Zeta Functions of Local Orders

DISSERTATION ZUR ERLANGUNG DES DOKTORGERADES DER-
NATURWISSENSCHAFTEN (DR. RER. NAT) DER FAKULTÄT
NWF I-MATHEMATIK DER UNIVERSITÄT REGENSBURG

vorgelegt von

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Regensburg, Januar 2006

Promotionsgesuch eingereicht am: 10. Januar 2006

Die Arbeit wurde angeleitet von Prof. Dr.Jannsen

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Introduction

The zeta-functions associated with algebraic curves over finite fields encode many arithmetic properties of the curves. In the non-singular case the theory is well-known. It is analogous to the theory of zeta-functions for number fields and culminates in the Hasse-Weil theorem about the Riemann hypothesis for curves. In the singular case, which will be the main topic of this thesis, the theory is more difficult and less explored. First of all, one does not deal with Dedekind rings anymore, but with orders, i.e., certain subring of them. The corresponding theory of (fractional) ideals becomes much more complicated. Secondly, there are various candidates for the zeta-function.

In 1973 Galkin[G] published a paper which deals with the zeta-function of a local ring $\mathcal{O}$ of a possibly singular, complete, geometrical irreducible algebraic curve X defined over a finite field $k = \mathbb{F}_q$ of q elements. His zeta-function is defined in the half-plane $\{s \in \mathbb{C} \mid Re(s) > 0\}$ by the absolutely convergent Dirichlet series

$$\zeta_{\mathcal{O}}(s) = \sum_{\mathfrak{a} \subseteq \mathcal{O}} \#(\mathcal{O}/\mathfrak{a})^{-s},$$

where the sum is taken over the (non-zero) ideals $\mathfrak{a}$ in the ring $\mathcal{O}$. Hence it is formally defined in the same way as the classical zeta-functions and it encodes the numbers of ideals with given norms. Galkin also treated the arithmetic case, where $\mathcal{O}$ is the local ring of an order of an algebraic number field. He also defined global zeta-functions this way, but it turns out that they do not have any functional equation, unless the considered ring is Gorenstein.

Green [Gr] defined another zeta-function which always satisfies a functional equation, but which is not defined in terms of local conditions. In particular, it does not possess an Euler product in the global case.

Finally, Stöhr [St1],[St2] defined a modified zeta-function which both has a functional equation and a purely local definition.The key point is to consider all (fractional) ideals $\mathfrak{a}$ that are *positive*, in the sense that they contain the ring $\mathcal{O}$, instead of considering the *integral* ideals, which are contained in $\mathcal{O}$, and so to define

$$\zeta(\mathcal{O}, s) := \sum_{\mathfrak{a} \supseteq \mathcal{O}} \#(\mathfrak{a}/\mathcal{O})^{-s} \quad , \quad Re(s) > 0.$$

It is this zeta function that we will mainly consider in this paper. We want to investigate its calculation and its properties, and for this it suffices to regard the local case, i.e., the case where $\mathcal{O}$ is a local ring. More precisely, $\mathcal{O}$ will be a local order, i.e., a local integral domain of dimension 1, whose normalization (integral closure) $\widetilde{\mathcal{O}}$ is finite over $\mathcal{O}$. This implies that $\widetilde{\mathcal{O}}$ is a semi-local Dedekind ring. Of course we have to assume that the residue field of $\mathcal{O}$ is finite (so that the groups $\mathfrak{a}/\mathcal{O}$ are finite). Moreover, as in Stöhr's paper we will restrict to the 'geometric' situation and assume that $\mathcal{O}$ is a k-algebra for a finite field k.

Now we discuss the plan of this thesis in more detail.

In the first section we will recall some facts from commutative algebra and algebraic geometry. These will be used later, in part also for the motivation of our investigation.

In section 2 we will introduce generalized zeta functions

$$\zeta(\mathfrak{d}, s) = \sum_{\mathfrak{a} \supseteq \mathfrak{d}} \#(\mathfrak{a}/\mathfrak{d})^{-s}$$

for every fractional ideal $\mathfrak{d}$ in an order $\mathcal{O}$, and associated partial zeta function

$$\zeta(\mathfrak{d}, \mathfrak{b}, s) = \sum_{\substack{\mathfrak{a} \supseteq \mathfrak{d} \\ \mathfrak{a} \sim \mathfrak{b}}} \#(\mathfrak{a}/\mathfrak{d})^{-s},$$

where $\mathfrak{b}$ is another fractional $\mathcal{O}$-ideal, and the sum is over all fractional ideals $\mathfrak{a}$ which contain $\mathfrak{d}$ and which are equivalent to $\mathfrak{b}$ ($\mathfrak{a} = \alpha \cdot \mathfrak{b}$ for some $\alpha \in K$). By introducing the degree of fractional ideals, we can write this zeta function as a power series in $\mathbb{Z}[[t]]$,

$$Z(\mathfrak{d}, \mathfrak{b}, t) = \sum_{\substack{\mathfrak{a} \supseteq \mathfrak{d} \\ \mathfrak{a} \sim \mathfrak{b}}} t^{\deg \mathfrak{a} - \deg \mathfrak{d}},$$

where $t = q^{-s}$. We deduce a simple reciprocity formula relating $Z(\mathfrak{d}, \mathfrak{b}, t)$ and $Z(\mathfrak{b}^*, \mathfrak{d}^*, t)$, where $\mathfrak{a}^* = \mathfrak{c} : \mathfrak{a}$ for a dualizing ideal $\mathfrak{c}$ of $\mathcal{O}$.
Here $\mathfrak{b} : \mathfrak{a} = \{x \in K \mid x\mathfrak{a} \subseteq \mathfrak{b}\}$ for fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$. We also relate $Z(\mathfrak{d}, \mathfrak{b}, t)$ to $Z(\mathcal{O}, \mathfrak{b} : \mathfrak{d}, t)$ by simple formula. Therefore it suffices to study the case $\mathfrak{d} = \mathcal{O}$. Most of this material is contained in Stöhr's paper [St1], but we have filled in some proofs.

In section 3 we introduce an important invariant of an order $\mathcal{O}$, the semigroup

$$S(\mathcal{O}) = \{(\mathrm{ord}_{\mathfrak{p}_1}(x), \ldots, \mathrm{ord}_{\mathfrak{p}_m}(x)) \mid x \in \mathcal{O} \backslash \{0\}\} \subseteq \mathbb{N}_0^m,$$

where $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$ are the maximal ideals of $\widetilde{\mathcal{O}}$, and $\mathrm{ord}_{\mathfrak{p}_i}$ in the normalized discrete valuation associated to $\mathfrak{p}_i$. We associate a similar set $S(\mathfrak{b}) \subseteq \mathbb{Z}^m$ to any fractional $\mathcal{O}$-ideal $\mathfrak{b}$, and use it to give a formula for the zeta function $Z(\mathcal{O}, \mathfrak{b}, t)$ (Theorem 3.10). We use this formula to show that (Theorem 3.6)

$$Z(\mathcal{O}, \mathfrak{b}, t) = \frac{L(\mathcal{O}, \mathfrak{b}, t)}{\Pi_{i=1}^m (1 - t^{d_i})} = L(\mathcal{O}, \mathfrak{b}, t) \cdot Z(\widetilde{\mathcal{O}}, t),$$

where $d_i = \dim_k \widetilde{\mathcal{O}}/\mathfrak{p}_i$ and $L(\mathcal{O}, \mathfrak{b}, t)$ is a polynomial in $\mathbb{Z}[t]$ of degree $\leq 2\delta$ ($\delta = \deg \widetilde{\mathcal{O}} = \dim_k \widetilde{\mathcal{O}}/\mathcal{O}$ the singularity degree of $\mathcal{O}$), which satisfies the functional equation

$$t^{-\delta} L(\mathcal{O}, \mathfrak{b}, t) = (1/qt)^{-\delta} L(\mathcal{O}, \mathfrak{b}^*, 1/qt).$$

We give some first properties of the polynomial $L(\mathcal{O}, \mathfrak{b}, t)$. By summing up over the (finitely many) representatives of the ideal classes $(\mathfrak{b})$ of $\mathcal{O}$, we get similar results for

$$Z(\mathcal{O}, t) = \frac{L(\mathcal{O}, t)}{\Pi_{i=1}^m (1 - t^{d_i})}$$

with

$$L(\mathcal{O}, t) = \sum_{(\mathfrak{b})} L(\mathcal{O}, \mathfrak{b}, t).$$

Again, these results are mostly contained in [St1], where we have added some proofs.

In the short section 4 we show that Galkin's zeta function can be related to Stöhr's (generalized, local) zeta functions. Therefore we will concentrate on the latter in the remaining part.

In section 5 we use the mentioned explicit formula of the previous section to calculate $Z(\mathcal{O}, t)$ (and hence $L(\mathcal{O}, t)$) for a first concrete example, namely $\mathcal{O} = k[[x, y]]/(y^2 - x^3)$, which is the singularity of a cusp.

In the remaining sections, which constitute the main part the thesis, we concentrate on the rational unibranch case, i.e. , the case where $m = 1$ ($\widetilde{\mathcal{O}}$ is again a local ring) and $d = 1$ (k is equal to the residue fields of $\mathcal{O}$ and $\widetilde{\mathcal{O}}$). (This situation arises, e.g., for a singularity of a curve at a totally rational point, which just has one branch.) In this case the semigroup $S(\mathcal{O})$ is a subsemigroup of $\mathbb{N}_0$, and it is determined by the finite set

$$\mathbb{N}_0 \backslash S(\mathcal{O})$$

of gaps of $\mathcal{O}$, i.e., the natural numbers not contained in $S(\mathcal{O})$.

In section 6 we develop further tools for the computation of the zeta functions. For any (fractional) ideal $\mathfrak{b}$ we define the numerical conductor $f(\mathfrak{b})$ and the conductor $\mathfrak{F}(\mathfrak{b}) = \mathfrak{p}^{f(\mathfrak{b})}$, and we prove a formula

$$L(\mathcal{O}, \mathfrak{b}, t) = \frac{(qt)^{\deg \mathfrak{b}}}{(U_{\mathfrak{b}} : U_{\mathcal{O}})} \sum_{i=0}^{f(\mathfrak{b})} n_i(\mathfrak{b})\, t^i$$

where the integers $n_i(\mathfrak{b})$ only depend on $S(\mathcal{O})$ (more precisely on the gaps of $S(\mathfrak{b})$) in a simple way. This generalizes a result of Stöhr, who treated the case $\mathfrak{b} = \mathcal{O}$. Next we introduce another invariant of $\mathfrak{b}$, the ring $\mathcal{O}_{\mathfrak{b}} = \mathfrak{b} : \mathfrak{b}$ (which is the biggest order $\mathcal{O}'$, $\mathcal{O} \subseteq \mathcal{O}' \subseteq \widetilde{\mathcal{O}}$, operating on $\mathfrak{b}$), and prove the useful formula

$$L(\mathcal{O}, \mathfrak{b}, t) = t^{\deg \mathcal{O}_{\mathfrak{b}}}\, L(\mathcal{O}_{\mathfrak{b}}, \mathfrak{b}, t).$$

We apply both results in section 7, where we calculate the zeta functions of the orders $\mathcal{O}_{12} = k[[x^3, x^4, x^5]] \subseteq k[[X]]$ and $\mathcal{O}_{13} = k[[x^2, x^5]] \subseteq k[[X]]$ with gaps $\{1, 2\}$ and $\{1, 3\}$, respectively.

In section 8 we develop further tools for the computation of the polynomials $L(\mathcal{O}, \mathfrak{b}, t)$ and the zeta polynomial $L(\mathcal{O}, t)$ of $\mathcal{O}$ itself. Our strategy is to deduce information just from the semigroup $S(\mathcal{O})$. We succeed in this in the case of orders with $S(\mathcal{O}) = S^{(n)} = \{0, 2, 4, 6, \cdots, 2n, 2n+1, \cdots\}$ (i.e., with gaps $\{1, 3, 5, \cdots, 2n-1\}$) which we call balanced. We prove for these

$$L(\mathcal{O}, t) = 1 + X + X^2 + \cdots + X^n$$

where $X = qt^2$.

In section 9 we come to the main objective of this thesis - the investigation when the considered zeta functions satisfy the Riemann hypothesis, i.e., have all zeroes on the line $Re(s) = 1/2$. For $Z(\mathcal{O}, t)$ this means that all zeroes α of $L(\mathcal{O}, t)$ have the property $|\alpha| = q^{-1/2}$. First of all, by the functional equation, this can only hold if $\mathcal{O}$ is Gorenstein (i.e., when $\mathcal{O}$ is a dualizing ideal). But Stöhr gave examples of Gorenstein orders which do not satisffy the Riemann hypothesis.
We study this more systematically. First of all we show (Theorems 9.5 and 9.6) that for balanced orders, the Riemann hypothesis holds for $Z(\mathcal{O}, t)$ and the 'principal zeta function' $Z(\mathcal{O}, \mathcal{O}, t)$ which was more often studied in the literature. $Z(\mathcal{O}, t)$ was studied less often, because in general it is difficult to find all equivalence classes of ideals. Here we study it for all orders of singularity degree ≤ 3 and find that the Riemann hypothesis for $Z(\mathcal{O}, t)$ only holds in the balanced cases. In the same vein, we show the following for the principal zeta function and arbitrary (rational, unibranch) orders $\mathcal{O}$ (Theorem 9.9): If $S(\mathcal{O})$ is not balanced, then $Z(\mathcal{O}, \mathcal{O}, t)$ does not satisfy the Riemann hypothesis for $q >> 0$.
We close with a speculation if this last condition on q is necessary. There is some evidence that both for $Z(\mathcal{O}, t)$ and $Z(\mathcal{O}, \mathcal{O}, t)$ the Riemann hypothesis holds if and only if $\mathcal{O}$ is balanced. Moreover, our investigations suggest that, like $Z(\mathcal{O}, \mathcal{O}, t)$ also $Z(\mathcal{O}, t)$ only depends on the semigroup $S(\mathcal{O})$.

Acknowledgment

I wish to thank my thesis advisor Prof. Dr. Uwe Jannsen, who has supported me continuously and kindly. A number of people have helped me during my studies in University-Regensburg, it is pleasure to acknowledge the helps of Dr. Lars Bruenjes, Dr. Marco Hien, Dr. Jens Hornbostel, Dr Ivan Kausz, David. J.C. Kwak and Dr. Christopher Rupprecht.

1 Some background from commutative algebra and algebraic geometry

In this section we recall briefly some topics in algebraic number theory and algebraic geometry, which we need later in our thesis.

Dedekind domains and orders

At first we introduce the class of Dedekind domains. It lies property between the class of principal ideal domains and the class of Noetherian integral domains. Dedekind domains are important in algebraic number theory and the algebraic theory of curves. The definition of a Dedekind domain is motivated by the following facts: Every principal ideal domain D is Noetherian. Consequently, every ideal ($\neq D$) has a primary decompositions, see [Hun].

Definition 1.1. *A Dedekind domain is an integral domain R in which every proper ideal is the product of a finite number of prime ideals.*

Every principal ideal domain is Dedekind. The converse, however is false, because the integral domain $Z[\sqrt{10}]$ is Dedekind domain but it is not a principal ideal domain, see [Hun].

Definition 1.2. *Let R be an integral domain with quotient field K. A fractional ideal of R is a nonzero R-submodule I of K such that $aI \subset R$ for some nonzero $a \in R$.*

Example 1.3. *Every ordinary nonzero ideal I in an integral domain R is a fractional ideal of R.*

Remark 1.4. *If I is a fractional ideal of a domain R and $aI \subset R$ $(0 \neq a \in R)$, then aI is an ordinary ideal in R and the map $I \to aI$ given by $x \mapsto ax$ is an R-module isomorphism.*

If R is an integral domain with quotient field K, then the set of all fractional ideals of R forms a commutative monoid, with identity R and multiplication given by $IJ = \{ \sum_{i=1}^{n} a_i b_i | a_i \in I \ b_i \in J; n \in \mathbb{N} \}$.

A fractional ideal I of an integral domain R is said to be invertible if $IJ = R$ for some fractional ideal J of R. Thus the invertible fractional ideals are precisely those that have inverses in the monoid of all fractional ideals.

Remark 1.5. *(i) The inverse of an invertible fractional ideal I is unique and is $I^{-1} = \{ a \in K | aI \subset R\}$. Indeed for any fractional ideal I the set $I^{-1} = \{a \in K | aI \subset R\}$ is easily seen to be a fractional ideal such that $I^{-1}I = II^{-1} \subset R$. If I is invertible and $IJ = JI = R$, then clearly $J \subset I^{-1}$. Conversely, since I^{-1} and J are R-submodules of K, $I^{-1} = RI^{-1} = (JI)I^{-1} = J(II^{-1}) \subset JR = RJ \subset J$, whence $J = I^{-1}$.*

(ii) If I, A, B are fractional ideals of R such that $IA = IB$ and I is invertible then $A = RA = (I^{-1}I)A = I^{-1}(IB) = RB = B$

(iii) If I is an ordinary ideal in R, then $R \subset I^{-1}$.

(iv) Multiplication and inversion behave property with respect to localization. That is, if P is a prime ideal of R and I a fractional ideal of R, then IR_P is a fractional ideal of R_P and $(IR_P)^{-1} = I^{-1} R_P$. Also $(IJ)R_P = (IR_P)(JR_P)$ for I, J fractional ideals of R.

We state some important properties of fractional ideals as follows, for more details see [Hun].

Let $I, I_1, I_2, \cdots, I_n$ be ideals in an integral domain R .

(i) The ideal $I_1 I_2 \cdots I_n$ is invertible if and only if each I_j is invertible.

(ii) If $P_1 \cdots P_m = I = Q_1 \cdots Q_n$, where the P_i and Q_j are prime ideals in R and every P_i is invertible , then m=n and (after reindexing) $P_i = Q_i$ for each $i = 1, \cdots, m$.

If R is a Dedekind domain, then every nonzero prime ideal of R is invertible and maximal.

Every invertible fractional ideal of an integral domain R with quotient field K is a finitely generated R-module.

Let R be an integral domain and I a fractional ideal of R . Then I is invertible if and only if I is a projective R-module .

Definition 1.6. *Let $A \subseteq R$, be a ring extension. By definition $x \in R$ is integral over A if there is a monic polynomial $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in A[X]\backslash\{0\}$ such that $f(x) = 0$.*

Remark 1.7. *(i) $x \in R$ is integral over A if only if $A[x] = \{g(x) \mid g \in A[X]\}$ is a finite generated A-module.*

(ii) If $x_1, \cdots, x_n \in R$ are integral over A, then $A[x_1, \cdots, x_n]$ is integral over A.

(iii) Let $A \subseteq B \subseteq C$ be rings. If C is integral over B and B is integral over A, then C is also integral over A.

(iv) Let R be a ring, A a subring , and let A' be the set of all elements $x \in R$ which are integral over A. Then A' is a subring of R, which is integrally closed in R, and integral over A.

Definition 1.8. *(i) Let R be a ring, A a subring. the ring $A' = \{x \in R \mid x \text{ is integral over } A\}$ is called the integral closure of A in R.*

(ii) A is called integrally closed in R if $A' = A$.

(iii) An integral domain A is said to be integrally closed if it is integrally closed in its quotient field.

Remark 1.9. *(i) The ring A' in 1.8 is integrally closed in R, and integral over A.*

(ii) Every principal ideal domain is integrally closed.

(iii) Let R be a ring integral over the subring A, let $\theta : R \longrightarrow R'$ be a homomorphism from R onto the ring R' and $\theta(A) = A'$. Then R' is integral over the subring A'.

Definition 1.10. *A discrete valuation ring is a principal ideal domain that has exactly one nonzero prime ideal.*

Theorem 1.11. *The following conditions on an integral domain R are equivalent.*

(i) R is a Dedekind domain;

(ii) Every proper ideal in R is uniquely a product of a finite number of prime ideals;

(iii) Every non zero ideal in R is invertible;

(iv) Every fractional ideal of R is invertible;

(v) The set of all fractional ideals of R is a group under multiplication;

(vi) Every ideal in R is projective;

(vii) Every fractional ideal of R is projective;

(viii) R is Notherian, integrally closed and every nonzero prime ideal is maximal;

(ix) R is Notherian and for every nonzero prime ideal P of R, the localization R_p of R at P is a discrete valuation ring.

Notation 1.12. *When I is a fractional ideal and n a positive integer, we shall write I^{-n} to mean $(I^{-1})^n$.*

Theorem 1.13. *If R is a Dedekind domain, then any fractional ideal I can be uniquely expressed as a product*

$$P_1^{a_1} \cdots P_n^{a_n}$$

with $P_1, \cdots, P_n$ distinct prime ideals of R and $a_1, \cdots, a_n$ integers (positive or negative).

Proof. Let I be a fractional ideal with generators $m_1, \cdots, m_k$. Each m_i is in K so there is a common denominator s in R such that $m_i s$ is also in R. It follows that $Is \subseteq R$. There exist factorizations of the ideals Rs and Is as

$$Rs = \Pi\, Q_j^{b_j} \quad , \quad Is = \Pi\, P_i^{a_i},$$

where the P_i and the Q_j are the prime ideals of R. It follows

$$IQ_1^{b_1} \cdots Q_t^{b_t} = P_1^{a_1} \cdots P_k^{a_k}.$$

We have seen that prime ideals are invertible, so we obtain

$$I = \Pi P_i^{a_i} \cdot \Pi Q_j^{-b_j}$$

This establishes the existence of a factorization of I as a product of prime ideals with integral exponents. Now we obtain uniqueness as follows. Suppose

$$I = \Pi P_i^{a_i} \, \Pi Q_j^{-b_j} = \Pi \, \mathfrak{P}_i^{c_i} \, \Pi \, \mathfrak{Q}_j^{-d_j},$$

where $P, Q, \mathfrak{P}, \mathfrak{Q}$ denote prime ideals and the a_i, b_j, c_i, d_j are positive integers. Then we have $\Pi P_i^{a_i} \, \Pi \mathfrak{Q}_j^{d_j} = \Pi \mathfrak{P}_i^{c_i} \, \Pi Q_j^{b_j}$. This is a factorization of ideals in R so the uniqueness statement for ideals in R can be used to get the uniqueness of the expression for I. □

The discussion to this point shows that the collection of all fractional ideals in a Dedekind ring forms a group under the rule of multiplication of fractional ideals . We denote this group by $\boldsymbol{I}(R)$ and call it simply the *ideal group* of R. The uniqueness statement of Theorem 1.13 implies that $\boldsymbol{I}(R)$ is a free abelian group with the collection of nonzero prime ideals as free generators. Generally, this is an infinitely generated group.

There is a subgroup of particular interest. Namely the collection of all principal fractional ideals Rx with x in K forms a subgroup of $\boldsymbol{I}(R)$ which is denoted by $\boldsymbol{P}(R)$. We let

$$\boldsymbol{C}(R) = \boldsymbol{I}(R)/\boldsymbol{P}(R)$$

and call $\boldsymbol{C}(R)$ the *class group* of R. The class group is an important invariant of the ring R. While $\boldsymbol{I}(R)$ and $\boldsymbol{P}(R)$ may be very large abelian groups, the class group can be very small. The following is an example:

Definition 1.14. *A number field K is a subfield of $\mathbb{C}$ having finite (degree as a vector space) over $\mathbb{Q}$. The integral closure of $\mathbb{Z}$ in K is denoted $\mathcal{O}_K$ and called the ring of integers of K.*

Remark 1.15. *Every number field has the form $\mathbb{Q}[\alpha]$ for some algebraic number $\alpha \in \mathbb{C}$. The structure of the integer rings is more complicated.*

Theorem 1.16. *The class group $\boldsymbol{C}(\mathcal{O}_K)$ (also called the class group of K) is finite.*

Definition 1.17. *An order is an integral domain A whose integral closure $\widetilde{A}$(in the quotient field) ia a Dedekind domain, and a finitely generated A-module. In this situation we also say that A is an order of $\widetilde{A}$.*

Valuation rings and orders
In this section we give a new view on valuation rings. Let K be a field.

Definition 1.18. *A (absolut or exponential) valuation on K is map*

$$| \, | : \; K \longrightarrow \mathbb{R}$$

with the properties,

(i) $|X| \geq 0$, *and* $|X| = 0$ *if and only if* $X = 0$

(ii) $|XY| = |X| \cdot |Y|$

(iii) $|X + Y| \leq |X| + |Y|$.

Example 1.19. *Let p be a prime number then one can define pe-adic valuation $| \, |_p$ on $\mathbb{Q}$ by $|m|_p = p^{-n_p}$ wether $m = \Pi q^{n_q}$ is prime decomposition with all prime q, and $| \, 0 \, |_p = 0$. The properties (i), (ii), (iii) are clear.*

A valuation field is a field with a valuation. Let $(K, | \, |)$ be a valuation field, then we have a metric on K by $d(X, Y) = |X - Y|$, addition and multiplication are continuous respected to this metric.

Definition 1.20. *Two valuations $| \, |_1$ and $| \, |_2$ on K are called equivalent, if there is a real number $t \geq 0$ such that for every $X \in K$, $|X|_1 = |X|_2^t$: This clearly an equivalence realation .*

Two valuations are equivalent if and only if define the same topologies.

Definition 1.21. *The valuation $| \, |$ is called non-archimedian, if $|n|$ is bounded for all $n \in \mathbb{N}$ and archimedian otherwise.*

Lemma 1.22. *A valuation $| \; |$ is non-archimedian if and only if satisfies in strong triangle equality that is*

$$|X + Y| \leq \; max \; (|X|, |Y|).$$

Definition 1.23. *A normalized discrete valuation v on a field K is a surjective group homomorphism of the multiplicative group $K^* = K \setminus \{0\}$ onto the additive group $\mathbb{Z}$ with the property*

$$v(x + y) \geq min(v(x), v(y)) \qquad (1)$$

for all $x, y \in K^*$ with $x \neq -y$.

By defining $|x| = \varepsilon^{v(x)}$ for any $0 < \varepsilon < 1$, this is the same as a non-archimedean valuation with value group $|K^\times|$ isomorphic to $\mathbb{Z}$.

Lemma 1.24. *Let v be a discrete valuation on the field K. Then the set*

$$\Lambda = \{x \in K^* \mid v(x) \geq 0\} \cup \{0\}$$

is a principal ideal domain with just one nonzero prime ideal

$$\mathfrak{m} = \{x \in K^* \mid v(x) > 0\} \cup \{0\}.$$

By choosing an element $\pi \in \Lambda$ with $v(\pi) = 1$, every element $x \in K^*$ can be written uniquely as $x = \pi^{v(x)}u$ where $u \in \Lambda^* = \{x \in K^* \mid v(x) = 0\}$.

Proof. See [Ke]. □

Definition 1.25. *Let v be a discrete valuation on K. The ring Λ in the above lemma is called the valuation ring of K relative to v, and every element $\pi \in \Lambda$ with $v(\pi) = 1$ is called a prime element.*

Some topics in commutative algebra

Let us first recall some notions and properties of topological groups. An (abelian) *topological group* is an abelian group G endowed with the structure of a topological space for which the homomorphism $G \times G \longrightarrow G$ defined by $(x, y) \longmapsto x - y$ is continuous. Such a structure is entirely determined by giving a fundamental system $\mathcal{V}$ of neighbourhoods of 0 such that for $V \in \mathcal{V}$ there are $V_1, V_2 \in \mathcal{V}$ with $V_1^{-1} \subseteq V$ and $V_2 \cdot V_2 \subseteq \mathcal{V}$.

