State Stability Analysis for the Fermionic Projector in the Continuum

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Nicht die Technik ist das Verhängnis, sondern die Verfilzung mit den gesellschaftlichen Verhältnissen, von denen sie umklammert wird.

Theodor W. Adorno (1903-1969)

Chapter 1

Introduction

It is an old dream of theoretical physics to find a theory that incorporates all physical phenomena in the sense that it provides a framework where all fundamental interactions are unified [ST90]. Today the standard model of particle physics can be seen as the state-of-the-art in this direction, at least if experimental verifiability is taken as a criterion. The standard model includes the electromagnetic, weak and strong interactions. It does not comprise gravity, which does not give notable effects until a very large length scale compared to that of particle physics.

A major disadvantage of this otherwise quite successful model is that it depends on at least 18 parameters: the coupling constant, the mass and vacuum expectation value of the Higgs boson field, the lepton and quark masses and the parameters in the so-called Kobayashi-Maskawa matrix [CG99]. These constants have to be put in by hand and are only obtainable from the experiment. It would be quite more satisfying if a theory that claims to be fundamental could actually predict at least some of these quantities.

Recently, another approach has been proposed [Fin06b]: the principle of the fermionic projector. In contrast to the standard model, it is not based on quantum field theory but on relativistic quantum mechanics, especially on a theory of Dirac seas, and the regularization procedure for high energies is justified by some *ad hoc* notion of discrete spacetime. The general framework is to take a projection operator, which in the continuum limit corresponds to a projector onto Dirac seas, as the basic object. Then set up a variational principle whose minimizers are the physical fermionic projectors. It is argued in [Fin06b] that, with some additional assumptions, a model similar to the standard model could be obtained.

If we forget about the discrete spacetime structure for the moment and use an effective continuum theory instead, this will still have consequences for some parameters: Consider a system of g Dirac seas of masses $m_1, \ldots, m_g$. It is not true that every mass configuration will be stable in the sense that the transition of a particle from one sea to another does not decrease the action. The following questions arise:

1. Do such stable configurations exist?

2. Is there a connection to the fact that elementary particles, e.g. the charged leptons appear

in three generations (electron, muon, tauon), where each of these has its fixed mass?

The first question cannot be answered in generality for an arbitrary number of Dirac seas. However, in this work we will have a look at the situation for $g = 1, 2, 3$. The last case is the most important because it reflects that, like in nature, the elementary particles appear in three generations. This immediately leads us to the second question, namely if the obtained stable mass configurations could give us an explanation why the elementary particles have got the masses they have. But this is beyond the scope of this work. Nevertheless, one can say that there is hope to find such configurations in the future, maybe by a more sophisticated numerics.

The thesis is organized as follows: Chapter 2 introduces the most important notions concerning the principle of the fermionic projector, chapter 3 shows how to treat Lorentz invariant distributions and gives formulae to calculate convolutions between them and chapter 4 explains state stability and how the preceding calculations can be applied in this framework. Chapter 5 gives a detailed exposition of how Lorentz invariant regularizations can be explicitly performed. A great part of the material of chapters 2–5 already appeared as a paper [FH07]. I decided to revise the argumentation again to explain some statements more thoroughly, while I put less emphasis on others. In Chapter 6 the algorithms and numerics are explained in detail. Several plots that show how some of the stable configurations look like will round off the work.

Let me seize the opportunity to express my gratitude to those people without whom this thesis could hardly be accomplished. It is impossible to enumerate them all. Let me first thank my supervisor, Prof. Dr. Felix Finster, for giving me as a physicist the opportunity to write a PhD in mathematics and the patience he had with me. Furthermore, thanks to Andreas Grotz for helpful comments on the text and to all my friends and colleagues, my parents and my sister for giving me encouragement all the time.

Chapter 2

The principle of the fermionic projector

2.1 Relativistic quantum mechanics

A quantum system is mathematically described by a Hilbert space $\mathfrak{H}$. The observables are expressed as self-adjoint linear operators on $\mathfrak{H}$, such that their spectrum is the set of possible measurement results. The usual choice in standard quantum mechanics is the replacement

$$\begin{array}{rcl} \text{classical system} & \longleftrightarrow & \text{quantum system} \\ \vec{x} & \longleftrightarrow & \vec{x}\,\cdot\,, \\ \vec{p} & \longleftrightarrow & -i\hbar\vec{\nabla}\,. \end{array}$$

Due to Planck's law, the energy of a quantum of radiation is $E = \hbar\omega$. For a plane wave $\psi(t,\vec{x}) \propto e^{i\omega t}$, we have

$$E\,\psi(t,\vec{x}) \;=\; \hbar\omega\,\psi(t,\vec{x}) \;=\; i\hbar\frac{\partial}{\partial t}\,\psi(t,\vec{x}),$$

giving the replacement rule

$$E \longleftrightarrow i\hbar\frac{\partial}{\partial t}.$$

If we now impose the nonrelativistic energy conservation condition

$$E \;=\; -\frac{\vec{p}^2}{2m} + V(x),$$

this will translate into quantum language as follows:

$$i\hbar\frac{\partial}{\partial t}\,\psi(t,\vec{x}) \;=\; -\frac{\hbar^2}{2m}\,\Delta\psi(t,\vec{x}) + V(x)\,\psi(t,\vec{x}) \tag{2.1}$$

The equality (2.1) is referred to as the **Schrödinger equation**.

In relativistic quantum mechanics, this construction is more difficult. We have to use the energy-momentum relation[1]

$$E^2 = p^2 + m^2. \tag{2.2}$$

Repeating the same steps as above, we will arrive at the **Klein-Gordon equation**,

$$\left(\frac{\partial^2}{\partial t^2} - \Delta + m^2\right)\psi(t,\vec{x}) = 0. \tag{2.3}$$

Since this equation does not admit a functional in ψ that may be interpreted as a positive definite probability density, this cannot be the suitable description of material particles like electrons. Another possibility is to quantize (2.2) in the form

$$E = \pm\sqrt{p^2 + m^2}, \tag{2.4}$$

yielding the famous **Dirac equation**,

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi(x) = 0, \tag{2.5}$$

where $x \equiv (t,\vec{x})$ and the γ_μ are matrices that fulfill the anticommutation relation

$$\left\{\gamma_\mu, \gamma_\nu\right\} \equiv \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} \cdot 1\!\!1.$$

But both the Klein-Gordon equation and the Dirac equation have a physical meaning: Quantum particles obeying (2.3) are named **bosons** and describe interaction fields, while the solution of Dirac's equation are matter fields called **fermions**.

The objects γ_μ have several representations in terms of 4×4-matrices. In our context, we always use the **Dirac representation**

$$\gamma_0 = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix},\ i = 1,2,3$$

where the σ_i are the **Pauli matrices**

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[1]From now on, we set $\hbar = c = 1$.

The solutions ψ of (2.5) are the **Dirac spinors**. An indefinite inner product can be introduced between them:

$$\langle \phi, \psi \rangle \equiv \int \sum_{\alpha=1}^{4N} s_\alpha \, \overline{\phi_\alpha(x)} \, \psi_\alpha(x) \, d^4x \,, \tag{2.6}$$

where $4N$ is the number of spinor components and

$$s_\alpha = \begin{cases} +1 & \text{if } 1 \le \alpha \le 2N \\ -1 & \text{if } 2N+1 \le \alpha \le 4N. \end{cases} \tag{2.7}$$

The space of all $4N$-component wavefunctions with this indefinite inner product is called $\mathcal{H}$. The motivation for the definition (2.6) is that the Dirac current $j_\mu = \left\langle \psi, \gamma_\mu \psi \right\rangle$ fulfills a continuity equation that can be derived from (2.5).

Naïvely, (2.5) has an instability problem. As (2.4) already indicates, the energy spectrum is not bounded from below. Any state of this system may make a transition to an eigenstate of less energy. Therefore, Dirac proposed that all negative-energy states are already filled. Due to Pauli's principle, every state can only be occupied once, so that states cannot fall to $E = -\infty$. The collection of all these filled states is called the **Dirac sea**.

Dirac's idea was abolished, because it naturally gives rise to an implausible multi-particle theory: The sea consists of infinitely many particles and would carry an infinite amount of mass, even in the vacuum. The modern formalism of QFT overcomes this situation by treating the collection of fermions as a quantum field. Particles and antiparticles appear as excitations of it.

But maybe it is not necessary to give up Dirac's pictorial idea. The Dirac sea turns out to be a good starting point for an alternative description of high energy physics. In order to generalize the discussion, it is useful to launch the projection operator onto the sea, i.e. the occupied states, as the basic object and put it on a spacetime that is discrete in some sense (see section 2.2). This will be called the **fermionic projector**. The main ideas are developed in detail in [Fin06b].

2.2 Discrete spacetime

One of the outstanding problems of modern physics is the interplay between quantum theory and gravity. There are many attempts to unify these two theories, e.g. string theory [Zwi04] and loop quantum gravity [Rov04]. We are not going to enter the discussion of these involved models, but content ourselves with the following naïve consideration.

Suppose we want to resolve physics on a very small length scale. Due to the uncertainty principle,

$$\Delta x \cdot \Delta p \ge \hbar,$$

we have to allow for a wide range of momenta. Light of momentum p has energy $E^2 = p^2c^2$

and therefore mass $m^2 = p^2/c^2$. In the Schwarzschild solution of Einstein's equation, for any point-like mass there is some region that no signal can escape from. This is called a **black hole**. Its radius is called the **Schwarzschild radius**

$$r_S = \frac{2Gm}{c^2},$$

where G is Newton's constant of gravity. That is, if we choose energy large enough for resolving smaller and smaller length scales, the Schwarzschild radius may grow. Thus the quantity l_P where

$$l_P := r_S = \Delta x$$

marks the minimum observable length in this model. It is called the **Planck length** and has the value

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.61624 \times 10^{-35}\text{m}.$$

We are thus led to the consequence that events have a minimal measurable distance from each other. Hence spacetime is seen not as a continuous but rather discrete entity. In the language of relativistic quantum theory we can say that the position/time operators X^i have discrete spectrum.

This motivates the following model. Let $(\mathcal{H}, \langle\cdot|\cdot\rangle)$ be an indefinite inner product space[2] where the corresponding inner product $\langle\cdot|\cdot\rangle$ has signature (p, q).

Consider operators $X^i : \mathcal{H} \to \mathcal{H}$ which represent the observables time and position and have purely discrete spectrum

$$M = \left\{x \in \mathbb{R}^4 : \exists\, u \in \mathcal{H} \text{ with } X^i u = x^i u,\ i = 0, \ldots, 3\right\}.$$

This M is the set of all possible spacetime position measurement outcomes[3] or, in short, spacetime events. It is useful to define the **joint eigenspaces**

$$e_x = \bigcap_{i=0}^{3} \text{Eig}(X^i, x^i), \qquad x \in M.$$

We assume that for every $x \in M$, $\dim e_x = 4N$, where N is the number of particles in the theory and the inner product $\langle\cdot|\cdot\rangle$ has signature $(2N, 2N)$. There is a basis $|x\alpha\rangle$, $x \in M$, $\alpha = 1, \ldots, 4N$

[2]In order to emphasize that this is not a Hilbert space, we write a calligraphic '$\mathcal{H}$' instead of the previously used Gothic '$\mathfrak{H}$'.

[3]Of course in quantum mechanics there is nothing like a "time operator", but we will assume so to build our model.

with

$$X^i \,|\, x\alpha\rangle \;=\; x^i \,|\, x\alpha\rangle\,,$$
$$\langle x\alpha \,|\, y\beta\rangle \;=\; s_\alpha\, \delta_{\alpha\beta}\, \delta_{xy},$$

and s_α as in (2.7). Projectors on the eigenspaces e_x are given by

$$E_x \;=\; \sum_{\alpha=1}^{4N} s_\alpha\, |x\alpha\rangle\, \langle x\alpha|\,.$$

As spectral projectors of X^i they satisfy

$$X^i E_x \;=\; x^i E_x\,.$$

They are selfadjoint with respect to $\langle\cdot|\cdot\rangle$, idempotent, and form a complete orthogonal family,

$$\begin{aligned} E_x^* &= E_x \\ E_x\, E_y &= \delta_{xy}\, E_x \\ \sum_{x\in M} E_x &= 1\,. \end{aligned}$$

With these notions in mind, we may stipulate:

Definition 2.1 The triple $(\mathcal{H}, \langle\cdot|\cdot\rangle\,, (E_x)_{x\in M})$ is called **discrete spacetime**.

2.3 The variational principle

Let $\Psi_1, \ldots, \Psi_n \in \mathcal{H}$ be the wave functions of Dirac particles. Given the subspace

$$\langle\Psi_1, \ldots, \Psi_n\rangle \;=\; \text{span}\,\{\Psi_1, \ldots, \Psi_n\} \subseteq \mathcal{H},$$

we have a full description of the corresponding many-particle quantum state. The projection operator on that subspace,

$$P \;=\; P_{\langle\Psi_1,\ldots,\Psi_n\rangle}, \tag{2.8}$$

is called the **fermionic projector**.

In theoretical physics, one often finds the following procedure: First, some basic object – phase space trajectory, quantum field etc. – is introduced. Then we set up an action principle via some Lagrangian functional. The corresponding Euler-Lagrange equations finally yield the physical behavior of the basic object. In our case this is summarized in (see [Fin06b]):

The Principle of the Fermionic Projector

A physical system is completely described by the fermionic projector in discrete space-time. The physical equations should be formulated exclusively with the fermionic projector in discrete space-time, i.e. they must be stated in terms of the operators P and $\left(E_p\right)_{p\in M}$ on $\mathcal{H}$.

The discussion how this action functional can be motivated is found in [Fin06b]. We will take it for granted and just repeat the construction. The **discrete kernel** of the fermionic projector is the expression

$$P(x,y) \;=\; E_x \, P \, E_y \tag{2.9}$$

The action is given in the form

$$\mathcal{S} \;=\; \sum_{x,y\in M} \mathcal{L}\left[A_{xy}\right] \,, \tag{2.10}$$

where $\mathcal{L}$ is some Lagrangian that depends on

$$A_{xy} \;=\; P(x,y)\, P(y,x)\,. \tag{2.11}$$

For a suitable Lagrangian, we need to introduce the following notion:

Definition 2.2 The **spectral weight** of a $K \times K$-Matrix A is the sum

$$|A| \;=\; \sum_{k=1}^{K} n_k \; |\lambda_k| \tag{2.12}$$

of its eigenvalues λ_k, counted with their multiplicities n_k.

We define the Lagrangian

$$\mathcal{L}\,[A] \;=\; \left|A^2\right| - \mu\, |A|^2 \,, \tag{2.13}$$

where $\mu \in \mathbb{R}$ may be seen as a Lagrange multiplier.

It turns out [Fin06b] that the first variation is written in the form

$$\delta\mathcal{S} \;=\; 2\mathrm{Tr}\;(Q\,\delta P) \tag{2.14}$$

with some matrix factor Q, and the Euler-Lagrange equations can be computed to be

$$[P, Q] = 0\,, \tag{2.15}$$

where $[\cdot, \cdot]$ is the commutator.

2.4 Continuum theory

2.4.1 Continuum version of the variational principle

Since today physics at the Planck scale is not experimentally accessible there is some kind of arbitrariness in the discrete space theory. But of course we have to demand that, from a coarse-grained point of view, structures of familiar high energy physics should emerge. We introduced the fermionic projector as a projection operator on occupied states. In the vacuum, this is nothing else than the projector on the Dirac sea with the kernel

$$P(\xi) = \int \frac{d^4k}{(2\pi)^4}\, \hat{P}(k)\, e^{ik\xi}, \tag{2.16}$$

where

$$\hat{P}(k) = \sum_{\alpha=1}^{g} \rho_\alpha\, (\not{k} + m_\alpha)\, \delta(k^2 - m_\alpha^2)\, \Theta(-k^0) \tag{2.17}$$

and $\xi \equiv y - x$. Let us introduce the notation

$$\begin{aligned} I^\vee &= \left\{\xi \in \mathbb{R}^4 : \xi^2 > 0 \text{ and } \xi^0 > 0\right\} \\ I^\wedge &= \left\{\xi \in \mathbb{R}^4 : \xi^2 > 0 \text{ and } \xi^0 < 0\right\} \\ L &= \left\{\xi \in \mathbb{R}^4 : \xi^2 = 0\right\} \\ S &= \left\{\xi \in \mathbb{R}^4 : \xi^2 < 0\right\}. \end{aligned}$$

For $\xi \in I^\vee$, we may form the closed chain A similar to (2.11) and its trace-free part A_0:

$$\begin{aligned} A(\xi) &= P(\xi)\, P(\xi)^*, \\ A_0 &= A - \frac{1}{4}\mathrm{Tr}(A). \end{aligned}$$

where we assumed[4] that $N = 1$ and $P(\xi)^* \equiv \gamma^0 P(\xi)^\dagger \gamma^0$ is the adjoint of P with respect to the spin scalar product (2.6).

[4] N is *not* the number of generations but the number of particle families *in* one generation!

As A_0 is trace-free, it has only got a vector part. Because of Lorentz invariance, it can be written as

$$A_0 \;=\; \frac{\not{\xi}}{2}\, f(\xi^2) \qquad \text{for } \xi \in I^\vee . \tag{2.18}$$

The relation $A_0^* = A_0$ requires f to be real. The Lagrangian

$$\mathcal{L} \;\equiv\; \mathrm{Tr}(A_0^2) \;=\; \xi^2\, f(\xi^2)^2 \tag{2.19}$$

is therefore non-negative and depends only on $z = \xi^2 > 0$. Thus the formal integral

$$\mathcal{S} \overset{\text{formally}}{=} \int_0^\infty \mathcal{L}(z)\, z\, dz \;=\; \int_0^\infty \mathrm{Tr}\,(A_0^2)\, z\, dz \tag{2.20}$$

gives a positive functional depending on the masses $m_1, \ldots, m_g$ and weight factors $\rho_1, \ldots, \rho_g$ as free parameters.

Now it remains to give the formal definition of $\mathcal{S}$ a precise mathematical meaning. In section 3 of [Fin06a] it is shown that P is singular on the light cone and there are v, $h \in C^\infty(\mathbb{R}^+)$ such that

$$P(\xi) \;=\; \not{\xi}\, v(\xi^2) + h(\xi^2) \qquad \text{for } \xi \in I^\vee . \tag{2.21}$$

Hence,

$$f(z) \;=\; \mathrm{Re}\Big(v(z)\,\overline{h(z)}\Big) \in C^\infty(\mathbb{R}^+) \tag{2.22}$$

and thus the integrand of (2.20) is smooth. Moreover, since v and h can be explicitly stated in terms of Bessel functions [Fin06a], one can derive the following facts:

1. For large z, the function f decays like $\mathcal{O}(z^{-2})$ This makes the integral in (2.20) absolutely convergent at infinity.

2. At $z = 0$, the Taylor expansion of f is

$$f(z) \;=\; \frac{\mathfrak{m}_3}{z^2} + \frac{\mathfrak{m}_5}{z} + \mathcal{O}(\log z) \tag{2.23}$$

with

$$\mathfrak{m}_3 \;=\; -\frac{1}{64\pi^5} \sum_{\alpha,\beta=1}^{g} \rho_\alpha\, \rho_\beta\, \Big(m_\alpha^3 + m_\beta^3\Big) \tag{2.24}$$

$$\mathfrak{m}_5 \;=\; \frac{1}{512\pi^5} \sum_{\alpha,\beta=1}^{g} \rho_\alpha\, \rho_\beta\, (m_\alpha - m_\beta)^2 (m_\alpha + m_\beta)^3 . \tag{2.25}$$

The integrand of (2.20) has got a non-integrable pole at $z = 0$,

$$\mathrm{Tr}(A_0^2)\, z \;=\; f(z)^2\, z^2 \;=\; \frac{\mathfrak{m}_3}{z^2} + \frac{2\mathfrak{m}_3\mathfrak{m}_5}{z} + O(\log z). \tag{2.26}$$

Thus the integral can be defined only after some kind of regularization. In our case, we subtract counter terms, i.e. indefinite integrals of the pole terms evaluated at $z = \varepsilon$, and take the limit $\varepsilon \searrow 0$,

$$\lim_{\varepsilon\searrow 0}\left(\int_\varepsilon^\infty \mathrm{Tr}(A_0^2)\, z\, dz - \frac{\mathfrak{m}_3}{\varepsilon} + 2\mathfrak{m}_3\mathfrak{m}_5 \log\varepsilon\right) \tag{2.27}$$

These indefinite integrals are defined up to additive constants of the form $C_1\mathfrak{m}_3$, $C_2\mathfrak{m}_3\mathfrak{m}_5$, or more general by adding a function $F(\mathfrak{m}_3, \mathfrak{m}_5)$. Altogether, we have

Definition 2.3 **(Lorentz invariant action)** For any given function $F \in C^\infty(\mathbb{R}\times\mathbb{R},\mathbb{R})$, we define the action $\mathcal{S} = \mathcal{S}(m_1,\ldots,m_g,\rho_1,\ldots,\rho_g)$ by

$$\mathcal{S} \;=\; \lim_{\varepsilon\searrow 0}\left(\int_\varepsilon^\infty \mathrm{Tr}(A_0^2)\, z\, dz - \frac{\mathfrak{m}_3}{\varepsilon} + 2\mathfrak{m}_3\mathfrak{m}_5 \log\varepsilon\right) + F(\mathfrak{m}_3, \mathfrak{m}_5). \tag{2.28}$$

Here A_0 is defined by (2.18) for any $\xi \in I^\vee$ and $z = \xi^2$. The parameters $\mathfrak{m}_3$ and $\mathfrak{m}_5$ are given by (2.24, 2.25).

Remark 2.4 The expression (2.28) is not necessarily positive, but bounded from below for fixed $\varepsilon > 0$. The action can be extended by certain additional summands.

Definition 2.5 **(Lorentz invariant variational principle)** We minimize the action $\mathcal{S}$, (2.20), varying the parameters $\rho_1,\ldots,\rho_g \geq 0$ and $m_1,\ldots,m_g \geq 0$ under the constraint

$$\mathcal{T} \;\equiv\; \sum_{\beta=1}^{g} \rho_\beta\, m_\beta^3 \;=\; 1. \tag{2.29}$$

Remark 2.6 The constraint (2.29) is introduced to avoid the uninteresting minimizer $\rho_1 = \ldots = \rho_g = m_1 = \ldots = m_g = 0$. The proof of Theorem 4.4 will motivate the special form of $\mathcal{T}$.

2.4.2 Connection to the discrete case

In this subsection we will discuss how the action principle (2.28, 2.29) is related to that of discrete spacetime (2.13).

Proposition 2.7 *$A_0(\xi)$ is reflection-symmetric, causal and Lorentz invariant. More precisely,* *it can be written as*

$$A_0(\xi) \;=\; \frac{\xi\!\!/}{2}\, f(\xi^2)\,\Theta(\xi^2)\,\epsilon(\xi^0)\,. \tag{2.30}$$

Proof. From the definition of P we obtain the rule

$$\forall \xi \in \mathbb{R}^4: \quad P(-\xi) \;=\; P(\xi)^*. \tag{2.31}$$

If we look at the decomposition of P into trace and trace-free part and take into account Lorentz invariance we may write $P(\pm\xi) = \alpha_\pm \xi\!\!/ + \beta_\pm$, from which we see that $[P(\xi), P(-\xi)] = 0$. This implies[5]

$$A(-\xi) \;=\; A(\xi) \quad \text{and} \quad A_0(-\xi) \;=\; A_0(\xi)\,. \tag{2.32}$$

Since for space-like ξ the component ξ_0 can always be Lorentz-transformed to zero, we may write there $A_0(\xi) = \xi\!\!/\, g(\xi^2)$, where $g \in C^\infty(]-\infty, 0[)$ – without an explicit dependence on the sign of ξ^0. But this gives $A_0(-\xi) = A_0(\xi) = -A_0(-\xi)$ and thus $A_0(\xi) = 0$ for $\xi \in S$.
If $\xi^2 > 0$, there are functions $f^\vee$, $f^\wedge$ such that

$$A_0(\xi) \;=\; \xi\!\!/ \cdot \begin{cases} f^\vee(\xi^2) & \text{for } \xi \in I^\vee \\ f^\wedge(\xi^2) & \text{for } \xi \in I^\wedge. \end{cases}$$

But then (2.32) implies

$$f^\vee(\xi^2) \;=\; -f^\wedge(\xi^2)\,,$$

which explains the structure of (2.30). □

Moreover, for timelike ξ the roots $\lambda_1, \ldots, \lambda_4$ of the characteristic polynomial of A (counted with multiplicities) are computed to be real. According to [Fin06a, Lemma 2.1], these roots all have the same sign. Combining these facts, we can write the the Lagrangian (2.19) in the alternative form

$$\mathcal{L} \;=\; \mathrm{Tr}(A^2) \;-\; \frac{1}{4}\,\mathrm{Tr}(A)^2 \;=\; |A^2| - \frac{1}{4}\,|A|^2 \qquad \text{if } \xi^2 \neq 0,$$

where $|\,.\,|$ is again the spectral weight. That is the Lagrangian (2.13) with $\mu = 1/4$ (the so-called critical case of the variational principle). But there are three main differences:

1. We have a regularization on the light cone that is Lorentz invariant.

2. The additional constraint (2.29) appears. In a certain sense it corresponds to the fact that

[5] Note that we have restricted the notion of Lorentz invariance to orthochronous Lorentz transforms.

the number of particles in discrete spacetime is fixed.

