresponds to a free particle, cf. our discussion in /2.2/. Accelerated particles as well as interacting oscillators can also be used as clocks, but their duration is not identical with that of an hypothetical absolute time.

In quantum mechanics single (free) particles are of limited use because of spreading wave packets[1]; anyway, they require space to be infinitely extended and a quantum field to exist, because without a quantum field (with infinitely many degrees of freedom) the (infinity of) positions at which the particle can be localized could not be distinguished endophysically.

We therefore conclude that *a non-cyclic physical time always requires infinitely many degrees of freedom* and make the hypothesis that *time is given by the infrared behaviour of fields, i.e. by the oscillation of field modes with arbitrary long wavelengths.*

If there is more than one field (or a field in more than one dimension), it may happen that one of them is able to separate instants, which another one cannot separate /1.3.3/. Consider e.g. a classical charge moving in an electromagnetic field and some other noninteracting field. Due to the long-range electromagnetic interaction the position of the charge is correlated with the infrared behaviour of the state of the electromagnetic field, but uncorrelated with the other field; the state of the other field does not allow to distinguish between states, i.e. positions, of the charge. – In finding a time observable /1.3.3/ in general one thus has to take into account all fields. If there are two fields serving as clocks (i.e., separating all instants), there is no problem with different durations being measured by them, since an absolute duration is meaningless from the relational point of view. – Our hypothesis implicitly assumes that every system with short-rage interactions is coupled (directly or indirectly) to the infrared modes of a field; otherwise the system would have no correlation with the clock(s).

4.1.2 Instantaneity

Already the quantized version of Rovelli's two-oscillator model /3.2.2/ shows severe problems with a relational interpretation: Arbitrary states in the tensor product of the Hilbert spaces of both oscillators, which fulfil the constraint condition (3.2.3,3.2.4), are allowed. On the one hand this allows to build quasi-localized states from the number states of oscillator 1, say, so that its position q_1 serves as a time for the motion $q_2(q_1)$ of oscillator 2; but on the other hand also

[1] There are models with infinitely massive particles, but this limit is beyond respected quantum mechanics (being useful for approximations, but of no avail in principle).

superpositions of different times are allowed, which is hard to interpret within a classical temporal logics.

In Rovelli's pioneering article [Rov90] therefore the assumption was made that the total system is in a semiclassical state in which the position observables of both oscillators (partial observables) are correlated sufficiently strong. From this Rovelli could derive an approximately unitary dynamics for the state of oscillator 2 when using the (simultaneous) position of oscillator 1 as clock time. – There is however no a priori reason to require the allowed states of a quantum endosystem to be semiclassical, let alone having a correlation between certain degrees of freedom (in this case q_1 and q_2). This lack of a fixed quantum evolution is the main problem with the relational approach and to date unsolved.

Part of the problem is that clock states can be coherently superposed, so that a clear separation of instants is impossible. For this problem we propose the following solution: If time is embodied in the infrared behaviour of a field, i.e. as an observable at infinity, different values of this ('time') observable at infinity require inequivalent[2] representations of the algebra of observables, and these representations correspond to states, which cannot be coherently superposed. (Every superposition of these states is equivalent to a mixture.) Moreover, observables at infinity are classical observables, i.e. they commute with all observables, so that time can always be considered as simultaneously measurable. Understanding time as an observable at infinity therefore reproduces the classical intuition with a definite value of the time observable.

The formalism in which this classicality can be achieved is algebraic quantum mechanics [Sau88, Emc72, Lan91, Haa96] of systems with infinitely many degrees of freedom. The starting point is the C*-algebra of observables with all possible states, i.e. linear functionals on it ascribing to each observable an expectation value. Each state induces a representation of the algebra through operators on a Hilbert space (via the GNS-construction). There is a universal mechanism for 'completion' of these representations (taking the bicommutant, or the closure in any of several topologies), which adds representation-dependent observables and results in a representation-dependent von Neumann-algebra. In particular, projection operators necessary for a classical interpretation (in terms of yes-no-alternatives) are in general added to the algebra.