A (desending) *filtration* $(G_n)_n$ (i.e., a descending chain of subgroups $(G_n)_n$ of G) defines a unique structure of topological group on G for which the G_n form a fundamental system of neighborhoods of 0. In this section we are essentially interested in topologies of this type. For this topology, G is *separated*(i.e., Hausdorff) if and only if $\cap_n G_n = \{0\}$. Two filtrations $(G_n)_n, (G'_n)_n$ define the same topology on G if and only if for every n, there exists an m such that $G'_m \subseteq G_n$, and vice versa.

Let G be a topological group defined by a filtration $(G_n)_n$. A sequence $(x_m)_m$ of elements of G is called a *Cauchy sequence*, if for every n there exists an m_0 such that $x_m - x_{m_0} \in G_n$ for every $m \geq m_0$. The topological group G is *complete* if every Cauchy sequence has a limit in G. One way to construct complete groups is to construct inverse limits.

An *inverse system* (of sets) consists of a collection of sets $(A_n)_{n\geq 0}$ and maps $\pi_n : A_{n+1} \longrightarrow A_n$ for every n. The *inverse limit* of the $(A_n)_n$ is the set

$$\underset{\leftarrow,n}{lim} A_n := \{(a_n)_n \in \Pi A_n | a_n = \pi_n(a_{n+1}) \text{ for all n}\}$$

For every m, the projection onto the m-th coordinate defines a canonical map

$$p_m : \underset{\leftarrow,n}{lim} A_n \longrightarrow A_m.$$

Let G be a topological group defined by a filtration $(G_n)_n$. We then have a natural inverse system $(G/G_n)_n$. Consider $\hat{G} := lim_{\leftarrow,n}(G/G_n)$. Let

$$\hat{G}_n := \{(a_m)_m \in \hat{G} \mid a_m = 0 \text{ for every } m \leq n\}$$

This defines a filtration $(\hat{G}_n)_n$ on $\hat{G}$ and makes latter into a topological group. Let X be a topological space, $Y \subset X$ a closed irreducible subset.

Remark 1.26. *If R is a ring and $\mathfrak{a} \subseteq R$ is an ideal, then*

$$\hat{R} = \underset{\infty\leftarrow n}{lim} R/\mathfrak{a}^n$$

becomes a ring in natural way and is called the $\mathfrak{a}$-adic completion of R.

Definition 1.27. *Let A be a commutative ring with unit. The ring of formal power series in one variable $A[[T]]$ is defined in the following way. Let $A^{\mathbb{N}}$ be the group of sequence with coefficient in A. To simplify we denote a sequence $(a_n)_{n\geq 0}$ by*

$$a_0 + a_1T + a_2T^2 + \cdots.$$

We endow $A^{\mathbb{N}}$ with a multiplicative law by setting

$$\left(\sum_{i\geq 0} a_iT^i\right)\left(\sum_{j\geq 0} b_jT^j\right) = \sum_{k\geq 0} c_kT^k,$$

where $c_k = \sum_{i+j=k} a_ib_j$. This ring clearly contains the polynomial ring $A[T]$. We define inductively the ring of formal power series

$$A[[T_1, \cdots, T_r]] = A[[T_1, \cdots, T_{r-1}]][[T_r]].$$

Proposition 1.28. *Let A be Noetherian ring; then the ring of formal power series $A[[T_1, \cdots, T_r]]$ is also Notherian.*

Definition 1.29. *Let K be a field. A ring A is called a K-algebra, if it is a K-vector space such that*

$$(\lambda a)b = a(\lambda b) = \lambda(ab) \qquad \forall \lambda \in K, \quad a, b \in A.$$

The dimension $dim_K A$ of a K-Algebra A is the dimension of A as K-vector space.

Definition 1.30. *A map of K-algebras $f : A \longrightarrow B$ is called a K- algebra homomorphism, if f is a K-linear ring homorphism.*

Some topics from algebraic geometry

In this subsection we briefly discuss the notion of schemes and algebraic curves which we will consider in our thesis. For more about this topic, see [Liu].

Zariski topology

Let A be a (commutative) ring (with unit). We let *SpecA* denote the set of prime ideals of A. It is called the *spectrum* of A. By convention, the unit ideal is not a prime ideal. Thus $Spec\, 0 = \emptyset$.

We will now endow *SpecA* with the structure of a topological space. For any ideal I of A, let $V(I) := \{\mathfrak{p} \in SpecA \mid I \subseteq \mathfrak{p}\}$. If $f \in A$, let $D(f) := SpecA \setminus V(fA)$.

Proposition 1.31. *Let A be a ring. We have the following properties:*

(i) For any pair of ideals I,J of A, we have $V(I) \cup V(J) = V(IJ)$.

(ii) Let $(I_\lambda)_\lambda$ be a family of ideals of A. Then $\cap_\lambda V(I_\lambda) = V(\sum_\lambda I_\lambda)$.

(iii) $V(A) = \emptyset$ and $V(0) = SpecA$.

Proof. See [Liu] □

Remark 1.32. *In particular, there exists a unique topology on SpecA whose closed subsets are the sets of the form $V(I)$ for an ideal I of A. Moreover, the sets of the form $D(f), f \in A$, constitute a base of open subsets of SpecA.*

Definition 1.33. *Let A be a ring. We call the topology defined by above proposition the* Zariski topology *on Spec A. An open set of the form $D(f)$ is called a* principal open subset*, while its complement $V(f) := V(fA)$ is called a* principal closed subset.

In the following, the set $SpecA$ will always be endowed with the Zariski topology.

Remark 1.34. *Let $\mathfrak{p} \in SpecA$. Then the singleton $\{\mathfrak{p}\}$ is closed for the Zariski topology if and only if $\mathfrak{p}$ is a maximal ideal of A. We will then say that $\mathfrak{p}$ is a closed point of $SpecA$. More generally, a point x of a topological space is said to be closed if the set $\{x\}$ is closed.*

Sheaves

Definition 1.35. *Let X be topological space. A presheaf $\mathcal{F}$ (of abelian groups) on X consists of the following datas*

(i) an abelian group $\mathcal{F}(U)$ for every open subset U of X, and

(ii) a group homomorphism (restriction map) $\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ for every pair of open subsets $V \subseteq U$

which verify the following conditions :

(iii) $\mathcal{F}(\emptyset) = 0$;

(iv) $\rho_{UU} = Id$;

(v) if we have three open subsets $W \subseteq V \subseteq U$, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

An element $s \in \mathcal{F}(U)$ is called a *section* of $\mathcal{F}$ over U. We let $s|_V$ denote the element $\rho_{UV}(s) \in \mathcal{F}(V)$ and we call it the *restriction* of s to V.

Definition 1.36. *We say that a presheaf $\mathcal{F}$ is a sheaf if it has the following properties:*

(i) (Uniqueness) Let U be an open subset of X, $s \in \mathcal{F}(U)$, $(U_i)_{i\in I}$ an open covering of U. If $s|_{U_i} = 0$ $\forall i \in I$ then $s = 0$.

(ii) (Glueing local sections) Let U be an open subset of X and let $(U_i)_{i\in I}$ be an open covering of U. Let $s_i \in \mathcal{F}(U_i)$, $(i \in I)$ be sections such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. Then there exists a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ $\forall i \in I$ (this section s is unique by condition (1)).

We can define in the same way *sheaves* of *rings*, *sheaves* of *algebras* over a fixed ring, etc. There is a natural notion of *(sub)sheaf* $\mathcal{F}'$ of $\mathcal{F}$: $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$, and the restriction ρ'_{UV} is induced by ρ_{UV}

Example 1.37. *Let X be a topological space. For any open subset U of X, let $\mathcal{C}(U) = C^0(U, \mathbb{R})$ be the set of continuous functions from U to $\mathbb{R}$. The restrictions ρ_{UV} are the usual restrictions of functions. Then $\mathcal{C}$ is a sheaf on X. If we let $\mathcal{F}(U) = \mathbb{R}^U$ be the set of functions on U with values in $\mathbb{R}$, this defines a sheaf $\mathcal{F}$ of which $\mathcal{C}$ is a (sub)sheaf.*

Example 1.38. *Let A be a ring. We want to define a sheaf on $X = SpecA$ as follows: As mentioned above, the collection of all subsets*

$$D(f) := \{\mathfrak{p} \in SpecR \mid f \notin \mathfrak{p}\} = X \setminus V((f)),$$

where f runs through all elements of A, is a base of Zariski topology on X. It is easy to see (cf. [Liu]) that a sheaf on X is already characterized by its values on the open subsets $D(f)$ and its restriction maps with respect to inclusions $D(fg) \subset D(f)$. We may therefore characterize a natural sheaf of rings, the structure sheaf $\mathcal{O}_X$ *on X by*

$$\mathcal{O}_X(D_f) := A_f \quad (\text{ where } A_f \text{ is the f-localization of } A) \text{ and}$$
$$\rho_{D(f)D(fg)} : A_f \to A_{fg} \quad \text{the canonical morphism.}$$

Definition 1.39. *Let $\mathcal{F}$, $\mathcal{G}$ be two (pre)sheaves on X. A morphism of (pre)sheaves $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ consists of a collection of homomorphisms $\alpha(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ where U runs through all open subsets of X, which is compatible with the restrictions ρ_{UV}.*

A morphism of (pre)sheaves α is called *injective* if for every open subset U of X, the homomorphism $\alpha(U)$ is injective (take care : a surjective morphism of sheaves is not defined in the same way). We can, of course, compose two morphisms of (pre)sheaves. An *isomorphism* is an invertible morphism α. This amounts to say that $\alpha(U)$ is an isomorphism for every open subset U of X .

Definition 1.40. *Let $f : X \to Y$ be a continuous map between topological spaces and let $\mathcal{F}$ be a sheaf on X. Then the* push forward *of $\mathcal{F}$ is a sheaf $f_* \mathcal{F}$ on Y which is defined by*

$$f_* \mathcal{F}(V) := \mathcal{F}\left(f^{-1}(V)\right) \quad \forall \text{ open } V \subset Y$$

together with the obvious restriction maps.

Definition 1.41. *A* locally ringed space *is a pair* $(X, \mathcal{O}_X)$*, where* X *is topological space and* $\mathcal{O}_X$ *is a sheaf of rings on* X *such that*

$$\mathcal{O}_{X,x} := \varinjlim_{U \ni x} \mathcal{O}_X(U)$$

is a local ring for all $x \in X$

Definition 1.42. *A* morphism of locally ringed spaces

$$(f, f^{\#}) \; : \; (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

consists of a continuous map $f : X \longrightarrow Y$ *and a morphism* $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ *of sheaves of rings which for each* $x \in X$ *induces a local homomorphism* $O_{Y,f(x)} \to O_{X,x}$ *of local rings. Such a morphism is an* isomorphism*, if* f *is a homeomorphism and* $f^{\#}$ *an isomorphism of sheaves.*

Remark 1.43. *For any* $\mathfrak{p} \in X = SpecA$*, the ring* $\mathcal{O}_{X,\mathfrak{p}}$ *is canonically isomorphic to the local ring* $A_{\mathfrak{p}}$*. In particular,* $(X, \mathcal{O}_X)$ *is a locally ringed space.*

Schemes

Definition 1.44. *We define an affine scheme to be a locally ringed space isomorphic to some* $(SpecA, \mathcal{O}_{SpecA})$ *constructed as above. By abuse of notation, the latter will often be denoted simply by* $SpecA$*.*

Definition 1.45. *A topological space* X *is called irreducible if for any decomposition* $X = A_1 \cup A_2$ *with closed subsets* $A_i \subset X (i = 1, 2)$ *we have* $X = A_1$ *or* $X = A_2$*. A subset* X' *of a topological space* X *is called irreducible if* X' *is irreducible as a space with the induced topology.*

Definition 1.46. *Let* X *be a topological space. By definition the Krull* dimension *or combinatorial dimension of* X *is the number*

$$\dim(X) := \max\{r \in \mathbb{N} \mid \exists \;\; Y_0 \subsetneq \cdots \subsetneq Y_r \subset X, \;\; Y_i \textit{ closed and irreducible}\}$$

The dimension of an affine scheme X is the dimension of its underlying topological space in the sense defined above.

Definition 1.47. *An* algebraic *affine k-scheme is an affine scheme* $X = SpecA$ *such that* A *is a finitely generated k-Algebra*

Definition 1.48. *An* integral *affine scheme is an affine scheme $X = SpecA$ such that A is an integral domain.*

Definition 1.49. *An affine algebraic curve over a field k is a one-dimensional integral algebraic affine k-scheme.*

Example 1.50. *$Spec(k[X])$ is an affine curve over k. $Spec\mathbb{Z}$ though being one-dimensional and integral is not an affine curve over any field k.*

Definition 1.51. *A scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that there is an open covering $X = \cup_{i \in I} U_i$ with the property that there are a family of rings $(A_i)_{i \in I}$ and isomorphisms*

$$(U_i, \mathcal{O}|_{U_i}) \simeq (Spec\ A_i\ , \mathcal{O}_{SpecA_i}) \quad \forall i \in I \quad .$$

Definition 1.52. *Let k be a field. A k-scheme* of finity type *is a scheme $(X, \mathcal{O}_X)$ such that there is a finite open covering $X = \cup_{i=1}^n U_i$ and isomorphisms*

$$(U_i, \mathcal{O}_X|u_i) \simeq (SpecA_i, \mathcal{O}_{SpecA_i}) \qquad \forall i \in \{1, \cdots, n\}$$

where the A_i are finitely generated k-Algebras.

Definition 1.53. *Let X be a k-scheme of finite type. X is said to be* separated *(resp.* complete*), if for every discrete valuation ring A with quotient field K and every commutative diagramm*

$$\begin{array}{ccc} Spec\ K & \longrightarrow & X \\ \downarrow & & \downarrow \\ Spec\ A & \longrightarrow & Spec\ k \end{array}$$

there exists a (resp. there exists a unique) morphism $SpecA \longrightarrow X$ such that following diagramm is commutative

$$\begin{array}{ccc} Spec\ K & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ Spec\ A & \longrightarrow & Spec\ k \end{array}$$

Example 1.54. *The k-scheme $X = Spec(k[T])$ is not complete: We may take $A = k[[t]]$ and $K = k((T))$.*

Definition 1.55. *Let k be a field. An* algebraic *k-scheme is a separated k-scheme of finite type.*

Definition 1.56. *Let k be a field. An* algebraic curve *over k is a one-dimensional integral algebraic k-scheme .*

Remark 1.57. *Since every affine scheme is separated, an affine algebraic curve over k is also an algebraic curve over k.*

Remark 1.58. *Fibre products exist in the category of schemes. Let X be a k-algebraic curve and let $\overline{k}$ be an algebraic closure of k. Then we often write $X \otimes_k \overline{k}$ instead of $X \ \times_{Speck} Spec\overline{k}$.*

Definition 1.59. *A local ring A with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$ is called regular, if A is noetherian, and $deim\mathfrak{m}/\mathfrak{m}^2 = dimA$.*

Definition 1.60. *Let $(X, \mathcal{O}_X)$ be a scheme, and let $x \in X$. We say that X is* regular *(or* non-singular*) at $x \in X$, or that x is a* regular point*, if $\mathcal{O}_{X,x}$ is a regular local ring. A point $x \in X$, which is not regular, is called a* singular *point of X. A scheme is called* regular*, if it is regular at all of its points. A scheme which is not regular is said to be* singular*.*

Definition 1.61. *Let X be an algebraic curve on k , and $x \in X$ be a point of X. A* branch *of X at x is a maximal ideal in the normalization $\widetilde{\mathcal{O}}_{X,x}$ of $\mathcal{O}_{X,x}$.*

Remark 1.62. *If x is a regular point of an algebraic curve X, then $\widetilde{\mathcal{O}}_{X,x} = \mathcal{O}_{X,x}$, and therefore at every regular point X has only one branch.*

Definition 1.63. *Let x be a singular point of an algebraic curve X , then we say the point x is* unibranch *if X has only one branch at x.*

Definition 1.64. *Let X be a topological space, a point $\xi \in X$ is called a generic point of X, if $\overline{\{\xi\}} = X$.*

Example 1.65. *Let R be an integral domain, then $\xi = (0) \in SpecR$ is a generic point of SpecR.*

Lemma 1.66. *Let X be an integral scheme with generic point ξ, then the local ring $\mathcal{O}_{X,\xi}$ is a field .*

Definition 1.67. *The field in the above lemma is said to be the rational functional field of X.*

2 Zeta functions of orders: definition and basic properties

Let k be a finite field of order q, and let $\mathcal{O}$ be an integral local k-algebra of dimension 1 whose residue field κ is finite. Let K be the quotient field of $\mathcal{O}$. We will assume that $\mathcal{O}$ is an order, i.e., that the integral closure $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ is a finitely generated $\mathcal{O}$ module.

Then, for fractional $\mathcal{O}$-ideals $\mathfrak{a}$ and $\mathfrak{b}$ with $\mathfrak{b} \subseteq \mathfrak{a}$ the $\mathcal{O}$-module $\mathfrak{a}/\mathfrak{b}$ has finite length. Hence it also has finite k-dimension, and we have

$$\#(\mathfrak{a}/\mathfrak{b}) = q^{\dim(\mathfrak{a}/\mathfrak{b})}.$$

We consider generalized zeta-functions by associating to each $\mathcal{O}$-ideal $\mathfrak{d}$ the Dirichlet series

$$\zeta(\mathfrak{d}, s) := \sum_{\mathfrak{a} \supseteq \mathfrak{d}} \#(\mathfrak{a}/\mathfrak{d})^{-s} \tag{2}$$

where $\mathfrak{a}$ runs through the (fractional) ideals containing $\mathfrak{d}$. We can write this series as follows as a power series in $t = q^{-s}$ with integer coefficients

$$\begin{aligned} Z(\mathfrak{d}, t) :&= \sum_{\mathfrak{a} \supseteq \mathfrak{d}} t^{\dim(\mathfrak{a}/\mathfrak{d})} \qquad (3) \\ &= \sum_{n=0}^{\infty} \#\{\mathcal{O}\text{-ideals that admit } \mathfrak{d} \text{ as subspace of codimension } n\}\, t^n. \end{aligned}$$

We define the *degree* $\deg(\mathfrak{a})$ of each $\mathcal{O}$-ideal $\mathfrak{a}$ by the properties $\deg(\mathcal{O}) = 0$ and

$$\dim(\mathfrak{a}/\mathfrak{b}) = \deg(\mathfrak{a}) - \deg(\mathfrak{b}), \tag{4}$$

whenever $\mathfrak{a} \supseteq \mathfrak{b}$. We will show that this is well-defined, and at the same time will deduce some properties:

If $\mathfrak{a} \subseteq \mathcal{O}$, then by (4) we must have $\dim(\mathcal{O}/\mathfrak{a}) = \deg\mathcal{O} - \deg\mathfrak{a}$ and thus define

$$\deg\, \mathfrak{a} = -\dim\, \mathcal{O}/\mathfrak{a}$$

Definition 2.1. *If $\alpha \in \mathcal{O}$, let by definition deg $\alpha = -$dim $\mathcal{O}/\alpha\mathcal{O}(=$ deg $\alpha\mathcal{O})$.*

Lemma 2.2. *If $\alpha \in \mathcal{O}$ and $\mathfrak{a} \subseteq \mathcal{O}$, then*

$$deg\ \alpha\ \mathfrak{a} = deg\ \mathfrak{a} + \ deg\ \alpha.$$

Proof. By the above, $\deg\alpha\mathfrak{a} = -\dim\mathcal{O}/\alpha\mathfrak{a}$, and clearly

$$\deg\mathfrak{a} = -\dim\mathcal{O}/\mathfrak{a} = -\dim\alpha\mathcal{O}/\alpha\mathfrak{a}.$$

On the other hand we have an exact sequence

$$0 \longrightarrow \alpha\mathcal{O}/\alpha\mathfrak{a} \hookrightarrow \mathcal{O}/\alpha\mathfrak{a} \twoheadrightarrow \mathcal{O}/\alpha\mathcal{O} \longrightarrow 0,$$

which implies

$$\dim\, \mathcal{O}/\alpha\mathfrak{a} = \dim\, \alpha\mathcal{O}/\alpha\mathfrak{a} + \dim\, \mathcal{O}/\alpha\mathcal{O} \quad \text{or} \quad \deg\, \alpha\mathfrak{a} = \deg\mathfrak{a} + \deg\alpha.$$

□

Lemma 2.3. *For every fractional ideal $\mathfrak{a}$ and every $\alpha \in \mathcal{O}$ such that $\alpha\mathfrak{a} \subseteq \mathcal{O}$, the integer deg $\alpha\mathfrak{a} -$ deg α is independent of α.*

Proof. Let $\beta \in \mathcal{O}$ with $\beta\mathfrak{a} \subseteq \mathcal{O}$ then

$$\deg\, \beta(\alpha\mathfrak{a}) = \deg\, \alpha(\beta\mathfrak{a}).$$

By Lemma 2.2 we conclude

$$\deg\, \alpha\mathfrak{a} + \deg\, \beta = \deg\, \beta\mathfrak{a} + \deg\alpha$$

or, $\deg\alpha\mathfrak{a} - \deg\alpha = \deg\beta\alpha - \deg\beta$. □

With this we get a well-defined degree for a fractional ideal in the general case.

Definition 2.4. *For every fractional ideal* $\mathfrak{a}$ *by definition*

$$deg\ \mathfrak{a} := deg\ \alpha\mathfrak{a} - deg\ \alpha,$$

where $\alpha \in \mathcal{O}$ *with* $\alpha\mathfrak{a} \subseteq \mathcal{O}$.

It remains to show that property (4) holds. But if $\mathfrak{b} \subseteq \mathfrak{a}$ are fractional ideals, and $\alpha \in \mathcal{O}$ with $\alpha\mathfrak{a} \subseteq \mathcal{O}$, then $\alpha\mathfrak{b} \subseteq \alpha\mathfrak{a} \subseteq \mathcal{O}$, and we have an exact sequence

$$0 \longrightarrow \alpha\mathfrak{a}/\alpha\mathfrak{b} \hookrightarrow \mathcal{O}/\alpha\mathfrak{b} \twoheadrightarrow \mathcal{O}/\alpha\mathfrak{a} \longrightarrow 0.$$

We get

$$\dim \mathcal{O}/\alpha\mathfrak{b} = \dim \alpha\mathfrak{a}/\alpha\mathfrak{b}\ +\ \dim \mathcal{O}/\alpha\mathfrak{a}$$

and thus

$$\deg \mathfrak{a}\ -\ \deg \mathfrak{b} = \dim \alpha\mathfrak{a}/\alpha\mathfrak{b} = \dim \mathfrak{a}/\mathfrak{b}$$

so wanted.

Remark 2.5. *Let* β *and* γ *be in* $\mathcal{O}$, *then* $deg\ \lambda\beta = deg\lambda + deg\beta$ *and* $deg\ \beta/\gamma = deg\ \beta - deg\ \gamma$.

In fact, by Lemma 2.2

$$\deg \lambda\beta = \deg \lambda(\beta\mathcal{O}) = \deg\lambda + \deg\beta\mathcal{O}$$

and

$$\deg \beta\mathcal{O} = \deg\gamma(\beta/\gamma\mathcal{O}) = \deg(\beta/\gamma\mathcal{O}) + \deg\gamma.$$

Lemma 2.6. *For every fractional ideal* $\mathfrak{a}$ *and* $\alpha \in K$

$$deg\ \alpha\mathfrak{a} = deg\ \alpha + deg\ \mathfrak{a}$$

Proof. There is a $\lambda \in \mathcal{O}$ such that $\lambda\alpha\mathfrak{a} \subseteq \mathcal{O}$. Let $\alpha = \beta/\gamma$ with $\beta, \alpha \in \mathcal{O}$. Then by Lemma 2.2

$$\begin{aligned} \deg\mathfrak{a} + \deg\lambda\beta &= \deg(\lambda\beta)\mathfrak{a} \\ = \deg \gamma(\lambda\beta/\gamma)\ \mathfrak{a} &= \deg\ (\lambda\beta/\gamma)\mathfrak{a} + \deg\gamma \end{aligned}$$

Together with remark 2.5 we have

$$\begin{aligned} \deg \mathfrak{a} + \deg \lambda + \deg \beta &= \deg\mathfrak{a} + \deg\lambda\beta \\ = \deg\ (\lambda\beta/\gamma)\ \mathfrak{a} + \deg \gamma &= \deg\lambda + \deg \alpha\mathfrak{a} + \deg \gamma, \end{aligned}$$

and the claim follows, again with remark 2.5. □

When $\mathfrak{d} = \mathcal{O}$ then the power series (3) encodes the numbers of positive $\mathcal{O}$-ideals of given degrees

$$\begin{aligned} Z(\mathcal{O},t) &= \sum_{\mathfrak{a}\supseteq\mathcal{O}} t^{\ \deg(\mathfrak{a})} \\ &= \sum_{n=0}^{\infty} \#\{\text{positive } \mathcal{O}\text{-ideals of degree } n\} t^n. \end{aligned}$$

Definition 2.7. *We let $r = dim\ (\mathcal{O}/\mathfrak{m})$, the degree of the residue field of $\mathcal{O}$ over the constant field k.*

The integral closure $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ is a semi local principal ideal domain, whose maximal ideals, say $\mathfrak{p}_1, ..., \mathfrak{p}_m$, correspond bijectively to the branches of $\mathcal{O}$. We denote by

$$d_i := \dim(\widetilde{\mathcal{O}}/\mathfrak{p}_i)$$

the degree of the residue field of $\mathfrak{p}_i$ over the constant field k (i=0,...,m). By 1.13 the $\widetilde{\mathcal{O}}$-ideals are just of the form

$$\mathfrak{p}^{\mathbf{n}} := \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_m^{n_m},$$

where $\mathbf{n} = (n_1, ..., n_m) \in \mathbb{Z}^m$. Clearly the $\mathbf{n}$ corresponding to $\widetilde{\mathcal{O}}$ is $\mathbf{0} = (0, ..., 0)$.