3. Instead of summing over spacetime , we have the correspondence

$$\sum_{x,y\in M} \cdots \longrightarrow \int_0^\infty \cdots \, z\, dz \tag{2.33}$$

This can at best be heuristically justified: With $\xi = y - x$,

$$\begin{aligned} \sum_{x,y\in M} \cdots \quad &\longrightarrow \quad \int_M \cdots d^4x \int_M \cdots d^4y \\ &\longrightarrow \quad \int_M \cdots d^4x \int_M \cdots d^4\xi \end{aligned} \tag{2.34}$$

The integrand, which is built up from position space kernels of the fermionic projector, depends on ξ only, so $\int_M \cdots d^4x$ gives an infinite constant, which is simply dropped. The final replacement

$$\int_M \cdots d^4\xi \longrightarrow \int_0^\infty \cdots \, z\, dz$$

is not just a continuum limit of the left hand side.[6] The only obvious connection between the the integration measures $z\,dz$ and $d^4\xi$ is the dimension. This means that in spite of the analogies, the variational principles (2.13) and (2.20, 2.28) are indeed different ones.

2.4.3 Euler-Lagrange equations

For the derivation of the Euler-Lagrange equations, we have to compute the variation of the Lagrangian

$$\delta\, \mathrm{Tr}\,(A_0^2) \;=\; 2\, \mathrm{Tr}\,(A_0\, \delta A_0) \;=\; 2\, \mathrm{Tr}\,(A_0\, \delta A)\,. \tag{2.35}$$

The last equality holds because A_0 has only got a vector component, so the scalar part of δA does not contribute to the trace. If we plug in the definition of A and use (2.32), we obtain

$$\delta\, \mathrm{Tr}\,(A_0^2) \;=\; \mathrm{Tr}\Big(A_0\, (\delta P(\xi)\, P(\xi)^* + P(\xi)\, \delta P(\xi)^*)\Big) \;=\; 2\, \mathrm{Re}\, \mathrm{Tr}(A_0\, P(\xi)\, \delta P(-\xi))\,. \tag{2.36}$$

[6]Note that Lorentz invariant integrands give necessarily $\int_M \cdots d^4\xi = \infty$ because one has to integrate over the hyperbolas $\xi^2 = \text{const.}$, whereas the right hand side may be bounded.

The full Euler-Lagrange equations are therefore

$$\lim_{\varepsilon \searrow 0} \left(\int_\varepsilon^\infty \mathrm{Re}\, \mathrm{Tr}\, (A_0\, P(\xi)\, \delta P(-\xi))\, z\, dz - \frac{\mathfrak{m}_3\, \delta\mathfrak{m}_3}{\varepsilon} + (\delta\mathfrak{m}_3\, \mathfrak{m}_5 + \mathfrak{m}_3\, \delta\mathfrak{m}_5) \log \varepsilon \right) + \delta F(\mathfrak{m}_3, \mathfrak{m}_5) - \lambda\, \delta\mathcal{T} \;=\; 0\,. \qquad (2.37)$$

Note that we have incorporated the constraint (2.29) with the help of the Lagrange multiplier λ. Since it is not obvious how to draw conclusions from this equation, we will use another method to understand the variational principle: the transformation from position to momentum space.

Chapter 3

Lorentz invariant distributions

3.1 Basic definitions

Definition 3.1 An **orthochronous Lorentz transform** is a map $\Lambda : \mathbb{R}^4 \to \mathbb{R}^4$ for which $\Lambda x \cdot \Lambda y = x \cdot y$ and $\operatorname{sgn} x^0 = \operatorname{sgn} (\Lambda x)^0$. The dot denotes the scalar product of Minkowski space. A distribution $F \in \mathcal{S}'(\hat{M})$ is called **Lorentz invariant** iff the equality[1]

$$F(k) \; = \; F(\Lambda k)$$

holds for every orthochronous Lorentz transform Λ. In other words, F only depends on $k^2 \equiv k \cdot k$ and the sign of k^0. Furthermore, F is said to be **negative (positive)** iff $\operatorname{supp} F \subseteq C^\wedge \;\; (C^\vee)$. It may then be written as

$$F(k) \; = \; f(k^2)\, \Theta(k^2)\, \Theta(\mp k^0) \tag{3.1}$$

for some $f \in L^2(\mathbb{R}^+, \mathbb{C})$.

Definition 3.2 Let F, G be distributions. The **convolution** of F and G is given by

$$(F * G)(q) \overset{\text{formally}}{\equiv} \int \frac{d^4k}{(2\pi)^4}\, F(k)\, G(q-k). \tag{3.2}$$

This will only work under certain additional assumptions or regularization procedures. We shall return to that point in section 3.3.

Convolutions naturally arise in Fourier theory, because the Fourier transform of a product is the convolution of Fourier transforms,

$$\widehat{F \cdot G} \; = \; \hat{F} * \hat{G}. \tag{3.3}$$

[1]From now on we are freely using the notation of distributions as generalized functions. This simplifies the pictorial understanding.

3.2 A Plancherel formula

The discussion up to now faces us with two problems: First, the expression of A_0 in position space involves Bessel functions, which make A_0 highly oscillatory for large ξ^2. This is of course a disadvantage for the numerical treatment. Second, there is no vivid image what the Euler-Lagrange equation (2.37) could mean. For this sake, we would like to transform our action principle to momentum space, i.e. the space of wave vectors. We will use the notations

$$\begin{array}{ll} M & \text{for position space,} \\ \hat{M} & \text{for momentum space.} \end{array}$$

Let $f, g : \mathbb{R}^+ \to \mathbb{C}$ measurable and complex-valued. Define

$$F(\xi) \;=\; f(\xi^2)\,\Theta(\xi^2)\,\epsilon(\xi^0), \qquad G(\xi) \;=\; g(\xi^2)\,\Theta(\xi^2)\,\epsilon(\xi^0). \tag{3.4}$$

We introduce the inner product

$$\langle F, G\rangle \equiv \int_0^\infty \overline{f(z)}\, g(z)\, z\, dz \tag{3.5}$$

and $L^2(M, z\,dz)$ is the space of functions where the integral on the right hand side converges absolutely. Now we pass over to momentum space. We invent the following notations for some important subsets of $\hat{M}$,

$$\begin{aligned} C^\vee &= \left\{k \in \hat{M} : \; k^2 > 0,\; k^0 > 0\right\} \\ C^\wedge &= \left\{k \in \hat{M} : \; k^2 > 0,\; k^0 < 0\right\} \\ C &= \left\{k \in \hat{M} : \; k^2 > 0\right\} \;=\; C^\vee \cup C^\wedge \,. \end{aligned} \tag{3.6}$$

Definition 3.3 The **Fourier transform** $\hat{f}$ of a function f is defined as

$$\hat{f}(k) \;=\; \int d^4\xi\, f(\xi)\, e^{-ik\xi}\,,$$

whereas the **inverse Fourier transform** is given by

$$f(\xi) \;=\; \int \frac{d^4k}{(2\pi)^4}\, \hat{f}(k)\, e^{ik\xi}\,.$$

The support of the Fourier transform of F has a similar shape to $\operatorname{supp} F$:

Proposition 3.4 *If F is a function as in (3.4), then $\hat{F}$ will have the form*

$$\hat{F}(k) \;=\; f(k^2)\,\Theta(k^2)\,\epsilon(k^0). \tag{3.7}$$

Proof. First, we have the symmetry

$$\begin{aligned}\hat{F}(-k) &= \int d^4\xi\; F(\xi)\, e^{ik\xi} \\ &= -\int d^4\xi\; F(-\xi)\, e^{ik\xi} \\ &= -\int d^4\xi\; F(\xi)\, e^{-ik\xi} \\ &= -\hat{F}(k),\end{aligned}$$

where we have used that $F(\xi) = -F(-\xi)$. Second, let $k \in \hat{M} \setminus C$. By Lorentz invariance, we may assume $k^0 = 0$. Then

$$\begin{aligned}\hat{F}(k) &= \int d^4\xi\; f(\xi^2)\,\Theta(\xi^2)\,\epsilon(\xi^0)\, e^{ik\xi} \\ &= \int_{-\infty}^{+\infty} dt\; \epsilon(t) \int d^3\xi\; f(\xi^2)\,\Theta(\xi^2)\, e^{i\vec{k}\cdot\vec{\xi}} \\ &= 0\end{aligned}$$

Therefore, $\operatorname{supp}\hat{F} \subset \overline{C}$. □

Similar to (3.5), we may introduce the inner product ($a = k^2$),

$$\left\langle \hat{F}, \hat{G} \right\rangle \equiv \frac{1}{(2\pi)^4} \int_0^\infty \overline{\hat{f}(a)}\, \hat{g}(a)\, a\, da, \tag{3.8}$$

and the corresponding L^2 space is denoted by $L^2(\hat{M}, a\,da)$.

An important relation between (3.5) and (3.8) is constituted by the following

Theorem 3.5 (Lorentz invariant Plancherel formula, scalar case) *For functions of the form (3.4), the Fourier transform is a unitary mapping from $L^2(M, z\,dz)$ to $L^2(\hat{M}, a\,da)$. In particular, for all $F, G \in L^2(M, z\,dz)$,*

$$\int_0^\infty \overline{f(z)}\, g(z)\, z\, dz \;=\; \frac{1}{(2\pi)^4} \int_0^\infty \overline{\hat{f}(a)}\, \hat{g}(a)\, a\, da. \tag{3.9}$$

Proof. The Fourier transform of $F(\xi)$ can be written as

$$\begin{aligned}\hat{F}(k) &= \int_0^\infty f(z)\,dz \int d^4\xi\, \delta(\xi^2 - z)\, \epsilon(\xi^0)\, e^{-ik\xi} \\ &= \hat{f}(k^2)\, \Theta(k^2)\, \epsilon(k^0)\end{aligned}$$

with

$$\hat{f}(a) = 2i\pi^2 \int_0^\infty f(z)\, a\, \frac{J_1(\sqrt{a\,z})}{\sqrt{a\,z}}\, dz.$$

where J_1 is the first-order Bessel function of the first kind. The final result is obtained by using the Parseval equation for the Hankel transform, which is proven in [Zem69]. □

There is also a proof that avoids special functions:

Alternative Proof of Theorem 3.5. By an approximation argument, it is sufficient to prove the theorem just for the case that F and G are such that the functions f and g belong to $C_0^\infty(\mathbb{R}^+)$. The wave operator on such functions is given by

$$-\Box F(\xi) = (Wf)(\xi^2)\, \Theta(\xi^2)\epsilon(\xi^0)$$

with

$$(Wf)(z) = -\frac{4}{z}\,\frac{d}{dz}\left(z^2\, \frac{d}{dz}\, f(z)\right), \qquad (3.10)$$

as can be easily verified by a straightforward calculation. By partial integration we can see that W is symmetric,

$$\begin{aligned}\int_0^\infty \overline{f(z)}\,(Wg)(z)\,z\,dz &= -4\int_0^\infty \overline{f(z)}\left(\frac{d}{dz}\,z^2\,\frac{d}{dz}g(z)\right)dz \\ &= 4\int_0^\infty \left(\frac{d}{dz}\,\overline{f(z)}\right)\left(z^2\,\frac{d}{dz}\,g(z)\right)dz \qquad \text{(boundary terms vanish)} \\ &= -4\int_0^\infty \left(\frac{d}{dz}\,z^2\,\frac{d}{dz}\overline{f(z)}\right) g(z)\,dz \\ &= \int_0^\infty \overline{(Wf)(z)}\, g(z)\, z\, dz\,,\end{aligned}$$

and non-negative,

$$\int_0^\infty \overline{f(z)}\,(Wf)(z)\,z\,dz \;=\; 4\int_0^\infty \overline{z\,\frac{d}{dz}\,f(z)}\left(z\,\frac{d}{dz}\,f(z)\right)dz$$
$$\;=\; \int_0^\infty \left|z\,\frac{d}{dz}\,f(z)\right|^2 dz \;\geq\; 0\,.$$

A self-adjoint extension of W an be constructed as follows. Note that

$$Wu = -\partial_z^2\tilde{u} \qquad \text{with} \qquad \tilde{u} \;\equiv\; z\,u(z) \tag{3.11}$$

and $-\partial_z^2$ is self-adjoint on $L^2(\mathbb{R}^+,\, dz)$ with domain $\mathbb{D}(-\partial_z^2) = H_0^{2,2}$. Set

$$\mathbb{D}(W) \;=\; \left\{u \text{ measurable with } z\,u(z) \in \mathbb{D}(-\partial_z^2)\right\}\,.$$

Let $v,\, w \in L^2(\mathbb{R}^+,\, z\,dz)$ and suppose that

$$\langle Wu,\, v\rangle_{L^2(\mathbb{R}^+,\,z\,dz)} \;=\; \langle u,\, w\rangle_{L^2(\mathbb{R}^+,\,z\,dz)} \qquad \forall\, u \in \mathbb{D}(W)\,.$$

The last equality can be rewritten as

$$\left\langle -\partial_z^2\tilde{u},\, \tilde{v}\right\rangle_{L^2(\mathbb{R}^+,\,dz)} \;=\; \langle \tilde{u},\, w\rangle_{L^2(\mathbb{R}^+,\,dz)} \qquad \forall\, \tilde{u} \in \mathbb{D}(-\partial_z^2)\,.$$

But self-adjointness of $-\partial_z^2$ implies $\tilde{v} \in \mathbb{D}(-\partial_z^2)$ and $-\partial_z^2\tilde{v} = w$. In other words, $v \in \mathbb{D}(W)$ and $Wv = w$. Hence, W with domain $\mathbb{D}(W)$ is self-adjoint.

It is easier to work in momentum space, since then the operator $-\Box$ and therefore W turn into the multiplication operator $\hat{f}(k) \mapsto k^2\,\hat{f}(k)$, which implies $\sigma(W) = \mathbb{R}^+ \cup \{0\}$ and that the spectral measure dE_a is absolutely continuous with respect to the Lebesgue measure da. From the spectral theorem we infer

$$\langle f,\, g\rangle_{L^2(\mathbb{R}^+,\,z\,dz)} \;=\; \int_{\sigma(W)} \langle f,\, dE_a\, g\rangle_{L^2(\mathbb{R}^+,\,z\,dz)}\,. \tag{3.12}$$

The functional calculus can be expressed by[2]

$$\left(\widehat{h(W)\,f}\right)(b) \;=\; h(b)\,\hat{f}(b)\,,$$

[2]Note that is a slight misuse of notation, since b is the *square* of a momentum variable.

and thus the spectral measure satisfies

$$\Big(\widehat{dE_a f}\Big)(b) \;=\; \delta(b-a)\,\hat{f}(b)\,da\,. \tag{3.13}$$

Hence the integrand in (3.12) can be written as

$$\langle f,\, dE_a\, g\rangle_{L^2(\mathbb{R}^+,\, z\,dz)} \;=\; \overline{\hat{f}(a)}\,\hat{g}(a)\,\rho(a)\,da \tag{3.14}$$

with a non-negative measurable function ρ, which can be determined by the following scaling argument. The left hand side of (3.14) is computed with the help of Fourier transformation and equation (3.13) to be

$$\begin{aligned}\overline{\hat{f}(a)}\,\hat{g}(a)\,\rho(a) \;&=\; \int_0^\infty z\,dz \int \frac{d^4k}{(2\pi)^4}\, e^{-ik^0\sqrt{z}}\,\overline{\hat{f}(k^2)}\,\Theta(k^2)\,\epsilon(k^0) \\ &\qquad \times \int \frac{d^4l}{(2\pi)^4}\, e^{-il^0\sqrt{z}}\,\hat{g}(l^2)\,\delta(l^2-a)\,\epsilon(l^0)\,.\end{aligned}$$

If we scale a by a factor λ^2 and transform the integration variables, $k \to \lambda k$, $l \to \lambda l$, and $z \to z/\lambda^2$, we obtain

$$\overline{\hat{f}(\lambda^2\,a)}\,\hat{g}(\lambda^2\,a)\,\rho(\lambda^2\,a) \;=\; \overline{\hat{f}(\lambda^2\,a)}\,\hat{g}(\lambda^2\,a)\,\lambda^2\,\rho(a)\,.$$

Hence $\rho(a) = c\,a$ with a constant $c > 0$. This is used in (3.14), which, in turn, is plugged into equation (3.12) to give

$$\int_0^\infty \overline{f(z)}\,g(z)\,z\,dz \;=\; c\int_0^\infty \overline{\hat{f}(a)}\,\hat{g}(a)\,a\,da\,.$$

To obtain the final result, we use the symmetry between position and momentum space as well as the fact that the Fourier transform and its inverse differ by a factor $(2\pi)^4$ (cf. Def. 3.3). □

Now let us turn to the vector case. Let

$$F(\xi) \;=\; \frac{\not\xi}{2}\,f(\xi^2)\,\Theta(\xi^2)\,\epsilon(\xi^0) \tag{3.15}$$

$$G(\xi) \;=\; \frac{\not\xi}{2}\,g(\xi^2)\,\Theta(\xi^2)\,\epsilon(\xi^0) \tag{3.16}$$

The inner product is defined by contracting the factors $\not\xi$ to z,

$$\langle F, G\rangle \equiv \int_0^\infty z\,\overline{f(z)}\,g(z)\,z\,dz. \tag{3.17}$$

The Fourier transform of F can be written as

$$\hat{F}(k) \;=\; \frac{i\not{k}}{2}\,\hat{f}(k^2)\,\Theta(k^2)\,\epsilon(k^0), \tag{3.18}$$

and the inner product in momentum space is given by

$$\left\langle \hat{F},\hat{G}\right\rangle \equiv \int_0^\infty a\,\overline{\hat{f}(a)}\,\hat{g}(a)\,a\,da. \tag{3.19}$$

The corresponding Hilbert spaces are also denoted by $L^2(M, z\,dz)$ and $L^2(\hat{M}, a\,da)$.

Corollary 3.6 (Lorentz invariant Plancherel formula, vector case) *For functions of the form (3.15, 3.16), the Fourier transform is a unitary mapping from $L^2(M, z\,dz)$ to $L^2(\hat{M}, a\,da)$. In particular, for all $F, G \in L^2(M, z\,dz)$,*

$$\int_0^\infty z\,\overline{f(z)}\,g(z)\,z\,dz \;=\; \int_0^\infty a\,\overline{\hat{f}(a)}\,\hat{g}(a)\,a\,da. \tag{3.20}$$

Proof. Define $F_s(\xi)$ such that $F(\xi) = \not{\xi}\,F_s(\xi)/2$. This translates to momentum space in the form

$$\hat{F}(k) \;=\; \frac{i\not{\partial}_k}{2}\,\hat{F}_s(k)\,.$$

and yields

$$\hat{f}(a) \;=\; 2\hat{f}_s'(a).$$

Furthermore, we have $\widehat{\xi^2 F_s(\xi)} = -\Box_k \hat{F}_s(k)$ or

$$\widehat{z\,f_s(z)} \;=\; -\hat{W}\hat{f}_s(a),$$

where $\hat{W}$ is the wave operator in momentum space. In a Lorentz invariant manner it is expressed as

$$\left(\hat{W}\hat{f}_s\right)(a) \;=\; \frac{4}{a}\,\frac{d}{da}\left(a^2\,\frac{d}{da}\,\hat{f}_s(a)\right). \tag{3.21}$$

We use Theorem 3.5 and (3.21) to get

$$\begin{aligned}
\int_0^\infty z\,\overline{f(z)}\,g(z)\,z\,dz &= \int_0^\infty \overline{z\,f_s(z)}\,g_s(z)\,z\,dz \\
&= \frac{1}{(2\pi)^4}\int_0^\infty \overline{\widehat{z\,f_s}(a)}\,\hat{g}_s(a)\,a\,da \\
&= \frac{1}{(2\pi)^4}\int_0^\infty \left(\hat{W}\overline{\hat{f}_s(a)}\right)\hat{g}_s(a)\,a\,da \\
&= \frac{1}{(2\pi)^4}\int_0^\infty 4\overline{\hat{f}_s'(a)}\,\hat{g}_s'(a)\,a^2\,da \\
&= \frac{1}{(2\pi)^4}\int_0^\infty a\,\overline{\hat{f}(a)}\,\hat{g}(a)\,a\,da.
\end{aligned}$$

□

3.3 Convolutions

Now we will have a closer look on convolutions of Lorentz invariant distributions. The aim is to obtain formulae that allow us to analyze our variational principle, which –in momentum space– consists of such convolutions. We have to distinguish two general cases: If both F and G are negative (or both positive) then the integration domain of $F * G$ is compact. This will in general not be the case if F is negative and G positive or vice versa. There the convolution integral only exists after a suitable regularization (see Fig. 3.1).

Notation. We will always write $\hat{F}(k)$ in the explicit form $f(a)\,\Theta(a)\,\epsilon(k^0)$ with the abbreviation $a \equiv k^2$ and without the hat, i.e. $f(a) = \hat{f}(k^2)$. Furthermore, f, $g, \ldots$ are from now on considered

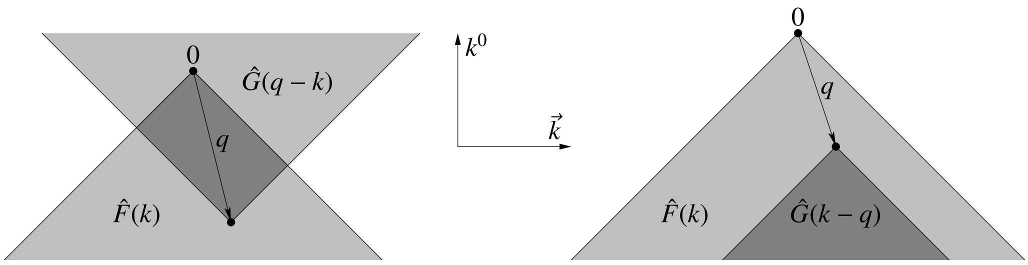


Figure 3.1: Regions of integration in the convolution formula (3.2) if F and G are both negative (left) and in the case that F is negative while G is positive (right)

to be real. Therefore it is possible to transfer the notation used in position space, that means

$$\begin{aligned} (f \cdot g) \quad &\hat{=} \quad (F * G)(k) \\ \overline{f}(a) \quad &\hat{=} \quad F(-k) \\ \partial\!\!\!/ f(a) \quad &\hat{=} \quad i k\!\!\!/ \, F(k). \end{aligned} \tag{3.22}$$

In particular, if f is negative then $\overline{f}$ is positive. Thus we may always assume that $f, g, \ldots$ are negative and write positive distributions in the form $\overline{f}, \overline{g}, \ldots$.

Before going into the details of the calculations, it is appropriate to prove two more general propositions.

Proposition 3.7 *Convolutions of Lorentz invariant distributions are Lorentz invariant.*

Proof. If Λ is any orthochronous Lorentz transform,

$$\begin{aligned} f * g(\Lambda q) &= \frac{1}{(2\pi)^4} \int f(k)\, g(\Lambda q - k)\, d^4 k \\ &= \frac{1}{(2\pi)^4} \int f(\Lambda^{-1} k)\, g(q - \Lambda^{-1} k)\, d^4 k \\ &= \frac{1}{(2\pi)^4} \int f(k')\, g(q - k')\, d^4 k', \end{aligned}$$

where $k' = \Lambda^{-1} k$. □

Proposition 3.8 *Convolutions of negative distributions are negative.*

Proof. It suffices to show that $f(k)\ g(q-k) \neq 0$ implies that q is backward-timelike. Indeed, $f(k) \neq 0$ only for $k_0 < -|\vec{k}| < 0$ and $g(q-k) \neq 0$ demands $q_0 - k_0 < -|\vec{q} - \vec{k}| < 0$. The triangle inequality yields $q_0 < -|\vec{q}|$. Hence $f * g$ is negative. □

3.3.1 Convolutions of Negative Distributions

In what follows, the shorthand notation

$$\Delta \;=\; \Delta(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc) \tag{3.23}$$

will be used. We remark that Δ is symmetric under exchange of arguments.

