Scattering Theory for Dirac Particles in the Kerr-Newman Geometry

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1 Introduction.

One of the most spectacular predictions of general relativity are black holes, which should form when a large mass is concentrated in a sufficiently small volume. The idea of a mass-concentration, which is so dense that even light would be trapped goes back to Laplace in the 18th century. Shortly after Einstein developed general relativity, Karl Schwarzschild discovered in 1916 a mathematical solution to the equations of the theory that describes such an object. It was only much later, with the work of people like Oppenheimer, Volkoff and Snyder in the 1930's, that the scientific community began to think seriously about the possibility that such objects might actually exist in the Universe. It was shown that when a sufficiently massive star runs out of fuel, it is unable to support itself against its own gravitational attraction and it should collapse into a black hole. Only in the 1960's and the 1970's, in the so-called Golden Era of black hole research, new interesting phenomena like the Hawking radiation (in 1975 [1]) and superradiance were discovered but for a rigorous mathematical description of them we have to wait until the 1990's and the beginning of the new century, when the rigorous analysis of the propagation and of the scattering properties of classical and quantum fields on black hole space-times was developed.

The time-dependent scattering theory for classical and quantum scalar fields in the Schwarzschild metric was first obtained in 1985 by Dimock [2] and in 1986-87 by Dimock and Kay [3, 4, 5]. Bachelot developed the scattering theory for classical fields (electromagnetic waves in 1991 [6] and Klein-Gordon in 1994 [7]) and gave a rigorous mathematical description of the Hawking radiation in a series of publications in 1997 [8], in 1999 [9] and in 2000 [10]. Concerning the time-dependent scattering theory of Dirac particles in a Coulomb field we find some early publications in 1964 by Dollard [11], in 1966 by Dollard and Velo [12] and in 1986 by Enss and Thaller [13]. Concerning Dirac fields in the Schwarzschild geometry, Nicolas developed in 1995 a scattering theory for classical massless Dirac particles [14] and in 1998 Jin constructed wave operators, classical at the event horizon and Dollard-modified at infinity, in the massive case [15]. He also showed that the long-range mass term in the Hamiltonian can be taken into account by a logarithmic phase shift in the free dynamics. Moreover, Melnyk gave in 2003 [16] a complete scattering theory for massive charged Dirac fields in the Reissner-Nordstrϕm metric and in 2004 [17] a proof of the Hawking effect for charged, massive spin 1/2 particles.

For the nonlinear Klein-Gordon equation on Schwarzschild like metrics partial scattering results by means of conformal methods have been given in 1995 by Nicolas [18]. A complete scattering theory for the wave equation, on stationary, asymptotically flat space-times, was obtained in 2001 by Häfner [19], using Mourre theory.

Whenever we attempt to analyze the scattering properties of fields in the more realistic framework of the Kerr-Newman black hole geometry, discovered in 1965 [20], we are faced with several difficulties, which are not present in the simplified picture of the Schwarzschild metric. First of all, the Kerr-Newman solution is only axisymmetric (cylindrical symmetry), since it possesses only two commuting Killing vector fields, namely the time coordinate vector field ∂_t and the longitude coordinate vector field ∂_φ. This implies that there is no decomposition in spin-weighted spherical harmonics and that artificial long-range terms at infinity will be present in the field equations. Moreover, another difficulty is due to fact that the Kerr-Newman space-time is not stationary. In particular it is impossible to find a Killing vector field which is time-like everywhere outside the black hole. In fact ∂_t becomes space-like in the ergosphere, a toroidal region around the horizon. This implies that for field equations describing particles of integral spin (wave equation, Klein-Gordon, Maxwell) there exists no positive definite conserved energy. For field equations describing particles with half integral spin (Weyl, Dirac) we can find a conserved L^2 norm with the usual interpretation of a conserved charge. Hence, the absence of stationarity in the Kerr-Newman metric is not really a difficulty for the scattering theory of classical Dirac fields.

For the reasons mentioned above there are only few analytical studies of the propagation of fields outside a Kerr-Newman black hole. Time-dependent scattering for the Klein-Gordon equation in the Kerr framework has been developed in 2003 by Häfner [21], while in 2004 Daudé [22] proved the existence and asymptotic completeness of wave operators, classical at the event horizon and Dollard-modified at infinity, for classical massive Dirac particles in the Kerr-Newman geometry.

In this work we develop a time-dependent scattering theory for massive Dirac particles outside a non-extreme Kerr-Newman background by giving explicit analytical expressions for wave operators classical at the event horizon and Dollard-modified at infinity. This is, to our knowledge, the first analytical result, since all previous works in this direction treat the problem of their existence and of the asymptotic completeness for the Dirac equation in the Kerr-Newman metric. Our work is organized as follows:

- in Section 2 we briefly present the Kerr-Newman metric, we give the expression of the Dirac equation like in [31] and we review the asymptotic behavior of its radial eigenfunctions. At the end of the section we introduce the integral representation for the Dirac propagator in the Kerr-Newman geometry (see [31]) and since by setting $M = Q = 0$ the Kerr-Newman metric goes over in the Minkowski metric in oblate spheroidal coordinates, we derive an integral representation for the Dirac propagator in this framework.

- In Section 3 we first recall the basic principles of classical and Dollard-modified wave operators. Then, we present a few results, which allow us later on to consider a simplified version of the wave operators.

- In Section 4 we summarize the main results obtained in Section 5, 6 and 7. In particular we present explicit analytical expressions for the wave operators at the event horizon and asymptotically at infinity and we state their asymptotic completeness, which will be proved in Section 7.

- In Section 5 we treat first classical wave operators asymptotically at infinity and we compute them explicitly. The main advantage is that we get an analytical expression of the time-dependent logarithmic phase shift, that we need in order to construct Dollard-modified wave operators asymptotically away from the black hole. After implementation of this phase shift in the free dynamics we evaluate the Dollard-modified wave operators and we obtain for them an integral representation in terms of the transmission coefficients and of the fundamental solutions of the Dirac equation.

- In Section 6 we give the expressions of the wave operators close to the event horizon.

- In Section 7 we prove the asymptotic completeness of the wave operators at the event horizon and asymptotically at infinity.

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2 The Dirac equation in the non-extreme Kerr-Newman black hole geometry.

2.1 Prolegomena.

The Kerr-Newman space-time is described in Boyer-Lindquist coordinates (t, r, θ, φ) with $r > 0$, $0 \leq \theta \leq \pi$, $0 \leq \varphi < 2\pi$ as the manifold

$$\mathcal{M} = \mathbb{R}_t \times \mathbb{R}_r^+ \times S^2_{\theta,\varphi} \ ,$$

equipped with the Lorentzian metric g_{jk} [23], such that $ds^2 = g_{jk}dx^j dx^k$ is given by

$$ds^2 = \frac{\Delta}{U}(dt - a\sin^2\theta d\varphi)^2 - U\left(\frac{dr^2}{\Delta} + d\theta^2\right) - \frac{\sin^2\theta}{U}(adt - (r^2+a^2)d\varphi)^2 \quad (2.1)$$

with

$$U(r,\theta) = r^2 + a^2\cos^2\theta, \quad \Delta(r) = r^2 - 2Mr + a^2 + Q^2,$$

where M, aM and Q are the mass, the angular momentum and the charge of the black hole, respectively. In the above coordinates the Kerr-Newman metric has two types of singularities. The set of points for which $U(r,\theta) = 0$, corresponding to the equatorial ring $r = 0$ and $\theta = \pi/2$ of the sphere $r = 0$, is a true curvature singularity. The zeros of the function $\Delta(r)$, called horizons, are coordinate singularities. Choosing appropriate coordinate systems, they can be interpreted as regular null hypersurfaces that can be crossed in one direction but would require speeds greater than that of light to be crossed the other way. For this reason they are called horizons. Depending on the number of horizons there are three types of Kerr-Newman space-times:

- Non-extreme Kerr-Newman space-time for $M^2 > a^2 + Q^2$. Δ has two distinct zeros, namely,

  $$r_0 = M - \sqrt{M^2 - a^2 - Q^2} \quad \text{and} \quad r_1 = M + \sqrt{M^2 - a^2 - Q^2},$$

  the first one corresponding to the Cauchy horizon and the second to the event horizon. For $a = Q = 0$ we obtain the Schwarzschild space-time.

- Extreme Kerr-Newman space-time for $M^2 = a^2 + Q^2$. M is then the double root of Δ and the sphere $r = M$ is the only horizon.

- For $M^2 < a^2 + Q^2$ the only singularity is the ring singularity at $z = 0$ and $x^2 + y^2 = a^2$, which is then a naked singularity.

We shall only consider the non-extreme case and restrict attention to the region $r > r_1$ outside the event horizon, i.e. to the so-called Boyer-Lindquist Block I, defined by $\mathcal{B}_I = \mathbb{R}_t \times \Sigma$, where the space-like slice Σ is $(r_1, +\infty) \times$

$S^2_{\theta,\varphi}$. Hence $\Delta > 0$ is always positive. Notice that Block I is not stationary, since there exists no globally defined time-like Killing vector field, and it contains a toroidal region, called ergosphere, surrounding the event horizon, where the vector ∂_t is space-like. Moreover, we assume that the charge of the black hole is so small that the gravitational attraction is the dominant force at a large distance from the black hole, that means

$$mM > |eQ|.$$

2.2 The Dirac Equation in the Kerr-Newman Geometry.

We briefly recall some elementary facts about the Dirac operator in curved space-time. The Dirac operator G is a differential operator of first order

$$G = iG^j(x)\frac{\partial}{\partial x^j} + B(x),$$

where B and the Dirac matrices G^j are 4×4 matrices. The Dirac matrices are related to the Lorentzian metric via the anticommutation relations

$$g^{jk}(x) = \frac{1}{2}\{G^j(x), G^k(x)\}. \tag{2.2}$$

The matrix B is determined by the spinor connection and the electromagnetic potential through minimal coupling. As such, it is determined by the Levi-Civita connection of the background metric (2.1) and the potential

$$A_j dx^j = -\frac{Qr}{U}\left(dt - a\sin^2\theta d\varphi\right).$$

The Dirac matrices are not uniquely determined by the anticommutation rules (2.2). The ambiguity in the choice of Dirac matrices adapted to a given metric is formulated naturally in terms of the spin and frame bundles [24]. A convenient method for the computation of the Dirac operator in this bundle formulation is provided by the Newman-Penrose formalism [25]. Furthermore, in [26] explicit formulas for the matrix B in terms of the Dirac matrices G^j are given. Thus, by combining the advantages of these different approaches it is possible like in [27] to choose first the Dirac matrices using a Newman-Penrose frame and then to construct the matrix B using the explicit formulas in [26].

The four-component wave function Ψ of a Dirac particle is a solution of the Dirac equation

$$(G - m)\,\Psi = 0, \tag{2.3}$$

which can be completely separated into ordinary differential equations. This was first shown for the Kerr metric by Chandrasekhar [28] and was later generalized to the Kerr-Newman geometry [29, 30]. Following [27], we introduce diagonal matrices $S(r,\theta)$ and $\Gamma(r,\theta)$ defined by

$$\begin{aligned} S &= \Delta^{\frac{1}{4}}\mathrm{diag}(\sqrt{r-ia\cos\theta}, \sqrt{r-ia\cos\theta}, \sqrt{r+ia\cos\theta}, \sqrt{r+ia\cos\theta}), \\ \Gamma &= -i\mathrm{diag}\left((r+ia\cos\theta), -(r+ia\cos\theta), -(r-ia\cos\theta), (r-ia\cos\theta)\right). \end{aligned}$$

Then the transformed wave function $\hat{\Psi} = S\Psi$ satisfies the Dirac equation

$$\Gamma S\,(G-m)\,S^{-1}\hat{\Psi} = 0. \tag{2.4}$$

This transformation allows to write the differential operator (2.4) as a sum of an operator $\mathcal{R}$, which depends only on the radius r, and an operator $\mathcal{A}$, which depends only on the angular variables θ, φ. More precisely, we have

$$\Gamma S\,(G-m)\,S^{-1} = \mathcal{R} + \mathcal{A}, \tag{2.5}$$

with

$$\begin{aligned} \mathcal{R} &= \begin{pmatrix} imr & 0 & \sqrt{\Delta}\mathcal{D}_+ & 0 \\ 0 & -imr & 0 & \sqrt{\Delta}\mathcal{D}_- \\ \sqrt{\Delta}\mathcal{D}_- & 0 & -imr & 0 \\ 0 & \sqrt{\Delta}\mathcal{D}_+ & 0 & imr \end{pmatrix}, \\ \mathcal{A} &= \begin{pmatrix} -am\cos\theta & 0 & 0 & \mathcal{L}_+ \\ 0 & am\cos\theta & -\mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ & -am\cos\theta & 0 \\ -\mathcal{L}_- & 0 & 0 & am\cos\theta \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{D}_\pm &= \frac{\partial}{\partial r} \mp \frac{1}{\Delta}\left[(r^2+a^2)\frac{\partial}{\partial t} + a\frac{\partial}{\partial\varphi} - ieQr\right] \\ \mathcal{L}_\pm &= \frac{\partial}{\partial\theta} + \frac{\cot\theta}{2} \mp i\left[a\sin\theta\frac{\partial}{\partial t} + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right]. \end{aligned}$$

By employing for $\hat{\Psi}$ the ansatz

$$\hat{\Psi}(t,r,\theta,\varphi) = e^{-i\omega t}e^{-i\left(k+\frac{1}{2}\right)\varphi}\begin{pmatrix} X_-(r)Y_-(\theta) \\ X_+(r)Y_+(\theta) \\ X_+(r)Y_-(\theta) \\ X_-(r)Y_+(\theta) \end{pmatrix}, \quad k\in\mathbb{Z}, \tag{2.6}$$

where $X_\pm(r)$ and $Y_\pm(\theta)$ denote respectively the radial and angular functions, we can substitute (2.6) into the transformed Dirac equation (2.4) and we obtain the eigenvalue problems

$$\mathcal{R}\hat{\Psi} = \lambda\hat{\Psi}, \qquad \mathcal{A}\hat{\Psi} = -\lambda\hat{\Psi}, \tag{2.7}$$

under which the Dirac equation decouples into the systems of ordinary differential equations

$$\begin{pmatrix} \sqrt{\Delta}\widetilde{\mathcal{D}}_+ & imr-\lambda \\ -imr-\lambda & \sqrt{\Delta}\widetilde{\mathcal{D}}_- \end{pmatrix} \begin{pmatrix} X_+ \\ X_- \end{pmatrix} = 0, \tag{2.8}$$

$$\begin{pmatrix} \widetilde{\mathcal{L}}_+ & -am\cos\theta+\lambda \\ am\cos\theta+\lambda & -\widetilde{\mathcal{L}}_- \end{pmatrix} \begin{pmatrix} Y_+ \\ Y_- \end{pmatrix} = 0,, \tag{2.9}$$

where the radial and the angular operators $\widetilde{\mathcal{D}}_\pm$ and $\widetilde{\mathcal{L}}_\pm$ are given by

$$\begin{aligned} \widetilde{\mathcal{D}}_\pm &= \frac{d}{dr} \pm \frac{i}{\Delta}\left[\omega(r^2+a^2) + \left(k+\frac{1}{2}\right)a + eQr\right] \\ \widetilde{\mathcal{L}}_\pm &= \frac{d}{d\theta} + \frac{\cot\theta}{2} \mp \left[a\omega\sin\theta + \frac{k+\frac{1}{2}}{\sin\theta}\right]. \end{aligned}$$

Notice that for $m=0$ the Dirac equation reduces to the Weyl neutrino and anti-neutrino equations. According to [31] we will use the vector notation

$$X = (X_+, X_-), \quad Y = (Y_-, Y_+)$$

and sometimes for sake of clarity add indices for the parameters involved, e.g. $X^{k\omega\lambda} \equiv X$. We finally remark that (2.6) is an eigenfunction of the angular operator $i\partial_\varphi$ with half odd integer eigenvalue $k+\frac{1}{2}$.

2.3 Hamiltonian Formulation.

Since both the radial and angular operators $\widetilde{\mathcal{D}}_\pm$ and $\widetilde{\mathcal{L}}_\pm$ depend on ω, the separation constant λ will also depend on ω, making the eigenvalue problem (2.7) rather complicated. Hence it is useful to bring the Dirac equation (2.5) into Hamiltonian form, in a way which is compatible with the separation of variables. For this purpose let us bring the time derivative in (2.5) to one side of the equation.Thus we get

$$i\left[\frac{r^2+a^2}{\sqrt{\Delta}}\begin{pmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix} - a\sin\theta\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\right]\frac{\partial\hat{\Psi}}{\partial t} = \left(\mathcal{R}^3 + \mathcal{A}^3\right)\hat{\Psi}, \tag{2.10}$$

where the operators $\mathcal{R}^3$ and $\mathcal{A}^3$ can be derived from $\mathcal{R}$ and $\mathcal{A}$ by setting the time derivatives equal to zero and $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ are Pauli matrices. Introducing a Regge-Wheeler-type coordinate $u \in (-\infty, +\infty)$ defined by

$$\frac{du}{dr} = \frac{r^2+a^2}{\Delta}, \tag{2.11}$$

which maps the event horizon r_1 to $-\infty$, (2.10) finally becomes

$$i\frac{\partial\hat{\Psi}}{\partial t} = H\hat{\Psi}, \tag{2.12}$$

where the Hamiltonian H is given by

$$H = \left[\left(1 - \frac{a^2 \Delta \sin^2\theta}{(r^2+a^2)^2} \right)^{-1} \left(\mathbb{1} - \frac{a\sqrt{\Delta}\sin\theta}{r^2+a^2} \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \right) \right] \left(\hat{\mathcal{R}} + \hat{\mathcal{A}} \right), \tag{2.13}$$

r now being an implicit function of u. Moreover, the radial and angular operators have the following form

$$\hat{\mathcal{R}} = -\frac{mr\sqrt{\Delta}}{r^2+a^2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\mathcal{E}_- & 0 & 0 & 0 \\ 0 & \mathcal{E}_+ & 0 & 0 \\ 0 & 0 & \mathcal{E}_+ & 0 \\ 0 & 0 & 0 & -\mathcal{E}_- \end{pmatrix}$$

$$\hat{\mathcal{A}} = \frac{am\cos\theta\sqrt{\Delta}}{r^2+a^2} \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\mathcal{M}_+ & 0 & 0 \\ -\mathcal{M}_- & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{M}_+ \\ 0 & 0 & \mathcal{M}_- & 0 \end{pmatrix}$$

with

$$\mathcal{E}_\pm = i\frac{\partial}{\partial u} \mp \left(\frac{ia}{r^2+a^2}\frac{\partial}{\partial\varphi} + \frac{eQr}{r^2+a^2} \right)$$

$$\mathcal{M}_\pm = \frac{\sqrt{\Delta}}{r^2+a^2} \left(i\frac{\partial}{\partial\theta} + i\frac{\cot\theta}{2} \pm \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} \right).$$

Since the Hamiltonian (2.13) is an operator acting on the wave functions on the hypersurfaces t =const., we would be tempted to choose the simplest scalar product on such a hypersurface,i.e.

$$(\hat{\Psi}|\hat{\Phi}) = \int_{\mathbb{R}} du \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\varphi \overline{\hat{\Psi}}(t,u,\theta,\varphi)\hat{\Phi}(t,u,\theta,\varphi), \tag{2.14}$$

where $\overline{\hat{\Psi}}$ is the complex conjugated, transposed spinor. Unfortunately, it turns out that (2.13) is not in general Hermitian with respect to the above scalar product, since we get derivatives with respect of r and θ of the square bracket in (2.13), when we consider the adjoint of H and integrate by parts. It is easy to check that the Hamiltonian H is indeed Hermitian in the special case $a = 0$. That means that hermiticity will be recovered again in the Schwarzschild background.
This problem was solved in [31], where it was shown that the Hamiltonian H is Hermitian with respect to the positive scalar product

$$< \hat{\Psi}|\hat{\Phi} >= \int_{\mathbb{R}} du \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\varphi < \hat{\Psi}|\hat{\Phi} >_{(t,u,\theta,\varphi)}, \tag{2.15}$$

with the inner product

$$< \hat{\Psi}|\hat{\Phi} >_{(t,u,\theta,\varphi)} = \overline{\hat{\Psi}}(t,u,\theta,\varphi) \left[\mathbb{1} + \frac{a\sqrt{\Delta}\sin\theta}{r^2+a^2} \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \right] \hat{\Phi}(t,u,\theta,\varphi). \tag{2.16}$$

Moreover, if we let $\mathcal{H}$ be the Hilbert space of wave functions with scalar product (2.15), then the operator H is essentially self-adjoint on $\mathcal{H}$ with domain of definition [31]

$$D(H) = C_0^\infty \left(\mathbb{R} \times S^2\right)^4 .$$

2.4 Asymptotic Behavior of the Radial Eigenfunctions.

In the Regge-Wheeler variable u, the radial equation (2.8) becomes

$$\left[\frac{d}{du} + i\Omega(u) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] X = \frac{\sqrt{\Delta}}{r^2+a^2} \begin{pmatrix} 0 & imr - \lambda \\ -imr - \lambda & 0 \end{pmatrix} X \tag{2.17}$$

with

$$\Omega(u) = \omega + \frac{\left(k+\frac{1}{2}\right)a + eQr}{r^2+a^2}. \tag{2.18}$$

Since the asymptotic behavior of the radial eigenfunctions has already been investigated in [31], we merely state here those results, that are of future relevance for the evaluation of the wave operator at the event horizon and asymptotically at infinity.
The following Lemma describes the asymptotics of $X(u)$ as $u \to -\infty$.

Lemma 2.1. *Every nontrivial solution X of (2.17) satisfying the boundary conditions $X_+(\hat{u}_2) = X_-(\hat{u}_2)$ with $u \in (-\infty, \hat{u}_2]$ has asymptotically at the event horizon the following form*

$$X(u) = \begin{pmatrix} e^{-i\Omega_0 u}\left(f_0^+ + R_0^+(u)\right) \\ e^{i\Omega_0 u}\left(f_0^- + R_0^-(u)\right) \end{pmatrix} \tag{2.19}$$

with $f_0 \neq 0$ and

$$|R_0| \leq c e^{du}, \quad \Omega_0 = \omega + \frac{\left(k+\frac{1}{2}\right)a + eQr_1}{r_1^2+a^2},$$

where $c, d > 0$ are suitable constants, which can be chosen locally uniformly in ω and $r_1 = M + \sqrt{M^2 - a^2 - Q^2}$.

Proof. Since the solutions of (2.17) go over for $r \to r_1$ into plane waves with frequency

$$\Omega_0 = \omega + \frac{\left(k+\frac{1}{2}\right)a + eQr_1}{r_1^2+a^2},$$

it is reasonable to make the ansatz

$$X(u) = B\begin{pmatrix} f_0^+(u) \\ f_0^-(u) \end{pmatrix}, \quad B = \begin{pmatrix} e^{-i\Omega_0 u} & 0 \\ 0 & e^{i\Omega_0 u} \end{pmatrix} \tag{2.20}$$

with B unitary matrix. Substitution of (2.20) into (2.17) yields to the following equation for f

$$\frac{df}{du} = \Bigg[i\left(\Omega_0 - \Omega(u)\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \\ + \frac{\sqrt{\Delta}}{r^2+a^2}\begin{pmatrix} 0 & e^{2i\Omega_0 u}\left(imr-\lambda\right) \\ e^{-2i\Omega_0 u}\left(-imr-\lambda\right) & 0 \end{pmatrix}\Bigg] f. \tag{2.21}$$

It is straightforward to verify that the square bracket in (2.21) vanishes on the event horizon $r = r_1$. Respect to the Regge-Wheeler variable u this leads to the exponential decay for $u \to -\infty$, in the sense that there exists constant $c_1, d > 0$ such that

$$\left|\frac{df}{du}\right| \le c_1 e^{du}\,|f|. \tag{2.22}$$

Notice that, $|f| \ne 0$ since X is a non trivial solution. This means, we can divide (2.22) by $|f| \ne 0$ and integrate from any $u < \hat{u}_2$ to $\hat{u}_2$ to obtain

$$\log\left|\frac{f(\hat{u}_2)}{f(u)}\right| \le c_2\left(e^{d\hat{u}_2} - e^{du}\right), \quad c_2 = \frac{c_1}{d}.$$

Since the right side of the above inequality remains finite for $u \to -\infty$, we can conclude that there exists for all $u < \hat{u}_2$ a constant $L > 0$ such that

$$\frac{1}{L} \le |f(u)| \le L. \tag{2.23}$$

Since the eigenvalues λ of the Chandrasekhar-Page angular equation depend smoothly on ω (see [32]), the constants c_1, c_2, d and L can be chosen locally uniformly in ω. Substitution of (2.23) into (2.22) gives

$$\left|\frac{df}{du}\right| \le c_1 L e^{du}. \tag{2.24}$$

The inequality (2.24) shows that the derivative of f respect to u is integrable, and thus $f(u)$ converges for $u \to -\infty$. Setting

$$f_0 = \lim_{u\to-\infty} f(u)$$

with $f_0 \ne 0$ because of (2.23), (2.24)can be integrated from $-\infty$ to u and we obtain

$$|f(u) - f_0| \le c e^{du}, \quad c = \frac{c_1 L}{d}.$$

□

Thus, from (2.19) we see that $X(u)$ does not decay to zero at the event horizon, but instead it behaves like a plane wave oscillating with frequency Ω_0. We now come to estimates which describe the asymptotics of solutions of the radial equation for large u. Asymptotically at infinity, (2.17) takes the form

$$\frac{dX}{du} = \left[\begin{pmatrix} -i\omega & im \\ -im & i\omega \end{pmatrix} + \frac{1}{u} \begin{pmatrix} -ieQ & -imM - \lambda \\ imM - \lambda & ieQ \end{pmatrix} + \mathcal{O}(u^{-2}) \right] X, \tag{2.25}$$

where the matrix potential on the right converges as $u \to +\infty$. Concerning the behavior of the solutions we can distinguish two cases:

- if $|\omega| < m$, the eigenvalues $\rho = \pm\sqrt{m^2 - \omega^2}$ of the matrix potential are real in the limit $u \to +\infty$. We have therefore one fundamental solution which decays exponentially like $\exp(-\sqrt{m^2-\omega^2}u)$ and a second one with exponential growth like $\exp(\sqrt{m^2-\omega^2}u)$. We denote these two fundamental solutions by $\hat{\Psi}_a^{k\omega n}$ with $a = 1, 2$. Moreover, they are normalized according to

$$\lim_{u \to -\infty} \left| \hat{\Psi}_a^{k\omega n}(u) \right| = 1, \quad \forall\, a = 1, 2.$$

- If $|\omega| > m$, the eigenvalues $\rho = \pm i\sqrt{\omega^2 - m^2}$ of the matrix potential are imaginary in the limit $u \to +\infty$. This means, we have two fundamental solutions $\hat{\Psi}_a^{k\omega n}$ which oscillate like $\exp(\pm i\sqrt{\omega^2 - m^2}u)$. Concerning the normalization, we are free to choose both the amplitude and the phase. In what follows we use the convention

$$f_{0,1}^{k\omega n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad f_{0,2}^{k\omega n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

  with $f_{0,a}^{k\omega n}$ as in the asymptotic expansion (2.19).

The next two Lemmas describe the asymptotics of the solutions of (2.17) as $u \to +\infty$, respectively for $|\omega| < m$ and $|\omega| > m$.

Lemma 2.2. *Every nontrivial solution X of (2.17) for $|\omega| < m$ behaves asymptotically for $u \to +\infty$ like*

$$X(u) = \hat{A} \begin{pmatrix} e^{-\tilde{\Phi}(u)} \, (f_\infty^+ + R_\infty^+(u)) \\ e^{\tilde{\Phi}(u)} \, (f_\infty^- + R_\infty^-(u)) \end{pmatrix} \tag{2.26}$$

with $f_\infty \neq 0$ and

$$\tilde{\Phi}(u) = \sqrt{m^2 - \omega^2}u - \tilde{\alpha} \log u, \quad \tilde{\alpha} = \frac{\omega e Q + M m^2}{\sqrt{m^2 - \omega^2}}$$

$$\hat{A} = \frac{\sqrt{2}}{2} \begin{pmatrix} \cosh\hat{\Theta} + i\sinh\hat{\Theta} & \sinh\hat{\Theta} + i\cosh\hat{\Theta} \\ \sinh\hat{\Theta} + i\cosh\hat{\Theta} & \cosh\hat{\Theta} + i\sinh\hat{\Theta} \end{pmatrix}, \quad \hat{\Theta} = \frac{1}{4}\log\left(\frac{m+\omega}{m-\omega}\right), \tag{2.27}$$

$$|R_\infty| \leq \frac{C'}{u},$$

where C' is a positive constant, which can be chosen locally uniformly in ω.

Proof. Let us write (2.17) as follows

$$\frac{dX}{du} = V(u)X \tag{2.28}$$

where the matrix potential $V(u)$ is given by

$$V(u) = i\Omega(u)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sqrt{\Delta}}{r^2+a^2}\begin{pmatrix} 0 & imr-\lambda \\ -imr-\lambda & 0 \end{pmatrix},$$

which asymptotically at infinity becomes

$$V(u) = \begin{pmatrix} -i\omega & im \\ -im & i\omega \end{pmatrix} + \frac{1}{u}\begin{pmatrix} -ieQ & -imM-\lambda \\ imM-\lambda & ieQ \end{pmatrix} + \mathcal{O}\left(\frac{1}{u^2}\right). \tag{2.29}$$

According to (2.29) and the hypothesis $|\omega| < m$, the eigenvalues of V are for sufficiently large u real. More precisely, there is a transformation matrix $B(u)$ with

$$B^{-1}VB = -\tilde{\Omega}(u)\sigma_3 = \begin{pmatrix} -\tilde{\Omega}(u) & 0 \\ 0 & \tilde{\Omega}(u) \end{pmatrix} \tag{2.30}$$

and a suitable function $\tilde{\Omega}(u)$. Since the matrix potential V converges for $u \to +\infty$ and has a regular expansion in powers of $1/u$, B can be chosen such that

$$|B(u)| \leq c_0, \quad \text{and} \quad |B'(u)| \leq \frac{c_0}{u^2} \tag{2.31}$$

with $c_0 > 0$. Let $\tilde{X}$ be the transformed radial spinor defined by

$$X = B\tilde{X}, \tag{2.32}$$

then we have

$$\frac{dX}{du} = B'\tilde{X} + B\frac{d\tilde{X}}{du}$$

and (2.28) becomes

$$B\frac{d\tilde{X}}{du} + B'\tilde{X} = VB\tilde{X}.$$

Multiplication of the above equation from the left by B^{-1} gives

$$\frac{d\tilde{X}}{du} = (B^{-1}VB - B^{-1}B')\tilde{X}$$

and by means of (2.32) we find

$$\frac{d}{du}(B^{-1}X) = (-\tilde{\Omega}\sigma_3 - B^{-1}B')(B^{-1}X). \tag{2.33}$$

Let us first compute the function $\tilde{\Omega}$. To this purpose we write the matrix potential V as

$$V(u) = V_1(u) + \mathcal{O}\left(\frac{1}{u^2}\right),$$

where

$$V_1(u) = \begin{pmatrix} -i\omega & im \\ -im & i\omega \end{pmatrix} + \frac{1}{u}\begin{pmatrix} -ieQ & -imM - \lambda \\ imM - \lambda & ieQ \end{pmatrix}.$$

In the limit $u \to +\infty$, the eigenvalues of the above matrix are

$$\tilde{\lambda}_1(u) = -\sqrt{m^2 - \omega^2} + \frac{\widetilde{\alpha}}{u} + \mathcal{O}\left(\frac{1}{u^2}\right),$$
$$\tilde{\lambda}_2(u) = \sqrt{m^2 - \omega^2} - \frac{\widetilde{\alpha}}{u} + \mathcal{O}\left(\frac{1}{u^2}\right)$$

with

$$\widetilde{\alpha} = \frac{\omega eQ + Mm^2}{\sqrt{m^2 - \omega^2}}.$$

Hence

$$B^{-1}VB = \begin{pmatrix} \tilde{\lambda}_1(u) & 0 \\ 0 & \tilde{\lambda}_2(u) \end{pmatrix}. \tag{2.34}$$

Comparing (2.34) with (2.30), we find that

$$\tilde{\Omega}(u) = \sqrt{m^2 - \omega^2} - \frac{\widetilde{\alpha}}{u} + \mathcal{O}\left(\frac{1}{u^2}\right). \tag{2.35}$$

Substituting the following ansatz

$$X(u) = B\begin{pmatrix} e^{-\widetilde{\Phi}} f^+(u) \\ e^{\widetilde{\Phi}} f^-(u) \end{pmatrix} \tag{2.36}$$

into (2.33) and neglecting terms of order $\mathcal{O}(u^{-2})$, we get

$$\widetilde{\Phi}'(u) = \sqrt{m^2 - \omega^2} - \frac{\widetilde{\alpha}}{u}.$$

Thus, $\widetilde{\Phi}(u)$ is given by

$$\widetilde{\Phi}(u) = \sqrt{m^2 - \omega^2}u - \alpha \log u.$$

Combining (2.30) with (2.35), we obtain

$$V = B\begin{pmatrix} -\sqrt{m^2 - \omega^2} & 0 \\ 0 & \sqrt{m^2 - \omega^2} \end{pmatrix} B^{-1} + \mathcal{O}\left(\frac{1}{u}\right). \tag{2.37}$$

Since

$$V(u) = V_\infty + \mathcal{O}\left(\frac{1}{u}\right), \quad V_\infty := \begin{pmatrix} -i\omega & im \\ -im & i\omega \end{pmatrix}$$

and the matrix B can be written as

$$B = \hat{A} + \mathcal{O}\left(\frac{1}{u}\right),$$

from (2.37) we have the following equation

$$V_\infty = \hat{A}\begin{pmatrix} -\sqrt{m^2-\omega^2} & 0 \\ 0 & \sqrt{m^2-\omega^2} \end{pmatrix}\hat{A}^{-1} \tag{2.38}$$

for the unknown matrix $\hat{A} = (\hat{a}_{ij})$ with $i, j = 1, 2$. The requirements

$$\hat{a}_{11} = \hat{a}_{22}, \quad \text{and} \quad \hat{a}_{12} = \hat{a}_{21}, \tag{2.39}$$

together with the condition $\det\hat{A} = 1$ yield (2.27).
At this point we notice that all terms of order $\mathcal{O}(u^{-1})$ has been absorbed into R_∞. By means of the ansatz (2.36) and using the bound (2.31), the following inequality can be obtained

$$\left|\frac{df}{du}\right| \leq \frac{c_0}{u^2}|f| \tag{2.40}$$

where for ease in notation we omitted the subscripts $\pm$. Since X is a non trivial solution, $|f| \neq 0$. Thus, we can divide (2.40) by $|f|$, integrate for sufficiently large u to obtain the bounds

$$\frac{1}{L} \leq |f(u)| \leq L. \tag{2.41}$$

The integrability of $f^{'}(u)$ can be seen directly by substituting the upper bound for $|f|$ into (2.40). Thus, f has a finite and, according to (2.41) non-zero limit

$$f_\infty := \lim_{u\to+\infty} f(u) \neq 0.$$

Finally, the $\frac{1}{u}$-decay of $R_\infty(u)$ is obtained by integrating (2.40) backwards from $u = +\infty$ and employing the resulting bound in the ansatz (2.36). □

The following Lemma is the analogue of Lemma 2.2 for $|\omega| > m$ and since the method of proof is very similar, we will omit it (see [31]).