Definition 2.8. *For $z \in K$ we define $ord_{\mathfrak{p}_i}(z) = n_i$ where $z\widetilde{\mathcal{O}} = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_m^{n_m}$.*

Lemma 2.9. *For $\boldsymbol{n} = (n_1, \cdots .n_m) \geq 0$ (i.e., $n_i \geq 0\ \forall\ i = 1, \cdots, m$) we have*

$$dim(\widetilde{\mathcal{O}}/\mathfrak{p}^{\boldsymbol{n}}) = \boldsymbol{n}.\boldsymbol{d} := \sum_{i=1}^{m} n_i d_i.$$

Proof. For every $n_i \geq 0$ and $\mathfrak{p}_i$ we have an exact sequence

$$0 \longrightarrow \mathfrak{p}_i^{n_i}/\mathfrak{p}_i^{n_i+1} \hookrightarrow \widetilde{\mathcal{O}}/\mathfrak{p}_i^{n_i+1} \twoheadrightarrow \widetilde{\mathcal{O}}/\mathfrak{p}_i^{n_i} \longrightarrow 0.$$

It yields

$$\dim_k \widetilde{\mathcal{O}}/\mathfrak{p}_i^{n_i+1} = \dim_k \widetilde{\mathcal{O}}/\mathfrak{p}_i^{n_i} + \dim_k \mathfrak{p}_i^{n_i}/\mathfrak{p}_i^{n_i+1}.$$

By induction $\dim_k \widetilde{\mathcal{O}}/\mathfrak{p}_i^{n_i} = n_i d_i$, since $\mathfrak{p}_i^{n_i}/\mathfrak{p}_i^{n_i+1} \simeq \widetilde{\mathcal{O}}/\mathfrak{p}_i$. On the other hand, for $\mathfrak{a} = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_m^{n_m}$ we have

$$\widetilde{\mathcal{O}}/\mathfrak{a} \simeq \widetilde{\mathcal{O}}/\mathfrak{p}_1^{n_i} \times \cdots \times \widetilde{\mathcal{O}}/\mathfrak{p}_m^{n_m}$$

by the Chinese remainder theorem. Therefore

$$\dim_k \widetilde{\mathcal{O}}/\mathfrak{p}^{\mathbf{n}} = \sum_{i=1}^{m} \dim_k \widetilde{\mathcal{O}}/\mathfrak{p}_i^{n_i} = \sum_{i=1}^{m} n_i d_i$$

□

Lemma 2.10.

$$Z(\widetilde{\mathcal{O}}, t) = \prod_{i=1}^{m} \frac{1}{1 - t^{d_i}}$$

Proof. By definition

$$\zeta(\widetilde{\mathcal{O}}, s) = \sum_{\mathfrak{a} \supseteq \widetilde{\mathcal{O}}} \#(\mathfrak{a}/\widetilde{\mathcal{O}})^{-s}.$$

Let $\mathfrak{a} = \mathfrak{p}^{\mathbf{n}} = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_m^{n_m}$. Because $\widetilde{\mathcal{O}} \subseteq \mathfrak{a} = \mathfrak{p}^{\mathbf{n}}$ we obtain $\mathbf{n} \leq 0$ or equivalently $-\mathbf{n} \geq 0$, but we have $\mathfrak{p}^{\mathbf{n}}/\widetilde{\mathcal{O}} \simeq \widetilde{\mathcal{O}}/\mathfrak{p}^{-\mathbf{n}}$ and so $\#(\mathfrak{p}^{\mathbf{n}}/\widetilde{\mathcal{O}}) = \#(\widetilde{\mathcal{O}}/\mathfrak{p}^{-\mathbf{n}})$. Therefore, by Lemma 2.9

$$\begin{aligned} Z(\widetilde{\mathcal{O}}, t) &= \sum_{\boldsymbol{n} \leq 0} = \#(\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_m^{n_m}/\widetilde{\mathcal{O}})^{-s} \\ &= \sum_{\boldsymbol{n} \geq 0} \#(\widetilde{\mathcal{O}}/\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_m^{n_m})^{-s} \\ &= \sum_{\boldsymbol{n} \geq 0} (q^{\sum_{i=1}^{m} n_i d_i})^{-s} \\ &= \sum_{\boldsymbol{n} \geq 0} \prod_{i=1}^{m} q^{-n_i d_i s} \\ &= \prod_{i=1}^{m} \sum_{n_i \geq 0} (q^{-s d_i})^{n_i} \\ &= \prod_{i=1}^{m} \frac{1}{1 - q^{-s d_i}} = \prod_{i=1}^{m} \frac{1}{1 - t^{d_i}}. \end{aligned}$$

□

Lemma 2.11. *Let R be an integral domain and $\widetilde{R}$ be the integral closure of R in the quotient field K. Then there exists an I-ideal $\mathfrak{F}$ such that:*
(i) $\mathfrak{F} \subseteq R$.
(ii) If I is an ideal of $\widetilde{R}$ such that $I \subseteq R$ then $I \subseteq \mathfrak{F}$.

Proof. We define $\mathfrak{F} = R : \widetilde{R} := \{\alpha \in K \ |\alpha\widetilde{R} \subseteq R\}$ and show that it satisfies (i) and (ii):
Since $1 \in \widetilde{R}$, clearly $\mathfrak{F} \subseteq R$, and it is obviously an ideal of $\widetilde{R}$. On the other hand, if I is an ideal of $\widetilde{R}$ contained in R, then $i\widetilde{R} \subseteq I \subseteq R$ for every $i \in I$, which yields $I \subseteq \mathfrak{F}$. □

Definition 2.12. *The ideal* $\mathfrak{F}$ *in the above lemma is called the conductor of* R*.*

For each i, let π_i be a generator of the maximal ideal $\mathfrak{p}_i$. Define $\boldsymbol{\pi^n} := \pi_1^{n_1} \cdots \pi_m^{n_m}$ for each $\mathbf{n} = (n_1, ..., n_m) \in \mathbb{Z}^m$. Then each $\mathcal{O}$-ideal is just of the form $\boldsymbol{\pi^n}\ \mathfrak{b}$ for some unique $\boldsymbol{n} \in \mathbb{Z}$ where $\mathfrak{b}$ is an $\mathcal{O}$-ideal satisfying $\mathfrak{b} \cdot \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}$. In fact:

Let $\mathfrak{a}$ be an $\mathcal{O}$-ideal, then $\mathfrak{a}\ \widetilde{\mathcal{O}}$ is an $\widetilde{\mathcal{O}}$-ideal, therefore

$$\mathfrak{a}\ \widetilde{\mathcal{O}} = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_m^{n_m} \qquad or \qquad \boldsymbol{\pi}^{\textbf{-n}}\ \mathfrak{a}\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}$$

for a unique $\boldsymbol{n} = (n_1, \cdots, n_m) \in \mathbb{Z}$. Now we take $\boldsymbol{\pi}^{-\mathbf{n}}\mathfrak{a} = \mathfrak{b}$.

An $\mathcal{O}$-ideal $\boldsymbol{\pi}^{\mathbf{n}}\mathfrak{b}$ contains $\mathfrak{d}$ if and only if $\boldsymbol{\pi}^{\mathbf{n}} \in \mathfrak{b} : \mathfrak{d}$, where for arbitrary fractional ideals $\mathfrak{a}, \mathfrak{b}$ we define

$$\mathfrak{b} : \mathfrak{a} = \{x \in K \mid x\mathfrak{a} \subseteq \mathfrak{b}\}$$

which is again a fractional ideal. Thus defining

$$\Gamma(\mathfrak{a}) := \{\mathbf{n} \in \mathbb{Z}^m \mid \boldsymbol{\pi}^{\mathbf{n}} \in \mathfrak{a}\ \}$$

and noting that

$$\dim \mathfrak{a}/\mathfrak{d} = \deg \mathfrak{a} - \deg \mathfrak{d} = \boldsymbol{n} \cdot \boldsymbol{d}\ +\ \deg \mathfrak{b}\ -\ \deg \mathfrak{d},$$

we obtain the partition

$$Z(\mathfrak{d}, t) = \sum_{\mathfrak{b}\cdot\widetilde{\mathcal{O}}=\widetilde{\mathcal{O}}} \left\{ \sum_{\mathbf{n}\in\Gamma(\mathfrak{b}:\mathfrak{d})} t^{\ \mathbf{n}\cdot\mathbf{d}+\deg(\mathfrak{b})-\deg(\mathfrak{b})} \right\},$$

where $\mathfrak{b}$ varies over the finitely many $\mathcal{O}$ - ideals satisfying $\mathfrak{b} \cdot \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}$. Note that the condition $\mathfrak{b} \cdot \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}$ implies $\mathfrak{F} \subseteq \mathfrak{b} \subseteq \widetilde{\mathcal{O}}$, where $\mathfrak{F}$ is the conductor ideal of $\mathcal{O}$, that is, the largest $\widetilde{\mathcal{O}}$-ideal contained in $\mathcal{O}$.

This partition depends on the choice of $\pi_1, \cdots, \pi_m$. To obtain a more natural partition we proceed as follows. Call two fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ equivalent ($\mathfrak{a} \sim \mathfrak{b}$), if $\mathfrak{a} \in K \backslash \{0\}$. Then define for every two $\mathcal{O}$-ideals $\mathfrak{d}$ and $\mathfrak{b}$ the *partial local zeta - function*

$$\zeta(\mathfrak{d}, \mathfrak{b}, s) := \sum_{\mathfrak{a} \supseteq \mathfrak{d}}^{\mathfrak{a} \sim \mathfrak{b}} \#(\mathfrak{a}/\mathfrak{d})^{-s},$$

where the sum is taken over the $\mathcal{O}$-ideals $\mathfrak{a}$ that contain $\mathfrak{d}$ and are equivalent to $\mathfrak{b}$. This Dirichlet series writes as follows as a power series in $t = q^{-s}(|t| < 1)$ with integer coefficients :

$$Z(\mathfrak{d}, \mathfrak{b}, t) = \sum_{\mathfrak{a} \supseteq \mathfrak{d}}^{\mathfrak{a} \sim \mathfrak{b}} t^{\dim(\mathfrak{a}/\mathfrak{d})} = \# \left\{ \begin{array}{l} \mathcal{O}\text{-ideals equivalent to } \mathfrak{b} \text{ that admit} \\ \mathfrak{d} \text{ as a subspace of codimension n} \end{array} \right\} t^n.$$

The partial zeta functions only depend on the classes of the ideals $\mathfrak{d}$ and $\mathfrak{b}$. Moreover the ideal class semigroup of $\mathcal{O}$ is finite, because as we have seen above, each $\mathcal{O}$-ideal is equivalent to an ideal $\mathfrak{b}$ with $\mathfrak{F} \subseteq \mathfrak{b} \subseteq \widetilde{\mathcal{O}}$, and $\widetilde{\mathcal{O}}/\mathfrak{F}$ is finite. We have the partition

$$Z(\mathfrak{d}, t) = \sum_{(\mathfrak{b})} Z(\mathfrak{d}, \mathfrak{b}, t).$$

Let $\mathfrak{c}$ be a dualizing $\mathcal{O}$-ideal, i.e., an ideal with $\mathfrak{c} : (\mathfrak{c} : \mathfrak{a}) = \mathfrak{a}$ for all (fractional) ideals, and abreviate $\mathfrak{a}^* = \mathfrak{c} : \mathfrak{a}$. For the following note that the set $U_{\mathfrak{b}} := \{u \in K \mid u\mathfrak{b} = \mathfrak{b}\}$ is a multiplicative group, which contains the group $U_{\mathcal{O}}$. See also 2.14 below.

Theorem 2.13. *(Reciprocity Formula for the Partial Local Zeta - Functions). For each pair of $\mathcal{O}$-ideals $\mathfrak{d}$ and $\mathfrak{b}$ we have*

$$(U_{\mathfrak{b}} : U_{\mathcal{O}})\ \zeta(\mathfrak{d}, \mathfrak{b}, s) = (U_{\mathfrak{d}} : U_{\mathcal{O}})\ \zeta(\mathfrak{b}^*, \mathfrak{d}^*, s).$$

The proof will occupy the rest of this section.

Lemma 2.14. *$U_{\mathfrak{b}}$ only depends on the equivalence class of the ideal $\mathfrak{b}$, and $U_{\mathfrak{b}}$ is the group of units of the ring $\mathcal{O}_{\mathfrak{b}} := \mathfrak{b} : \mathfrak{b}$.*

Proof. The second statement follows from the definitions. Since $\mathfrak{b} : \mathfrak{b} = \alpha\mathfrak{b} : \alpha\mathfrak{b}$, for every $\alpha \in K$ clearly $U_{\alpha\mathfrak{b}} = U_{\mathfrak{b}}$ and this yields the first assertion. □

Remark 2.15. *If $\mathcal{O}'$ is a subring of K, then $\mathcal{O}' : \mathcal{O}' = \mathcal{O}'$. In particular, $\widetilde{\mathcal{O}} : \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}$ and $\mathcal{O} : \mathcal{O} = \mathcal{O}$, $U_{\widetilde{\mathcal{O}}} = U_{\widetilde{\mathcal{O}}:\widetilde{\mathcal{O}}}$ and $U_{\mathcal{O}} = U_{\mathcal{O}:\mathcal{O}}$.*

Lemma 2.16. *If $\mathfrak{b}$ is an $\mathcal{O}$-ideal then $U_{\mathcal{O}} \subseteq U_{\mathfrak{b}} \subseteq U_{\widetilde{\mathcal{O}}}$.*

Proof. At first we show

$$\mathcal{O} \subseteq \mathfrak{b} : \mathfrak{b} \subseteq \widetilde{\mathcal{O}}. \tag{5}$$

The first inclusion is obvious. For the second, let $b_1, \cdots, b_r$ be generators of $\mathfrak{b}$ as $\mathcal{O}$-module, and let $x \in \mathfrak{b} : \mathfrak{b}$ be arbitrary. Then we have

$$x \cdot \mathfrak{b}_j = \sum_{i=1}^{r} a_{ij} b_i \qquad i, j \in \{1, \cdots, r\} \quad , \quad a_{ij} \in \mathcal{O}.$$

Let $A = (a_{ij})$. Then we have

$$\chi_A(t) = det\,(tE - A) \in \mathcal{O}[\,t\,]$$

and $\chi_A(A) = O$, where O represents the zero matrix. Hence $\chi_A(x) = 0$ and this yields $x \in \widetilde{\mathcal{O}}$. Thus we have (4) and

$$U_{\mathcal{O}} \subseteq U_{\mathfrak{b}:\mathfrak{b}} \subseteq U_{\widetilde{\mathcal{O}}},$$

and this yields the lemma by 2.14 and 2.15. □

Since by [St2] $\mathfrak{d}^* : \mathfrak{b}^* = \mathfrak{b} : \mathfrak{d}$, we have $\mathfrak{d}^* : \mathfrak{d}^* = \mathfrak{d} : \mathfrak{d}$ and therefore

$$U_{\mathfrak{b}^*} = U_{\mathfrak{b}}$$

for each $\mathcal{O}$-ideal $\mathfrak{d}$. Thus the reciprocity formula claims that its left hand side remains invariant, when $\mathfrak{d}$ and $\mathfrak{b}$ are replaced by $\mathfrak{b}^*$ and $\mathfrak{d}^*$, respectively.

The ideals $\mathfrak{a}$ satisfying $\mathfrak{a} \supseteq \mathfrak{d}$ and $\mathfrak{a} \sim \mathfrak{b}$ are just of the form $z^{-1}\mathfrak{b}$, where z varies over a complete system of representatives of $(\mathfrak{b} : \mathfrak{d})\backslash\{0\}$ by the action of multiplicative group $U_{\mathfrak{b}}$. Thus the partial local zeta - function can be written as follows

$$\begin{aligned}\zeta(\mathfrak{d}, \mathfrak{b}, s) &= \sum_{z \in (\mathfrak{b}:\mathfrak{d}\setminus 0)/U_{\mathfrak{b}}} q^{-s\dim(z^{-1}\mathfrak{b}\setminus\mathfrak{d})} \\ &= \frac{1}{(U_{\mathfrak{b}} : U_{\mathcal{O}})} \sum_{z \in (\mathfrak{b}:\mathfrak{d}\setminus 0)/U_{\mathcal{O}}} q^{-s\dim(z^{-1}\mathfrak{b}/\mathfrak{d})}.\end{aligned} \tag{6}$$

In the second sum we restricted the action of $U_{\mathfrak{b}}$ on $(\mathfrak{b} : \mathfrak{d})$ to the action of $U_{\mathcal{O}}$, and in order to compensate this we had to divide the infinite sum by the index $(U_{\mathfrak{b}} : U_{\mathcal{O}})$.

Lemma 2.17. *Let $z \in K$, then*

$$deg(z\mathcal{O}) = deg(z\widetilde{\mathcal{O}}) - deg(\widetilde{\mathcal{O}}) = -\sum_{i=1}^{m} d_i \ ord_{\mathfrak{p}_i}(z).$$

Proof. At first we show the left equation. By definition

$$\dim z\widetilde{\mathcal{O}}/z\mathcal{O} = \deg z\widetilde{\mathcal{O}} - \deg z\mathcal{O}.$$

Since $z\widetilde{\mathcal{O}}/z\mathcal{O} \simeq \widetilde{\mathcal{O}}/\mathcal{O}$ we have

$$\begin{aligned}\deg z\widetilde{\mathcal{O}} - \deg z\mathcal{O} &= \dim z\widetilde{\mathcal{O}}/z\mathcal{O} \\ &= \dim \widetilde{\mathcal{O}}/\mathcal{O} = \deg \widetilde{\mathcal{O}} - \deg \mathcal{O} = \deg\widetilde{\mathcal{O}},\end{aligned}$$

because $\deg\mathcal{O} = 0$. For second equation, we prove it for $z \in \mathcal{O}$, then we prove the general case $z \in K$.
Let $z \in \mathcal{O}$. Then $z\widetilde{\mathcal{O}} = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_m^{n_m} \subseteq \widetilde{\mathcal{O}}$, and by lemma 2.9

$$\begin{aligned}\deg \widetilde{\mathcal{O}} - \deg z\widetilde{\mathcal{O}} &= \dim \widetilde{\mathcal{O}}/z\widetilde{\mathcal{O}} \\ &= \sum_{i=0}^{m} \dim \widetilde{\mathcal{O}}/\mathfrak{p}_i^{n_i} = \sum_{i=1}^{m} n_i d_i \ .\end{aligned}$$

This yields, by definition 2.8,

$$\deg z\widetilde{\mathcal{O}} - \deg \widetilde{\mathcal{O}} = -\sum_{i=1}^{m} d_i \ ord_{\mathfrak{p}_i} \ . \tag{7}$$

Now let $z \in K$, and $z = \alpha/\beta$, where α and β are in $\mathcal{O}$. By the first case we have,

$$\begin{aligned}\deg z\widetilde{\mathcal{O}} - \deg \widetilde{\mathcal{O}} &= \deg z\mathcal{O} = \deg \alpha\mathcal{O} - \deg \beta\mathcal{O} \\ &= (\deg \alpha\widetilde{\mathcal{O}} - \deg\widetilde{\mathcal{O}}) - (\deg \beta\widetilde{\mathcal{O}} - \deg \widetilde{\mathcal{O}}) \\ &= -\sum_{i=0}^{m} d_i \operatorname{ord}_{\mathfrak{p}_i}(\alpha) + \sum_{i=0}^{m} d_i \operatorname{ord}_{\mathfrak{p}_i}(\beta) \\ &= -\sum_{i=0}^{m} d_i \operatorname{ord}_{\mathfrak{p}_i}(z) \ .\end{aligned} \tag{8}$$

□

Remark 2.18. *Let $z \in K$ and $\mathfrak{a}$ be an $\mathcal{O}$-ideal, then*

$$deg\ z\mathfrak{a} \ + \sum_{i=0}^{m} d_i\ ord_{\ \mathfrak{p}_i}(z) = deg\ \mathfrak{a}\ .$$

Proof. By definition,

$$\deg z\mathfrak{a} = \deg z\mathcal{O} \ + \ \deg \mathfrak{a}.$$

By Lemma 2.17

$$\deg z\mathcal{O} \ + \sum_{i=1}^{m} d_i \operatorname{ord}_{\ \mathfrak{p}_i}(z) = 0\ .$$

This yields the assertion .

□

Definition 2.19. *Let $z \in K$. By definition the absolute valuation of z is*

$$|z| := \begin{cases} q^{deg\ (\mathcal{O}z)} & , \quad z \neq 0 \\ |z| = 0 & , \quad z = 0\ . \end{cases}$$

Remark 2.20. *Clearly one deduces from the definition*

(i) $|z_1 \cdot z_2 \cdots z_n| = |z_1| \cdot |z_2| \cdots |z_n|; \quad (z_i \in K), \quad (n \in \mathbb{N})$

(ii) $|z| = |\alpha|/|\beta|; \quad (\alpha, \beta \in \mathcal{O})$.

Lemma 2.21.

$$\zeta(\mathfrak{d}, \mathfrak{b}, s) = \frac{q^{-s(deg(\mathfrak{b}) - deg(\mathfrak{d}))}}{(U_{\mathfrak{b}} : U_{\mathcal{O}})} \sum_{z \in (\mathfrak{b}:\mathfrak{d})/U_{\mathcal{O}}} |z|^s \tag{9}$$

Proof. According to the definition

$$\dim\,(z^{-1}\mathfrak{b}/\mathfrak{d}) = \deg\, z^{-1}\mathfrak{b} \; - \; \deg\, \mathfrak{d}, \tag{10}$$

and by remark 2.18, we have

$$\deg\, z^{-1}\mathfrak{b} = \deg\, \mathfrak{b} \; + \sum_{i=1}^{m} d_i\, \mathrm{ord}_{\mathfrak{p}_i}(z). \tag{11}$$

Now (10) and (11) imply

$$\dim(z^{-1}\mathfrak{b}/\mathfrak{d}) = \deg \mathfrak{b} \; - \; \deg \mathfrak{d} \; + \sum_{i=1}^{m} d_i\, \mathrm{ord}_{\mathfrak{p}_i}(z)\;. \tag{12}$$

Obviously (12) and (6) imply the lemma.

□

Observing now, that the integer $\deg(\mathfrak{b}) - \deg(\mathfrak{d})$ and the ideal $\mathfrak{b} : \mathfrak{d}$ do not change, when $\mathfrak{d}$ and $\mathfrak{b}$ are replaced by $\mathfrak{b}^*$ and $\mathfrak{d}^*$ respectively, see [St2], we have shown the reciprocity formula.

From lemma 2.21 we also deduce :

$$\zeta(\mathfrak{d}, \mathfrak{b}, s) = (U_{\mathfrak{b}:\mathfrak{d}} : U_{\mathfrak{b}})\; q^{s(\deg(\mathfrak{d}) - \deg(\mathfrak{b}) + \deg(\mathfrak{b}:\mathfrak{d}))}.\; \zeta(\mathcal{O}, \mathfrak{b} : \mathfrak{d}, s)$$

This identity justifies that from now on we will assume that the ideal $\mathfrak{d}$ is equal to the ring $\mathcal{O}$.

3 A formula for the zeta function and the functional equation

Let $\mathcal{O}$ be an order, and keep the notation from the previous section. The following is an important invariant of the order, and is often studied in the literature.

Definition 3.1. *The set*

$$S(\mathcal{O}) = \{(ord_{\mathfrak{p}_1}(z), \cdots, ord_{\mathfrak{p}_m}(z)) \in \mathbb{Z}^m \mid z \in \mathcal{O}\backslash\{0\}\}$$

is called the semigroup associated to $\mathcal{O}$*.*

It is clear that $S(\mathcal{O})$ is a subsemigroup of $\mathbb{Z}^m$. we generalize this to ideals.

Definition 3.2. *Let S be a semigroup. An S-module is a set M with a binary operation*

$$S \times M \longrightarrow M \quad , \quad (s, m) \longrightarrow s + m \, ,$$

such that

(i) $0 + m = m$,

(ii) $s_1 + (s_2 + m) = (s_1 + s_2) + m$.

Now to each $\mathcal{O}$-ideal $\mathfrak{b}$ we associate the set of integer vectors,

$$S(\mathfrak{b}) := \{(ord_{p_i}(z), ..., ord_{p_m}(z)) \in \mathbb{Z}^n \mid z \in \mathfrak{b}\backslash\{0\}\}.$$

When $\mathfrak{b}$ is equal to the local ring $\mathcal{O}$, then we get the semigroup $S(\mathcal{O})$ associated to $\mathcal{O}$. We will show that the sets $S(\mathfrak{b})$ may provide important information about the zeta function.

Lemma 3.3. *For every fractional $\mathcal{O}$-ideal $\mathfrak{b}$, $S(\mathfrak{b})$ is a $S(\mathcal{O})-$module.*

Proof. Let $\boldsymbol{n} \in S(\mathcal{O})$ and $\boldsymbol{m} \in S(\mathfrak{b})$. Then there exists $x \in \mathcal{O}$ and $y \in \mathfrak{b}$ such that $Ord(x) := (Ord_{\mathfrak{p}_1}(x), \cdots, Ord_{\mathfrak{p}_m}(x) = \boldsymbol{n}$ and $Ord(y) = \boldsymbol{m}$. Since $\mathfrak{b}$ is a fractional $\mathcal{O}$-ideal we have $x.y \in \mathfrak{b}$, so that

$$Ord(x.y) = Ord(x) + Ord(y) = \boldsymbol{n} + \boldsymbol{m} \in S(\mathfrak{b}).$$

It is easy to verify *(i)* and *(ii)*. □

Proposition 3.4. *For any fractional $\mathcal{O}$-ideal $\mathfrak{b}$*

$$\sum_{z\in\mathfrak{b}/U_{\mathcal{O}}} |z|^s = \sum_{\boldsymbol{n}\in s(\mathfrak{b})} \varepsilon_{\boldsymbol{n}}(\mathfrak{b}) t^{\boldsymbol{n}\cdot d}.$$

Here z varies over a complete system of representatives of $\mathfrak{b}$ by the action of $U_{\mathcal{O}}$ and

$$\varepsilon_{\boldsymbol{n}}(\mathfrak{b}) := \frac{q^r}{q^r-1} \sum_{i=0}^{(1,\dots,1} (-1)^{|i|} q^{\boldsymbol{n}\cdot d + deg(\mathfrak{b}\cap\mathfrak{p}^{\boldsymbol{n}+i})},$$

the sum is taken over the vectors $i = (i_1, ... i_m) \in (0,1)^m$, and we abbreviate $|i| = i_1 + ... + i_m$.

Before proving the proposition, we need some lemma and tools from measure theory as follows.

Let $\mathcal{M}$ be the system of subsets $\mathbf{M}$ of the field K, obtained from the (fractional) $\mathcal{O}$-ideals by translations $z + \mathbf{M}$ $(z \in K)$ and the operations of union $\mathbf{M} \cup \mathbf{N}$, intersection $\mathbf{N} \cap \mathbf{N}$ and complementation $\mathbf{M}\backslash\mathbf{N}$. It can be seen in an elementary way, or by using a Haar measure $\hat{\mu}$ on the locally compact total quotient ring of the completion $\hat{\mathcal{O}}$ of $\mathcal{O}$, that one can attribute to each $\mathbf{M} \in \mathcal{M}$ a volume $\mu(\mathbf{M}) \geq 0$ uniquely determined by the three axioms:

(i) *Normalization :* $\mu(\mathcal{O}) = 1$

(ii) *Invariance under translation :* $\mu(z + \mathbf{M}) = \mu(\mathbf{M})$

(iii) *Additivity :* $\mu(\mathbf{M} \cup \mathbf{N}) = \mu(\mathbf{M}) + \mu(\mathbf{N})$ whenever $\mathbf{M} \cap \mathbf{N} = \emptyset$.