Lemma 3.9 *Suppose that f and g are negative distributions. Then the following convolutions are well-defined and given explicitly by*

$$(f \cdot g)(a) = \frac{1}{32\pi^3} \int_0^a dc\, f(c) \int_0^{(\sqrt{a}-\sqrt{c})^2} db\, g(b)\, \frac{\sqrt{\Delta}}{a} \tag{3.24}$$

$$\begin{aligned} (\partial\!\!\!/ f \cdot g)(a) &= (\partial\!\!\!/ \alpha)(a)\,, \\ \alpha(a) &\equiv \frac{1}{32\pi^3} \int_0^a dc\, f(c) \int_0^{(\sqrt{a}-\sqrt{c})^2} db\, g(b)\, \sqrt{\Delta}\, \frac{a-b+c}{2a^2} \end{aligned} \tag{3.25}$$

$$\begin{aligned} (f \cdot \partial\!\!\!/ g)(a) &= (\partial\!\!\!/ \beta)(a)\,, \\ \beta(a) &\equiv \frac{1}{32\pi^3} \int_0^a dc\, f(c) \int_0^{(\sqrt{a}-\sqrt{c})^2} db\, g(b)\, \sqrt{\Delta}\, \frac{a+b-c}{2a^2} \end{aligned} \tag{3.26}$$

$$(\partial_k f \cdot \partial^k g)(a) = \frac{1}{32\pi^3} \int_0^a dc\, f(c) \int_0^{(\sqrt{a}-\sqrt{c})^2} db\, g(b)\, \sqrt{\Delta}\, \frac{c+b-a}{2a}\,. \tag{3.27}$$

Proof. Let $q \in C^\wedge$ with $q^2 = a$. Then

$$\begin{aligned} (f \cdot g)(a) &= \\ &= \int_0^\infty dc \int_0^\infty db\, f(c)\, g(b) \int \frac{d^4k}{(2\pi)^4}\, \delta(k^2-c)\, \Theta(-k^0)\, \delta((q-k)^2-b)\, \Theta(k^0-q^0) \\ &= \int_0^\infty dc \int_0^\infty db\, f(c)\, g(b) \int \frac{d^4k}{(2\pi)^4}\, \delta(k^2-c)\, \Theta(-k^0)\, \delta(a-b+c-2qk)\, \Theta(k^0-q^0)\,. \end{aligned}$$

Since $(f \cdot g)(a)$ is Lorentz invariant by Proposition 3.7, we may assume that q points in 0-direction, i.e. $q = (-\sqrt{a}, \vec{0})$. The last integral is transformed by using polar coordinates $\omega = k^0$, $p = |\vec{k}|$,

$$\begin{aligned} &\int \frac{d^4k}{(2\pi)^4}\, \delta(k^2-c)\, \Theta(-k^0)\, \delta(a-b+c-2qk)\, \Theta(k^0-q^0) = \\ &= \frac{1}{4\pi^3} \int_{-\sqrt{a}}^0 d\omega \int_0^\infty dp\, p^2\, \delta(\omega^2-p^2-c)\, \delta(a-b+c+2\omega\sqrt{a}) \\ &= \frac{1}{8\pi^3} \int_{-\sqrt{a}}^0 d\omega\, \Theta(\omega^2-c)\, \sqrt{\omega^2-c}\, \delta(a-b+c+2\omega\sqrt{a}) \\ &= \frac{1}{16\pi^3\sqrt{a}} \Theta(\omega_0^2-c)\, \sqrt{\omega_0^2-c}\, \chi_{[-\sqrt{a},0]}(\omega_0) \end{aligned}$$

with

$$\omega_0 = \frac{b-a-c}{2\sqrt{a}}.$$

Now $\omega_0^2 - c = \Delta/4a$. Therefore $\Theta(\omega_0^2 - c) = \Theta(\Delta)$, which is nonzero for

$$a < \left(\sqrt{b} - \sqrt{c}\right)^2 \qquad \text{or} \qquad a > \left(\sqrt{b} + \sqrt{c}\right)^2 . \tag{3.28}$$

The characteristic function $\chi_{[-\sqrt{a},0]}(\omega_0)$ is equal to 1 if

$$-a \le b - c \le a \tag{3.29}$$

is fulfilled. If $b < c$, then we infer from the right inequality in (3.29)

$$\begin{aligned} a &\ge b - c \\ &= \left(\sqrt{b} + \sqrt{c}\right)\left(\sqrt{b} - \sqrt{c}\right) \\ &> \left(\sqrt{b} - \sqrt{c}\right)^2 , \end{aligned}$$

contradicting the left relation of (3.28). For $b > c$ one uses the left inequality in (3.29) and interchanges b and c in the previous calculation. Thus it is enough to consider the second inequality of (3.28). If it holds, then both

$$\Delta > 0 \quad \text{and} \quad a > \left(\sqrt{b} + \sqrt{c}\right)\left|\left(\sqrt{b} - \sqrt{c}\right)\right| = |b - c|$$

are satisfied. Hence

$$(f \cdot g)(a) = \frac{1}{16\pi^3 \sqrt{a}} \int_0^\infty dc \int_0^\infty db\, f(c)\, g(b)\, \Theta(\sqrt{a} - \sqrt{b} - \sqrt{c}) \sqrt{\frac{\Delta}{4a}} ,$$

which is equivalent to the desired result (3.24).

In order to derive (3.25), we remark that for a distribution $\psi(k^0, |k^2|)$ we have the formula

$$\begin{aligned} &\int_{-\infty}^{+\infty} d\omega \int_0^\infty p^2\, dp \int_{-\pi/2}^{+\pi/2} \sin\theta\, d\theta \int_0^{2\pi} d\phi\, \not{k}\, \psi(\omega, p) = \\ &= \int_{-\infty}^{+\infty} d\omega \int_0^\infty p^2\, dp \int_{-\pi/2}^{+\pi/2} \sin\theta\, d\theta \int_0^{2\pi} d\phi \\ &\qquad \times (\omega\gamma_0 - p\,(\sin\theta\, \cos\phi\, \gamma_1 - \sin\theta\, \sin\phi\, \gamma_2 - \cos\theta\, \gamma_3))\; \psi(\omega, p) \\ &= 4\pi\, \gamma_0 \int_{-\infty}^{+\infty} \omega\, d\omega \int_0^\infty p^2\, dp\, \psi(\omega, p) . \end{aligned}$$

For this reason, we can repeat the calculation of (3.24), but with an extra factor

$$i\gamma_0\,\omega_0 \;=\; i\,\frac{\not{q}}{-\sqrt{a}}\,\frac{b-a-c}{2\sqrt{a}} \;=\; i\not{q}\,\frac{a-b+c}{2a}\,,$$

where in our notation $i\not{q}$ will be represented by $\not{\partial}$. The equality (3.26) can be seen as replacing $\not{k}$ by $\not{q}-\not{k}$ and

$$i(\not{q}-\gamma_0\,\omega_0) \;=\; i\not{q}\left(1-\frac{a-b+c}{2a}\right) \;=\; i\not{q}\,\frac{a+b-c}{2a}\,.$$

In (3.27), the additional factor is

$$k^2-\not{q}\not{k} \;=\; c-\not{q}\left(\not{q}\,\frac{a-b+c}{2a}\right) \;=\; \frac{c+b-a}{2}\,.$$

□

3.3.2 Mixed convolutions

We already saw that convolutions of distributions of mixed type are a delicate issue. But in the case where $f(c)$ vanishes for large c we have at least statements for $q \in C$:

Lemma 3.10 *Let f and g be negative distributions where $f(c) = 0$ for every $c > c_{max}$ for some $c_{max} > 0$. Then for $q \in C^\wedge$ and $a \equiv q^2 \geq 0$, the following convolutions are well-defined and given by*

$$(f\cdot\overline{g})(a) \;=\; \frac{1}{32\pi^3}\int_a^\infty dc\, f(c)\int_0^{(\sqrt{c}-\sqrt{a})^2} db\, g(b)\,\frac{\sqrt{\Delta}}{a} \tag{3.30}$$

$$\begin{aligned}(\not{\partial}f\cdot\overline{g})(a) &= (\not{\partial}\alpha)(a)\,,\\ \alpha(a) &\equiv \frac{1}{32\pi^3}\int_a^\infty dc\, f(c)\int_0^{(\sqrt{c}-\sqrt{a})^2} db\, g(b)\,\sqrt{\Delta}\,\frac{a-b+c}{2a^2}\end{aligned} \tag{3.31}$$

$$\begin{aligned}(f\cdot\not{\partial}\overline{g})(a) &= (\not{\partial}\beta)(a)\,,\\ \beta(a) &\equiv \frac{1}{32\pi^3}\int_a^\infty dc\, f(c)\int_0^{(\sqrt{c}-\sqrt{a})^2} db\, g(b)\,\sqrt{\Delta}\,\frac{a+b-c}{2a^2}\end{aligned} \tag{3.32}$$

$$(\partial_k f\cdot\partial^k\overline{g})(a) \;=\; \frac{1}{32\pi^3}\int_a^\infty dc\, f(c)\int_0^{(\sqrt{c}-\sqrt{a})^2} db\, g(b)\,\sqrt{\Delta}\,\frac{c+b-a}{2a}\,. \tag{3.33}$$

Proof. For $q \in C^\wedge$, we have $q^0 < 0$ and thus

$$
\begin{aligned}
f \cdot \overline{g}(a) &= \int \frac{d^4k}{(2\pi)^4} \hat{F}(k)\, \hat{G}(k-q) \\
&= \int_0^\infty dc \int_0^\infty db\, f(c)\, g(b) \int \frac{d^4k}{(2\pi)^4}\, \delta(k^2 - c)\, \delta((k-q)^2 - b)\, \Theta(-k^0)\, \Theta(q^0 - k^0) \\
&= \int_0^\infty dc \int_0^\infty db\, f(c)\, g(b) \int \frac{d^4k}{(2\pi)^4}\, \delta(k^2 - c)\, \delta(c - 2qk + a - b)\, \Theta(q^0 - k^0).
\end{aligned}
$$

Again, we may assume that $q = (-\sqrt{a}, \vec{0})$ and choose polar coordinates. Hence the last integral is equal to

$$
\begin{aligned}
&\frac{1}{4\pi^3} \int_{-\infty}^{-\sqrt{a}} d\omega \int_0^\infty dp\, p^2\, \delta(\omega^2 - p^2 - c)\, \delta(c + 2\omega\sqrt{a} + a - b) \\
&= \frac{1}{8\pi^3} \int_{-\infty}^{-\sqrt{a}} d\omega\, \Theta(\omega^2 - c)\, \sqrt{\omega^2 - c}\, \delta(c + 2\omega\sqrt{a} + a - b) \\
&= \frac{1}{8\pi^3} \sqrt{\frac{\omega_0^2 - c}{4a}}\, \Theta(\omega_0^2 - c)\, \Theta(c - b - a)
\end{aligned}
$$

with

$$
\omega_0 = \frac{a - b + c}{-2\sqrt{a}}.
$$

The rest of the proof is analogous to that of Lemma 3.9. □

Remark 3.11 For $q \in C^\vee$, we may also use the preceding lemma because

$$
\begin{aligned}
f \cdot \overline{g}(a) &= \int \frac{d^4k}{(2\pi)^4} \hat{F}(k)\, \hat{G}(k-q) \\
&= \int \frac{d^4k}{(2\pi)^4} \hat{F}(k+q)\, \hat{G}(k),
\end{aligned}
$$

i.e. changing the sign of q is the same as exchanging f and g on the right hand side of the formulae (3.30)–(3.33).

Up to now there does not arise a problem because the distributions we are going to consider are sums of finite-mass Dirac seas and therefore vanish for large k^2 (and so do convolutions of them). The difficulty appears if one wants to calculate mixed convolutions for q outside the mass cone, i.e. $q^2 < 0$. There the intersection of the integration regions is the set

$$
\left\{k :\ k^2 = c,\ k^0 < 0\right\} \cap \left\{k :\ (k-q)^2 = b,\ k^0 - q^0 < 0\right\}.
$$

This set is non-compact and Lorentz invariant distributions have to be constant on it. Therefore the convolution integral does not exist for nonzero F and G. It can only be defined after some regularization that necessarily breaks Lorentz invariance. We will see that, after subtracting suitable counter terms that are supported on the light-cone, we can remove the regularization again – without destroying Lorentz invariance in the result away from the light-cone.
The regularization of $\hat{F}$ is performed by the definitions

$$f^{\varepsilon}(k) \equiv f(k^2)\, e^{\varepsilon k^0} \tag{3.34}$$

$$\hat{F}^{\varepsilon}(k) \equiv f^{\varepsilon}(k)\, \Theta(k^2)\, \Theta(-k^0). \tag{3.35}$$

Lemma 3.12 *Suppose that $f(k^2)$ and $g(k^2)$ are negative distributions which vanish identically for large k^2. Then for $q \in \hat{M} \setminus C$ and setting $a = q^2 \leq 0$, the following formulae hold for the products of the corresponding regularized distributions (3.34, 3.35),*

$$(f^{\varepsilon} \cdot \overline{g^{\varepsilon}})(q) = \frac{1}{32\pi^3} \int_0^{\infty} dc\, f(c) \int_0^{\infty} db\, g(b)\, H_{\varepsilon}(q,b,c) \tag{3.36}$$

$$(\partial_k f^{\varepsilon} \cdot \partial^k \overline{g^{\varepsilon}})(q) = \frac{1}{32\pi^3} \int_0^{\infty} dc\, f(c) \int_0^{\infty} db\, g(b)\, \frac{b+c-a}{2}\, H_{\varepsilon}(q,b,c)\,, \tag{3.37}$$

where the function H_{ε} is given by

$$H_{\varepsilon}(q,b,c) = \frac{1}{2\varepsilon|\vec{q}|}\, \exp\left(\varepsilon|\vec{q}|\, \frac{\sqrt{\Delta}}{a} + \varepsilon q^0\, \frac{c-b}{a}\right). \tag{3.38}$$

Proof. Let $u = (\varepsilon, \vec{0})$. Then

$$f^{\varepsilon} \cdot \overline{g^{\varepsilon}}(q) = \int_0^{\infty} dc\, f(c) \int_0^{\infty} db\, g(b)\, I(q,b,c)$$

with

$$I(q,b,c) = \int \frac{d^4k}{(2\pi)^4}\, \delta(k^2 - c)\, \delta((k-q)^2 - b)\, \Theta(-k^0)\, \Theta(q^0 - k^0)\, e^{ku+(k-q)u}.$$

Choose a frame where $q = (0, x, 0, 0)$ and $u = (\alpha, \beta, 0, 0)$ with $x > 0$ and $\alpha > |\beta|$. We transform the integral into cylindrical coordinates $k = (\omega, p, r\cos\phi, r\sin\phi)$ and obtain

$$\begin{aligned}
I(q,b,c) &= \frac{1}{8\pi^3}\int_{-\infty}^{0} d\omega \int_{-\infty}^{\infty} dp \int_0^\infty dr\, r \\
&\quad \cdot\delta(\omega^2 - p^2 - r^2 - c)\,\delta(\omega^2 - (p-x)^2 - r^2 - b)\, e^{2\omega\alpha - 2p\beta + x\beta} \\
&= \frac{1}{16\pi^3}\int_{-\infty}^{0} d\omega \int_{-\infty}^{\infty} dp\, \Theta(\omega^2 - p^2 - c)\,\delta(2px - x^2 + c - b)\, e^{2\alpha\omega - \beta(2p-x)} \\
&= \frac{1}{32\pi^3}\int_{-\infty}^{0} d\omega \int_{-\infty}^{\infty} \frac{dP}{x}\, \Theta\left(\omega^2 - \frac{P^2}{4x^2}\right)\delta(P - x^2 + c - b)\, e^{2\alpha\omega - \beta(\frac{P}{x} - x)} \\
&= \frac{1}{32\pi^3 x}\int_{-\infty}^{0} d\omega\, \Theta(\omega^2 - K^2)\, e^{2\alpha\omega - \beta(2K-x)} \qquad \text{with } K = \frac{x^2 - c + b}{2x} \\
&= \frac{1}{64\pi^3}\frac{1}{\alpha\, x}\, e^{-2\alpha\sqrt{K^2+c} - \beta(2K-x)} \\
&= \frac{1}{64\pi^3}\frac{1}{\alpha\, x}\, \exp(-\alpha\, x\, A - \beta\, x\, B)
\end{aligned}$$

with

$$\begin{aligned}
A &= \frac{2\sqrt{K^2+c}}{x} = -\frac{\sqrt{(-a-c+b)^2 - 4ac}}{a} = -\frac{\sqrt{\Delta}}{a} \\
B &= \frac{2K - x}{x} = \frac{c-b}{a}.
\end{aligned}$$

Note that A and B are Lorentz invariant and independent of the regularization scale ε. The back-transformation to the reference frame where $q = (q^0, \vec{q})$ and $u = (\varepsilon, \vec{0})$ gives the following substitution rules

$$\begin{aligned}
\beta\, x &= -u\, q = -\varepsilon\, q^0 \\
\alpha\, x &= \sqrt{(\varepsilon^2 + \beta^2)\, x^2} = \sqrt{-u^2\, q^2 + (u\, q)^2} = \varepsilon\, |\vec{q}|
\end{aligned}$$

This yields (3.38). The equation (3.37) is obtained with a procedure similar to that used in the proofs of Lemmata 3.9 and 3.12. □

Note that the function H_ε becomes singular for $\varepsilon \searrow 0$. But if we may subtract contributions on the light cone then the limit $\varepsilon \searrow 0$ exists and is Lorentz invariant. That shall be the content of the following Lemma.

Lemma 3.13 *Suppose that $f(k^2)$ and $g(k^2)$ are negative distributions which vanish identically for large k^2. Then the products of the corresponding regularized distributions (3.34) have the* *decomposition*

$$\begin{aligned}(f^\varepsilon \cdot \overline{g^\varepsilon})(q) &= S_1^\varepsilon(q) + R_1^\varepsilon(q)\\ (\partial_k f^\varepsilon \cdot \partial^k \overline{g^\varepsilon})(q) &= S_2^\varepsilon(q) + R_2^\varepsilon(q)\,,\end{aligned}$$

where the S_i^ε *are distributions which are supported on the light cone,*

$$\operatorname{supp} S_i^\varepsilon \subset \{\xi^2 = 0\}\,,$$

and the R_i^ε *are regular as* $\varepsilon \searrow 0$*. The limits*

$$R_i = \lim_{\varepsilon \searrow 0} R_i^\varepsilon$$

are the Lorentz invariant distributions

$$R_i(q) = \int_0^\infty dc\, f(c) \int_0^\infty db\, g(b)\, K_i(q,b,c) \tag{3.39}$$

with

$$K_1(q,b,c) = \frac{1}{32\pi^3} \begin{cases} \dfrac{\sqrt{\Delta}}{a}\,\Theta(\sqrt{b}-\sqrt{a}-\sqrt{c}) - \dfrac{|b-c|}{a}\,\Theta(b-c) & \text{if } q \in C^\vee \\ \dfrac{\sqrt{\Delta}}{a}\,\Theta(\sqrt{c}-\sqrt{a}-\sqrt{b}) - \dfrac{|b-c|}{a}\,\Theta(c-b) & \text{if } q \in C^\wedge \\ \dfrac{\sqrt{\Delta}-|b-c|}{2a} & \text{if } q \notin C \end{cases} \tag{3.40}$$

$$K_2(q,b,c) = \frac{1}{32\pi^3} \times \begin{cases} \dfrac{\sqrt{\Delta}\,(b+c-a)}{2a}\,\Theta(\sqrt{b}-\sqrt{a}-\sqrt{c}) - \dfrac{|b-c|\,(b+c)}{2a}\,\Theta(b-c) & \text{if } q \in C^\vee \\ \dfrac{\sqrt{\Delta}\,(b+c-a)}{2a}\,\Theta(\sqrt{c}-\sqrt{a}-\sqrt{b}) - \dfrac{|b-c|\,(b+c)}{2a}\,\Theta(c-b) & \text{if } q \in C^\wedge \\ \dfrac{\sqrt{\Delta}\,(b+c-a) - |b-c|\,(b+c)}{4a} & \text{if } q \notin C\,, \end{cases} \tag{3.41}$$

where we again set $a = q^2$*.*

Note that for $q \in C^\wedge$ the formulae (3.40) and (3.41) are almost equal to the results of Lemma 3.12 – up to new additional summands. These may be considered as counter terms coming from the regularization procedure; they remove the poles of the convolution integral on the cone $\{q : q^2 = 0\}$. Indeed K_1 and K_2 do not have poles at $a = 0$, since

$$\sqrt{\Delta} \;=\; |\,b - c\,| + O(a).$$

Before we begin the proof of the lemma, we have to derive a useful criterion.

Proposition 3.14 *For any distribution h, the expression*

$$f(q) \;=\; \frac{1}{|\vec{q}|}\left(h(q^0 + |\vec{q}|) - h(q^0 - |\vec{q}|)\right) \tag{3.42}$$

is supported on the light cone.

Proof. Let G be a spherical symmetric distribution supported on the light cone, i.e.

$$G(\xi) \;=\; g(\xi^0)\, \delta(\xi^2).$$

Compute the Fourier transform in polar coordinates (t, r, θ, ϕ),

$$\begin{aligned}
f(q) \;&:=\; \int d^4\xi\; g(\xi^0)\, \delta(\xi^2)\, e^{-iq\xi} \\
&=\; 2\pi \int_{-\infty}^{\infty} dt\; g(t)\, e^{-iq^0 t} \int_0^\infty r^2\, dr\; \delta(t^2 - r^2) \int_{-1}^{1} d\cos\theta\; e^{i|\vec{q}|r\cos\theta} \\
&=\; \frac{2\pi}{i|\vec{q}|} \int_{-\infty}^{\infty} dt\; g(t)\, e^{-iq^0 t} \int_0^\infty r\, dr\; \delta(t^2 - r^2) \left(e^{i|\vec{q}|r} - e^{-i|\vec{q}|r}\right) \\
&=\; \frac{\pi}{i|\vec{q}|} \int_{-\infty}^{\infty} dt\; g(t)\, e^{-iq^0 t} \left(e^{i|\vec{q}||t|} - e^{-i|\vec{q}||t|}\right) \\
&=\; \frac{\pi}{i|\vec{q}|} \int_{-\infty}^{\infty} dt\; g(t)\, \epsilon(t)\, e^{-iq^0 t} \left(e^{i|\vec{q}|t} - e^{-i|\vec{q}|t}\right) \;=\; \frac{1}{2|\vec{q}|}\left(h(q^0 + |\vec{q}|) - h(q^0 - |\vec{q}|)\right),
\end{aligned}$$

where h is the Fourier transform of the function $2i\pi g(t)\epsilon(t)$. The function g was arbitrary, and so is the distribution h. □

Proof of Lemma 3.13. Consider the product $f^\varepsilon \cdot \overline{g^\varepsilon}$. Extend the function H_ε in (3.36, 3.38) by zero for all $q \in C$ and expand it in ε,

$$H_\varepsilon(q, b, c) \;=\; \frac{\Theta(-a)}{2\varepsilon|\vec{q}|} + \frac{q^0}{2|\vec{q}|}\,\frac{\Theta(-a)}{a}\,(c - b) + \frac{\sqrt{\Delta}}{2a}\,\Theta(-a) + O(\varepsilon). \tag{3.43}$$

Apply Proposition 3.14 to $h(x) = \epsilon(x)$ and use the formula

$$\Theta(-yz) \;=\; \frac{1}{2}\,(\epsilon(y) - \epsilon(z)) \qquad \text{for } y > z.$$

Then we see that the first summand of (3.43) is supported on the light cone. Dealing with the second summand is more complicated. Set

$$h(x) \;=\; (c-b)\,\frac{\epsilon(x)}{4x},$$

hence

$$f(q) \;=\; \frac{c-b}{2a} \times \begin{cases} q^0/|\vec{q}| & \text{for } q \notin C \\ -1 & \text{for } q \in C^{\vee} \\ 1 & \text{for } q \in C^{\wedge} \end{cases}$$

is also supported on the light cone. Similarly, by choosing $h(x) = -1/x$ one can derive the same support property for $\frac{\mathrm{PP}}{a}$. Consequently, for

$$S_1^{\varepsilon} \;=\; \frac{\Theta(-a)}{2\varepsilon|\vec{q}|} + f(q) + \frac{|b-c|}{2}\,\frac{\mathrm{PP}}{a},$$

we have $\operatorname{supp} S_1^{\varepsilon} = \left\{\xi :\ \xi^2 = 0\right\}$, and $H_\varepsilon - S_1^{\varepsilon}$ is Lorentz invariant in the limit,

$$\lim_{\varepsilon\searrow 0}\left(H_\varepsilon - S_1^{\varepsilon}\right) \;=\; \begin{cases} -\dfrac{|b-c|}{a}\,\Theta(b-c) & \text{for } q \in C^{\vee} \\ -\dfrac{|b-c|}{a}\,\Theta(c-b) & \text{for } q \in C^{\wedge} \\ \dfrac{\sqrt{\Delta}-|b-c|}{2a} & \text{for } q \notin C. \end{cases}$$

Then for $q \in C^{\wedge}$, we have to take the contribution from Lemma 3.10. If $q \in C^{\vee}$, we rewrite $f \cdot \overline{g}$ by double conjugation and apply again Lemma 3.10,

$$(f\cdot\overline{g}) \;=\; \overline{(g\cdot\overline{f})(-q)} \;=\; \frac{1}{32\pi^3}\int_a^{\infty} dc\, g(c) \int_0^{(\sqrt{c}-\sqrt{a})^2} db\, f(b)\,\frac{\sqrt{\Delta}}{a}.$$

Summing up, we get the result (3.40).