Lemma 2.3. *Every nontrivial solution X of (2.25) for $|\omega| > m$ has asymptotically at infinity the form*

$$X(u) = A\begin{pmatrix} e^{-i\Phi(u)}(f^+_\infty + R^+_\infty(u)) \\ e^{i\Phi(u)}(f^-_\infty + R^-_\infty(u)) \end{pmatrix} \tag{2.42}$$

with

$$f_\infty \neq 0, \quad |R_\infty| \leq \frac{C}{u}$$

$$\Phi(u) = \epsilon(\omega)\left(\sqrt{\omega^2-m^2}u + \frac{\omega eQ + Mm^2}{\sqrt{\omega^2-m^2}}\log u\right),$$

$$A = \begin{pmatrix} \cosh\Theta & \sinh\Theta \\ \sinh\Theta & \cosh\Theta \end{pmatrix}, \quad \Theta = \frac{1}{4}\log\left(\frac{\omega+m}{\omega-m}\right), \tag{2.43}$$

where $C > 0$ is a constant, which can be chosen locally uniformly in ω and $\epsilon(\omega)$ is a sign function such that $\epsilon(\omega) = 1$ if $\omega > m$ and $\epsilon(\omega) = -1$ if $\omega < -m$.

In analogy to potential wall problems for Schrödinger operators, we call the function $f_\infty^\pm$ in (2.42) corresponding to the fundamental solutions $\hat{\Psi}_a^{k\omega n}$ with $a = 1, 2$ the **transmission coefficients**, and denote them by $f_{\infty,a}^{k\omega n}$. In view of these results we have the following qualitative properties of the wave function $\hat{\Psi}_a^{k\omega n}$. The solutions $\hat{\Psi}_{a=1,2}^{k\omega n}$ of the Dirac equation go over asymptotically near the event horizon to spherical waves. In particular in the region $|\omega| > m$, the solutions for $a = 1$ are the incoming waves, i.e. asymptotically near the event horizon they are waves moving towards the black hole. On the other hand the solutions for $a = 2$ are the outgoing waves, which near the event horizon move outwards away from the black hole. Asymptotically near infinity, $\hat{\Psi}_a^{k\omega n}$ goes over to spherical waves too. In the region $|\omega| < m$, the fundamental solutions for $a = 1, 2$, near the event horizon are both linear combinations of incoming and outgoing waves, taken in such a way that $\hat{\Psi}_1^{k\omega n}$ and $\hat{\Psi}_2^{k\omega n}$ at infinity have exponential decay and growth, respectively.

2.5 Integral Representation for the Dirac Propagator.

The spectral decomposition for the propagator e^{-itH} is given by

$$e^{-itH} = \int_{-\infty}^{\infty} e^{-i\omega t} dE_\omega$$

with dE_ω the spectral measure of the Hamiltonian H. Moreover, we remind that $\sigma(H)$ is continuous. Thus, it is not clear how dE_ω can be written in terms of the solutions of the radial and angular equations. In what follows we will shortly describe how this problem has been solved in [31]. The main strategy is to consider the operator H on a given spatial interval, let us say H_{u_1,u_2}. In the end, the integral representation for e^{-itH} can be obtained by taking suitable limits for $u_1 \to -\infty$ and $u_2 \to +\infty$.
From [33] it is known, that $\sigma(H_{u_1,u_2})$ is purely discrete with finite-dimensional eigenspaces, since H_{u_1,u_2} is an elliptic operator on a bounded domain. The next step is to choose an eigenvector basis compatible with the separation of

variables (2.6). First of all, we observe that the basis vectors can be taken as eigenvectors of the operator $i\partial_\varphi$ with eigenvalues $k+\frac{1}{2}$, $k \in \mathbb{Z}$. Let $\mathcal{H}^k_{u_1,u_2}$ be this eigenspace of the azimuthal operator and $H^k_{u_1,u_2}$ the restriction of H_{u_1,u_2} to the eigenspace $\mathcal{H}^k_{u_1,u_2}$. Furthermore, the basis vectors can be chosen as eigenvectors of the angular operator $\mathcal{A}$, having the property that the spectrum of $H^k_{u_1,u_2}$ is purely discrete. The eigenvalues are non-degenerate and depend smoothly on ω (see appendix in [31]). Let us denote the eigenvalues of $\mathcal{A}$ by $\lambda_n(\omega)$ with $n \in \mathbb{Z}$. It is clear that $\lambda_n < \lambda_{n+1}$ for all $n \in \mathbb{Z}$ and $\lambda_n(\cdot) \in C^\infty(\mathbb{R})$. The radial equation (2.8) has for any given $k \in \mathbb{Z}$, $\omega \in \sigma(H^k_{u_1,u_2})$ and $n \in \mathbb{Z}$ at most one solution satisfying the boundary conditions

$$\hat{\Psi}_1(u_1) = \hat{\Psi}_3(u_1), \quad \hat{\Psi}_2(u_1) = \hat{\Psi}_4(u_1)$$

and

$$\hat{\Psi}_1(u_2) = \hat{\Psi}_3(u_2), \quad \hat{\Psi}_2(u_2) = \hat{\Psi}_4(u_2).$$

Thus, it results that for any k, ω and n there is at most one eigenstate $\hat{\Psi}^{k\omega n}_{u_1,u_2}$ of H_{u_1,u_2}. Let $N(k,\omega)$ be the set of n for which such an eigenvector exists. The eigenvector basis is

$$\left(\hat{\Psi}^{k\omega n}_{u_1,u_2}\right)_{k\in\mathbb{Z},\omega\in\sigma(H^k_{u_1,u_2}),n\in N(k,\omega)}. \tag{2.44}$$

Moreover, the eigenfunctions can be normalized with respect to the scalar product (2.14). In particular the radial and angular parts satisfy the normalization conditions

$$(X^{k\omega n}_{u_1,u_2}|X^{k\omega n}_{u_1,u_2}) = 1 \quad \text{and} \quad (Y^{k\omega n}|Y^{k\omega n}) = 1,$$

where X and Y are the same as in (2.6). The self-adjointness of the angular operator $\mathcal{A}$ with respect to (2.14) implies that its eigenvectors are orthogonal, hence, the eigenfunctions are for given k and ω orthonormal, namely

$$(\hat{\Psi}^{k\omega n}_{u_1,u_2}|\hat{\Psi}^{k\omega n'}_{u_1,u_2}) = \delta_{nn'}, \quad n,n' \in N(k,\omega).$$

However, for different values of ω the eigenfunctions are in general not orthogonal with respect to (2.14), but since H_{u_1,u_2} is self-adjoint with respect to (2.15), its eigenspaces are orthogonal with respect to the latter scalar product. Thus, we get for $\omega \neq \omega'$

$$< \hat{\Psi}^{k\omega n}_{u_1,u_2}|\hat{\Psi}^{k'\omega'n'}_{u_1,u_2} >= 0.$$

The two scalar products (2.14) and (2.15) coincide in the special case $a = 0$. With the help of the basis (2.44) the spectral decomposition for H_{u_1,u_2} becomes

$$e^{-itH_{u_1,u_2}} = \sum_{k\in\mathbb{Z}} \sum_{\omega\in\sigma(H^k_{u_1,u_2})} e^{-i\omega t} \sum_{n,n'\in N(k,\omega)} c_{nn'}\, \hat{\Psi}^{k\omega n}_{u_1,u_2} < \hat{\Psi}^{k\omega n'}_{u_1,u_2}|\hat{\Psi} >, \tag{2.45}$$

where the coefficients $c_{nn'}$ are chosen such that

$$\sum_{n,n' \in N(k,\omega)} c_{nn'} \hat{\Psi}^{k\omega n}_{u_1,u_2} < \hat{\Psi}^{k\omega n'}_{u_1,u_2} | \hat{\Psi} > \tag{2.46}$$

is the projection of $\hat{\Psi}$ onto the eigenspace of $H^k_{u_1,u_2}$, corresponding to the eigenvalue ω, i.e.

$$c_{nn'} = (A^{-1})_{nn'} \quad \text{with} \quad A_{nn'} =< \hat{\Psi}^{k\omega n}_{u_1,u_2} | \hat{\Psi}^{k\omega n'}_{u_1,u_2} > .$$

The first two sums in (2.45) can be interpreted as a decomposition of $\hat{\Psi}$ into the orthogonal eigenstates of the azimuthal operator $i\partial_\varphi$ and the Hamiltonian H, respectively. Thus, the convergence in norm in $\mathcal{H}_{u_1,u_2}$ is assured. Moreover, the term (2.46) is the basis representation of the projector on the respective eigenspace.
The next step is to take the limit $u_1 \to -\infty$ in the above spectral decomposition, since it would be expected that the energy gaps $\Delta\omega_{kn}$ between neighboring eigenvalues, given by $\Delta\omega_{kn} = \min\{\tilde{\omega}_{kn} - \omega_{kn} | \tilde{\omega}_{kn} > \omega_{kn}\}$ with $\tilde{\omega}_{kn}, \omega_{kn} \in \sigma(H^k_{u_1,u_2})$ and $N(k, \tilde{\omega}_{kn}), N(k, \omega_{kn}) \neq 0$ should tend to zero. The main goal is to rewrite the sums in (2.45) as Riemann sums which converge to integrals for $u_1 \to -\infty$. This will yield a formula for the propagator of the Hamiltonian H_{u_2}. In order to make this mathematically precise, in [31] Finster et al. derived estimates for $\Delta\omega_{kn}$ and they related the eigenvectors $\hat{\Psi}^{k\omega n}_{u_1,u_2}$ in (2.45) to solutions $\hat{\Psi}^{k\omega n}_{u_2}(u)$ with $k, n \in \mathbb{Z}$, $\omega \in \mathbb{R}$, $u \in (-\infty, u_2]$ of the Dirac equation with boundary conditions

$$\hat{\Psi}_1(u_2, \theta, \varphi) = \hat{\Psi}_3(u_2, \theta, \varphi) \quad \text{and} \quad \hat{\Psi}_2(u_2, \theta, \varphi) = \hat{\Psi}_4(u_2, \theta, \varphi).$$

The radial and angular functions corresponding to $\hat{\Psi}^{k\omega n}_{u_2}$ will be denoted by $X^{k\omega n}_{u_2}$ and $Y^{k\omega n}$, respectively. From Lemma 2.3 it follows that $X(u)$ does not decay to zero for $u \to -\infty$. Thus, it results that $\hat{\Psi}^{k\omega n}_{u_2}(u)$ cannot have finite norm and hence is not a vector in the Hilbert space $\mathcal{H}_{u_2}$. This shows that the Hamiltonian H_{u_2} has no point spectrum. The functions $\hat{\Psi}^{k\omega n}_{u_2}(u)$ will be normalized according to

$$\lim_{u \to -\infty} |X^{k\omega n}_{u_2}| = 1 \quad \text{and} \quad (Y^{k\omega n} | Y^{k\omega n}) = 1.$$

The next two lemmas describe the behavior of the normalization factors and the energy gaps as $u_1 \to -\infty$. For the proofs see [31].

Lemma 2.4. *For fixed u_2 and asymptotically for $u_1 \to -\infty$, it results*

$$X^{k\omega n}_{u_1,u_2} = g(u_1) X^{k\omega n}_{u_2}|_{[u_1,u_2]} \quad \textit{with} \quad g(u_1) = (u_2 - u_1) + \mathcal{O}(1).$$

Moreover,

$$| < \hat{\Psi}^{k\omega n}_{u_1,u_2} | \hat{\Psi}^{k\omega n'}_{u_1,u_2} > -\delta_{nn'} | \leq \frac{c}{u_2 - u_1} < Y^{k\omega n} | \sin\theta \sigma_1 | Y^{k\omega n'} >,$$

where the constant c can be chosen locally uniformly in ω.

Lemma 2.5. *The following estimate holds asymptotically as* $u_1 \to -\infty$

$$\Delta\omega_{kn} = \frac{\pi}{u_2 - u_1} + \mathcal{O}(1)$$

for fixed u_2 *locally uniformly in* ω.

The next lemma gives the integral representation for the propagator of H_{u_2}.

Lemma 2.6. *For every* $\hat{\Psi} \in C_0^\infty((-\infty, u_2] \times S^2)^4$ *and* $x = (u, \theta, \varphi)$ *it results*

$$\left(e^{-itH_{u_2}}\right)(x) = \frac{1}{\pi} \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \sum_{n\in\mathbb{Z}} \hat{\Psi}_{u_2}^{k\omega n}(x) < \hat{\Psi}_{u_2}^{k\omega n} | \hat{\Psi} > \tag{2.47}$$

The last step in order to get an integral representation for e^{-itH} is to take in (2.47) a suitable limit for $u_2 \to +\infty$. This is done in the following theorem, which gives the representation of the Dirac propagator for a Dirac particle of mass m and charge e. For a detailed proof we refer to [31]. Here we limit us just to state it.

Theorem 2.7. *For every* $\hat{\Psi} \in C_0^\infty\left(\mathbb{R} \times S^2\right)^4$ *the Dirac propagator has the following integral representation*

$$\hat{\Psi}(t,x) = \left(e^{-iHt}\hat{\Psi}\right)(x) = \frac{1}{\pi} \sum_{k,n\in\mathbb{Z}} \int_{-\infty}^{+\infty} d\omega\ e^{-i\omega t} \sum_{a,b=1}^{2} t_{ab}^{k\omega n} \hat{\Psi}_a^{k\omega n}(x) \langle \hat{\Psi}_b^{k\omega n} | \hat{\Psi} \rangle, \tag{2.48}$$

where the coefficients $(t_{ab})_{a,b=1,2}$ *are given by*

$$t_{ab} = \begin{cases} \delta_{a1}\delta_{b1} & \text{if } |\omega| \le m \\ \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \frac{t_a \bar{t}_b}{|t_1|^2 + |t_2|^2}\, d\alpha & \text{if } |\omega| > m \end{cases} \tag{2.49}$$

and the functions t_a *are related to the transmission coefficients by the following relations*

$$t_1(\alpha) = f_{\infty,2}^+ e^{-i\alpha} - f_{\infty,2}^- e^{i\alpha}, \quad t_2(\alpha) = -f_{\infty,1}^+ e^{-i\alpha} + f_{\infty,1}^- e^{i\alpha}. \tag{2.50}$$

The integral and the series in (2.48) *converge in norm in the Hilbert space* $\mathcal{H}$.

2.6 The Dirac Propagator in the Oblate Spheroidal Coordinates.

For $M = Q = 0$ the Kerr-Newman metric in Boyer-Lindquist coordinates takes the form of the Minkowski metric expressed in oblate spheroidal coordinates, namely

$$ds^2 = dt^2 - \left(\frac{U}{\widetilde{\Delta}}\, dr^2 + U\, d\theta^2 + \widetilde{\Delta} \sin^2\theta\, d\varphi^2\right) \tag{2.51}$$

with

$$U(r,\theta) := r^2 + a^2\cos^2\theta \quad \text{and} \quad \widetilde{\Delta} := r^2 + a^2.$$

Notice that the expression (2.51) can be reduced to the Minkowski metric in Cartesian coordinates by means of the transformations

$$x = \sqrt{r^2+a^2}\sin\theta\cos\varphi, \quad y = \sqrt{r^2+a^2}\sin\theta\sin\varphi, \quad z = r\cos\theta.$$

It is interesting to observe that in the oblate spheroidal coordinates the surface given by $r = 0$ and $\theta = \frac{\pi}{2}$ is a circular disk of radius $|a|$ which lies in the x, y-plane and is centered at the origin. It is clear that in order to derive the integral representation of the Dirac propagator in these new coordinates it will be necessary to impose some regularity conditions on the spinors at $r = 0$.

Since it has been already shown in [25] that the Dirac equation separates in the oblate spheroidal coordinates, we may set $M = Q = 0$ in the system of ODEs (2.9) for the radial spinors to obtain

$$\begin{pmatrix} \sqrt{\widetilde{\Delta}}\widehat{\mathcal{D}}_+ & imr - \lambda \\ -imr - \lambda & \sqrt{\widetilde{\Delta}}\widehat{\mathcal{D}}_- \end{pmatrix} \begin{pmatrix} X_+ \\ X_- \end{pmatrix} = 0, \tag{2.52}$$

where the radial operators $\widehat{\mathcal{D}}_\pm$ are given by

$$\widehat{\mathcal{D}}_\pm = \frac{d}{dr} \pm i\left(\omega + \frac{\left(k+\frac{1}{2}\right)a}{\widetilde{\Delta}}\right). \tag{2.53}$$

Notice that the system (2.9) governing the angular functions $Y_\pm$ is unaffected by going to flat space. Proceeding as in Section 2.3 it can be verified that the the Hamiltonian H_∞ is given by

$$H_\infty = \frac{\widetilde{\Delta}}{U}\left[\left(\mathbb{1} - \frac{a\sin\theta}{\sqrt{\widetilde{\Delta}}}\begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}\right)\right]\left(\hat{\mathcal{R}}_\infty + \hat{\mathcal{A}}_\infty\right), \tag{2.54}$$

where the radial and angular operators have the following form

$$\hat{\mathcal{R}}_\infty = -\frac{mr}{\sqrt{\widetilde{\Delta}}}\begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix} + \begin{pmatrix} -\widehat{\mathcal{E}}_- & 0 & 0 & 0 \\ 0 & \widehat{\mathcal{E}}_+ & 0 & 0 \\ 0 & 0 & \widehat{\mathcal{E}}_+ & 0 \\ 0 & 0 & 0 & -\widehat{\mathcal{E}}_- \end{pmatrix}$$

$$\hat{\mathcal{A}}_\infty = \frac{am\cos\theta}{\sqrt{\widetilde{\Delta}}}\begin{pmatrix} 0&0&i&0 \\ 0&0&0&i \\ -i&0&0&0 \\ 0&-i&0&0 \end{pmatrix} + \begin{pmatrix} 0 & -\widehat{\mathcal{M}}_+ & 0 & 0 \\ -\widehat{\mathcal{M}}_- & 0 & 0 & 0 \\ 0 & 0 & 0 & \widehat{\mathcal{M}}_+ \\ 0 & 0 & \widehat{\mathcal{M}}_- & 0 \end{pmatrix}$$

with

$$\widehat{\mathcal{E}}_\pm = i\frac{\partial}{\partial r} \mp \frac{ia}{\widetilde{\Delta}}\frac{\partial}{\partial\varphi}, \qquad \widehat{\mathcal{M}}_\pm = \frac{1}{\sqrt{\widetilde{\Delta}}}\left(i\frac{\partial}{\partial\theta} + i\frac{\cot\theta}{2} \pm \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right).$$

According to [31], the Hamiltonian H_∞ is Hermitian with respect to the positive scalar product

$$< \hat{\Psi}|\hat{\Phi} >_\infty = \int_0^\infty dr \int_{-1}^1 d(\cos\theta) \int_0^{2\pi} d\varphi < \hat{\Psi}|\hat{\Phi} >^\infty_{(t,r,\theta,\varphi)}, \tag{2.55}$$

with the inner product

$$< \hat{\Psi}|\hat{\Phi} >^\infty_{(t,u,\theta,\varphi)} = \overline{\hat{\Psi}}(t,u,\theta,\varphi)\left[\mathbb{I} + \frac{a\sin\theta}{\sqrt{\widetilde{\Delta}}}\begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}\right]\hat{\Phi}(t,r,\theta,\varphi). \tag{2.56}$$

Let $\mathcal{H}_\infty$ be the Hilbert space of wave functions endowed with the scalar product (2.55). Then the operator H_∞ is essentially self-adjoint on $\mathcal{H}_\infty$ with domain of definition

$$D(H_\infty) = C_0^\infty\left([0,+\infty) \times S^2\right)^4.$$

Since the solutions of the radial equation must be continuous, we need now to find an appropriate boundary condition for the radial spinors at $r = 0$. Notice that the radial system (2.52) can also be considered for $r < 0$ and its coefficients are continuous at $r = 0$. This makes possible to extend the radial solution continuously through the disk $r = 0$. Taking into account that the radial spinors can be transformed into each other by means of the transformation

$$X(-r) \longrightarrow \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} X(r),$$

the requirement of continuity on the surface $\theta = \frac{\pi}{2}$, $r > 0$ leads us to the following boundary condition

$$X_+(0) = iX_-(0). \tag{2.57}$$

Hence by imposing (2.57) and proceeding similar to [31], we obtain the following formula for the propagator $e^{-iH_\infty t}$

$$\left(e^{-iH_\infty t}\hat{\Psi}\right)(x) = \frac{1}{\pi}\sum_{k,n\in\mathbb{Z}} \int_{\sigma(H_\infty)} d\omega\ e^{-i\omega t} t^\infty_{kn}(\omega)\hat{\Psi}^\infty_{k\omega n}(x)\langle\hat{\Psi}^\infty_{k\omega n}|\hat{\Psi}\rangle_\infty\ , \tag{2.58}$$

where $\sigma(H_\infty) := \mathbb{R}\backslash[-m,m]$ denotes the spectrum of the free Dirac operator in Minkowski metric and the transmission coefficients $t^\infty_{k\omega n}$ are positive smooth functions of the frequency ω.

3 Classical and Dollard-modified wave operators.

Let H_0 be the formal limit of the Hamiltonian operator H for $r \to r_1$ and H_∞ the Hamiltonian operator obtained from H by setting $M = Q = 0$. From Section 2.6 we know that H_∞ corresponds to the Hamiltonian operator for the Dirac equation on the Minkowski space-time in oblate spheroidal coordinates. Let $\hat{\Psi}_0^{(0,\infty)}$ be some initial data. Furthermore, let the Hamiltonian operators H_0 and H_∞ act respectively on the Hilbert spaces defined by $\mathcal{H}_0 = L^2\left((r_1,\infty)\times S^2\right)^4$ and $\mathcal{H}_\infty = L^2\left([0,\infty)\times S^2\right)^4$.
Close to the event horizon we consider the classical wave operators in the future and past infinity

$$W_0^{\pm}\hat{\Psi}_0^{(0)} = s - \lim_{t\to\pm\infty} e^{-iHt}\mathcal{I}_0 e^{iH_0 t}\hat{\Psi}_0^{(0)}, \quad \hat{\Psi}_0^{(0)} \in \mathcal{H}_0 \tag{3.1}$$

in the Hilbert space $\mathcal{H} = L^2\left((r_1,\infty)\times S^2\right)^4$ together with a smooth bounded identifying operator $\mathcal{I}_0$ from $\mathcal{H}_0$ to $\mathcal{H}$ defined as follows

$$\begin{cases} (\mathcal{I}_0\Psi)|_{u>\hat{u}_1} = 0 \\ (\mathcal{I}_0\Psi)|_{u\le\hat{u}_1} = \chi_0\Psi \end{cases}, \quad \Psi \in \mathcal{H}_0. \tag{3.2}$$

Notice that the limit in (3.1) has to be understood in the strong sense. We introduce a cut-off function $\chi_0 \in C^\infty(\mathbb{R})$ such that

$$\chi_0 = \begin{cases} 1 & \text{if } u < \hat{u}_0 \\ 0 & \text{if } u > \hat{u}_1 \\ 0 \le \chi_0 \le 1 & \text{otherwise} \end{cases} \tag{3.3}$$

with $\hat{u}_0$, $\hat{u}_1 \in \mathbb{R}$ and $\hat{u}_0 < \hat{u}_1$, since we want to compare H with H_0 in the neighborhood of the horizon.
Generally, the classical wave operators at infinity are defined by

$$W_\infty^{\pm}\hat{\Psi}_0^{(\infty)} = s - \lim_{t\to\pm\infty} e^{-iHt}\mathcal{I}_\infty e^{iH_\infty t}\hat{\Psi}_0^{(\infty)}, \quad \hat{\Psi}_0^{(\infty)} \in \mathcal{H}_\infty \tag{3.4}$$

in the Hilbert space $\mathcal{H}$, where $\mathcal{I}_\infty$ is a smooth bounded identifying operator from $\mathcal{H}_\infty$ to $\mathcal{H}$ specified by

$$\begin{cases} \left.(\mathcal{I}_\infty\tilde{\Psi})\right|_{u\ge u_0} = \chi_\infty\tilde{\Psi} \\ \left.(\mathcal{I}_\infty\tilde{\Psi})\right|_{u<u_0} = 0 \end{cases}, \quad \tilde{\Psi} \in \mathcal{H}_\infty \tag{3.5}$$

and $\chi_\infty \in C^\infty(\mathbb{R})$ is a radially symmetric cut-off function such that

$$\chi_\infty = \begin{cases} 1 & \text{if } u > u_1 \\ 0 & \text{if } u < u_0 \\ 0 \le \chi_\infty \le 1 & \text{otherwise} \end{cases} \tag{3.6}$$

with u_0, $u_1 \in \mathbb{R}$ and $u_0 < u_1$. A mathematical tool to prove the existence of wave operators such as (3.1) and (3.4) is given by the Cook's method [34]. This method relies on the observation that if $f \in C^1(\mathbb{R})$ and $f' \in L^1(\mathbb{R})$, then the limit of the function $f(t)$ for $t \to \infty$ exists, since it holds

$$|f(t) - f(s)| = \left| \int_s^t d\tau f'(\tau) \right| \leq \int_s^t d\tau |f'(\tau)| \to 0$$

as $s < t$ both go to ∞. Going back to (3.1) and (3.4) and defining

$$\eta_{0,\infty}(t) = e^{-iHt} \mathcal{I}_{0,\infty} e^{iH_{0,\infty} t} \hat{\Psi}_0^{(0,\infty)},$$

we observe that

$$\begin{aligned} \| \eta_{0,\infty}(t) - \eta_{0,\infty}(s) \|_{\mathcal{H}} &\leq \int_s^t d\tau \| \eta_{0,\infty}(\tau) \|_{\mathcal{H}} \\ &= \int_s^t d\tau \| \left(-H\mathcal{I}_{0,\infty} + \mathcal{I}_{0,\infty} H_{0,\infty} \right) e^{iH_{0,\infty}\tau} \hat{\Psi}_0^{(0,\infty)} \|_{\mathcal{H}}, \end{aligned}$$

since e^{-iHt} is unitary. By means of the Cook's method the problem of the existence of the wave operators (3.1) and (3.4) reduces to show that

$$\| \left(H\mathcal{I}_{0,\infty} - \mathcal{I}_{0,\infty} H_{0,\infty} \right) e^{iH_{0,\infty}\tau} \hat{\Psi}_0^{(0,\infty)} \|_{\mathcal{H}} \in L^1.$$

As we will see in the next section, this method is made impossible by the long-range nature of the gravitational and Coulomb force acting on the fermion and the above condition can never be satisfied. This problem appears already in non-relativistic quantum mechanics. For a Coulomb potential, solutions of the Schrödinger equation do not go over asymptotically to solutions of the free Schrödinger equation. Instead, they tend towards free solutions modified by a phase shift logarithmic in time. This problem was first solved by Dollard [11]. Since our problem is similar, we can introduce certain new wave operators which compare the full dynamics with a free dynamics modified by a logarithmic phase shift. Here the modified wave operators are defined by

$$\widetilde{W}_\infty^\pm \hat{\Psi}_0 = s - \lim_{t \to \pm\infty} e^{-iHt} \mathcal{I}_\infty e^{iH_\infty t} e^{i\delta(t)} \hat{\Psi}_0^{(\infty)}, \quad \hat{\Psi}_0^{(\infty)} \in \mathcal{H}_\infty \tag{3.7}$$

with the requirements that the phase shift operator $\delta(t)$ commutes with H_∞, and that the limit exists in the strong sense in $\mathcal{H}$. Jin [15] proved the existence of modified wave operators for massive Dirac fields on the Schwarzschild black hole space-time by means of Cook's method. Finally, notice that if $m = 0$ in the Schwarzschild geometry there is no long-range force, and thus it suffices to show the existence of classical wave operators at infinity. This was done in [14], where asymptotic completeness of the

classical wave operators was also proved.
In our problem we deal with massive fermions in the Kerr-Newman metric, and our main goal is to show that classical wave operators and modified wave operators do exist at the horizon and at infinity, respectively. Asymptotic completeness will also be proved. Instead of applying Cook's method, we shall use as main technical tool the integral representation for the Dirac propagator e^{-iHt} given by Theorem 2.7. The main advantage is that we can compute explicitly the wave operators at the event horizon and at infinity and asymptotic completeness does not need to be proved, since it can be directly read off from the analytical results for the wave operators at the event horizon and at infinity.
In preparation, we now prove some results, which will considerably simplify the computation of the wave operators in Sections 5 and 6.

Lemma 3.1. *For every $\hat{\Psi}_0^{(\infty)}$, $\hat{\Psi}_0^{(0)} \in C_c^\infty$ the following identities hold*

$$s - \lim_{t\to\pm\infty} e^{-iHt}(\mathcal{I}_\infty - \mathcal{I}_\infty^{'})e^{iH_\infty t}\hat{\Psi}_0^{(\infty)} = 0 \tag{3.8}$$

$$s - \lim_{t\to\pm\infty} e^{-iHt}(\mathcal{I}_0 - \mathcal{I}_0^{'})e^{iH_0 t}\hat{\Psi}_0^{(0)} = 0, \tag{3.9}$$

where $\mathcal{I}_{0,\infty}$ are identifying operators already defined at the beginning of the present Section.

Proof. For ease in notation we omit to write explicitly the superscripts 0 and ∞ attached to the initial data $\hat{\Psi}_0^{(0,\infty)}$. Let us first show (3.8). We can always choose $\mathcal{I}_\infty^{'}$ such that $\mathcal{I}_\infty - \mathcal{I}_\infty^{'}$ defines a new identifying operator with cut-off function χ having compact support. We set $K := \text{supp}\chi$. Since e^{-iHt} is a unitary operator, it results that

$$\|e^{-iHt}(\mathcal{I}_\infty - \mathcal{I}_\infty^{'})e^{iH_\infty t}\hat{\Psi}_0\|_{\mathcal{H}} = \|(\mathcal{I}_\infty - \mathcal{I}_\infty^{'})e^{iH_\infty t}\hat{\Psi}_0\|_{\mathcal{H}} \le \int_K du|\Psi(t,u,\theta,\varphi)|^2$$

with

$$\Psi(t,u,\theta,\varphi) = e^{iH_\infty t}\hat{\Psi}_0(u,\theta,\varphi).$$

Since asymptotically at infinity the spinor Ψ goes over into the solution of the massive Dirac equation in Minkowski space, which decays at the rate $t^{-\frac{3}{2}}$ [35], we conclude that $|\Psi|^2$ decay at the rate t^{-3}. Hence, for $t \to \pm\infty$ it follows that

$$\lim_{t\to\pm\infty} \|e^{-iHt}(\mathcal{I}_\infty - \mathcal{I}_\infty^{'})e^{iH_\infty t}\hat{\Psi}_0\|_{\mathcal{H}} = 0.$$

In order to prove (3.9), let us choose $\mathcal{I}_0^{'}$ such that $\mathcal{I}_0 - \mathcal{I}_0^{'}$ defines an identifying operator with cut-off function $\hat{\chi}$ having compact support. We set $\hat{K} = \text{supp}\hat{\chi}$. At this point we can proceed like we did for the case at infinity.

More precisely, since a massive Dirac particle behaves near the event horizon like a solution of the wave equation in Minkowski space, then according to Huygens principle it has rapid decay in time and we can conclude that

$$\lim_{t\to\pm\infty} \|e^{-iHt}(\mathcal{I}_0 - \mathcal{I}_0^{'})e^{iH_0 t}\hat{\Psi}_0\|_{\mathcal{H}} = 0 \ .$$

□

As a consequence of the above Lemma, we get no contribution to the wave operators at infinity and at the event horizon, when we integrate over a compact interval in the spatial variable u. The next Lemma controls the contribution of a small neighborhood of the points $\omega = \pm m$ to the wave operators in the limit $t \to \pm\infty$.

Lemma 3.2. *For every $\kappa > 0$ and $\hat{\Psi}_0^{(\infty)} \in C_c^\infty$ there exist constants μ, ϵ, $T > 0$ and an appropriate identifying operator $\mathcal{I}_\infty$ such that for all t with $|t| > T$,*

$$\|e^{-iHt}\mathcal{I}_\infty e^{iH_\infty t} e^{i\delta(t)}\hat{\Psi}_0^{(\infty)} - e^{-iHt}(\mathbb{1}-E_\epsilon)\mathcal{I}_\infty e^{iH_\infty t}(\mathbb{1}-E_\mu^\infty)e^{i\delta(t)}\hat{\Psi}_0^{(\infty)}\|_{\mathcal{H}} < \kappa.$$

Here the projectors E_ϵ and E_μ^∞ are defined in terms of the spectral measures of the Hamiltonians H and H_∞ as follows

$$E_\epsilon := \int\limits_{B_\epsilon} dE_{\omega'}\,, \qquad E_\mu^\infty := \int\limits_{B_\mu} dE_\omega^\infty$$

with $B_\epsilon := B_\epsilon(-m) \cup B_\epsilon(m)$ and $B_\mu := B_\mu(-m) \cup B_\mu(m)$. Moreover, $e^{i\delta(t)}$ is a phase shift operator which commutes with H_∞.