It is a generally accepted interpretation that the choice of representation is arbitrary in principle and depends on the experimental situation to be described, but is by far not determined through it. There is a necessity to make a choice for a particular description of the quantum system, which introduces new, contextual properties. Since one can never know the whole of an infinite system, such a choice is unavoidable, if one wants to describe the whole system as an endosystem [Pri94c, AP02, Pri98, Pri97, Pri94b, Pri87].

For the free field we had identified the behaviour of q_1 in the limit of infinite box length as a clock time in chapter /3/. If we want to describe quantum

[2] If we do not restrict ourselves to pure states, disjoint representations must be used.

dynamics with this clock time, then we must choose only representations in which[3] $Q(q_1) = \sqrt{\hbar}\frac{a_1+a_1^+}{\sqrt{2\omega_1}}$ converges in the limit. A change of representation can then effect a change of the value of $Q(q_1)$, i.e. alter the time of the system. Time emerges as a contextual property, corresponding to the fact that one can never know the phase of all oscillators from quasi-local measurements only.

Even if in the limit of infinite box length the observable $Q(q_1)$ is a classical observable and dispersion-free in the chosen state with value t, this does not impose any constraints on the local observables $Q(q_n)$, since their expectation values can be chosen at will within the equivalence class of the chosen state. Within this class the coherent superposability[4] of local states remains untouched. On the other hand states from different classes having different values of the time observable cannot be coherently superposed. (In traditional quantum mechanics this time superselection rule has to be postulated.)

The above reasoning does not apply directly to relational quantum mechanics, because a relational algebraic quantum mechanics is still to be developed: One has to find a relational analogue of the Weyl quantization, and it must be shown that the algebra of constants of motion thus obtained is a C*-algebra. Choosing a representation of this C*-algebra then corresponds to a choice of evolution. If one wants to interpret the evolution in terms of partial observables, one has to embed this C*-algebra into the 'usual' C*-algebra of algebraic quantum mechanics, the algebra of canonical (anti-)commutation relations, which may allow for a partial 'time' observable in suitable states as discussed above.

4.2 General dynamics and quantum measurement

In traditional dynamics the expectation values of all observables depend continuously on a time parameter t. Unitary time evolution is a basic symmetry in quantum theory, derived from an automorphism group of the algebra of observables [Sim76]. Unitary dynamics cannot account for wavefunction collapse during measurement (quantum measurement problem). There are three options w.r.t. the reality of collapse:

(a) One can argue that collapses do never occur and resort to a many-worlds interpretation of quantum mechanics. This option does not appreciate the empirical evidence for collapse.

[3] Here Q is the quantization map in the canonical quantization of positions and momenta.

[4] This has been called *kinematical superposition principle* (stating that the superposition of two states is a valid state), in distinction from the *dynamical superposition principle*, which claims that the result of a time evolution of a superposition of two states is the same as the superposition of respective results of their time evolutions.

(b) One can argue that collapses do occur approximately, but never 'exactly'. In this case it can however be proved that coherence can be re-established through suitable later interactions/measurements. This would in turn require to find reasons why such interactions do rarely take place. Explanations in this direction tend to end up with problems similar to von Neumann's infinite chain of measurements.
(c) One can try to solve the measurement problem and explain collapse. Since the insolubility under reasonable assumptions in the traditional Hilbert space setting is well known [BLM96], one must change these assumptions at least partially.

Within algebraic quantum mechanics an approach of the last kind is possible: The measurement apparatus is modelled as having infinitely[5] many degrees of freedom and is coupled to an object system having finitely many degrees of freedom. For suitable interactions definite pointer states can be obtained, but this takes infinitely long [Hep72], in contrast with empirical facts. For this reason it has been argued that the traditional automorphic dynamics cannot be correct [Bre98]; the alternative is a non-automorphic dynamics, which means that observable quantities are mapped into unobservable ones during time evolution.