To get (3.41), remember that applying a derivative on each of f and g corresponds to multiplying of the integrand with the factor $(b+c-a)/2$, or $(b+c-\Box_\xi)/2$ in position space. Since $\Box_\xi S_1^{\varepsilon}$ is also supported on the light cone, we can proceed analogously to the calculation of K_1 and get[3]

$$\tilde{K}_2(q,b,c) \;=\; K_1(q,b,c)\,\frac{b+c-a}{2}.$$

[3]The tilde is just there to distinguish the naïve guess from the final K_2.

Now we subtract the contribution

$$\triangle K_2(q,b,c) \;\equiv\; \frac{|b-c|}{4} + \frac{b-c}{4}\,\Theta(q^2)\,\epsilon(q^0) \;=\; \begin{cases} \dfrac{|b-c|}{2}\,\Theta(b-c) & \text{for } q \in C^\vee \\ \dfrac{|b-c|}{2}\,\Theta(c-b) & \text{for } q \in C^\wedge \\ \dfrac{|b-c|}{4} & \text{for } q \notin C. \end{cases}$$

But $\triangle K_2$ is supported on the light cone, since the Fourier transform of a constant is $\propto\ \delta^4(\xi)$ and $\Box_q^2\left(\Theta(q^2)\,\epsilon(q^0)\right) = 0$ in the distributional sense, so the term $\Theta(q^2)\,\epsilon(q^0)$ is supported on the light cone as well (cf. Lemma 5.2). Now $K_2 = \tilde{K}_2 - \triangle K_2$. □

We restate the central results of Lemma 3.13: The products $f \cdot \overline{g}$ and $\partial_k f \cdot \partial^k \overline{g}$ are singular only on the light cone. The regularization is needed to make this singular contribution finite. The remaining regular part does not depend on the regularization. We may therefore write the result in the compact form

$$(f \cdot \overline{g})\,(q) \;=\; (\text{l.c.}) + \int_0^\infty dc\; f(c) \int_0^\infty db\; g(b)\; K_1(q,b,c) \tag{3.44}$$

$$\left(\partial_k f \cdot \partial^k \overline{g}\right)(q) \;=\; (\text{l.c.}) + \int_0^\infty dc\; f(c) \int_0^\infty db\; g(b)\; K_2(q,b,c), \tag{3.45}$$

where “(l.c.)” marks the singular contribution on the light cone.

Now we turn to the case where only one derivative is involved.

Lemma 3.15 *Suppose that f and g are negative, Lorentz invariant distributions. Then, using the short notation just introduced before,*

$$(\partial\!\!\!/ f \cdot \overline{g})(q) \;=\; (\text{l.c.}) + \int_0^\infty dc\; f(c) \int_0^\infty db\; g(b)\; L_1(q,b,c) \tag{3.46}$$

$$(f \cdot \partial\!\!\!/ \overline{g})(q) \;=\; (\text{l.c.}) + \int_0^\infty dc\; f(c) \int_0^\infty db\; g(b)\; L_2(q,b,c) \tag{3.47}$$

with

$$L_1(q,b,c) = \frac{i\not{q}}{32\pi^3} \times \begin{cases} \sqrt{\Delta}\,\frac{a-b+c}{2a^2}\,\Theta(\sqrt{b}-\sqrt{a}-\sqrt{c}) + \frac{(b-c)^2-2ab}{2a^2}\,\Theta(b-c) & \text{if } q \in C^\vee \\ \sqrt{\Delta}\,\frac{a-b+c}{2a^2}\,\Theta(\sqrt{c}-\sqrt{a}-\sqrt{b}) - \frac{(b-c)^2-2ab}{2a^2}\,\Theta(c-b) & \text{if } q \in C^\wedge \\ \sqrt{\Delta}\,\frac{a-b+c}{4a^2} + \frac{(b-c)^2-2ab}{4a^2}\,\epsilon(b-c) & \text{if } q \notin C \end{cases} \tag{3.48}$$

$$L_2(q,b,c) = \frac{i\not{q}}{32\pi^3} \times \begin{cases} \sqrt{\Delta}\,\frac{a+b-c}{2a^2}\,\Theta(\sqrt{b}-\sqrt{a}-\sqrt{c}) - \frac{(b-c)^2-2ac}{2a^2}\,\Theta(b-c) & \text{if } q \in C^\vee \\ \sqrt{\Delta}\,\frac{a+b-c}{2a^2}\,\Theta(\sqrt{c}-\sqrt{a}-\sqrt{b}) + \frac{(b-c)^2-2ac}{2a^2}\,\Theta(c-b) & \text{if } q \in C^\wedge \\ \sqrt{\Delta}\,\frac{a+b-c}{4a^2} - \frac{(b-c)^2-2ac}{4a^2}\,\epsilon(b-c) & \text{if } q \notin C\,, \end{cases} \tag{3.49}$$

where again $a = q^2$.

Proof. Define

$$h^\varepsilon(k) = h(k^2)\,e^{\varepsilon k^0} \qquad \text{with } h(k^2) = \int_0^{k^2} f(e)\,de.$$

Then

$$\begin{aligned} \not{\partial}_\xi f^\varepsilon(k) &= i\not{k}\, f(k^2)\,e^{\varepsilon k^0} \\ &= \frac{i}{2}\left(\not{\partial}_k h(k^2)\right) e^{\varepsilon k^0} \\ &= \frac{i}{2}\not{\partial}_k\, e^{\varepsilon k^0} - \frac{i\varepsilon}{2}\gamma^0\, h(k^2)\,e^{\varepsilon k^0} \\ &= \frac{1}{2}\left(-\xi - i\varepsilon\gamma^0\right) h^\varepsilon(k). \end{aligned}$$

By Lemma 3.13,

$$\begin{aligned} \not{\partial} f \cdot \overline{g} &= \lim_{\varepsilon\searrow 0} \frac{1}{2}\left(-\xi - i\varepsilon\gamma^0\right) h^\varepsilon\,\overline{g^\varepsilon} \\ &= \text{(l.c.)} + \int_0^\infty dc\, h(c) \int_0^\infty db\, g(b)\, \frac{-\xi}{2}\, K_1(q,b,c) \end{aligned}$$

Since we can write $\not{\xi} = -i\not{\partial}_q$ and

$$\begin{aligned}\int_0^\infty dc\, h(c)\, K_1(q,b,c) &= \int_0^\infty dc \int_0^c de\, f(e)\, K(q,b,c) \\ &= \int_0^\infty de\, f(e) \int_e^\infty dc\, K_1(q,b,c),\end{aligned}$$

we get the result (3.46) with

$$L_1(q,b,e) = \int_e^\infty \frac{i}{2}\, \not{\partial}_q K_1(q,b,c)\, dc. \tag{3.50}$$

Now we are going to derive an explicit expression for L_1. We use the antiderivatives

$$\begin{aligned}\int \frac{d}{da}\frac{\sqrt{\Delta}}{a}\, dc &= \sqrt{\Delta}\, \frac{b-c-a}{2a^2} \\ \int \frac{d}{da}\frac{|b-c|}{a}\, dc &= |b-c|\, \frac{b-c}{2a^2}.\end{aligned}$$

For large c, we may expand

$$\begin{aligned}\sqrt{\Delta} &= (c-b) - a\,\frac{c+b}{c-b} + O(c^{-2}) \\ \sqrt{\Delta}\,(b-c-a) &= -(c-b)^2 + 2ab + a^2 + O(c^{-1}),\end{aligned}$$

hence the difference of the indefinite integrals from above is finite in the limit $c \to \infty$,

$$\lim_{c\to\infty} \left(\sqrt{\Delta}\, \frac{b-c-a}{2a^2} - |b-c|\, \frac{b-c}{2a^2} \right) = \frac{b}{a} + \frac{1}{2}.$$

Plugging these results into (3.50), we obtain for L_1 the formula

$$\frac{i\not{q}}{32\pi^3} \begin{cases} \sqrt{\Delta}\, \dfrac{a-b+c}{2a^2}\, \Theta(\sqrt{b}-\sqrt{a}-\sqrt{c}) + \dfrac{(b-c)^2}{2a^2}\, \Theta(b-c) & \text{for } q \in C^\vee \\ \sqrt{\Delta}\, \dfrac{a-b+c}{2a^2}\, \Theta(\sqrt{c}-\sqrt{a}-\sqrt{b}) + \dfrac{(b-c)^2}{2a^2}\, \Theta(c-b) + \dfrac{b}{a} + \dfrac{1}{2} & \text{for } q \in C^\wedge \\ \sqrt{\Delta}\, \dfrac{a-b+c}{4a^2} + \dfrac{|b-c|\,(b-c)}{4a^2} + \dfrac{b}{2a} + \dfrac{1}{4} & \text{for } q \notin C\,. \end{cases}$$

Finally, we subtract the distribution

$$b\,\Theta(b-c)\,\frac{\mathrm{PP}}{a} + \frac{1}{4}\Theta(a)\,\epsilon(-q^0) + \frac{1}{4},$$

which is supported on the light cone due to Proposition 3.14. This gives the asserted formula for L_1.

The formula for L_2 is obtained via the identity $f \cdot \overline{\not{\partial} g} = \overline{\not{\partial} g} \cdot \overline{\overline{f}}$, which yields

$$L_2(q,b,c) = L_1(-q,c,b).$$

□

Chapter 4

State stability

4.1 The variational principle in momentum space

In this chapter, we will use our previous results in order to analyze the stability of Dirac seas. First, we will transform the variational principle to momentum space via the Plancherel formula of section 3.2. After that, we will give a criterion for state stability. This can be done by calculating certain expressions with the help of some convolution formulae of section 3.3.

There is a difficulty in translating A_0 to momentum space because of its singularities on the light cone. Thus A_0 has to be regularized. However a simple Heaviside cutoff is not useful because the Fourier transform of such a discontinuous function would depend on the regularization scale in a highly oscillatory way.

For this reason we will use a different method. Introduce an extension of A_0 called $\tilde{\mathcal{M}}$ by

$$\tilde{\mathcal{M}} \in \mathcal{S}'(M) \qquad \text{and} \qquad \tilde{\mathcal{M}}(\xi) \;=\; 2A_0(\xi) \text{ for every } \xi \notin L, \tag{4.1}$$

where $\mathcal{S}'(M)$ is the space of tempered distributions. Additionally, we impose that for every $\xi \in M$,

$$\tilde{\mathcal{M}}(-\xi) \;=\; \tilde{\mathcal{M}}(\xi). \tag{4.2}$$

The Fourier transform of $\hat{\mathcal{M}}$ is denoted by $\hat{\mathcal{M}} \in \mathcal{S}'(\hat{M})$. By a symmetry argument like that in Proposition 3.4,

$$\operatorname{supp} \hat{\mathcal{M}} \subseteq \overline{C} \qquad \text{and} \qquad \hat{\mathcal{M}}(-k) = \hat{\mathcal{M}}(k). \tag{4.3}$$

Now we will describe a Lorentz invariant regularization procedure for $\tilde{\mathcal{M}}$ such that there is a regularized quantity $\tilde{\mathcal{M}}^\varepsilon$ which is in $L^2(M, z\,dz)$. We may then apply the Plancherel formula and write the action (2.28) as

$$\mathcal{S} \;=\; \frac{1}{(2\pi)^4} \lim_{\varepsilon \searrow 0} \left(\int_0^\infty \frac{1}{4} \mathrm{Tr}\left(\hat{\mathcal{M}}^\varepsilon(k)^2 \right) a\, da + F_\varepsilon(\mathfrak{m}_3, \mathfrak{m}_5) \right), \tag{4.4}$$

where $k \in \hat{M}$ is any vector with $k^2 = a$ and F_ε is some function that, amongst others, contains the counter-terms. The first variation is then

$$\delta\mathcal{S} \;=\; \frac{1}{(2\pi)^4} \lim_{\varepsilon \searrow 0} \left(\int_0^\infty \frac{1}{2} \mathrm{Tr}\left(\hat{\mathcal{M}}^\varepsilon \, \delta\hat{\mathcal{M}}^\varepsilon \right) a\, da + \delta F_\varepsilon(\mathfrak{m}_3, \mathfrak{m}_5) \right). \tag{4.5}$$

Note that

$$\begin{aligned} \forall \xi \notin L: \; \mathcal{M}(\xi) \;&=\; 2\,(P(\xi)\,P(-\xi))_0 \\ &=\; 2\left(\int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4}\, \hat{P}(p)\,\hat{P}(q)\, e^{i(p-q)\xi} \right)_0 \\ &=\; 2 \int \frac{d^4k}{(2\pi)^4} \left(\int \frac{d^4q}{(2\pi)^4}\, \hat{P}(k+q)\,\hat{P}(q) \right)_0 e^{ik\xi}, \end{aligned}$$

where the subscript zero denotes taking the trace-free part of the term in brackets. This motivates to split up the *regularized* quantity $\hat{\mathcal{M}}^\varepsilon$ as

$$\hat{\mathcal{M}}^\varepsilon(k) \;=\; \hat{\mathcal{F}}(k) + \hat{\mathcal{N}}^\varepsilon(k) \tag{4.6}$$

with

$$\hat{\mathcal{F}}(k) \;\equiv\; 2 \int \frac{d^4q}{(2\pi)^4} \left(\hat{P}(k+q)\,\hat{P}(q) \right)_0, \qquad \text{defined for } k^2 > 0. \tag{4.7}$$

The function $\hat{\mathcal{N}}^\varepsilon$ contains all the dependence on the regularization. It arises as a consequence when passing over from $2A_0$ via $\hat{\mathcal{M}}$ to $\hat{\mathcal{M}}^\varepsilon$. The poles of $\hat{\mathcal{M}}$ are characterized by $\mathfrak{m}_3$ and $\mathfrak{m}_5$. This is reflected in the fact that $\hat{\mathcal{N}}^\varepsilon(k)$ will depend only on $\mathfrak{m}_3$, $\mathfrak{m}_5$ and ε. The integral in (4.7) exists because $\hat{P}$ is supported on the lower mass shells and hence the integrand in (4.7) has compact support. For $\varepsilon \searrow 0$ we have $\hat{\mathcal{M}}^\varepsilon \to \hat{\mathcal{M}}$ and therefore for $k^2 > 0$ there is a limit function $\hat{\mathcal{N}}(k) = \lim_{\varepsilon \searrow 0} \hat{\mathcal{N}}^\varepsilon(k)$. With this considerations in mind, the variation can be written as

$$\delta\mathcal{S} \;=\; \frac{1}{(2\pi)^4} \lim_{\varepsilon \searrow 0} \int_0^\infty \mathrm{Tr}\left(\hat{\mathcal{M}}^\varepsilon \, \delta\hat{\mathcal{F}}^\varepsilon \right) a\, da \tag{4.8}$$

$$+ \frac{1}{(2\pi)^4} \lim_{\varepsilon \searrow 0} \left(\int_0^\infty \frac{1}{2} \mathrm{Tr}\left(\hat{\mathcal{M}}^\varepsilon \, \delta\hat{\mathcal{N}}^\varepsilon \right) a\, da + \delta F_\varepsilon(\mathfrak{m}_3, \mathfrak{m}_5) \right). \tag{4.9}$$

The regularization in (4.8) can be removed, as we will see soon. The divergence of the integral in (4.9) is cured by the counter-terms in δF_ε. This yields the following

Lemma 4.1 *The first variation of the action (2.28) can be written in the form*

$$\delta \mathcal{S} = \frac{1}{(2\pi)^4} \int_0^\infty a\,da \int \frac{d^4q}{(2\pi)^4}\, \mathrm{Tr}\Big(\hat{\mathcal{M}}(k)\, \delta\Big(\hat{P}(k+q)\, \hat{P}(q)\Big)_0\Big) + \delta F(\mathfrak{m}_3, \mathfrak{m}_5), \tag{4.10}$$

where $k \in \hat{M}$ is any vector with $k^2 = a$.

This lemma is proven at the end of chapter 5.

The variation of the product

$$\delta\Big(\hat{P}(k+q)\, \hat{P}(q)\Big)_0 \tag{4.11}$$

in (4.10) can be expressed in another way. Note that

$$\int d^4q\, \Big(\delta\hat{P}(k+q)\Big)\, \hat{P}(q) = \int d^4q \Big(\delta\hat{P}(q)\Big)\, \hat{P}(-k+q) \tag{4.12}$$

by changing variables and commutation of the two factors leaves the vector part of the product (4.11) invariant. Due to (4.3) we may rewrite the variation of the action in the formula

$$\delta \mathcal{S} = \frac{1}{(2\pi)^4} \int_0^\infty a\,da \int \frac{d^4q}{(2\pi)^4} \mathrm{Tr}\Big(\Big[\hat{\mathcal{M}}(k)\, \hat{P}(k+q) + \hat{\mathcal{M}}(-k)\, \hat{P}(-k+q)\Big]\, \delta\hat{P}(q)\Big) + \delta F(\mathfrak{m}_3, \mathfrak{m}_5) \tag{4.13}$$

where $k \in \hat{M}$ with $k^2 = a$ is arbitrary. We dropped the subscript zero because anyway $\hat{\mathcal{M}}$ has only got a vector part.

To get a deeper understanding of the Euler-Lagrange equations, we have a closer look on the first summand in the trace of (4.13). Since $\mathrm{supp}\, \hat{\mathcal{M}} \subseteq \overline{C}$, we may insert a δ-distribution,

$$\int_0^\infty a\,da \int d^4q\, \hat{\mathcal{M}}(k)\, \hat{P}(k+q)\, \delta\hat{P}(q) = \int_0^\infty a\,da \int_0^\infty db \int d^4q\, \delta(q^2-b)\, \hat{\mathcal{M}}(k)\, \hat{P}(k+q)\, \delta\hat{P}(q). \tag{4.14}$$

In order to convert the q-integral into a k-integral we note that an integral over Lorentz invariant distributions can be simplified to

$$\int d^4q\, \delta(q^2-b) \cdots = 4\pi \int_{-\infty}^{+\infty} dq^0 \int_0^\infty d|\vec{q}|\, \delta\Big((q^0)^2 - |\vec{q}|^2 - b\Big). \tag{4.15}$$

By using hyperbolic coordinates ($q^0 = \pm\sqrt{n}\ \cosh\beta$, $|\vec{q}| = \sqrt{b}\ \sinh\beta$) and assuming $k = (\pm\sqrt{a}, \vec{0})$ in the beginning, we can motivate a transformation rule by keeping q fixed and varying k instead,

$$\int_0^\infty a\,da \int_{C^\wedge} d^4q \cdots = 4\pi \int_0^\infty a\,da \int_0^\infty d\beta\,(\sinh\beta)^2 \int_0^\infty b\,db = \int_0^\infty b\,db \int_{C^\wedge \text{ or } C^\vee} d^4k \cdots . \tag{4.16}$$

On the right hand side, one can choose any vector $q \in C^\wedge$ with $q^2 = b$. The k-integral goes over $C^\vee$ or $C^\wedge$ if we started with a vector $k = (+\sqrt{a}, \vec{0})$ or $(-\sqrt{a}, \vec{0})$, respectively. When we also take into account the second summand of the bracket in (4.13), the integration region of the k-integral extends to C. Since $\operatorname{supp} \hat{\mathcal{M}} \subseteq \overline{C}$ one can extend the integral over all $k \in \hat{M}$. Hence,

$$\delta\mathcal{S} = \int_0^\infty b\,db \int \frac{d^4k}{(2\pi)^4}\, \mathrm{Tr}\left(\hat{\mathcal{M}}(k)\, \hat{P}(k+q)\, \delta\hat{P}(q)\right) + \delta F(\mathfrak{m}_3, \mathfrak{m}_5). \tag{4.17}$$

With the definition of the convolution and using the symmetry of $\hat{\mathcal{M}}$, namely

$$\left(\hat{\mathcal{M}} * \hat{P}\right)(q) \equiv \int \frac{d^4q}{(2\pi)^4}\, \hat{\mathcal{M}}(k)\, \hat{P}(q-k) = \int \frac{d^4q}{(2\pi)^4}\, \hat{\mathcal{M}}(k)\, \hat{P}(k+q), \tag{4.18}$$

we obtain the variation of the action in the form

$$\delta\mathcal{S} = \int_0^\infty \mathrm{Tr}\left(\left(\hat{\mathcal{M}} * \hat{P}\right)(q)\, \delta\hat{P}(q)\right)\, b\,db + \delta F(\mathfrak{m}_3, \mathfrak{m}_5). \tag{4.19}$$

The variation of F can be computed with the chain rule

$$\delta F(\mathfrak{m}_3, \mathfrak{m}_5) = D_1 F(\mathfrak{m}_3, \mathfrak{m}_5)\, \delta\mathfrak{m}_3 + D_2 F(\mathfrak{m}_3, \mathfrak{m}_5)\, \delta\mathfrak{m}_5. \tag{4.20}$$

Now remember that $\tilde{\mathcal{M}}(\xi)$ was only defined up to contributions on the light cone. We are free to modify $\tilde{\mathcal{M}}(\xi)$ by additive terms of the form

$$c_0\, \not{\xi}\, \delta'(\xi^2)\, \epsilon(\xi^0) + c_1\, \not{\xi}\, \delta(\xi^2)\, \epsilon(\xi^0). \tag{4.21}$$

We will see in Lemma 4.11 that a suitable choice of the real constants c_0, c_1 cancels the term coming from the variation of F. Thus choosing a function F or giving certain arbitrary values to c_0 and c_1 is equivalent. In the following we will always omit the δF-term.

Proposition 4.2 *The first variation of the action $\mathcal{S}$, (2.28), can be written in momentum space as*

$$\delta\mathcal{S} \;=\; 2\int_0^\infty \mathrm{Tr}\Big(\hat{Q}(q)\,\delta\hat{P}(q)\Big)\,b\,db\,, \tag{4.22}$$

where $q \in C^\wedge$ is any vector in the lower mass cone with $q^2 = b$ and

$$\hat{Q}(q) \;=\; \frac{1}{2}\,(\hat{\mathcal{M}} * \hat{P})(q) \equiv \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\,\hat{\mathcal{M}}(p)\,\hat{P}(q-p)\,. \tag{4.23}$$

Here $\hat{\mathcal{M}}$ is the Fourier transform of the distribution $\tilde{\mathcal{M}}$ as introduced by (4.1), for specific values of the free parameters c_0 and c_1 in (4.21).

Since $\delta\hat{P}(q)$ as a variation of a sum of Dirac seas has its support in $C^\wedge$ and $q \in C^\wedge$ implies that the integrand of (4.23) is compactly supported, the integral in (4.23) exists. The inverse Fourier transform of $\hat{Q}$ is

$$Q(\xi) \;=\; \frac{1}{2}\,\tilde{\mathcal{M}}(\xi)\,P(\xi). \tag{4.24}$$

Thus $Q(y-x)$ is the kernel of the operator Q that appears in the Euler-Lagrange equations $[P,Q] = 0$ of the variational principle in §3.5 of [Fin06b].

The Fourier transformed function $\hat{Q}(q)$ turns out to be useful in defining a notion of stability (see Def. 5.6.2 of [Fin06b]):

Definition 4.3 The fermionic projector of the vacuum is called **state stable** if the corresponding operator $\hat{Q}(k)$ is well-defined in the lower mass cone $C^\wedge$ and can be written as

$$\hat{Q}(k) \;=\; a\,\frac{\not{k}}{|k|} + b \tag{4.25}$$

where $|k| \equiv \sqrt{k^2}$, and a, b are continuous real functions on $C^\wedge$ having the following properties:

(i) a and b are Lorentz invariant,

$$a = a(k^2)\,, \qquad b = b(k^2).$$

(ii) a is non-negative.

(iii) The function $a + b$ is minimal on the mass shells,

$$(a+b)(m_\alpha^2) \;=\; \inf_{q\in C^\wedge}(a+b)(q^2) \qquad \forall\alpha \in \{1,\ldots,g\}. \tag{4.26}$$

Note that in our case the representation (4.25) and property **(i)** are clearly fulfilled. To derive conditions that imply **(ii)** and **(iii)**, we state the following theorem. Its proof will also elucidate the constraint (2.29).