By the additivity we have $\mu(\mathbf{M} \cup \mathbf{N}) = \mu(\mathbf{M}) + \mu(\mathbf{N}) - \mu(\mathbf{M} \cap \mathbf{N})$, and more generally, as follows by induction,

$$\mu(\bigcup_{i=1}^{n} \mathbf{M}_i) = \sum_{j=1}^{n} (-1)^{j-1} \sum_{i_1<\dots<i_j} \mu(\mathbf{M}_{i_1} \cap \dots \cap \mathbf{M}_{i_j}).$$

Lemma 3.5. *For every (fractional) ideal* $\mathfrak{a}$ *and elements* $z_1, \cdots, z_n \in K$ *we have*

$$(i) \quad \mu(\mathfrak{a}) = q^{deg(\mathfrak{a})} \quad , \quad (ii) \quad \mu(z_1...z_n\boldsymbol{M}) = |z_1|...|z_n|\mu(\boldsymbol{M}).$$

Proof. (i) First we suppose $\mathfrak{a} \subseteq \mathcal{O}$. Then $\mathcal{O} = \bigcup_{x \in R}(x + \mathfrak{a})$, where R is a system of representatives for $\mathcal{O}/\mathfrak{a}$. Clearly $|R| = |\mathcal{O}/\mathfrak{a}|$, so by axiom(i), $\mu(\mathcal{O}) = \sum_{x\in R} \mu(x+\mathfrak{a})$ and by axioms (i) and (ii) $1 = \sum_{x\in R} \mu(\mathfrak{a}) = |\mathcal{O}/\mathfrak{a}|\mu(\mathfrak{a})$, and this yields

$$\mu(\mathfrak{a}) = 1/|\mathcal{O}/\mathfrak{a}| = 1/q^{(dim \mathcal{O}/\mathfrak{a})} = 1/q^{-deg\mathfrak{a}} = q^{deg\mathfrak{a}}.$$

Now we consider an arbitrary fractional-$\mathcal{O}$ ideal $\mathfrak{a}$. Then there exists $\alpha \in \mathcal{O}$ such that $\alpha\mathfrak{a} \subseteq \mathcal{O}$, and $deg\mathfrak{a} = -dim(O/\alpha\mathfrak{a}) + dim(\mathcal{O}/\alpha\mathcal{O})$. On the other hand if $\mathfrak{a} \subseteq \mathfrak{b}$ are fractional-$\mathcal{O}$ ideal, by a similar argument as for case one, one obtains $\mu(\mathfrak{a}) = \mu(\mathfrak{b})/|\mathfrak{b}/\mathfrak{a}|$. Applied to $\alpha\mathfrak{a} \subseteq \mathfrak{a}$, we get $\mu(\mathfrak{a}) = |\mathfrak{a}/\alpha\mathfrak{a}| \cdot \mu(\alpha\mathfrak{a})$. By case 1

$$\mu(\mathfrak{a}) = q^{dim\ \mathfrak{a}\alpha/\mathfrak{a}}.q^{deg\alpha\mathfrak{a}} = q^{deg\mathfrak{a} - deg\alpha\mathfrak{a} + deg\alpha\mathfrak{a}}$$

and (i) is proved.
(ii)We fix z and show that $\mu'(\mathbf{M}) := \mu(z\mathbf{M})/|z|$ satisfies in three above axioms and by unicity of the measure μ clearly $\mu(z\mathbf{M})/|z| = \mu(\mathbf{M})$

(i) $\mu'(\mathcal{O}) = \mu(z\mathcal{O})/|z| = q^{degz\mathcal{O}}/|z| = |z|/|z| = 1.$

(ii) $\mu'(z'+\mathbf{M}) = \mu(z(z'+\mathbf{M}))/|z| = \mu(zz'+z\mathbf{M})/|z| = \mu(z\mathbf{M})/|z| = \mu'(\mathbf{M})$ for every $\mathbf{M} \in \mathcal{M}$ and $z' \in K$.

(iii)

$$\begin{aligned}\mu'(\mathbf{M} \cup \mathbf{N}) =& \mu(z(\mathbf{M} \cup \mathbf{N}))/|z| \\ =& \mu(z\mathbf{M})/|z| + \mu(z\mathbf{N})/|z| \\ =& \mu'(\mathbf{M}) + \mu'(\mathbf{N}).\end{aligned}$$

Therefore μ' satisfies in three axioms and we have $\mu'(\mathbf{M}) = \mu(\mathbf{M}), \forall \mathbf{M} \in \mathcal{M}$ or $\mu(z(\mathbf{M})/|z| = \mu(\mathbf{M})$. □

Proof. (proposition 3.4) Note that $\mathfrak{b}\backslash\{0\}$ is the disjoint union of the sets

$$\mathfrak{b}_{\boldsymbol{n}} := \mathfrak{b} \cap \boldsymbol{\pi}^n U_{\tilde{\mathcal{O}}} = \{z \in \mathfrak{b} \mid ord_{\mathfrak{p}_i}(z) = n_i \text{ for each } \ i = 1, \cdots, m\},$$

for all $\boldsymbol{n} = (n_1, \cdots, n_m) \in \mathbb{Z}^m$. Since $|z|$ assumes on $\mathfrak{b}_{\boldsymbol{n}}$ the constant value $q^{-\boldsymbol{n}\cdot\boldsymbol{d}}$, and since $t = q^{-s}$ we obtain

$$\sum_{z\in\mathfrak{b}/U_{\mathcal{O}}} |z|^s = \sum_{\boldsymbol{n}\in s(\mathfrak{b})} \#(\mathfrak{b}_{\boldsymbol{n}} \,/U_{\mathcal{O}})\; t^{\boldsymbol{n}\,\boldsymbol{d}}. \tag{13}$$

We now study $\#(\mathfrak{b}_{\boldsymbol{n}}/U_{\mathcal{O}})$.

Since $\mathcal{O} = \mathfrak{m} \cap U_{\mathcal{O}} = \emptyset$, where $\mathfrak{m}$ is maximal ideal in $\mathcal{O}$, we have

$$\mu(U_{\mathcal{O}}) = \mu(\mathcal{O}) - \mu(\mathfrak{m}) \quad \text{or} \quad \mu(U_{\mathcal{O}}) = 1 - q^{-\dim \mathcal{O}/\mathfrak{m}} = 1 - q^{-r} \ .$$

By Lemma 3.5 (ii) we have $\mu(zU_{\mathcal{O}}) = |z| \cdot \mu(U_{\mathcal{O}}) = q^{-\boldsymbol{n}\cdot\boldsymbol{d}}\mu(U_{\mathcal{O}})$ for every $z \in \mathfrak{b}_{\boldsymbol{n}}$. Since $b_n = \cup_{z\in R}(zU_{\mathcal{O}})$ where R is a system of representatives for $\mathfrak{b}_{\boldsymbol{n}}/U_{\mathcal{O}}$, we obtain

$$\mu(\mathfrak{b}_{\boldsymbol{n}}) = \#(\mathfrak{b}_{\boldsymbol{n}}/U_{\mathcal{O}})\; |z|\; \mu(U_{\mathcal{O}}) = q^{-\boldsymbol{n}\boldsymbol{d}}\mu(\mathfrak{b}_{\boldsymbol{n}})(1 - q^{-r}).$$

Obviously $z \in \mathfrak{p}^{\boldsymbol{n}}$ if and only if $Ordz \geq \boldsymbol{n}$, therefore we have

$$\mathfrak{b}_{\boldsymbol{n}} = (\mathfrak{b} \cap \mathfrak{p}^{\boldsymbol{n}}) \backslash \cup_{i=1}^{m} (\mathfrak{b} \cap \mathfrak{p}^{\boldsymbol{n}}\mathfrak{p}_i) \ .$$

Now, by the additivity of the measure, we obtain

$$\mu(\mathfrak{b}_{\boldsymbol{n}}) = \mu(\mathfrak{b} \cap \mathfrak{p}^{\boldsymbol{n}}) - \sum_{j=0}^{m} (-1)^{j-1} \sum_{i_1<\ldots<i_j}^{(1,\ldots,1)} \mu(\mathfrak{b} \cap \mathfrak{p}^{\boldsymbol{n}}\mathfrak{p}_{i_1} \cap \ldots \cap \mathfrak{p}^{\boldsymbol{n}}\mathfrak{p}_{i_j}) =$$

$$\sum_{j=0}^{m} (-1)^j \sum_{i_1<\ldots<i_j} q^{deg(\mathfrak{b}\cap\mathfrak{p}^{\boldsymbol{n}}\mathfrak{p}_{i_1}\cap\ldots\cap\mathfrak{p}^{\boldsymbol{n}}\mathfrak{p}_{i_j})} = \sum_{|\boldsymbol{i}|=0}^{(1,\ldots,1)} (-1)^{|\boldsymbol{i}|} q^{deg(\mathfrak{b}\cap\mathfrak{p}^{\boldsymbol{n}+\boldsymbol{i}})}$$

and therefore

$$|\mathfrak{b}_{\boldsymbol{n}} \,/U_{\mathcal{O}}| = \varepsilon_{\boldsymbol{n}}(\mathfrak{b}).$$

By (13) we have the proof. □

Theorem 3.6. *The partial zeta-function has the expansion*

$$Z(\mathcal{O}, \mathfrak{b}, t) = \frac{1}{(U_{\mathfrak{b}} : U_{\mathcal{O}})} \sum_{\boldsymbol{n}\in S(\mathfrak{b})} \varepsilon_{\boldsymbol{n}}(\mathfrak{b})\; t^{\boldsymbol{n}\cdot\boldsymbol{d} + deg(\mathfrak{b})}$$

where

$$\varepsilon_{\boldsymbol{n}}(\mathfrak{b}) := \frac{q^r}{q^r - 1} \sum_{\boldsymbol{i}=0}^{(1,\ldots,1)} (-1)^{|\boldsymbol{i}|} q^{\boldsymbol{n}\cdot\boldsymbol{d} + deg(\mathfrak{b}\cap\mathfrak{p}^{\boldsymbol{n}+\boldsymbol{i}})}.$$

Here the sum is taken over the vectors $\boldsymbol{i} = (i_1, ... i_m) \in (0,1)^m$*, and we abbreviate* $|\boldsymbol{i}| = i_1 + ... + i_m$.

Proof. By Lemma 2.21 we have

$$\zeta(\mathcal{O}, \mathfrak{b}, s) = \frac{q^{-s\deg(\mathfrak{b})}}{(U_{\mathfrak{b}} : U_{\mathcal{O}})} \sum_{z\in\mathfrak{b}/U_{\mathcal{O}}} |z|^s.$$

Now proposition 3.4 implies the theorem. □

Remark 3.7. *By the proof of proposition 3.4 the coefficients* $\varepsilon_{\boldsymbol{n}}(\mathfrak{b})$ *are positive for all* $\boldsymbol{n} \in S(\mathfrak{b})$*, and they vanish for all other integer vectors. Thus, instead of summing up over the vectors* $\boldsymbol{n} \in S(\mathfrak{b})$ *in the expansion of* $Z(\mathcal{O}, \mathfrak{b}, t)$*, we may sum up over all integer vectors* $\boldsymbol{n} \in \mathbb{Z}^m$.

.

Lemma 3.8. *For every fractional* $\mathcal{O}$*-ideal there is an* $\boldsymbol{n}_0$*, such that* $\mathfrak{p}^{\boldsymbol{n}} \subseteq \mathfrak{b}$ *for all* $\boldsymbol{n} \geq \boldsymbol{n}_0$.

Proof. We choose an $\alpha \in \mathfrak{b}\backslash\{0\}$. Then $\alpha\mathcal{O} \subseteq \mathfrak{b}$, and hence $\mathfrak{F} \subseteq \mathcal{O} \subseteq \alpha^{-1}\mathfrak{b}$, where $\mathfrak{F}$ is the conductor of $\mathcal{O}$. Let

$$\alpha\,\mathfrak{F} = \mathfrak{p}_1^{n_0^1} ... \mathfrak{p}_m^{n_0^m},$$

then $\mathfrak{p}^{\boldsymbol{n}} \subseteq \mathfrak{p}^{\boldsymbol{n}_0} \subseteq \mathfrak{b}$ for every $\boldsymbol{n} \geq \boldsymbol{n}_0 = (n_0^1, \cdots, n_0^m)$.

□

.

Corollary 3.9. *If the vector* $\boldsymbol{n}$ *is so large that* $\mathfrak{p}^{\boldsymbol{n}} \subseteq \mathfrak{b}$*, then*

$$\varepsilon_{\boldsymbol{n}}(\mathfrak{b}) = \frac{q^\delta}{1 - q^{-r}}\, \Pi_{i=1}^m (1 - q^{-d_i}).$$

.

Proof. We have
$$dim_k\ \widetilde{\mathcal{O}}/\mathfrak{p}^{\mathbf{n}+\mathbf{i}} = deg\ \widetilde{\mathcal{O}} - deg\ \mathfrak{p}^{\mathbf{n}+\mathbf{i}}$$
and
$$dim_k\ \widetilde{\mathcal{O}}/\mathfrak{p}^{\mathbf{n}+\mathbf{i}} = \sum_{j=1}^{m}(n_j + i_j).$$
Therefore, as $\delta = \deg\ \widetilde{\mathcal{O}}$,

$$\begin{aligned}
\varepsilon_{\mathbf{n}}(\mathfrak{b}) &= \frac{q^r}{q^r-1}\sum_{\boldsymbol{i}=0}^{(1,\dots,1)}(-1)^{|\mathbf{i}|}q^{\mathbf{n}.\mathbf{d}+deg\mathfrak{p}^{\mathbf{n}+\mathbf{i}}}\\
&= \frac{q^r}{q^r-1}\sum_{i=0}^{(1,\dots,1)}(-1)^{(i_1+\dots i_m)}q^{\delta-(i_1d_1+\dots+i_md_m)}\\
&= \frac{q^{r+\delta}}{q^r-1}\sum_{i=0}^{(1,\dots,1)}(-1)^{(i_1+\dots i_m)}q^{-i_1d_1-\dots-i_md_m}\\
&= \frac{q^{r+\delta}}{q^{r-1}}\Pi_{i=1}^m(1-q_i^{d_i})
\end{aligned}$$

□

.

In particular, in the special case where $\mathfrak{b} = \widetilde{\mathcal{O}}$ and $\mathbf{n} = (0,...,0)$, we obtain the well known formula
$$(U_{\widetilde{\mathcal{O}}} : U_{\mathcal{O}}) = \frac{q^\delta}{1-q^{-r}}\prod_{i=1}^{m}(1-q^{-d_i}).$$

Theorem 3.10. *For each $\mathcal{O}$-ideal $\mathfrak{b}$, we can write*
$$Z(\mathcal{O},\mathfrak{b},t) = \frac{L(\mathcal{O},\mathfrak{b},t)}{\prod_{i=1}^m(1-t^{d_i})} \tag{14}$$
where $L(\mathcal{O},\mathfrak{b},t)$ is a polynomial with integer coefficients of degree not larger than 2δ in t, satisfying the functional equation
$$t^{-\delta}L(\mathcal{O},\mathfrak{b},t) = (1/qt)^{-\delta}L(\mathcal{O},\mathfrak{b}^*,(1/qt)). \tag{15}$$

Proof. When we multiply the power series $Z(\mathcal{O}, \mathfrak{b}, t)$ by the product

$$\prod_{i=0}^{m}(1-t^{d_i}) = \sum_{j=0}^{(1,\ldots,1)} (-1)^{|\mathbf{j}|}\ t^{\mathbf{j}.\mathbf{d}}, \tag{16}$$

then by Theorem 3.6 and Remark 3.7 we get the power series

$$L(\mathcal{O}, \mathfrak{b}, t) = \frac{q^r}{(U_{\mathfrak{b}} : U_{\mathcal{O}})(q^r-1)} \sum_{\mathbf{n}\in\mathbb{Z}^m} \gamma_{\mathbf{n}}(\mathfrak{b})\ t^{\mathbf{n}.\mathbf{d}+\deg(\mathfrak{b})}, \tag{17}$$

where

$$\gamma_{\mathbf{n}}(\mathfrak{b}) := \sum_{\mathbf{i},\mathbf{j}=\mathbf{0}}^{(1,\ldots,1)} (-1)^{|\mathbf{i}|+|\mathbf{j}|} q^{\mathbf{n}.(\mathbf{d}-\mathbf{j})+\deg(\mathfrak{b}\cap\mathfrak{p}^{\mathbf{n}+\mathbf{i}-\mathbf{j}})}. \tag{18}$$

Let $\mathfrak{a}$ be another $\mathcal{O}$-ideal. Note that $\mathfrak{c} : \mathfrak{b}^* = \mathfrak{b},\ \deg(\mathfrak{c} : \mathfrak{b}) = \deg(\mathfrak{c}) - \deg(\mathfrak{b})$ and

$$\begin{aligned}\deg(\mathfrak{b}^* \cap \mathfrak{a}) &= \deg((\mathfrak{c} : \mathfrak{b}) \cap (\mathfrak{c} : \mathfrak{a}^*))\\ &= \deg(\mathfrak{c} : (\mathfrak{b} + \mathfrak{a}^*)) = \deg(\mathfrak{c}) - \deg(\mathfrak{b} + \mathfrak{a}^*).\end{aligned}$$

Since by the isomorphism theorem, dim $(\mathfrak{b} + \mathfrak{a}^*/\mathfrak{a}^*)$= dim $(\mathfrak{b}/\mathfrak{b} \cap \mathfrak{a}^*)$, this implies

$$\begin{aligned}\deg(\mathfrak{b}^* \cap \mathfrak{a}) &= \deg(\mathfrak{c}) - \deg(\mathfrak{b} + \mathfrak{a}^*)\\ &= \deg(\mathfrak{c}) - (\dim(\mathfrak{b} + \mathfrak{a}^*)/\mathfrak{a}^*) + \deg\mathfrak{a}^*)\\ &= \deg\mathfrak{c} - (\deg\mathfrak{b} - \deg(\mathfrak{b} \cap \mathfrak{a}^*) + \deg\mathfrak{a}^*).\end{aligned}$$

Since $\deg\mathfrak{a}^* = \deg\mathfrak{c} - \deg\mathfrak{a}$, we obtain

$$\deg(\mathfrak{b}^* \cap \mathfrak{a}) = \deg\mathfrak{c} - \deg\mathfrak{b} + \deg(\mathfrak{b} \cap \mathfrak{a}^*) - \deg\mathfrak{c} + \deg\mathfrak{a}.$$

This implies

$$\deg(\mathfrak{b}^* \cap \mathfrak{a}) = \deg\mathfrak{a} - \deg\mathfrak{b} + \deg(\mathfrak{b} \cap \mathfrak{a}^*). \tag{19}$$

Note that $\deg(\mathfrak{c}) = \deg(\mathfrak{c} : \mathcal{O}) = \deg(\mathfrak{c} : \tilde{\mathcal{O}}) + \dim(\tilde{\mathcal{O}}/\mathcal{O}) = \deg(\mathfrak{c} : \tilde{\mathcal{O}}) + \delta$. Multiplying the canonical ideal $\mathfrak{c}$ by a suitable element of K we can assume that $\mathfrak{c} : \tilde{\mathcal{O}} = \tilde{\mathcal{O}}$. In fact, $\mathfrak{c} : \tilde{\mathcal{O}}$ is a fractional $\mathcal{O}$-ideal, so we have $\mathfrak{c} : \tilde{\mathcal{O}} = \tilde{\mathcal{O}}\cdot\pi^{\boldsymbol{n}}$, for some $\boldsymbol{n} \in \mathbb{Z}^m$. This implies

$$(\pi^{-\boldsymbol{n}}\mathfrak{c}) : \tilde{\mathcal{O}} = \tilde{\mathcal{O}}.$$

In this case $\deg(\mathfrak{c}) = 2\delta$. Taking $\mathfrak{a} = \mathfrak{p}^{\boldsymbol{n}}$ we have

$$\mathfrak{a}^* = (\mathfrak{c} : \mathfrak{p}^{\boldsymbol{n}}) = (\mathfrak{c} : \pi^{\boldsymbol{n}}\tilde{\mathcal{O}}) = \pi^{-\boldsymbol{n}}\,(\mathfrak{c} : \tilde{\mathcal{O}}) = \pi^{-\boldsymbol{n}}\tilde{\mathcal{O}} = \mathfrak{p}^{-\boldsymbol{n}}. \tag{20}$$

On the other hand we have

$$\mathbf{n}\cdot\mathbf{d} = \dim_k \mathcal{O}/\mathfrak{a} = \deg\tilde{\mathcal{O}} - \deg\mathfrak{a} = \delta - \deg\mathfrak{a},$$

or

$$\deg\mathfrak{a} = \delta - \mathbf{n}\cdot\mathbf{d}\ .$$

By substitution in (19) we have

$$\deg(\mathfrak{b}^* \cap \mathfrak{p}^{\boldsymbol{n}}) = \delta - \boldsymbol{n}.\mathbf{d} - \deg(\mathfrak{b}) + \deg(\mathfrak{b} \cap \mathfrak{p}^{-\boldsymbol{n})}. \tag{21}$$

Replacing the vector **n** by **n+i-j** we obtain the formula

$$\gamma_{\mathbf{n}}(\mathfrak{b}^*) = q^{\mathbf{n}.\mathbf{d}+\delta-\deg(\mathfrak{b})}\,\gamma_{-\mathbf{n}}(\mathfrak{b}), \tag{22}$$

which immediately implies the functional equation. By the functional equation the power series $L(\mathcal{O}, \mathfrak{b}, t) \in \mathbb{Z}[[t]]$ is a polynomial in t of degree at most 2δ; for

$$L(\mathcal{O}, \mathfrak{b}, t) = q^{\delta}\ t^{2\delta} L(\mathcal{O}, \mathfrak{b}, 1/tq) = q^{\delta} t^{2\delta}(a_0 + a_1/qt + ...a_{2\delta}/(qt)^{2\delta} + ...)$$

or

$$L(\mathcal{O}, \mathfrak{b}, t)) = a_0 q^{\delta} t^{2\delta} + a_1 q^{\delta-1} t^{2\delta-1} + ... + a_{2\delta} q^{-\delta}.$$

□

.

Proposition 3.11. *The degree of the polynomial $L(\mathcal{O}, \mathfrak{b}, t)$ is smaller than 2δ if and only if the ideal $\mathfrak{b}$ is non-dualizing.*

Proof. Writing

$$L(\mathcal{O}, \mathfrak{b}, t) = \sum_{i=0}^{2\delta} n_i(\mathfrak{b}) t^i$$

where the coefficients $n_i(\mathfrak{b})$ are integers, we can rephrase the functional equation 15 of theorem 3.10 as follows:

$$t^{-\delta} \sum_{i=0}^{2\delta} n_i(\mathfrak{b}) t^i = (1/qt)^{-\delta} \sum_{i=0}^{2\delta} n_i(\mathfrak{b}^*)(1/qt)^i,$$

or

$$\sum_{i=0}^{2\delta} n_i(\mathfrak{b}) t^i = q^{\delta} t^{2\delta} \sum_{i=0}^{2\delta} n_i(\mathfrak{b}^*) q^{-i} t^{-i}$$

or

$$\sum_{i=0}^{2\delta} n_i(\mathfrak{b})t^i = \sum_{i=0}^{2\delta} q^{\delta-i} n_i(\mathfrak{b}^*)t^{2\delta-i},$$

or equivalently

$$n_{2\delta-i}(\mathfrak{b}^*) = q^{\delta-i} n_i(\mathfrak{b}) \quad (i = 0, ..., 2\delta). \tag{23}$$

On the other hand for $t = 0$ we have $Z(\mathcal{O}, \mathfrak{b}, 0) = L(\mathcal{O}, \mathfrak{b}, 0)$. By definition of the partial zeta function $Z(\mathcal{O}, \mathfrak{b}, 0) = 1$ if and only if $\mathfrak{b} = \alpha\mathcal{O}$ for some α in K. Therefore we obtain

$$n_0(\mathfrak{b}) = \begin{cases} 1 & \text{when } \mathfrak{b} \text{ is principal,} \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

By (23) and (24) and definition of dualizing ideal in section 2, we see $\mathfrak{b}$ is dualizing if and only if $\mathfrak{b}^*$ is principal, in which case we have

$$n_{2\delta}(\mathfrak{b}) = q^\delta n_0(\mathfrak{b}^*) = q^\delta.$$

This implies

$$n_{2\delta}(\mathfrak{b}) = \begin{cases} q^\delta & \text{when } \mathfrak{b} \text{ is dualizing} \\ 0 & \text{otherwise,} \end{cases} \tag{25}$$

and hence the claim. □

Lemma 3.12. *If $\mathfrak{b}$ is an $\mathcal{O}$-ideal, then the cardinality of the set $\{\ \mathfrak{a} \mid \mathfrak{a}.\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}$ and $\mathfrak{a} \sim \mathfrak{b}\}$ is equal to the order of the group $U_{\widetilde{\mathcal{O}}}/U_{\mathfrak{b}}$.*

Proof. We fix an $\mathfrak{a} = \alpha\mathfrak{b}$ with $\mathfrak{a}\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}$ and $\mathfrak{a} \sim \mathfrak{b}$. Then the following map is a bijection:

$$\{\mathfrak{a} | \mathfrak{a}\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}, \mathfrak{a} \sim \mathfrak{b}\} \xrightarrow{\Phi} U_{\widetilde{\mathcal{O}}}/U_{\mathfrak{b}}\ .$$

$$\mathfrak{a}' = \alpha'\mathfrak{b} \longrightarrow [\alpha'/\alpha]$$

□

.

Proposition 3.13.

$$\sum_{i=0}^{2\delta} n_i(\mathfrak{b}) = (U_{\widetilde{\mathcal{O}}} : U_{\mathfrak{b}})$$

Proof. Since

$$\lim_{t\to 1} Z(\mathcal{O},\mathfrak{b},t)/Z_{\widetilde{\mathcal{O}}}(t) = \#\,\{\mathfrak{a} : \mathfrak{a}.\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}, \mathfrak{a}\sim\mathfrak{b}\},$$

see [St2], the value of $L(\mathcal{O},\mathfrak{b},t)$ at t=1 is equal to the number of $\mathcal{O}$-ideals $\mathfrak{a}$ that satisfies $\mathfrak{a}.\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}$ and $\mathfrak{a}\sim\mathfrak{b}$, and by Lemma 3.12 is equal to the index $(U_{\widetilde{\mathcal{O}}} : U_{\mathfrak{b}})$.

□

Proposition 3.14. *The coefficients of $L(\mathcal{O},\mathfrak{b},t)$ satisfy the linear relation*

$$\sum_{i=0}^{2\delta}(q^{\delta-i}-1)n_i(\mathfrak{b}) = 0. \tag{26}$$

Proof. Since $U_{\mathfrak{b}} = U_{\mathfrak{b}^*}$, by proposition 3.13

$$\sum_{i=0}^{2\delta} n_i(\mathfrak{b}) = \sum_{i=0}^{2\delta} n_i(\mathfrak{b}^*).$$

Now by (23) in the proof of Proposition 3.11

$$\sum_{i=0}^{2\delta} q^{\delta-i}n_i(\mathfrak{b}) = \sum_{i=0}^{2\delta} n_i(\mathfrak{b}), \tag{27}$$

which implies the claim. □

Theorem 3.15. *We can write*

$$Z(\mathcal{O},t) = \frac{L(\mathcal{O},t)}{\Pi_{i=0}^{m}(1-t^{d_i})} \tag{28}$$

where $L(\mathcal{O},t)$ is a polynomial with integer coefficient of degree 2δ in t, which satisfies the property that the Laurent polynomial $t^{-\delta}L(\mathcal{O},t)$ remains invariant when t is replaced by $1/qt$.

Proof. Since the assignment $\mathfrak{b}\longmapsto\mathfrak{b}^*$ permutes the ideal classes, and since the local zeta-function $Z(\mathcal{O},t)$ is the sum of the partial local zeta-functions $Z(\mathcal{O},\mathfrak{b},t)$, theorem 3.10 and the functional equation (15) immediately imply the theorem.

□

Proposition 3.16. *The sum of the coefficients of $L(\mathcal{O},t)$ is equal to the number of the $\mathcal{O}$-ideal $\mathfrak{a}$ satisfying $\mathfrak{a}.\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}$.*

Proof. Since

$$Z_{\widetilde{\mathcal{O}}}(t) = \prod_{i=0}^{m} \frac{1}{1-t^{d_i}}, \tag{29}$$

the following equation implies the proof

$$\lim_{t\to 1} L(\mathcal{O}, \mathfrak{b}, t) = \lim_{t\to 1} Z(\mathfrak{d}, \mathfrak{b}, t)/Z_{\widetilde{\mathcal{O}}}(t) = \#\{\mathfrak{a} : \mathfrak{a}.\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}, \mathfrak{a} \sim \mathfrak{b}\}.$$

□

4 Comparison with Galkin's zeta function

In this section we consider another generalized zeta-function by associating to each $\mathcal{O}$-ideal $\mathfrak{d}$ the Dirichlet series

$$\zeta_G(\mathfrak{d}, s) := \sum_{\mathfrak{a}\subseteq\mathfrak{d}} \#(\mathfrak{d}/\mathfrak{a})^{-s} \ .$$

(For $\mathfrak{d} = \mathcal{O}$ this is Galkin's zeta function). Let us also define corresponding partial zeta functions by means of negative ideals, that is the ideals contained in $\mathfrak{d}$.