This latter view finds support from our idea of different time instants as inequivalent representations of the algebra of observables. Choosing a representation with a given value of a time observable may create new facts, i.e., classical observables being not defined in 'earlier' representations may emerge. If there is a one-parameter family of such representations describing time evolution, the question naturally arises as to when a new fact does emerge (cmp. /2.2.1/).

The answer depends on the choice of non-automorphic dynamics, but which one is the correct one? For sufficiently weak short-range interactions it certainly has to reproduce unitary dynamics *locally*. This requirement does however not fix a unique 'dynamics at infinity'. Following the general philosophy we speculate that such a 'dynamics at infinity' might be not unique. This would allow for a non-deterministic 'evolution' of local states having sufficiently strong long-range interactions.

If a relational algebraic quantum mechanics can be formulated, a choice of representation would fix constants of motion and thus evolution. Since many representations are compatible with a given local evolution, the evolution 'at infinity' could be chosen at will. I.e., the indeterminism is introduced not at the level of instantaneous states, but on the level of dynamical evolutions. We think that this idea deserves further study. The technical problem are however enormous, because the constants of motion of appropriate models would have to be explicitly known.

Radical relationalism takes quite an opposite point of view: dynamics is a set of instantaneous descriptions of a system, which contain a time observable /1.3.3/ taking different values for different descriptions. The main question then is:

[5]Without this infinity coherence could be re-established by a suitable later measurement [Bel75].

Which descriptions may be combined into a set to obtain a valid dynamics? – For conformity with traditional dynamics the set would have to be a continuous family of descriptions parametrized by the value of the time observable, and the expectation values of all other observables would have to depend continuously on this parameter within the family. The concept of evolving constants of motion adheres to this traditional dynamics. E.g., in the two-oscillator model /3.1.1/ such a family is given by specifying the values of all constants of motion and choosing q_1 as parameter; then e.g. the dependence $q_2(q_1)$ is continuous.

A purely instantaneous description is however an ambitious project, since it has to account for our knowledge of the past as a property of the instantaneous state of the system, i.e. it amounts to a theory of consciousness.

In conclusion, the relational concept of time leads to a proper understanding of time as a physical quantity. The concept of evolving constants of motion and relational quantization are worth to be further explored (e.g., a momentum constraint instead of a Hamiltonian might lead to a relational understanding of space in the quantum domain), especially in an algebraic framework still to be created. The possibility to choose time evolutions too some degree at will might account for the indeterminism of measurement collapse and the 'emergence' (albeit not within 'time') of classical observables.

We close with a quote of Einstein:

> *The hardest thing to understand is why we can understand anything at all.*

Summary

The nature of time has long been debated in human history and nowadays is considered of central importance in understanding quantum gravity. This thesis focuses on and advocates the relational concept of time, which was put forward in the 17th century in opposition to Newton's absolute time, and only in 1990 explored in a quantum mechanical framework by Carlo Rovelli.

After a historical introduction the mathematical models of time are carefully analyzed in chapter 1, followed by a discussion of the role of time played in fundamental theories. Using as an example nonrelativistic mechanics, the process of parametrization is explained, leading to a separation of a 'canonical time coordinate' from an arbitrary evolution parameter. The discussion of the role of time in special and general relativity as well as in quantum mechanics shows that more fundamental theories use less structure of time. This is followed by an exposition of the history of the relational concept of time, which negates the existence of an absolute duration and therefore often is called "timeless". Next it is shown how fundamental theories can be formulated and re-interpreted using this concept. We put emphasis on the hitherto neglected connection between relationalism and non-extensibility, while absolute time is shown to be unproblematic in classical mechanics just because of the possibility to extend the system without changing its nature. We conclude chapter 1 with a new axiomatic basis for the construction of time observables based on a simultaneity relation between 'observations', which are treated as a primitive concept and intuitively correspond to measurement events, but without knowing 'when' these events occur. There is no fundamental time observable; any observable qualifies as a time observable, if it allows to separate all instants.