Theorem 4.4 *The fermionic projector is a critical point of the variational principle of Definition (2.5) if and only if the functions a and b in the representation (4.25) have the following properties:*

(a) $(a+b)(m_\alpha^2) = (a+b)(m_\beta^2) \qquad \forall \alpha, \beta \in \{1, \ldots, g\}$

(b) $(a+b)'(m_\alpha^2) = 0 \qquad \forall \alpha \in \{1, \ldots, g\}\,.$

Proof. Assuming $\hat{P}$ in the form (2.17), we find

$$\frac{1}{2}\int_0^\infty \mathrm{Tr}\Big(\hat{Q}(q)\,\hat{P}(q)\Big)\; b\,db \;=\; \frac{1}{2}\sum_{\beta=1}^{g} \rho_\beta\, m_\beta^3\, \mathrm{Tr}\left(\hat{Q}(q_\beta)\,\frac{\not{q}_\beta + m_\beta}{m_\beta}\right), \tag{4.27}$$

where q, $q_\beta \in C^\wedge$ are any vectors with $q^2 = b$ and $q_\beta^2 = m_\beta^2$. By (4.25),

$$\frac{1}{2}\int_0^\infty \mathrm{Tr}\Big(\hat{Q}(q)\,\hat{P}(q)\Big)\; b\,db \;=\; 2\sum_{\beta=1}^{g} \rho_\beta\, m_\beta^3 (a+b)(m_\beta^2). \tag{4.28}$$

According to Proposition 4.2 we see that $\delta\mathcal{S}$ is calculated by varying ρ_β and m_β while keeping $\hat{Q} = a + b$ fixed. The constraint (2.29) is incorporated via a Lagrange multiplier λ. The variation of the weights ρ_β yields

$$(a+b)(m_\beta^2) \;=\; \frac{\lambda}{2} \qquad \forall \beta \in \{1, \ldots, g\}\,, \tag{4.29}$$

which proves (a). Varying the masses m_β and using (4.29) gives (b). □

This theorem shows strong connection to state stability. But it is weaker in the sense that

- it does not tell anything about the non-negativity condition **(ii)**,
- (a) and (b) only imply that the points m_β^2 are *critical points*, while a state-stable configuration is required to be a *global minimum* of the action.

If we want to get stronger criteria, we have to consider more general variations $\delta\hat{P}$. For instance, we might think of perturbing the Dirac sea by adding an light-weighted test Dirac sea that may even have negative mass[1]. That is, we take the **perturbed fermionic projector**

$$\hat{P}(k) \;=\; \sum_{\beta=1}^{g+1} \rho_\beta \, (\not{k} + m_\beta) \, \delta(k^2 - m_\beta^2) \, \Theta(-k^0),$$

where ρ_{g+1} is to vanish for the unperturbed fermionic projector.

Definition 4.5 The function

$$V(m_{g+1}) \;=\; \frac{1}{2m_{g+1}^3} \, \frac{\partial}{\partial \rho_{g+1}} \, \mathcal{S} \Big|_{\rho_{g+1}=0}$$

is called the **variation density**.

Theorem 4.6 *The functions a and b in (4.25) are related to the variation density by*

$$V(m) \;=\; \epsilon(m)\, a(m^2) + b(m^2).$$

Proof. Since m_{g+1} can be negative, (4.28) is changed to

$$\frac{1}{2} \int_0^\infty \mathrm{Tr}\left(\hat{Q}(q)\, \hat{P}(q)\right) \, b\, db \;=\; 2 \sum_{\beta=1}^{g+1} \rho_\beta \, m_\beta^3 \left(\epsilon(m_\beta)\, a(m_\beta^2) + b(m_\beta^2)\right). \tag{4.30}$$

To get the result, differentiate with respect to ρ_{g+1}. □

Corollary 4.7 *The conditions **(ii)** and **(iii)** can be expressed in terms of the variation density by*

(ii') $V(m) \geq V(-m) \qquad \forall m \in \mathbb{R}^+,$

(iii') $V(m_\beta) \leq \inf_{\mathbb{R}^+} V \qquad \forall \beta \in \{1, \ldots, g\}.$

[1]The test Dirac sea does not have a physical meaning, it is rather a mathematical tool.

4.2 Convolutions with Dirac seas

Theorem 4.8 *For any $k \in C$ with $k^2 > 0$,*

$$\hat{\mathcal{F}}(k) = \frac{1}{64\,\pi^3}\,\frac{\not{k}}{k^4}\,\epsilon(k^0) \sum_{\alpha,\beta=1}^{g} \rho_\alpha\,\rho_\beta\,J(k^2, m_\alpha, m_\beta)\,, \tag{4.31}$$

where

$$J(a,x,y) = -\sqrt{\Delta(a,x^2,y^2)}\,(x-y)\,\epsilon(x^2-y^2)\left[(x+y)^2-a\right]\Theta\Big((|x|-|y|)^2-a\Big) \tag{4.32}$$

with Δ according to (3.23).

Proof. The fermionic projector can be written in the form $P = -i\not{\partial} f + g$ with

$$f(a) = \sum_{\beta=1}^{g} \rho_\beta\,\delta(a-m_\beta^2) \qquad \text{and} \qquad g(a) = \sum_{\beta=1}^{g} \rho_\beta\,m_\beta\,\delta(a-m_\beta^2), \tag{4.33}$$

then

$$(P \cdot \overline{P})_0 = -i\Big((\not{\partial} f)\cdot\overline{g} - g\cdot(\not{\partial}\overline{f})\Big)\,.$$

For $k \in C^\wedge$, we may apply Lemma 3.12 to get

$$\begin{aligned} (P\cdot\overline{P})_0 &= -i\,\not{\partial}\,\alpha(a) \\ \alpha(a) &= \frac{1}{32\pi^3}\int_0^\infty dc\,f(c)\int_0^\infty db\,g(b)\,\sqrt{\Delta}\,\frac{a-b+c}{2a^2} \\ &\quad\times\Big(\Theta(\sqrt{c}-\sqrt{a}-\sqrt{b})-\Theta(\sqrt{b}-\sqrt{a}-\sqrt{c})\Big)\,, \end{aligned}$$

where $a \equiv k^2$. Now we plug in the special form (4.33) of f and g. This yields (4.31) with

$$\begin{aligned} J(a,x,y) &= -2x\sqrt{\Delta(a,x^2,y^2)}\,(a-x^2+y^2)\Big(\Theta(|y|-\sqrt{a}-|x|)-\Theta(|y|-\sqrt{a}-|y|)\Big) \\ &= -2x\sqrt{\Delta(a,x^2,y^2)}\,(a-x^2+y^2)\,\epsilon(|y|-|x|)\,\Theta\Big((|y|-|x|)^2-a\Big)\,. \end{aligned}$$

But (4.31) is symmetric in the indices α and β, so J can be symmetrized in x and y. This gives (4.32). The case $k \in C^\vee$ can be treated by exchanging f and g. □

Now we are going to introduce the distribution $\hat{\mathcal{M}}$ after regularization.

Theorem 4.9 *The function*

$$\hat{\mathcal{M}}(k) \;=\; \frac{1}{64\pi^3}\,\frac{k\!\!\!/}{k^4}\,\Theta(k^2)\,\epsilon(k^0)\sum_{\alpha,\beta=1}^{g}\rho_\alpha\,\rho_\beta\Big(J(k^2,m_\alpha,m_\beta)\;+\;K(k^2,m_\alpha,m_\beta)\Big) \tag{4.34}$$

with J according to (4.32) and

$$K(a,x,y) \;=\; (x-y)^2(x+y)^3\;-\;2a\,(x^3+y^3) \tag{4.35}$$

defines a tempered distribution, if the pole on the cone $\{k:\;k^2=0\}$ is understood as the distributional derivative of a logarithm, i.e.

$$\frac{k\!\!\!/}{k^2}\,\Theta(k^2)\,\epsilon(k^0) \qquad \textit{stands for} \qquad \frac{1}{2}\,\partial\!\!\!/_k\Big(\log(k^2)\,\Theta(k^2)\,\epsilon(k^0)\Big). \tag{4.36}$$

Its Fourier transform $\mathcal{M}(\xi)$ satisfies away from the light cone the relation

$$\tilde{\mathcal{M}}(\xi) \;=\; 2A_0(\xi) \qquad \forall\xi\notin L \tag{4.37}$$

with A_0 as given by (2.18).

Proof. We calculate (4.34) by choosing a fixed pair of indices $\alpha,\beta\in\{1,\dots,g\}$ instead of summing over them.

For $m_\alpha\neq m_\beta$, (4.37) is written as

$$\hat{\mathcal{M}}(k) = -2i\Big((\partial\!\!\!/ f)\cdot\overline{g} - g\cdot(\partial\!\!\!/\overline{f})\Big).$$

Now we can use Lemma 3.15 to get the distribution K in the required form. Since

$$\sqrt{\Delta(a,b,c)} \;=\; |b-c| - a\,\frac{b+c}{|b-c|} + O(a^2), \tag{4.38}$$

a straightforward calculation yields that the distribution $\hat{\mathcal{M}}(k)$ has no poles on the cone $\{k:\;k^2=0\}$. Therefore $\hat{\mathcal{M}}(k)\in L^1_{\text{loc}}(\hat{M})$, thus it defines a tempered distribution.

For $m_\alpha = m_\beta$ we have $J=0$ and hence

$$\hat{\mathcal{M}} \;=\; -\frac{1}{16\pi^3}\,\frac{k\!\!\!/}{k^2}\,\Theta(k^2)\,\epsilon(k^0)\,\rho_\alpha\,\rho_\beta m_\alpha^3\,.$$

Treating the term $\frac{\not k}{k^2}\,\Theta(k^2)\,\epsilon(k^0)$ as in (4.36) shows that $\hat{\mathcal{M}}$ is a tempered distribution again. The identity (4.37) can be calculated directly by Fourier transform. □

Remark 4.10 An asymptotic analysis of the distribution $\hat{\mathcal{F}}$ yields that

$$\hat{\mathcal{F}}(k) \;=\; 0 \qquad \text{if } k^2 > \max_{\beta\in\{1,\ldots,g\}} m_\beta^2, \tag{4.39}$$

while for $k^2 \searrow 0$ the distribution $\hat{\mathcal{F}}$ has the expansion (see (4.38))

$$\begin{aligned}\hat{\mathcal{F}}(k) \;&=\; -\frac{\not k\,\epsilon(k^0)}{64\,\pi^3} \sum_{\alpha,\beta \text{ with } m_\alpha\neq m_\beta} \rho_\alpha\,\rho_\beta \left(\frac{(m_\alpha - m_\beta)^2(m_\alpha+m_\beta)^3}{k^4} - 2\,\frac{m_\alpha^3+m_\beta^3}{k^2}\right) \\ &\quad + \not k\, O(k^0)\end{aligned} \tag{4.40}$$

and thus $\hat{\mathcal{F}}$ has poles for $k^2 = 0$, so the distribution $\hat{\mathcal{M}}$, which differs from $\hat{\mathcal{F}}$ by the appearance of K, has also got poles there. But these have a meaning in the distributional sense as told in (4.36) and thus the order of the pole is smaller than that of $\hat{\mathcal{F}}$,

$$\hat{\mathcal{M}}(k) = -\frac{1}{32\pi^3}\,\not k\,\Theta(k^2)\,\epsilon(k^0) \sum_{\alpha,\beta \text{ with } m_\alpha = m_\beta} \rho_\alpha\,\rho_\beta\,\frac{m_\alpha^3+m_\beta^3}{k^2} + \not k\,O(k^0). \tag{4.41}$$

For large k^2, we have

$$\hat{\mathcal{M}}(k) \asymp 2\pi^2\,\not k\,\Theta(k^2)\,\epsilon(k^0)\left(\frac{\mathfrak{m}_3}{k^2} + \frac{4\mathfrak{m}_5}{k^4}\right) \qquad \text{if } k^2 > \max_{\beta\in\{1,\ldots,g\}} m_\beta^2 \tag{4.42}$$

Finally, we will clarify the role of c_0 and c_1 in (4.21):

Lemma 4.11 *The Fourier transforms of the distributions*

$$A(\xi) \;\equiv\; \not\xi\,\delta(\xi^2)\,\epsilon(\xi^0)\,, \qquad B(\xi) \;\equiv\; \not\xi\,\delta'(\xi^2)\,\epsilon(\xi^0)$$

satisfy the relations

$$\int_0^\infty \mathrm{Tr}\Big((\hat{A} * \hat{P})(q)\,\delta\hat{P}(q)\Big)\,a\,da \;=\; 32\pi^4\,\delta\mathfrak{m}_3 \tag{4.43}$$

$$\int_0^\infty \mathrm{Tr}\Big((\hat{B} * \hat{P})(q)\,\delta\hat{P}(q)\Big)\,a\,da \;=\; -32\pi^4\,\delta\mathfrak{m}_5\,, \tag{4.44}$$

where $\mathfrak{m}_3$ and $\mathfrak{m}_5$ are defined by (2.24, 2.25), and where again $a = q^2$.

Proof. As before, we represent the fermionic projector in the form $\hat{P} = -i\partial\!\!\!/ f + g$, where f and g are given by (4.33). This allows us to represent $\hat{A}$ and $\hat{B}$ as

$$i\partial\!\!\!/ \left(h(a) - \overline{h}(a)\right)$$

with

$$h(a) = \begin{cases} 8\pi^2\delta'(a) & \text{for } \hat{A} \\ 2\pi^2\delta(a) & \text{for } \hat{B}\,. \end{cases}$$

The convolutions $\hat{P} * \hat{A}$ and $\hat{P} * \hat{B}$ can be written in the form

$$(\partial\!\!\!/ f + ig)\cdot\left(\partial\!\!\!/ h - \partial\!\!\!/ \overline{h}\right)\,.$$

By Lemmata 3.9 and 3.10 we obtain the explicit expressions

$$\begin{aligned}
(\partial\!\!\!/ f\cdot(\partial\!\!\!/ h - \partial\!\!\!/ \overline{h}))(a) &= \alpha(a)\,, \\
\alpha(a) &= \frac{1}{32\pi^3}\int_0^\infty dc\, f(c)\,\epsilon(a-c)\int_0^{(\sqrt{a}-\sqrt{c})^2} db\, h(b)\,\sqrt{\Delta}\,\frac{c+b-a}{2a} \\
(ig\cdot(\partial\!\!\!/ h - \partial\!\!\!/ \overline{h}))(a) &= i\partial\!\!\!/ \beta(a)\,, \\
\beta(a) &= \frac{1}{32\pi^3}\int_0^\infty dc\, g(c)\,\epsilon(a-c)\int_0^{(\sqrt{a}-\sqrt{c})^2} db\, h(b)\,\sqrt{\Delta}\,\frac{a+b-c}{2a^2}\,.
\end{aligned}$$

This can be further simplified using the expansion

$$\sqrt{\Delta} = |a-c| - b\,\frac{a+c}{|a-c|} + O(b^2)$$

For the case of $\hat{A}$ we get

$$\begin{aligned}
\alpha(a) &= -\frac{1}{4\pi}\int_0^\infty dc\, f(c)\,\epsilon(a-c)\frac{d}{db}\left(\sqrt{\Delta}\,\frac{c+b-a}{2a}\right)\Big|_{b=0} \\
&= -\frac{1}{4\pi}\int_0^\infty dc\, f(c) = -\frac{1}{4\pi}\sum_{\alpha=1}^{g}\rho_\alpha \\
\beta(a) &= -\frac{1}{4\pi}\int_0^\infty dc\, g(c)\,\epsilon(a-c)\frac{d}{db}\left(\sqrt{\Delta}\,\frac{a+b-c}{2a^2}\right)\Big|_{b=0} \\
&= \frac{1}{4\pi}\int_0^\infty dc\, g(c)\,\frac{c}{a^2} = \frac{1}{4\pi}\sum_{\alpha=1}^{g}\rho_\alpha\,\frac{m_\alpha^3}{a^2}.
\end{aligned}$$

Hence,

$$(\hat{P} * \hat{A})(q) \;=\; -\frac{1}{4\pi} \sum_{\alpha=1}^{g} \frac{\rho_\alpha}{a^2}\,(m_\alpha \not{q} + a) \;.$$

The variation $\delta\hat{P}$ is the sum of g test Dirac seas via

$$\begin{aligned} \delta\hat{P} \;&=\; \sum_{\beta=1}^{g} \frac{\delta\hat{P}}{\delta\rho_\beta}\,\delta\rho_\beta + \frac{\delta\hat{P}}{\delta m_\beta^3}\,\delta m_\beta^3 \\ &=\; \sum_{\beta=1}^{g} \frac{\delta R}{\delta\rho}\,\delta\rho_\beta + \frac{\delta R}{\delta m^3}\,\delta m_\beta^3\,, \end{aligned} \tag{4.45}$$

where

$$R(q) = \rho\,(\not{q} + m)\,\delta(q^2 - m^2).$$

But the integral (4.43) with such a test Dirac sea R is calculated to be

$$\int_0^\infty \mathrm{Tr}\Big((\hat{P} * \hat{A})(q)\,R(q)\Big)\,a\,da \;=\; -\frac{1}{\pi} \sum_{\alpha=1}^{g} \rho\,\rho_\alpha\,\left(m_\alpha^3 + m^3\right)\,,$$

and (4.45) as well as the symmetry in the indices α and β gives

$$\begin{aligned} \int_0^\infty \mathrm{Tr}\Big((\hat{P} * \hat{A})(q)\,\delta\hat{P}(q)\Big)\,a\,da \;&=\; -\frac{1}{2\pi}\,\delta\left(\sum_{\alpha,\beta=1}^{g} \rho_\alpha\,\rho_\beta\,\left(m_\alpha^3 + m_\beta^3\right)\right) \\ &=\; 32\pi^4\,\delta\mathfrak{m}_3, \end{aligned}$$

where in the last line we have used (2.24). The computation for $\hat{B}$ is similar. ◻

Chapter 5

A Lorentz invariant regularization

In this chapter we will explain how a sensible Lorentz invariant regularization of the poles of $\tilde{\mathcal{M}}$ can be performed such that it is explicit both in position and momentum space. The regularization is necessary in order to define the action rigorously because $\tilde{\mathcal{M}}$ has got a pole on the light cone,

$$\tilde{\mathcal{M}} \;=\; \mathfrak{m}_3 \, \frac{\not{\xi}}{\xi^4} \, \Theta(\xi^2) \, \epsilon(\xi^0) + O(\xi^{-2}) \,. \tag{5.1}$$

The subsequent Fourier integral is the starting point of our analysis.

Lemma 5.1 *The following identity holds in the sense of distributions:*

$$\int \frac{d^4k}{(2\pi)^4} \, e^{-\frac{\varepsilon k^2}{2}} \, \Theta(k^2) \, \epsilon(k^0) \, e^{ik\xi} \;=\; -\frac{i}{4\pi^2\varepsilon^2} \, e^{-\frac{\xi^2}{2\varepsilon}} \, \Theta(\xi^2) \, \epsilon(\xi^0) \;+\; \frac{i}{2\pi^2\varepsilon} \, \delta(\xi^2) \, \epsilon(\xi^0) \tag{5.2}$$

Proof. Spherical symmetry allows us to write $\xi = (t, r, 0, 0)$ with $r \geq 0$. In polar coordinates $k = (\omega,\; p\cos\theta,\; p\sin\theta\cos\phi,\; p\sin\theta\sin\phi)$ the Fourier integral can be evaluated to

$$\begin{aligned}
A \;&\equiv\; \int \frac{d^4k}{(2\pi)^4} \, e^{-\frac{\varepsilon k^2}{2}} \, \Theta(k^2) \, \epsilon(k^0) \, e^{ik\xi} \\
&=\; \frac{1}{8\pi^3} \int_{-\infty}^{\infty} d\omega \, \epsilon(\omega) \, e^{i\omega t} \int_0^{|\omega|} p^2 \, dp \, e^{-\frac{\varepsilon}{2}(\omega^2 - p^2)} \int_{-1}^{1} d\cos\vartheta \, e^{-ipr\cos\vartheta} \\
&=\; \frac{i}{8\pi^3 r} \int_{-\infty}^{\infty} d\omega \, \epsilon(\omega) \, e^{i\omega t} \int_0^{|\omega|} p \, dp \, e^{-\frac{\varepsilon}{2}(\omega^2 - p^2)} \left(e^{-ipr} - e^{ipr} \right) \\
&=\; \frac{i}{8\pi^3 r} \int_{-\infty}^{\infty} d\omega \, \epsilon(\omega) \int_{-|\omega|}^{|\omega|} p \, dp \, e^{-\frac{\varepsilon}{2}(\omega^2 - p^2) + i(\omega t - pr)} \,.
\end{aligned}$$

Introduce mass cone coordinates

$$u \;=\; \frac{1}{2}(\omega + p) \,, \qquad v \;=\; \frac{1}{2}(\omega - p) \,,$$

and light cone coordinates

$$s = \frac{1}{2}(t-r)\,, \qquad l = \frac{1}{2}(t+r)\,.$$

Then

$$A = \frac{i}{4\pi^3 r}\left(\int_0^\infty \int_0^\infty - \int_{-\infty}^0 \int_{-\infty}^0\right) du\, dv\, (u-v)\, e^{-2\varepsilon uv + 2ius + 2ivl}\,.$$

Now carry out the v-integral, which for $u > 0$ gives

$$\int_0^\infty (u-v)\, e^{-2\varepsilon uv+2ivl}\, dv = \left(u + \frac{i}{2}\frac{\partial}{\partial l}\right)\int_0^\infty e^{-2\varepsilon uv+2ivl}\, dv = \left(u + \frac{i}{2}\frac{\partial}{\partial l}\right)\frac{1}{2\varepsilon u - 2il}\,,$$

For $u < 0$ we obtain the same formula up to an extra minus sign. Therefore

$$\begin{aligned} A &= \frac{i}{4\pi^3 r}\int_{-\infty}^\infty \left[\left(u + \frac{i}{2}\frac{\partial}{\partial l}\right)\frac{1}{2\varepsilon u - 2il}\right] e^{2ius}\, du \\ &= \frac{1}{8\pi^3 r}\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial l}\right)\int_{-\infty}^\infty \frac{1}{2\varepsilon u - 2il}\, e^{2ius}\, du \\ &= \frac{i}{8\pi^2 r\varepsilon}\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial l}\right)\left[\epsilon(s)\,\Theta(sl)\, e^{-\frac{2sl}{\varepsilon}}\right], \end{aligned}$$

In the last step we used the calculus of residues to calculate the integral (e.g. see Theorem VI, 2.2 in [Lan93]). Transforming back to polar coordinates,

$$A = -\frac{i}{4\pi^2\varepsilon}\,\frac{1}{r}\frac{\partial}{\partial r}\left[\epsilon(t)\,\Theta(t^2-r^2)\, e^{-\frac{t^2-r^2}{2\varepsilon}}\right],$$

one may compute the distributional derivatives to get (5.2). □

Using this result we can give formulae for some important Fourier integrals:

Lemma 5.2 *The following equations hold in the sense of distributions:*

$$\int \delta'(\xi^2)\,\epsilon(\xi^0)\, e^{-ik\xi}\, d^4\xi = -i\pi^2\,\Theta(k^2)\,\epsilon(k^0) \tag{5.3}$$

$$\int \delta(\xi^2)\,\epsilon(\xi^0)\, e^{-ik\xi}\, d^4\xi = -4i\pi^2\,\delta(k^2)\,\epsilon(k^0) \tag{5.4}$$

$$\int \not{\xi}\,\delta'(\xi^2)\,\epsilon(\xi^0)\, e^{-ik\xi}\, d^4\xi = 2\pi^2\,\not{k}\,\delta(k^2)\,\epsilon(k^0) \tag{5.5}$$

$$\int \not{\xi}\,\delta(\xi^2)\,\epsilon(\xi^0)\, e^{-ik\xi}\, d^4\xi = 8\pi^2\,\not{k}\,\delta'(k^2)\,\epsilon(k^0) \tag{5.6}$$

Proof. Let ϕ be a test function. By partial integration,

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_0^\infty e^{-\frac{z}{2\varepsilon}}\,\phi(z)\,dz &= \frac{2}{\varepsilon}\,\phi(0)+\frac{2}{\varepsilon}\int_0^\infty e^{-\frac{z}{2\varepsilon}}\,\phi'(z)\,dz \\ &= \frac{2}{\varepsilon}\,\phi(0)+4\phi'(0)+O(\varepsilon)\,.\end{aligned}$$

Hence,

$$\frac{1}{\varepsilon^2}\,e^{-\frac{z}{2\varepsilon}}\,\Theta(z)-\frac{2}{\varepsilon}\,\delta(z)\ \stackrel{\varepsilon\to 0}{\longrightarrow}\ -4\,\delta'(z)\qquad \text{in } \mathcal{S}'(\mathbb{R}). \tag{5.7}$$

But this implies that for $\varepsilon \searrow 0$ equation (5.2) converges in $\mathcal{S}'(\hat{M})$ to

$$\int \frac{d^4k}{(2\pi)^4}\,\Theta(k^2)\,\epsilon(k^0)\,e^{ik\xi} \;=\; \frac{i}{\pi^2}\,\delta'(\xi^2)\,\epsilon(\xi^0) \tag{5.8}$$

if $\xi \neq 0$. In the case $\xi = 0$, we take an arbitrary test function $\eta \in C_0^\infty(M)$ and use a scaling argument to obtain

$$\lim_{\delta\searrow 0}\,\lim_{\varepsilon\searrow 0}\int \eta\Big(\frac{\xi}{\delta}\Big)\left[\frac{1}{\varepsilon^2}\,e^{-\frac{\xi^2}{2\varepsilon}}\,\Theta(\xi^2)\,\epsilon(\xi^0)\;-\;\frac{1}{2\varepsilon}\,\delta(\xi^2)\,\epsilon(\xi^0)\right] \;=\; 0\,. \tag{5.9}$$

An inverse Fourier transform applied to (5.8) gives (5.3).