Proof. Without risk of confusion we can omit to write explicitly the superscript ∞ attached to the initial data. First of all let us define

$$\Delta W := e^{-iHt}\mathcal{I}_\infty e^{iH_\infty t}e^{i\delta(t)}\hat{\Psi}_0 - e^{-iHt}(\mathbb{1} - E_\epsilon)\mathcal{I}_\infty e^{iH_\infty t}(\mathbb{1} - E_\mu^\infty)e^{i\delta(t)}\hat{\Psi}_0.$$

Then, by adding and subtracting the term $e^{-iHt}\mathcal{I}_\infty e^{iH_\infty t}(\mathbb{1} - E_\mu^\infty)e^{i\delta(t)}\hat{\Psi}_0$ to ΔW we get

$$\|\Delta W\|_{\mathcal{H}} \le \|e^{-iHt}\mathcal{I}_\infty e^{iH_\infty t}e^{i\delta(t)}\hat{\Psi}_0 - e^{-iHt}\mathcal{I}_\infty e^{iH_\infty t}(\mathbb{1} - E_\mu^\infty)e^{i\delta(t)}\hat{\Psi}_0\|_{\mathcal{H}}$$
$$+\|e^{-iHt}\mathcal{I}_\infty e^{iH_\infty t}(\mathbb{1}-E_\mu^\infty)e^{i\delta(t)}\hat{\Psi}_0 - e^{-iHt}(\mathbb{1}-E_\epsilon)\mathcal{I}_\infty e^{iH_\infty t}(\mathbb{1}-E_\mu^\infty)e^{i\delta(t)}\hat{\Psi}_0\|_{\mathcal{H}},$$

which simplifies to

$$\|\Delta W\|_{\mathcal{H}} \le \|e^{-iHt}\mathcal{I}_\infty e^{iH_\infty t}E_\mu^\infty e^{i\delta(t)}\hat{\Psi}_0\|_{\mathcal{H}} + \\ + \|e^{-iHt}E_\epsilon\mathcal{I}_\infty e^{iH_\infty t}(\mathbb{1} - E_\mu^\infty)e^{i\delta(t)}\hat{\Psi}_0\|_{\mathcal{H}}. \quad (3.10)$$

Let us analyze the first term in (3.10). Since e^{-iHt} and $e^{iH_\infty t}$ are unitary operators, $\mathcal{I}_\infty$ is a bounded operator with $\|\mathcal{I}_\infty\|_\mathcal{H} \leq 1$ and the phase shift operator $e^{i\delta(t)}$ commutes with E_μ^∞, we have

$$\|e^{-iHt}\mathcal{I}_\infty e^{iH_\infty t} E_\mu^\infty e^{i\delta(t)} \hat{\Psi}_0\|_\mathcal{H} \leq \|E_\mu^\infty \hat{\Psi}_0\|_\mathcal{H}. \tag{3.11}$$

Since the term $E_\mu^\infty \hat{\Psi}_0$ is given by

$$E_\mu^\infty \hat{\Psi}_0 = \int\limits_{\sigma(H_\infty)} \chi_{B_\mu}(\omega) dE_\omega^\infty \hat{\Psi}_0$$

and the Hamiltonian H_∞ has a purely continuous spectrum, we conclude that $\|E_\mu^\infty \hat{\Psi}_0\|_\mathcal{H}$ converges monotonically to zero as $\mu \to 0$. Hence, we can by choosing μ sufficiently small arrange that

$$\|E_\mu^\infty \hat{\Psi}_0\|_\mathcal{H} < \frac{\kappa}{2}. \tag{3.12}$$

It remains to show that

$$\|e^{-iHt} E_\epsilon \mathcal{I}_\infty e^{iH_\infty t}(\mathbb{1} - E_\mu^\infty) e^{i\delta(t)} \hat{\Psi}_0\|_\mathcal{H} < \frac{\kappa}{2} \tag{3.13}$$

for every t such that $|t| > T$. To this purpose, let us decompose the projector E_ϵ as follows

$$E_\epsilon = E_\epsilon^+ + E_\epsilon^- \quad \text{with} \quad E_\epsilon^\pm = \int\limits_{B_\epsilon(\pm m)} dE_{\omega'}.$$

Again, since e^{-iHt} is unitary, the left side of (3.13) can be estimated by

$$\|E_\epsilon \mathcal{I}_\infty e^{iH_\infty t}(\mathbb{1} - E_\mu^\infty) e^{i\delta(t)} \hat{\Psi}_0\|_\mathcal{H} \leq \sum_\pm \|E_\epsilon^\pm \mathcal{I}_\infty e^{iH_\infty t}(\mathbb{1} - E_\mu^\infty) e^{i\delta(t)} \hat{\Psi}_0\|_\mathcal{H}.$$

The next step is to prove that

$$\|E_\epsilon^+ \mathcal{I}_\infty e^{iH_\infty t}(\mathbb{1} - E_\mu^\infty) e^{i\delta(t)} \hat{\Psi}_0\|_\mathcal{H} < \frac{\kappa}{4}. \tag{3.14}$$

First of all, we represent the vector $(\mathbb{1} - E_\mu^\infty)\hat{\Psi}_0$ in the form

$$(\mathbb{1} - E_\mu^\infty)\hat{\Psi}_0 = (H_\infty - m)\Phi_0$$

with

$$\Phi_0 = \int\limits_\Sigma \frac{dE_\omega^\infty}{\omega - m} \hat{\Psi}_0, \qquad \Sigma := \mathbb{R} \backslash \{B_\mu(-m) \cup B_\mu(m)\}. \tag{3.15}$$

Then clearly

$$\|\Phi_0\| \leq \frac{\|\hat{\Psi}_0\|}{\mu}. \tag{3.16}$$

Taking into account that the operator $(H_\infty - m)$ commutes with $e^{iH_\infty t}$, adding and subtracting the term H to H_∞ and making use of the relation

$$\mathcal{I}_\infty(H-m) = [\mathcal{I}_\infty, H-m] + (H-m)\mathcal{I}_\infty,$$

we obtain

$$\|E_\epsilon^+ \mathcal{I}_\infty e^{iH_\infty t}(\mathbb{1} - E_\mu^\infty)e^{i\delta(t)}\hat{\Psi}_0\|_{\mathcal{H}} \leq \|E_\epsilon^+[\mathcal{I}_\infty, H-m]e^{i\delta(t)}e^{iH_\infty t}\Phi_0\|_{\mathcal{H}} + \\ + \|E_\epsilon^+\mathcal{I}_\infty(H_\infty - H)e^{i\delta(t)}e^{iH_\infty t}\Phi_0\|_{\mathcal{H}} + \|E_\epsilon^+(H-m)\mathcal{I}_\infty e^{i\delta(t)}e^{iH_\infty t}\Phi_0\|_{\mathcal{H}}. \tag{3.17}$$

Let us consider the last term in (3.17). Since the operator $(H-m)$ commutes with E_ϵ^+, then with the help of (3.16), we get the following estimates

$$\|E_\epsilon^+(H-m)\mathcal{I}_\infty e^{i\delta(t)}e^{iH_\infty t}\Phi_0\|_{\mathcal{H}} \leq \frac{\epsilon}{\mu}\|\hat{\Psi}_0\|.$$

By choosing ϵ sufficiently small, it can be achieved that

$$\|E_\epsilon^+(H-m)\mathcal{I}_\infty e^{i\delta(t)}e^{iH_\infty t}\Phi_0\|_{\mathcal{H}} < \frac{\kappa}{12}. \tag{3.18}$$

Since the commutator appearing in (3.17) is simply a multiplication operator, we may choose a suitable identifying operator $\mathcal{I}_\infty$ such that

$$\|E_\epsilon^+[\mathcal{I}_\infty, H-m]e^{i\delta(t)}e^{iH_\infty t}\Phi_0\|_{\mathcal{H}} < \frac{\kappa}{12}. \tag{3.19}$$

It remains now to show that

$$\|E_\epsilon^+\mathcal{I}_\infty(H_\infty - H)e^{i\delta(t)}e^{iH_\infty t}\Phi_0\|_{\mathcal{H}} < \frac{\kappa}{12} \tag{3.20}$$

for every t such that $|t| > T$. Since H tends asymptotically to H_∞, we can always find a suitable $\mathcal{I}_\infty$ such that for every $\nu > 0$ it results

$$\|E_\epsilon^+\mathcal{I}_\infty(H_\infty - H)e^{i\delta(t)}e^{iH_\infty t}\Phi_0\|_{\mathcal{H}} \leq \nu\|e^{iH_\infty t}\Phi_0\|_{H^{1,2}}.$$

Garding inequality [36] together with (3.15) yields that

$$\|e^{iH_\infty t}\Phi_0\|_{H^{1,2}} \leq \|\Phi_0\|_{\mathcal{H}} + \|H_\infty\Phi_0\|_{\mathcal{H}} \leq \frac{2+m}{\mu}\|\hat{\Psi}_0\|.$$

Thus, we can choose ν so small that (3.20) holds. Putting together the estimates (3.14), (3.19) and (3.20), we finally obtain (3.14). With the same method it can be shown that

$$\|E_\epsilon^-\mathcal{I}_\infty e^{iH_\infty t}(\mathbb{1} - E_\mu^\infty)e^{i\delta(t)}\hat{\Psi}_0\|_{\mathcal{H}} < \frac{\kappa}{4}. \tag{3.21}$$

Combining together (3.14), (3.21), (3.11) and (3.12), we get that

$$\|\Delta W\|_{\mathcal{H}} < \kappa$$

and this completes the proof. Notice that the method we used to prove this Lemma, applies also to the case of classical wave operators at infinity, i.e. when we consider (3.7) without phase shift operator $e^{i\delta(t)}$. □

Lemma 3.3. *For every $\kappa > 0$ and $\hat{\Psi}_0^{(0)} \in C_c^\infty$ there exist constants μ, ϵ, $T > 0$ and a suitable identifying operator $\mathcal{I}_0$ such that for all t with $|t| > T$,*

$$\|e^{-iHt}\mathcal{I}_0 e^{iH_0 t}\hat{\Psi}_0^{(0)} - e^{-iHt}(\mathbb{1} - E_\epsilon)\mathcal{I}_0 e^{iH_0 t}(\mathbb{1} - E_\mu^0)\hat{\Psi}_0^{(0)}\|_{\mathcal{H}} < \kappa.$$

Here the projectors E_ϵ and E_μ^0 are defined as in Lemma 3.2.

Proof. The above statement can be proved with the same method we used to show Lemma 3.2. □

When the integral representation of the Dirac propagator is applied in order to evaluate the wave operators at infinity and at the event horizon, we first obtain a triple integral over the frequency $\omega^{'}$, the spatial variable u and the frequency ω and then perform the limits $t \to \pm\infty$. The next result allows us to interchange the limits $t \to \pm\infty$ with the integral over the frequency $\omega^{'}$, thus reducing our problem to the computation of an asymptotic double integral.

Lemma 3.4. *For every $\kappa > 0$ and $\hat{\Psi}_0^{(\infty)} \in C_c^\infty$ there exist constants L, $T > 0$ and an appropriate identifying operator $\mathcal{I}_\infty$ such that for all t with $|t| > T$,*

$$\|e^{-iHt}(\mathbb{1} - E_\epsilon)\mathcal{I}_\infty e^{iH_\infty t}(\mathbb{1} - E_\mu^\infty)e^{i\delta(t)}\hat{\Psi}_0^{(\infty)} - \\ - e^{-iHt}E_L\mathcal{I}_\infty e^{iH_\infty t}(\mathbb{1} - E_\mu^\infty)e^{i\delta(t)}\hat{\Psi}_0^{(\infty)}\|_{\mathcal{H}} < \kappa.$$

Here the projectors E_ϵ, E_μ^∞ are defined as in Lemma 3.2 and E_L is given by

$$E_L := \int_{-L}^{-m-\epsilon} dE_{\omega'} + \int_{m+\epsilon}^{L} dE_\omega^{'} + \int_{-m+\epsilon}^{m-\epsilon} dE_{\omega'}.$$

Proof. For ease in notation we omit to write explicitly the superscript ∞ attached to the initial data. Let us define the vector

$$\Delta S := e^{-iHt}(\mathbb{1} - E_\epsilon)\mathcal{I}_\infty e^{iH_\infty t}(\mathbb{1} - E_\mu^\infty)e^{i\delta(t)}\hat{\Psi}_0 - \\ - e^{-iHt}E_L\mathcal{I}_\infty e^{iH_\infty t}(\mathbb{1} - E_\mu^\infty)e^{i\delta(t)}\hat{\Psi}_0.$$

By introducing

$$\psi(t) := e^{-iHt}\mathcal{I}_\infty e^{iH_\infty t}(\mathbb{1} - E_\mu^\infty)e^{i\delta(t)}\hat{\Psi}_0$$

and taking into account that $\mathbb{1} - E_\epsilon - E_L$ commutes with e^{-iHt}, we observe that

$$\Delta S = \int_{\mathbb{R}\setminus[-L,L]} dE_{\omega'} \psi(t) = \int_{\mathbb{R}\setminus[-L,L]} \frac{1}{\omega'} dE_{\omega'} H\psi(t).$$

Thus we conclude that

$$\|\Delta S\|_{\mathcal{H}} \leq \frac{1}{L} \|H\psi(t)\|_{\mathcal{H}}.$$

Applying the relation

$$H\mathcal{I}_\infty = \mathcal{I}_\infty H + [H, \mathcal{I}_\infty],$$

employing the definition of $\psi(t)$ and adding and subtracting H_∞ to H, we find that

$$\|H\psi(t)\|_{\mathcal{H}} \leq \|\mathcal{I}_\infty (H - H_\infty) e^{iH_\infty t} (\mathbb{1} - E_\mu^\infty) e^{i\delta(t)} \hat{\Psi}_0\|_{\mathcal{H}} + \\ + \|\mathcal{I}_\infty H_\infty e^{iH_\infty t} (\mathbb{1} - E_\mu^\infty) e^{i\delta(t)} \hat{\Psi}_0\|_{\mathcal{H}} + \|[H, \mathcal{I}_\infty] e^{iH_\infty t} (\mathbb{1} - E_\mu^\infty) e^{i\delta(t)} \hat{\Psi}_0\|_{\mathcal{H}}.$$

At this point we can estimate the above terms exactly as in Lemma 3.2. Choosing L sufficiently large, we conclude that

$$\|\Delta S\|_{\mathcal{H}} < \kappa.$$

Clearly the above result holds also when the phase shift operator $e^{i\delta(t)}$ is not present in the definition (3.7) of the wave operator, i.e. when we treat classical wave operators at infinity. □

Lemma 3.5. *For every $\kappa > 0$ and $\hat{\Psi}_0^{(0)} \in C_c^\infty$ there exist constants $L, T > 0$ and an appropriate identifying operator $\mathcal{I}_0$ such that for all t with $|t| > T$,*

$$\|e^{-iHt} (\mathbb{1} - E_\epsilon) \mathcal{I}_0 e^{iH_0 t} (\mathbb{1} - E_\mu^0) \hat{\Psi}_0^{(0)} - \\ - e^{-iHt} E_L \mathcal{I}_0 e^{iH_0 t} (\mathbb{1} - E_\mu^0) \hat{\Psi}_0^{(0)}\|_{\mathcal{H}} < \kappa.$$

Here the projectors E_ϵ, E_μ^0, E_L are defined as in Lemma 3.2, 3.3 and 3.4.

Proof. It can be shown with the same method we used to prove Lemma 3.4. □

4 Main results.

We present here the main results obtained in Sections 5, 6 and 7. For ease of notation we omit to write explicitly the sums over the indices $a^{'}, b^{'} = 1, 2$ in the expressions of the wave operators at infinity and at the event horizon.

Theorem 4.1. ***(Wave Operators at Infinity)***
Let $\widetilde{W}_\infty^\pm$ be given as in (3.7) with phase shift operator $e^{i\delta(t)}$ specified by

$$\delta(t) := -\mathrm{sgn}(t)\alpha(\omega)\log(\frac{\kappa}{\omega}|t|) \quad \textit{with} \quad \alpha(\omega) := \epsilon(\omega)\frac{\omega eQ + Mm^2}{\sqrt{\omega^2 - m^2}},$$

where κ is defined according to $\kappa := \epsilon(\omega)\sqrt{\omega^2 + m^2}$,

$$\mathrm{sgn}(t) := \left\{ \begin{array}{ll} +1 & \textit{if } t > 0 \\ -1 & \textit{if } t < 0 \end{array} \right. \quad \textit{and} \quad \epsilon(\omega) := \left\{ \begin{array}{ll} +1 & \textit{if } \omega > m \\ -1 & \textit{if } \omega < -m \end{array} \right.$$

Then for every $\hat{\Psi}_0^{(\infty)} \in C_c^\infty$,

$$\left(\widetilde{W}_\infty^+ \hat{\Psi}_0^{(\infty)}\right)_{(x)} = \sum_{k,n\in\mathbb{Z}} \int_{\sigma(H_\infty)} d\omega\; t^{kn}_{a^{'}b^{'}}(\omega)\hat{\Psi}^{k\omega n}_{a^{'}}(x)\widehat{f}_{kn}(\omega) f^+_\infty \overline{f}^{\,+}_{\infty,b^{'}},$$

$$\left(\widetilde{W}_\infty^- \hat{\Psi}_0^{(\infty)}\right)_{(x)} = \sum_{k,n\in\mathbb{Z}} \int_{\sigma(H_\infty)} d\omega\; t^{kn}_{a^{'}b^{'}}(\omega)\hat{\Psi}^{k\omega n}_{a^{'}}(x)\widehat{f}_{kn}(\omega) f^-_\infty \overline{f}^{\,-}_{\infty,b^{'}},$$

where $\sigma(H_\infty)$ denotes the spectrum of the free Dirac operator in the Minkowski metric. The functions $f^\pm_{\infty,b^{'}}$ and $f^\pm_\infty$ corresponding to the fundamental solutions $\hat{\Psi}^{k\omega n}_{b^{'}}$ and $\hat{\Psi}^\infty_{k\omega n}$, respectively, are the transmission coefficients, $t^{kn}_{a^{'}b^{'}}$ are given by (2.49) and the function $\widehat{f}_{kn}(\omega)$ is defined by

$$\widehat{f}_{kn}(\omega) = 8\pi\; t^\infty_{kn}(\omega) \int_0^\infty du \int_{-1}^1 d(\cos\theta) < X^\infty_{k\omega n}(u) Y^{k\omega n}(\theta) | \hat{\Psi}_0^{k,(\infty)}(u,\theta) >_\infty$$

with the inner product

$$\begin{aligned} &< X^\infty_{k\omega n}(u) Y^{k\omega n}(\theta) | \hat{\Psi}_0^{k,(\infty)}(u,\theta) >_\infty \\ &\quad = \overline{X}^\infty_{k\omega n}(u)\overline{Y}^{k\omega n}(\theta)\left[\mathbb{1} + \frac{a\sin\theta}{\sqrt{u^2+a^2}}\begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}\right]\hat{\Psi}_0^{k,(\infty)}(u,\theta). \end{aligned}$$

Theorem 4.2. ***(Wave Operators at the Event Horizon)***
Let $W_0^\pm$ be as defined in (3.1). Then for every $\hat{\Psi}_0^{(0)} \in C_c^\infty$,

$$\begin{aligned} W_0^+ &= W_{0,I}^+ + W_{0,II}^+, \\ W_0^- &= W_{0,I}^- + W_{0,II}^- \end{aligned}$$

with

$$\left(W^{+}_{0,I}\hat{\Psi}^{(0)}_{0}\right)_{(x)} = \sum_{k,n\in\mathbb{Z}} \int\limits_{\sigma(H_\infty)} d\omega\ t^{kn}_{a'2}(\omega)\ \hat{\Psi}^{k\omega n}_{a'}(x)\widehat{f}^{\,0,kn}_{2}(\omega),$$

$$\left(W^{-}_{0,I}\hat{\Psi}^{(0)}_{0}\right)_{(x)} = \sum_{k,n\in\mathbb{Z}} \int\limits_{\sigma(H_\infty)} d\omega\ t^{kn}_{a'1}(\omega)\ \hat{\Psi}^{k\omega n}_{a'}(x)\widehat{f}^{\,0,kn}_{1}(\omega),$$

$$\left(W^{+}_{0,II}\hat{\Psi}^{(0)}_{0}\right)_{(x)} = \sum_{k,n\in\mathbb{Z}} \int_{-m}^{m} d\omega\ \hat{\Psi}^{k\omega n}_{1}(x)\overline{C}_{-}\widehat{f}^{\,0,kn}_{2}(\omega),$$

$$\left(W^{-}_{0,II}\hat{\Psi}^{(0)}_{0}\right)_{(x)} = \sum_{k,n\in\mathbb{Z}} \int_{-m}^{m} d\omega\ \hat{\Psi}^{k\omega n}_{1}(x)\overline{C}_{+}\widehat{f}^{\,0,kn}_{1}(\omega)$$

where $\sigma(H_\infty)$ *denotes the spectrum of the free Dirac propagator and for* $i = 1,\ 2$ *the functions* $\widehat{f_i}^{\,0,kn}(\omega)$ *are defined as follows*

$$\widehat{f_i}^{\,0,kn}(\omega) = 2\pi \int_{-\infty}^{\infty} du \int_{-1}^{1} d(\cos\theta)\ \overline{X}^{0,k\omega n}_{i}(u)\ \overline{Y}^{k\omega n}(\theta)\hat{\Psi}^{k,(0)}_{0}(u,\theta).$$

Moreover, the $\hat{\Psi}^{k\omega n}_{a'}$*'s are the fundamental solutions, the coefficients* $t^{kn}_{a'1}$ *and* $t^{kn}_{a'2}$ *are as defined in Theorem 2.7 and the terms* $\overline{C}_{\pm}$ *satisfy the relation* (6.32).

Theorem 4.3. ***(Asymptotic Completeness)***
Let the wave operators W^+ *and* W^- *be defined by*

$$W^{\pm}(\hat{\Psi}^{(0)}_{0}, \hat{\Psi}^{(\infty)}_{0}) = W^{\pm}_{0}\hat{\Psi}^{(0)}_{0} + \widetilde{W}^{\pm}_{\infty}\hat{\Psi}^{(\infty)}_{0}$$

with $\hat{\Psi}^{(0)}_{0} \in \mathcal{H}_0$ *and* $\hat{\Psi}^{(\infty)}_{0} \in \mathcal{H}_\infty$*. The range of* $W^{\pm}$ *is dense in the Hilbert space* $\mathcal{H}$*.*

5 Modified wave operators asymptotically at infinity.

5.1 Classical Wave Operators at Infinity $W_\infty^\pm$.

Due to the long-range nature of the gravitational and Coulomb forces, we must modify the free dynamics in the definition of the wave operator asymptotically at infinity in order to compensate a logarithmic phase shift. To this purpose, we use the following strategy. First we attempt to define the wave operator asymptotically at infinity in a classical way. By direct computation, we get the necessary information needed to implement a time-dependent logarithmic phase shift. This will then make it possible to introduce a well-defined wave operator at infinity.

Let us start with the classical definition of the wave operator asymptotically at infinity as given by (3.4). By means of the integral representation for the propagator of the Dirac equation in the Kerr-Newman geometry (see Theorem 2.7) and making use of the formula (2.58), the definition (3.4) can be written as

$$\left(W_\infty^\pm \hat{\Psi}_0^{(\infty)}\right)_{(x)} = \frac{1}{\pi^2} \lim_{t\to\pm\infty} \sum_{k',n',k,n} \int_{\mathbb{R}} d\omega' \int_{\mathbb{R}} du \int_{\sigma(H_\infty)} d\omega \; e^{i(\omega-\omega')t} P_{kn}^{k'n'}(\omega,\omega',u) \quad (5.1)$$

with $k', n', k, n \in \mathbb{Z}$ and

$$P_{kn}^{k'n'}(\omega,\omega',u) = \sum_{a',b'=1}^{2} t_{a'b'}^{k'n'}(\omega') t_{kn}^{\infty}(\omega) \hat{\Psi}_{a'}^{k'\omega'n'}(x) < \hat{\Psi}_{b'}^{k'\omega'n'} | \chi_\infty \hat{\Psi}_{k\omega n}^{\infty} >_{(u)} \langle \hat{\Psi}_{k\omega n}^{\infty} | \hat{\Psi}_0^{(\infty)} \rangle_\infty .$$

There $\hat{\Psi}^\infty$ denotes the eigenfunctions of the Hamiltonian at infinity H_∞ and $\hat{\Psi}_0^{(\infty)}$ is some smooth initial data with compact support defined as follows

$$\hat{\Psi}_0^{(\infty)}(u,\theta,\varphi) = \sum_{\hat{k}\in\mathbb{Z}} \hat{\Psi}_0^{\hat{k},(\infty)}(u,\theta) e^{-i(\hat{k}+\frac{1}{2})\varphi}. \quad (5.2)$$

Notice that $\hat{\Psi}_0^{\hat{k},(\infty)}(u,\theta)$ could also be defined analogously as in (2.6). Since the indexes ω and ω' attached to the spinors and to the transmission coefficients can always be recovered by looking at the component indexes a' and b', we can avoid to write them explicitly. For ease of notation, we will in what follows omit the indexes k, n and n'. This causes no confusion, because in view of (5.5) the index k is the same in all factors, whereas the distinction between n and $n^{'}$ can be made by looking at the corresponding indices ∞ or $a^{'}$, $b^{'}$. Moreover, the expression of the wave operator can be further simplified by omitting the sums over k, n, $n^{'}$ and $a^{'}$, $b^{'}$. Furthermore, we omit the superscript ∞ for the initial data $\hat{\Psi}_0^{(\infty)}$. With this in mind (5.1) becomes

$$\left(W_\infty^{\pm}\hat{\Psi}_0\right)_{(x)} = \frac{1}{\pi^2}\lim_{t\to\pm\infty}\int_{\mathbb{R}} d\omega^{'}\int_{\mathbb{R}} du\int_{\sigma(H_\infty)} d\omega\; e^{i(\omega-\omega^{'})t}\,t_{a^{'}b^{'}}\hat{\Psi}_{a^{'}}(x)$$
$$< \hat{\Psi}_{b^{'}}|\chi_\infty\hat{\Psi}^\infty >_{(u)}\langle\hat{\Psi}^\infty|\hat{\Psi}_0\rangle_\infty.$$

Moreover, since the matrix potential in (2.25) converges for $u\to+\infty$, we can pass to the limit $u\to+\infty$ in (2.25) and solve the correspondent system of ordinary differential equations together with the requirement that at infinity both outgoing plane waves and incoming ones are present. Thus, we have the following analytical expression for the radial 2-spinor $X^\infty(u)$

$$X^\infty(u) = \begin{pmatrix}\cosh\Theta & \sinh\Theta\\ \sinh\Theta & \cosh\Theta\end{pmatrix}\begin{pmatrix}e^{-i\Phi_\infty(u)}f_\infty^+\\ e^{i\Phi_\infty(u)}f_\infty^-\end{pmatrix},\quad \Phi_\infty(u) := \kappa u \tag{5.3}$$

with $\kappa := \epsilon(\omega)\sqrt{\omega^2-m^2}$, Θ as in (2.43) and $\epsilon(\omega)$ as defined in Lemma 2.3. Concerning the angular components of the spinor at infinity, we observe that, as the angular system of ODEs (2.9) does not depend on u, at infinity the angular eigenfunctions $Y(\theta)$ are solutions of the Chandrasekhar-Page equation, whose properties were studied in [32].

Furthermore, notice that for two spinors Ψ and Φ with radial components $X^{k\omega n}$ and $X^{k^{'}\omega^{'}n^{'}}$ and angular components $Y^{k\omega n}$ and $Y^{k^{'}\omega^{'}n^{'}}$, the scalar product $<\cdot|\cdot>$ as given by (2.15) does not split into a product, more precisely it was found in [35] that

$$< \Psi|\Phi >= (X^{k\omega n}|X^{k^{'}\omega^{'}n^{'}})(Y^{k\omega n}|Y^{k^{'}\omega^{'}n^{'}})+$$
$$+\, a(X^{k\omega n}|\frac{\sqrt{\Delta}\sigma_2}{r^2+a^2}|X^{k^{'}\omega^{'}n^{'}})(Y^{k\omega n}|\sigma_1\sin\theta|Y^{k^{'}\omega^{'}n^{'}}) \tag{5.4}$$

where σ_1 is the Pauli matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and

$$(X^{k\omega\lambda}|X^{k^{'}\omega^{'}n^{'}}) = \int_{\mathbb{R}} du\,\overline{X}^{k\omega n}(u)\;X^{k^{'}\omega^{'}n^{'}}(u)$$
$$(Y^{k\omega n}|Y^{k^{'}\omega^{'}n^{'}}) = 2\pi\delta_{kk^{'}}\int_{-1}^{1} d(\cos\theta)\;\overline{Y}^{k\omega n}(\theta)\;Y^{k^{'}\omega^{'}n^{'}}(\theta).$$

This mixing of the radial and angular parts in the scalar product can be understood from the fact that the Kerr-Newman solution is only axisymmetric.

By employing (5.4) and integrating over the azimuthal variable φ, it can be checked that $<\hat{\Psi}_{b'}|\chi_\infty \hat{\Psi}^\infty>_{(u)}$ and $\langle \hat{\Psi}^\infty | \hat{\Psi}_0 \rangle_\infty$ are given respectively by

$$<\hat{\Psi}_{b'}|\chi_\infty \hat{\Psi}^\infty>_{(u)} = 2\pi \delta_{kk'} g_{b'}(\omega, \omega', u) \chi_\infty(u), \tag{5.5}$$

$$\langle \hat{\Psi}^\infty | \hat{\Psi}_0 \rangle_\infty = 2\pi f(\omega) \tag{5.6}$$

with

$$g_{b'}(\omega,\omega',u) := A(\omega,\omega')\overline{X}_{b'}(u) X^\infty(u) + \frac{a\sqrt{\Delta}}{r^2+a^2} B(\omega,\omega')\overline{X}_{b'}(u)\sigma_2 X^\infty(u) \tag{5.7}$$

and

$$f(\omega) := t^\infty(\omega) \int_0^\infty du \int_{-1}^1 d(\cos\theta) < X^\infty(u) Y(\theta) | \hat{\Psi}_0^k(u,\theta) >_\infty \tag{5.8}$$

with inner product defined by (2.56). Here the functions $A(\omega,\omega')$ and $B(\omega,\omega')$ are given by

$$A(\omega,\omega') := \int_{-1}^1 d(\cos\theta) \; \overline{Y}^{k'\omega' n'}(\theta) \; Y^{k\omega n}(\theta), \tag{5.9}$$

$$B(\omega,\omega') := \int_{-1}^1 d(\cos\theta) \;\; \sin\theta \, \overline{Y}^{k'\omega' n'}(\theta) \sigma_1 Y^{k\omega n}(\theta). \tag{5.10}$$

Notice that since the angular functions $Y^{k\omega n}(\theta)$ depend smoothly on ω and ω' (see [32]) and the angular initial data are smooth functions with compact support, it results that (5.8), (5.9) and (5.10) will depend smoothly on ω and ω' too. Furthermore, since the angular components are normalized and square integrable, it can be easily shown by means of Hölder inequality that (5.9) and (5.10) are bounded with

$$|A(\omega,\omega')| \le 1 \quad \text{and} \quad |B(\omega,\omega')| \le 1. \tag{5.11}$$

By means of (5.5) and (5.6) and with our notation the wave operator can be written in the compact form

$$\left(W_\infty^\pm \hat{\Psi}_0\right)_{(x)} = 4 \lim_{t\to\pm\infty} \int_{\mathbb{R}} d\omega' \; t_{a'b'}(\omega') \hat{\Psi}_{a'}(x) \int_{\mathbb{R}} du \; \chi_\infty(u) \int_{\sigma(H_\infty)} d\omega \; e^{i(\omega-\omega')t} \, g_{b'}(\omega,\omega',u) f(\omega).$$

In view of the fact that the regions $\omega \approx \pm m$, $\omega' \approx \pm m$ and $|\omega'| > L$ can be controlled with Lemma 3.1, 3.2 and 3.4, our task is to analyze the expression

$$\int_{\Omega'} d\omega' \; t_{a'b'}(\omega') \hat{\Psi}_{a'}(x) \lim_{t\to\pm\infty} \int_{u_1}^\infty du \int_{\Omega} d\omega \; e^{i(\omega-\omega')t} \, g_{b'}(\omega,\omega',u) f(\omega), \tag{5.12}$$

where

$$\begin{aligned}\Omega^{'} &:= [-L,-m-\epsilon]\cup[-m+\epsilon,m-\epsilon]\cup[m+\epsilon,L],\\ \Omega &:= (-\infty,-m-\delta]\cup[m+\delta,+\infty)\end{aligned}$$

for constants ϵ, δ and $L > 0$. Let us now analyze which kind of integrands enter in (5.12). Looking back at (5.7), we observe that we have to deal with the following expressions

$$\overline{X}_{b'}(u)X^{\infty}(u)f(\omega) \tag{5.13}$$

and

$$\overline{X}_{b'}(u)\sigma_2 X^{\infty}(u)f(\omega). \tag{5.14}$$

Since the terms arising from (5.14) by substitution of (5.3) and (2.42) are of similar structure as those in (5.13), we may restrict attention to (5.13). Moreover, by choosing u_1 large enough, we can replace the radial 2-spinors $X_{b'}(u)$ appearing in (5.13) with the analytical expressions given respectively in Lemma 2.2 and Lemma 2.3, which are smooth for every $\omega^{'} \in \{\omega^{'} \in \mathbb{R} : |\omega^{'}| \geq m+\epsilon \text{ and } |\omega^{'}| \leq m-\epsilon\}$. Notice that the integral over $\omega^{'}$ in (5.12) splits into two integrals over the regions

$$\Omega_1^{'} : = [-L,-m-\epsilon]\cup[m+\epsilon,L] \tag{5.15}$$

$$\Omega_2^{'} : = [-m+\epsilon,m-\epsilon]. \tag{5.16}$$

The next step is to analyze the integrands entering in the double integral over the spatial variable u and the frequency ω given by

$$\int_{u_1}^{\infty} du \int_{\Omega} d\omega\, e^{i(\omega-\omega^{'})t}\, g_{b'}(\omega,\omega^{'},u)f(\omega), \tag{5.17}$$

in the limits $t \to \pm\infty$, when we treat the regions $\Omega_1^{'}$ and $\Omega_2^{'}$, respectively.