Chapter 2 gives a brief account of three main problems connected with time: The arrow of time, time measurement and quantum gravity. The problem of the arrow of time actually has to be disentangled from the problem of irreversibility: a solution of the latter essentially excludes cyclic motions and is required for a solution of the former, which consists in showing that a fundamental direction between any two non-identical instants is physically meaningful. We give a formal definition of the arrow of time. This classical analysis is followed by a review of the problem of the arrow of time in quantum theory, where the situation becomes more complicated because of indeterminism. The discussion shows that there is no experimental evidence for a fundamental arrow of time, so that no contradiction with the relational concept of time arises. The second problem, time measurement, is of particular importance for the relational approach, in

which time has no reality except if measured by a clock. In quantum theory time observables have long been considered to be impossible in the traditional Hilbert space formalism according to a 'theorem' of Pauli. Notwithstanding a fault in the proof of Pauli's 'theorem', useful time operators seem to be possible only within the more general formalism of positive operator valued measures (POVM). Clocks based on an oscillation mechanism do however require phase measurements; quantum phase operators can be defined as certain positive operator valued measures. Phase difference operators do also exist in the traditional Hilbert space formalism, if another quantization is used, as is done with relational quantization. The third problem with time, quantum gravity, is sketched only briefly. The diffeomorphism invariance of canonical quantum gravity entails that the Wheeler-DeWitt equation is devoid of time and cannot be understood as an evolution equation.

Chapter 3 introduces and discusses a model of Rovelli [Rov90] consisting of two oscillators with no external time. In this model one oscillator is considered as a clock and defines a relational time for the other one. In the first section we introduce this model and generalize it to a free massless scalar field in one dimension. We establish the relation between a single field mode and the infrared behaviour of the field through constants of motion. In the second section after a short review of canonical quantization we review Rovelli's quantization and generalize it to the free field. We could not prove the existence of the quantization map, but calculations using computer algebra indicate that the quantization does exist. For an enlarged algebra (containing also the products of the energies of two field modes) we prove the nonexistence of a quantization map. (For the canonical quantization of position and momentum a similar theorem has been proved by Groenewold and van Hove.)

In chapter 4 we observe that a clock time always requires infinitely many degrees of freedom and we make the hypothesis that a time observable is given by the infrared behaviour of quantum fields, leading to a classical notion of time when using algebraic quantum mechanics. This does however not solve the main problem of quantum relationalism: Which conditions determine a particular evolution? – In a second section we give an informal discussion of some implications of a physical time for possible solutions to the quantum measurement problem.

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Acknowledgements

Invaluable thanks go to my open minded advisor, Prof. em. Gustav M. Obermair, for giving me the opportunity of studying the most interesting theories and problems in the foundations of physics to which I had been inclined since my school years. I am indebted to him for his patience with time and for being my advisor even after becoming an Emeritus. I thank him and the Fakultät für Physik at Regensburg University for the possibility of visiting interesting conferences. I thank Prof. em. Ernst Werner for agreeing to write the second referee's report on this thesis.

I gratefully acknowledge continued support from the Hans-Böckler foundation, without whose continued backing my broad studies would not have been possible.

It was a pleasure to have few short, but inspiring conversations with Prof. Carlo Rovelli, whose writings not only show his lively desire to learn and to grasp the concealed meaning behind the obvious, but also his concern for a human, peaceful world.

Thanks to all the members of the former chair Obermair for the friendly ambience, especially Lizy Lazar, Gerhard Meier and Jakob Lamey, with whom I had the opportunity to discuss several controversial issues.

Last but not least I would like to thank the developer communities of LyX and pybliographic, the software which made it delightful to author this thesis.