In order to derive (5.4), apply the operator $\Box_k$ to both sides of (5.3). On the left hand side the resulting term is

$$\begin{aligned}\Box_k \int \delta'(\xi^2)\,\epsilon(\xi^0)\,e^{-ik\xi}\,d^4\xi &= \int \xi^2\,\delta'(\xi^2)\,\epsilon(\xi^0)\,e^{-ik\xi}\,d^4\xi \\ &= -\int \delta(\xi^2)\,\epsilon(\xi^0)\,e^{-ik\xi}\,d^4\xi\,,\end{aligned} \tag{5.10}$$

where in the last line we used the equality $z\,\delta'(z) = -\delta(z)$. On the right hand side of (5.2), application of $\Box_k$ yields

$$\begin{aligned}-i\pi^2\Box_k\Big(\Theta(k^2)\,\epsilon(k^0)\Big) &= -i\pi^2\Big(\partial_\omega^2-\nabla_{\vec{k}}\cdot\nabla_{\vec{k}}\Big)\Big(\Theta(k^2)\,\epsilon(k^0)\Big) \\ &= -i\pi^2\Big(4\,\omega^2\,\delta'(k^2)\,\epsilon(\omega)+2\,\delta(k^2)\,\epsilon(\omega)+2\,\Theta(k^2)\,\delta'(\omega)\Big) \\ &\qquad +i\pi^2\Big(4\,\vec{k}^2\,\delta'(k^2)-6\,\delta(k^2)\Big)\,\epsilon(\omega) \\ &= -4i\pi^2\,\delta(k^2)\,\epsilon(\omega)-2i\pi^2\,\Theta(k^2)\,\delta'(\omega)\,.\end{aligned} \tag{5.11}$$

But when applied to a test function ϕ, the second summand vanishes almost everywhere,

$$\int \Theta(k^2)\, \delta'(\omega)\, \phi(\omega)\, d\omega \;=\; -\,\Theta(-\vec{k}^2)\, \phi'(0) \;=\; 0 \qquad \text{a.e.}$$

Relation (5.4) then follows from the equality of (5.10) and (5.11).

The formulae (5.5) and (5.6) can be proven by applying the operator $i\partial\!\!\!/_k$ to (5.3) and (5.4), respectively. □

By integration over ε we obtain Fourier transformations of regularized poles:

Proposition 5.3 *The following equations hold in the sense of distributions:*

$$\int \frac{d^4k}{(2\pi)^4}\, e^{-\frac{\varepsilon k^2}{2}}\, \Box_k\Big(\log(k^2)\, \Theta(k^2)\, \epsilon(k^0)\Big)\, e^{ik\xi}$$
$$= \;-\frac{i}{16\pi^2}\left[\frac{1}{\xi^2}\left(1 - e^{-\frac{\xi^2}{2\varepsilon}}\right)\Theta(\xi^2)\, \epsilon(\xi^0) \;+\; \delta(\xi^2)\, \epsilon(\xi^0)\, (c - 1 + \log\varepsilon)\right] \tag{5.12}$$
$$\int \frac{d^4k}{(2\pi)^4}\, e^{-\frac{\varepsilon k^2}{2}}\, \partial\!\!\!/_k\Big(\log(k^2)\, \Theta(k^2)\, \epsilon(k^0)\Big)\, e^{ik\xi}$$
$$= \;-\frac{1}{2\pi^2}\, \partial\!\!\!/_\xi\left[\frac{1}{\xi^2}\left(1 - e^{-\frac{\xi^2}{2\varepsilon}}\right)\Theta(\xi^2)\, \epsilon(\xi^0) \;+\; \delta(\xi^2)\, \epsilon(\xi^0)\, (c + \log\varepsilon)\right] \tag{5.13}$$

Here the constant c is given by

$$c \;=\; \gamma + \log 2\,, \tag{5.14}$$

and γ is the Euler-Mascheroni constant.

Proof. With a slight misuse of notation, we integrate (5.2) as a function of ε over the compact interval $[\varepsilon, L]$ with $L \in \mathbb{R}$. This gives

$$\int \frac{d^4k}{(2\pi)^4}\, \frac{2}{k^2}\left(e^{-\frac{\varepsilon k^2}{2}} - e^{-\frac{L k^2}{2}}\right)\Theta(k^2)\, \epsilon(k^0)\, e^{ik\xi}$$
$$= \;-\frac{i}{4\pi^2}\, \frac{2}{\xi^2}\left(e^{-\frac{\xi^2}{2L}} - e^{-\frac{\xi^2}{2\varepsilon}}\right)\Theta(\xi^2)\, \epsilon(\xi^0) \;+\; \frac{i}{2\pi^2}\, \delta(\xi^2)\, \epsilon(\xi^0)\, (\log L - \log\varepsilon)\;. \tag{5.15}$$

Subtracting the $\log L$-term and using (5.4), the last formula turns into

$$\int \frac{d^4k}{(2\pi)^4}\left\{\frac{1}{k^2}\left(e^{-\frac{\varepsilon k^2}{2}} - e^{-\frac{L k^2}{2}}\right)\Theta(k^2)\, \epsilon(k^0) \;-\; \delta(k^2)\, \epsilon(k^0)\, (c + \log L)\right\} e^{ik\xi}$$
$$= \;-\frac{i}{4\pi^2}\left[\frac{1}{\xi^2}\left(e^{-\frac{\xi^2}{2L}} - e^{-\frac{\xi^2}{2\varepsilon}}\right)\Theta(\xi^2)\, \epsilon(\xi^0) \;+\; \delta(\xi^2)\, \epsilon(\xi^0)\, (c + \log\varepsilon)\right]. \tag{5.16}$$

For $a = k^2 > 0$ we have

$$\begin{aligned}
-\frac{1}{4}\,\hat{W}\,(\log a\,\Theta(a)) &= \frac{1}{a}\,\frac{d}{da}\left(a^2\,\frac{d}{da}\log a\right) \\
&= \frac{1}{a} \\
&= \lim_{L\to\infty}\frac{1}{a}\,\left(1-e^{La/2}\right)\Theta(a)\,, \qquad (5.17)
\end{aligned}$$

where $\hat{W}$ is the wave operator in the form (3.10).

In order to see what happens around $a = 0$, take any test function $\eta \in C_0^\infty(\mathbb{R})$. If we choose $x > 0$ so small that $\eta(a)$ approaches a constant in the interval $[0, x]$ we obtain the expansion

$$\begin{aligned}
\int_{-\infty}^{x}\eta(a)\,\frac{1}{a}\,\left(1-e^{La/2}\right)\,\Theta(a)\,da &= \eta(0)\int_0^x\frac{1}{a}\,\left(1-e^{La/2}\right)\,da+O(x) \\
&= \eta(0)\left(\gamma+\Gamma\left(0,\frac{L\,x}{2}\right)+\log\frac{L\,x}{2}\right)+O(x)\,. \qquad (5.18)
\end{aligned}$$

Here Γ is the upper incomplete Gamma function. Now subtract $\eta(0)\log L$ on both sides to get the limit

$$\lim_{L\to\infty}\int_{-\infty}^{x}\eta(a)\left(\frac{1}{a}\,\left(1-e^{La/2}\right)\,\Theta(a)-\delta(a)\,\log L\right)\,da \;=\; \eta(0)\,(\log x+\gamma-\log 2)+O(x)\,. \quad (5.19)$$

On the other hand,

$$\begin{aligned}
\int_{-\infty}^{x}\eta(a)\,\frac{1}{a}\,\frac{d}{da}\,\left(a^2\,\frac{d}{da}\,(\Theta(a)\log a)\right)\,da &= \eta(x)-(x\,\eta'(x)-\eta(x))\log x+\int_0^x a\,\eta''(a)\log a\,da \\
&= \eta(0)\;(1+\log x)+O(x)\,,
\end{aligned}$$

where we have used partial integration. Comparison with (5.19) gives the approximation

$$\lim_{Lx\to\infty}\int_0^x\eta(a)\left(\frac{1}{a}\,\left(1-e^{La/2}\right)\,\Theta(a)-\delta(a)\,\left(\log L+c-1\right)+\frac{1}{4}\hat{W}\,(\Theta(a)\,\log a)\right)\,da \;=\; O(x)\,.$$

By (5.17) multiplying both sides with $e^{-\varepsilon a/2}$, the following distributional equation can be derived,

$$\lim_{L\to\infty}\left[\frac{1}{a}\,\left(e^{-\varepsilon a/2}-e^{-La/2}\right)\,\Theta(a)-\delta(a)\,\left(\log L+c-1\right)\right] \;=\; -\frac{1}{4}\hat{W}\,(\log a\,\Theta(a))\;e^{-\varepsilon a/2}\,.$$

Finally we take the limit $L \to \infty$ in (5.16). The assertion (5.12) is then verified by means of (5.4).

In order to prove (5.13), we apply the operator $-i\partial\!\!\!/_\xi$ to (5.16). Using (5.18) together with the distributional identity

$$\int_{-\infty}^{x} \eta(a)\, \frac{d}{da}\, (\Theta(a)\, \log a)\, da \;=\; \eta(0) \log(x) \;+\; O(x)\,,$$

which is derived by a similar method as (5.12), we conclude that

$$\lim_{L\to\infty} \left\{ \frac{1}{a} \left(e^{-\varepsilon a/2} - e^{-La/2} \right) \Theta(a) - \delta(a)(c + \log L) \right\} k\!\!\!/\, \epsilon(k^0) \;=\; \frac{1}{2}\, \partial\!\!\!/_k \left(\log(a)\, \Theta(a)\, \epsilon(k^0) \right) e^{-\varepsilon a/2}\,, \tag{5.20}$$

giving (5.13). □

Remark 5.4 The differentiation of $\epsilon(k^0)$ does not give a contribution,

$$\begin{aligned} \partial\!\!\!/_k \left(\Theta(k^2)\, \epsilon(k^0) \right) &= 2k\!\!\!/\, \delta(k^2)\, \epsilon(k^0) + 2\gamma^0\, \delta(k^0)\, \Theta(k^2) \\ &= 2k\!\!\!/\, \delta(k^2)\, \epsilon(k^0)\,, \end{aligned}$$

since for any test function $\phi \in C_0^\infty(\mathbb{R})$ the identity

$$\begin{aligned} \int d^4k\, \delta(k^0)\, \Theta(k^2) &= 4\pi \int_{-\infty}^{+\infty} d\omega \int_0^\infty p^2\, dp\, \delta(\omega)\, \Theta(\omega^2 - p^2)\, \phi(\omega^2 - p^2) \\ &= 4\pi \int_0^\infty p^2\, dp\, \Theta(-p^2)\, \phi(-p^2) \;=\; 0 \end{aligned}$$

holds.

Instead of bringing in (5.15) the $\log L$-term to the left, one can do so with the $\log \varepsilon$-term and take the limit $\varepsilon \searrow 0$:

Corollary 5.5 *The following equations hold in the sense of distributions:*

$$\int \frac{d^4k}{(2\pi)^4}\, \Box_k \Big(\log(k^2)\, \Theta(k^2)\, \epsilon(k^0) \Big)\, e^{ik\xi} \;=\; -\frac{i}{4\pi^2}\, \Box_\xi \Big(\log(\xi^2)\, \Theta(\xi^2)\, \epsilon(\xi^0) \Big) \tag{5.21}$$

$$\int \frac{d^4k}{(2\pi)^4}\, \partial\!\!\!/_k \Big(\log(k^2)\, \Theta(k^2)\, \epsilon(k^0) \Big)\, e^{ik\xi} \;=\; -\frac{i}{4\pi^2}\, \partial\!\!\!/_\xi \Box_\xi \Big((\log(\xi^2) - 1)\, \Theta(\xi^2)\, \epsilon(\xi^0) \Big)\,. \tag{5.22}$$

How do these insights help us in regularizing $\tilde{\mathcal{M}}$ near the light cone? As already indicated in section 4.1, we want to find a regularization, explicit both in position and in momentum space that makes the action finite,

$$\int_0^\infty \left(\tilde{\mathcal{M}}^\varepsilon \right)^2 a\, da \;<\; \infty\,. \tag{5.23}$$

If we carry out the derivatives on the right hand side of (5.22) we will see that this distribution has a pole of the form (5.1),

$$(5.22) \;=\; \frac{2i}{\pi^2}\,\frac{\not\xi}{\xi^4}\,\Theta(\xi^2)\,\epsilon(\xi^0) \qquad \text{if } \xi \notin L\,.$$

Thus (5.22) extends this pole in the distributional sense across the light cone and gives an explicit formula for the corresponding Fourier transform. The regularization is performed by introducing in (5.22) a smooth momentum cutoff via a

$$\text{regularizing factor} \quad e^{-\varepsilon k^2/2} \quad \text{in momentum space}\,, \tag{5.24}$$

which leads us to (5.13). the new $\delta(\xi^2)$-term is supported on the light cone and can be dropped by setting the lower bound of integration in (5.23) to some $\delta > 0$ and taking the limit $\delta \to 0$,

$$\int_{0^+}^{\infty} (\tilde{\mathcal{M}}^{\varepsilon})^2\, a\, da \;\equiv\; \lim_{\delta\searrow 0} \int_{\delta}^{\infty} (\tilde{\mathcal{M}}^{\varepsilon})^2\, a\, da\,. \tag{5.25}$$

Then according to (5.13, 5.22) the regularization effect can be described by the

$$\text{replacement} \quad \frac{1}{z^2} \;\longrightarrow\; -\frac{d}{dz}\left[\frac{1}{z}\left(1 - e^{-z/2\varepsilon}\right)\right] \quad \text{in position space}\,. \tag{5.26}$$

Since

$$\frac{1}{z}\left(1 - e^{-z/2\varepsilon}\right) \;=\; \int_0^{1/2\varepsilon} e^{-\lambda z}\, d\lambda\,,$$

the square bracket in (5.26) is smooth in z. The decay of the factor $e^{-z/2\varepsilon}$ localizes the regularization in a strip $z \lesssim \varepsilon$ around the light cone. Altogether, our regularization method is convenient both in momentum and position space by performing (5.24) or (5.26), respectively.

We are now able to complete the constructions in section 4.1 by proving Lemma 4.1. Note that due to (5.22), the pole $\sim \not\xi/\xi^4$ corresponds in momentum space to a term of the form $\not k/k^2$. By the expansion (4.42) it makes sense to regularize $\hat{\mathcal{M}}$ in the following way:

Definition 5.6 For any $\varepsilon > 0$ and $k \in C$ with $k^2 > 0$, we introduce the function $\hat{\mathcal{N}}^\varepsilon$ by

$$\hat{\mathcal{N}}^\varepsilon(k) \;=\; \frac{1}{64\pi^3}\, \frac{\not k}{k^4}\, \epsilon(k^0) \sum_{\alpha,\beta=1}^{g} \rho_\alpha\, \rho_\beta\, K(k^2, m_\alpha, m_\beta) \;+\; 2\pi^2\, \mathfrak{m}_3\, \frac{\not k}{k^2}\, \left(e^{-\varepsilon k^2/2} - 1\right) \epsilon(k^0)\,,$$

where K is the function (4.35). For any $\varepsilon > 0$, we define the distribution $\hat{\mathcal{M}}^\varepsilon$ by

$$\hat{\mathcal{M}}^\varepsilon(k) \;=\; \hat{\mathcal{M}}(k) + 2\pi^2\, \mathfrak{m}_3\, \frac{\not k}{k^2}\, \left(e^{-\varepsilon k^2/2} - 1\right) \Theta(k^2)\, \epsilon(k^0) \tag{5.27}$$

with $\hat{\mathcal{M}}$ according to (4.34). The Fourier transform of $\hat{\mathcal{M}}^\varepsilon$ is denoted by $\mathcal{M}^\varepsilon$.

Proof of Lemma 4.1. For any $\varepsilon > 0$ the pole of $\hat{\mathcal{M}}^\varepsilon(k)$ on the mass cone is integrable in the $L^2(\hat{M},\, a\,da)$-norm, see equation (4.41). Definition 5.6 and (4.42) imply that, for large k^2,

$$\hat{\mathcal{M}}^\varepsilon(k) \;=\; 2\pi^2\, \not k\, \Theta(k^2)\, \epsilon(k^0) \left(\frac{\mathfrak{m}_3}{k^2}\, e^{-\frac{\varepsilon k^2}{2}} + \frac{4\mathfrak{m}_5}{k^4} \right) \qquad \text{if } k^2 > \max_{\beta \in \{1,\ldots,g\}} m_\beta^2\,. \tag{5.28}$$

Therefore $\hat{\mathcal{M}}^\varepsilon \in L^2(\hat{M},\, a\,da)$ and with our Plancherel formula we get

$$\int_{0^+}^\infty \mathrm{Tr}\left(\left(\tilde{\mathcal{M}}^\varepsilon\right)^2 \right) z\, dz \;=\; \frac{1}{(2\pi)^4} \int_{0^+}^\infty \mathrm{Tr}\left(\left(\hat{\mathcal{M}}^\varepsilon\right)^2 \right) a\, da$$

when the contributions at $z = 0$ or $a = 0$ are disregarded as before. The counter-term needed in (4.4) can be determined by plugging (5.28) into (4.4) and integrating out: By the choice

$$F_\varepsilon(\mathfrak{m}_3, \mathfrak{m}_5) \;=\; G(\mathfrak{m}_3, \mathfrak{m}_5) - \frac{4\pi^4 \mathfrak{m}_3}{\varepsilon} - 32\pi^4\, \mathfrak{m}_3 \mathfrak{m}_5 \log \varepsilon$$

the limit $\varepsilon \searrow 0$ exists.[1] Then (4.6) follows from Definition 5.6.

It remains to show the existence of the limits in (4.8) and (4.9). For large a, the distribution $\mathcal{F}(a)$ vanishes because of (4.39). If a is small, we will use (4.40) to rewrite (4.8) as

$$\int_0^\infty \left[\frac{\not k}{a}\, \hat{\mathcal{M}}^\varepsilon \right] \left(c + O(a) \right) da\,, \tag{5.29}$$

where c is the leading coefficient in (4.40). The square bracket converges in $\mathcal{S}'(\mathbb{R})$ due to Definition 5.6 and (4.34). If we interpret the round brackets as test function, the limit $\varepsilon \searrow 0$ in (5.29) will make sense.

[1] Note the coincidence of the ε-dependent counter-term with that of (2.28).

In (4.9) we split up the integral according to

$$\int_0^\infty = \int_0^1 + \int_1^\infty .$$

The first integral can be written again similar to (5.29) by plugging in $\hat{\mathcal{N}}^\varepsilon$ explicitly and is thereby for the same reason convergent for $\varepsilon \searrow 0$. On the other hand, the second integral is analyzed similar to that in (4.4) since $\hat{\mathcal{N}}^\varepsilon$ and $\hat{\mathcal{M}}^\varepsilon$ coincide for large a because of (4.6) and (4.39). □

Chapter 6

Numerical analysis

The notion of state stability leads us to the speculation that configurations with stable masses correspond to a family of elementary particles with otherwise equal properties. For instance, if we could find an arrangement where the masses $m_1 = 511$ keV, $m_2 = 106$ MeV and $m_3 = 1.78$ GeV are stable we would be able to interpret this as the family of charged leptons e (electron), μ (muon), τ (tauon), respectively. Of course this program is quite ambitious, and maybe it is not so easy because mass renormalization effects have to be taken into account. However, it is interesting to study if there are any stable configurations with one, two or three masses at all or at least approximations to those. The variation density is nonlinear in the masses and weight factors and the appearing integrals cannot be computed analytically. Thus a numerical treatment is necessary.

There are two methods to apply. The first one tries to find a minimum of the action with respect to the masses and weight factors. It will be described in section 6.1. The other one seeks to satisfy the Euler-Lagrange equations via the properties of the variation density explained in section 4.1 and shall be presented in section 6.2.

6.1 Minimizing the action

6.1.1 Basic method

First we have to calculate the function $\hat{\mathcal{M}}$ in the form (4.34). We decompose the sum $J + K$ into its continuous parts

$$(J + K)(a, x, y) = \begin{cases} J^{\text{l}}(a, x, y) & \text{if } (|x| - |y|)^2 \geq a \\ J^{\text{r}}(a, x, y) & \text{if } (|x| - |y|)^2 < a \end{cases} \tag{6.1}$$

Furthermore, define

$$M^{\mathtt{l}}(a,x,y) = a^{-3/2} J^{\mathtt{l}}(a,x,y) \quad (6.2)$$
$$M^{\mathtt{r}}(a,x,y) = a^{-3/2} J^{\mathtt{r}}(a,x,y) \quad (6.3)$$

and

$$dM^{\mathtt{l}}(a,x,y) \equiv \left(\frac{\partial}{\partial y} M^{\mathtt{l}}(a,x,y)\right)_{\text{smooth}} \quad (6.4)$$
$$dM^{\mathtt{r}}(a,x,y) \equiv \frac{\partial}{\partial y} M^{\mathtt{r}}(a,x,y)\,, \quad (6.5)$$

where “smooth” means that we removed the δ-peaks arising from the distributional differentiation of the modulus-term. We define the vectors

$$\boldsymbol{m} \equiv \left(m_1,\ldots,m_g\right) \quad (6.6)$$
$$\boldsymbol{\rho} \equiv \left(\rho_1,\ldots,\rho_g\right) \quad (6.7)$$

and introduce the notation

$$D \equiv \frac{\partial}{\partial \rho_{g+1}} \quad (6.8)$$
$$D_0 \equiv \left.\frac{\partial}{\partial \rho_{g+1}}\right|_{\rho_{g+1}=0} \quad (6.9)$$
$$q \equiv m_{g+1}\,. \quad (6.10)$$

Hence we may write (4.5) in the compact form

$$V = \frac{D_0 \mathcal{S}}{a^{3/2}}\,. \quad (6.11)$$

Now let

$$L^{\sigma\tau}_{ijk}(a) \equiv M^{\sigma}(a,m_i,m_j)\, M^{\tau}(a,m_k,m_l)\, a \quad (6.12)$$

with $i,j,k,l = 1,\ldots,g+1$; $\sigma,\tau = \mathtt{l}$ or $\mathtt{r}$ and $m_l^2 = a$. Then the regular part of the action is equal to

$$\mathcal{S}_{\text{reg}}(\boldsymbol{m},\boldsymbol{\rho},q) = \sum_{ijkl} \rho_i\,\rho_j\,\rho_k\,\rho_l\, s_{ijkl} \quad (6.13)$$

with

$$
s_{ijkl} \equiv \begin{cases}
\int\!\!\!\!\!\!\mathrm{N}_{0}^{\delta a} L^{\mathtt{ll}}_{ijk}(a)\,da + \int\!\!\!\!\!\!\mathrm{N}_{\delta a}^{\delta b} L^{\mathtt{rl}}_{ijk}(b)\,db + \int_{\delta b}^{\infty} L^{\mathtt{rr}}_{ijk}(b)\,db & \text{if } \delta b > \delta a > 0 \\
\int\!\!\!\!\!\!\mathrm{N}_{\delta a}^{\delta b} L^{\mathtt{rl}}_{ijk}(b)\,db + \int_{\delta b}^{\infty} L^{\mathtt{rr}}_{ijk}(b)\,db & \text{if } \delta b > \delta a = 0 \\
\int\!\!\!\!\!\!\mathrm{N}_{0}^{\delta b} L^{\mathtt{ll}}_{ijk}(b)\,db + \int\!\!\!\!\!\!\mathrm{N}_{\delta b}^{\delta a} L^{\mathtt{lr}}_{ijk}(a)\,da + \int_{\delta a}^{\infty} L^{\mathtt{rr}}_{ijk}(a)\,da & \text{if } \delta a > \delta b > 0 \\
\int\!\!\!\!\!\!\mathrm{N}_{\delta b}^{\delta a} L^{\mathtt{lr}}_{ijk}(a)\,da + \int_{\delta a}^{\infty} L^{\mathtt{rr}}_{ijk}(a)\,da & \text{if } \delta a > \delta b = 0\,,
\end{cases} \tag{6.14}
$$

where $\delta a = (m_i - m_j)^2$ and $\delta b = (m_k - m_l)^2$. Here the symbol $\int\!\!\!\!\!\!\mathrm{N}$ indicates that this integral is performed numerically. Note that $L^{\mathtt{rr}}$ is just a polynomial and can thus be analytically integrated. The contribution to the variation density defined in (4.5) is simply

$$
V_{\mathrm{reg}}(\boldsymbol{m}, \boldsymbol{\rho}, q) \;=\; \frac{4}{q^3} \sum_{ijk} \rho_i\, \rho_j\, \rho_k\, s_{ijkl} \Big|_{m_l = q} \tag{6.15}
$$

if we set $q = m_{g+1}$ The additional function F in (4.10) for the extended action of Appendix A is given by

$$
F(\tilde{\mathfrak{m}}_3, \tilde{\mathfrak{m}}_5, \tilde{\mathfrak{r}}_4, \tilde{\mathfrak{r}}_5) \;\equiv\; c_0\, \tilde{\mathfrak{m}}_3 + c_1\, \tilde{\mathfrak{m}}_5 + c_4\, \tilde{\mathfrak{r}}_4 + c_5\, \tilde{\mathfrak{r}}_5 . \tag{6.16}
$$