5.1.1 The Region $\Omega_1^{'}$.

Let $\omega^{'} \in \Omega_1^{'}$ be fixed. With the help of (2.42) and (5.3), the product $\overline{X}_{b'}(u)X^{\infty}(u)$ becomes

$$\begin{aligned}\overline{X}_{b'}(u)X^{\infty}(u) &= \left(\overline{f}^{+}_{\infty,b'} + \overline{R}^{+}_{\infty,b'}\right)\left(f^{+}_{\infty}\rho(\omega)e^{i(\Phi-\Phi_\infty)} + f^{-}_{\infty}\mu(\omega)e^{i(\Phi+\Phi_\infty)}\right)\\ &+ \left(\overline{f}^{-}_{\infty,b'} + \overline{R}^{-}_{\infty,b'}\right)\left(f^{+}_{\infty}\mu(\omega)e^{-i(\Phi+\Phi_\infty)} + f^{-}_{\infty}\rho(\omega)e^{-i(\Phi-\Phi_\infty)}\right)\end{aligned} \tag{5.18}$$

with

$$\rho(\omega) = \cosh\widetilde{\Theta}\cosh\Theta + \sinh\widetilde{\Theta}\sinh\Theta, \tag{5.19}$$

$$\mu(\omega) = \sinh\widetilde{\Theta}\cosh\Theta + \cosh\widetilde{\Theta}\sinh\Theta, \tag{5.20}$$

where $\widetilde{\Theta}$ and Θ are defined according to

$$\widetilde{\Theta} = \frac{1}{4}\log\left(\frac{\omega^{'}+m}{\omega^{'}-m}\right), \quad \Theta = \frac{1}{4}\log\left(\frac{\omega+m}{\omega-m}\right). \tag{5.21}$$

A careful inspection of the expression for $\overline{X}_{b'}(u)X^{\infty}(u)f(\omega)$ shows that all terms in (5.18) are of the form

$$f_{\infty}^{\pm}(\omega)f(\omega)\{\begin{array}{c}\rho(\omega)\\ \mu(\omega)\end{array}\}\left(\overline{f}_{\infty,b'}^{\pm}+\overline{R}_{\infty,b'}^{\pm}(u)\right)e^{i(\pm\Phi\pm\Phi_{\infty})} \tag{5.22}$$

with $\Phi := \Phi(u)$ as defined in Lemma 2.3. Turning back to (5.7), we notice that (5.22) will be multiplied by the smooth and bounded function $A(\omega)$ defined by (5.9). Moreover, since we can always choose u_1 large enough, according to Lemma 2.3 the following asymptotic expansion for the rest function $R_{\infty,b'}(u)$ holds, namely

$$R_{\infty,b'}(u) = \frac{c}{u} + \mathcal{O}\left(\frac{1}{u^2}\right), \tag{5.23}$$

with $c \in \mathbb{C}$ depending smoothly on $\omega^{'}$, since the asymptotic solution (2.42) is smooth in $\omega^{'}$ such that $|\omega^{'}| > m$. Like in [35] the notation $\mathcal{O}(u^{-n})$ means that error terms depend smoothly on $\omega^{'}$ and that their u-derivatives have the natural scaling behavior

$$\partial_{\omega'}\mathcal{O}\left(u^{-n}\right) = \mathcal{O}\left(u^{-n}\right) \quad \text{and} \quad \partial_u\mathcal{O}\left(u^{-n}\right) = \mathcal{O}\left(u^{-n-1}\right).$$

Hence, all integrands arising in the computation of the wave operator (5.12) are of the type

$$T(u,\omega) := A(\omega)f_{\infty}^{\pm}(\omega)\{\begin{array}{c}\rho(\omega)\\ \mu(\omega)\end{array}\}f(\omega)\left(\overline{f}_{\infty,b'}^{\pm}+\frac{\overline{c}}{u}+\mathcal{O}\left(\frac{1}{u^2}\right)\right)e^{i(\pm\Phi\pm\Phi_{\infty})}. \tag{5.24}$$

Furthermore, since we considered smooth initial data with compact support, the function $f(\omega)$ is smooth and has rapid decay in ω. Thus, it is reasonable to replace the term $A(\omega)f_{\infty}^{\pm}(\omega)\{\begin{array}{c}\rho(\omega)\\ \mu(\omega)\end{array}\}f(\omega)$ appearing in (5.22) with some function $F(\omega)$ in C^1 such that $F(\omega)$ and its derivative decay at least quadratically as $\omega \to \infty$. Our task is now to compute the following limit integral

$$\lim_{t\to\pm\infty}\int_{u_1}^{\infty} du \int_{\Omega} d\omega \;\; e^{i(\omega-\omega^{'})t}\, T(u,\omega)$$

with $T(u,\omega)$ given by

$$T(u,\omega) := F(\omega)\left(\overline{f}_{\infty,b'}^{\pm}+\frac{\overline{c}}{u}+\mathcal{O}\left(\frac{1}{u^2}\right)\right)e^{i(\pm\Phi\pm\Phi_{\infty})}.$$

By changing the integration variable ω according to $\kappa = \epsilon(\omega)\sqrt{\omega^2 - m^2}$, we end up with the evaluation of the following limit integrals

$$I_1 = \lim_{t \to \pm\infty} \int_{u_1}^{\infty} du \;\; u^{\pm i\alpha'} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa \;\; F(\kappa) e^{i\varphi_{(t)}(\kappa,u)}, \tag{5.25}$$

$$I_2 = \lim_{t \to \pm\infty} \int_{u_1}^{\infty} du \;\; u^{\pm i\alpha' - 1} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa \;\; F(\kappa) e^{i\varphi_{(t)}(\kappa,u)}, \tag{5.26}$$

$$I_3 = \lim_{t \to \pm\infty} \int_{u_1}^{\infty} du \;\; u^{\pm i\alpha'} h(u) \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa \;\; F(\kappa) e^{i\varphi_{(t)}(\kappa,u)}, \quad h(u) := \mathcal{O}\left(\frac{1}{u^2}\right), \tag{5.27}$$

where

$$\varphi_{(t)}(\kappa, u) := (\omega - \omega')t + (\pm\kappa' \pm \kappa)u \tag{5.28}$$

$$F(\kappa) := \frac{k}{\omega(\kappa)} F(\omega(\kappa)), \tag{5.29}$$

$$\alpha' := \epsilon(\omega') \frac{\omega' eQ + Mm^2}{\sqrt{\omega'^2 - m^2}}. \tag{5.30}$$

5.1.2 The Region Ω_2'.

Let be now $\omega' \in \Omega_2'$ be fixed. Then the analytical expression for the 2-spinor $X_{b'}(u)$ asymptotically at infinity is given by Lemma 2.2. Moreover, from Theorem 2.7 we know that in this region the transmission coefficients $t_{a'b'}$ have the simple form

$$t_{a'b'} = \delta_{a'1}\delta_{b'1}.$$

As a consequence it can be easily checked that (5.13) and (5.14) simplify to

$$\overline{X}_1(u) X^\infty(u) f(\omega) \tag{5.31}$$

and

$$\overline{X}_1(u) \sigma_2 X^\infty(u) f(\omega). \tag{5.32}$$

Since $\Psi_1^{k\omega' n'}$ for $|\omega'| < m$ is the fundamental solution, that at infinity has exponential decay, it follows from Lemma 2.2, that $X_1(u)$ is given by

$$X_1(u) = \hat{A} \begin{pmatrix} e^{-\tilde{\Phi}(u)} \left(f^+_{\infty,1} + R^+_{\infty,1}(u) \right) \\ 0 \end{pmatrix}.$$

Analogously to the case $\omega' \in \Omega_1'$ we begin by treating the term (5.31). After some straightforward calculations it can be shown that all terms arising from (5.31) are of the form

$$f^\pm_\infty(\omega) f(\omega) \left\{ \begin{array}{c} \tilde{\rho} \\ (i\tilde{\rho})^* \end{array} \right\} \left(\overline{f}^+_{\infty,1} + \overline{R}^+_{\infty,1} \right) e^{-\tilde{\Phi} \pm i\Phi_\infty}, \tag{5.33}$$

where $*$ denotes complex conjugation, the function $\widetilde{\rho} := \widetilde{\rho}(\omega)$ is simply a linear combination in terms of $\cosh\Theta$ and $\sinh\Theta$ and $f_a(\omega)$ is as defined in (5.8). Going back to (5.7), we notice that (5.33) has again to be multiplied by the smooth and bounded function $A(\omega)$ given by (5.9). Furthermore, since we may choose u_1 sufficiently large, it is reasonable to consider the following asymptotic expansion for the rest function $R^{+}_{\infty,1}(u)$, namely

$$R^{+}_{\infty,1}(u) = \frac{\hat{c}}{u} + \mathcal{O}\left(\frac{1}{u^2}\right), \tag{5.34}$$

with $\hat{c} \in \mathbb{C}$ depending smoothly on $\omega^{'}$, since the asymptotic solution given by (2.26) depends smoothly on $\omega^{'}$ such that $|\omega^{'}| < m$. Hence, all integrands arising in the computation of the wave operator (5.12) are of the type

$$S(u,\omega) := A(\omega) f^{\pm}_{\infty}(\omega) \left\{ \begin{array}{c} \widetilde{\rho} \\ (i\widetilde{\rho})^{*} \end{array} \right\} f(\omega) \left(\overline{f}^{+}_{\infty,1} + \frac{\overline{\hat{c}}}{u} + \mathcal{O}\left(\frac{1}{u^2}\right) \right) e^{-\widetilde{\Phi} \pm i\Phi_{\infty}}. \tag{5.35}$$

Since the function $f(\omega)$ is smooth and has rapid decay in ω, we may replace the term $A(\omega) f^{\pm}_{\infty}(\omega) \{ \begin{array}{c} \widetilde{\rho} \\ (i\widetilde{\rho})^{*} \end{array} \} f(\omega)$ appearing in (5.22) with some function $\hat{F}(\omega)$ in C^1 such that $\hat{F}(\omega)$ and its derivative decay at least quadratically as $\omega \to \infty$. Our task is now to compute the following limit integral

$$\lim_{t\to\pm\infty} \int_{u_1}^{\infty} du \int_{\Omega} d\omega \;\; e^{i(\omega-\omega^{'})t}\, S(u,\omega)$$

with $S(u,\omega)$ given by

$$S(u,\omega) := \hat{F}(\omega) \left(\overline{f}^{+}_{\infty,1} + \frac{\overline{\hat{c}}}{u} + \mathcal{O}\left(\frac{1}{u^2}\right) \right) e^{-\widetilde{\Phi} \pm i\Phi_{\infty}}.$$

By changing the integration variable ω according to $\kappa = \epsilon(\omega)\sqrt{\omega^2 - m^2}$, we end up with the evaluation of the following limit integrals

$$\hat{I}_1 = \lim_{t\to\pm\infty} \int_{u_1}^{\infty} du \;\; e^{-\beta u} u^{\widetilde{\alpha}} \int_{\mathbb{R}\setminus B_\delta(0)} d\kappa \;\; \hat{F}(\kappa) e^{i\widetilde{\varphi}_{(t)}(\kappa,u)}, \tag{5.36}$$

$$\hat{I}_2 = \lim_{t\to\pm\infty} \int_{u_1}^{\infty} du \;\; e^{-\beta u} u^{\widetilde{\alpha}-1} \int_{\mathbb{R}\setminus B_\delta(0)} d\kappa \;\; \hat{F}(\kappa) e^{i\widetilde{\varphi}_{(t)}(\kappa,u)}, \tag{5.37}$$

$$\hat{I}_3 = \lim_{t\to\pm\infty} \int_{u_1}^{\infty} du \; e^{-\beta u} u^{\widetilde{\alpha}}\, \hat{h}(u) \int_{\mathbb{R}\setminus B_\delta(0)} d\kappa \; \hat{F}(\kappa) e^{i\widetilde{\varphi}_{(t)}(\kappa,u)},\; \hat{h}(u) := \mathcal{O}\left(\frac{1}{u^2}\right), \tag{5.38}$$

where $\widetilde{\alpha}$ as defined in Lemma 2.2, $\beta := \sqrt{m^2 - \omega'^2}$ and

$$\widetilde{\varphi}_{(t)}(\kappa, u) := (\omega - \omega')t \pm i\kappa u, \tag{5.39}$$

$$\hat{F}(\kappa) := \frac{k}{\omega(\kappa)}\hat{F}(\omega(\kappa)). \tag{5.40}$$

5.2 Theorems for the Evaluation of $W_\infty^\pm$.

The next two lemmas allow us to evaluate (5.25) over the interval $[u_1, +\infty)$.

Lemma 5.1. *For every function $f \in C^1(\mathbb{R})$ satisfying the bound*

$$|f|, |f'| \leq \frac{1}{1+\kappa^2}, \tag{5.41}$$

and for every given $\omega' \in \Omega_1'$ defined by (5.153), it results

$$\mathfrak{T}_1 := \lim_{t\to\pm\infty} \int_{u_1}^{\infty} du\, u^{\pm i\alpha'} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i\varphi_1(\kappa,u)} = 0 \quad \text{if} \quad f(\kappa') = 0 \tag{5.42}$$

$$\mathfrak{T}_2 := \lim_{t\to\pm\infty} \int_{u_1}^{\infty} du\, u^{\pm i\alpha'} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i\varphi_2(\kappa,u)} = 0 \quad \text{if} \quad f(-\kappa') = 0 \tag{5.43}$$

with fixed $\alpha' \in \mathbb{R}$ and $u_1 > 0$, where

$$\varphi_1(\kappa, u) = (\omega - \omega')t \pm (\kappa' - \kappa)u, \quad \text{and} \quad \varphi_2(\kappa, u) = (\omega - \omega')t \pm (\kappa' + \kappa)u.$$

Proof. We show (5.42) with $u^{i\alpha'}$, since the case with $u^{-i\alpha'}$ and (5.43) can be proved analogously. After introducing a convergence generating factor $e^{-\sigma u}$ with $\sigma > 0$, we can apply Fubini theorem to obtain

$$\lim_{t\to\pm\infty} \lim_{\sigma\to 0^+} \int_{-\infty}^{\infty} d\kappa\, f(\kappa)\, e^{i(\omega-\omega')t} \int_{u_1}^{\infty} du\, u^{i\alpha'} e^{-\tilde{z}\frac{u}{u_1}} \tag{5.44}$$

with

$$\tilde{z} := \tilde{z}(\kappa) = u_1(\sigma - iK) \quad \text{and} \quad K := \pm(\kappa' - \kappa). \tag{5.45}$$

We denote the integral over u in (5.44) by $\mathfrak{S}$. Then, introducing the new integration variable $\tau := u/u_1$, we find

$$\mathfrak{S} = u_1^{1+i\alpha'} \int_1^{\infty} d\tau\, \tau^{i\alpha'} e^{-\tilde{z}\tau}.$$

The above integral can be related to the Kummer function of the second kind $U(a, b; \tilde{z})$, also known in the literature on special functions under the name of confluent hypergeometric function of the second kind. Indeed, if the conditions $\Re a > 0$ and $\Re \tilde{z} > 0$ are satisfied, the following integral representation (see 13.2.6 in [37]) holds, namely

$$\Gamma(a) U(a, b; \tilde{z}) = e^{\tilde{z}} \int_1^\infty d\tau \; e^{-\tilde{z}\tau} (\tau - 1)^{a-1} \tau^{b-a-1}.$$

Thus, by choosing $a = 1$ and $b = 2 + i\alpha'$, we obtain

$$\mathfrak{S} = u_1^{1+i\alpha'} e^{-\tilde{z}} U(1, 2 + i\alpha'; \tilde{z}).$$

Moreover, $U(a, b; \tilde{z})$ can be expressed in terms of the Kummer function of the second kind $M(a, b; \tilde{z})$ as follows (13.1.3 in [37])

$$U(a, b; \tilde{z}) = \frac{\pi}{\sin(\pi b)} \left[\frac{M(a, b; \tilde{z})}{\Gamma(1 + a - b)\Gamma(b)} - \tilde{z}^{1-b} \frac{M(1 + a - b, 2 - b; \tilde{z})}{\Gamma(a)\Gamma(2 - b)} \right]. \tag{5.46}$$

Taking into account that for $b = 2 + i\alpha'$

$$\frac{\pi}{\sin(\pi b)} = -\pi i \operatorname{csch}(\pi\alpha'),$$

with the help of (5.46) the integral $\mathfrak{S}$ can be equivalently written as

$$\mathfrak{S} = -\pi i u_1^{1+i\alpha'} \operatorname{csch}(\pi\alpha') e^{-\tilde{z}} \left[\frac{M(1, 2 + i\alpha'; \tilde{z})}{\Gamma(-i\alpha')\Gamma(2 + i\alpha')} - \tilde{z}^{-1-i\alpha'} \frac{M(-i\alpha', -i\alpha'; \tilde{z})}{\Gamma(-i\alpha')} \right].$$

Now, using that $M(a, a; \tilde{z}) = e^{-\tilde{z}}$ (13.6.12 in [37]) and the following properties of the complex Gamma function

$$\Gamma(1 + \beta) = \beta\Gamma(\beta), \quad \Gamma(\beta)\Gamma(1 - \beta) = \frac{\pi}{\sin(\pi\beta)}, \tag{5.47}$$

which imply that

$$\Gamma(-i\alpha')\Gamma(2 + i\alpha') = \pi i (1 + i\alpha') \operatorname{csch}(\pi\alpha'), \tag{5.48}$$

$\mathfrak{S}$ becomes

$$\mathfrak{S} = u_1^{1+i\alpha'} \left[\pi i \frac{\operatorname{csch}(\pi\alpha')}{\Gamma(-i\alpha')} \tilde{z}^{-1-i\alpha'} - \frac{e^{-\tilde{z}}}{1 + i\alpha'} M(1, 2 + i\alpha'; \tilde{z}) \right]. \tag{5.49}$$

From the second equation in (5.47) with $\beta = -i\alpha'$ it results

$$\Gamma(-i\alpha')\Gamma(1 + i\alpha') = \pi i \operatorname{csch}(\pi\alpha'), \tag{5.50}$$

and therefore it holds for every real α'

$$\pi i \frac{\operatorname{csch}(\pi\alpha')}{\Gamma(-i\alpha')} = \Gamma(1 + i\alpha'). \tag{5.51}$$

Substitution of (5.51) into (5.49) finally gives

$$\mathfrak{S} = u_1^{1+i\alpha'} \left[\Gamma(1+i\alpha')\tilde{z}^{-1-i\alpha'} - \frac{e^{-\tilde{z}}}{1+i\alpha'} M(1, 2+i\alpha'; \tilde{z}) \right]. \tag{5.52}$$

By means of (5.52) the integral (5.44) splits into the following two integrals

$$u_1^{1+i\alpha'}\Gamma(1+i\alpha') \lim_{t\to\pm\infty} \lim_{\sigma\to 0^+} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa \, \frac{f(\kappa)}{\tilde{z}^{1+i\alpha'}(\kappa)} \, e^{i(\omega-\omega')t} +$$
$$- \frac{u_1^{1+i\alpha'}}{1+i\alpha'} \lim_{t\to\pm\infty} \lim_{\sigma\to 0^+} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa \, f(\kappa) \, e^{-\tilde{z}(\kappa)} M(1, 2+i\alpha'; \tilde{z}) \, e^{i(\omega-\omega')t}, \tag{5.53}$$

which will be treated separately. We begin with the second integral on the r.h.s. of (5.53). The limit for $\sigma \to 0^+$ can be taken inside the integral sign by applying dominated convergence theorem. To this purpose we have to show that there exists an integrable function $\mathfrak{H}(\kappa)$ on the real line, such that for every σ it results

$$|G_\sigma(\kappa)| \leq \mathfrak{H}(\kappa)$$

with

$$G_\sigma(\kappa) := f(\kappa) e^{-\tilde{z}(\kappa)} M(1, 2+i\alpha'; \tilde{z}) e^{i(\omega-\omega')t}. \tag{5.54}$$

From (5.54) it follows that

$$|G_\sigma(\kappa)| = e^{-u_1\sigma} \, |f(\kappa)| \left| M(1, 2+i\alpha'; \tilde{z}) \right| \tag{5.55}$$

In order to estimate $\left| M(1, 2+i\alpha'; \tilde{z}) \right|$ we can use the following integral representation for the Kummer function of the first kind (13.2.1 [37]), namely

$$M(a, b; \tilde{z}) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 d\tau \, e^{\tilde{z}\tau} (1-\tau)^{b-a-1} \tau^{a-1} \quad \text{for} \quad \Re b > \Re a > 0. \tag{5.56}$$

Considering that for $a = 1$ and $b = 2 + i\alpha'$ the prefactor on the r.h.s. of (5.56) is simply $1 + i\alpha'$, it can be shown that

$$\left| M(1, 2+i\alpha'; \tilde{z}) \right| \leq \sqrt{1+\alpha'^2} \int_0^1 d\tau \, e^{u_1\sigma\tau}. \tag{5.57}$$

Insertion of (5.57) into (5.55) gives

$$|G_\sigma(\kappa)| \leq \sqrt{1+\alpha'^2} \, \frac{(1-e^{-u_1\sigma})}{u_1\sigma}.$$

Moreover, since $u_1\sigma > 0$, there exists a constant $C \in \mathbb{R}^+$ such that

$$\frac{1-e^{-u_1\sigma}}{u_1\sigma} \leq C.$$

Finally, we can estimate (5.54) as follows

$$|G_\sigma(\kappa)| \leq C\sqrt{1+\alpha^{'2}}\ |f(\kappa)| =: \mathfrak{H}(\kappa).$$

Hence, dominated convergence theorem applies and we can take the limit for $\sigma \to 0^+$ inside the integral sign in the second term on the r.h.s. of (5.53), which simplifies to

$$\lim_{t\to\pm\infty} \int\limits_{\mathbb{R}\backslash B_\delta(0)} d\kappa\ \hat{f}(\kappa) M(1, 2+i\alpha^{'}; \mp iu_1(\kappa^{'}-\kappa))\ e^{i(\omega-\omega^{'})t}, \tag{5.58}$$

where we set

$$\hat{f}(\kappa) := f(\kappa)\ e^{\pm i(\kappa^{'}-\kappa)u_1}.$$

Since $\hat{f}(\kappa)$ satisfies (5.41) and the Kummer function of the first kind in (5.58) is bounded, after a change of the variable of integration from κ into ω according to the relation $d\kappa = \frac{\omega}{\kappa} d\omega$ Riemann-Lebesgue lemma tells us that in the limit as $t \to \pm\infty$ (5.58) gives no contribution. Therefore, (5.53) reduces to

$$\Gamma(1+i\alpha^{'}) \lim_{t\to\pm\infty} \lim_{\sigma\to 0^+} \int\limits_{\mathbb{R}\backslash B_\delta(0)} d\kappa\ \mathfrak{F}_\sigma(\kappa)\ e^{i(\omega-\omega^{'})t} \tag{5.59}$$

where the function $\mathfrak{F}_\sigma(\kappa)$ is defined by

$$\mathfrak{F}_\sigma(\kappa) := \hat{g}(\kappa) \left(\frac{\kappa^{'}-\kappa}{\sigma \mp i(\kappa^{'}-\kappa)}\right) [\sigma \mp i(\kappa^{'}-\kappa)]^{-i\alpha^{'}}, \quad \hat{g}(\kappa) := \frac{f(\kappa)}{\kappa^{'}-\kappa}.$$

Since it holds

$$|\mathfrak{F}_\sigma(\kappa)| = |\hat{g}(\kappa)| \frac{|\kappa^{'}-\kappa|}{\sqrt{\sigma^2+(\kappa^{'}-\kappa)^2}} \leq |\hat{g}(\kappa)|$$

with $\hat{g} \in C(\mathbb{R})$ and decaying at least cubically for $|\kappa| \to \infty$, dominated convergence theorem allows us to take the limit for $\sigma \to 0^+$ inside the integral sign in (5.59), which reduces to

$$\lim_{t\to\pm\infty} \int\limits_{\mathbb{R}\backslash B_\delta(0)} d\kappa\ \hat{g}(\kappa)(\kappa^{'}-\kappa)^{-i\alpha^{'}} e^{i(\omega-\omega^{'})t}. \tag{5.60}$$

Again we can change the variable of integration from κ into ω, according to the relation $d\kappa = \frac{\omega}{\kappa} d\omega$ and use Riemann-Lebesgue lemma in order to show that the above integral limit is zero. □

Lemma 5.2. *Let $f \in C^1(\mathbb{R})$ satisfy* (5.41) *and let us consider*

$$A^{(I)} := \int_{u_1}^{\infty} du\ u^{i\alpha'} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i(\omega-\omega')t+i(\kappa'-\kappa)u} \tag{5.61}$$

$$A^{(II)} := \int_{u_1}^{\infty} du\ u^{-i\alpha'} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i(\omega-\omega')t-i(\kappa'-\kappa)u} \tag{5.62}$$

$$A^{(III)} := \int_{u_1}^{\infty} du\ u^{-i\alpha'} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i(\omega-\omega')t-i(\kappa'+\kappa)u} \tag{5.63}$$

$$A^{(IV)} := \int_{u_1}^{\infty} du\ u^{i\alpha'} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i(\omega-\omega')t+i(\kappa'+\kappa)u}. \tag{5.64}$$

Then, for every given $\omega' \in \Omega_1'$, $u_1 > 0$ and $\alpha' \in \mathbb{R}$ defined like in (5.30) *it holds*

$$A^{(I)} = \begin{cases} 2\pi f(\kappa') e^{i\alpha' \log\left(\frac{\kappa'}{\omega'}t\right)} & \text{if } t \to +\infty \\ 0 & \text{if } t \to -\infty \end{cases}, \tag{5.65}$$

$$A^{(II)} = \begin{cases} 0 & \text{if } t \to +\infty \\ 2\pi f(\kappa') e^{-i\alpha' \log\left(-\frac{\kappa'}{\omega'}t\right)} & \text{if } t \to -\infty \end{cases}, \tag{5.66}$$

$$A^{(III)} = \begin{cases} 2\pi f(-\kappa') e^{-i\alpha' \log\left(\frac{\kappa'}{\omega'}t\right)} & \text{if } t \to +\infty \\ 0 & \text{if } t \to -\infty \end{cases}, \tag{5.67}$$

$$A^{(IV)} = \begin{cases} 0 & \text{if } t \to +\infty \\ 2\pi f(-\kappa') e^{i\alpha' \log\left(-\frac{\kappa'}{\omega'}t\right)} & \text{if } t \to -\infty \end{cases}. \tag{5.68}$$

Proof. We begin with $A^{(I)}$ and the case where $\omega' \in [m+\epsilon, L]$. Moreover, we rewrite the function $f(\kappa)$ as follows

$$f(\kappa) = \tilde{F}(\kappa) + f(|\kappa'|)\frac{\omega'^2+1}{\omega^2+1},$$

with

$$\tilde{F}(\kappa) := f(\kappa) - f(|\kappa'|)\frac{\omega'^2+1}{\omega^2+1}.$$

and we define according to (5.30)

$$\alpha_+' := \frac{\omega' eQ + Mm^2}{\sqrt{\omega'^2 - m^2}}.$$

As a consequence

$$A_{>}^{(I)} := \int_{u_1}^{\infty} du\ u^{i\alpha'_+} \int\limits_{\mathbb{R}\backslash B_\delta(0)} d\kappa f(\kappa)\ e^{i(\omega-|\omega'|)t+i(|\kappa'|-\kappa)u}$$

splits into two terms. Since $\tilde{F}(\kappa)$ satisfies the requirements of Lemma 5.1, it follows immediately that in the limit $t \to \pm\infty$

$$\int_{u_1}^{\infty} du\ u^{i\alpha'_+} \int\limits_{\mathbb{R}\backslash B_\delta(0)} d\kappa\ \tilde{F}(\kappa)\ e^{i(\omega-|\omega'|)t+i(|\kappa'|-\kappa)u} = 0.$$

Hence, $A_{>}^{(I)}$ reduces to

$$A_{>}^{(I)} = f(|\kappa'|)(\omega'^2+1) \int_{u_1}^{\infty} du\ u^{i\alpha'_+} \int\limits_{\mathbb{R}\backslash B_\delta(0)} d\kappa\ \frac{e^{i(\omega-|\omega'|)t+i(|\kappa'|-\kappa)u}}{\omega^2+1}.$$

Introducing a convergence generating factor $e^{-\sigma u}$ with $\sigma > 0$ in the above expression, we apply Fubini theorem to interchange the order of integration. Thus, we get

$$A_{>}^{(I)} = f(|\kappa'|)(\omega'^2+1) \lim_{\sigma\to 0^+} \int\limits_{\mathbb{R}\backslash B_\delta(0)} d\kappa \frac{e^{i(\omega-|\omega'|)t}}{\omega^2+1} \int_{u_1}^{\infty} du\ u^{i\alpha'_+} e^{i(|\kappa'|-\kappa)u-\sigma u}.$$

According to (5.52) the integral over u in the above expression gives

$$\int_{u_1}^{\infty} du\ u^{i\alpha'_+} e^{i(|\kappa'|-\kappa)u-\sigma u} = u_1^{1+i\alpha'_+} \Big[\Gamma(1+i\alpha'_+)\tilde{z}^{-1-i\alpha'_+} + \\ - \frac{e^{-\tilde{z}}}{1+i\alpha'_+} M(1, 2+i\alpha'_+; \tilde{z})\Big]$$

with $\tilde{z}(\kappa)$ defined as in Lemma 5.1. Thus, $A_{>}^{(I)}$ becomes

$$A_{>}^{(I)} = \frac{\Gamma(1+i\alpha'_+)}{i^{1+i\alpha'_+}} f(|\kappa'|)(\omega'^2+1) \lim_{\sigma\to 0^+} \int\limits_{\mathbb{R}\backslash B_\delta(0)} d\kappa \frac{e^{i(\omega-|\omega'|)t}}{(\omega^2+1)(\kappa-\kappa_p)^{1+i\alpha'_+}} + \\ - \frac{u_1^{1+i\alpha'_+}}{1+i\alpha'_+} f(|\kappa'|)(\omega'^2+1) \lim_{\sigma\to 0^+} \int\limits_{\mathbb{R}\backslash B_\delta(0)} d\kappa\ \frac{e^{i(\omega-|\omega'|)t-\tilde{z}(\kappa)}}{\kappa^2+m^2+1} M(1, 2+i\alpha'_+; \tilde{z}) \tag{5.69}$$

with $\kappa_p := |\kappa'| + i\sigma$. Since the second term on the r.h.s. of (5.69) can be evaluated by means of the same method we used to compute the second limit integral in (5.53), where now $f(\kappa) = (\kappa^2 + m^2 + 1)^{-1}$, we can conclude immediately that for $t \to \pm\infty$

$$\lim_{\sigma \to 0^+} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa \, \frac{e^{i(\omega - |\omega'|)t - \tilde{z}(\kappa)}}{\kappa^2 + m^2 + 1} M(1, 2 + i\alpha'_+; \tilde{z}) = 0.$$

Therefore, (5.69) reduces to

$$A^{(I)}_> = \frac{\Gamma(1 + i\alpha'_+)}{i^{1+i\alpha'_+}} f(|\kappa'|)(\omega'^2 + 1) \lim_{\sigma \to 0^+} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa \, \frac{e^{i(\omega - |\omega'|)t}}{(\omega^2 + 1)(\kappa - \kappa_p)^{1+i\alpha'_+}}. \tag{5.70}$$

Let S denote the integral over κ in the above expression. With the help of the dispersion relation $\kappa^2 = \omega^2 - m^2$ we can change the variable of integration from κ to ω and we obtain

$$S = \int\limits_{\widehat{\Omega}} d\omega \frac{\omega}{\epsilon(\omega)\sqrt{\omega^2 - m^2}} \frac{e^{i(\omega - |\omega'|)t}}{(\omega^2 + 1)(\epsilon(\omega)\sqrt{\omega^2 - m^2} - \kappa_p)^{1+i\alpha'_+}} \tag{5.71}$$

with $\widehat{\Omega} := (-\infty, -m - \delta] \cup [m + \delta, +\infty)$ and $\epsilon(\omega)$ a sign function as defined in Lemma 2.3. Since the integrand function in (5.71) is integrable both at infinity and at $\pm m$, the above integral exists in $(-\infty, -m] \cup [m, +\infty)$. We consider first the case $t \to +\infty$ and we compute (5.71) by means of the method of contour integrals. Since for $t \to +\infty$ the imaginary part of e^{izt} decays exponentially in the complex upper half-plane, we may close the contour there. Furthermore, we extend on the complex plane the integrand in (5.71) as follows

$$F(z) := \frac{z}{\sqrt{z^2 - m^2}} \frac{(\sqrt{z^2 - m^2} + \kappa_p)^{1+i\alpha'_+}}{(z^2 + 1)\,[(z - \omega_1)(z - \omega_2)]^{1+i\alpha'_+}} e^{i(z - |\omega'|)t}, \tag{5.72}$$

where $F(z)$ has two branch points at $\pm m$ and ω_1, ω_2 denote the complex roots of the equation $z^2 - m^2 - \kappa_p^2 = 0$. In order to make (5.72) single-valued on the interval of integration, we choose $\theta_1 \in [0, 2\pi)$ and $\theta_2 \in [-\pi, \pi)$ with $\theta_1 := \mathrm{Arg}(z - m)$ and $\theta_2 := \mathrm{Arg}(z + m)$ and we prescribe that $\sqrt{z^2 - m^2} = \sqrt{\omega^2 - m^2}$ for $\theta_1 = 0$ and $\sqrt{z^2 - m^2} = -\sqrt{\omega^2 - m^2}$ for $\theta_2 = \pi$. Moreover, we observe that $F(z)$ has two simple poles at $\pm i$. To apply the Residue Theorem, we have to choose a contour $\mathcal{C}$ in such a way that it circumvents the point ω_1 on the upper half-plane and $F(z)$ is analytic within and on $\mathcal{C}$ except for the simple pole at $z = i$. This can be done by fixing the contour $\mathcal{C}$ as in Figure 1. Thus, we get

$$\int\limits_{\mathcal{C}} dz \, F(z) = 2\pi i \, \mathrm{Res}(F(z), z = i). \tag{5.73}$$

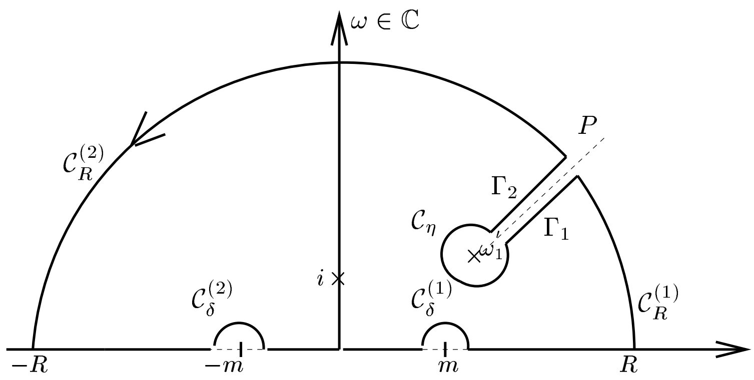

Figure 1: Integration around the closed contour $\mathcal{C}$.