Definition 4.1. *For arbitrary fractional $\mathcal{O}$-ideals $\mathfrak{d}$ and $\mathfrak{b}$ define*

$$\zeta_G(\mathfrak{d}, \mathfrak{b}, s) := \sum_{\mathfrak{a}\subseteq\mathfrak{d}}^{\mathfrak{a}\sim\mathfrak{b}} (\mathfrak{d}/\mathfrak{a})^{-s} \ .$$

Here the sum is over all fractional $\mathcal{O}$-ideals $\mathfrak{a}$, which are contained in $\mathfrak{d}$ and equivalent to $\mathfrak{b}$.

Lemma 4.2. *One has*

$$\zeta_G(\mathfrak{d}, \mathfrak{b}, s) = \frac{q^{-s(deg\mathfrak{d}-deg\mathfrak{b})}}{U_{\mathfrak{b}} : U_{\mathcal{O}}} \sum_{\alpha\in(\mathfrak{d}:\mathfrak{b}/U_{\mathcal{O}})} \mid \alpha \mid^s \ . \qquad (30)$$

Proof. We have $\mathfrak{a} \sim \mathfrak{b}$ if and only if $\mathfrak{a} = \alpha\mathfrak{b}$ for some $\alpha \in K^*$, and $\alpha\mathfrak{b} \subseteq \mathfrak{d}$ if and only if $\alpha \in (\mathfrak{d} : \mathfrak{b})\backslash\{0\}$. Moreover $\alpha\mathfrak{b} = \alpha'\mathfrak{b}$ if and only if $\alpha.(\alpha')^{-1}\mathfrak{b} = \mathfrak{b}$ if and only if $\alpha.(\alpha')^{-1} \in U_{\mathfrak{b}}$ if and only if $\alpha = \lambda.\alpha'$ where $\lambda \in U_{\mathfrak{b}}$. This yields a bijection between $\{\mathfrak{a} : \mathfrak{a} \subseteq \mathfrak{d}, \mathfrak{a} \sim \mathfrak{b}\}$ and $(\mathfrak{d} : \mathfrak{b})/U_{\mathfrak{b}}$,and thus the partial local zeta function can be written as follows

$$\zeta_G(\mathfrak{d},\mathfrak{b},s) = \sum_{\alpha\in(\mathfrak{d}:\mathfrak{b}\setminus\{0\})/U_\mathfrak{b}} q^{-s.dim\frac{\mathfrak{d}}{\alpha\mathfrak{b}}} = \frac{1}{(U_\mathfrak{b}:U_\mathcal{O})} \sum_{\alpha\in(\mathfrak{d}:\mathfrak{b}\setminus\{0\})/U_\mathcal{O}} q^{-s.dim\frac{\mathfrak{d}}{\alpha\mathfrak{b}}} \quad .$$

In the second sum we restricted the action of $U_\mathfrak{b}$ on $(\mathfrak{b}:\mathfrak{d})$ to the action of $U_\mathcal{O}$, and in order to compensate this we had to divide the infinite sum by the index $(U_\mathfrak{b}:U_\mathcal{O})$. By definition of the *absolute value* $|z|$ of $z\in K$, we obtain the formula,

$$\zeta_G(\mathfrak{d},\mathfrak{b},s) = \frac{q^{-s(deg\mathfrak{d}-deg\mathfrak{b})}}{U_\mathfrak{b}:U_\mathcal{O}} \sum_{\alpha\in(\mathfrak{d}:\mathfrak{b})/U_\mathcal{O}} |\ \alpha\ |^s \qquad (31)$$

□

Lemma 4.3.

$$\zeta_G(\mathfrak{d},\mathfrak{b},s) = (U_\mathfrak{b}:U_\mathcal{O})\ q^{-s(deg\ \mathfrak{d}-deg\ \mathfrak{b}+deg\ \mathfrak{d}:\mathfrak{b})}\ \zeta_G(\mathfrak{d}:\mathfrak{b},\mathcal{O},s).$$

Proof. We have

$$\zeta_G\ (\mathfrak{d},\mathfrak{b},s) = \frac{q^{-s(deg\mathfrak{d}-deg\mathfrak{b})}}{(U_\mathfrak{b}:U_\mathcal{O})} \sum_{z\in(\mathfrak{d}:\mathfrak{b})/U_\mathcal{O}} |z|^s$$

and

$$\zeta_G\ ((\mathfrak{d}:\mathfrak{b}),\mathcal{O},s) = \frac{q^{-s(deg(\mathfrak{d}:\mathfrak{b})-deg\mathcal{O})}}{(U_\mathcal{O}:U_\mathcal{O})} \sum_{z\in((\mathfrak{d}:\mathfrak{b}):\mathcal{O})/U_\mathcal{O}} |z|^s = \frac{q^{-s(deg\ \mathfrak{d}:\mathfrak{b})}}{1} \sum_{z\in(\mathfrak{d}:\mathfrak{b})/U_\mathcal{O}} |z|^s,$$

which implies the claim. □

Now we want to compare Z_G with Stöhr's zeta function Z defined in section 3. Let $t = q^{-s}$ and define $Z_G(\mathfrak{d}, \mathfrak{b}, t)$ as the power series in t with $\zeta_G(\mathfrak{d}, \mathfrak{b}, s) = Z_G(\mathfrak{d}, \mathfrak{b}, t)$.

Let $\mathfrak{c}$ be a dualizing ideal, and write $\mathfrak{a}^* = \mathfrak{c} : \mathfrak{a}$ as before. Then $(\mathfrak{a}^*)^* = \mathfrak{a}$, and the assignment $\mathfrak{a} \longmapsto \mathfrak{a}^* = \mathfrak{c} : \mathfrak{a} = \{z \in K \mid z\mathfrak{a} \subseteq \mathfrak{c}\}$ defines an anti-monotonous, codimension preserving, bijective involution of the set of $\mathcal{O}$ -ideals. Moreover, $\mathfrak{a} \sim \mathfrak{b}$ implies $\mathfrak{a}^* \sim \mathfrak{b}^*$. From this we obtain

$$Z(\mathfrak{b}^*, \mathfrak{d}^*, t) = \sum_{\mathfrak{a} \supseteq \mathfrak{b}^*}^{\mathfrak{a} \sim \mathfrak{d}^*} t^{\dim(\mathfrak{a}/\mathfrak{b}^*)} = \sum_{\mathfrak{a}^* \subseteq \mathfrak{b}}^{\mathfrak{a}^* \sim \mathfrak{d}} t^{\dim(\mathfrak{b}/\mathfrak{a}^*)} = Z_G\ (\mathfrak{b}, \mathfrak{d}, t)$$

In particular, we have

$$Z_G(\mathcal{O}, \mathfrak{b}, t) = Z(\mathfrak{c}, \mathfrak{b}^*, t)$$

thus the Galkin type zeta functions appear as special Stöhr zeta functions.

5 A concrete example

In this section k is a finite field. Let $A := k[y,z]/(y^3 - z^2)$. A is not local a ring and has many maximal ideals, for example for every ordered pair $(x_0, y_0) \in k^2$ with $y_0^3 = z_0^2$, the ideal $(y - y_0, z - z_0)$ is a maximal ideal of A. For example $(y,z), (y-1, z-1), \cdots$, are maximal ideals in A.
Let the ideal $\mathfrak{p} := (y,z) \in SpecA$, then $A_{\mathfrak{p}}$ is a local ring. If we consider the algebraic curve $X = SpecA$ over k, the point $\mathfrak{p} := (y,z) \in X$ is singular (it is a cusp singularity [Ha] Ex. I 5.14), and

$$\mathcal{O}_{X.x} = A_{\mathfrak{p}} \quad \text{and} \quad \widehat{\mathcal{O}}_{X,x} = \varprojlim_{n} A_{\mathfrak{p}}/\mathfrak{p}^n A_{\mathfrak{p}}$$

Here $\widehat{\mathcal{O}}_{X,x}$ is the completion of $\mathcal{O}_{X,x}$. The mapping

$$\mathfrak{a} \longmapsto \hat{a} = \widehat{O}$$

defines a bijective monotonous correspondence that preserves degrees and quotients, between the ideals of the local ring $\mathcal{O}$ and the ideals of its completion $\widehat{\mathcal{O}}$, see [G]. Two $\mathcal{O}$–ideals $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent if and only if $\hat{\mathfrak{a}}$ and $\hat{\mathfrak{b}}$ are equivalent in $\widehat{\mathcal{O}}$ that is $\hat{\mathfrak{a}} = z\hat{\mathfrak{b}}$ for some unit z of the total quotient ring of $\widehat{\mathcal{O}}$. Thus the zeta-function $\zeta(\mathcal{O}, s)$ and the partial zeta-functions $\zeta(\mathcal{O}, \mathfrak{b}, s)$ will not change when $\mathcal{O}$ and $\mathfrak{b}$ are replaced by the corresponding completions $\widehat{\mathcal{O}}$ and $\hat{\mathfrak{b}}$, respectively.

According to the above comment, we consider $\widehat{A}_{\mathfrak{p}} = k[[y,z]]/(y^3 - z^2)$ instead of $\mathcal{O}_{X,x}$, where $k[[y,z]]$ is the formal power series in the variables y and z. By mapping $y \mapsto X^2$ and $z \mapsto X^3$ one can easily see

$$R := k[[X^2, X^3]] \simeq k[[y,z]]/(y^3 - z^2) = \widehat{A}_{\mathfrak{p}} \ .$$

Therefore we can consider the zeta-function of the ring R instead of $\mathcal{O}_{X.x}$. Now we have

$$R = k[[X^2, X^3]] \subseteq k[[X]] = \widetilde{R}, \tag{32}$$

where $\widetilde{R}$ is the integral closure of R, see Proposition 5.2 below. Since X^{2k+3l} is in $k\ [[X^2, X^3]]$ for k and l in $\mathbb{N}$, we conclude that

$$R = \{a_0 + a_1X + a_2X^2 + ... + a_nX^n + ... \mid a_1 = 0 \text{ and } a_i \in k\} \ .$$

By multiplication in $\widetilde{R}$

$$(a_0 + a_1X + ...+)(b_0 + b_1X + ...+) = \sum_{i\in\mathbb{Z}} c_iX^i \ ,$$

where $c_i = \sum_{\mu+\nu=i} a_\mu b_\nu$. When $\sum_{i\in\mathbb{Z}} c_iX^i = 1$ we see $a_0b_0 = 1$ and we easily see that the group of units of R is

$$U(\widetilde{R}) = \{a_0 + a_1X + ... + a_nX^n + ... \mid a_0 \neq 0\ \} \ .$$

Clearly $\widetilde{R}$ is an integral domain, and we find $K((X))$ the quotient field of $\widetilde{R}$, as follows:
Every element of $K((X))$ is of the form f/g, such that g is not zero. Therefore one can write $g = X^i.\widetilde{g}$, such that $i \geq 0$ and $\widetilde{g}$ is unit, this means

$$\frac{f}{g} = \frac{f}{X^i\widetilde{g}} = \frac{f.(\widetilde{g})^{-1}}{X^i} \ .$$

Therefore we can write the elements of $K((X))$ as bounded Laurent series, that is:

$$K((X)) = \left\{ \sum_{i\in\mathbb{Z}} a_iX^i \mid \exists\ j \in \mathbb{Z} \text{ s.t. } a_i = 0 \text{ for } i < j \right\}.$$

We can define a discrete value on $K((X))$ as follows:

$$v\left(\sum_{i\in\mathbb{Z}} a_iX^i\right) = n, \qquad \text{if } a_n \neq 0 \text{ and } \quad a_i = 0 \text{ for all } i < n.$$

$\widetilde{R}$ is the discrete valuation ring of $K((X))$ because

$$\widetilde{R} = \{f \in K((X)) \ \mid v(f) \geq 0\}$$

and $U(\widetilde{R}) = \{f \in K((X)) \ \mid v(f) = 0\}$. $\widetilde{R}$ is local ring and the unique maximal ideal is

$$\mathfrak{M} = \left\{ \sum_{i \in \mathbb{Z}} a_i X^i \mid a_i = 0 \quad \text{for} \quad i \leq 0 \right\} .$$

This implies $\widetilde{R}/\mathfrak{M} \simeq k$, so the singularity is rational and unibranched. Every ideal I of $\widetilde{R}$ is of the form (X^n), where $n = \text{Min}\{\nu(f) \mid f \in I\} \in \mathbb{N}_0$.

Proposition 5.1. *Let $\mathfrak{F}$ be the conductor of R. Then $\mathfrak{F} = \widetilde{R}X^2$ and as an Ideal in R it is generated by $\{X^2, X^3\}$.*

Proof. Since $\mathfrak{F} \subseteq R \subseteq \widetilde{R}$ and $\mathfrak{F}$ is an ideal of $\widetilde{R}$, we must have

$$\mathfrak{F} = \left\{ \sum_{i \in \mathbb{Z}} a_i X^i \mid a_i = 0 \text{ for all } i < 2 \right\} = \widetilde{R}X^2.$$

For the second part, let $I = (X^2, X^3)$ be the R-ideal generated by X^2 and X^3. By definition, $\mathfrak{F}$ is an R-ideal, containing X^2 and X^3, so that $I \subseteq \mathfrak{F}$. Conversely, let x be an arbitrary element of $\mathfrak{F}$ then

$$x = \sum_{i \geq 2} a_i X^i = X^2(a_2 + a_4 X^2 + ...) + a_3 X^2 \in I \ .$$

This shows $\mathfrak{F} \subseteq I$ and hence the claim.

□

Proposition 5.2.

(i) R is an order of $\widetilde{R}$

(ii) $\widetilde{R}$ is the integral closure of R.

Proof. **(i)**: Let K and $\widetilde{K}$ be the quotient fields of R and $\widetilde{R}$ respectively. Clearly $K \subseteq \widetilde{K}$ and since X^3 and $\frac{1}{X^2}$ are in K one concludes that $X \in K$. Therefore $\widetilde{K} \subseteq K$ and this yields $K = \widetilde{K}$.
(ii)$\widetilde{R}$ is integral over R, because it is generated by 1, X as an R-module. On the other hand, since $\widetilde{R}$ is discrete valuation ring, it is integrally closed.
Let R' be the integral closure of R in the common quotient field K; then $\widetilde{R} \subseteq R'$. Since R' is integral over R and $R \subseteq \widetilde{R} \subseteq R'$, we deduce that R' is integral over $\widetilde{R}$. But $\widetilde{R}$ is integrally closed and it yields $\widetilde{R} = R'$. □

Lemma 5.3. *For any ideal I of $\widetilde{R}$, $\widetilde{R}/I$ is finite.*

Proof. Every ideal of $\widetilde{R}$ is of the form (X^n) for some $n \in \mathbb{N}$. Now we introduce a homomorphism Φ from $\widetilde{R}$ to k^n as follows

$$\Phi(\sum_{i=0}^{\infty} a_i X^i) = (a_0, a_1, ..., a_n) .$$

Clearly I is the kernel of Φ and this yields $\widetilde{R}/I \simeq k^n$, which shows the claim since k is finite. □

Lemma 5.4. *Every fractional R-ideal $\mathfrak{a}$ of R is equivalent to a fractional R-ideal $\mathfrak{a}'$ such that $\mathfrak{F} \subseteq \mathfrak{a}' \subseteq \widetilde{R}$*

Proof. We know that every fractional R-ideal is of the form αI such that $\alpha \in K$ and I is an integral ideal in R. Now $\widetilde{R}$ is a valuation ring, therefore it is a principal ideal domain. Hence $\mathfrak{a}\widetilde{R} = a\widetilde{R}$ for some $a \in \widetilde{R}$. If we take $\mathfrak{a}' = \frac{1}{a}\mathfrak{a}$ it is obviously a fractional R-ideal, and

$$\mathfrak{F} = (\mathfrak{a}'\widetilde{R})\mathfrak{F} = \mathfrak{a}'(\widetilde{R}\mathfrak{F}) = \mathfrak{a}'\mathfrak{F} \subseteq \mathfrak{a}' .$$

On the other hand, $\mathfrak{a}'\widetilde{R} = \widetilde{R}$ and hence $\mathfrak{a}' \subseteq \widetilde{R}$ These imply the desired statement.

□

Proposition 5.5. *There are only finitely many fractional R-ideals $\mathfrak{a}'$ with $\mathfrak{F} \subseteq \mathfrak{a}' \subseteq \widetilde{R}$.*

Proof. It suffices to show that there are only finitely many subgroups between $\mathfrak{F}$ and $\widetilde{R}$. For every $\mathfrak{a}'$, $\mathfrak{a}'/\mathfrak{F}$ is a subgroup of $\widetilde{R}/\mathfrak{F}$. This correspondence is one to one. This shows the claim, since there are only finitely many subgroups in $\widetilde{R}/\mathfrak{F}$. □

Let all notations be as in section 2. In our case $r = \dim(R/\mathfrak{m}_R) = 1$ and since $\widetilde{R}$ is local ring, we have $m = 1$; that is the singularity is rational and unibranch. Now we want to determine the equivalence classes of integral R-ideals in order to calculate the partial zeta functions and then to find $Z(R,t)$. Since

$$\mathfrak{F} \subseteq k\,[[X^2, X^3]] \subseteq k\,[[X]]$$

and

$$\begin{aligned}\widetilde{R}/\mathfrak{F} &= k\,[[X]]]\,/X^2\,k\,[[X]] \\ &= k\,\bar{1} \oplus\ k\,\overline{X},\end{aligned}$$

we conclude that dim $_k$ $\widetilde{R}/\mathfrak{F} = 2$ with the k-base $\bar{1}$ and $\overline{X}$.

Proposition 5.6. *To give a k-subvectorspace of $\widetilde{R}/\mathfrak{F}$ is same as giving an R-submodule of $\widetilde{R}/\mathfrak{F}$.*

Proof. For the non-trivial direction let $\mathfrak{a}/\mathfrak{F}$ be a k-subvectorspace of $\widetilde{R}/\mathfrak{F}$. Every element of $\mathfrak{a}/\mathfrak{F}$ can be represented as $y = \mathfrak{b}_0 + \mathfrak{b}_1 X + \mathfrak{F}$ with $\mathfrak{b}_0, \mathfrak{b}_1 \in k$. then, for $x = \sum_{i=0}^{\infty}\ a_i X$ in R we have

$$\begin{aligned}xy =&(a_0 + a_2 X^2 + \cdots)[(b_0 + b_1 X) + \mathfrak{F}\,] \\ =&[(b_0 a_0 + b_0 a_2 X^2 + \cdots) + (b_1 a_0 X^{+} b_1 a_2 X^3 + \cdots) + \mathfrak{F}\,] \\ =&(b_0 a_0 + b_1 a_0 X) + \mathfrak{F} \\ =&a_0[(b_0 + b_1 X) + \mathfrak{F}].\end{aligned}$$

Since $\mathfrak{a}/\mathfrak{F}$ is a k - submodule, $x(\mathfrak{a}/\mathfrak{F}) \subseteq \mathfrak{a}/\mathfrak{F}$. □

By this, the R-submodules of $\widetilde{R}/\widetilde{\mathfrak{F}}$ are $\widetilde{R}/\mathfrak{F}$, (0) and the 1-dimensional k-subvectorspaces

$$< \bar{1} > \qquad < \overline{X} > \qquad < \alpha\bar{1} + \beta\overline{X} > \qquad (\alpha, \beta \neq 0)\ .$$

Clearly $< \alpha\bar{1} + \beta\overline{X} > = < \bar{1} + \gamma\overline{X} >$ such that $\gamma \in k^*$ and therefore there are $q - 1 = \#k^*$ such subvectorspaces. The corresponding R-ideals are:

$$\mathfrak{F} \Longleftrightarrow < 0 >$$

$$\widetilde{R} \Longleftrightarrow \widetilde{R}/\mathfrak{F}$$

$$R \Longleftrightarrow < \bar{1} >$$

$$\mathfrak{F} + RX \Longleftrightarrow < \overline{X} >$$

$$\mathfrak{F} + R(1 + \beta X) \Longleftrightarrow < \overline{1 + \beta x} >$$

By above diagram we found $q+3$, R-submodules between $\mathfrak{F}$ and $\widetilde{R}$. Now we try to determine which R- submodules in the left column are equivalent, to determine all equivalence classes of fractional R-modules which contain in R.

Lemma 5.7. *$\widetilde{R}$ and R are not equivalent, in other words $[\widetilde{R}] \neq [R]$.*

Proof. Assume $R = \alpha\widetilde{R}$ for some $\alpha \in K$. This yields $\alpha \in \widetilde{R}$ or $\alpha = \sum_{i=0}^{\infty} a_i X^i$ for some $a_i \in k$:
(i) If $a_0 = a_1 = 0$, then

$$R = (\sum_{i=2}^{\infty} a_{i=0} X^i)\ \widetilde{R} = (X^2 \sum_{i=0}^{\infty} a_{i+2} X^i)\widetilde{R}\,.\,\widetilde{R} = \mathfrak{F}$$

(ii) If $a_1 \neq 0$ then $\alpha = \alpha \cdot 1 \notin R$
(iii) If $a_0 \neq 0$ then $\alpha.X = \sum_{i=0}^{\infty} a_i X^{i+1} \notin R$
Since all above cases imply contradiction, we conclude that $\widetilde{R} \nsim R$.

□

Lemma 5.8. *$\mathfrak{F}$ is equivalent to $\widetilde{R}$ that is $[\,\mathfrak{F}\,] = [\,\widetilde{R}\,]$.*

Proof. By 5.1 we have $\mathfrak{F} = X^2 \cdot \widetilde{R}$. □

Remark 5.9. *By Lemmas 5.7 and 5.8, $\mathfrak{F} \nsim R$*

Lemma 5.10. *$\widetilde{R}$ is equivalent to $\mathfrak{F} + RX$, that is $[\,\widetilde{R}\,] = [\mathfrak{F} + RX]$.*

Proof. Obviously we have

$$\mathfrak{F} + R\,X = \{f \in k[\,[X]\,] \mid a_0 = 0\} = \widetilde{\mathfrak{m}} = X\widetilde{R}$$

□

Lemma 5.11. *$\mathfrak{F} + R(1+\beta X)$ and R are equivalent, that is*
$[\,\mathfrak{F}I + R(1+\beta X)\,] = [R]$.

Proof. In other words, taking an arbitrary element in $\mathfrak{F} + R(1+\beta X)$ and $\alpha = 1 + \beta X$, we try to find an element in R such as follows:

Let $a_2X^2 + a_3X^3 + ... + X^n + ... \in \mathfrak{F}$ and $b_0 + b_2X^2 + ...b_nX^n + ... \in R$ we look for the element $c_0 + c_2X^2 + c_3X^3 + ... + c_nX^n + ... \in R$ such that

$$\sum_{i=2}^{\infty} a_i X^i + (\sum_{i=0}^{\infty} b_i X^i)(1+\beta X) = (1+\beta X)\sum_{i=0}^{\infty} c_i X^i. \tag{33}$$

Here $b_1, c_1 = 0$ and a_i and b_i are arbitrary elements of k. We solve c_i in term of a_i and b_i as follows.

The left hand side of equation (33) is:

$(a_2X^2 + a_3X^3 + \cdots + a_nX^n + \cdots)+$
$(b_0 + b_2X^2 + \cdots + b_nX^n + \cdots)+$
$(b_0\beta X + b_2\beta X^3 + \cdots + \beta b_{n-1}X^n + \cdots) =$
$b_0 + b_0\beta X + (a_2 + b_2)X^2 + (a_3 + b_3 + b_2\beta)X^3 + \cdots + (a_n + b_n + \beta b_{n-1})X^n + \cdots$

We claim that $\mathfrak{F} + R(1 + \beta X) = \alpha R$ for $\alpha = 1 + \beta X$.
The right hand side of equation(33):

$$\begin{aligned}
&(c_0 + c_2X^2 + c_3X^3 + \cdots + c_nX^n + \cdots)+ \\
&(\beta c_0X + \beta c_2X^3 + \beta c_3X^4 + \cdots + \beta c_{n-1}X^n \cdots) = \\
&c_0 + \beta c_0X + c_2X^2 + (c_3 + \beta c_2)X^3 + \cdots + c_n + \beta c_{n-1})X^n + \cdots
\end{aligned}$$

By comparing we have the following system of equations

$$\begin{aligned}
b_0 &= c_0 \\
b_0\beta &= \beta c_0 \\
a_2 + b_2 &= c_2 \\
a_3 + b_3 + b_2\beta &= (c_3 + c_2\beta) \\
a_4 + b_4 + b_3\beta &= c_4 + \beta c_3 \\
\ldots &= \ldots \\
a_n + b_n + b_{n-1}\beta &= c_n + \beta c_{n-1}
\end{aligned} \tag{34}$$

We can solve c_i recursively in terms of a_i and b_i, this implies

$$\mathfrak{F} + R(1 + \beta X) \subseteq (1 + \beta X)\ R.$$

Conversely, if we choose c_i and a_i arbitrary, then by the system of equations (34) we find suitable b_i. This yields that

$$\alpha R \subseteq \mathfrak{F} + R(1 + \beta X).$$

□

By lemmas 5.7 up to 5.11

$$[\mathfrak{F}] = [\widetilde{R}] = [\mathfrak{F} + RX]$$

$$[R] = [\mathfrak{F} + R(1 + \beta X].$$

That is, we have two equivalence classes of ideals. In other words we have to consider two partial zeta function in the process of calculating $Z(R, t)$.

Lemma 5.12.

$$Z(R, R, t) = 1 + \frac{q\, t^2}{1 - t}$$

Before proving the lemma we need to identify the invariants according to section 2 .