It contributes to V via the additional summand

$$
V_F(\boldsymbol{m}, \boldsymbol{\rho}, q) \;=\; \frac{1}{q^3} D_0 F(\boldsymbol{m}, \boldsymbol{\rho}, q)\,, \tag{6.17}
$$

where the derivative at $\rho_{g+1} = 0$ is given by

$$
D_0 F \;=\; \frac{\partial F}{\partial \tilde{\mathfrak{m}}_3}\, D_0 \tilde{\mathfrak{m}}_3 + \frac{\partial F}{\partial \tilde{\mathfrak{m}}_5}\, D_0 \tilde{\mathfrak{m}}_5 + \frac{\partial F}{\partial \tilde{\mathfrak{r}}_4}\, D_0 \tilde{\mathfrak{r}}_4 + \frac{\partial F}{\partial \tilde{\mathfrak{r}}_5}\, D_0 \tilde{\mathfrak{r}}_5\,. \tag{6.18}
$$

We set (cf. (2.24), (2.25))

$$\tilde{\mathfrak{m}}_3 = \frac{-1}{32\pi^5} \sum_{i,j=1}^{g+1} \rho_i \rho_j m_i^3 \tag{6.19}$$

$$\tilde{\mathfrak{m}}_5 = \frac{1}{256\pi^5} \sum_{i,j=1}^{g+1} \rho_i \rho_j \left(m_j^5 + m_i m_j^4 - 2 m_i^2 m_j^3 \right) \tag{6.20}$$

$$\tilde{\mathfrak{r}}_3 = \sum_{i=1}^{g+1} \rho_i m_i^4 \tag{6.21}$$

$$\tilde{\mathfrak{r}}_5 = \sum_{i=1}^{g+1} \rho_i m_i^5 \,. \tag{6.22}$$

Note that without the test Dirac sea of mass m_{g+1} we would have

$$\mathfrak{m}_3 = \tilde{\mathfrak{m}}_3 \Big|_{\rho_{g+1}=0} \tag{6.23}$$

$$\mathfrak{m}_5 = \tilde{\mathfrak{m}}_5 \Big|_{\rho_{g+1}=0} \tag{6.24}$$

$$\mathfrak{r}_4 = \tilde{\mathfrak{r}}_4 \Big|_{\rho_{g+1}=0} \tag{6.25}$$

$$\mathfrak{r}_5 = \tilde{\mathfrak{r}}_5 \Big|_{\rho_{g+1}=0} \,. \tag{6.26}$$

Now it just remains to assemble the full variation density,

$$V = V_{\mathrm{reg}} + V_F \,. \tag{6.27}$$

In order to check state stability, we also need to calculate the derivative of V,

$$\begin{aligned} \frac{dV}{dq} &= \frac{dV_{\mathrm{reg}}}{dq} + \frac{dV_F}{dq} \\ &= \frac{3}{q} V_{\mathrm{reg}} + \frac{4}{q^3} \sum_{ijk} \rho_i \rho_j \rho_k \, \frac{d}{dq} \left(s_{ijkl} \Big|_{m_l=q} \right) + \frac{dV_F}{dq} \\ &= \frac{3}{q} V_{\mathrm{reg}} + \frac{4}{q^3} \sum_{ijk} \rho_i \rho_j \rho_k \left(d s_{ijk} \right) + \frac{dV_F}{dq} \,, \end{aligned} \tag{6.28}$$

where

$$ds_{ijk} \equiv \begin{cases} \fint_0^{\delta a} dL^{\mathtt{ll}}_{ijk}(a)\,da + \fint_{\delta a}^{\delta b} dL^{\mathtt{rl}}_{ijk}(b)\,db + \int_{\delta b}^{\infty} dL^{\mathtt{rr}}_{ijk}(b)\,db & \text{if } \delta b > \delta a > 0 \\ \fint_{\delta a}^{\delta b} dL^{\mathtt{rl}}_{ijk}(b)\,db + \int_{\delta b}^{\infty} dL^{\mathtt{rr}}_{ijk}(b)\,db & \text{if } \delta b > \delta a = 0 \\ \fint_0^{\delta b} dL^{\mathtt{ll}}_{ijk}(b)\,db + \fint_{\delta b}^{\delta a} dL^{\mathtt{lr}}_{ijk}(a)\,da + \int_{\delta a}^{\infty} dL^{\mathtt{rr}}_{ijk}(a)\,da & \text{if } \delta a > \delta b > 0 \\ \fint_{\delta b}^{\delta a} dL^{\mathtt{lr}}_{ijk}(a)\,da + \int_{\delta a}^{\infty} dL^{\mathtt{rr}}_{ijk}(a)\,da & \text{if } \delta a > \delta b = 0\,, \end{cases} \tag{6.29}$$

with $\delta a = (m_i - m_j)^2$, $\delta b = (m_k - q)^2$ and

$$dL^{\sigma\tau}_{ijk}(a) \;\equiv\; M^{\sigma}(a, m_i, m_j)\, dM^{\tau}(a, m_k, q)\, a \qquad \text{with } i, j, k, l = 1, \ldots, g+1;\ \sigma, \tau = \mathtt{l} \text{ or } \mathtt{r}\,. \tag{6.30}$$

with $i, j, k, l = 1, \ldots, g+1$; $\sigma, \tau = \mathtt{l}$ or $\mathtt{r}$.

6.1.2 One Sea

We begin our analysis with the case $g = 1$. Without restriction we can set $m_1 = \rho_1 = 1$. The action component away from the light cone has got a maximum at $q = 1$ because $V(q)$ reduces to

$$\int_{\varepsilon}^{(1-|q|)^2} L^{\mathtt{rl}}_{111}(b)\Big|_{m_l=q}\, db + \int_{(1-|q|)^2}^{\infty} L^{\mathtt{rr}}_{111}\Big|_{m_l=q}\, db \;=\; (16 - 64\,\ln|q-1|)\,(q-1)^2 + O((q-1)^3). \tag{6.31}$$

By Taylor expansions of the lightcone contributions it can be shown that $c_1 = 0$ has to hold in order to keep the horizontal tangent[1] at $q = 1$. For $q \to 0$, the regular contribution behaves like $-q^{-3}$, while the c_0-term, which is given by

$$C_0(q) = \frac{c_0 (q-1)^2 (1+q)^3}{256\pi^5\, q^3}, \tag{6.32}$$

has the expansion

$$\frac{c_0}{256\pi^5\, q^3} + O(q^{-2})\,.$$

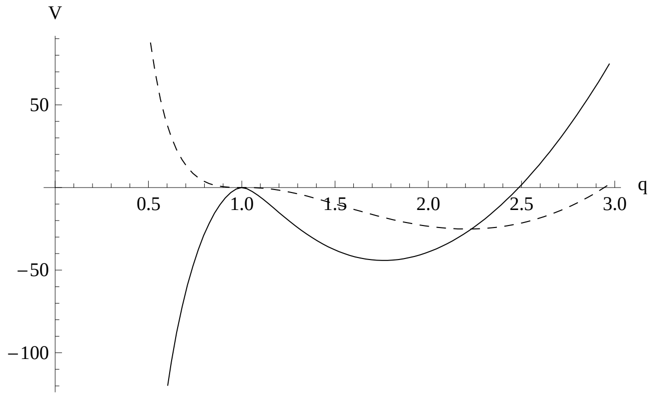
If we let $q \to \infty$ then the regular term will have the asymptotics $\asymp 8q^2 \ln q$ and $C_0(q) \asymp \frac{c_0}{256\pi^5} q^2$. Furthermore, since

$$C_0(q) - C_0(-q) \;=\; \frac{c_0\,(q^2-1)^2}{128\pi^5} \geq 0,$$

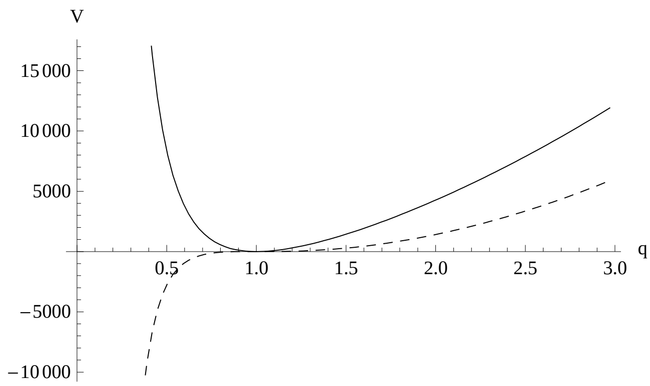
[1]This is no longer true if one uses the extended action of appendix A.

any local and bounded violation of condition **(ii')** in Corollary 4.7 can be cured by a sufficiently large $c_0 > 0$. Unfortunately, the ln-term in (6.31) cannot be neutralized by any choice of c_0 and thus one always has got a maximum at $q = 1$. We refer to this as the **small maximum problem** since with a larger c_0 this effect can be made arbitrarily small. But in summary, we can say that a state-stable configuration does – at least approximately – exist.

Let us show some plots which perhaps make the situation clearer. First we draw $V(q)$ (solid line) and $V(-q)$ (dashed) when $c_0 = 0$:



This is far from state stability: For $q \geq 0$ the function $V(q)$ is not bounded from below and there is a region where $V(q) < V(-q)$ holds. But with an increased value of c_0, e.g. $c_0 = 10^8$, we get the following picture:



Here the conditions of Corollary 4.7 are clearly satisfied except for the small maximum problem.

6.1.3 Two Seas

Because of the nontrivial coupling of two different masses in our model, calculations get more and more complex for increasing g. For that reason, we will set up an algorithm (see Algorithm A in Fig. 6.1) that tries to give the lightcone parameters and the weight factors values in such a way that the configuration is state stable. There are no rigorous proofs any longer so we essentially have to base our argumentation on a few plots.

For instance, we may set $m_1 = 1$, $m_2 = 10$. Without the extension of the action (i.e. $c_4 = c_5 = 0$), we obtain amongst others the following solution:

Figure 6.1:

Algorithm A: Finding Suitable Lightcone Parameters and Weight Factors for $g = 2, 3$

1. Express ρ_g as a function of $m_1, \ldots, m_g$, $\rho_1, \ldots \rho_{g-1}$ and $\mathcal{T}$:

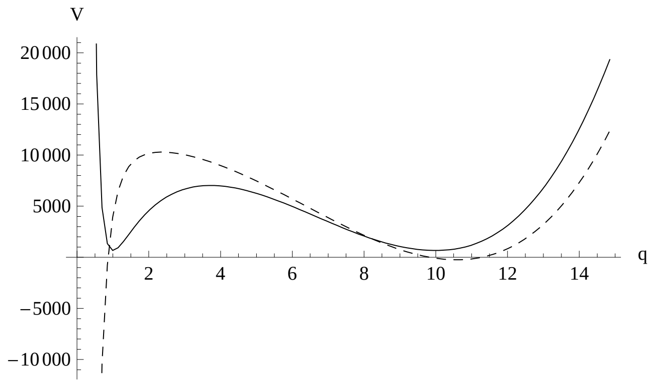
$$\rho_g = \frac{\mathcal{T} - \sum_{i=1}^{g-1} \rho_i \, m_i^3}{m_g^3}$$

2. Calculate $\mathcal{S}$, $\partial_{m_i}\mathcal{S}$ $(i = 1, \ldots, g)$ and $\partial_{\rho_i}\mathcal{S}$ $(i = 1, \ldots, g-1)$.

3. Solve the respective system of equations:

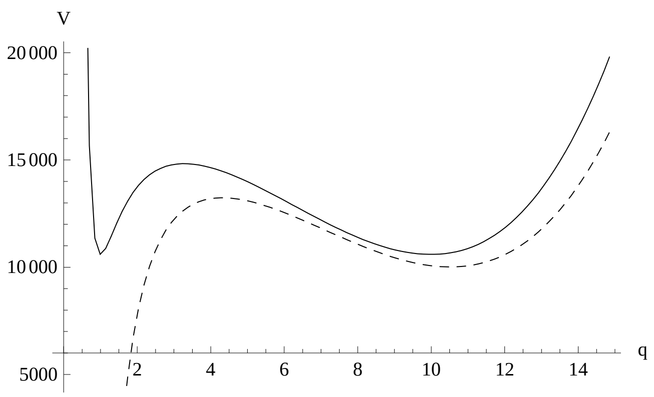
$g = 2$	$g = 3$
$\partial_{m_1}\mathcal{S} = 0$ $\partial_{m_2}\mathcal{S} = 0$	$\partial_{m_1}\mathcal{S} = 0$ $\partial_{m_2}\mathcal{S} = 0$ $\partial_{m_3}\mathcal{S} = 0$ $\partial_{\rho_2}\mathcal{S} = 0$
for $c_0,\ c_1$	for $c_0,\ c_1,\ c_4,\ c_5$.

4. Solve the equation $\partial_{\rho_2}\mathcal{S} = 0$ for real $\mathcal{T}$. In general, solutions will not be unique. Check the plots for these different $\mathcal{T}$'s.



$\rho_1 = 1, \rho_2 = 0.04405, c_0 = -2.91675 \cdot 10^8, c_1 = 2.1995 \cdot 10^9$

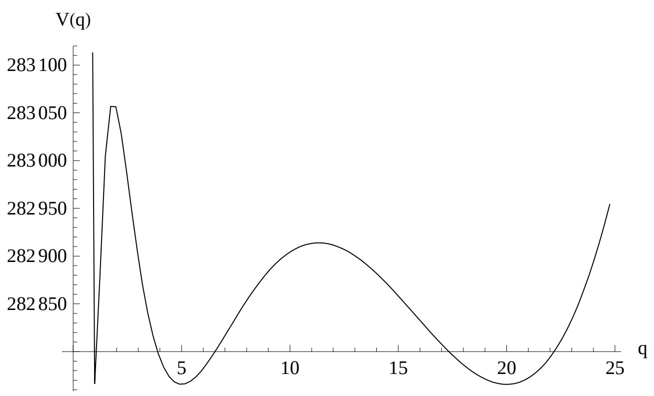
To achieve condition **(ii')**, one may set $c_4 = 100$ and $c_5 = 100$ before applying algorithm A. This leads to the following configuration, which is apparently state-stable:



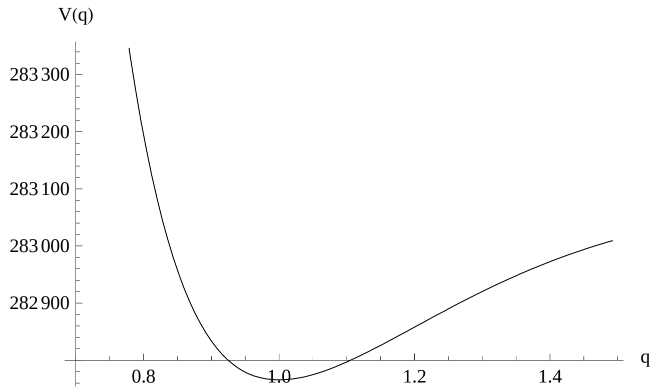
$\rho_1 = 1, \rho_2 = 0.0213623, c_0 = -1.49378 \cdot 10^8, c_1 = 9.69472 \cdot 10^8$

6.1.4 Three Seas

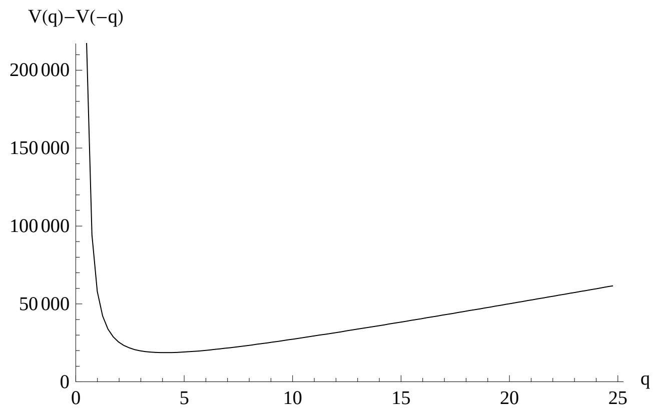
The algorithm A was set up in such a way that it can be used in the case $g = 3$. However, it fixes all light cone parameters. Thus apart from the masses there is no variable left that one can choose in order to satisfy condition **(ii')** of Corollary 4.7. But in spite of this difficulty, state-stable configurations do exist. Here is an example with $m_1 = 1$, $m_2 = 5$, $m_3 = 20$, $\rho_1 = 1$, $\rho_2 = 10^{-4}$, $\rho_3 = 9.69598 \cdot 10^{-6}$, $c_0 = -6.69221 \cdot 10^8$, $c_1 = -2.51578 \cdot 10^9$, $c_4 = 9658.25$, $c_5 = 8416.56$:



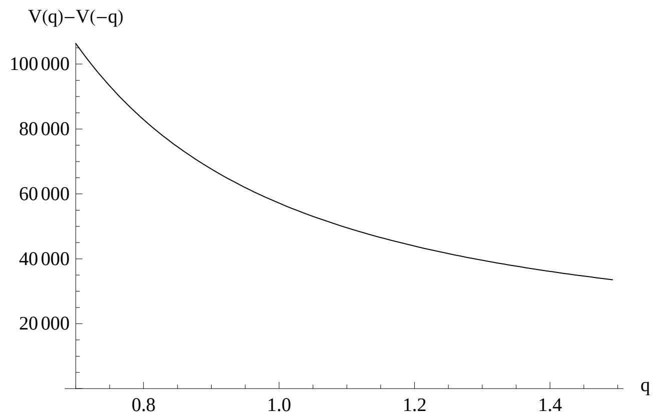
By zooming into the plot we verify the minimum at $q = 1$:



Instead of $V(-q)$ we draw the graph of $V(q) - V(-q)$. This function is obviously positive:



Again we have a closer look at the neighborhood of $q = 1$:



6.2 Variation density method

The preceding paragraphs provided us with a well-working method to obtain solutions. In what follows, we will show the problems that arise when one tries to get solutions per hand. To keep things simple, we set $c_5 = 0$ and do not care about the condition $V(q) \geq V(-q)$. The construction

Figure 6.2:

Algorithm B: Lightcone Parameters and Weight Factors

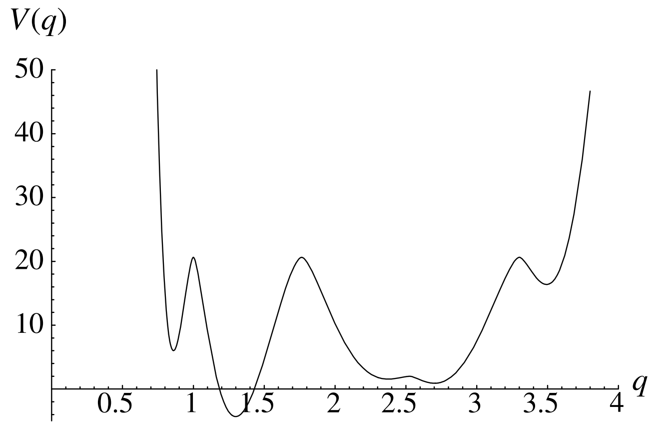
1. Fix the masses $m_1 = \mu$, $m_2 = 1$, $m_3 = M$ for some given μ and M.
2. Solve the equations
$$\left.\frac{dV}{dq}\right|_{q=1} = 0, \qquad \left.\frac{dV}{dq}\right|_{q=M} = 0\,. \tag{6.33}$$
for c_0 and c_1 analytically. This is possible since (6.33) is an inhomogeneous linear system in c_0 and c_1.
3. As far as the weight factors are concerned, set $\rho_2 = 1$ and find ρ_1 and ρ_3 such that the conditions (cf. Theorem 4.4 (a))
$$\left\{ \begin{array}{l} V(m_1) = V(m_2) \\ V(m_1) = V(m_3) \end{array} \right\}$$
are fulfilled.

Algorithm C: Adjusting the mass m_1

1. Choose two starting values μ_0 and μ_1.
2. Set
$$\begin{aligned} G_0 &:= \left.\frac{dV}{dq}\right|_{q=m_1} \qquad \text{where } m_1 = \mu_0 \\ G_1 &:= \left.\frac{dV}{dq}\right|_{q=m_1} \qquad \text{where } m_1 = \mu_1\,. \end{aligned}$$
3. Calculate
$$\mu_{\text{new}} := \mu_0 - \frac{G_0}{G_1 - G_0}\,(\mu_1 - \mu_0)\;.$$
and
$$G_{\text{new}} := \left.\frac{dV}{dq}\right|_{q=m_1} \qquad \text{where } m_1 = \mu_{\text{new}}\,.$$
4. If $|\mu_0 - \mu_{\text{new}}| + |\mu_1 - \mu_{\text{new}}| < \delta$ for some small δ, STOP and $\mu := \mu_{\text{new}}$.
5. Otherwise set
$$\left\{ \begin{array}{l} \mu_0 = \mu_{\text{new}} \\ \mu_1 = \mu_{\text{new}} \end{array} \right\} \text{ if } \left\{ \begin{array}{l} |\mu_0 - \mu_{\text{new}}| \geq |\mu_1 - \mu_{\text{new}}| \\ |\mu_0 - \mu_{\text{new}}| < |\mu_1 - \mu_{\text{new}}| \end{array} \right\}$$
and return to step 2.

of a configuration consists of the steps enumerated in Fig. 6.2B. It is important to mention that Algorithm B does not ensure that dV/dq vanishes at $m_1 = \mu$. In general this will not be the case. But if we interpret $\left.\frac{dV}{dq}\right|_{q=\mu}$ as a function of μ, we will be able to find zeros by using the secant method described in Fig. 6.2C.

It turns out that there are certain difficulties in actally obtaining stable configurations. For instance, our algorithms only look for horizontal tangents of $V(q)$, not necessarily minima, and there is no constraint that forbids the weight factors ρ_i to become negative. We illustrate that by a typical example:

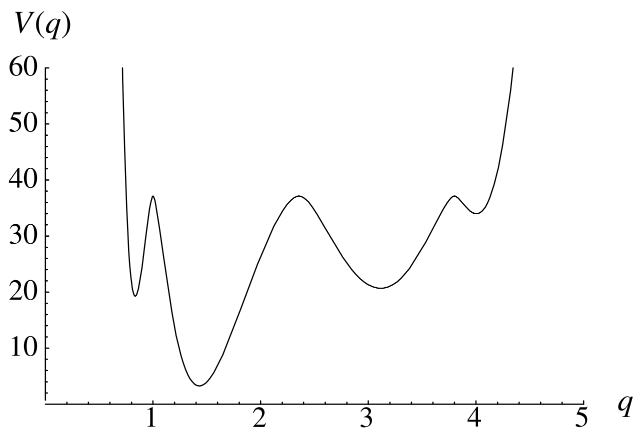


$m_1 = 1.764397,$
$m_2 = 1,$
$m_3 = 3.3,$
$\rho_1 = 0.137872,$
$\rho_2 = 1,$
$\rho_3 = 0.209065,$
$c_0 = 6470798,$
$c_1 = -37715903,$
$c_4 = 0\,.$

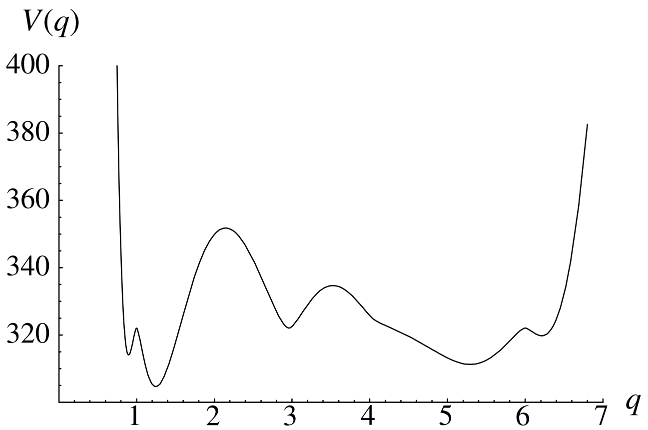
This configuration is not stable as the variation density has got local maxima at the masses.