Since $F(z)$ on the intervals $[m+\delta, R]$ and $[-R, -m-\delta]$ becomes respectively

$$F(z) = \begin{cases} F_+(\omega) = \dfrac{\omega}{\sqrt{\omega^2 - m^2}} \dfrac{e^{i(\omega - |\omega^{'}|)t}}{(\omega^2+1)(\sqrt{\omega^2-m^2} - \kappa_p)^{1+i\alpha^{'}_+}} & \text{if } z \in N_1, \\ F_-(\omega) = \dfrac{\omega}{-\sqrt{\omega^2 - m^2}} \dfrac{e^{i(\omega - |\omega^{'}|)t}}{(\omega^2+1)(-\sqrt{\omega^2-m^2} - \kappa_p)^{1+i\alpha^{'}_+}} & \text{if } z \in N_2 \end{cases}$$

with $N_1 := [m+\delta, R]$ and $N_2 := [-R, -m-\delta]$, it results that

$$\lim_{R\to\infty} \left(\int_{m+\delta}^{R} d\omega\; F_+(\omega) + \int_{-R}^{-m-\delta} d\omega\; F_-(\omega) \right) = S.$$

Hence, (5.73) becomes

$$S + \sum_{i=1}^{2} \int_{\mathcal{C}_\delta^{(i)}} dz F(z) + \lim_{R\to\infty} \sum_{i=1}^{2} \int_{\mathcal{C}_R^{(i)}} dz F(z) + \int_{-m+\delta}^{m-\delta} dz F(z) +$$

$$+ \lim_{R\to\infty} \int_{\Gamma} dz F(z) = 2\pi i \operatorname{Res}(F(z), z = i) \quad (5.74)$$

with $\Gamma := \Gamma_1 \cup \mathcal{C}_\eta \cup \Gamma_2$. Let us first consider the integral on $\mathcal{C}_\delta^{(1)}$. To this purpose we introduce the parameterization $z := m + \delta e^{i\theta}$ with $\theta \in [0, \pi]$ and we observe that the integrand $F(z)$ can be rearranged as follows

$$F(z) = \frac{\Psi(z)}{\sqrt{z - m}}$$

with $\Psi(z)$ bounded on $\mathcal{C}_\delta^{(1)}$. Thus, taking into account that $\delta t > 0$ and $\sin\theta \geq 0$ for every $\theta \in [0, \pi]$ we have

$$\Big| \int_{\mathcal{C}_\delta^{(1)}} dz F(z) \Big| \leq c\sqrt{\delta} \int_0^{\pi} d\theta\; e^{-\delta t \sin\theta} \leq c\sqrt{\delta}.$$

An analogous relation holds for the integral on $\mathcal{C}_\delta^{(2)}$. Hence, we can make the contributions coming from the intervals $[-m-\delta, -m+\delta]$ and $[m-\delta, m+\delta]$ arbitrary small by choosing $\delta > 0$ sufficiently small. We consider now only the integral on $\mathcal{C}_R^{(1)}$ in (5.74), the case $i = 2$ being similar. Let us introduce the following parameterization $z = Re^{i\theta}$ on $\mathcal{C}_R^{(1)}$ with $\theta \in [0, \hat{\theta}]$, $\hat{\theta} = \theta_0 - \tilde{\epsilon}$ and $0 < \tilde{\epsilon} < \theta_0 < \pi/2$, where θ_0 is the slope of the ray P in Figure 1. Moreover, $F(z)$ can be rearranged as follows

$$F(z) = \frac{z\hat{\Psi}(z)}{z^2+1}\, e^{izt}$$

with $\hat{\Psi}(z)$ bounded on $\mathcal{C}_R^{(1)}$.Then, we have

$$\lim_{R\to\infty}\Big| \int_{\mathcal{C}_R^{(1)}} dz F(z)\Big| \leq c \lim_{R\to\infty}\int_0^{\hat{\theta}} d\theta\; e^{-Rt\sin\theta}.$$

Since the inequality $\sin\theta \geq \frac{2\theta}{\pi}$ holds for every $\theta \in [0, \frac{\pi}{2}]$, we obtain

$$\lim_{R\to\infty}\Big| \int_{\mathcal{C}_R^{(1)}} dz F(z)\Big| \leq \frac{\pi c}{2t}\lim_{R\to\infty}\frac{1}{R}\left(1 - e^{-\frac{2Rt\hat{\theta}}{\pi}}\right) = 0$$

and we conclude that

$$\lim_{R\to\infty}\int_{\mathcal{C}_R^{(1)}} dz F(z) = 0.$$

Let us consider now in (5.74)

$$\lim_{t\to\pm\infty}\lim_{\sigma\to 0^+}\int_{-m+\delta}^{m-\delta} dz F(z).$$

Taking into account that on $[-m+\delta, m-\delta]$ we have $\theta_1 = \pi$ and $\theta_2 = 0$, respectively, the function $F(z)$ becomes there

$$F_\sigma(\omega) := \frac{\omega}{\sqrt{m^2-\omega^2}}\frac{e^{i(\omega-|\omega^{'}|)t}}{(\omega^2+1)(\sqrt{m^2-\omega^2}-\kappa_p)^{1+i\alpha_+^{'}}}.$$

Since the above function is integrable on $[-m+\delta, m-\delta]$ and moreover, we have $|\sqrt{m^2-\omega^2}-\kappa_p| \leq |\sqrt{m^2-\omega^2}-|\kappa^{'}||$, it results for every $\sigma > 0$ and given $\omega^{'} \in [m+\epsilon, L]$, that $|F_\sigma(\omega)| \leq \mathfrak{U}(\omega)$, with $\mathfrak{U}(\omega)$ integrable on $[-m+\delta, m-\delta]$. Thus, according to dominated convergence theorem we can take the limit $\sigma \to 0^+$ inside the integral and we obtain

$$\lim_{t\to\pm\infty}\int_{-m+\delta}^{m-\delta} d\omega \frac{\omega}{\sqrt{m^2-\omega^2}}\frac{(\sqrt{m^2-\omega^2}-|\kappa^{'}|)^{1+i\alpha_+^{'}}\; e^{i(\omega-|\omega^{'}|)t}}{(\omega^2+1)(2m^2-\omega^2-\omega^{'2})^{1+i\alpha_+^{'}}}.$$

Since $\omega' > m$ and $\omega \neq \pm m$, the integrand above is Lebesgue-integrable on the interval $[-m+\delta, m-\delta]$. Hence, we can apply Riemann-Lebesgue lemma and conclude that

$$\lim_{t\to\pm\infty}\lim_{\sigma\to 0^+}\int_{-m+\delta}^{m-\delta} dzF(z) = 0.$$

for every $\delta > 0$. The above considerations imply that (5.74) reduces asymptotically for $t \to +\infty$ and in the limit as $\sigma \to 0^+$ to

$$\lim_{\sigma\to 0^+} S = 2\pi i \lim_{\sigma\to 0^+} \mathrm{Res}(F(z), z = i) + \lim_{\sigma\to 0^+}\lim_{R\to\infty}\int\limits_{\Gamma'} dzF(z),$$

where the integral on the r.h.s. of the above expression is taken now over the counterclockwise contour $\Gamma' := \Gamma_1 \cup \mathcal{C}_\eta \cup \Gamma_2$. By a direct calculation, it is easy to verify that the residue at $z = i$ is dominated by the exponential e^{-t}, which vanishes for $t \to +\infty$. Thus, we get

$$\lim_{\sigma\to 0^+} S = \lim_{\sigma\to 0^+}\lim_{R\to\infty}\int\limits_{\Gamma'} dzF(z). \tag{5.75}$$

The complex integral in (5.75) can be reduced to the Hankel contour integral for the reciprocal gamma function. To this scope let us rewrite the second term in (5.75) as follows

$$\lim_{R\to\infty}\int\limits_{\Gamma'} dzF(z) = e^{i(\omega_1 - |\omega'|)t}\lim_{R\to\infty}\int\limits_{\Gamma'} dz\ (z-\omega_1)^{\beta-1}\tilde{F}(z)\ e^{i(z-\omega_1)t} \tag{5.76}$$

with

$$\beta := -i\alpha'_+ \quad \text{and} \quad \tilde{F}(z) := \frac{z}{\sqrt{z^2-m^2}}\left(\frac{\sqrt{z^2-m^2}+\kappa_p}{z+\omega_1}\right)^{1-\beta}\frac{1}{z^2+1}.$$

Moreover, in the limit for $R \to \infty$ the contour Γ' becomes the gamma function contour (see Figure 1), which encloses ω_1 and embraces a ray P from the point ω_1 along which $\mathfrak{Im} z > \mathfrak{Im}\omega_1$. Let $\mathcal{C}_{\Gamma'}$ be the gamma function contour. Then, (5.76) becomes

$$\lim_{R\to\infty}\int\limits_{\Gamma'} dzF(z) = e^{i(\omega_1 - |\omega'|)t}\int\limits_{\mathcal{C}_{\Gamma'}} dz\ (z-\omega_1)^{\beta-1}\tilde{F}(z)\ e^{i(z-\omega_1)t}. \tag{5.77}$$

According to [38], since the function $\tilde{F}(z)$ is continuous on, and analytic within, the contour $\mathcal{C}_{\Gamma'}$, the integral on the r.h.s. in (5.77) is convergent and $\tilde{F}(z)$ admits a convergent expansion around ω_1, namely

$$\tilde{F}(z) = \sum_{n=0}^{\infty} c_n (z-\omega_1)^n \quad \text{for} \quad |z-\omega_1| < \eta. \tag{5.78}$$

Therefore, we can substitute (5.78) into (5.77) and take the sum outside the integral and we get

$$\lim_{R\to\infty}\int_{\Gamma'} dzF(z) = e^{i(\omega_1-|\omega^{'}|)t}\sum_{n=0}^{\infty} c_n \int_{\mathcal{C}_{\Gamma'}} dz\ (z-\omega_1)^{\beta+n-1}\ e^{i(z-\omega_1)t}. \quad (5.79)$$

Setting $\hat{z} := -i(z-\omega_1)t$ and $\tilde{\alpha} := \beta+n$, in order to bring (5.79) to the form of the Hankel contour integral of the reciprocal gamma function (see [39], Vol.1, p.14), we obtain

$$\lim_{R\to\infty}\int_{\Gamma'} dzF(z) = \frac{2\pi t^{-\beta}}{i^{\beta-1}} e^{i(\omega_1-|\omega^{'}|)t}\sum_{n=0}^{\infty}\frac{c_n}{i^n\Gamma(1-\beta-n)}t^{-n}.$$

Taking into account that $\beta = -i\alpha^{'}_{+}$, we get uniformly in t

$$\lim_{\sigma\to 0^+} S = 2\pi i^{1+i\alpha^{'}_{+}} e^{i\alpha^{'}_{+}\log t}\lim_{\sigma\to 0^+} e^{i(\omega_1-|\omega^{'}|)t}\sum_{n=0}^{\infty}\frac{c_n t^{-n}}{i^n\Gamma(1+i\alpha^{'}_{+}-n)}. \quad (5.80)$$

Since $\omega_1 \to |\omega^{'}|$ for $\sigma\to 0^+$ and the coefficients c_n depend analytically on σ, (5.80) becomes

$$\lim_{\sigma\to 0^+} S = 2\pi i^{1+i\alpha^{'}_{+}} e^{i\alpha^{'}_{+}\log t}\sum_{n=0}^{\infty}\frac{\tilde{c}_n t^{-n}}{i^n\Gamma(1+i\alpha^{'}_{+}-n)} \quad \text{with} \quad \tilde{c}_n := \lim_{\sigma\to 0^+} c_n.$$

Recalling that $\sqrt{\omega_1^2-m^2} \to +\sqrt{\omega^{'2}-m^2} = |\kappa^{'}|$ for $\sigma\to 0^+$, a short calculation shows that

$$\tilde{c}_0 = \left(\frac{\kappa^{'}}{\omega^{'}}\right)^{i\alpha^{'}_{+}}\frac{1}{\omega^{'2}+1}$$

and (5.80) becomes asymptotically for $t\to+\infty$

$$\lim_{\sigma\to 0^+} S = 2\pi i^{1+i\alpha^{'}_{+}}\frac{e^{i\alpha^{'}_{+}\log\left(\frac{\kappa^{'}}{\omega^{'}}t\right)}}{\Gamma(1+i\alpha^{'}_{+})(\omega^{'2}+1)}. \quad (5.81)$$

Substitution of (5.81) into (5.70) gives

$$A_{>}^{(I)} = 2\pi f(|\kappa^{'}|)\, e^{i\alpha^{'}_{+}\log\left(\frac{\kappa^{'}}{\omega^{'}}t\right)} \quad \text{for} \quad t\to+\infty.$$

For $t\to-\infty$ we close the contour in the lower half-plane and we choose $F(z)$ as follows

$$F(z) := \frac{z}{-\sqrt{z^2-m^2}}\frac{(-\sqrt{z^2-m^2}+\kappa_p)^{1+i\alpha^{'}_{+}}}{(z^2+1)\left[(z-\omega_1)(z-\omega_2)\right]^{1+i\alpha^{'}_{+}}}e^{i(z-|\omega^{'}|)t}$$

with $\theta_1 \in (0, 2\pi]$, $\theta_2 \in (-\pi, \pi]$ and $\sqrt{z^2 - m^2} = -\sqrt{\omega^2 - m^2}$ for $\theta_1 = 2\pi$ and $\sqrt{z^2 - m^2} = \sqrt{\omega^2 - m^2}$ for $\theta_2 = -\pi$. Moreover, $F(z)$ has only one simple pole at $z = -i$. Also in this case, in order to apply the Residue Theorem, we have to fix a contour $\widetilde{\mathcal{C}}$ in such a way that it circumvents the point ω_2 laying on the complex lower half-plane and $F(z)$ is analytic within and on $\widetilde{\mathcal{C}}$ except for the simple pole at $z = -i$. This can be achieved by fixing the contour $\widetilde{\mathcal{C}}$ as in Figure 2. Thus, by applying the Residue Theorem we find

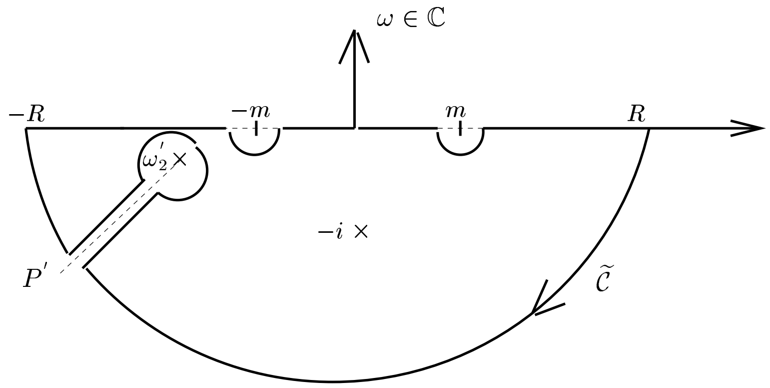

Figure 2: Integration around the closed contour $\widetilde{\mathcal{C}}$.

that

$$\int_{\widetilde{\mathcal{C}}} dz F(z) = -2\pi i \,\mathrm{Res}(F(z), z = -i),$$

where $\mathrm{Res}(F(z), z = -i)$ is dominated by the exponential term e^t and goes to zero for $t \to -\infty$. Employing the same method we used in order to compute $A_>^{(I)}$ for $t \to +\infty$ with the only exception that in this case $\tilde{F}(z)$ is defined as follows

$$\tilde{F}(z) := \frac{z}{-\sqrt{z^2 - m^2}} \left(\frac{-\sqrt{z^2 - m^2} + \kappa_p}{z + \omega_2} \right)^{1-\beta} \frac{1}{z^2 + 1}$$

and taking into account that $\sqrt{\omega_2^2 - m^2} \to +\sqrt{\omega'^2 - m^2} = |\kappa'|$ for $\sigma \to 0^+$, a short computation shows that the coefficient $\tilde{c}_0$ is now zero and this implies that

$$A_>^{(I)} = 0 \quad \text{for} \quad t \to -\infty. \tag{5.82}$$

Concerning the case $\omega' \in [-L, -m - \epsilon]$, we have

$$A_<^{(I)} := \int_{u_1}^{\infty} du\, u^{i\alpha'_-} \int_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa)\, e^{i(\omega + |\omega'|)t - i(|\kappa'| + \kappa)u},$$

where α'_- is defined according to (5.30) as follows

$$\alpha'_- := -\frac{\omega' eQ + Mm^2}{\sqrt{\omega'^2 - m^2}}$$

and proceeding analogously as we did for the evaluation of $A^{(I)}_>$, we obtain that $A^{(I)}_< = 0$ for $t \to -\infty$ and

$$A^{(I)}_< = 2\pi f(-|\kappa'|)\, e^{i\alpha'_- \log\left(\frac{\kappa'}{\omega'}t\right)} \quad \text{for} \quad t \to +\infty.$$

Thus, we can conclude that

$$A^{(I)} = \begin{cases} 2\pi f(\kappa')\, e^{i\alpha' \log\left(\frac{\kappa'}{\omega'}t\right)} & \text{if } t \to +\infty \\ 0 & \text{if } t \to -\infty \end{cases}. \tag{5.83}$$

Notice that $A^{(II)}$ can be immediately obtained from $A^{(I)}$ by means of the transformation

$$\kappa' \to -\kappa', \quad \kappa \to -\kappa \quad \text{and} \quad t \to -t, \tag{5.84}$$

taking care that $\alpha(\kappa') \to -\alpha(\kappa')$ under the transformation $\kappa' \to -\kappa'$. Moreover, $A^{(III)}$ can be evaluated by means of the same methods like those adopted in the computation of $A^{(I)}$. Finally, the results for $A^{(IV)}$ come from those one for $A^{(III)}$ with the help of the transformation (5.84). □

The next Lemma allows us to evaluate (5.26).

Lemma 5.3. *For every function $f \in C^1(\mathbb{R})$, sharing the property* (5.41) *and for every given $\omega' \in \Omega'_1$ it holds*

$$\mathfrak{S}_1 := \lim_{t\to\pm\infty} \int_{u_1}^{\infty} du\, u^{\pm i\alpha' - 1} \int_{\mathbb{R}\backslash B_\delta(0)} d\kappa f(\kappa)\, e^{i\varphi_1(\kappa,u)} = 0, \tag{5.85}$$

$$\mathfrak{S}_2 := \lim_{t\to\pm\infty} \int_{u_1}^{\infty} du\, u^{\pm i\alpha' - 1} \int_{\mathbb{R}\backslash B_\delta(0)} d\kappa f(\kappa)\, e^{i\varphi_2(\kappa,u)} = 0, \tag{5.86}$$

for fixed $\alpha' \in \mathbb{R}$ like in (5.30) *and some $u_1 > 0$. Moreover, $\varphi_1(\kappa,u)$ and $\varphi_2(\kappa,u)$ are as defined in Lemma 5.1.*

Proof. We give here the proof of (5.85) in the case $u^{i\alpha'-1}$; in the other cases the proof is similar. Let us first suppose that $\alpha' \neq 0$. Proceeding like in Lemma 5.1, we can introduce a convergence generating factor $e^{-\sigma u}$ with $\sigma > 0$ and after applying Fubini theorem, we obtain

$$\mathfrak{S}_1 := \lim_{t\to\pm\infty} \lim_{\sigma\to 0^+} \int_{\mathbb{R}\backslash B_\delta(0)} d\kappa f(\kappa)\, e^{i(\omega-\omega')t} \int_{u_1}^{\infty} du\, u^{i\alpha'-1} e^{-\sigma u \pm i(\kappa'-\kappa)u}.$$

The integral over u in the above expression can be again computed by means of the Kummer function of the second kind and we get

$$\int_{u_1}^{\infty} du\, u^{i\alpha' - 1} e^{-\sigma u \pm i(\kappa' - \kappa)u} = u_1^{i\alpha'} e^{-\tilde{z}}\, U(1, 1 + i\alpha'; \tilde{z}).$$

with $\tilde{z}$ given by (5.45). With the help of (5.46) and taking into account the following relations

$$\Gamma(1 + i\alpha')\Gamma(1 - i\alpha') = \frac{i\pi\alpha'}{\sin(i\pi\alpha')} \quad \text{and} \quad \frac{1}{\Gamma(1 - i\alpha')} = \frac{\sin(i\pi\alpha')}{\pi}\Gamma(i\alpha'),$$

we get

$$\int_{u_1}^{\infty} du\, u^{i\alpha' - 1} e^{-\sigma u \pm i(\kappa' - \kappa)u} = u_1^{i\alpha'} \left[\frac{i}{\alpha'} e^{-\tilde{z}} M(1, 1 + i\alpha'; \tilde{z}) + \Gamma(i\alpha')\tilde{z}^{-i\alpha'} \right], \tag{5.87}$$

where $M(1, 1 + i\alpha'; \tilde{z})$ is the Kummer function of the first kind. Substitution of (5.87) into the expression for $\mathfrak{S}_1$ gives

$$\begin{aligned} \mathfrak{S}_1 := i\frac{u_1^{i\alpha'}}{\alpha'} \lim_{t\to\pm\infty} \lim_{\sigma\to 0^+} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) M(1, 1 + i\alpha'; \tilde{z})\, e^{i(\omega - \omega')t - \tilde{z}(\kappa)} + \\ + u_1^{i\alpha'}\Gamma(i\alpha') \lim_{t\to\pm\infty} \lim_{\sigma\to 0^+} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa \frac{f(\kappa)}{\tilde{z}^{i\alpha'}(\kappa)}\, e^{i(\omega - \omega')t}. \end{aligned} \tag{5.88}$$

Let us consider the first term in (5.88) and let us define

$$H_\sigma(\kappa) := f(\kappa) M(1, 1 + i\alpha'; \tilde{z})\, e^{i(\omega - \omega')t - \tilde{z}(\kappa)}.$$

In order to take the limit for $\sigma \to 0^+$ inside the integral, we have to verify that dominated convergence theorem can be applied, i.e., we have to find an integrable function $\mathfrak{J}(\kappa)$, such that

$$|H_\sigma(\kappa)| \le \mathfrak{J}(\kappa)$$

for every $\sigma > 0$. We notice that the integral representation (5.56) does not hold any more, since $\Re a = \Re b$ with $a = 1$ and $b = 1 + i\alpha'$ in this case and thus, we cannot use it in order to show that $|M(1, 1 + i\alpha'; \tilde{z})|$ is bounded. On the other hand $\Re \tilde{z} > 0$ and 13.1.4 in [37] implies that for $C > 0$ there exists a $\rho > 0$ such that

$$|M(1, 1 + i\alpha'; \tilde{z})| \le |\Gamma(1 + i\alpha')| e^{u_1 \sigma} \left(1 + \frac{C}{|\tilde{z}|}\right) \tag{5.89}$$

for every $\tilde{z} \in \mathbb{C}\backslash K$ with $K := \{\tilde{z} \in \mathbb{C} : |\Re\tilde{z}| \leq \rho, |\Im\tilde{z}| \leq \rho\}$. Moreover, $M(1, 1 + i\alpha^{'}; \tilde{z})$ is bounded for every $\tilde{z} \in K$. Without loss of generality we can choose $C = 1$, $\rho > 1$ and (5.89) becomes

$$|M(1, 1 + i\alpha^{'}; \tilde{z})| \leq 2|\Gamma(1 + i\alpha^{'})|e^{u_1\sigma}.$$

Hence, we get

$$|H_\sigma(\kappa)| \leq |\Gamma(1 + i\alpha^{'})||f(\kappa)| =: \mathfrak{J}(\kappa).$$

Taking the limit for $\sigma \to 0^+$ and defining $f_1(\kappa) := f(\kappa)e^{\pm i(\kappa^{'}-\kappa)u_1}$, we observe that $f_1(\kappa)$ satisfies (5.41). Moreover, $|M(1, 1+i\alpha^{'}; \mp iu_1(\kappa^{'} - \kappa))|$ is bounded. Changing the variable of integration from κ into ω according to $d\kappa = \frac{\omega}{\kappa}d\omega$, Riemann-Lebesgue lemma yields that in the limit for $t \to \pm\infty$ the first term in (5.88) gives no contribution. Concerning the second term in (5.88), we can first apply dominated convergence theorem to get rid of the limit $\sigma \to 0^+$, then we can change the variable of integration from κ into ω and finally, we can use the Riemann-Lebesgue lemma in order to show that such a limit integral is zero.

Let be now $\alpha^{'} = 0$. We consider for simplicity the case with positive sign in $\varphi_1(\kappa, u)$. By introducing a convergence generating factor $e^{-\sigma u}$ with $\sigma > 0$ to interchange the order of integration in (5.85), we find that

$$\mathfrak{S}_1 = \lim_{t\to\pm\infty} \lim_{\sigma\to 0^+} \int\limits_{\mathbb{R}\backslash B_\delta(0)} d\kappa f(\kappa)\, e^{i(\omega-\omega^{'})t} \int_{u_1}^{\infty} du\, \frac{e^{-i(\kappa-\kappa_p)u}}{u} \tag{5.90}$$

with $\kappa_p = \kappa^{'} + i\sigma$. Since with the help of 5.1.4 in [37] the integral over the spatial variable u in (5.90) can be related to the exponential integral E_1, namely

$$\int_{u_1}^{\infty} du\, \frac{e^{-i(\kappa-\kappa_p)u}}{u} = E_1(iu_1(\kappa - \kappa_p)),$$

$\mathfrak{S}_1$ can be written as

$$\mathfrak{S}_1 = \lim_{t\to\pm\infty} \lim_{\sigma\to 0^+} \int\limits_{\mathbb{R}\backslash B_\delta(0)} d\kappa f(\kappa) E_1(iu_1(\kappa - \kappa_p))\, e^{i(\omega-\omega^{'})t}.$$

Moreover, it holds

$$E_1(iu_1(\kappa - \kappa_p)) = e^{-\sigma u_1} E_1(i(\kappa - \kappa^{'}))$$

and after a short computation $\mathfrak{S}_1$ becomes

$$\mathfrak{S}_1 = \lim_{t\to\pm\infty} \lim_{\sigma\to 0^+} e^{-\sigma u_1} \int\limits_{\mathbb{R}\backslash B_\delta(0)} d\kappa f(\kappa) E_1(iu_1(\kappa - \kappa^{'}))\, e^{i(\omega-\omega^{'})t}.$$

Thus, by taking the limit for $\sigma \to 0^+$ we obtain

$$\mathfrak{S}_1 = \lim_{t\to\pm\infty} \int_{\mathbb{R}\backslash B_\delta(0)} d\kappa f(\kappa) E_1(iu_1(\kappa-\kappa^{'}))\, e^{i(\omega-\omega^{'})t}.$$

Since the function $E_1(iu_1(\kappa-\kappa^{'}))$ diverges logarithmically at $\kappa=\kappa^{'}$ (see [37]), we introduce the following decomposition over the interval of integration, namely $(-\infty,-\delta]\cup[\delta,\kappa^{'}-\eta]\cup[\kappa^{'}-\eta,\kappa^{'}+\eta]\cup[\kappa^{'}+\eta,+\infty)$ with $\eta>0$ and without loss of generality $\kappa^{'}>0$, such that for $|\kappa-\kappa^{'}|>\eta$ the exponential integral E_1 has the asymptotic expansion (see 5.1.51 [37])

$$E_1(iu_1(\kappa-\kappa^{'})) \sim \frac{e^{-iu_1(\kappa-\kappa^{'})}}{iu_1(\kappa-\kappa^{'})}\left(1+\sum_{n=1}^{m}\frac{(-)^n n!}{i^n u_1^n(\kappa-\kappa^{'})^n}\right),$$

while for $|\kappa-\kappa^{'}|\le\eta$ the function E_1 admits the series expansion (compare 5.1.11 [37])

$$E_1(iu_1(\kappa-\kappa^{'})) \sim -\gamma-\log\left(iu_1(\kappa-\kappa^{'})\right)-\sum_{n=1}^{m}\frac{(-iu_1^n)^n(\kappa-\kappa')^n}{nn!}$$

where γ is the Euler constant. Notice that the function $f(\kappa)E_1(iu_1(\kappa-\kappa^{'}))$ is continuous on the intervals $(-\infty,\kappa^{'}-\eta]$ and $[\kappa^{'}+\eta,+\infty)$ and decay at least cubically. Thus, after a transformation of the variable of integration from κ into ω, Riemann-Lebesgue lemma implies that

$$\lim_{t\to\pm\infty}\int_{-\infty}^{-\delta} d\kappa f(\kappa)E_1(iu_1(\kappa-\kappa^{'}))\, e^{i(\omega-\omega^{'})t}=0, \tag{5.91}$$

$$\lim_{t\to\pm\infty}\int_{\delta}^{\kappa^{'}-\eta} d\kappa f(\kappa)E_1(iu_1(\kappa-\kappa^{'}))\, e^{i(\omega-\omega^{'})t}=0, \tag{5.92}$$

$$\lim_{t\to\pm\infty}\int_{\kappa^{'}+\eta}^{+\infty} d\kappa f(\kappa)E_1(iu_1(\kappa-\kappa^{'}))\, e^{i(\omega-\omega^{'})t}=0. \tag{5.93}$$

Since the function $f(\kappa)E_1(iu_1(\kappa-\kappa^{'}))$ has only a logarithmic divergence at $\kappa=\kappa^{'}$ in $[\kappa^{'}-\eta,\kappa^{'}+\eta]$, it is there integrable and after transformation of the variable of integration from κ into ω Riemann-Lebesgue lemma gives

$$\lim_{t\to\pm\infty}\int_{\kappa^{'}-\eta}^{\kappa^{'}+\eta} d\kappa f(\kappa)E_1(iu_1(\kappa-\kappa^{'}))\, e^{i(\omega-\omega^{'})t}=0. \tag{5.94}$$

Putting together (5.91), (5.92), (5.93) and (5.94) we obtain the desired result. Notice that if $\kappa^{'}\in B_\delta(0)$, then the function $f(\kappa)E_1(iu_1(\kappa-\kappa^{'}))$ is continuous on the whole interval $\mathbb{R}\backslash B_\delta(0)$ and from Riemann-Lebesgue lemma it follows again that $\mathfrak{S}_1=0$. □

The next result allows us to show that the limit integral I_3 does not give any contribution.

Lemma 5.4. *For every function $f \in C^1(\mathbb{R})$, sharing the property* (5.41) *and for every given $\omega' \in \Omega_1'$ it holds*

$$\lim_{t\to\pm\infty} \int_{u_1}^{\infty} du\, u^{\pm i\alpha'} h(u) \int_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa)\, e^{i(\omega-\omega')t+i(\pm\kappa\pm\kappa')u} = 0 \qquad (5.95)$$

for fixed $\alpha' \in \mathbb{R}$ as in (5.30)*, some $u_1 > 0$ and $h(u) = \mathcal{O}(u^{-2})$.*

Proof. Due to the presence of the function $h(u) = \mathcal{O}(u^{-2})$ we can immediately apply Fubini theorem, interchange the order of integration and perform the integration over the variable u. Hence (5.95) becomes

$$\lim_{t\to\pm\infty} \int_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) N(\kappa)\, e^{i(\omega-\omega')t} \qquad (5.96)$$

with

$$N(\kappa) := \int_{u_1}^{\infty} du\, u^{\pm i\alpha'} h(u) e^{i(\pm\kappa\pm\kappa')u}.$$

Notice that

$$|N(\kappa)| \le \int_{u_1}^{\infty} \frac{du}{u^2} = \frac{1}{u_1}.$$

At this point we can apply Riemann-Lebesgue lemma to (5.137) and conclude that (5.95) does not give any contribution. □

By means of the next result we can evaluate (5.36), (5.37) and (5.38).

Lemma 5.5. *For every $f \in C^1(\mathbb{R})$ satisfying* (5.41)*, it holds for every real positive β and given $\omega' \in \Omega_2'$*

$$\lim_{t\to\pm\infty} \int_{u_1}^{\infty} du\, e^{-\beta u} u^{\widetilde{\alpha}}\, Z(u) \int_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa)\, e^{i(\omega-\omega')t\pm i\kappa u} = 0, \qquad (5.97)$$

with the function $Z(u)$ given by

$$Z(u) = \left\{ \begin{array}{c} Z_1(u) \\ Z_2(u) \\ Z_3(u) \end{array} \right\} := \left\{ \begin{array}{c} 1 \\ u^{-1} \\ \hat{h}(u) \end{array} \right\}, \quad \hat{h}(u) := \mathcal{O}\left(\frac{1}{u^2}\right)$$

and $\widetilde{\alpha} \in \mathbb{R}$ as defined in Lemma 2.2.

Proof. Due to the presence of the term $e^{-\beta u}$ decaying for $u \to +\infty$ we can apply Fubini theorem to interchange the order of integration and we get

$$\lim_{t\to\pm\infty} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) G(\kappa)\, e^{i(\omega-\omega')t} \tag{5.98}$$

with

$$G(\kappa) := \int_{u_1}^{\infty} du\, u^{\widetilde{\alpha}} Z(u)\, e^{-\beta u \pm i\kappa u}$$

By means of the variable transformation $u = u_1 x$, we obtain

$$G(\kappa) = u^{1+\widetilde{\alpha}} \int_1^{\infty} dx\, x^{\widetilde{\alpha}} Z(x)\, e^{-\hat{\beta}x \pm i u_1 \kappa x} \quad \text{with } \hat{\beta} := u_1\beta.$$

Moreover, from [39] it is known that for $n \in \mathbb{N}$ and positive real b

$$\int_1^{\infty} dx\, x^{\widetilde{\alpha}-n} e^{-bx} = \frac{\Gamma(1+\widetilde{\alpha}-n, b)}{b^{1+\widetilde{\alpha}-n}}, \tag{5.99}$$

where $\Gamma(1+\widetilde{\alpha}-n, b)$ is the incomplete Gamma function. Notice that for $\widetilde{\alpha} = n-1$ the incomplete Gamma function gives rise to the exponential integral function

$$E_1(b) = \Gamma(0, b),$$

which remains finite, since $b > 0$. We show that $G(\kappa)$ is bounded. Let us first consider the cases $Z_1(u) = 1$ and $Z_2(u) = u^{-1}$. Then by means of (5.99) it can be verified that

$$|G(\kappa)| \le \begin{cases} \dfrac{\Gamma(1+\widetilde{\alpha}, \hat{\beta})}{\beta^{1+\widetilde{\alpha}}} & \text{for } Z_1(u) = 1 \\ \dfrac{\Gamma(\widetilde{\alpha}, \hat{\beta})}{\beta^{\widetilde{\alpha}}} & \text{for } Z_2(u) = \dfrac{1}{u} \end{cases}.$$

Finally for $Z_3(u) = \hat{h}(u)$ we obtain the bound

$$|G(\kappa)| \le u_1^{1+\widetilde{\alpha}} \int_1^{\infty} dx\, x^{\widetilde{\alpha}-2} e^{-\hat{\beta}x} = \frac{\Gamma(\widetilde{\alpha}-1, \hat{\beta})}{\beta^{\widetilde{\alpha}-1}}.$$

Let us rewrite (5.98) as follows

$$\lim_{t\to\pm\infty} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) G(\kappa) e^{i(\omega-\omega')t}.$$

After changing the variable of integration from κ to ω and applying Riemann-Lebesgue lemma to (5.98), we obtain the desired result. □

5.3 Modified Wave Operators at infinity $\widetilde{W}_{\infty}^{\pm}$.

We consider now the modified wave operator

$$\widetilde{W}_{\infty}^{\pm}\hat{\Psi}_0^{(\infty)} = s - \lim_{t\to\pm\infty} e^{-iHt}\mathcal{I}_{\infty}e^{iH_{\infty}t}e^{i\delta(t)}\hat{\Psi}_0^{(\infty)}, \quad \hat{\Psi}_0^{(\infty)} \in \mathcal{H}_{\infty},$$

where

$$e^{i\delta(t)} := e^{-i\mathrm{sgn}(t)\alpha(\kappa)\log(\frac{\kappa}{\omega}|t|)}, \quad \alpha(\kappa) = \epsilon(\omega)\frac{\omega eQ + Mm^2}{\sqrt{\omega^2 - m^2}}.$$

We show that the above defined phase shift offsets the logarithmic divergence, which appears in Lemma 5.2. Proceeding like we did for $W_{\infty}^{\pm}$, we just need to restrict our analysis to the term

$$\overline{X}_{b'}(u)X^{\infty}(u)e^{i\delta(t)}f(\omega) \tag{5.100}$$

taking care to distinguish between the cases $\omega' \in \Omega_1'$ and $\omega' \in \Omega_2'$.

5.3.1 The Region Ω_1'.

We can start with (5.18) multiplied by the logarithmic phase shift in order to compute the terms arising from (5.100). Analogously to the case $\omega' \in \Omega_1'$ for the classical wave operator, we end up after a change of the variable of integration from ω into κ with the evaluation of the following limit integrals

$$\lim_{t\to\pm\infty}\int_{u_1}^{\infty} du\, u^{i\alpha'} \int_{\mathbb{R}\backslash B_\delta(0)} d\kappa F(\kappa)e^{i\delta(t)}e^{i(\omega-\omega')t\pm i(\kappa'\pm\kappa)u}, \tag{5.101}$$

$$\lim_{t\to\pm\infty}\int_{u_1}^{\infty} du\, u^{\pm i\alpha'-1} \int_{\mathbb{R}\backslash B_\delta(0)} d\kappa F(\kappa)e^{i\delta(t)}e^{i(\omega-\omega')t\pm i(\kappa'\pm\kappa)u}, \tag{5.102}$$

$$\lim_{t\to\pm\infty}\int_{u_1}^{\infty} du\, u^{\pm i\alpha'}h(u) \int_{\mathbb{R}\backslash B_\delta(0)} d\kappa F(\kappa)e^{i\delta(t)}e^{i(\omega-\omega')t\pm i(\kappa'\pm\kappa)u}, \tag{5.103}$$

$$\lim_{t\to\pm\infty}\int_{u_0}^{u_1} du\, u^{i\alpha'} \int_{\mathbb{R}\backslash B_\delta(0)} d\kappa F(\kappa)e^{i\delta(t)}e^{i(\omega-\omega')t\pm i(\kappa'\pm\kappa)u}, \tag{5.104}$$

$$\lim_{t\to\pm\infty}\int_{u_0}^{u_1} du\, u^{\pm i\alpha'-1} \int_{\mathbb{R}\backslash B_\delta(0)} d\kappa F(\kappa)e^{i\delta(t)}e^{i(\omega-\omega')t\pm i(\kappa'\pm\kappa)u}, \tag{5.105}$$

$$\lim_{t\to\pm\infty}\int_{u_0}^{u_1} du\, u^{\pm i\alpha'}h(u) \int_{\mathbb{R}\backslash B_\delta(0)} d\kappa F(\kappa)e^{i\delta(t)}e^{i(\omega-\omega')t\pm i(\kappa'\pm\kappa)u}, \tag{5.106}$$

with $F(\kappa)$ as in (5.29), $h(u) = \mathcal{O}(u^{-2})$ and given $\kappa' \in \mathbb{R}\backslash\{0\}$. By means of Theorem 5.7 and Lemma 5.2 we can evaluate (5.101) and show that the result does not contain anymore the logarithmic divergence in time. In Theorem 5.11 we show that (5.104), (5.105) and (5.106) do not give any contribution.