Remark 5.13. *$S(R) = \{v(x)$ such that $x \in R\} = \{\ 0, 2, 3, \cdots\ \}$, we have $d = dim_k\ \widetilde{R}/(X) = 1$ and since $R/\mathfrak{m} \simeq k$, we conclude that $r = dim_k\ R/\mathfrak{m} =$ 1.*

Proof. By theorem 3.6

$$Z(R, R, t) = \frac{1}{(U_R : U_R)} \sum_{n \in S(R)} \varepsilon_n(R)\, t^{n.1+0}$$

where

$$\varepsilon_n(R) = \frac{q}{q-1} \cdot \sum_{i=0}^{1} (-1)^{\,i}\, q^{\,n + \deg(R \cap (X)^{n+i})} \quad , \quad n = \{0, 2, \cdots\}.$$

For $i = 0$

$$\deg(R \cap (X)^{n+i}) \quad = \deg(R \cap (X)^n) \quad = \begin{cases} 1 - n & if \quad n \geq 2 \\ 0 & if \quad n = 0\ , \end{cases}$$

for $i = 1$

$$\deg(R \cap (X)^{n+i}) = \deg(R \cap (X)^{n+1} \quad = \begin{cases} -n & if \quad n \geq 2 \\ -1 & if \quad n = 0\ . \end{cases}$$

Therefore we have

$$\begin{aligned} \varepsilon_0(R) &= \frac{q}{q-1}(1-q^{-1}) = 1 \\ \varepsilon_n(R) &= \frac{q}{q-1}\,(q-1) \quad = q, \ \text{ if } \ n \geq 2. \end{aligned} \tag{35}$$

Since $S(R) = \{0, 2, 3, \cdots\}$, we have

$$Z(R, R, t) = 1 + \sum_{n=2}^{\infty} q\, t^n = 1 + q \sum_{n=2}^{\infty} t^n = 1 + \frac{qt^2}{1-t} \tag{36}$$

□

Lemma 5.14.

$$(U_{\mathfrak{F}} : U_R) = q = \#\, k$$

Proof. Clearly

$$U_{\,\mathfrak{F}} = \{\, \alpha \in k \quad s.t \quad \alpha \cdot \mathfrak{F} = \mathfrak{F} \,\} = \widetilde{R}^{\,*}.$$

By the following exact, commutative diagrams

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U^1_{\mathfrak{F}} & \longrightarrow & U_{\mathfrak{F}} & \twoheadrightarrow & k^* & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 1 & \longrightarrow & U^1_R & \longrightarrow & U_R & \longrightarrow & k^* & \longrightarrow & 0 \end{array}$$

and

$$1 \longrightarrow U^1_R \longrightarrow U^1_{\mathfrak{F}} \longrightarrow k \longrightarrow 0$$

one obtains $U_{\mathfrak{F}} \,/\, U_R \simeq U^1_{\mathfrak{F}} /\, U^1_R \simeq k.$

□

Lemma 5.15.

$$Z(R, \mathfrak{F}, t\,) = \frac{t}{1-t}$$

Proof. Since $\deg \mathfrak{F} = -\dim R/\,\mathfrak{F} = -1$, we have by theorem 3.6

$$Z(R, \mathfrak{F}, t) = \frac{1}{(U_{\mathfrak{F}} : U_R)} \sum_{n \in S(\,\mathfrak{F}\,)} \varepsilon_n(\,\mathfrak{F}\,)\, t^{n-1},$$

where

$$\varepsilon_n(\mathfrak{F}) = \frac{q}{q-1} \cdot \sum_{i=0}^{1} (-1)^i \; q^{\,n \cdot 1 + \deg(\mathfrak{F} \cap (X)^{n+i})}.$$

On the other hand

$$\deg(\mathfrak{F} \cap (X^{n+i}) = \begin{cases} 1-n & if \quad i=0 \\ -n & if \quad i=1. \end{cases}$$

Therefore

$$\varepsilon_n(\mathfrak{F}) = \frac{q}{q-1}(q-1) = q,$$

and since $S(\mathfrak{F}) = \{2, 3, \cdot, \cdot, \cdot\}$ we have

$$Z(R, \mathfrak{F}, t) = \frac{1}{q} \sum_{n=2}^{\infty} q\, t^{n-1} = \sum_{n=2}^{\infty} t^{\,n-1} = \frac{t}{1-t}. \tag{37}$$

□

Theorem 5.16.

$$Z(R, t) = \frac{1+qt^2}{1-t}$$

Proof. Summing up the two equations 35 and 36 of Lemma 5.12 and 5.12 respectively, gives the result . □

To calculate more difficult cases, we need some further tools which will be developed in the next section.

6 The rational unibranch case I

In this section we assume that $\mathcal{O}$ is rational.This means that $d_i = 1$ for each $i = 1, ..., m$. This implies that r=1.
In this special situation the principal partial zeta-function $Z(\mathcal{O}, \mathcal{O}, t)$ by theorem 3.5 has the expansion

$$Z(\mathcal{O}, \mathcal{O}, t) = \sum_{n \in S} \varepsilon_n t^{n_1 + ... n_m}, \tag{38}$$

where

$$\varepsilon_n = \frac{q}{q-1} \sum_{i=0}^{m} (-1)^j \sum_{i_1 < ... < i_j} q^{n_1 + ... n_m - \dim(\mathcal{O}/\mathfrak{p}^{\boldsymbol{n}} \mathfrak{p}_{i_1} ... \mathfrak{p}_{i_j})} \tag{39}$$

for each $\boldsymbol{n} = (n_1, ... n_m) \in \mathbb{N}^m$. Thus $Z(\mathcal{O}, \mathcal{O}, t)$ can be expressed in term of the integers

$$\ell(\boldsymbol{n}) := \dim(\mathcal{O}/\mathcal{O} \cap \mathfrak{p}^{\boldsymbol{n}}).$$

Lemma 6.1. *For each $i = 1, ..., m$ we have;*

$$\ell(n_1, ..., n_i + 1, ..., n_m) \leq \ell(n_1, ..., n_m) + 1. \tag{40}$$

Proof. **(i)** Let $\mathfrak{b} := \mathcal{O} \cap \mathfrak{p}_1^{n_1} ... \mathfrak{p}_m^{n_m}$ and $\mathfrak{a} := \mathcal{O} \cap \mathfrak{p}_1^{n_1} ... \mathfrak{p}_i^{n_i+1} ... \mathfrak{p}_m^{n_m}$. Then $\mathfrak{a} \subseteq \mathfrak{b}$, and we have an exact sequence

$$0 \longrightarrow \mathfrak{b}/\mathfrak{a} \longrightarrow \mathcal{O}/\mathfrak{a} \longrightarrow \mathcal{O}/\mathfrak{b} \longrightarrow 0$$

Hence

$$\dim(\mathcal{O}/\mathcal{O}\mathfrak{p}_1^{n_1}...\mathfrak{p}_1^{n_i+1}...\mathfrak{p}_m^{n_m}) = \dim(\mathcal{O}\cap\mathfrak{p}_1^{n_1}...\mathfrak{p}_m^{n_m}/\mathcal{O}\cap\mathfrak{p}_1^{n_1}...\mathfrak{p}_i^{n_i+1}...\mathfrak{p}_m^{n_m}) + \dim(\mathcal{O}/\mathcal{O}\cap\mathfrak{p}_1^{n_1}...\mathfrak{p}_m^{n_m}). \tag{41}$$

On the other hand, we have a k-monomorphism

$$\mathcal{O}\cap\mathfrak{p}_1^{n_1}...\mathfrak{p}_m^{n_m}/\mathcal{O}\cap\mathfrak{p}_1^{n_1}...\mathfrak{p}_i^{n_i+1}...\mathfrak{p}_m^{n_m} \longrightarrow \mathfrak{p}_1^{n_1}...\mathfrak{p}_m^{n_m}/\mathfrak{p}_1^{n_1}...\mathfrak{p}_i^{n_i+1}...\mathfrak{p}_m^{n_m}$$

and

$$\begin{aligned} 1 = d_i &= \dim(\widetilde{\mathcal{O}}/\pi_i\widetilde{\mathcal{O}}) \\ &= \dim(\pi^{(n_1,\dots,n_m)}\widetilde{\mathcal{O}}/\pi^{(n_1,\dots,n_i+1,\dots,n_m)}\widetilde{\mathcal{O}}) \\ &= \dim(\mathfrak{p}_1^{n_1}...\mathfrak{p}_m^{n_m}/\mathfrak{p}_1^{n_1}...\mathfrak{p}_i^{n_i+1}...\mathfrak{p}_m^{n_m}.) \end{aligned} \tag{42}$$

This yields

$$\dim(\mathcal{O}\cap\mathfrak{p}_1^{n_1}...\mathfrak{p}_m^{n_m}/\mathcal{O}\cap\mathfrak{p}_1^{n_1}...\mathfrak{p}_i^{n_i+1}...\mathfrak{p}_m^{n_m}) \le 1,$$

and hence the claim by (41).

□

Now we restrict to the unibranch case, i.e., the case where $m = 1$. In this case the semigroup $S = S(\mathcal{O}) \subseteq \mathbb{N}$ is a numerical semigroup, whose genus $\#(\mathbb{N}\backslash S)$ is equal to the singularity degree δ and whose conductor is the exponent f of the conductor ideal $\mathfrak{F} = \mathcal{O} : \widetilde{\mathcal{O}}$. The δ positive integers that do not belong to S, are called the *gaps* of S.

Theorem 6.2. *Let $\mathcal{O}$ be a rational unibranch order. Then*

$$L(\mathcal{O},\mathcal{O},t) = \sum_{i=0}^{f} n_i t^i,$$

where

$$n_i = \begin{cases} q^{\#\{gaps\le i\}} & \text{if } \ i\in S \ \text{ and } \ i-1\notin S \\ -q^{\#\{gaps\le i\}} & \text{if } \ i\notin S \ \text{ and } i-1\in S \\ 0 & \text{otherwise}. \end{cases}$$

Proof. By (39) we have

$$Z(\mathcal{O},\mathcal{O},t) = \sum_{n\in S}\varepsilon_n t^n \quad \text{where} \quad \varepsilon_n = \frac{q}{q-1}(q^{n-\ell(n)} - q^{(n-\ell(n+1))}) \tag{43}$$

and $\ell(n) = \dim(\mathcal{O}/\mathfrak{p}^n \cap \mathcal{O})$. By the following monomorphism

$$\mathcal{O} \cap \mathfrak{p}^i/\mathcal{O} \cap \mathfrak{p}^{i+1} \hookrightarrow \mathfrak{p}^i/\mathfrak{p}^{i+1} \tag{44}$$

and $\dim(\mathfrak{p}^i/\mathfrak{p}^{i+1}) = 1$ we conclude that

$$\dim(\mathcal{O} \cap \mathfrak{p}^i/\mathcal{O} \cap \mathfrak{p}^{(i+1)}) = \begin{cases} 1\,; & \exists\, \alpha \in \mathcal{O} \cap \mathfrak{p}^i \backslash \mathcal{O} \cap \mathfrak{p}^{i+1} \\ 0\,; & \nexists\, \alpha \in \mathcal{O} \cap \mathfrak{p}^i \backslash \mathcal{O} \cap \mathfrak{p}^{i+1}\ , \end{cases} \tag{45}$$

or equivalently

$$\dim(\mathcal{O} \cap \mathfrak{p}^i/\mathcal{O} \cap \mathfrak{p}^{(i+1)}) = \begin{cases} 1\,; & \exists\, \alpha \in \mathcal{O}\ |\upsilon(\alpha) = i \\ 0\,; & \nexists\, \alpha \in \mathcal{O}\ |\upsilon(\alpha) = i\ . \end{cases} \tag{46}$$

By looking at the descending chain of ideals

$$\mathcal{O} \supseteq \mathcal{O} \cap \mathfrak{p}^1 \supseteq \mathcal{O} \cap \mathfrak{p}^2 \supseteq \ldots \supseteq \mathcal{O} \cap \mathfrak{p}^n \tag{47}$$

and (46) we see that $\ell(n) = \dim(\mathcal{O}/\mathfrak{p}^n \cap \mathcal{O})$ is equal to the number of elements in S smaller than n.
That is

$$n - \ell(n) = \#\{\text{ the gaps of S smaller than n.}\} \tag{48}$$

If $n \in S$, then we have

$$\begin{aligned} n - \ell(n) &= \#\{\text{ the gaps of S smaller than}\} \\ &= \#\{\text{ the gaps of S smaller than n+1}\} = n + 1 - \ell(n+1), \end{aligned}$$

which implies

$$\ell(n+1) = \ell(n) + 1\ . \tag{49}$$

Thus we conclude

$$\varepsilon_n = \frac{q}{q-1}(q^{n-\ell(n)} - q^{n-\ell(n)-1}) = q^{n-\ell(n)}\ .$$

By (48)

$$Z(\mathcal{O}, \mathcal{O}, t) = \sum_{n \in S} q^{\#\{\text{gaps}<n\}}\, t^n\ . \tag{50}$$

We multiply(50) by $1-t$ and calculate n_i as follows
At first we define:

$$\eta(n) = \begin{cases} q^{\#\ \mathrm{gap}\leq n} & \text{if } n \in S \\ 0 & \text{if } n \notin S\ . \end{cases} \tag{51}$$

Now we have

$$\begin{aligned} L(\mathcal{O},\mathcal{O},t) = (1-t)\sum_{n\in S} q^{\#\mathrm{gaps}\leq n}\, t^n &= (1-t)\sum_{n\in\mathbb{N}_0} \eta(n)\, t^n \\ &= \sum_{n\in\mathbb{N}_0} \eta(n)\, t^n \ - \sum_{n\in\mathbb{N}_0} \eta(n)\, t^{n+1} \\ &= \sum_{n\in\mathbb{N}_0} (\eta(n)-\eta(n-1))\, t^n\ , \end{aligned}$$

By (51) and defining $\eta(-1) := 0$ we have

$$\eta(n)-\eta(n{-}1) = \begin{cases} q^{\#\mathrm{gaps}\leq n} - q^{\#\mathrm{gaps}\leq n-1} = 0 & \text{if } \ n\in S\ ,\ n-1\in S \\ q^{\#\mathrm{gaps}\leq n} - 0 \qquad = q^{\#\mathrm{gaps}\leq n} & \text{if } \ n\in S\ ,\ n-1\notin S \\ 0\ -\ q^{\#\mathrm{gaps}\leq (n-1)} \quad = -q^{\#\mathrm{gaps}\leq n} & \text{if } \ n\notin S\ ,\ n-1\in S \\ 0 & \text{if } \ n\notin S\ ,\ n-1\notin S \end{cases}$$

and this implies the theorem.

□

Now we generalize the above theorem, that is instead of $L(\mathcal{O},\mathcal{O},t)$ we consider $L(\mathcal{O},\mathfrak{b},t)$ for every $\mathcal{O}$-ideal $\mathfrak{b}$.

Let $f(\mathfrak{b}) := \min\{n \mid \mathfrak{p}^n \subseteq \mathfrak{b}\}$; this $f(\mathfrak{b})$ exists by Lemma 4.9.

Definition 6.3. *By definition the conductor of the fractional $\mathcal{O}$-ideal $\mathfrak{b}$ is $\mathfrak{F}(\mathfrak{b}) = \mathfrak{p}^{f(\mathfrak{b})}$.*

Lemma 6.4. *Let $\mathfrak{b}$ be a fractional$\mathcal{O}$-ideal. Then*

$$f(\mathfrak{b}) = min\,\{n \mid \{n, n+1, \cdots\} \subseteq S(\mathfrak{b})\}\ .$$

Proof. Let $\mathfrak{p}^n \subseteq \mathfrak{b}$ and $n-1 \in S(\mathfrak{b})$; then $\exists\, \beta \in \mathfrak{b}$ such that $v(b) = n-1$. This implies $\beta \in \mathfrak{p}^{n-1} \setminus \mathfrak{p}^n$. Since dim $\mathfrak{p}^{n-1}/\mathfrak{p}^n = 1$, we conclude $\mathfrak{p}^{n-1} = k\beta + \mathfrak{p}^n$ and this implies $\mathfrak{p}^{n-1} \subseteq \mathfrak{b}$. On the other hand let $m = f(\mathfrak{b})$. Then by definition $\mathfrak{p}^m \subseteq \mathfrak{b}$ and hence

$$\{m, m+1, m+2, \cdots\} = S(\mathfrak{p}^m) \subseteq S(\mathfrak{b}),$$

so that

$$m \geq \min\,\{n \mid \{n, n+1, \cdots\} \subseteq S(\mathfrak{b})\}\ .$$

Suppose $\{m-1, m, \cdots\} \subseteq S(\mathfrak{b})$. By the above argument we have $\mathfrak{p}^{m-1} \subseteq \mathfrak{b}$ and this is a contradiction.

□

Remark 6.5. $\mathfrak{F}(\mathcal{O}) = \mathcal{O} : \widetilde{\mathcal{O}}$

Theorem 6.6. *Let $\mathcal{O}$ be a local rational unibranch order and $\mathfrak{b}$ be a fractional $\mathcal{O}$-ideal with $\mathcal{O} \subseteq \mathfrak{b} \subseteq \widetilde{\mathcal{O}}$ (compare lemma 7.1 below). Then*

$$L(\mathcal{O}, \mathfrak{b}, t) = \frac{(t \cdot q)^{\deg \mathfrak{b}}}{(U_{\mathfrak{b}} : U_{\mathcal{O}})} \sum_{i=0}^{f(\mathfrak{b})} n_i(\mathfrak{b})\, t^i\ . \tag{52}$$

Here $f(\mathfrak{b})$ is a positive integer smaller or equal to 2δ, and

$$n_i(\mathfrak{b}) = \begin{cases} q^{\#\{\text{gaps} \leq i\}} & \text{if} \quad i \in S(\mathfrak{b}) \ \text{ and } \ i-1 \notin S(\mathfrak{b}) \\ -q^{\#\{\text{gaps} \leq i\}} & \text{if} \quad i \notin S(\mathfrak{b}) \ \text{ and } i-1 \in S(\mathfrak{b}) \\ 0 & \text{otherwise}\ . \end{cases}$$

Proof. We define $\ell(n) := \dim \mathfrak{b}/(\mathfrak{b} \cap \mathfrak{p}^n)$. By theorem 3.6

$$Z(\mathcal{O}, \mathfrak{b}, t) = \frac{1}{(U_{\mathfrak{b}} : U_{\mathcal{O}})} \sum_{n \in S(\mathfrak{b})} \varepsilon_n(\mathfrak{b})\; t^{\,n+\deg \mathfrak{b}}, \tag{53}$$

where

$$\varepsilon_n = \frac{q}{q-1} \left(q^{n+\deg\mathfrak{b}-\ell(n)} - q^{n+\deg\mathfrak{b}-\ell(n+1)} \right) .$$

According to the the following descending chain of ideals

$$\mathfrak{b} \supseteq \mathfrak{b} \cap \mathfrak{p} \supseteq \mathfrak{b} \cap \mathfrak{p}^2 \supseteq \cdots \supseteq \mathfrak{b} \cap \mathfrak{p}^n,$$

with a similar argument as for theorem 5.2 we see that $\ell(n)$ is equal to the numbers of elements of S smaller than n, that is

$$n - \ell(n) = \#\{\text{gaps of } S(\mathfrak{b}) \text{ smaller than } n, \}.$$

If $n \in S(\mathfrak{b})$ then $\ell(n+1) = \ell(n) + 1$ and therefore

$$\begin{aligned}
\varepsilon_n &= \frac{q}{q-1} \cdot q^{\deg\mathfrak{b}} \left(q^{n-\ell(n)} - q^{n-\ell(n+1)} \right) \\
&= \frac{q^{\deg\mathfrak{b}+1}}{q-1} \left(q^{n-\ell(n)} - q^{n-\ell(n)-1} \right) \\
&= q^{\deg\mathfrak{b}} \cdot q^{n-\ell(n)}
\end{aligned}$$

By (53) we have

$$\begin{aligned}
Z(\mathcal{O}, \mathfrak{b}, t) &= \frac{t^{\ \deg\mathfrak{b}}}{(U_\mathfrak{b} : U_\mathcal{O})} \sum_{n \in S(\mathfrak{b})} \varepsilon_n(\mathfrak{b})\ t^n \\
&= \frac{t^{\ \deg\mathfrak{b}}}{(U_\mathfrak{b} : U_\mathcal{O})} \sum_{n \in S(\mathfrak{b})} q^{\ \deg\mathfrak{b}} \cdot q^{\ n-\ell(n)}\ t^n \\
&= \frac{(t \cdot q)^{\ \deg\mathfrak{b}}}{(U_\mathfrak{b} : U_\mathcal{O})} \sum_{n \in S(\mathfrak{b})} q^{\ n-\ell(n)}\ t^n.
\end{aligned}$$

It yields

$$Z(\mathcal{O}, \mathfrak{b}, t) = \frac{(t \cdot q)^{\deg\mathfrak{b}}}{(U_\mathfrak{b} : U_\mathcal{O})} \sum_{n \in S\mathfrak{b}} q^{\#\{\text{gaps}<n\}}\ t^n. \tag{54}$$

Multiply equation (54) by $1 - t$ and define

$$\eta(n) = \begin{cases} q^{\#\ \text{gap} \leq n} & \text{if } n \in S(\mathfrak{b}) \\ 0 & \text{if } n \notin S(\mathfrak{b}). \end{cases} \tag{55}$$

With the same argument as for theorem 6.2 we then have

$$L(\mathcal{O}, \mathfrak{b}, t) = \frac{(t \cdot q)^{\deg \mathfrak{b}}}{U_{\mathfrak{b}} : U_{\mathcal{O}}} \sum_{i=0}^{\mathfrak{f}(\mathfrak{b})} n_i(\mathfrak{b})\, t^i \ . \tag{56}$$

□

Since $\mathcal{O} \subseteq \mathfrak{b}$ by assumption, we have $f(\mathfrak{b}) \leq f(\mathcal{O})$; but it is well-known that $f(\mathcal{O}) \leq 2\delta$.

Lemma 6.7. *If $\mathfrak{b}$ is an $\mathcal{O}$-ideal with $\mathcal{O} \subset \mathfrak{b}$, then*

$$deg_{\mathcal{O}}(\mathfrak{b}) = deg_{\mathcal{O}_{\mathfrak{b}}}(\mathfrak{b}) \ + \ dim \mathcal{O}_{\mathfrak{b}}/\mathcal{O}, \tag{57}$$

where $\mathcal{O}_{\mathfrak{b}} = \mathfrak{b} : \mathfrak{b}$.

Proof. By the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathfrak{b}}/\mathcal{O} \hookrightarrow \mathfrak{b}/\mathcal{O} \twoheadrightarrow \mathfrak{b}/\mathcal{O}_{\mathfrak{b}} \longrightarrow 0,$$

we obtain

$$\begin{aligned} \deg_{\mathcal{O}}(\mathfrak{b}) = \dim \mathfrak{b}/\mathcal{O} &= \dim \mathfrak{b}/\mathcal{O}_{\mathfrak{b}} \ + \ \dim \mathcal{O}_{\mathfrak{b}}/\mathcal{O} \\ &= \deg_{\mathcal{O}_{\mathfrak{b}}}(\mathfrak{b}) + \dim \mathcal{O}_{\mathfrak{b}}/\mathcal{O} \ . \end{aligned} \tag{58}$$

□

Lemma 6.8. *Let $\mathfrak{b}$ be an $\mathcal{O}$-ideal, then*

$$Z(\mathcal{O}_{\mathfrak{b}}, \mathfrak{b}, t) = t^{-dim(\mathcal{O}_{\mathfrak{b}}/\mathcal{O})} \cdot Z(\mathcal{O}, \mathfrak{b}, t). \tag{59}$$

Proof. By equation (6) in section 2, we have

$$Z(\mathcal{O}, \mathfrak{b}, t) = t^{\deg_{\mathcal{O}}(\mathfrak{b})} \ \cdot \sum_{z \in \mathfrak{b} \setminus \{0\}/U_{\mathfrak{b}}} |z|^s \ .$$

If we consider $\mathfrak{b}$ as an $\mathcal{O}_{\mathfrak{b}}$-ideal, then

$$Z(\mathcal{O}_{\mathfrak{b}}, \mathfrak{b}, t) = t^{\deg_{\mathcal{O}_{\mathfrak{b}}}(\mathfrak{b})} \ \cdot \sum_{z \in \mathfrak{b} \setminus \{0\}/U_{\mathfrak{b}}} |z|^s \ .$$

Equation (57) implies the lemma. □

Corollary 6.9. *Let* $\mathfrak{b}$ *be an* $\mathcal{O}$*-ideal, then*

$$L(\mathcal{O}_{\mathfrak{b}}, \mathfrak{b}, t) = t^{-dim \mathcal{O}_{\mathfrak{b}}/\mathcal{O}} \cdot L(\mathcal{O}, \mathfrak{b}, t)\ . \tag{60}$$

Remark 6.10. *One can deduce the above corollary by means of theorem 6.6 as follows:*

$$L(\mathcal{O}, \mathfrak{b}, t) = \frac{(t \cdot q)^{deg(\mathfrak{b})}}{U_{\mathfrak{b}} : U_{\mathcal{O}}} \sum_{i=0}^{\mathfrak{f}(\mathfrak{b})} n_i(\mathfrak{b})\, t^i\ , \tag{61}$$

and

$$L(\mathcal{O}_{\mathfrak{b}}, \mathfrak{b}, t) = \frac{(t \cdot q)^{deg_{\mathcal{O}_{\mathfrak{b}}}(b)}}{U_{\mathfrak{b}} : U_{\mathcal{O}_{\mathfrak{b}}}} \sum_{i=0}^{\mathfrak{f}(\mathfrak{b})} n_i(\mathfrak{b})\, t^i, \tag{62}$$

so the equations $(U_{\mathfrak{b}} : U_{\mathcal{O}}) = q^{\dim \mathcal{O}_{\mathfrak{b}}/\mathcal{O}}$, $(U_{\mathfrak{b}} : U_{\mathcal{O}_{\mathfrak{b}}}) = 1$ yield equation (60).

7 Two more examples

As first example in this section, we consider the order $\mathcal{O} = k[[X^2, X^5]]$ and its integral closure $\widetilde{\mathcal{O}} = k[[X]]$.
From the argument in the proof of lemma 5.4, we see that every ideal class contains an ideal $\mathfrak{b}$ such that

$$\mathfrak{F} \subseteq \mathfrak{b} \subseteq \widetilde{\mathcal{O}} \qquad \text{and} \qquad \mathfrak{b} \cdot \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}. \tag{63}$$

Now we use another method to finde the representative of every equivalence class as follows:

Lemma 7.1. *Let $\mathcal{O}$ be an order of $\widetilde{\mathcal{O}}$, then every ideal class contains an ideal $\mathfrak{b}$ such that*

$$\mathfrak{b} \cdot \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}} \qquad \textit{and} \qquad \mathcal{O} \subseteq \mathfrak{b} \subseteq \widetilde{\mathcal{O}}.$$

Proof. By (63) there is an ideal $\mathfrak{b}$ such that $\mathfrak{b} \cdot \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}$. Now we claim that

$$\exists\, \alpha \in \mathfrak{b} \cap U_{\widetilde{\mathcal{O}}}$$

In fact, otherwise $\mathfrak{b} \subseteq \widetilde{\mathfrak{m}}$, where $\widetilde{\mathfrak{m}}$ is the maximal ideal in $\widetilde{\mathcal{O}}$ and this implies $\widetilde{\mathcal{O}} = \mathfrak{b} \cdot \widetilde{\mathcal{O}} \subseteq \widetilde{\mathfrak{m}}$ which is a contradiction. Then we have

$$\alpha^{-1} \cdot \mathfrak{b} \subseteq \alpha^{-1}\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}.$$

On the other hand, $1 \in \alpha^{-1}\mathfrak{b}$, and since $\alpha^{-1}\mathfrak{b}$ is $\mathcal{O}$-ideal we have $\mathcal{O} = \mathcal{O} \cdot 1 \subseteq \alpha^{-1}\mathfrak{b}$. Therefore

$$\mathcal{O} \subseteq \alpha^{-1}\mathfrak{b} \subseteq \widetilde{\mathcal{O}} \quad \text{and} \quad \alpha^{-1}\mathfrak{b} \cdot \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}, \tag{64}$$

and we take $\alpha^{-1}\mathfrak{b}$ as the desired $\mathcal{O}$-ideal.

□

Remark 7.2. *Let $\mathcal{O} \subseteq \mathfrak{b} \subseteq \widetilde{\mathcal{O}}$, and define $\mathcal{O}_{\mathfrak{b}} := \mathfrak{b} : \mathfrak{b}$. By definition of $\mathfrak{b} : \mathfrak{b}$, this is a ring, and $\mathfrak{b}$ is an $\mathcal{O}_{\mathfrak{b}}$-module. Moreover, $1 \in \mathfrak{b}$ yields $\mathcal{O}_{\mathfrak{b}} = \mathcal{O}_{\mathfrak{b}} \cdot 1 \subseteq \mathfrak{b}$, therefore we have*

$$\mathcal{O}_{\mathfrak{b}} \subseteq \mathfrak{b} \subseteq \widetilde{\mathcal{O}}. \tag{65}$$

Note $\mathcal{O}_{\alpha\mathfrak{b}} = \mathcal{O}_{\mathfrak{b}}$, for every $\alpha \in K$.

Remark 7.3. *Let $\mathfrak{b}$ be an order and $\mathcal{O} \subseteq \mathfrak{b} \subseteq \widetilde{\mathcal{O}}$. Then $\mathcal{O}_{\mathfrak{b}} = \mathfrak{b}$, because obviously $\mathfrak{b} \subseteq \mathfrak{b} : \mathfrak{b}$, and by the above we have*

$$\mathcal{O}_{\mathfrak{b}} \subseteq \mathfrak{b}.$$

Now we come back to our example. Clearly

$$\mathcal{O} = \{\ a_o + a_2X^2 + a_4X^4 + a_5X^5 + ... \quad s.t \quad a_i \in k\ \}.$$

Since $S(\mathcal{O}) = \{0, 2, 4, 5, ...\}$, i.e., has gaps 1 and 3, henceforth we also use the notation $\mathcal{O}_{13}$ instead of $\mathcal{O}$.
Obviously $\dim\widetilde{\mathcal{O}}/\mathcal{O} = 2$. Now we use the following correspondence

$$\{\mathcal{O}_{13} - \text{submodules } \ \mathfrak{b}\ s.t\ \mathcal{O} \subseteq \mathfrak{b} \subseteq \widetilde{\mathcal{O}}\} \Leftrightarrow \{\mathcal{O}_{13} - \text{submodules } \ \overline{\mathfrak{b}} \subseteq \widetilde{\mathcal{O}}/\mathcal{O}\ \}$$

As k-vector space

$$\widetilde{\mathcal{O}}/\mathcal{O} \simeq k\overline{X} \oplus k\overline{X}^3.$$

Therefore $\dim \overline{\mathfrak{b}} = 1$ or $\mathfrak{b} = \mathcal{O}_{13}$ or $\mathfrak{b} = \widetilde{\mathcal{O}}$, In the first case:

(i) $\overline{\mathfrak{b}} = k(\overline{X} + \beta\overline{X}^3)$, or

(ii) $\overline{\mathfrak{b}} = k\overline{X}^3$.

Here β varies in k. The case (i) is not possible, because

$$X^2(\overline{X} + \beta\overline{X}^3) = \overline{X}^3 \notin \overline{\mathfrak{b}},$$

which shows that $k(\overline{X} + \mathfrak{b}\overline{X}^3)$ is not a $\mathcal{O}_{13}$- submodule of $\widetilde{\mathcal{O}}/\mathcal{O}_{13}$. Therefore we are left with case (ii) where

$$\begin{aligned}\mathfrak{b} = <1, X^3>_{\mathcal{O}_{13}} &= \mathcal{O}_{13} + \mathcal{O}_{13}X^3 \\ &= \{\ b_0 + b_2X^2 + b_3X^3 + \cdots \quad s.t \quad b_i \in k\} \\ &= \mathcal{O}_1 := k[[X^2, X^3]].\end{aligned}$$

Therefore we have three equivalence classes of $\mathcal{O}_{13}$-modules represented by

$$\mathcal{O}_{13} \quad , \quad \mathcal{O}_1 \quad , \quad \widetilde{\mathcal{O}}.$$

(it follows from 2.14 and 7.3 that these ideals-being orders- are pairwise inequivalent.) So we need to compute

$$L(\mathcal{O}_{13}, \mathcal{O}_{13}, t) \quad , \quad L(\mathcal{O}_{13}, \mathcal{O}_1, t) \quad , \quad L(\mathcal{O}_{13}, \widetilde{\mathcal{O}}, t). \tag{66}$$

$\boldsymbol{L(\mathcal{O}_{13}, \mathcal{O}_{13}, t)}$:

Obviously $\mathfrak{F} := \mathcal{O} : \widetilde{\mathcal{O}} = X^4 \cdot \widetilde{\mathcal{O}}$ and $S(\mathcal{O}_{13}) = \{0, 2, 4, 5, \cdots\}$. These yield $f = 4$, therefore we have

$$L(\mathcal{O}_{13}, \mathcal{O}_{13}, t) = \sum_{i=0}^{4} n_i \, t^i$$

with

$$n_0 = 1 \quad , \quad n_1 = -1 \quad , \quad n_2 = q \quad , \quad n_3 = -q \quad , \quad n_4 = q^2$$

by theorem 6.2, i.e.,

$$L(\mathcal{O}_{13}, \mathcal{O}_{13}, t) = 1 - t + qt^2 - qt^3 + q^2t^4. \tag{67}$$

$\boldsymbol{L(\mathcal{O}_{13}, \mathcal{O}_1, t)}$:

By corollary 6.9 we have

$$L(\mathcal{O}_{13}, \mathcal{O}_1, t) = t^{\dim \mathcal{O}_{\mathcal{O}_1}/\mathcal{O}_{13}} L(\mathcal{O}_{\mathcal{O}_1}, \mathcal{O}_1, t)),$$

and, by Remark 7.3 and an easy computation,

$$\mathcal{O}_{\mathcal{O}_1} := \mathcal{O}_1 : \mathcal{O}_1 = \mathcal{O}_1 \quad \text{and} \quad \dim \mathcal{O}_{\mathcal{O}_1}/\mathcal{O}_{13} = 1.$$

Therefore we have

$$L(\mathcal{O}_{13}, \mathcal{O}_1, t) = t \cdot L(\mathcal{O}_1, \mathcal{O}_1, t).$$

From the example in section 5 we have $L(\mathcal{O}_1, \mathcal{O}_1, t) = 1 - t + q\,t^2$, and this yields

$$L(\mathcal{O}_{13}, \mathcal{O}_1, t) = t \cdot (1 - t + q\, t^2). \tag{68}$$

$\boldsymbol{L(\mathcal{O}_{13}, \widetilde{\mathcal{O}}, t)}$:

As above we have

$$L(\mathcal{O}_{13}, \widetilde{\mathcal{O}}, t) = t^{\dim \mathcal{O}_{\widetilde{\mathcal{O}}}/\mathcal{O}_{13}} L(\mathcal{O}_{\widetilde{\mathcal{O}}}, \widetilde{\mathcal{O}}, t)$$

and

$$\mathcal{O}_{\widetilde{\mathcal{O}}} := \widetilde{\mathcal{O}} : \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}} \quad , \quad \dim \mathcal{O}_{\widetilde{\mathcal{O}}}/\mathcal{O}_{13} = 2,$$

so that

$$L(\mathcal{O}_{13}, \widetilde{\mathcal{O}}, t) = t^2 \cdot L(\widetilde{\mathcal{O}}, \widetilde{\mathcal{O}}, t).$$

By Lemma 2.10 and equation (14) in theorem 3.10 , $L(\widetilde{\mathcal{O}}, \widetilde{\mathcal{O}}, t) = 1$ and this implies

$$L(\mathcal{O}_{13}, \widetilde{\mathcal{O}}, t) = t^2. \tag{69}$$

By summing, we have

$$\begin{aligned} L(\mathcal{O}_{13}, t) &= L(\mathcal{O}_{13}, \mathcal{O}_{13}, t) \;+\; L(\mathcal{O}_{13}, \mathcal{O}_1, t) \;+\; L(\mathcal{O}_{13}, \widetilde{\mathcal{O}}, t) \\ &= 1 - t + qt^2 - qt^3 + q^2t^4 \;+\; t \cdot (1 - t + q\, t^2) \;+\; t^2 \\ &= 1 + qt^2 + q^2t^4. \end{aligned} \tag{70}$$

By Theorem 3.15

$$Z(\mathcal{O}_{13}, t) = \frac{1 + qt^2 + q^2t^4}{1 - t}. \tag{71}$$

Example 2

In this example, we consider the Ring

$$\mathcal{O} = \mathcal{O}_{12} := \left\{ \sum_{i=0}^{\infty} a_i X^i \mid a_i \in k \quad \text{and} \quad a_1, a_2 = 0 \right\}$$

and its integral closure $\widetilde{\mathcal{O}} := K[[X]]$. Like in the last example, we use the correspondence between $\mathcal{O}_{12}$-ideals between $\mathcal{O}_{12}$ and $\widetilde{\mathcal{O}}$ and $\mathcal{O}_{12}$-submodule of $\widetilde{\mathcal{O}}/\mathcal{O}_{12}$. Clearly as k vector space $\widetilde{\mathcal{O}}/\mathcal{O}_{12} = k\overline{X} \bigoplus k\overline{X}^2$. By considering subvector spaces with dimension $0, 1, 2$, since

$$\mathcal{O} + \mathcal{O}X \quad , \quad \mathcal{O} + \mathcal{O}(X + \beta X^2)$$

are equivalent: $\mathcal{O} + \mathcal{O}X = (1 - \beta X) \cdot (\mathcal{O} + \mathcal{O}(X + \beta X)$, we consider the following list of $\mathcal{O}_{12}$-ideals

$$\mathcal{O}_{12} \quad , \quad \mathcal{O}_{12} + \mathcal{O}_{12} \cdot X \quad , \quad \mathcal{O}_1 = \mathcal{O}_{12} + \mathcal{O}_{12} \cdot X^2 \quad , \quad \widetilde{\mathcal{O}}.$$

So

$$\begin{aligned} L(\mathcal{O}, t) = L(\mathcal{O}_{12}, \mathcal{O}_{12}, t) \ + \ L(\mathcal{O}_{12}, \mathcal{O}_{12} + \mathcal{O}_{12}X, t) \\ + L(\mathcal{O}_{12}, \mathcal{O}_1, t) \quad + L(\mathcal{O}_{12}, \widetilde{\mathcal{O}}, t). \end{aligned} \tag{72}$$

By Remark 7.3, $\mathcal{O}_{12} = \mathcal{O}_{\mathcal{O}_{12}}$, $\mathcal{O}_1 = \mathcal{O}_{\mathcal{O}_1}$ and $\widetilde{\mathcal{O}} = \mathcal{O}_{\widetilde{\mathcal{O}}}$, therefore Corollary 6.27 yields

$$\begin{aligned} L(\mathcal{O}_{12}, t) = L(\mathcal{O}_{12}, \mathcal{O}_{12}, t) \ + \ L(\mathcal{O}_{12}, \mathcal{O}_{12} + \mathcal{O}_{12}X, t) \\ + t \cdot L(\mathcal{O}_1, \mathcal{O}_1, t) \quad + t^2 \cdot L(\widetilde{\mathcal{O}}, \widetilde{\mathcal{O}}, t). \end{aligned} \tag{73}$$

$\boldsymbol{L(\mathcal{O}_{12}, \mathcal{O}_{12}, t)}$:

Clearly $S(\mathcal{O}_{12}) = \{0, 3, 4, \cdots\}$ and $\mathfrak{F} := \mathcal{O}_{12} : \widetilde{\mathcal{O}} = X^3\widetilde{\mathcal{O}}$. Therefore $f = 3$ and

$$L(\mathcal{O}_{12}, \mathcal{O}_{12}, t) = \sum_{i=0}^{3} n_i t^i, \tag{74}$$

where

$$n_0 = 1 \qquad , \qquad n_1 = -q^{\#\{\text{gaps}<1\}} = -q^0 = -1$$

$$n_2 = 0 \qquad , \qquad n_3 = \quad q^{\#\{\text{gaps}<3\}} = \quad q^2,$$

i.e. ,

$$L(\mathcal{O}_{12}, \mathcal{O}_{12}, t) = 1 - t + q^2 t^3 \tag{75}$$

$\boldsymbol{L(\mathcal{O}_{12}, \mathcal{O}_{12} + \mathcal{O}_{12}X, t)}$:

Note that $\mathcal{O}_{12} + X \cdot \mathcal{O}_{12} = \mathfrak{b} := \{a_0 + a_1 X + a_3 X^3 + \cdots \,|a_i \in k\}$ so $S(\mathcal{O}_{12} + \mathcal{O}_{12}X) = \{0, 1, 3, 4 \cdots\}$, $\mathfrak{F}(\mathfrak{b}) = X^3 \cdot \widetilde{\mathcal{O}}$ and $\mathfrak{f}(\mathfrak{b}) = 3$. According to Theorem 6.6

$$L(\mathcal{O}_{12}, \mathfrak{b}, t) = \frac{(qt)^{\deg \mathfrak{b}}}{U_{\mathfrak{b}} : U_{\mathcal{O}_{12}}} \sum_{i=0}^{3} n_i(\mathfrak{b})\, t^i. \tag{76}$$

Since $\mathcal{O}_{\mathfrak{b}} = \mathcal{O}_{12}$ we conclude $(U_{\mathfrak{b}} : U_{\mathcal{O}_{12}}) = 1$. Clearly deg $\mathfrak{b} =$ dim $\mathfrak{b}/\mathcal{O}_{12} = 1$ and

$$n_0 = 1 \qquad\qquad n_1 = 0$$

$$n_2 = -q^{\#\{\text{gaps}<2\}} = -1 \qquad\qquad n_3 = \quad q^{\#\{\text{gaps}<3\}} = \; q.$$

Therefore (75) yields

$$L(\mathcal{O}_{12}, \mathcal{O}_2, t) = q\, t\, (1 - t^2 + qt^3).$$

$\boldsymbol{L(\mathcal{O}_1, \mathcal{O}_1, t)}$:

We have

$$n_0 = 1 \qquad n_1 = -q^{\;\#\text{gaps}\leq 1} = -1 \qquad n_2 = q^{\;\#\text{gaps}\leq 2} = q.$$

Therefore

$$L(\mathcal{O}_1, \mathcal{O}_1, t) = 1 - t + q\, t^2$$

$\boldsymbol{L(\widetilde{\mathcal{O}},\widetilde{\mathcal{O}},t)}$

As we have seen above, $L(\widetilde{\mathcal{O}},\widetilde{\mathcal{O}},t)=1$

(72) implies

$$\begin{aligned}L(\mathcal{O},t) &= (1-t+q^2t^3) + \ (\,q\,t\,(1-t^2+qt^3))\ +\ t\,(1-t+qt^2)\ +\ t^2(1)\\ &= 1+qt^2+q^2t^3+q^2t^4.\end{aligned}$$

$\boldsymbol{Z(\mathcal{O},t)}$

By Theorem 3.15

$$Z(\mathcal{O},t)=\frac{1+qt+q^2t^3+q^2t^4}{1-t}.$$

8 The rational unibranch case II

In this section, we develop some tools which will be used later for calculating zeta-functions of more complicate examples. Again $\mathcal{O}$ will be a local rational unibranch order.

Lemma 8.1. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be* $\mathcal{O}$*-ideals and* $\mathcal{O} \subseteq \mathfrak{a} \subseteq \mathfrak{b}$*, then*

$$dim\ \mathfrak{b}/\mathfrak{a} = \#(S(\mathfrak{b})\backslash S(\mathfrak{a})).$$

Proof. Since for $n >> 0$ $\mathfrak{p}^n \subseteq \mathfrak{a}$, we have

$$0 \longrightarrow \mathfrak{a}/\mathfrak{p}^n \hookrightarrow \mathfrak{b}/\mathfrak{p}^n \twoheadrightarrow \mathfrak{b}/\mathfrak{a} \longrightarrow 0\ .$$

This yields $\dim \mathfrak{b}/\mathfrak{a} = \dim \mathfrak{b}/\mathfrak{p}^n - \dim \mathfrak{a}/\mathfrak{p}^n$. Using the filtrations

$$\begin{array}{c} \mathfrak{b} \supseteq \mathfrak{b}\cap\mathfrak{p} \supseteq \mathfrak{b}\cap\mathfrak{p}^2 \supseteq \cdots \supseteq \mathfrak{b}\cap\mathfrak{p}^n = \mathfrak{p}^n \\ \\ \mathfrak{a} \supseteq \mathfrak{a}\cap\mathfrak{p} \supseteq \mathfrak{a}\cap\mathfrak{p}^2 \supseteq \cdots \supseteq \mathfrak{a}\cap\mathfrak{p}^n = \mathfrak{p}^n, \end{array} \tag{77}$$

we conclude as in the proof of Theorem 6.6

$$\begin{aligned} \dim \mathfrak{b}/\mathfrak{a} &= \dim \mathfrak{b}/\mathfrak{p}^n - \dim \mathfrak{a}/\mathfrak{p}^n \\ &= \ell_{\mathfrak{b}}(n) - \ell_{\mathfrak{a}}(n) \\ &= \#(S(\mathfrak{b})\backslash S(\mathfrak{a}))\ . \end{aligned}$$

□

Lemma 8.2.

$$deg\, L(\mathcal{O}, \mathfrak{b}, t) \leq f(\mathfrak{b}) + deg\, \mathfrak{b}\ .$$

Proof. We have

$$\deg L(\mathcal{O}, \mathfrak{b}, t) \leq \delta + \dim(\mathfrak{b}/(\mathfrak{b} : \widetilde{\mathcal{O}})) \tag{78}$$

see [St1]. Since $\mathfrak{b} : \widetilde{\mathcal{O}} = \mathfrak{F}(\mathfrak{b})$, we obtain $\dim(\mathfrak{b}/(\mathfrak{b} : \widetilde{\mathcal{O}})) = \deg \mathfrak{b} - \deg \mathfrak{F}(\mathfrak{b})$ and this yields

$$\delta - \deg \mathfrak{F}(\mathfrak{b}) = \deg \widetilde{\mathcal{O}} - \deg \mathfrak{F}(\mathfrak{b}) = \dim \widetilde{\mathcal{O}}/\mathfrak{F}(\mathfrak{b}) = \mathfrak{f}(\mathfrak{b})$$

because by definition $\mathfrak{F}(\mathfrak{b}) = \mathfrak{p}^{\mathfrak{f}(\mathfrak{b})}$.
By substitution in (78)

$$\deg L(\mathcal{O}, \mathfrak{b}, t) \leq \mathfrak{f}(\mathfrak{b}) + \deg \mathfrak{b}$$

□

Remark 8.3. *(see[St2]) For $m \leq 5$ and $q \geq m$*

$$deg\, L(\mathcal{O}, \mathfrak{b}, t) = \mathfrak{f}(\mathfrak{b}) + deg\, \mathfrak{b}$$

We can generalize Corollary 6.9 as follows

Lemma 8.4. *Let $\mathcal{O}'$ be a ring such that $\mathcal{O} \subseteq \mathcal{O}' \subseteq \mathcal{O}_{\mathfrak{b}} \subseteq \mathfrak{b}$, then*

$$L(\mathcal{O}, \mathfrak{b}, t) = t^{deg \mathcal{O}'} L(\mathcal{O}', \mathfrak{b}, t).$$

Proof. The proof is similar to the proof of 6.8. □

Notation 8.5.

$$S^{(n)} := \{0, 2, 4, \cdots, 2n, 2n+1, \cdots\}$$

$$S(n) := \{0, \cdots, n, n+1. \cdots\}$$

and $\mathcal{O}(n) := k \oplus \mathfrak{p}^n$, where $\mathfrak{p}$ is the maximal ideal of $\widetilde{\mathcal{O}}$.

We note that by the notation in section 7, $S(\mathcal{O}_{13}) = S^{(2)}$ and $S(\mathcal{O}_{12}) = S(3)$.

Definition 8.6. *By definition, an order $\mathcal{O}$ is balanced of degree n, if $S(\mathcal{O}) = S^{(n)}$.*

Lemma 8.7. *Let $\mathcal{O}$ be balanced of degree $n \in \mathbb{N}$, and let $\mathcal{O}' = \mathcal{O} + \mathfrak{p}^{2n-1}$. Then $\mathcal{O}'$ is balanced of degree $n-1$, and*

$$L(\mathcal{O}, t) = L(\mathcal{O}, \mathcal{O}, t) + t\, L(\mathcal{O}', t)$$

Proof. By definition

$$L(\mathcal{O}, t) = L(\mathcal{O}, \mathcal{O}, t) + \sum_{(\mathfrak{b}) \neq (\mathcal{O})} L(\mathcal{O}, \mathfrak{b}, t).$$

Now let $\mathcal{O} \subseteq \mathfrak{b} \subseteq \widetilde{\mathcal{O}}$ with $(\mathfrak{b}) \neq (\mathcal{O})$. Then $\deg \mathfrak{b} \geq 1$ and by Remark 8.3

$$f(\mathfrak{b}) + \deg(\mathfrak{b}) = \deg L(\mathcal{O}, \mathfrak{b}, t) \leq 2\delta = 2n.$$

This yields

$$f(\mathfrak{b}) \leq 2n - 1.$$

We conclude $\mathfrak{p}^{2n-1} \subseteq \mathfrak{p}^{f(\mathfrak{b})} \subseteq \mathfrak{b}$. Thus $\mathcal{O}' = \mathcal{O} + \mathfrak{p}^{2n-1}$ is a ring such that

$$\mathcal{O} \subseteq \mathcal{O}' \subseteq \mathfrak{b} \subseteq \widetilde{\mathcal{O}}. \tag{79}$$

Clearly

$$S(\mathcal{O}') = S(\mathcal{O} + \mathfrak{p}^{2n-1}) = \{0, 2, 4, \cdots, 2n-2, 2n-1, \cdots\},$$

so $\mathcal{O}'$ is balanced of order $n-1$, and obviously $\mathfrak{b}$ is a $\mathcal{O}'$-module. This shows that we have

$$\sum_{\substack{(\mathfrak{b}) \neq (\mathcal{O}) \\ \mathfrak{b}\ \mathcal{O}-module}} L(\mathcal{O}, \mathfrak{b}, t) = \sum_{\mathfrak{b}\ \mathcal{O}'-module} L(\mathcal{O}, \mathfrak{b}, t).$$

By Lemma 8.4 we have

$$\sum_{\mathfrak{b}\ \mathcal{O}'-module} L(\mathcal{O}, \mathfrak{b}, t) = \sum_{\mathfrak{b}\ \mathcal{O}'-module} t\, L(\mathcal{O}', \mathfrak{b}, t)$$

$$= t\, L(\mathcal{O}', t).$$

This yields the lemma.

□

We use this to give explicit formulae for the zeta functions of balanced orders.

Theorem 8.8. *Let $\mathcal{O}$ be balanced of degree n. Then*

$$L(\mathcal{O},\mathcal{O},t) = 1 - t + qt^2 - qt^3 + \cdots + q^{n-1}t^{2n-2} - q^{n-1}t^{2n-1} + q^n t^{2n}$$

$$L(\mathcal{O},t) = 1 + qt^2 + q^2t^4 + \cdots + q^n t^{2n}.$$

Proof. The gaps of $S(\mathcal{O})$ are $1, 3, 5, \cdots 2n-1$. Therefore, by theorem 6.2, we have $L(\mathcal{O},\mathcal{O},t) = \sum_{i=0}^{2n} n_i t$, with

$$n_i = \begin{cases} q^k & ,\text{ if } \quad i = 2k \\ -q^k & ,\text{ if } \quad i = 2k+1. \end{cases} \tag{80}$$

This shows the first claim. For the second we proceed by induction on n. For $n = 0$ we have $S(\mathcal{O}) = \mathbb{N}_0$ and hence $\mathcal{O} = \widetilde{\mathcal{O}}$, where $L(\widetilde{\mathcal{O}},t) = 1$ as claimed. Now let $\mathcal{O}$ be balanced of degree $n \geq 1$, and let $\mathcal{O}' = \mathcal{O} + \mathfrak{p}^{n-1}$ as in Lemma 8.7. Then $\mathcal{O}'$ is balanced of degree $n-1$ and hence

$$L(\mathcal{O}',t) = 1 + qt^2 + q^2t^4 + \cdots + q^{n-1}t^{2n-2}$$

by induction assumption. By Lemma 8.7 again, and the result on $L(\mathcal{O},\mathcal{O},t)$, we get

$$\begin{aligned} L(\mathcal{O},t) = 1 - t + q\,t^2 - q\,t^3 + \cdots + q^{n-1}\,t^{2n-2} - q^{n-1}\,t^{2n-1} + q^n\,t^{2n} \\ + t(1 + q\,t^2 + q^2\,t^4 + q^3\,t^6 + \cdots + q^{n-1}\,t^{2n-2}) \end{aligned}$$

This yields the claim for $\mathcal{O}$. □

Lemma 8.9. *Let $\mathcal{O} \subsetneqq \mathcal{O}'$ be orders of $\widetilde{\mathcal{O}}$ and let $n = deg\,\mathcal{O}'$. Assume that all $\mathcal{O}$-ideals $\mathfrak{b}$ with $\mathcal{O} \subseteq \mathfrak{b} \subseteq \widetilde{\mathcal{O}}$ and $deg\,\mathfrak{b} \geq n$ have the following property*

(i) $\mathfrak{b}$ is an $\mathcal{O}'$- ideal

Then we have

$$L(\mathcal{O},t) = \sum_{\substack{(\mathfrak{b}) \\ \mathcal{O}\subseteq\mathfrak{b}\subseteq\widetilde{\mathcal{O}} \\ deg\,\mathfrak{b}<n}} L(\mathcal{O},\mathfrak{b},t) + t^{deg\,\mathcal{O}'} L(\mathcal{O}',t)$$

In other words

$$\sum_{\substack{(\mathfrak{b}) \\ \mathcal{O}\subseteq\mathfrak{b}\subseteq\widetilde{\mathcal{O}} \\ deg\,\mathfrak{b}>n}} L(\mathcal{O},\mathfrak{b},t) = t^{deg\,\mathcal{O}'} L(\mathcal{O}',t)$$

Proof. We have

$$L(\mathcal{O},t) = \sum_{\substack{(\mathfrak{b}) \\ \mathcal{O}\subseteq\mathfrak{b}\subseteq\widetilde{\mathcal{O}} \\ \deg\mathfrak{b}<n}} L(\mathcal{O},\mathfrak{b},t) + \sum_{\substack{(\mathfrak{b}) \\ \mathcal{O}\subseteq\mathfrak{b}\subseteq\widetilde{\mathcal{O}} \\ \deg\mathfrak{b}\geq n}} L(\mathcal{O},\mathfrak{b},t) \tag{81}$$

Let $(\mathfrak{b})$ be one term of the right summand of (80). Then by (i) $\mathfrak{b}$ is an $\mathcal{O}'$ - ideal, and $\mathcal{O}' \subseteq \mathfrak{b}$ because $1 \in \mathfrak{b}$. On the other hand, if $\mathfrak{b}$ is an $\mathcal{O}'$- ideal and $\mathcal{O}' \subseteq \mathfrak{b} \subseteq \widetilde{\mathcal{O}}$, then $\mathfrak{b}$ is $\mathcal{O}$- ideal by restriction and $\deg\mathfrak{b} \geq \deg\mathcal{O}' = n$.Therefore we can write

$$\begin{aligned}
\sum_{\substack{(\mathfrak{b}),\mathfrak{b}\ \mathcal{O}-ideal \\ \mathcal{O}\subseteq\mathfrak{b}\subseteq\widetilde{\mathcal{O}} \\ \deg\mathfrak{b}\geq n}} L(\mathcal{O},\mathfrak{b},t) &= \sum_{\substack{(\mathfrak{b}),\mathfrak{b}\ \mathcal{O}'-ideal \\ \mathcal{O}\subseteq\mathfrak{b}\subseteq\widetilde{\mathcal{O}}}} L(\mathcal{O},\mathfrak{b},t) \\
&= \sum_{(\mathfrak{b}),\mathfrak{b}\ \mathcal{O}'-ideal} t^{\deg\mathcal{O}'}\, L(\mathcal{O}',\mathfrak{b},t) \\
&= t^{\deg\mathcal{O}'}\, L(\mathcal{O}',t)
\end{aligned}$$

□

Lemma 8.10. *(i) Let $\mathcal{O} \subseteq \mathfrak{b}$ and $S(n) = S(\mathfrak{b})$ then*

$$\mathfrak{b} = \mathcal{O}(n)$$

(ii) Let $S(n) \subseteq S(\mathfrak{b})$, then $\mathcal{O}(n) \subseteq \mathfrak{b}$

Proof. (i) Since $f(\mathfrak{b}) = n$, and $1 \in \mathfrak{b}$ we have $k\oplus\mathfrak{p}^n \subseteq \mathfrak{b}$, and by Lemma(8.1)

$$\dim\ \mathfrak{b}/(k \oplus \mathfrak{p}^n) = \#\{S(n)\backslash S(\mathfrak{b})\} = 0,$$

and this yields $\mathfrak{b} = \mathcal{O}(n)$
(ii) If $S(n) \subseteq S(\mathfrak{b})$ it yields $f(\mathfrak{b}) \leq n$ and consequently $\mathfrak{p}^n \subseteq \mathfrak{p}^{f(\mathfrak{b})} \subseteq \mathfrak{b}$. We conclude $\mathcal{O}(n) = k \oplus \mathfrak{p}^n \subseteq \mathfrak{b}$.