5.3.2 The Region $\Omega_2^{'}$.

Proceeding like in Section 5.1.2 and applying Theorem 5.11, after a change of the variable of integration from ω to κ we only need to evaluate the following limit integral

$$\lim_{t\to\pm\infty}\int_{u_1}^{\infty} du\, e^{-\beta u}u^{\widetilde{\alpha}}Z(u)\int\limits_{\mathbb{R}\backslash B_\delta(0)} d\kappa \widehat{F}(\kappa)e^{i\delta(t)}e^{i(\omega-\omega^{'})t\pm i\kappa u}, \tag{5.107}$$

where

$$Z(u) := \left\{ \begin{array}{c} Z_1(u) \\ Z_2(u) \\ Z_3(u) \end{array} \right\} := \left\{ \begin{array}{c} 1 \\ u^{-1} \\ \hat{h}(u) \end{array} \right\}, \quad \hat{h}(u) := \mathcal{O}\left(\frac{1}{u^2}\right),$$

$\beta := \sqrt{m^2 - \omega^{'2}}$ and $\widetilde{\alpha}$, $\delta(t)$ as defined in Lemma 2.2 and Theorem 5.7, respectively. In Theorem 5.12 we will prove that (5.107) gives no contribution.

5.4 Theorems for the Evaluation of $\widetilde{W}_{\infty}^{\pm}$.

Lemma 5.6. *Let $\kappa \in \mathbb{R}\backslash B_\delta(0)$ and $t \in \mathbb{R}$ with $|t| \gg 1$. Moreover, let the phase $\delta(\kappa,t)$ be defined by*

$$\delta(\kappa,t) := -i\mathrm{sgn}(t)\alpha(\kappa)\log(\frac{\kappa}{\omega}|t|)$$

with $\alpha(\kappa)$ given by (5.3). Then, the following estimate holds

$$\left|\frac{e^{i\delta(\kappa,t)} - e^{i\delta(\kappa^{'},t)}}{\kappa-\kappa^{'}}\right| \le \frac{2Mm^2}{\min(\kappa^2,\kappa^{'2})}\log|t|$$

for every given $\kappa^{'} \in \mathbb{R}\backslash\{0\}$.

Proof. Since

$$\left|e^{i\delta(\kappa,t)} - e^{i\delta(\kappa^{'},t)}\right| = \left|\int_{\kappa}^{\kappa^{'}} dx\frac{de^{i\delta(x,t)}}{dx}\right| = \left|\int_{\kappa^{'}}^{\kappa} dx\frac{de^{i\delta(x,t)}}{dx}\right| \le |\kappa-\kappa^{'}|\sup_{x\in I}\left|\frac{de^{i\delta(x,t)}}{dx}\right|$$

with $I = [\kappa,\kappa^{'}]$ or $I = [\kappa^{'},\kappa]$, it results that

$$\left|\frac{e^{i\delta(\kappa,t)} - e^{i\delta(\kappa^{'},t)}}{\kappa-\kappa^{'}}\right| \le \sup_{x\in I}\left|\frac{d\delta(x,t)}{dx}\right|. \tag{5.108}$$

Moreover, after some elementary calculations we obtain

$$\frac{d\delta}{dx} = \mathrm{sgn}(t)\epsilon(\omega)\left[-\frac{m^2}{x^2}\left(\frac{eQ}{\omega}+M\right)\left(\log|t|+\log\left(\frac{\kappa}{\omega}\right)\right)+\frac{m^2(\omega eQ+m^2M)}{\omega^2x^2}\right]$$

with $\omega = \omega(x) = \sqrt{x^2+m^2}$. Since $|t| \gg 1$, we can restrict our analysis to the dominant term in the above expression. Taking into account that $|eQ| < Mm$, we obtain that

$$\left|\frac{d\delta}{dx}\right| \leq \frac{2Mm^2}{x^2}\log|t|.$$

Inspection of the cases $\kappa < \kappa^{'}$ and $\kappa \geq \kappa^{'}$ together with the above estimates and (5.108) completes the proof. □

Theorem 5.7. *For every function $f \in C^1(\mathbb{R})$, sharing the property* (5.41) *and for every given $\omega^{'} \in \Omega_1^{'}$ it results*

$$\hat{\mathfrak{T}}_1 := \lim_{t\to\pm\infty}\int_{u_1}^{\infty} du u^{\pm i\alpha^{'}} \int_{\mathbb{R}\backslash B_\delta(0)} d\kappa f(\kappa)\left(e^{i\delta(t)} - e^{i\delta_1^{'}(t)}\right)e^{i\varphi_{(t)}^{(1)}(\kappa,u)} = 0 \tag{5.109}$$

$$\hat{\mathfrak{T}}_2 := \lim_{t\to\pm\infty}\int_{u_1}^{\infty} du u^{\pm i\alpha^{'}} \int_{\mathbb{R}\backslash B_\delta(0)} d\kappa f(\kappa)\left(e^{i\delta(t)} - e^{i\delta_2^{'}(t)}\right)e^{i\varphi_{(t)}^{(2)}(\kappa,u)} = 0, \tag{5.110}$$

with

$$\delta(t) = -\mathrm{sgn}(t)\alpha(\kappa)\log(\frac{\kappa}{\omega}|t|), \tag{5.111}$$

$$\delta_1^{'}(t) := -\mathrm{sgn}(t)\alpha(\kappa^{'})\log(\frac{\kappa^{'}}{\omega^{'}}|t|), \tag{5.112}$$

$$\delta_2^{'}(t) := \mathrm{sgn}(t)\alpha(\kappa^{'})\log(\frac{\kappa^{'}}{\omega^{'}}|t|), \tag{5.113}$$

(5.114)

$u_1 > 0$ *and $\alpha^{'}$ defined by* (5.3)*. Moreover,*

$$\varphi_{(t)}^{(1)}(\kappa,u) = (\omega-\omega^{'})t \pm (\kappa^{'}-\kappa)u \tag{5.115}$$

$$\varphi_{(t)}^{(2)}(\kappa,u) = (\omega-\omega^{'})t \pm (\kappa^{'}+\kappa)u. \tag{5.116}$$

Proof. We just need to show (5.109) for the case $u^{i\alpha^{'}}$ since all other cases can be proved by means of the same method. Proceeding exactly like in

Lemma 5.1 we end up with

$$\hat{\mathfrak{T}}_1 := u_1^{1+i\alpha'}\Gamma(1+i\alpha')\lim_{t\to\pm\infty}\lim_{\sigma\to 0^+}\int\limits_{\mathbb{R}\backslash B_\delta(0)} d\kappa\frac{\hat{f}_{(t)}(\kappa)}{\tilde{z}^{1+i\alpha'}(\kappa)}e^{i(\omega-\omega')t}+$$
$$-\frac{u_1^{1+i\alpha'}}{1+i\alpha'}\lim_{t\to\pm\infty}\lim_{\sigma\to 0^+}\int\limits_{\mathbb{R}\backslash B_\delta(0)} d\kappa\hat{f}_{(t)}(\kappa)e^{-\tilde{z}(\kappa)}M(1,2+i\alpha';\tilde{z})e^{i(\omega-\omega')t}, \tag{5.117}$$

where $\tilde{z}$ is given by (5.45) and $\hat{f}_{(t)}(\kappa) := f(\kappa)(e^{i\delta(t)} - e^{i\delta_1'(t)})$. Let us begin with the first integral in (5.117). Since we would like to perform the limit $\sigma \to 0^+$, we rewrite the integrand in the above expression as follows

$$f(\kappa)\frac{e^{i\delta(t)} - e^{i\delta_1'(t)}}{\kappa-\kappa'}\frac{\kappa-\kappa'}{\sigma\mp(\kappa'-\kappa)}\frac{e^{i(\omega-\omega')t}}{[\sigma\mp(\kappa'-\kappa)]^{i\alpha'}}.$$

Notice that the function

$$f(\kappa)\frac{e^{i\delta(t)} - e^{i\delta_1'(t)}}{\kappa-\kappa'}$$

is continuous at $\kappa=\kappa'$ and decay at least like $1/k^3$ for $|\kappa|\to\infty$. Moreover, it holds

$$\left|\frac{\kappa-\kappa'}{\sigma\mp(\kappa'-\kappa)}\right| \le 1.$$

Hence, we can apply dominated convergence theorem to take the limit $\sigma \to 0^+$ inside the integral and we obtain

$$\lim_{t\to\pm\infty}\int\limits_{\mathbb{R}\backslash B_\delta(0)} d\kappa\frac{f(\kappa)}{(\kappa'-\kappa)^{i\alpha'}}\frac{e^{i\delta(t)} - e^{i\delta_1'(t)}}{\kappa-\kappa'}e^{i(\omega-\omega')t}. \tag{5.118}$$

Without loss of generality we suppose that $\kappa' > 0$ and we consider for some fixed $\delta > 0$ and $\epsilon(t) = 1/\log^2|t|$ with $|t| \gg 1$ the decomposition $(-\infty,-\delta]\cup[\delta,\kappa'-\epsilon(t)]\cup[\kappa'-\epsilon(t),\kappa'+\epsilon(t)]\cup[\kappa'+\epsilon(t),\infty)$ of $\mathbb{R}\backslash B_\delta(0)$. with $|\kappa'|>\delta$. Let us rewrite $e^{i(\omega-\omega')t}$ as follows

$$e^{i(\omega-\omega')t} = \frac{1}{1+i\frac{\kappa}{\omega}t}(1+\partial_\kappa)e^{i(\omega-\omega')t}. \tag{5.119}$$

We begin with $(-\infty,-\delta]$. Notice that because of our assumptions on $f(\kappa)$ integrating by parts gives no boundary terms at infinity. Moreover, making use of Lemma 5.6 and applying dominated convergence theorem it can be verified that

$$\lim_{t\to\pm\infty}\int_{-\infty}^{-\delta} d\kappa \frac{f(\kappa)}{(\kappa^{'}-\kappa)^{i\alpha^{'}}}\frac{e^{i\delta(t)}-e^{i\delta_1^{'}(t)}}{\kappa-\kappa^{'}}e^{i(\omega-\omega^{'})t}=0.$$

Concerning the interval $[\delta,\kappa^{'}-\epsilon(t)]$, we use again (5.119) and we obtain

$$\int_{\delta}^{\kappa^{'}-\epsilon(t)} d\kappa G(\kappa,t)(1+\partial_\kappa)e^{i(\omega-\omega^{'})t} = \int_{\delta}^{\kappa^{'}-\epsilon(t)} d\kappa G(\kappa,t)e^{i(\omega-\omega^{'})t}+$$
$$+\left.G(\kappa,t)\right|_{\delta}^{\kappa^{'}-\epsilon(t)}-\int_{\delta}^{\kappa^{'}-\epsilon(t)} d\kappa\partial_\kappa\left(G(\kappa,t)\right)e^{i(\omega-\omega^{'})t} \quad (5.120)$$

with

$$G(\kappa,t):=\frac{f(\kappa)}{(\kappa^{'}-\kappa)^{i\alpha^{'}}}\frac{e^{i\delta(t)}-e^{i\delta_1^{'}(t)}}{\kappa-\kappa^{'}}\frac{1}{1+i\frac{\kappa}{\omega}t}.$$

Regarding the boundary term in (5.120), a direct computation shows that

$$\lim_{t\to\pm\infty}\left.G(\kappa,t)\right|_{\delta}^{\kappa^{'}-\epsilon(t)}=0. \quad (5.121)$$

Notice that the following estimate holds, namely

$$\left|\frac{1}{1+i\frac{\kappa}{\omega}t}\right|\leq\frac{\omega}{\kappa}\frac{1}{|t|}. \quad (5.122)$$

Hence, Lemma 5.6 implies that

$$|G(\kappa,t)|\leq 2Mm^2\frac{\log|t|}{|t|}\frac{|f|}{k^2}\frac{\omega}{\kappa}.$$

Let us analyze the first integral on r.h.s. of (5.120). Making use of Lemma 5.6 and (5.122), we find that

$$\left|\int_{\delta}^{\kappa^{'}-\epsilon(t)} d\kappa G(\kappa,t)e^{i(\omega-\omega^{'})t}\right|\leq 2Mm^2(\kappa^{'}-\delta-\epsilon(t))\frac{\log|t|}{|t|}\sup_{\kappa\in I_1}\left(\frac{|f|}{\kappa^2}\frac{\omega}{\kappa}\right)$$

with $I_1:=[\delta,\kappa^{'}-\epsilon(t)]$. Thus, we obtain that

$$\lim_{t\to\pm\infty}\int_{\delta}^{\kappa^{'}-\epsilon(t)} d\kappa G(\kappa,t)e^{i(\omega-\omega^{'})t}=0. \quad (5.123)$$

We treat now the last integral on the r.h.s. of (5.120). To this purpose it can be checked by means of Lemma 5.6 and (5.122) that

$$\left|\int_{\delta}^{\kappa^{'}-\epsilon(t)} d\kappa\partial_\kappa\left(G(\kappa,t)\right)e^{i(\omega-\omega^{'})t}\right|$$
$$\leq 2Mm^2(\kappa^{'}-\delta-\epsilon(t))\left(C_1\frac{\log|t|}{|t|}+C_2\frac{\log^3|t|}{|t|}\right)$$

with positive constants C_1 and C_2. Hence, we conclude that

$$\lim_{t\to\pm\infty}\int_{\delta}^{\kappa'-\epsilon(t)} d\kappa\partial_\kappa\left(G(\kappa,t)\right)e^{i(\omega-\omega')t}=0. \tag{5.124}$$

Putting together (5.121), (5.123) and (5.124) implies that

$$\lim_{t\to\pm\infty}\int_{\delta}^{\kappa'-\epsilon(t)} d\kappa\frac{f(\kappa)}{(\kappa'-\kappa)^{i\alpha'}}\frac{e^{i\delta(t)}-e^{i\delta_1'(t)}}{\kappa-\kappa'}e^{i(\omega-\omega')t}=0.$$

Similarly it can be shown that

$$\lim_{t\to\pm\infty}\int_{\kappa'+\epsilon(t)}^{\infty} d\kappa\frac{f(\kappa)}{(\kappa'-\kappa)^{i\alpha'}}\frac{e^{i\delta(t)}-e^{i\delta_1'(t)}}{\kappa-\kappa'}e^{i(\omega-\omega')t}=0.$$

Concerning the interval $[\kappa'-\epsilon(t),\kappa'+\epsilon(t)]$, by employing Lemma 5.6 we find that

$$\left|\lim_{t\to\pm\infty}\int_{\kappa'-\epsilon(t)}^{\kappa'+\epsilon(t)} d\kappa\frac{f(\kappa)}{(\kappa'-\kappa)^{i\alpha'}}\frac{e^{i\delta(t)}-e^{i\delta_1'(t)}}{\kappa-\kappa'}e^{i(\omega-\omega')t}\right|\leq$$

$$\frac{4Mm^2}{(\kappa'-\epsilon)^2\log|t|}\sup_{\kappa\in[\kappa'-\epsilon,\kappa']}|f|+\frac{4Mm^2}{\kappa'^2\log|t|}\sup_{\kappa\in[\kappa',\kappa'+\epsilon]}|f|.$$

Clearly, the above integral can be made arbitrarily small by choosing $|t|$ sufficiently large. Therefore, we have proved that

$$\lim_{t\to\pm\infty}\int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa\frac{f(\kappa)}{(\kappa'-\kappa)^{i\alpha'}}\frac{e^{i\delta(t)}-e^{i\delta_1'(t)}}{\kappa-\kappa'}e^{i(\omega-\omega')t}=0. \tag{5.125}$$

Let us analyze the second integral on the r.h.s. of (5.117). We can again apply dominated convergence theorem to take the limit for $\sigma\to 0^+$ inside the integral, which simplifies to

$$\lim_{t\to\pm\infty}\int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa\hat{f}_{(t)}(\kappa)e^{\pm i(\kappa'-\kappa)u_1}M(1,2+i\alpha';\mp iu_1(\kappa'-\kappa))e^{i(\omega-\omega')t}. \tag{5.126}$$

By means of (5.119) (5.126) splits into two integrals, namely

$$\lim_{t\to\pm\infty}\int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa\frac{\hat{f}_{(t)}(\kappa)}{1+i\frac{\kappa}{\omega}t}e^{\pm i(\kappa'-\kappa)u_1}M(1,2+i\alpha';\mp iu_1(\kappa'-\kappa))e^{i(\omega-\omega')t}+$$

$$\lim_{t\to\pm\infty}\int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa\frac{\hat{f}_{(t)}(\kappa)}{1+i\frac{\kappa}{\omega}t}e^{\pm i(\kappa'-\kappa)u_1}M(1,2+i\alpha';\mp iu_1(\kappa'-\kappa))\partial_\kappa e^{i(\omega-\omega')t}$$

Concerning the first one, notice that we already showed in Lemma 5.1 that the Kummer function of the first kind $M(1, 2+i\alpha^{'}; \tilde{z})$ is bounded. Hence we can use dominated convergence theorem to show that such integral is zero. Regarding the second integral, we integrate by parts to obtain

$$\lim_{t\to\pm\infty} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa \frac{d}{d\kappa}\left(\frac{\hat{f}_{(t)}(\kappa)}{1+i\frac{\kappa}{\omega}t} e^{\pm i(\kappa^{'}-\kappa)u_1} M(1, 2+i\alpha^{'}; \mp iu_1(\kappa^{'}-\kappa))\right) \cdot \\ \cdot e^{i(\omega-\omega^{'})t}. \quad (5.127)$$

We do not get any boundary term at infinity since $f(\kappa)$ has at least quadratic decay for $|\kappa| \to \infty$ and on the boundary of $B_\delta(0)$ since the term $(1+i\frac{\kappa}{\omega}t)^{-1}$ decays for $t \to \pm\infty$. Moreover, according to 13.4.12 in [37]

$$\frac{dM(1, 2+i\alpha^{'}; z)}{dz} = M(1, 2+i\alpha^{'}; z) - \frac{1+i\alpha^{'}}{2+i\alpha^{'}} M(1, 3+i\alpha^{'}; z)$$

with $z = \mp iu_1(\kappa^{'} - \kappa)$ is bounded, since $M(1, 2+i\alpha^{'}; z)$ is bounded (see Lemma 5.1) and $M(1, 3+i\alpha^{'}; z)$ satisfies the condition $\Re b > \Re a > 0$ with $a = 1$ and $b = 3+i\alpha^{'}$, which yields with the help of the integral representation 13.2.1 in [37] to the estimate

$$\left|M(1, 3+i\alpha^{'}; z)\right| \leq \sqrt{1+\left(\frac{\alpha^{'}}{2}\right)^2},$$

Hence, by exploiting the derivative respect to κ in (5.127) and employing dominated convergence theorem we find that (5.127) is zero in the limit $t \to \pm\infty$. Hence, we can conclude that (5.126) gives no contribution. Putting together (5.126) and (5.125) completes the proof. □

Theorem 5.8. *Let $f \in C^1(\mathbb{R})$ satisfy (5.41), then it holds for every given $\omega^{'} \in \Omega_1^{'}$*

$$\int_{u_1}^{\infty} du\, u^{i\alpha^{'}} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i\delta(t)} e^{i(\omega-\omega^{'})t+i(\kappa^{'}-\kappa)u} = \begin{cases} 2\pi f(\kappa^{'}) & \text{if } t\to+\infty \\ 0 & \text{if } t\to-\infty \end{cases},$$

$$\int_{u_1}^{\infty} du\, u^{-i\alpha^{'}} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i\delta(t)} e^{i(\omega-\omega^{'})t-i(\kappa^{'}-\kappa)u} = \begin{cases} 0 & \text{if } t\to+\infty \\ 2\pi f(\kappa^{'}) & \text{if } t\to-\infty \end{cases},$$

$$\int_{u_1}^{\infty} du\, u^{-i\alpha^{'}} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i\delta(t)} e^{i(\omega-\omega^{'})t-i(\kappa^{'}+\kappa)u} = \begin{cases} 2\pi f(-\kappa^{'}) & \text{if } t\to+\infty \\ 0 & \text{if } t\to-\infty \end{cases},$$

$$\int_{u_1}^{\infty} du\, u^{i\alpha^{'}} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i\delta(t)} e^{i(\omega-\omega^{'})t+i(\kappa^{'}+\kappa)u} = \begin{cases} 0 & \text{if } t\to+\infty \\ 2\pi f(-\kappa^{'}) & \text{if } t\to-\infty \end{cases}$$

with u_1, $\alpha^{'}$ and $\delta(t)$ defined like in Theorem 5.7.

Proof. We give here a proof of the first result, the others being similar. By subtracting and adding to $e^{i\delta(t)}$ the term $e^{i\delta_1^{'}(t)}$ with $\delta_1^{'}(t)$ already defined in Theorem 5.7 we obtain

$$\int_{u_1}^{\infty} du\ u^{i\alpha^{'}} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i\delta(t)} e^{i(\omega-\omega^{'})t+i(\kappa^{'}-\kappa)u}$$
$$= \int_{u_1}^{\infty} du\ u^{i\alpha^{'}} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) \left(e^{i\delta(t)} - e^{i\delta_1^{'}(t)}\right) e^{i(\omega-\omega^{'})t+i(\kappa^{'}-\kappa)u} +$$
$$+ e^{i\delta_1^{'}(t)} \int_{u_1}^{\infty} du\ u^{i\alpha^{'}} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i(\omega-\omega^{'})t+i(\kappa^{'}-\kappa)u}.$$

Theorem 5.7 implies that the first term on the r.h.s. of the above expression does not give any contribution in the limit for $t \to \pm\infty$. Application of Lemma 5.2 to the remaining term completes the proof. Notice that in order to recover the last two results it is necessary to add and subtract a term $e^{i\delta_2^{'}(t)}$ as given in Theorem 5.7. □

In the following we prove that (5.102) is zero in the limit $t \to \pm\infty$.

Theorem 5.9. *For every function $f \in C^1(\mathbb{R})$ satisfying (5.41) and for every given $\omega^{'} \in \Omega_1^{'}$ it holds*

$$\hat{\mathfrak{S}}_1 := \lim_{t\to\pm\infty} \int_{u_1}^{\infty} du\ u^{\pm i\alpha^{'}-1} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i\delta(t)} e^{i\varphi_{(t)}^{(1)}(\kappa,u)} = 0, \tag{5.128}$$

$$\hat{\mathfrak{S}}_2 := \lim_{t\to\pm\infty} \int_{u_1}^{\infty} du\ u^{\pm i\alpha^{'}-1} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i\delta(t)} e^{i\varphi_{(t)}^{(2)}(\kappa,u)} = 0 \tag{5.129}$$

with $\alpha^{'}$, $u_1 > 0$, $\delta(t)$, $\varphi_{(t)}^{(1)}(\kappa,u)$ and $\varphi_{(t)}^{(2)}(\kappa,u)$ like in Theorem 5.7.

Proof. Here we show (5.128) for $u^{i\alpha^{'}-1}$; in the other cases the proof is similar. Let us first suppose that $\alpha^{'} \neq 0$. Proceeding like in Lemma 5.3, we obtain

$$\hat{\mathfrak{S}}_1 := i\frac{u_1^{i\alpha^{'}}}{\alpha^{'}} \lim_{t\to\pm\infty} \lim_{\sigma\to 0^+} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i\delta(t)} M(1, 1+i\alpha^{'}; \tilde{z}) e^{i(\omega-\omega^{'})t - \tilde{z}(\kappa)} +$$
$$+ u_1^{i\alpha^{'}} \Gamma(i\alpha^{'}) \lim_{t\to\pm\infty} \lim_{\sigma\to 0^+} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa \frac{f(\kappa) e^{i\delta(t)}}{\tilde{z}^{i\alpha^{'}}(\kappa)} e^{i(\omega-\omega^{'})t} \tag{5.130}$$

with $\tilde{z}(\kappa)$ defined by (5.45). Let us consider the first term in (5.130). Notice that $\left|e^{i\delta(t)}\right| = 1$. Again dominated convergence theorem can be applied to take the limit $\sigma \to 0^+$ inside the integral sign. Hence, we have

$$\lim_{t\to\pm\infty} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f_1(\kappa) e^{i\delta(t)} M(1, 1+i\alpha^{'}; \tilde{z}) e^{i(\omega-\omega^{'})t} \tag{5.131}$$

with $f_1(\kappa) := f(\kappa) e^{\pm i(\kappa^{'}-\kappa)u_1}$. Notice that $M(1, 1+i\alpha^{'}; \tilde{z})$ together with its derivative respect to κ is bounded. Since the above limit integral is similar to (5.126), by proceeding like we did for (5.126), we can conclude that (5.131) is zero in the limit for $t \to \pm\infty$. Concerning the second term in (5.130), we can again apply dominated convergence theorem to get rid of the limit as $\sigma \to 0^+$ and we find that

$$\hat{\mathfrak{S}}_1 = (\mp u_1)^{i\alpha^{'}} \Gamma(i\alpha^{'}) \lim_{t\to\pm\infty} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa \frac{f(\kappa) e^{i\delta(t)}}{(\kappa^{'}-\kappa)^{i\alpha^{'}}} e^{i(\omega-\omega^{'})t}.$$

Adding and subtracting the term $e^{i\delta_1^{'}(t)}$ to $e^{i\delta(t)}$ with $\delta_1^{'}(t)$ like in Theorem 5.7, we get

$$\begin{aligned}\int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa \frac{f(\kappa) e^{i\delta(t)}}{(\kappa^{'}-\kappa)^{i\alpha^{'}}} e^{i(\omega-\omega^{'})t} = &\int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa \frac{f(\kappa)\left(e^{i\delta(t)} - e^{i\delta_1^{'}(t)}\right)}{(\kappa^{'}-\kappa)^{i\alpha^{'}}} e^{i(\omega-\omega^{'})t} \\ &+ e^{i\delta_1^{'}(t)} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa \frac{f(\kappa)}{(\kappa^{'}-\kappa)^{i\alpha^{'}}} e^{i(\omega-\omega^{'})t}.\end{aligned}$$

Concerning the second term, Riemann-Lebesgue lemma implies that it is zero for $t \to \pm\infty$. Regarding the first one we can use (5.119), Lemma 5.6 and then apply dominated convergence theorem in order to show that no contribution comes from it for $t \to \pm\infty$.
Let be now $\alpha^{'} = 0$ and for simplicity we consider the case with positive sign in $\varphi_{(t)}^{(1)}(\kappa, u)$. Proceeding like in Lemma 5.3 we obtain

$$\hat{\mathfrak{S}}_1 = \lim_{t\to\pm\infty} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i\delta(t)} E_1(iu_1(\kappa-\kappa^{'})) e^{i(\omega-\omega^{'})t}. \tag{5.132}$$

Adding and subtracting the term $e^{i\delta_1^{'}(t)}$ to $e^{i\delta(t)}$, applying Lemma 5.3 and defining the function $F(\kappa) := f(\kappa) E_1(iu_1(\kappa-\kappa^{'}))$ (5.132) becomes

$$\hat{\mathfrak{S}}_1 = \lim_{t\to\pm\infty} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa F(\kappa) \left(e^{i\delta(t)} - e^{i\delta_1^{'}(t)}\right) e^{i(\omega-\omega^{'})t}.$$

Notice that since the exponential integral has only a logarithmic divergence at $\kappa = \kappa^{'}$, it results that $F(\kappa)$ is there integrable. Since for $z \in \mathbb{C}$ it holds (see 5.1.23 and 5.1.26 [37])

$$\frac{dE_1(z)}{dz} = -E_0(z) = -\frac{e^{-z}}{z},$$

using (5.119) together with Lemma 5.6 and dominated convergence theorem we can conclude that the above limit integral is zero. □

By means of the next result we are able to compute (5.103).

Lemma 5.10. *For every function* $f \in C^1(\mathbb{R})$*, sharing the property* (5.41) *and for every given* $\omega' \in \Omega_1'$ *it holds*

$$\lim_{t\to\pm\infty} \int_{u_1}^{\infty} du\, u^{\pm i\alpha'} h(u) \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i\delta(t)} e^{i(\omega-\omega')t \pm i(\kappa'-\kappa)u} = 0, \quad (5.133)$$

$$\lim_{t\to\pm\infty} \int_{u_1}^{\infty} du\, u^{\pm i\alpha'} h(u) \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i\delta(t)} e^{i(\omega-\omega')t \pm i(\kappa'+\kappa)u} = 0, \quad (5.134)$$

with $\alpha' \in \mathbb{R}$*,* $u_1 > 0$*,* $h(u) = \mathcal{O}(u^{-2})$ *and the phase* $\delta(t)$ *as defined in Theorem5.7.*

Proof. Here we show (5.133) for $u^{i\alpha'}$; in the other cases the proof is similar. Due to the presence of the function $h(u) = \mathcal{O}(u^{-2})$ we can immediately apply Fubini theorem, interchange the order of integration and perform the integration over the variable u. Hence (5.133) becomes

$$\lim_{t\to\pm\infty} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i\delta(t)} \widehat{N}(\kappa)\, e^{i(\omega-\omega')t} \quad (5.135)$$

with

$$\widehat{N}(\kappa) := \int_{u_1}^{\infty} du\, u^{\pm i\alpha'} h(u) e^{\pm i(\kappa'-\kappa)u}.$$

Notice that

$$\left|\widehat{N}(\kappa)\right| \leq \int_{u_1}^{\infty} \frac{du}{u^2} = \frac{1}{u_1}.$$

By adding and subtracting a term $e^{i\delta_1'(t)}$ as defined in Theorem 5.7 to $e^{i\delta(t)}$ and applying Lemma 5.4, (5.137) reduces to

$$\lim_{t\to\pm\infty} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa \hat{f}_{(t)}(\kappa) \widehat{N}(\kappa)\, e^{i(\omega-\omega')t}$$

with $\hat{f}_{(t)}(\kappa) := f(\kappa)(e^{i\delta(t)} - e^{i\delta_1(t)})$. Since the above integrand is of the same kind as that in (5.126), we can proceed as we did in Theorem 5.7 and conclude that

$$\lim_{t\to\pm\infty} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa \hat{f}_{(t)}(\kappa) \widehat{N}(\kappa)\, e^{i(\omega-\omega')t} = 0.$$

□

The next theorem shows that no contribution comes from the limit integrals (5.104), (5.105) and (5.106).

Theorem 5.11. *For every* $\hat{\Psi}_0 \in C_c^\infty$ *the following identity holds*

$$s - \lim_{t\to\pm\infty} e^{-iHt}(\mathcal{I}_\infty - \mathcal{I}_\infty^{'})e^{iH_\infty t}e^{i\delta(t)}\hat{\Psi}_0 = 0 \tag{5.136}$$

with $\mathcal{I}_\infty$ *identifying operator as defined in* (3.5) *and* $\delta(t)$ *given by* (5.111).

Proof. We denote the cut-off function corresponding to $\mathcal{I}_\infty^{'}$ by $\chi_\infty^{'}$, i.e.

$$\chi_\infty^{'} = \begin{cases} 1 & \text{if } u > u_3 \\ 0 & \text{if } u < u_2 \\ 0 \leq \chi_\infty^{'} \leq 1 & \text{otherwise} \end{cases} .$$

Then $\mathcal{I}_\infty - \mathcal{I}_\infty^{'}$ defines a new identifying operator with cut-off function χ having compact support. Let us set $K := \operatorname{supp}\chi$. We show now that

$$\lim_{t\to\pm\infty} \int_K du\, u^{\pm i\alpha^{'}} \widetilde{Z}(u) \int_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa)(e^{i\delta(t)} - e^{i\delta_1^{'}(t)})e^{i(\omega-\omega^{'})t\pm i(\kappa^{'}-\kappa)u} = 0$$

for $\delta_1^{'}(t)$ as defined in Theorem 5.7 and

$$\widetilde{Z}(u) := \begin{Bmatrix} \widetilde{Z}_1(u) \\ \widetilde{Z}_2(u) \\ \widetilde{Z}_3(u) \end{Bmatrix} := \begin{Bmatrix} 1 \\ u^{-1} \\ h(u) \end{Bmatrix} \chi(u), \quad h(u) := \mathcal{O}\left(\frac{1}{u^2}\right).$$

Let us apply Fubini theorem to interchange the order of integration. Then performing the integration over the variable u, we obtain

$$\lim_{t\to\pm\infty} \int_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa)e^{i\delta(t)}Q(\kappa)\, e^{i(\omega-\omega^{'})t} \tag{5.137}$$

with

$$Q(\kappa) := \int_{u_1}^{\infty} du\, u^{\pm i\alpha^{'}} \widetilde{Z}(u)e^{\pm i(\kappa^{'}-\kappa)u}.$$

A simple calculation shows that we can always find a positive real constant C such that

$$|Q(\kappa)| \leq C.$$

By adding and subtracting a term $e^{i\delta_1^{'}(t)}$ as defined in Theorem 5.7 to $e^{i\delta(t)}$ and applying Lemma 5.4, (5.137) reduces to

$$\lim_{t\to\pm\infty} \int_{\mathbb{R}\setminus B_\delta(0)} d\kappa \hat{f}_{(t)}(\kappa)Q(\kappa)\, e^{i(\omega-\omega^{'})t}$$

with $\hat{f}_{(t)}(\kappa) := f(\kappa)(e^{i\delta(t)} - e^{i\delta_1(t)})$. Since the above integrand is of the same kind as that in (5.126), we can proceed as we did in Theorem 5.7 and conclude that

$$\lim_{t\to\pm\infty} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa \hat{f}_{(t)}(\kappa) Q(\kappa)\ e^{i(\omega-\omega^{'})t} = 0.$$

This implies that

$$\begin{aligned}&\lim_{t\to\pm\infty} e^{-iHt}(\mathcal{I}_\infty - \mathcal{I}_\infty^{'}) e^{iH_\infty t} e^{i\delta(t)} \hat{\Psi}_0 \\ &= \lim_{t\to\pm\infty} e^{-iHt}(\mathcal{I}_\infty - \mathcal{I}_\infty^{'}) e^{iH_\infty t} e^{i\delta_1^{'}(t)} \hat{\Psi}_0 = \lim_{t\to\pm\infty} e^{-iHt+i\delta_1^{'}(t)}(\mathcal{I}_\infty - \mathcal{I}_\infty^{'}) e^{iH_\infty t} \hat{\Psi}_0.\end{aligned} \tag{5.138}$$

Since the operator $e^{-iHt+i\delta_1^{'}(t)}$ is unitary, we can now apply Lemma 3.1 to (5.138) and conclude that

$$\lim_{t\to\pm\infty} e^{-iHt+i\delta_1^{'}(t)}(\mathcal{I}_\infty - \mathcal{I}_\infty^{'}) e^{iH_\infty t} \hat{\Psi}_0 = 0.$$

□

Theorem 5.12. *For every function $f \in C^1(\mathbb{R})$ satisfying the bound*(5.41) *it holds*

$$\lim_{t\to\pm\infty} \int_{u_1}^{\infty} du\ e^{-\beta u} u^{\tilde{\alpha}} Z(u) \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i\delta(t)} e^{i(\omega-\omega^{'})t \pm i\kappa u} = 0 \tag{5.139}$$

for given $\omega^{'} \in \Omega_2^{'}$ and positive real β.