□

9 On the Riemann hypothesis

Let k be a finite field of order q. Let X be a (possibly singular) complete integral algebraic curve over k and let $\widetilde{X}$ be its normalization. Then Stöhr [St2] defined a global zeta function $\zeta(X,s)$ counting the positive divisors on X, and showed that it has an Euler product decomposition

$$\zeta(X,s) = \prod_{x \in |X|} \zeta(\mathcal{O}_{X,x}, s),$$

where $|X|$ is the set of closed points of X, and $\zeta(\mathcal{O}_{X,x}, s)$ is Stöhr's zeta function of the order $\mathcal{O}_{X,x}$. For $\widetilde{X}$ this gives the classical zeta function of the non-singular complete curve $\widetilde{X}$, and by the Hasse-Weil theorem, it satisfies the Riemann hypothesis, i.e., the zeroes of $\zeta(\widetilde{X}, s)$ lie on the line $Re(s) = 1/2$. If x is a regular point of X, then there is only one point $\widetilde{x}$ of $\widetilde{X}$ lying above x, and one has $\mathcal{O}_{X,x} \cong \mathcal{O}_{\widetilde{X},\widetilde{x}}$. Since $\zeta(\widetilde{X}, s)$ has a similar Euler product, Theorem 3.15 shows that

$$\zeta(X,s) = \zeta(\widetilde{X}, s) \cdot \prod_{x \text{ singular}} L(\mathcal{O}_{X,x}, t)$$

where $t = q^{-s}$ and the product is over the finitely many singular points of X. Therefore $\zeta(X,s)$ satisfies the Riemann hypothesis if and only if all $L(\mathcal{O}_{X,x}, q^{-s})$ satisfy the Riemann hypothesis.

This motivates to investigate for which orders $\mathcal{O}$ the Riemann hypothesis holds for $\zeta(\mathcal{O}, s)$. Since $|q^{-s}| = q^{-Re(s)}$, the latter holds if and only if the zeroes β of $Z(\mathcal{O}, t)$ satisfy $|\beta| = q^{-1/2}$. By Theorem 3.15 this means that the zeroes of $L(\mathcal{O}, t)$ have this property.

First we study the examples in sections 5 and 7.

In the example of section 5 we have $Z(R,R,t)=0$ if and only if $L(R,R,t)=0$ if and only $t^2-\frac{1}{q}t+\frac{1}{q}=0$. This equation easily yields

$$|t_{1,2}|=q^{-1/2}.$$

Moreover, $Z(R,t)=0$ if and only if $L(R,t)=0$ if and only if $t^2+\frac{1}{q}=0$. This yields $t=\pm iq^{-1/2}$. That is the Riemann hypothesis is valid both for $Z(R,R,t)=0$ and $Z(R,t)$.

In the first example of section 7 we have

$$L(\mathcal{O},t)=1+qt^2+q^2t^4.$$

Obviously by taking $X=q\,t^2$, we have

$$X^2+X+1=0 \Longleftrightarrow X=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}.$$

This gives $|X|=1$ and hence $|t|=q^{-1/2}$. Hence $L(\mathcal{O},t)$ satisfies the Riemann hypothesis.
For further examples we need lemmas as follows

Lemma 9.1. *Let X be an unknown and $X_i\in K$, then*

$$(X-X_1)(X-X_2)\cdots(X-X_n)=\sum_{i=0}^{n}(-1)^i\,\sigma_i(X_1,\cdots,X_n)\,X^{n-i}. \tag{82}$$

Here

$$\sigma_i(X_1,\cdots,X_n)\ =\sum_{1\le\mu_1\cdots\le\mu_i\le n} X_{\mu_1}\cdot X_{\mu_2}\cdots X_{\mu_i}\quad,\quad i\in\{0,...,n\}.$$

Proof. See[Fi,Sa] □

Lemma 9.2.

$$(1-\alpha_1\,t)(1-\alpha_2\,t)\cdots(1-\alpha_n\,t)=\sum_{i=0}^{n}(-1)^i\,\sigma_i(\alpha_1,\cdots,\alpha_i)\,t^i \tag{83}$$

Proof.

$$\begin{aligned}&(1-\alpha_1t)(1-\alpha_2t)\cdots(1-\alpha_nt) \\ =&(\alpha_1\cdots\alpha_n)(\alpha_1^{-1}-t)(\alpha_2^{-1}-t)\cdots(\alpha_n^{-1}-t)\\ =&(\alpha_1\cdots\alpha_n)\cdot(-1)^n(t-\alpha_1^{-1})\cdots(t-\alpha_n^{-1})\end{aligned} \tag{84}$$

According to Lemma 9.1

$$(t-\alpha_1^{-1})\cdots(t-\alpha_n^{-1}) = \sum_{i=0}^{n} (-1)^i\ \sigma_i(\alpha_1^{-1},\cdots,\alpha_n^{-1})\ t^{n-i}.$$

Then (84) yields

$$(1-\alpha_1 t)(1-\alpha_2 t)\cdots(1-\alpha_n t) = \sum_{i=0}^{n}(-1)^{n+i}(\alpha_1\cdots\alpha_n)\ \cdot\ \sigma_i(\alpha_1^{-1},\cdots,\alpha_n^{-1})\ t^{n-i}$$

$$= \sum_{i=0}^{n}(-1)^{n+i}\ (\alpha_1\cdots\alpha_n) \sum_{1\le\mu_1\cdots\le\mu_i\le n} (\alpha_{\mu_1}^{-1}\cdots\alpha_{\mu_i}^{-1})\ t^{n-i}$$

$$= \sum_{i=0}^{n}\ (-1)^{n-i} \sum_{1\le\nu_1\le\cdots\le\nu_{n-i}\le n} \alpha_{\nu_1}\cdots\alpha_{\nu_{n-i}}\ t^{n-i}$$

$$= \sum_{i=0}^{n}\ (-1)^i\ \sigma_i(\alpha_1,\cdots,\alpha_i)\ t^i$$

□

Remark 9.3. *If* $L(\mathcal{O},t)=\prod_{i=1}^{m}(1-\alpha_i\ t)$ *satisfies the Riemann hypothesis, then* $|\alpha_i| = q^{1/2}$ *for all* $1\le i\le m$.

Proof. Let $t_i = q^{-s_i}$ with $Re\,(s_i)=1/2$ be a root of $L(\mathcal{O},t)=0$, then $\alpha_i = q^{s_i}$ and it implies $|\alpha_i| = q^{Re\,(s_i)} = q^{1/2}$. □

Lemma 9.4. *If* $L(\mathcal{O},t)=\sum_{i=0}^{m}\ n_i\,t^i$ *satisfies the Riemann hypothesis, then*

$$|n_i| \le \binom{m}{i}\ q^{i/2} \qquad , \qquad 0\le i\le m. \tag{85}$$

Proof. $L(\mathcal{O},t)$ has constant term 1 and hence the form in Remark 9.3. By Lemma 9.2 and Remark 9.3 we get

$$n_i = (-1)^i\ \sigma_i(\alpha_1,\cdots,\alpha_i).$$

and

$$|n_i| \le \sum_{1\le\nu_1\le\cdots\le\nu_{n-i}\le n} |\alpha_{\nu_1}|\cdots|\alpha_{\nu_{n-i}}|$$

$$\le \binom{m}{i} q^{i/2}. \tag{86}$$

□

We use inequality (85) to show that the following example does not satisfy the Riemann hypothesis.
In the second example of section 7 we have

$$L(\mathcal{O},t) = 1 + q\,t + q^2\,t^3 + q^2\,t^4 = \sum_{i=0}^{4} n_i t^i. \tag{87}$$

Clearly $\delta = 2$ and $f = 3$. Note that since $f < 2\delta$, $\mathcal{O}$ is not Gorenstein.

According to Lemma 9.4 for $i = 1$ the Riemann hypothesis would imply

$$q = |n_1| \leq 4\,q^{1/2}.$$

Therefore for $q \geq 17$ the Riemann hypothesis is not valid.
On the positive side we show:

Theorem 9.5. *Let $\mathcal{O}$ be a balanced order. Then the Riemann hypothesis holds for $L(\mathcal{O},t)$.*

Proof. Let $\mathcal{O}$ be balanced of degree n. By Theorem 8.8 we have

$$L(\mathcal{O},t) = 1 + q\,t^2 + q^2\,t^4 + q^3\,t^6 + \cdots + q^n\,t^{2n}. \tag{88}$$

Since for $q\,t^2 \neq 1$

$$L(\mathcal{O},t) = \frac{(q\,t^2)^{n+1} - 1}{q\,t^2 - 1}, \tag{89}$$

we conclude that the roots of $L(\mathcal{O},t) = 0$ are $(n+1)-$th roots of unity that is

$$q\,t^2 = \zeta_{n+1}\ ,$$

equivalently

$$t = q^{-1/2}\,\zeta_{2(n+1)}\ .$$

This implies $|t| = q^{-1/2}$.

□

Theorem 9.6. *Let $\mathcal{O}$ be a balanced order. Then the Riemann hypothesis holds for $L(\mathcal{O}, \mathcal{O}, t)$.*

Proof. Let $\mathcal{O}$ be balanced of degree n, that is $S(\mathcal{O}) = \{0, 2, 4, 6, \cdots, 2n, 2n+1, \cdots\}$. By Lemma 8.7 and Theorem 8.8 we have

$$L(\mathcal{O}, \mathcal{O}, t) = \frac{X^{n+1}-1}{X-1} - t\,\frac{X^n - 1}{X-1}. \tag{90}$$

where $X = qt^2$. It therefore suffices to show:

Lemma 9.7. *Let $X = q\,t^2$ and $n \in \mathbb{N}$, then for the polynomial*

$$X^{n+1} - 1 - t\,(X^n - 1) = q^{n+1}\,t^{2n+2} - q^n\,t^{2n+1} + t - 1,$$

all roots t have $|t| = q^{-1/2}$.

Proof. If $|t| = q^{-1/2}$, then $t = q^{-1/2} \cdot e^{i\alpha}$ with $\alpha \in \mathbb{R}$. It is sufficient to find $2n+2$ different values α, such that $t = q^{-1/2} \cdot e^{i\alpha}$ solves the equation. We have $X = q\,t^2 = e^{i2\alpha}$, so the equation means

$$e^{i(2n+2)\alpha} - 1 - q^{-1/2}(e^{i(2n+1)\alpha} + e^{i\alpha}) = 0.$$

By multiplying with $q^{1/2}e^{-i(n+1)\alpha}$, this yields

$$q^{1/2}(e^{i(n+1)\alpha} - e^{-i(n+1)\alpha}) - (e^{i\,n\alpha} - e^{-i\,n\alpha}) = 0.$$

This yields

$$f(\alpha) = q^{1/2}\mathrm{Sin}(n+1)\alpha - \mathrm{Sin}\;n\alpha = 0.$$

Look at α in the interval $[0, 2\pi]$. For $i \in \{0, \cdots, 2n+1\}$ we have

$$f(\frac{1}{n+1}(\frac{\pi}{2} + j\pi)) = q^{1/2}\;\mathrm{Sin}(\frac{\pi}{2} + j\pi) - \mathrm{Sin}(\frac{n}{n+1}(\frac{\pi}{2} + j\pi)),$$

clearly

$$q^{1/2}\mathrm{Sin}(\frac{\pi}{2} + j\pi) = \begin{cases} q^{1/2} & , \text{ if } \quad j \text{ is even} \\ -q^{1/2} & , \text{ if } \quad j \text{ is odd}\;. \end{cases}$$

Because $q^{1/2} > 1$, we obtain

$$f(\frac{1}{n+1}(\frac{\pi}{2} + j\pi)) = \begin{cases} > 0 & , \text{ if } \quad j \text{ is even} \\ < 0 & , \text{ if } \quad j \text{ is odd}\;. \end{cases}$$

This means that $f(\alpha)$ has $2n+1$ zeros on the interval $[\frac{1}{n+1}\,\frac{\pi}{2}, 2\pi - \frac{1}{n+1}\,\frac{\pi}{2}]$

(Note that $\frac{1}{n+1}(\frac{\pi}{2}+(2n+1)\pi) = 2\pi - \frac{1}{n+1}\ \frac{\pi}{2})$. Since we also have $f(0) = 0$ the Lemma is proved.

Lemma 7.10 implies $L(\mathcal{O}, \mathcal{O}, t)$ satisfies Riemann Hypothesis .

□

Lemma 9.8. *For $\mathcal{O}$ which is not Gorenstein, $L(\mathcal{O}, \mathcal{O}, t)$ does not satisfy The Riemann hypothesis.*

Proof. Since $\mathcal{O}$ is not Gorenstein, deg $L(\mathcal{O}, \mathcal{O}, t) = f < 2\delta$. Then $n_f = q^\delta \neq (q^{1/2})^f$ implies $L(\mathcal{O}, \mathcal{O}, t)$ does not satisfy the Riemann hypothesis. □

The counterpart of theorem 9.6 is:

Theorem 9.9. *If $\mathcal{O}$ is not balanced, then for $q >> 0$, $L(\mathcal{O}, \mathcal{O}, t)$ does not satisfy the Riemann hypothesis.*

Proof. Note that $\mathcal{O}$ is balanced if and only if $2 \in S(\mathcal{O})$ (because then all even natural numbers are contained in $S(\mathcal{O})$). If $\mathcal{O}$ is not balanced, then 1 and 2 are gaps, i.e., $1, 2 \subseteq \mathbb{N}_0 \backslash S(\mathcal{O})$ (if $1 \in S(\mathcal{O})$, then $\mathcal{O} = \widetilde{\mathcal{O}}$, which is balanced). Let $m \in \mathbb{N}$ be such that $1, 2, 3, \cdots, m$ are gaps, and $m+1 \in S(\mathcal{O})$. Then by theorem 6.2 we have

$$L(\mathcal{O}, \mathcal{O}, t) = \sum_{i=0}^{2\delta} n_i t^i$$

with $n_{m+1} = q^m$. By Lemma 9.4, the Riemann hypothesis would imply

$$q^m \leq \binom{f}{m+1} q^{(m+1)/2},$$

which cannot hold for $q >> 0$.

□

Studying all orders with singularity degree $\delta \leq 3$

First we classify all orders with $\boldsymbol{\delta}$ less or equal than 3.

$\boldsymbol{\delta = 1}$

Since S is a semigroup and $S \neq \mathbb{N}$ we have $1 \notin S$. Therefore there is only one possibility, that is

(i) $S = S(2) = \{0, 2, 3, \cdots\}$.

$\boldsymbol{\delta = 2}$

Since $\{2, 3\} \subseteq S$ implies $\delta = 1$, we have $\{1, 2\} \subseteq \mathbb{N}\backslash S$ or $\{1, 3\} \subseteq \mathbb{N}\backslash S$, therefore we have two possibilities as follows;

(i) $S = \{0, 3, 4, \cdots\} = S(3)$,

(ii) $S = \{0, 2, 4, 5 \cdots\} = S^{(2)}$.

$\boldsymbol{\delta = 3}$

In this case, by similar arguments concerning the gaps , we have four possibilities

(i) $S = S(4) = \{0, 4, 5, \cdots\}$,

(ii) $S = \{0, 3, 5, 6, \cdots\}$,

(iii) $S = \{0, 3, 4, 6, \cdots\}$,

(iv) $S = S^{(3)} = \{0, 2, 4, 6, 7, \cdots\}$.

Now we calculate the $L(\mathcal{O}, t)$ in each of the above cases as follows:

In the case $\boldsymbol{\delta = 1}$, we have $S(\mathcal{O}) = S(2) = \{0, 2, 3, \cdots\}$ and hence $\mathcal{O} = \mathcal{O}(2)$ by Lemma 8.10. We have already calculated $L(\mathcal{O}(2), t) = 1 + qt^2$ which satisfies the Riemann hypothesis.

Let $\boldsymbol{\delta = 2}$. In the case (i) $\mathcal{O} = \mathcal{O}(3)$ and we have already calculated $L(\mathcal{O}(3), t) = 1 + qt + q^2t^3 + q^2t^4$, which clearly does not satisfy the Riemann hypothesis for $q >> 0$.

In the case (ii) the order is balanced of degree two and we have already calculated $L(\mathcal{O},t) = 1 + qt^2 + q^2t^4$ which satisfies the Riemann hypothesis.

Let $\boldsymbol{\delta} = \mathbf{3}$ In the case (i) we use the formula

$$L(\mathcal{O},t) = \sum_{i=0}^{\delta-1} N_i^{\delta-1}(q^i t^i + q^\delta t^{2\delta-i}),$$

where N_i^r is the number of i-dimensional subspace of k^r(See[St1]). We have

$$N_0^0 = 1$$

$$N_0^1 = 1 \qquad N_1^1 = 1$$

$$N_0^2 = 1 \qquad N_1^2 = \tfrac{q^2-1}{q-1} = q+1 \qquad N_2^2 = 1.$$

Therefore

$$\begin{aligned} L(\mathcal{O},t) &= 1(1 + q^3t^6) + (q+1)(qt + q^3t^5) + 1(q^2t^2 + q^3t^4) \\ &= 1 + (q+q^2)t + q^2t^2 + q^3t^4 + (q^3+q^4)t^5 + q^3t^6\ . \end{aligned}$$

$n_1 = q + q^2$ shows that $L(\mathcal{O},t)$ does not satisfy the Riemann hypothesis for $q >> 0$.

In the case (ii) we have $\mathcal{O} = k \oplus k\alpha \oplus \mathfrak{p}^5$, where $\upsilon(\alpha) = 3$. $\overline{\alpha} \in \mathfrak{p}^3/\mathfrak{p}^5$ yields $\overline{\alpha} = k_1\overline{\pi}^3 + k_2\overline{\pi}^4$ $k_1 \neq 0$; therefore we may assume $\alpha \equiv \pi^3 + a\pi^4 \mod \mathfrak{p}^5$. Accordingly we may take $\alpha = \pi^3 + a\pi^4$, where $a \in k$.

We now study the $\mathcal{O}$-ideals $\mathfrak{b}$, such that $\mathcal{O} \subseteq \mathfrak{b} \subseteq \widetilde{\mathcal{O}}$.

Let deg $\mathfrak{b} = 1$ then there are two possibilities: $S(\mathfrak{b}) = S(3)$ and hence $\mathcal{O} = \mathcal{O}(3)$, or $S(\mathfrak{b}) = \{0, 2, 3, 5, 6, \cdots\}$. We study the second possibility as follows:
Clearly $\mathfrak{b} = k \oplus k\beta \oplus k\alpha \oplus \mathfrak{p}^5 = k\beta \oplus \mathcal{O} =: \mathfrak{b}_\beta$ where $\upsilon(\beta) = 2$, and $\mathcal{O}_\mathfrak{b} = \mathcal{O}(3)$. We claim that all ideals in this case are equivalent. By a similar argument as above we may take $\beta = \pi^2 + d\pi^4$ where $d \in k$. In this stage we suppose there is $\delta \in \widetilde{\mathcal{O}}^\times$ with $\delta^{-1} \in \mathfrak{b}_\beta$ such that $\delta \cdot \beta \equiv \beta'(\text{mod } \mathfrak{p}^5)$ and prove our

claim. In the next stage we will prove existence of such a δ.

We want to show $\mathfrak{b}_\beta \sim \mathfrak{b}_{\beta'}$ where $\beta' = \pi^2 + d'\pi^4$ for some $d' \in k$.
$\delta^{-1} \in \mathfrak{b}_\beta$ shows that $1 \in \delta \cdot \mathfrak{b}_\beta$ and it implies $\mathcal{O} \subseteq \delta\mathfrak{b}_\beta$; $\delta \cdot \beta = \beta'$ yields $\beta' \in \delta \cdot \mathfrak{b}_\beta$ and consequently $\mathfrak{b}_{\beta'} = \mathcal{O} + k\beta' \subseteq \delta \cdot \mathfrak{b}_\beta$. Since

$$\dim\,(\delta \cdot \mathfrak{b}_\beta)/\mathfrak{b}_{\beta'} = \#\{S(\delta \cdot \mathfrak{b}_\beta)\backslash S(\mathfrak{b}_{\beta'})\} = 0,$$

this implies $\mathfrak{b}_{\beta'} = \delta \cdot \mathfrak{b}_\beta$ or $\mathfrak{b}_\beta \sim \mathfrak{b}_{\beta'}$.

Now in the second stage we prove the existence of δ which we have used in the above argument. We choose $\gamma \in \widetilde{\mathcal{O}}^\times \cap \mathfrak{b}_\beta$; clearly

$$\gamma \equiv 1 + v(\pi^2 + d\pi^4) + w(\pi^3 + a\pi^4)(\text{mod}\ \mathfrak{p}^5)\ .$$

Since

$$\begin{aligned}(1 + v\beta + w\alpha)(1 - v\beta + v^2\beta^2 - w\alpha) &\equiv 1 - v\beta + v^2\beta^2 - w\alpha + v\beta - v^2\beta^2 + w\alpha \\ &\equiv 1(\text{mod}\ \mathfrak{p}^5),\end{aligned}$$

this implies

$$\gamma^{-1}\beta \equiv (1 - v\beta + v^2\beta^2 - w\alpha)\beta \equiv \beta - v\ \beta^2 \equiv \beta - v\ \pi^4(\text{mod}\ \mathfrak{p}^5)\ .$$

For every $\beta' = \pi^2 + d'\pi^4$ we can take $v = d - d'$ and we have $\gamma^{-1}\mathfrak{b}_\beta = \mathfrak{b}_{\beta'}$.
Now we take $\delta = \gamma^{-1}$.
Let deg $\mathfrak{b} = 2$. Then we have $S(\mathfrak{b}) = S(2)$ and hence $\mathfrak{b} = \mathcal{O}(2)$.
Now

$$L(\mathcal{O}, t) = L(\mathcal{O}, \mathcal{O}, t) + L(\mathcal{O}, \mathcal{O}(3), t) + L(\mathcal{O}, \mathfrak{b}, t) + \sum_{\substack{(\mathfrak{b}) \\ \deg \mathfrak{b} \geq 2}} L(\mathcal{O}, \mathfrak{b}, t)\ .$$

We have

$$L(\mathcal{O}, \mathcal{O}, t) = 1 - t + q^2t^3 - q^2t^4 + q^3t^5$$

$$L(\mathcal{O}, \mathcal{O}(3), t) = t(L(\mathcal{O}(3), \mathcal{O}(3), t)) = t(1 + qt + q^2t^3 + q^2t^4)$$

$$L(\mathcal{O}, \mathfrak{b}, t) = q\ t(1 - t + qt^2 - qt^4 + q^2t^5)$$

and by Lemma 8.9

$$\sum_{\substack{(\mathfrak{b}) \\ \deg \mathfrak{b} \geq 2}} L(\mathcal{O}, \mathfrak{b}, t) = t^2 \, L(\mathcal{O}(2), t) = t^2(1 + qt^2).$$

We have finally

$$L(\mathcal{O}, t) = 1 + q \, t + t^2 + 2q^2 \, t^3 + q \, t^4 + q^3 \, t^5 + q^3 \, t^5 + q^3 \, t^6.$$

$n_3 = 2q^2$ implies that $L(\mathcal{O}, t)$ does not satisfy the Riemann hypothesis for $q >> 0$.

In the case (iii), if deg $\mathfrak{b} \geq 1$ then $\mathcal{O} \subseteq \mathcal{O}(3) \subseteq \mathfrak{b}$ therefore we have

$$\begin{aligned} L(\mathcal{O}, t) &= L(\mathcal{O}, \mathcal{O}, t) + t \, L(\mathcal{O}(3), t) \\ &= 1 - t + q^2 t^3 - q^2 t^5 + q^3 t^6 + t(1 + qt + q^2 t^3 + q^2 t^4) \\ &= 1 + qt^2 + q^2 t^3 + q^2 t^4 + q^3 t^6 \end{aligned}$$

The $n_3 = q^2$ shows that $L(\mathcal{O}, t)$ does not satisfy the Riemann hypothesis for $q >> 0$.

In the case (iv), $\mathcal{O}$ is balanced of type n which we have already studied in the general case: We have $L(\mathcal{O}, t) = 1 + qt^2 + q^2 t^4 + q^3 t^6$ which satisfies the Riemann hypothesis.

We conclude this section and this paper with some comments and questions. Let $\mathcal{O}$ be a rational unibranch order over the finite field k .

Question.9.10 Does $L(\mathcal{O}, t)$ satisfy the Riemann hypothesis if and only if $\mathcal{O}$ is balanced?
All examples give evidence for this. By theorem 9.5 the question is to show that for non-balanced $\mathcal{O}$ the Riemann hypothesis never holds.

Question 9.11 Does $L(\mathcal{O}, \mathcal{O}, t)$ satisfy the Riemann hypothesis if and only if $\mathcal{O}$ is balanced?
By Theorems 9.6 and 9.9 the question is to remove the hypothesis $q = \#k >> 0$ in Theorem 9.9. Examples give evidence that this is possible.

Question 9.12 Does $L(\mathcal{O}, t)$ only depend on the semigroup $S(\mathcal{O})$?
By Theorem 6.2 this holds for $L(\mathcal{O}, \mathcal{O}, t)$, and by Theorem 6.6 the analogue also holds for any partial $L-$polynomial $L(\mathcal{O}, \mathfrak{b}, t)$, $\mathfrak{b}$ any ideal, and the $S(\mathcal{O})-$module $S(\mathfrak{b})$. The problem is to show that semigroup of ideal classes $(\mathfrak{b})$, together with the $S(\mathcal{O})-$modules $S(\mathfrak{b})$, only depend on $S(\mathcal{O})$. Again all examples give evidence for this.

Notation

$S(\mathcal{O})$	Page.7
$\boldsymbol{C}(R)$	Page.15
$\boldsymbol{I}(R)$	Page.15
$\boldsymbol{P}(R)$	Page.15
K^*	Set of non zero elements of K,Page.16
$\mid\ \mid$	absolut value , Page.16
Λ	Page.17
$\mathfrak{m}$	Page.17
π	Page.17
$(G_n)_n$	Page.17
$dimX$	Page.22
$A[[t]]$	Page.18
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