Proof. Due to the presence of the term $e^{-\beta u}$ we can use Fubini theorem, interchange the order of integration and finally perform the integration over the variable u. Hence we obtain

$$\lim_{t\to\pm\infty} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) W(\kappa)\ e^{i\delta(t)} e^{i(\omega-\omega^{'})t} \tag{5.140}$$

with

$$W(\kappa) := \int_{u_1}^{\infty} du\ e^{-\beta u} u^{\tilde{\alpha}} Z(u) e^{\pm i\kappa u}.$$

Using (5.99) in Lemma 5.5 we always find a positive real constant C such that

$$|W(\kappa)| \leq C.$$

By adding and subtracting a term $e^{i\delta_1'(t)}$ as defined in Theorem 5.7 to $e^{i\delta(t)}$ and applying Lemma 5.4, (5.140) reduces to

$$\lim_{t\to\pm\infty} \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa \hat{f}_{(t)}(\kappa) W(\kappa)\, e^{i(\omega-\omega')t}$$

with $\hat{f}_{(t)}(\kappa) := f(\kappa)(e^{i\delta(t)} - e^{i\delta_1(t)})$. Since the above integrand is of the same kind as that in (5.126), we can proceed as we did in Theorem 5.7 and conclude that

$$\lim_{t\to\pm\infty} \int_{u_1}^{\infty} du\, e^{-\beta u} u^{\tilde{\alpha}} Z(u) \int\limits_{\mathbb{R}\setminus B_\delta(0)} d\kappa f(\kappa) e^{i\delta(t)} e^{i(\omega-\omega')t \pm i\kappa u} = 0.$$

□

5.5 Final Theorem for $\widetilde{W}^{\pm}_{\infty}$.

For ease of notation in the expressions for the wave operators we omit to write explicitly the sums over the indices $a', b' = 1, 2$.

Theorem 5.13. *Let $\widetilde{W}^{\pm}_{\infty}$ be given as in (3.7) with phase shift operator $e^{i\delta(t)}$ specified by*

$$\delta(t) := -\mathrm{sgn}(t)\alpha(\omega)\log(\frac{\kappa}{\omega}|t|) \quad \textit{with} \quad \alpha(\omega) := \epsilon(\omega)\frac{\omega e Q + M m^2}{\sqrt{\omega^2 - m^2}},$$

where k is defined according to $k := \epsilon(\omega)\sqrt{\omega^2 + m^2}$,

$$\mathrm{sgn}(t) := \begin{cases} +1 & \textit{if } t > 0 \\ -1 & \textit{if } t < 0 \end{cases} \quad \textit{and} \quad \epsilon(\omega) := \begin{cases} +1 & \textit{if } \omega > m \\ -1 & \textit{if } \omega < -m \end{cases}$$

Then for every $\hat{\Psi}_0^{(\infty)} \in C_c^\infty$,

$$\left(\widetilde{W}^{+}_{\infty}\hat{\Psi}_0^{(\infty)}\right)_{(x)} = \sum_{k,n\in\mathbb{Z}} \int\limits_{\sigma(H_\infty)} d\omega\, t^{kn}_{a'b'}(\omega) \hat{\Psi}^{k\omega n}_{a'}(x) \widehat{f}_{kn}(\omega) f^{+}_{\infty} \overline{f}^{+}_{\infty,b'},$$

$$\left(\widetilde{W}^{-}_{\infty}\hat{\Psi}_0^{(\infty)}\right)_{(x)} = \sum_{k,n\in\mathbb{Z}} \int\limits_{\sigma(H_\infty)} d\omega\, t^{kn}_{a'b'}(\omega) \hat{\Psi}^{k\omega n}_{a'}(x) \widehat{f}_{kn}(\omega) f^{-}_{\infty} \overline{f}^{-}_{\infty,b'},$$

where $\sigma(H_\infty)$ denotes the spectrum of the free Dirac operator in the Minkowski metric. The functions $f^{\pm}_{\infty,b'}$ and $f^{\pm}_{\infty}$ corresponding to the fundamental solutions $\hat{\Psi}^{k\omega n}_{b'}$ and $\hat{\Psi}^{\infty}_{k\omega n}$, respectively, are the transmission coefficients, $t^{kn}_{a'b'}$ are given by (2.49) and the function $\widehat{f}_{kn}(\omega)$ is defined by

$$\widehat{f}_{kn}(\omega) = 8\pi t^{\infty}_{k\omega n} \int_0^\infty du \int_{-1}^{1} d(\cos\theta) < X^{\infty}_{k\omega n}(u) Y^{k\omega n}(\theta) | \hat{\Psi}_0^{k,(\infty)}(u,\theta) >_\infty$$

with the inner product

$$< X^{\infty}_{k\omega n}(u) Y^{k\omega n}(\theta) | \hat{\Psi}_0^{k,(\infty)}(u,\theta) >_{\infty}$$
$$= \overline{X}^{\infty}_{k\omega n}(u) \overline{Y}^{k\omega n}(\theta) \left[\mathbb{1} + \frac{a \sin\theta}{\sqrt{r^2+a^2}} \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \right] \hat{\Psi}_0^{k,(\infty)}(u,\theta).$$

Proof. For ease of notation we omit to write the superscript ∞ attached to the initial data $\hat{\Psi}_0^{(\infty)}$. Employing the integral representation for the propagator of the Dirac equation in the Kerr-Newman geometry (see Theorem 2.7), the Dollard-modified wave operator $\widetilde{W}^{\pm}_{\infty}$ can be written as

$$\left(\widetilde{W}^{\pm}_{\infty} \hat{\Psi}_0 \right)_{(x)} = \frac{1}{\pi^2} \lim_{t \to \pm\infty} \sum_{k^{'},n^{'},k,n} \int_{\mathbb{R}} d\omega^{'} \int_{\mathbb{R}} du \int_{\sigma(H_\infty)} d\omega \; e^{i(\omega-\omega^{'})t} e^{i\delta(t)} P^{k^{'}n^{'}}_{kn}(\omega,\omega^{'},u) \quad (5.141)$$

with $k^{'}, n^{'}, k, n \in \mathbb{Z}$ and

$$P^{k^{'}n^{'}}_{kn}(\omega,\omega^{'},u) = \sum_{a^{'},b^{'}=1}^{2} t^{k^{'}\omega^{'}n^{'}}_{a^{'}b^{'}} t^{\infty}_{k\omega n} \hat{\Psi}^{k^{'}\omega^{'}n^{'}}_{a^{'}}(x)$$
$$< \hat{\Psi}^{k^{'}\omega^{'}n^{'}}_{b^{'}} | \chi_\infty \hat{\Psi}^{\infty}_{k\omega n} >_{(u)} \langle \hat{\Psi}^{\infty}_{k\omega n} | \hat{\Psi}_0 \rangle_{\infty}.$$

$\hat{\Psi}^{\infty}$ denotes the eigenfunctions of the Hamiltonian at infinity H_∞ and $\hat{\Psi}_0$ is some smooth initial data with compact support defined as follows

$$\hat{\Psi}_0(u,\theta,\varphi) = \sum_{\hat{k}\in\mathbb{Z}} \hat{\Psi}_0^{\hat{k}}(u,\theta) e^{-i\left(\hat{k}+\frac{1}{2}\right)\varphi}. \quad (5.142)$$

Analogously to the treatment of the classical wave operator at infinity, since the indexes ω and $\omega^{'}$ attached to the spinors and to the transmission coefficients can always be recovered by looking at the component indexes $a^{'}$, $b^{'}$, we can avoid to write them explicitly. For ease of notation, we will in what follows omit the indexes k, n and $n^{'}$. This causes no confusion, because in view of (5.144) the index k is the same in all factors, whereas the distinction between n and $n^{'}$ can be made by looking at the corresponding indices ∞ or $a^{'}$, $b^{'}$. Moreover, the expression of the wave operator can be further simplified by omitting the sums over k, n, $k^{'}$, $n^{'}$ and $a^{'}$, $b^{'}$. Hence, (5.141) becomes

$$\left(\widetilde{W}^{\pm}_{\infty} \hat{\Psi}_0 \right)_{(x)} = \frac{1}{\pi^2} \lim_{t \to \pm\infty} \int_{\mathbb{R}} d\omega^{'} \int_{\mathbb{R}} du \int_{\sigma(H_\infty)} d\omega \; e^{i(\omega-\omega^{'})t} \, e^{i\delta(t)} t_{a^{'}b^{'}} \hat{\Psi}_{a^{'}}(x)$$
$$< \hat{\Psi}_{b^{'}} | \chi_\infty \hat{\Psi}^{\infty} >_{(u)} \langle \hat{\Psi}^{\infty} | \hat{\Psi}_0 \rangle_{\infty}.$$

Since the matrix potential in (2.25) converges for $u \to +\infty$, we can pass to the limit $u \to +\infty$ in (2.25) and solve the correspondent system of ODEs together with the requirement that at infinity both outgoing plane waves and incoming ones are present. Thus, we have the following analytical expression for the radial 2-spinor $X^{\infty}(u)$

$$X^{\infty}(u) = \begin{pmatrix} \cosh\Theta & \sinh\Theta \\ \sinh\Theta & \cosh\Theta \end{pmatrix} \begin{pmatrix} e^{-i\Phi_{\infty}(u)} f_{\infty}^{+} \\ e^{i\Phi_{\infty}(u)} f_{\infty}^{-} \end{pmatrix}, \quad \Phi_{\infty}(u) := \kappa u \quad (5.143)$$

with $\kappa := \epsilon(\omega)\sqrt{\omega^2 - m^2}$, Θ as in (2.43) and $\epsilon(\omega)$ as defined in Lemma 2.3. Concerning the angular components of the spinor at infinity, we observe that, as the angular system of ODEs (2.9) does not depend on u, at infinity the angular eigenfunctions $Y(\theta)$ are solutions of the Chandrasekhar-Page equation.
By employing (5.4) and integrating over the azimuthal variable φ, it can be checked that $< \hat{\Psi}_{b'} | \chi_{\infty} \hat{\Psi}^{\infty} >_{(u)}$ and $\langle \hat{\Psi}^{\infty} | \hat{\Psi}_0 \rangle_{\infty}$ are given respectively by

$$< \hat{\Psi}_{b'} | \chi_{\infty} \hat{\Psi}^{\infty} >_{(u)} = 2\pi \delta_{kk'} g_{b'}(\omega, \omega', u) \chi_{\infty}(u), \quad (5.144)$$
$$\langle \hat{\Psi}^{\infty} | \hat{\Psi}_0 \rangle_{\infty} = 2\pi f(\omega) \quad (5.145)$$

with

$$g_{b'}(\omega, \omega', u) := A(\omega, \omega') \overline{X}_{b'}(u) X^{\infty}(u) + \frac{a\sqrt{\Delta}}{r^2 + a^2} B(\omega, \omega') \overline{X}_{b'}(u) \sigma_2 X^{\infty}(u) \quad (5.146)$$

and

$$f(\omega) := t^{\infty}(\omega) \int_0^{\infty} du \int_{-1}^{1} d(\cos\theta) < X^{\infty}(u) Y(\theta) | \hat{\Psi}_0^k(u, \theta) >_{\infty} \quad (5.147)$$

with the inner product

$$< X^{\infty}(u) Y(\theta) | \hat{\Psi}_0^k(u, \theta) >_{\infty} = \overline{X}^{\infty}(u) \overline{Y}(\theta) \left[\mathbb{1} + \frac{a \sin\theta}{\sqrt{r^2 + a^2}} \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \right] \hat{\Psi}_0^k(u, \theta).$$

Here the functions $A(\omega, \omega')$ and $B(\omega, \omega')$ are defined as in (5.9) and (5.10), respectively. Moreover, since the angular functions $Y^{k\omega n}(\theta)$ depend smoothly on ω and ω' (see [32]) and the angular initial data are smooth functions with compact support, it results that (5.147), (5.9) and (5.10) will depend smoothly on ω and ω' too. Furthermore, since the angular components are normalized and square integrable, it can be easily shown by means of Hölder inequality that (5.9) and (5.10) are bounded with

$$|A(\omega, \omega')| \leq 1 \quad \text{and} \quad |B(\omega, \omega')| \leq 1. \quad (5.148)$$

By means of (5.144) and (5.145) and with our notation the wave operator can be written in the compact form

$$\left(\widetilde{W}_{\infty}^{\pm}\hat{\Psi}_0\right)_{(x)} = 4 \lim_{t\to\pm\infty} \int_{\mathbb{R}} d\omega' \; t_{a'b'}(\omega')\hat{\Psi}_{a'}(x) \int_{\mathbb{R}} du \; \chi_{\infty}(u) \int_{\sigma(H_{\infty})} d\omega \; e^{i(\omega-\omega')t} \, e^{i\delta(t)} g_{b'}(\omega,\omega',u) f(\omega). \quad (5.149)$$

Since the regions $\omega \approx \pm m$, $\omega' \approx \pm m$ and $|\omega'| > L$ can be controlled with Lemma 3.2 and 3.4, after application of Theorem 5.11 our task is to compute the expression

$$\int_{\Omega'} d\omega' \; t_{a'b'}(\omega')\hat{\Psi}_{a'}(x) \lim_{t\to\pm\infty} \int_{u_1}^{\infty} du \int_{\Omega} d\omega \; e^{i(\omega-\omega')t} \, e^{i\delta(t)} g_{b'}(\omega,\omega',u) f(\omega), \quad (5.150)$$

where

$$\begin{aligned} \Omega' &:= [-L,-m-\epsilon] \cup [-m+\epsilon, m-\epsilon] \cup [m+\epsilon, L], \\ \Omega &:= (-\infty,-m-\delta] \cup [m+\delta,+\infty) \end{aligned}$$

for constants ϵ, δ and $L > 0$. Without loss of generality we can now consider ω' as given. The next step is to analyze the integrands entering in (5.150). Looking back at (5.146), we observe that we have to deal with the following expressions

$$\overline{X}_{b'}(u) X^{\infty}(u) f(\omega) \quad (5.151)$$

and

$$\frac{a\sqrt{\Delta}}{r^2+a^2}\overline{X}_{b'}(u)\sigma_2 X^{\infty}(u) f(\omega). \quad (5.152)$$

Moreover, by choosing u_1 large enough, we can replace the radial 2-spinors $X_{b'}(u)$ appearing in (5.151) and (5.152) with the analytical expressions given in Lemma 2.2 and Lemma 2.3, respectively, which are smooth for every $\omega' \in \{\omega' \in \mathbb{R} : |\omega'| \geq m+\epsilon \text{ and } |\omega'| \leq m-\epsilon\}$. Furthermore, for u_1 large enough we have

$$\frac{a\sqrt{\Delta}}{r^2+a^2} = \frac{a}{u} + \mathcal{O}\left(\frac{1}{u^2}\right),$$

where r depends on the Regge-Wheeler variable u as defined in (2.11). Notice that the integral over ω' in (5.150) splits into two integrals over the regions

$$\begin{aligned} \Omega_1' : &= [-L,-m-\epsilon] \cup [m+\epsilon, L] \\ \Omega_2' : &= [-m+\epsilon, m-\epsilon]. \end{aligned}$$

Moreover, for $\omega' \in \Omega_2'$ according to Theorem2.7 we have to take care to choose $t_{a'b'} = \delta_{a'1}\delta_{b'1}$ in (5.150). Our initial task has now been reduced to the analysis of the integrands entering in the double integral over the spatial variable u and the frequency ω given by

$$\int_{u_1}^{\infty} du \int_{\Omega} d\omega\; e^{i(\omega-\omega')t}\; e^{i\delta(t)} g_{b'}(\omega,\omega',u) f(\omega) \tag{5.153}$$

in the limits $t \to \pm\infty$, when we treat the regions Ω_1' and Ω_2', respectively. Since the terms coming from (5.152) by substitution of (5.143) and application of Lemma 2.2 and Lemma 2.3 (depending whether $\omega' \in \Omega_2'$ or $\omega' \in \Omega_2'$) can all be controlled by means of Theorem 5.9, Lemma 5.10 and Theorem5.12, we can conclude that such terms do not give any contribution in the evaluation of the modified wave operator at infinity. Thus we may restrict our attention to the integrands arising from (5.151).
In the region Ω_1' such integrands can be evaluated by Theorem 5.8, 5.9 and Lemma 5.10. As a consequence it can be checked that the modified wave operators $\widetilde{W}^{\pm}_{\infty,\Omega_1'}$ simplify to

$$\left(\widetilde{W}^{+}_{\infty,\Omega_1}\hat{\Psi}_0\right)_{(x)} = \sum_{a',b'=1}^{2} \int_{\Omega_1} d\omega\; t_{a'b'}(\omega)\hat{\Psi}_{a'}(x)\widehat{f}(\omega) f^{+}_{\infty}\overline{f}^{+}_{\infty,b'}, \tag{5.154}$$

$$\left(\widetilde{W}^{-}_{\infty,\Omega_1}\hat{\Psi}_0\right)_{(x)} = \sum_{a',b'=1}^{2} \int_{\Omega_1} d\omega\; t_{a'b'}(\omega)\hat{\Psi}_{a'}(x)\widehat{f}(\omega) f^{-}_{\infty}\overline{f}^{-}_{\infty,b'} \tag{5.155}$$

with $\widehat{f}(\omega) := 8\pi f(\omega)$ where $f(\omega)$ is defined by (5.147). Conversely, in the region Ω_2' we can make use of Theorem 5.12 to get control of the integrals arising from (5.151). Hence we find that

$$\left(\widetilde{W}^{\pm}_{\infty,\Omega_2'}\hat{\Psi}_0\right)_{(x)} = 0. \tag{5.156}$$

By taking the limits $L \to \infty$ and $\epsilon \to 0^+$ the modified wave operators at infinity (5.154) and (5.155) are given by

$$\left(\widetilde{W}^{+}_{\infty}\hat{\Psi}_0\right)_{(x)} = \sum_{a',b'=1}^{2} \int_{\sigma(H_\infty)} d\omega\; t_{a'b'}(\omega)\hat{\Psi}_{a'}(x)\widehat{f}(\omega) f^{+}_{\infty}\overline{f}^{+}_{\infty,b'}, \tag{5.157}$$

$$\left(\widetilde{W}^{-}_{\infty}\hat{\Psi}_0\right)_{(x)} = \sum_{a',b'=1}^{2} \int_{\sigma(H_\infty)} d\omega\; t_{a'b'}(\omega)\hat{\Psi}_{a'}(x)\widehat{f}(\omega) f^{-}_{\infty}\overline{f}^{-}_{\infty,b'}., \tag{5.158}$$

where $\sigma(H_\infty)$ denotes the spectrum of the free Dirac operator in the Minkowski metric. Finally, notice that (5.157) and (5.158) do not contain anymore the logarithmic divergence in time. □

6 Wave operators at the event horizon.

6.1 Classical Wave Operators at the Event Horizon $W_0^{\pm}$.

Let H be the Hamiltonian as defined in (2.13). Moreover, let H_0 denote the Hamiltonian close to the event horizon and $\hat{\Psi}_0^{(0)}$ some initial data with compact support. Furthermore, let the operator H_0 act on the Hilbert space $\mathcal{H}_0$. Close to the event horizon we consider the classical wave operators in the future and past infinity

$$W_0^{\pm}\hat{\Psi}_0^{(0)} = s - \lim_{t\to\pm\infty} e^{-iHt}\mathcal{I}_0 e^{iH_0 t}\hat{\Psi}_0^{(0)}, \quad \hat{\Psi}_0^{(0)} \in \mathcal{H}_0 \tag{6.1}$$

in the Hilbert space $\mathcal{H} = L^2\left((r_1,\infty)\times S^2\right)^4$ together with a smooth bounded identifying operator $\mathcal{I}_0$ defined as follows

$$\mathcal{I}_0 : \begin{cases} \mathcal{H}_0 \to \mathcal{H} \\ \hat{\Psi} \to \chi_0\hat{\Psi} \end{cases} .$$

We introduce a cut-off function $\chi_0 \in C^{\infty}(\mathbb{R})$ such that

$$\chi_0 = \begin{cases} 1 & \text{if } u < \hat{u}_0 \\ 0 & \text{if } u > \hat{u}_1 \\ 0 \le \chi_0 \le 1 & \text{otherwise} \end{cases} \quad \text{with} \quad \hat{u}_0, \hat{u}_1 \in \mathbb{R} \quad \text{and} \quad \hat{u}_0 < \hat{u}_1 < 0, \tag{6.2}$$

since we want to compare H with H_0 in the neighborhood of the horizon. Moreover, let us denote the spectrum of H_0 with $\sigma(H_0) = \mathbb{R}$ and for ease of notation let us omit the superscript 0 attached to the initial data $\hat{\Psi}_0^{(0)}$. Our first task is to give an explicit integral representation for (6.1). This can be achieved with the help of the integral representation for the propagator of the Dirac equation in the Kerr-Newman geometry (see Theorem 2.7). Hence we find that (6.1) can be written as

$$\left(W_0^{\pm}\hat{\Psi}_0\right)_{(x)} = \frac{1}{\pi^2}\lim_{t\to\pm\infty}\sum_{k',n',k,n}\int_{\mathbb{R}} d\omega' \int_{\mathbb{R}} du \int_{\sigma(H_0)} d\omega e^{i(\omega-\omega')t}\widetilde{P}_{kn}^{k'n'}(\omega,\omega',u) \tag{6.3}$$

with $k', n', k, n \in \mathbb{Z}$ and

$$\widetilde{P}_{kn}^{k'n'}(\omega,\omega',u) = \sum_{a',b',a,b=1}^{2} t_{a'b'}^{k'\omega'n'} t_{ab}^{0,k\omega n}\hat{\Psi}_{a'}^{k'\omega'n'}(x)$$
$$< \hat{\Psi}_{b'}^{k'\omega'n'}|\chi_0\hat{\Psi}_a^{0,k\omega n} >_{(u)} \langle\hat{\Psi}_b^{0,k\omega n}|\hat{\Psi}_0\rangle_0,$$

where $\hat{\Psi}^0$ denotes the eigenfunctions H_0 and $\hat{\Psi}_0$ are some smooth initial data with compact support defined as follows

$$\hat{\Psi}_0(u,\theta,\varphi) = \sum_{\hat{k}\in\mathbb{Z}} \hat{\Psi}_0^{\hat{k}}(u,\theta) e^{-i\left(\hat{k}+\frac{1}{2}\right)\varphi}. \tag{6.4}$$

Notice that $\hat{\Psi}_0^{\hat{k}}(u,\theta)$ could also be defined analogously as in (2.6). Since the indexes ω and $\omega^{'}$ attached to the spinors and to the transmission coefficients can always be recovered by looking at the component indexes a, b, $a^{'}$, $b^{'}$, we can avoid to write them explicitly. For ease of notation, we will in what follows omit the indexes k, n and $n^{'}$. This causes no confusion, because in view of (6.9) the index k is the same in all factors, whereas the distinction between n and $n^{'}$ can be made by looking at the corresponding indices a, b or $a^{'}$, $b^{'}$.Moreover, the expression of the wave operator can be further simplified by omitting the sums over k, n, $k^{'}$, $n^{'}$ and a, b, $a^{'}$, $b^{'}$.
Since the matrix potential in (2.25) converges in the limit $u \to -\infty$, we can pass to the limit $r \to r_1$ in (2.17) and solve the correspondent system of ODEs together with the requirement that at the event horizon there are either only outgoing plane waves or incoming ones. Thus, we obtain the following analytical expression for $X_a^0(u)$

$$X_a^0(u) = \begin{pmatrix} e^{-i\Omega_0 u} f_{0,a}^{+} \\ e^{i\Omega_0 u} f_{0,a}^{-} \end{pmatrix}, \quad \Omega_0 := \omega + \omega_0, \quad \omega_0 := \frac{\left(k+\frac{1}{2}\right)a + eQr_1}{r_1^2+a^2} \tag{6.5}$$

with

$$f_{0,1}^{+} = 1 \qquad f_{0,1}^{-} = 0 \tag{6.6}$$
$$f_{0,2}^{+} = 0 \qquad f_{0,2}^{-} = 1. \tag{6.7}$$

Making use of the above choice together with (2.49) and (2.50) the transmission coefficients at the event horizon t_{ab}^0 are computed to be

$$t_{ab}^0 = \frac{1}{2}\delta_{ab}. \tag{6.8}$$

Notice that, since the angular system of ODEs (2.9) does not depend on the spatial variable u, at the event horizon the angular eigenfunctions $Y(\theta)$ are again solutions of the Chandrasekhar-Page equation.
Hence, (6.3) becomes

$$\left(W_0^{\pm}\hat{\Psi}_0\right)_{(x)} = \frac{1}{2\pi^2} \lim_{t\to\pm\infty} \int_{\mathbb{R}} d\omega^{'} \int_{\mathbb{R}} du \int_{\sigma(H_0)} d\omega e^{i(\omega-\omega^{'})t} t_{a^{'}b^{'}} \hat{\Psi}_{a^{'}}(x)$$
$$< \hat{\Psi}_{b^{'}} | \chi_0 \hat{\Psi}_a^0 >_{(u)} \langle \hat{\Psi}_a^0 | \hat{\Psi}_0 \rangle_0.$$

Employing (5.4) and integrating over the azimuthal variable φ, it can be verified that

$$< \hat{\Psi}_{b'} | \chi_0 \hat{\Psi}_a^0 >_{(u)} = 2\pi \delta_{kk'} \widetilde{g}_{b'a}(\omega, \omega', u) \chi_0(u), \tag{6.9}$$

$$\langle \hat{\Psi}_a^0 | \hat{\Psi}_0 \rangle_0 = 2\pi f_a^0(\omega) \tag{6.10}$$

with

$$\widetilde{g}_{b'a}(\omega, \omega', u) = A(\omega, \omega') \overline{X}_{b'}(u) X_a^0(u) + \frac{a\sqrt{\Delta}}{r^2 + a^2} B(\omega, \omega') \overline{X}_{b'}(u) \sigma_2 X_a^0(u) \tag{6.11}$$

and

$$f_a^0(\omega) = \int_{-\infty}^{\infty} du \int_{-1}^{1} d(\cos\theta) \ \overline{X}_a^0(u) \ \overline{Y}(\theta) \hat{\Psi}_0^k(u, \theta), \tag{6.12}$$

where $A(\omega, \omega')$ and $B(\omega, \omega')$ are given respectively by (5.9) and (5.10), depend smoothly on ω and ω' and are bounded according to (5.11). By means of (6.9) and (6.10) and with our notation the wave operator at the event horizon can be written in the compact form

$$\left(W_0^{\pm} \hat{\Psi}_0 \right)_{(x)} = 2 \lim_{t \to \pm\infty} \int_{\mathbb{R}} d\omega' t_{a'b'}(\omega') \hat{\Psi}_{a'}(x) \int_{\mathbb{R}} du \chi_0(u) \int_{\sigma(H_0)} d\omega e^{i(\omega - \omega')t} \widetilde{g}_{b'a}(\omega, \omega', u) f_a^0(\omega).$$

Since the regions $\omega \approx \pm m$, $\omega' \approx \pm m$ and $|\omega'| > L$ can be controlled with Lemma 3.1, 3.3 and 3.5, our task is to analyze the expression

$$\int_{\Omega'} d\omega' t_{a'b'}(\omega') \hat{\Psi}_{a'}(x) \lim_{t \to \pm\infty} \int_{-\infty}^{\hat{u}_0} du \int_{\widetilde{\Omega}} d\omega e^{i(\omega - \omega')t} \widetilde{g}_{b'a}(\omega, \omega', u) f_a^0(\omega) \tag{6.13}$$

where

$$\begin{aligned} \Omega' &:= [-L, -m - \epsilon] \cup [-m + \epsilon, m - \epsilon] \cup [m + \epsilon, L], \\ \widetilde{\Omega} &:= (-\infty, -m - \delta] \cup [-m + \delta, m - \delta] \cup [m + \delta, +\infty) \end{aligned}$$

for constants ϵ, δ and $L > 0$. Let us now analyze which kind of integrands enter in (6.13). Looking back at (6.11), we observe that we have to deal with the following expressions

$$\overline{X}_{b'}(u) X_a^0(u) f_a^0(\omega) \tag{6.14}$$

and

$$\frac{a\sqrt{\Delta}}{r^2 + a^2} \overline{X}_{b'}(u) \sigma_2 X_a^0(u) f_a^0(\omega). \tag{6.15}$$

It is interesting to observe that at the event horizon (6.14) and (6.15) give rise to integrands with different phases. By choosing $|\hat{u}_0|$ large enough, we can replace $X_{b'}(u)$ entering in (6.14) and (6.15) with the analytical expressions derived in Lemma 2.1. In particular for (6.15) employing (6.5) and Lemma 2.1 a direct computation shows that all integrands have the following phase

$$\widetilde{\varphi}^{\pm}_{(t)}(\omega, u) = (\Omega_0(\omega) - \Omega_0^{'})t \pm (\Omega_0(\omega) + \Omega_0^{'})u \tag{6.16}$$

with $\Omega_0(\omega)$ like in (6.5) and $\Omega_0^{'} = \omega^{'} + \omega_0$, whereas for (6.15) we get a phase

$$\widehat{\varphi}^{\pm}_{(t)}(\omega, u) = (\omega - \omega^{'})(t \pm u). \tag{6.17}$$

The next step is to analyze the integrands entering in the double integral over the spatial variable u and the frequency ω given by

$$\int_{-\infty}^{\hat{u}_0} du \int_{\widetilde{\Omega}} d\omega\; e^{i(\omega-\omega^{'})t}\widetilde{g}_{b'\,a}(\omega, \omega^{'}, u) f_a^0(\omega), \tag{6.18}$$

in the limits $t \to \pm\infty$, when we treat the regions $\Omega_1^{'}$ and $\Omega_2^{'}$, respectively.

6.1.1 The Region $\Omega_1^{'}$.

Let $\omega^{'} \in \Omega_1^{'}$ be fixed. With the help of Lemma 2.1 and (6.5) the product $\overline{X}_{b'}(u)X_a^0(u)$ can be written as

$$\overline{X}_{b'}(u)X_a^0(u) = \Big[f^{+}_{0,a}\left(\overline{f}^{+}_{0,b'} + \overline{R}^{+}_{0,b'}\right) e^{i(\omega^{'}-\omega)u} + f^{-}_{0,a}\left(\overline{f}^{-}_{0,b'} + \overline{R}^{-}_{0,b'}\right) e^{-i(\omega^{'}-\omega)u}\Big]. \tag{6.19}$$

Hence all terms arising from (6.19) are of the form

$$f^{\pm}_{0,a} f_a^0(\omega)\left(\overline{f}^{\pm}_{0,b'} + \overline{R}^{\pm}_{0,b'}\right) e^{\pm i(\omega^{'}-\omega)u}. \tag{6.20}$$

Going back to (6.11), we observe that (6.20) will be multiplied by the smooth and bounded function $A(\omega, \omega^{'})$ defined by (5.9). Moreover, since we can always choose $|\hat{u}_0|$ large enough, according to Lemma 2.1 the following inequality holds for the rest functions $R^{\pm}_{0,b'}(u)$

$$\left|R^{\pm}_{0,b'}(u)\right| \le c e^{du} \tag{6.21}$$

with c and d positive constants. Hence, in the computation of the wave operator (6.13) all integrands arising from the first term in (6.11) are of the type

$$T_0(u, \omega) := f^{\pm}_{0,a} A(\omega) f_a^0(\omega)\left(\overline{f}^{\pm}_{0,b'} + \overline{R}^{\pm}_{0,b'}(u)\right) e^{\pm i(\omega^{'}-\omega)u}. \tag{6.22}$$

Furthermore, since we considered smooth initial data with compact support, the function $f_a^0(\omega)$ is smooth and has rapid decay in ω. Thus, also in this case it is reasonable to replace the term $f_{0,a}^{\pm} A(\omega) f_a^0(\omega)$ in (6.22) with some function $F_0(\omega)$ in C^1 such that $F_0(\omega)$ and its derivative decay at least quadratically as $\omega \to \infty$. Our task is now to compute the following limit integral

$$\lim_{t\to\pm\infty} \int_{-\infty}^{\hat{u}_0} du \int_{\widetilde{\Omega}} d\omega e^{i(\omega-\omega')t} T_0(u,\omega)$$

with $T_0(u,\omega)$ given by

$$T_0(u,\omega) = F_0(\omega)\left(\overline{f}_{0,b'}^{\pm} + \overline{R}_{0,b'}^{\pm}(u)\right) e^{\pm i(\omega'-\omega)u}.$$

Finally, we end up with the evaluation of the following limit integrals

$$\mathfrak{M}_1 = \lim_{t\to\pm\infty} \int_{-\infty}^{\hat{u}_0} du \int_{\widetilde{\Omega}} d\omega\ F_0(\omega)\ e^{i(\omega-\omega')(t\pm u)} \tag{6.23}$$

$$\mathfrak{M}_2 = \lim_{t\to\pm\infty} \int_{-\infty}^{\hat{u}_0} du\ \overline{R}_{0,b'}^{\pm}(u) \int_{\widetilde{\Omega}} d\omega\ F_0(\omega)\ e^{i(\omega-\omega')(t\pm u)}, \tag{6.24}$$

where we set

$$F_0(\omega) := f_{0,a}^{\pm} A(\omega) f_a^0(\omega). \tag{6.25}$$

Concerning (5.14), a similar analysis shows that we have to compute the limit integrals

$$\mathfrak{M}_3 = \lim_{t\to\pm\infty} \int_{-\infty}^{\hat{u}_0} du\ \frac{a\sqrt{\Delta}}{r^2+a^2} \int_{\widetilde{\Omega}} d\omega\ F_1(\omega)\ e^{i\widetilde{\varphi}_{(t)}^{\pm}(\omega,u)} \tag{6.26}$$

$$\mathfrak{M}_4 = \lim_{t\to\pm\infty} \int_{-\infty}^{\hat{u}_0} du\ \frac{a\sqrt{\Delta}}{r^2+a^2} \overline{R}_{0,b'}^{\pm}(u) \int_{\widetilde{\Omega}} d\omega\ F_1(\omega)\ e^{i\widetilde{\varphi}_{(t)}^{\pm}(\omega,u)} \tag{6.27}$$

with $\widetilde{\varphi}_{(t)}^{\pm}(\omega,u)$ as defined in (6.16) and

$$F_1(\omega) := -i f_{0,a}^{\pm} B(\omega) f_a^0(\omega). \tag{6.28}$$

6.1.2 The Region Ω_2'.

Let be now $\omega' \in \Omega_2'$ be fixed. Theorem 2.7 implies that the coefficients $t_{a'b'}$ have the simple form

$$t_{a'b'} = \delta_{a',1}\delta_{b',1}.$$

As a consequence the terms (6.14) and (6.15) simplifies to

$$\overline{X}_1(u) X_a^0(u) f_a^0(\omega) \tag{6.29}$$

and

$$\overline{X}_1(u)\sigma_2 X_a^0(u) f_a^0(\omega). \tag{6.30}$$

Notice that close to the event horizon X_1 must be a linear combination of ingoing and outgoing plane waves, in order that the requirement of exponential decay for the radial spinor X_1 in the limit $u \to \infty$ be fulfilled. Hence we have the following analytical expression for X_1

$$X_1^0(u) = \begin{pmatrix} e^{-i\Omega_0 u}(C_+ + R_{0,1}^+(u)) \\ e^{i\Omega_0 u}(C_- + R_{0,1}^-(u)) \end{pmatrix}, \tag{6.31}$$

where according to (3.24) and (3.31) in [31] the coefficients C_+ and C_- satisfy the equations

$$|C_+|^2 - |C_-|^2 = 0,$$
$$|C_+|^2 + |C_-|^2 = 1.$$

A straightforward calculation gives

$$|C_+|^2 = |C_-|^2 = \frac{1}{2}. \tag{6.32}$$

Proceeding as in Section 6.1.1 it can be checked that also in this case we end up with the computation of limit integrals of the type $\mathfrak{M}_1, \cdots, \mathfrak{M}_4$ as defined by (6.23), (6.24), (6.26) and (6.27).

6.2 Theorems for the Evaluation of $W_0^\pm$.

The next result shows us that (6.24) gives no contribution in the evaluation of the wave operator at the event horizon.

Theorem 6.1. *For every function $f \in C^1(\mathbb{R})$ satisfying the bound*

$$|f|,|f^{'}| \leq \frac{1}{1+\omega^2}, \tag{6.33}$$

and for every given $\omega^{'}$, it results

$$\lim_{t\to\pm\infty} \int_{-\infty}^{\hat{u}_0} du \overline{R}_0^\pm(u) \int_{\tilde{\Omega}} d\omega f(\omega) e^{i(\omega-\omega^{'})(t\pm u)} = 0, \tag{6.34}$$

where $\overline{R}_0^\pm(u)$ is such that $\left|\overline{R}_0^\pm(u)\right| \leq ce^{du}$ with positive constants c and d.

Proof. Notice that $\overline{R}_0^\pm(u) \in L^1((-\infty, \hat{u}_0)$. In order to apply Fubini theorem to interchange the order of integration in (6.24), we introduce a convergence generating factor $e^{\sigma u}$ and we obtain

$$\lim_{t\to\pm\infty} \lim_{\sigma\to 0^+} \int_{\tilde{\Omega}} d\omega f(\omega) G_\sigma(\omega) e^{i(\omega-\omega^{'})t}$$

with

$$G_\sigma(\omega) = \int_{-\infty}^{\hat{u}_0} du \overline{R}_0^{\pm}(u) e^{\sigma u} e^{\pm i(\omega-\omega^{'})u}.$$

Since

$$|G_\sigma(\omega)| \leq \frac{c}{d} e^{d\hat{u}_0},$$

we can apply dominated convergence theorem to take the limit $\sigma \to 0^+$ inside the integral and we find that

$$\lim_{t\to\pm\infty} \int_{\widetilde{\Omega}} d\omega f(\omega) G(\omega) e^{i(\omega-\omega^{'})t}. \tag{6.35}$$

Because of our assumptions on $f(\omega)$ and of the boundedness of the function $G(\omega)$ Riemann-Lebesgue lemma implies that (6.35) is zero. □

The next two Theorems allow us to compute (6.23).

Theorem 6.2. *For every function $f \in C^1(\mathbb{R})$ satisfying* (6.33) *and for every given $\omega^{'}$ it holds*

$$\lim_{t\to\pm\infty} \int_{-\infty}^{\hat{u}_0} du \int_{\widetilde{\Omega}} d\omega f(\omega) e^{i(\omega-\omega^{'})(t\pm u)} = 0 \quad \textit{if} \quad f(\omega^{'}) = 0. \tag{6.36}$$

with $\hat{u}_0 < 0$.

Proof. By introducing a convergence generating factor $e^{\sigma u}$ with $\sigma > 0$, we can apply Fubini theorem and we get

$$\lim_{t\to\pm\infty} \lim_{\sigma\to 0^+} \int_{\widetilde{\Omega}} d\omega f(\omega) e^{i(\omega-\omega^{'})t} \int_{-\infty}^{\hat{u}_0} du \; e^{[\sigma\pm i(\omega-\omega^{'})]u}.$$

Integration over u yields to

$$\lim_{t\to\pm\infty} \lim_{\sigma\to 0^+} e^{\sigma\hat{u}_0} \int_{\widetilde{\Omega}} d\omega \frac{f(\omega) e^{\pm i(\omega-\omega^{'})\hat{u}_0}}{\sigma \pm i(\omega-\omega^{'})} e^{i(\omega-\omega^{'})t},$$

which can be rewritten as follows

$$\lim_{t\to\pm\infty} \lim_{\sigma\to 0^+} e^{\sigma\hat{u}_0} \int_{\widetilde{\Omega}} d\omega g(\omega) \frac{\omega-\omega^{'}}{\sigma \mp i(\omega-\omega^{'})} e^{i(\omega-\omega^{'})t}$$

with

$$g(\omega) := \frac{f(\omega) e^{\mp i(\omega-\omega^{'})\hat{u}_0}}{\omega-\omega^{'}}.$$

Since $g \in C(\mathbb{R})$ and moreover

$$\left| \frac{\omega - \omega'}{\sigma \mp i(\omega - \omega')} \right| \leq 1,$$

we can apply dominated convergence theorem in order to take the limit inside the integral and we obtain

$$\lim_{t \to \pm\infty} \int\limits_{\widetilde{\Omega}} d\omega g(\omega) e^{i(\omega - \omega')t}.$$

By applying Riemann-Lebesgue lemma we can conclude that the above expression is zero and this completes the proof. □

Theorem 6.3. *Let $f \in C^1(\mathbb{R})$ satisfy (6.33) and $\hat{u}_0 < 0$. Then it holds*

$$H^{(I)} := \int_{-\infty}^{\hat{u}_0} du \int\limits_{\widetilde{\Omega}} d\omega f(\omega) e^{i(\omega-\omega')(t-u)} = \begin{cases} 0 & \text{if } t \to +\infty \\ 2\pi f(\omega') & \text{if } t \to -\infty \end{cases} \quad (6.37)$$

$$H^{(II)} := \int_{-\infty}^{\hat{u}_0} du \int\limits_{\widetilde{\Omega}} d\omega f(\omega) e^{i(\omega-\omega')(t+u)} = \begin{cases} 2\pi f(\omega') & \text{if } t \to +\infty \\ 0 & \text{if } t \to -\infty \end{cases} \quad (6.38)$$

for every given ω'.

Proof. Since from Lemma 3.3 we already know that there is no contribution coming from the intervals $B_\delta(-m)$ and $B_\delta(m)$, we can extend the interval of integration from $\widetilde{\Omega}$ to the whole real axis. Let us begin with (6.37) and let us rewrite the function $f(\omega)$ as follows

$$f(\omega) = F(\omega) + f(\omega') \frac{\omega'^2 + 1}{\omega^2 + 1} \quad \text{with} \quad F(\omega) = f(\omega') - f(\omega') \frac{\omega'^2 + 1}{\omega^2 + 1}.$$

Since $F(\omega') = 0$ and $F(\omega)$ decays quadratically in ω for $\omega \to \infty$, we can apply Theorem 6.2 and we obtain

$$H^{(I)} = f(\omega')(\omega'^2 + 1) \int_{-\infty}^{\hat{u}_0} du \int_{\mathbb{R}} d\omega \frac{e^{i(\omega-\omega')(t-u)}}{\omega^2 + 1}.$$

By introducing a convergence factor $e^{\sigma u}$, we can apply Fubini theorem and integrate over the spatial variable u. Hence, we find that

$$H^{(I)} = i f(\omega')(\omega'^2 + 1) \lim_{\sigma \to 0^+} e^{\sigma \hat{u}_0} \int\limits_{\mathbb{R}} d\omega \frac{e^{i(\omega-\omega')(t-\hat{u}_0)}}{(\omega^2 + 1)(\omega - \omega_p)}$$

with $\omega_p = \omega' - i\sigma$. Defining the complex function

$$\mathfrak{B}(z) := \frac{e^{i(z-\omega')(t-\hat{u}_0)}}{(z^2+1)(z-\omega_p)},$$

the integral over ω can be evaluated by means of the Residue theorem. Notice that for $t \to +\infty$ the contour has to be closed on the complex upper half-plane containing only one simple pole at $z = i$, while for $t \to -\infty$ on the lower half-plane, where we have two simple poles respectively at $z = -i$ and $z = \omega_p$. A straightforward calculation gives the result in (6.37). Concerning (6.38), it can be immediately obtained from $H^{(I)}$ by means of the transformation

$$t \to -t, \quad \omega \to -\omega, \quad \omega' \to -\omega'.$$

□

The next result shows that (6.26) and (6.27) are zero. This implies that (6.15) does not contribute at all in the evaluation of the wave operator.

Theorem 6.4. *For every function $f \in C^2(\mathbb{R})$ satisfying the bounds*

$$|f|, |f'|, |f''| \leq \frac{1}{1+\omega^2}, \tag{6.39}$$

it holds

$$\lim_{t\to\pm\infty} \int_{-\infty}^{\hat{u}_0} du \frac{a\sqrt{\Delta}}{r^2+a^2} \int_{\widetilde{\Omega}} d\omega f(\omega) e^{i\widetilde{\varphi}^{\pm}_{(t)}(\omega,u)} = 0, \tag{6.40}$$

$$\lim_{t\to\pm\infty} \int_{-\infty}^{\hat{u}_0} du \frac{a\sqrt{\Delta}}{r^2+a^2} \overline{R_0^{\pm}}(u) \int_{\widetilde{\Omega}} d\omega f(\omega) e^{i\widetilde{\varphi}^{\pm}_{(t)}(\omega,u)} = 0, \tag{6.41}$$

where $\overline{R_0^{\pm}}(u)$ is such that $\left|\overline{R_0^{\pm}}(u)\right| \leq ce^{du}$ with positive constants c, d and $\widetilde{\varphi}^{\pm}_{(t)}(\omega,u)$ as defined in (6.16).

Proof. Let us define

$$\mathfrak{G}(u) = \left\{ \begin{array}{c} \mathfrak{G}_1(u) \\ \mathfrak{G}_2(u) \end{array} \right\} := \left\{ \begin{array}{c} 1 \\ \overline{R_0^{\pm}}(u) \end{array} \right\} \frac{a\sqrt{\Delta}}{r^2+a^2}.$$

Since from Lemma 3.3 we already know that there is no contribution coming from the intervals $[-m-\delta, -m+\delta]$ and $[m-\delta, m+\delta]$, we can extend the interval of integration from $\widetilde{\Omega}$ to the whole real axis. Let us introduce a new variable of integration $\Omega_0 := \omega + \omega_0$ with ω_0 as defined in Lemma 2.1. Our task is to compute the following limit integral

$$\lim_{t\to\pm\infty} \int_{-\infty}^{\hat{u}_0} du\ \mathfrak{G}(u) \int_{-\infty}^{\infty} d\Omega_0 f(\Omega_0) e^{i\widetilde{\varphi}^{\pm}_{(t)}(\Omega_0,u)}. \tag{6.42}$$

Making use of the identity

$$\left(1-\frac{\partial^2}{\partial\Omega_0^2}\right)e^{i\widetilde{\varphi}^{\pm}_{(t)}(\omega,u)}=\left[1+(t\pm u)^2\right]e^{i\widetilde{\varphi}^{\pm}_{(t)}(\Omega_0,u)},$$

we can integrate by parts (6.42) to obtain

$$\lim_{t\to\pm\infty}\int_{-\infty}^{\hat{u}_0}du\,\mathfrak{G}(u)\int_{-\infty}^{\infty}d\Omega_0\frac{g(\Omega_0)}{1+(t\pm u)^2}e^{i\widetilde{\varphi}^{\pm}_{(t)}(\Omega_0,u)}\tag{6.43}$$

with

$$\left(1-\frac{\partial^2}{\partial\Omega_0^2}\right)f(\Omega_0)=:g(\Omega_0)\in C(\mathbb{R})\quad\text{and}\quad|g|\leq\frac{1}{1+\Omega_0^2}.$$

Notice that, due to our hypothesis on the function f, after partial integration in (6.42) we do not get any boundary term. Since $\mathfrak{G}(u)\left[1+(t\pm u)^2\right]^{-1}$ is integrable in u, we can apply Fubini theorem to interchange the order of integration and we get

$$\lim_{t\to\pm\infty}\int_{-\infty}^{\infty}d\Omega_0\,g(\Omega_0)\int_{-\infty}^{\hat{u}_0}du\,\frac{\mathfrak{G}(u)e^{i\widetilde{\varphi}^{\pm}_{(t)}(\Omega_0,u)}}{1+(t\pm u)^2}.\tag{6.44}$$

Let us define $r^*:=r(\hat{u}_0)$ with $r^*>r_1$, where $r_1=M+\sqrt{M^2-a^2-Q^2}$ corresponds to the event horizon. Since $r_1>M$, in the non-extreme case ($M>a,Q$), it results that $\Delta<r^2$. As a consequence the following estimate holds

$$\frac{a\sqrt{\Delta}}{r^2+a^2}<\frac{a}{r}\frac{\sqrt{\Delta}}{r}<1.$$

In view of the above considerations the following estimates hold

$$\left|\int_{-\infty}^{\hat{u}_0}du\frac{a\sqrt{\Delta}}{r^2+a^2}e^{i\widetilde{\varphi}^{\pm}_{(t)}(\Omega_0,u)}\right|\leq|a|\int_{r_1}^{r^*}\frac{dr}{\sqrt{\Delta}}=C_1\tag{6.45}$$

$$\left|\int_{-\infty}^{\hat{u}_0}du\frac{a\sqrt{\Delta}}{r^2+a^2}\overline{R}_0^{\pm}(u)e^{i\widetilde{\varphi}^{\pm}_{(t)}(\Omega_0,u)}\right|\leq c\int_{-\infty}^{\hat{u}_0}du\,e^{du}=C_2\tag{6.46}$$

with positive constants C_1 and C_2. Since

$$\left|g(\Omega_0)\int_{-\infty}^{\hat{u}_0}du\,\frac{\mathfrak{G}(u)e^{i\widetilde{\varphi}^{\pm}_{(t)}(\Omega_0,u)}}{1+(t\pm u)^2}\right|\leq\begin{cases}C_1|g(\Omega_0)| & \text{if }\mathfrak{G}=\mathfrak{G}_1\\ C_2|g(\Omega_0)| & \text{if }\mathfrak{G}=\mathfrak{G}_2\end{cases},$$

we may use dominated convergence theorem to take the limit $t\to\pm\infty$ inside the integrand in (6.44). Hence we obtain

$$\int_{-\infty}^{\infty}d\Omega_0\,g(\Omega_0)\lim_{t\to\pm\infty}\int_{-\infty}^{\hat{u}_0}du\,\frac{\mathfrak{G}(u)e^{i\widetilde{\varphi}^{\pm}_{(t)}(\Omega_0,u)}}{1+(t\pm u)^2}.\tag{6.47}$$

According to the estimates (6.45) and (6.46) and to the fact that the function $\left[1+(t\pm u)^2\right]^{-1}$ is integrable in u, we may again apply dominated convergence theorem to perform the limit $t \to \pm\infty$ inside the integral over you. This completes the proof. $\square$

6.3 Final Theorem for $W_0^{\pm}$.

For ease of notation in the expressions for the wave operators $W_{0,I}^{\pm}$ we omit to write explicitly the sum over the index $a^{'} = 1, 2$.

Theorem 6.5. *Let $W_0^{\pm}$ be as defined in* (3.1). *Then for every $\hat{\Psi}_0^{(0)} \in C_c^{\infty}$,*

$$\begin{aligned} W_0^+ &= W_{0,I}^+ + W_{0,II}^+, \\ W_0^- &= W_{0,I}^- + W_{0,II}^- \end{aligned}$$

with

$$\begin{aligned}
\left(W_{0,I}^+ \hat{\Psi}_0^{(0)}\right)_{(x)} &= \sum_{k,n\in\mathbb{Z}} \int_{\sigma(H_\infty)} d\omega\; t_{a'2}^{kn}(\omega)\; \hat{\Psi}_{a'}^{k\omega n}(x) \widehat{f}_2^{0,kn}(\omega), \\
\left(W_{0,I}^- \hat{\Psi}_0^{(0)}\right)_{(x)} &= \sum_{k,n\in\mathbb{Z}} \int_{\sigma(H_\infty)} d\omega\; t_{a'1}^{kn}(\omega)\; \hat{\Psi}_{a'}^{k\omega n}(x) \widehat{f}_1^{0,kn}(\omega), \\
\left(W_{0,II}^+ \hat{\Psi}_0^{(0)}\right)_{(x)} &= \sum_{k,n\in\mathbb{Z}} \int_{-m}^{m} d\omega\; \hat{\Psi}_1^{k\omega n}(x) \overline{C}_- \widehat{f}_2^{0,kn}(\omega), \\
\left(W_{0,II}^- \hat{\Psi}_0^{(0)}\right)_{(x)} &= \sum_{k,n\in\mathbb{Z}} \int_{-m}^{m} d\omega\; \hat{\Psi}_1^{k\omega n}(x) \overline{C}_+ \widehat{f}_1^{0,kn}(\omega)
\end{aligned}$$

where $\sigma(H_\infty)$ denotes the spectrum of the free Dirac propagator and for $i = 1,\ 2$ the functions $\widehat{f}_i^{\,0,kn}(\omega)$ are defined as follows

$$\widehat{f}_i^{\,0,kn}(\omega) = 2\pi \int_{-\infty}^{\infty} du \int_{-1}^{1} d(\cos\theta)\; \overline{X}_i^{0,k\omega n}(u)\; \overline{Y}^{k\omega n}(\theta) \hat{\Psi}_0^{k,(0)}(u,\theta).$$

Moreover, the $\hat{\Psi}_{a'}^{k\omega n}$'s are the fundamental solutions, the coefficients $t_{a'1}^{kn}$ and $t_{a'2}^{kn}$ are as defined in Theorem 2.7 and the terms $\overline{C}_{\pm}$ are given by (6.31).

Proof. For ease of notation we omit to write explicitly the superscript 0 attached to the initial data $\hat{\Psi}_0^{(0)}$. By means of the integral representation for the propagator of the Dirac equation in the Kerr-Newman geometry (see Theorem 2.7) the wave operator at the event horizon as defined in (6.1) can be written as

$$\left(W_0^{\pm}\hat{\Psi}_0\right)_{(x)} = \frac{1}{\pi^2}\lim_{t\to\pm\infty}\sum_{k',n',k,n}\int_{\mathbb{R}} d\omega' \int_{\mathbb{R}} du \int_{\sigma(H_0)} d\omega e^{i(\omega-\omega')t}\widetilde{P}_{kn}^{k'n'}(\omega,\omega',u) \quad (6.48)$$

with $k', n', k, n \in \mathbb{Z}$ and

$$\widetilde{P}_{kn}^{k'n'}(\omega,\omega',u) = \sum_{a',b',a,b=1}^{2} t_{a'b'}^{k'\omega'n'} t_{ab}^{0,k\omega n}\hat{\Psi}_{a'}^{k'\omega'n'}(x) < \hat{\Psi}_{b'}^{k'\omega'n'}|\chi_0\hat{\Psi}_a^{0,k\omega n} >_{(u)} \langle\hat{\Psi}_b^{0,k\omega n}|\hat{\Psi}_0\rangle_0.$$

There $\hat{\Psi}^0$ denotes the eigenfunctions of the Hamiltonian at the event horizon H_0 and $\hat{\Psi}_0$ is some smooth initial data with compact support defined as follows

$$\hat{\Psi}_0(u,\theta,\varphi) = \sum_{\hat{k}\in\mathbb{Z}}\hat{\Psi}_0^{\hat{k}}(u,\theta)e^{-i\left(\hat{k}+\frac{1}{2}\right)\varphi}. \quad (6.49)$$

Analogously to the treatment of the modified wave operators at infinity,we introduce the same convention in the notation, i.e., we avoid to write explicitly the indexes ω and ω' attached to the spinors and to the transmission coefficients and we omit the indexes k, n and n'. Also in this case there is no risk of confusion, because in view of (6.54) the index k is the same in all factors, whereas the distinction between n and n' can be made by looking at the corresponding indices a, b or a', b'.Moreover, the expression of the wave operator can be further simplified by omitting the sums over k, n, k', n' and a, b, a', b'.
Since the matrix potential in (2.25) converges in the limit $u \to -\infty$, we can pass to the limit $r \to r_1$ in (2.17) and solve the correspondent system of ODEs together with the requirement that at the event horizon there are either only outgoing plane waves or incoming ones. Thus, we obtain the following analytical expression for $X_a^0(u)$

$$X_a^0(u) = \begin{pmatrix} e^{-i\Omega_0 u}f_{0,a}^+ \\ e^{i\Omega_0 u}f_{0,a}^- \end{pmatrix}, \quad \Omega_0 := \omega+\omega_0, \quad \omega_0 := \frac{\left(k+\frac{1}{2}\right)a + eQr_1}{r_1^2+a^2} \quad (6.50)$$

with

$$f_{0,1}^+ = 1 \qquad f_{0,1}^- = 0 \quad (6.51)$$

$$f_{0,2}^+ = 0 \qquad f_{0,2}^- = 1. \quad (6.52)$$

Making use of the above choice together with (2.49) and (2.50) the transmission coefficients at the event horizon t^0_{ab} are computed to be

$$t^0_{ab} = \frac{1}{2}\delta_{ab}. \tag{6.53}$$

Hence, (6.48) becomes

$$\left(W_0^{\pm}\hat{\Psi}_0\right)_{(x)} = \frac{1}{2\pi^2}\lim_{t\to\pm\infty}\int\limits_{\mathbb{R}} d\omega^{'}\int\limits_{\mathbb{R}} du \int\limits_{\sigma(H_0)} d\omega e^{i(\omega-\omega^{'})t} t_{a^{'}b^{'}}\hat{\Psi}_{a^{'}}(x)$$
$$< \hat{\Psi}_{b'}|\chi_0\hat{\Psi}^0_a >_{(u)} \langle\hat{\Psi}^0_a|\hat{\Psi}_0\rangle_0.$$

Employing (5.4) and integrating over the azimuthal variable φ, it can be verified that $< \hat{\Psi}_{b'}|\chi_0\hat{\Psi}^0_a >_{(u)}$ and $\langle\hat{\Psi}^0_a|\hat{\Psi}_0\rangle_0$ are given respectively by

$$< \hat{\Psi}_{b'}|\chi_0\hat{\Psi}^0_a >_{(u)} = 2\pi\delta_{kk'}\widetilde{g}_{b'a}(\omega,\omega^{'},u)\chi_0(u), \tag{6.54}$$
$$\langle\hat{\Psi}^0_a|\hat{\Psi}_0\rangle_0 = 2\pi f^0_a(\omega) \tag{6.55}$$

with

$$\widetilde{g}_{b'a}(\omega,\omega^{'},u) = A(\omega,\omega^{'})\overline{X}_{b'}(u)X^0_a(u) + \frac{a\sqrt{\Delta}}{r^2+a^2}B(\omega,\omega^{'})\overline{X}_{b'}(u)\sigma_2 X^0_a(u) \tag{6.56}$$

and

$$f^0_a(\omega) = \int_{-\infty}^{\infty} du \int_{-1}^{1} d(\cos\theta)\ \overline{X}^0_a(u)\ \overline{Y}(\theta)\hat{\Psi}^k_0(u,\theta). \tag{6.57}$$

Here the functions $A(\omega,\omega^{'})$ and $B(\omega,\omega^{'})$ are given respectively by (5.9) and (5.10), depend smoothly on ω and $\omega^{'}$ and are bounded according to (5.11). By means of (6.54) and (6.55) and with our notation the wave operator at the event horizon can be written in the compact form

$$\left(W_0^{\pm}\hat{\Psi}_0\right)_{(x)} = 2\lim_{t\to\pm\infty}\int\limits_{\mathbb{R}} d\omega^{'} t_{a^{'}b^{'}}(\omega^{'})\hat{\Psi}_{a^{'}}(x)\int\limits_{\mathbb{R}} du\chi_0(u)$$
$$\int\limits_{\sigma(H_0)} d\omega e^{i(\omega-\omega^{'})t}\widetilde{g}_{b'a}(\omega,\omega^{'},u)f^0_a(\omega).$$

Since the regions $\omega \approx \pm m$, $\omega^{'} \approx \pm m$ and $|\omega^{'}| > L$ can be controlled with Lemma 3.1, 3.3 and 3.5, our task is to compute the expression

$$\int\limits_{\Omega'} d\omega^{'} t_{a^{'}b^{'}}(\omega^{'})\hat{\Psi}_{a^{'}}(x)\lim_{t\to\pm\infty}\int_{-\infty}^{\hat{u}_0} du \int\limits_{\widetilde{\Omega}} d\omega e^{i(\omega-\omega^{'})t}\widetilde{g}_{b'a}(\omega,\omega^{'},u)f^0_a(\omega), \tag{6.58}$$

where

$$\Omega^{'} := [-L, -m-\epsilon] \cup [-m+\epsilon, m-\epsilon] \cup [m+\epsilon, L],$$
$$\widetilde{\Omega} := (-\infty, -m-\delta] \cup [-m+\delta, m-\delta] \cup [m+\delta, +\infty)$$

for constants ϵ, δ and $L > 0$. Without loss of generality we can now consider $\omega^{'}$ as given. The next step is to analyze the integrands entering in (6.58). Looking back at (6.56), we observe that we have to deal with the following expressions

$$\overline{X}_{b'}(u) X^0_a(u) f^0_a(\omega) \tag{6.59}$$

and

$$\frac{a\sqrt{\Delta}}{r^2+a^2}\overline{X}_{b'}(u)\sigma_2 X^0_a(u) f^0_a(\omega). \tag{6.60}$$

At the event horizon (6.59) and (6.60) give rise to integrands with different phases. By choosing $|\hat{u}_0|$ large enough, we can replace $X_{b'}(u)$ entering in (6.59) and (6.60) with the analytical expressions derived in Lemma 2.1 if $|\omega^{'}| > m$, whereas for $|\omega^{'}| < m$, since $t_{a'b'} = \delta_{a',1}\delta_{b',1}$, the radial spinor close to the event horizon X_1 must be a linear combination of ingoing and outgoing plane waves, in order that the requirement of exponential decay for the radial spinor X_1 in the limit $u \to \infty$ be fulfilled. In particular X_1 will be given by (6.31). Concerning (6.59) a direct computation shows that all integrands have the following phase

$$\widehat{\varphi}^{\pm}_{(t)}(\omega, u) = (\omega - \omega^{'})(t \pm u), \tag{6.61}$$

whereas for (6.60) we get a different phase, namely

$$\widetilde{\varphi}^{\pm}_{(t)}(\omega, u) = (\Omega_0(\omega) - \Omega^{'}_0)t \pm (\Omega_0(\omega) + \Omega^{'}_0)u \tag{6.62}$$

with $\Omega_0(\omega)$ as in (6.50) and $\Omega^{'}_0 = \omega^{'} + \omega_0$. This is due to the presence of the Pauli matrix σ_2 in (6.60). The next step is to analyze the integrands entering in the double integral over the spatial variable u and the frequency ω given by

$$\int_{-\infty}^{\hat{u}_0} du \int_{\widetilde{\Omega}} d\omega \; e^{i(\omega-\omega^{'})t}\widetilde{g}_{b'a}(\omega, \omega^{'}, u) f^0_a(\omega), \tag{6.63}$$

in the limits $t \to \pm\infty$, when we treat the regions $\Omega^{'}_1$ and $\Omega^{'}_2$, respectively. In the region $\Omega^{'}_2$ application of Theorems 6.1, 6.3 and 6.4 gives

$$\left(W^{+}_{0,\Omega_2}\hat{\Psi}_0\right)_{(x)} = \int_{\Omega_2} d\omega \hat{\Psi}_1(x)\,\overline{C}_{-}\widehat{f_2}^{\,0}(\omega), \tag{6.64}$$

$$\left(W^{-}_{0,\Omega_2}\hat{\Psi}_0\right)_{(x)} = \int_{\Omega_2} d\omega \hat{\Psi}_1(x)\,\overline{C}_{+}\widehat{f_1}^{\,0}(\omega). \tag{6.65}$$

Concerning the region Ω_1' we may again use Theorems 6.1, 6.3 and 6.4 to obtain the following results

$$\left(W^{+}_{0,\Omega_1}\hat{\Psi}_0\right)_{(x)} = \sum_{a'=1}^{2}\int\limits_{\Omega_1} d\omega\; t_{a'2}(\omega)\hat{\Psi}_{a'}(x)\widehat{f_2}^{\,0}(\omega), \tag{6.66}$$

$$\left(W^{-}_{0,\Omega_1}\hat{\Psi}_0\right)_{(x)} = \sum_{a'=1}^{2}\int\limits_{\Omega_1} d\omega\; t_{a'1}(\omega)\hat{\Psi}_{a'}(x)\widehat{f_1}^{\,0}(\omega), \tag{6.67}$$

where $\widehat{f_a}^{\,0}(\omega) := 2\pi f_a^0(\omega)$ for $a = 1,\, 2$ and $f_a^0(\omega)$ is given by (6.57). Let us take the limits $L \to \infty$ and $\epsilon \to 0^+$ and let us add (6.64) to (6.66) and (6.65) to (6.67), then we end up with the following analytical expressions for the wave operator at the event horizon, namely

$$\left(W^{+}_{0}\hat{\Psi}_0\right)_{(x)} = \sum_{a'=1}^{2}\int\limits_{\sigma(H_\infty)} d\omega\; t_{a'2}(\omega)\hat{\Psi}_{a'}(x)\widehat{f_2}^{\,0}(\omega) + \int_{-m}^{m} d\omega\hat{\Psi}_1(x)\,\overline{C}_{-}\widehat{f_2}^{\,0}(\omega), \tag{6.68}$$

$$\left(W^{-}_{0}\hat{\Psi}_0\right)_{(x)} = \sum_{a'=1}^{2}\int\limits_{\sigma(H_\infty)} d\omega\; t_{a'1}(\omega)\hat{\Psi}_{a'}(x)\widehat{f_1}^{\,0}(\omega) + \int_{-m}^{m} d\omega\hat{\Psi}_1(x)\,\overline{C}_{+}\widehat{f_1}^{\,0}(\omega), \tag{6.69}$$

where $\sigma(H_\infty)$ denotes the spectrum of the free Dirac propagator in the Minkowski metric. □

7 Asymptotic completeness.

We define the inverse wave operators at the horizon and at infinity for $\hat{\Psi}_0 \in \mathcal{H}$ by

$$\begin{aligned} \widehat{W}_0^{\pm}\hat{\Psi}_0 &= s - \lim_{t\to\pm\infty} e^{-iH_0 t}\mathcal{I}_0^* e^{iHt}\hat{\Psi}_0, \\ \widehat{W}_\infty^{\pm}\hat{\Psi}_0 &= s - \lim_{t\to\pm\infty} e^{-iH_\infty t}\mathcal{I}_\infty^* e^{iHt} e^{i\delta(t)}\hat{\Psi}_0, \end{aligned}$$

where $\mathcal{I}_0^*$ and $\mathcal{I}_\infty^*$ are respectively the adjoints of $\mathcal{I}_0$ and $\mathcal{I}_\infty$. We introduce the wave operators W^+ and W^- given by

$$W^{\pm}(\hat{\Psi}_0^{(0)}, \hat{\Psi}_0^{(\infty)}) = W_0^{\pm}\hat{\Psi}_0^{(0)} + \widetilde{W}_\infty^{\pm}\hat{\Psi}_0^{(\infty)}$$

with $\hat{\Psi}_0^{(0)} \in \mathcal{H}_0$ and $\hat{\Psi}_0^{(\infty)} \in \mathcal{H}_\infty$, as well as the inverse wave operators $\widehat{W}^+$ and $\widehat{W}^-$

$$\widehat{W}^{\pm}\hat{\Psi}_0 = \widehat{W}_0^{\pm}\hat{\Psi}_0 + \widehat{W}_\infty^{\pm}\hat{\Psi}_0$$

with $\hat{\Psi}_0 \in \mathcal{H}$. Finally the scattering operator S is defined by

$$S = \widehat{W}^+ W^-.$$

Notice that if the operators $W^{\pm}$ are asymptotically complete then the scattering matrix S is unitary. In the following we prove the asymptotic completeness for $W^{\pm}$.

Theorem 7.1. ***(Asymptotic Completeness)***
Let the wave operators W^+ and W^- be defined by

$$W^{\pm}(\hat{\Psi}_0^{(0)}, \hat{\Psi}_0^{(\infty)}) = W_0^{\pm}\hat{\Psi}_0^{(0)} + \widetilde{W}_\infty^{\pm}\hat{\Psi}_0^{(\infty)}$$

with $\hat{\Psi}_0^{(0)} \in \mathcal{H}_0$ and $\hat{\Psi}_0^{(\infty)} \in \mathcal{H}_\infty$. The range of $W^{\pm}$ is dense in the Hilbert space $\mathcal{H}$.

Proof. Here we limit us to show the asymptotic completeness for the wave operator W^+, since the same method applies also to W^-. Let us consider the vectors

$$\frac{1}{\pi}\sum_{k,n\in\mathbb{Z}}\int_{-\infty}^{+\infty} d\omega \sum_{a,b=1}^{2} \Psi_a^{k\omega n} t_{ab}^{k\omega n} G_b^{kn}(\omega) \tag{7.1}$$

with functions $G_b^{kn} \in C_0^\infty(\mathbb{R}\backslash\{-m,m\})$. We assume that only a finite number of the G_b^{kn} are non-zero. Notice that, according to Theorem 2.7, the vectors given by (7.1) span the whole Hilbert space $\mathcal{H}$. From Theorems 5.13

and 6.5, we know that

$$\left(\widetilde{W}^+_\infty \hat{\Psi}^{(\infty)}_0\right)_{(x)} = \sum_{k,n\in\mathbb{Z}} \int\limits_{\sigma(H_\infty)} d\omega\; t^{kn}_{a'b'}(\omega)\hat{\Psi}^{k\omega n}_{a'}(x)\widehat{f}_{kn}(\omega) f^+_\infty \overline{f}^+_{\infty,b'},$$

$$\left(W^+_{0,I}\hat{\Psi}^{(0)}_0\right)_{(x)} = \sum_{k,n\in\mathbb{Z}} \int\limits_{\sigma(H_\infty)} d\omega\; t^{kn}_{a'2}(\omega)\;\hat{\Psi}^{k\omega n}_{a'}(x)\widehat{f}^{0,kn}_{2}(\omega),$$

$$\left(W^+_{0,II}\hat{\Psi}^{(0)}_0\right)_{(x)} = \sum_{k,n\in\mathbb{Z}} \int_{-m}^{m} d\omega\;\hat{\Psi}^{k\omega n}_1(x)\overline{C}_-\widehat{f}^{0,kn}_{2}(\omega).$$

Since from [31] it holds

$$|f^+_{\infty,1}|^2 - |f^-_{\infty,1}|^2 = 1,$$
$$|f^+_\infty|^2 - |f^-_\infty|^2 = 0$$

with $f^+_\infty > 0$ and according to (6.32) $|C_+|^2 = |C_-|^2 = 1/2$, for $|\omega| > m$ we may choose $\widehat{f}_{kn}(\omega)$ and $\widehat{f}^{0,kn}_{2}(\omega)$ in $C^\infty_0(\mathbb{R}\backslash\{-m,m\})$ such that

$$\widehat{f}_{kn}(\omega) f^+_\infty \overline{f}^+_{\infty,1} \;=\; G^{kn}_1(\omega), \tag{7.2}$$
$$\widehat{f}^{0,kn}_{2}(\omega) \;=\; G^{kn}_2(\omega) - \widehat{f}_{kn}(\omega) f^+_\infty \overline{f}^+_{\infty,2}, \tag{7.3}$$

whereas for $|\omega| < m$ we choose $\widehat{f}^{0,kn}_{2}(\omega)$ such that

$$\widehat{f}^{0,kn}_{2}(\omega)\overline{C}_- = G^{kn}_1(\omega). \tag{7.4}$$

As $\widehat{f}_{kn}(\omega)$ is the spectral density of $H_\infty \hat{\Psi}^{(\infty)}_0$, whereas $\widehat{f}^{0,kn}_{2}(\omega)$ is the Fourier transform of $\hat{\Psi}^{(0)}_0$, it is clear that there are vectors $\hat{\Psi}^{(\infty)}_0 \in \mathcal{H}_\infty$ and $\hat{\Psi}^{(0)}_0 \in \mathcal{H}_0$, which realize the choice of $\widehat{f}_{kn}(\omega)$ and $\widehat{f}^{0,kn}_{2}(\omega)$. □

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