But there are some possibilities left. First, we may try to vary the masses. This is a delicate issue because $V(q)$ depends on the masses in a highly nonlinear way, which makes it almost impossible to predict what happens then to the shape of $V(q)$. Second, we can use the extended action of appendix A and thus get further adjustable parameters.

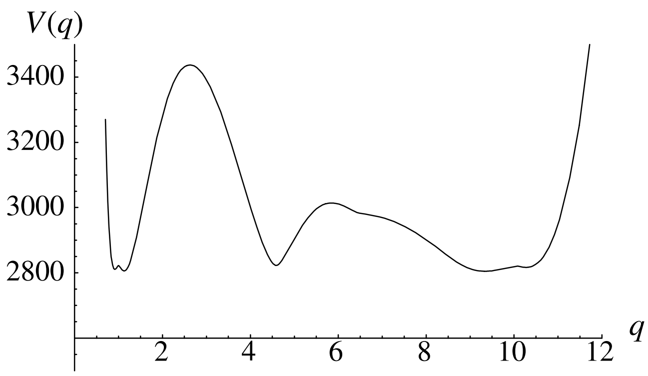
We will now attempt to obtain state stability by increasing the mass m_3 and choosing m_1 such that $V'(m_1) = 0$. The following plots represent $m_3 = 3.8,\ 6,\ 10,\ 15$, respectively.



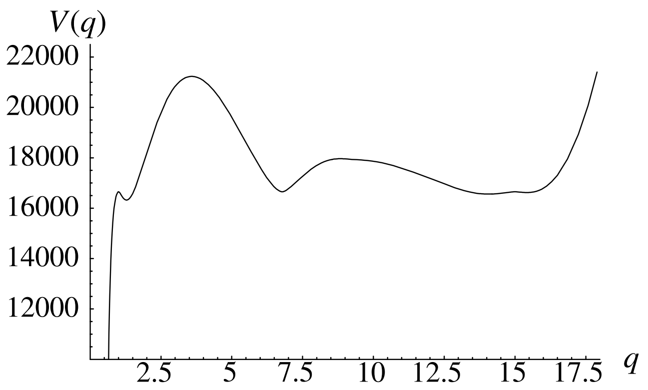
$m_1 = 2.354406,$
$m_2 = 1,$
$m_3 = 3.8,$
$\rho_1 = 0.0315924,$
$\rho_2 = 1,$
$\rho_3 = 0.153781,$
$c_0 = 17760248,$
$c_1 = 41096122,$
$c_4 = 0\,.$



$$m_1 = 2.959266,\quad m_2 = 1,\quad m_3 = 6,\quad \rho_1 = -0.0745943,\quad \rho_2 = 1,\quad \rho_3 = 0.0862904,\quad c_0 = 210974143,\quad c_1 = -88749542,\quad c_4 = 0\,.$$

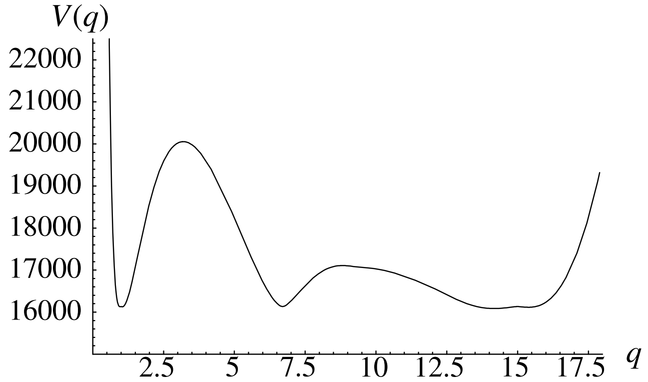


$$m_1 = 4.587525,\quad m_2 = 1,\quad m_3 = 10,\quad \rho_1 = -0.137099,\quad \rho_2 = 1,\quad \rho_3 = 0.0554986,\quad c_0 = 2.472462 \cdot 10^9,\quad c_1 = -2.553788 \cdot 10^8,\quad c_4 = 0\,.$$



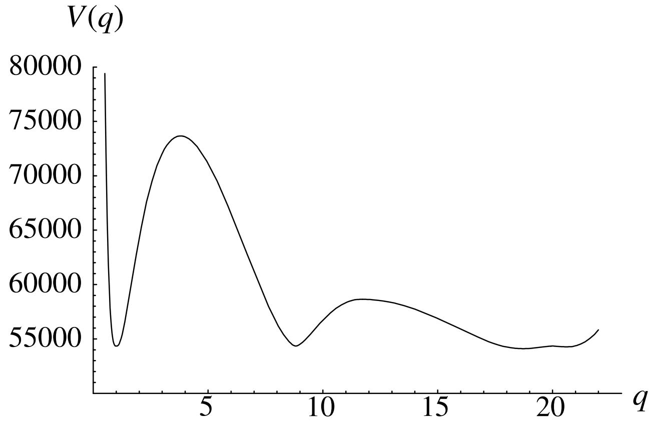
$$m_1 = 6.780356,\quad m_2 = 1,\quad m_3 = 15,\quad \rho_1 = -0.152159,\quad \rho_2 = 1,\quad \rho_3 = 0.0479747,\quad c_0 = 1.959969 \cdot 10^{10},\quad c_1 = -7.773376 \cdot 10^8,\quad c_4 = 0\,.$$

The last plot indicates that $V(q)$ is not bounded from below as $q \searrow 0$. This is where the extended action and its additional parameters come into play. Namely, we may assign a nonzero value to c_4. By (6.17) this will add a constant to the first derivative of V and thus shift the horizontal tangents to other values. In order to repair this, the other light cone parameters and masses have to be readjusted. Their contributions then change their singular behavior at $q \searrow 0$. The next plot, in which we set $c_4 = 400$, demonstrates that.



$$\begin{aligned} m_1 &= 6.702723, \\ m_2 &= 1, \\ m_3 &= 15, \\ \rho_1 &= -0.143466, \\ \rho_2 &= 1, \\ \rho_3 &= 0.0380002, \\ c_0 &= 1.565925 \cdot 10^{10}, \\ c_1 &= 5.856810 \cdot 10^{8}, \\ c_4 &= 400\,. \end{aligned}$$

With this method we can even try $m_3 = 20$:

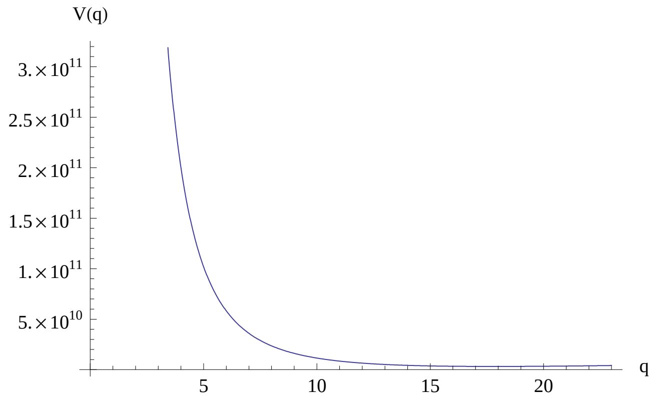


$$\begin{aligned} m_1 &= 8.822033, \\ m_2 &= 1, \\ m_3 &= 20, \\ \rho_1 &= -0.156397, \\ \rho_2 &= 1, \\ \rho_3 &= 0.0367561, \\ c_0 &= 7.100854 \cdot 10^{10}, \\ c_1 &= -1.385740 \cdot 10^{9}, \\ c_4 &= 1000\,. \end{aligned}$$

As already mentioned, the result is not satisfying because ρ_1 is always negative and it is not clear what this should physically mean.

6.3 Further remarks

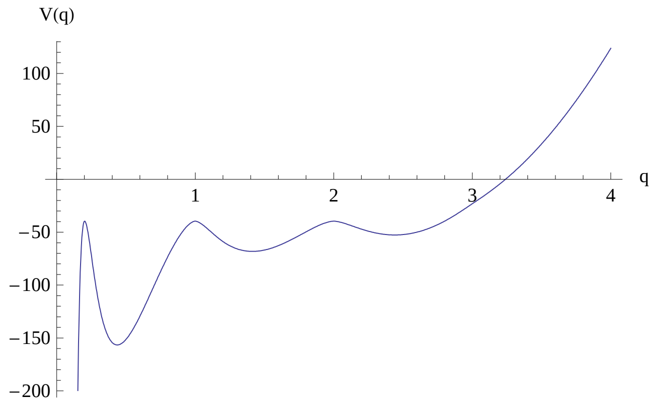
The plots drawn here only cover a small number of possible shapes of variation densities. For arbitrary values of the weights and light cone variables c_0, c_1, c_4, c_5, one does not even get critical points at the masses:



$$m_1 = 10.7939,$$
$$m_2 = 1,$$
$$m_3 = 20,$$
$$\rho_1 = \rho_2 = \rho_3 = 1,$$
$$c_0 = c_1 = c_4 = c_5 = 0.$$

This implies that the existence of stable configurations is deeply connected to what happens on the light cone and the density of states in a certain sea.[2]

Even when applying the method of section 6.2 it is possible that there exist variation densities which are not bounded from below.



$$m_1 = 0.201665,$$
$$m_2 = 1,$$
$$m_3 = 2,$$
$$\rho_1 = -0.867637$$
$$\rho_2 = 1,$$
$$\rho_3 = 0.253132,$$
$$c_0 = -351960,$$
$$c_1 = 653013,$$
$$c_4 = c_5 = 0.$$

We already considered such a case in section 6.2. There we had to modify the light cone variables in order to stabilize the system.

[2]This shows that the discrete spacetime structure may have drastic consequences for physics in its experimentally accessible range.

Chapter 7

Conclusion

The principle of the fermionic projector in the continuum gives an indication that there might be a deeper reason why elementary particles only appear with a few definite masses. Even though we could not obtain physical relevant mass ratios, we showed the approximate existence of state-stable configurations. In order to achieve that, we made use of certain contributions supported on the light cone. In doing so, there seems to be some arbitrariness here. However, in some sense these parameters contain the structure of the underlying discrete spacetime.

We studied Lorentz invariant distributions and their convolutions. Some of these are well-defined because the convolution integrals have compactly supported integrands. Other convolutions can be regularized such that the property of being ill-defined only plays a role on the light cone.

These results were used to build a variational principle and to give criteria for state stability, which could be numerically analyzed. Some plots were presented to decide about state stability and to show how possible configurations could look like.

Appendix A

The extended action

In chapter 6, we repeatedly made use of a more general action principle than that of Definition 2.5. Employing a different regularization scheme one can show [Fin06a] that the distribution Q in (4.23) can be replaced by

$$Q(\xi) \;=\; \frac{1}{2}\,\tilde{\mathcal{M}}(\xi)\,P(\xi) + c_2\,\delta^4(\xi) - c_4\,i\partial\!\!\!/\,\delta^4(\xi) - c_5\Box\,\delta^4(\xi) \tag{A.1}$$

with arbitrary real parameters c_2, c_4 and c_5. In the definition of state stability, Definition 4.3 the functions a and b have to be replaced by

$$a(k^2) \longrightarrow a(k^2) + c_4\,|k|, \qquad b(k^2) \longrightarrow b(k^2) + c_2 + c_5\,k^2\;. \tag{A.2}$$

The parameter c_2 is just an additive constant and hence does not contribute to the variation of the action. Repeating the calculation in Theorem 4.4, one may conclude that the action can be supplemented as follows:

Definition A.1 The **extended action** $\mathcal{S}_{\text{ext}}$ is defined by

$$\mathcal{S}_{\text{ext}} \;\equiv\; \mathcal{S} + c_4 \sum_{\beta=1}^{g} \rho_\beta\, m_\beta^4 + c_5 \sum_{\beta=1}^{g} \rho_\beta\, m_\beta^5$$

with $\mathcal{S}$ as in (2.20) and the free parameters $c_4,\, c_5 \in \mathbb{R}$. The corresponding variational principle under the constraint (2.29) is called the **extended variational principle**.

Appendix B

Code listings

In this appendix we present the Mathematica™ code that we used to obtain the plots.

Code for sections 6.1.2, 6.1.3 and 6.1.4

Basic definitions:

```
Δ[a_, b_, c_] = a^2 + b^2 + c^2 - 2 (a b + b c + a c);
J[q_, x_, y_] = -√Δ[q^2, x^2, y^2] (x - y) ((x + y)^2 - q^2) Sign[Abs[x] - Abs[y]]
    UnitStep[(Abs[x] - Abs[y])^2 - q^2] + (x + y) ((x^2 - y^2)^2 - 2 q^2 (x^2 - x y + y^2));
MI[q_, x_, y_] = (q J[q, x, y]) / q^4;
J1[q_, x_, y_] = -√Δ[q^2, x^2, y^2] (x - y) ((x + y)^2 - q^2) Sign[Abs[x] - Abs[y]] +
   (x + y) ((x^2 - y^2)^2 - 2 q^2 (x^2 - x y + y^2));
J2[q_, x_, y_] = (x + y) ((x^2 - y^2)^2 - 2 q^2 (x^2 - x y + y^2));
M1[q_, x_, y_] = q / q^4 J1[q, x, y];
M2[q_, x_, y_] = q / q^4 J2[q, x, y];
Zero[m_] = 0;
qrep = {q^2 → a, 1/q^2 → a^-1, 1/q^4 → a^-2, q^4 → a^2};
L11[m1_, m2_, m3_, m4_, a_] = M1[q, m1, m2] M1[q, m3, m4] q^2 /. qrep;
L21[m1_, m2_, m3_, m4_, a_] = M2[q, m1, m2] M1[q, m3, m4] q^2 /. qrep;
L12[m1_, m2_, m3_, m4_, a_] = M1[q, m1, m2] M2[q, m3, m4] q^2 /. qrep;
L22[m1_, m2_, m3_, m4_, a_] = M2[q, m1, m2] M2[q, m3, m4] q^2 /. qrep;
dL11[m1_, m2_, m3_, m4_, a_] = ∂m4 L11[m1, m2, m3, m4, a] /. {Abs' → Sign, Sign' → Zero};
dL12[m1_, m2_, m3_, m4_, a_] = ∂m4 L12[m1, m2, m3, m4, a] /. {Abs' → Sign, Sign' → Zero};
dL21[m1_, m2_, m3_, m4_, a_] = ∂m4 L21[m1, m2, m3, m4, a] /. {Abs' → Sign, Sign' → Zero};

S11[m1_, m2_, a_] = Integrate[L11[m1, m2, m1, m2, a], a];
S22[m1_, m2_, m3_, m4_, a_] = Integrate[L22[m1, m2, m3, m4, a], a];
S21[m1_, m2_, m3_, m4_, a_] = Integrate[L21[m1, m2, m3, m4, a], a];
S12[m1_, m2_, m3_, m4_, a_] = Integrate[L12[m1, m2, m3, m4, a], a];

dS11[m1_, m2_, a_] = 1/2 ∂m2 S11[m1, m2, a] /. {Abs' → Sign, Sign' → Zero};
dS22[m1_, m2_, m3_, m4_, a_] = ∂m4 S22[m1, m2, m3, m4, a] /. {Abs' → Sign, Sign' → Zero};
dS21[m1_, m2_, m3_, m4_, a_] = ∂m4 S21[m1, m2, m3, m4, a] /. {Abs' → Sign, Sign' → Zero};
dS12[m1_, m2_, m3_, m4_, a_] = ∂m4 S12[m1, m2, m3, m4, a] /. {Abs' → Sign, Sign' → Zero};
```

```
gen = 3;
μ3tot = 0; μ5tot = 0; μ3 =.; μ5 =.;
r4tot = 0; r5tot = 0;
genrep = {ρ[gen + 1] → 0, m[gen + 1] → mvar};
For[i = 1, i ≤ gen + 1, i++,
  For[j = 1, j ≤ gen + 1, j++,
   μ3tot += -1/(32 π^5) ρ[i] (ρ[j] m[j]^3);
   μ5tot += 1/(256 π^5) ρ[i] ρ[j] (m[j]^5 + m[i] m[j]^4 - 2 m[i]^2 m[j]^3);
  ];
  r4tot += ρ[i] m[i]^4;
  r5tot += ρ[i] m[i]^5];
μ3rep = μ3 → Simplify[μ3tot /. genrep];
μ5rep = μ5 → Simplify[μ5tot /. genrep];
r4rep = r4 → Simplify[r4tot /. genrep];
r5rep = r5 → Simplify[r5tot /. genrep];

Dμ3 = Simplify[D[μ3tot, ρ[gen + 1]] /. genrep];
Dμ5 = Simplify[D[μ5tot, ρ[gen + 1]] /. genrep];
Dr4 = Simplify[D[r4tot, ρ[gen + 1]] /. genrep];
Dr5 = Simplify[D[r5tot, ρ[gen + 1]] /. genrep];
dρμ3 = {}; dρμ5 = {}; dρr4 = {}; dρr5 = {};
dmμ3 = {}; dmμ5 = {}; dmr4 = {}; dmr5 = {};
F[μ3_, μ5_, r4_, r5_] = c1 μ3 + c0 μ5 + c4 r4 + c5 r5;
For[i = 1, i ≤ gen, i++,
 dρμ3 = Join[dρμ3, {D[μ3 /. μ3rep, ρ[i]]}];
 dρμ5 = Join[dρμ5, {D[μ5 /. μ5rep, ρ[i]]}];
 dρr4 = Join[dρr4, {D[r4 /. r4rep, ρ[i]]}];
 dρr5 = Join[dρr5, {D[r5 /. r5rep, ρ[i]]}];

 dmμ3 = Join[dmμ3, {D[μ3 /. μ3rep, m[i]]}];
 dmμ5 = Join[dmμ5, {D[μ5 /. μ5rep, m[i]]}];
 dmr4 = Join[dmr4, {D[r4 /. r4rep, m[i]]}];
 dmr5 = Join[dmr5, {D[r5 /. r5rep, m[i]]}];
]
Faction[m_, ρ_] =
  {F[μ3, μ5, r4, r5], D[F[μ3, μ5, r4, r5], μ3] dmμ3 + D[F[μ3, μ5, r4, r5], μ5] dmμ5 +
     D[F[μ3, μ5, r4, r5], r4] dmr4 + D[F[μ3, μ5, r4, r5], r5] dmr5,
    D[F[μ3, μ5, r4, r5], μ3] dρμ3 + D[F[μ3, μ5, r4, r5], μ5] dρμ5 + D[F[μ3, μ5, r4, r5], r4]
      dρr4 + D[F[μ3, μ5, r4, r5], r5] dρr5} /. {μ3rep, μ5rep, r4rep, r5rep};
DFaction[m_, ρ_, mvar_] = (D[F[μ3, μ5, r4, r5], μ3] Dμ3 +
     D[F[μ3, μ5, r4, r5], μ5] Dμ5 + D[F[μ3, μ5, r4, r5], r4] Dr4 +
     D[F[μ3, μ5, r4, r5], r5] Dr5) /. {μ3rep, μ5rep, r4rep, r5rep};
```

This routine calculates the action with function F and its gradients of the action with respect to the masses and weight factors:

```
action[m_, ρ_] := (
  Module[{m1, m2, m3, m4},
   Stot = 0; dρStot = {}; dmStot = {};
   For[n4 = 1, n4 ≤ gen,
    dρS = 0; dmS = 0;
    For[n1 = 1, n1 ≤ gen,
     For[n2 = n1, n2 ≤ gen,
       For[n3 = 1, n3 ≤ gen,
         S = 0; dS = 0;
         combi = 1; If[n2 > n1, combi *= 2];
         m1 = m[n1]; m2 = m[n2]; m3 = m[n3]; m4 = m[n4];
         δa = (m2 - m1)^2; δb = (m4 - m3)^2;
         If[δb > δa,
          (* first case *)
          If[δa > 0,
           S += Re[NIntegrate[L11[m1, m2, m3, m4, a], {a, 0, δa}, AccuracyGoal → 4]];
           dS += Re[NIntegrate[dL11[m1, m2, m3, m4, a], {a, 0, δa - amin}, AccuracyGoal → 4]];
          ];
          If[δb > 0,
           S +=
            Re[NIntegrate[L21[m1, m2, m3, m4, b], {b, δa, δb}] - N[S22[m1, m2, m3, m4, δb]]];
           dS += Re[NIntegrate[dL21[m1, m2, m3, m4, b], {b, δa, δb - amin}] - N[
               dS22[m1, m2, m3, m4, δb]]];
          ],
          (* second case *)
          If[δb > 0,
           S += Re[NIntegrate[L11[m1, m2, m3, m4, a], {a, 0, δb}, AccuracyGoal → 4]];
           dS +=
            Re[NIntegrate[dL11[m1, m2, m3, m4, a], {a, amin, δb - amin}, AccuracyGoal → 4]];
          ];
          If[δa > 0,
           S +=
            Re[NIntegrate[L12[m1, m2, m3, m4, b], {b, δb, δa}] - N[S22[m1, m2, m3, m4, δa]]];
           dS += Re[NIntegrate[dL12[m1, m2, m3, m4, b], {b, δb, δa - amin}] - N[
               dS22[m1, m2, m3, m4, δa]]]];
         ];
         dρS += combi ρ[n1] ρ[n2] ρ[n3] S;
         dmS += combi ρ[n1] ρ[n2] ρ[n3] ρ[n4] dS;
         n3++] n2++] n1++];
    Stot += ρ[n4] dρS;
    dρStot = Join[dρStot, {4 dρS}];
    dmStot = Join[dmStot, {4 dmS}];
    n4++];];
  {Stot, dmStot, dρStot} + Faction[m, ρ]);
```

This subprogram computes $q^3\, V_{\mathrm{reg}}(q)$:

```
taction[m_, ρ_, m4_] := (
  Module[{m1, m2, m3},
   ρS = 0;
   For[n1 = 1, n1 ≤ gen,
    For[n2 = n1, n2 ≤ gen,
      For[n3 = 1, n3 ≤ gen,
        S = 0;
        combi = 1;
        If[n2 > n1, combi *= 2];
        m1 = m[n1]; m2 = m[n2]; m3 = m[n3];
        δa = (Abs[m2] - Abs[m1])^2; δb = (Abs[m4] - Abs[m3])^2;
        If[δb > δa,
         (* first case *)
         If[δa > 0,
          S += Re[NIntegrate[L11[m1, m2, m3, m4, a], {a, 0, δa}, AccuracyGoal → 4]]];
         If[δb > 0,
          S +=
           Re[NIntegrate[L21[m1, m2, m3, m4, b], {b, δa, δb}] - N[S22[m1, m2, m3, m4, δb]]]],
         (* second case *)
         If[δb > 0,
          S += Re[NIntegrate[L11[m1, m2, m3, m4, a], {a, 0, δb}, AccuracyGoal → 4]]];
         If[δa > 0,
          S +=
           Re[NIntegrate[L12[m1, m2, m3, m4, b], {b, δb, δa}] - N[S22[m1, m2, m3, m4, δa]]]]];
        ρS += combi ρ[n1] ρ[n2] ρ[n3] S;
        n3++] n2++] n1++];];
  ρS);
```

Next we obtain the variation density (T $\hat{=}$ V):

```
T[m_, ρ_, m4_] := ((4 taction[m, ρ, m4] + DFaction[m, ρ, m4]) / m4^3);
DT[m_, ρ_, m4_] := ((T[m, ρ, m4 + Δq] - T[m, ρ, m4]) / Δq);
```

With this function we can plot the positive and negative part of V:

```
PlotV[mmin_, mmax_, steps_] := (mstep = (mmax - mmin)/steps;
  li = {};
  mli = {};
  For[m4 = mmin, m4 < mmax,
   li = Join[li, {{m4, T[m, ρ, m4]}}];
   mli = Join[mli, {{m4, T[m, ρ, -m4]}}];
   m4 += mstep];
  ListLinePlot[{li, mli}, PlotStyle → {Black, {Black, Dashed}}, AxesLabel → {"q", "V"}]);
```

The constraint (2.29) is built in as follows:

```
Clear[m]; NC =.;
neben = 0;
For[i = 1, i ≤ gen,
  neben += ρ[i] m[i]^3;
   i++];
ρgenrep = Solve[neben == NC, ρ[gen]][[1]];
ρgen[m_, ρ_] = ρ[gen] /. ρgenrep;

dρgen = {};
For[i = 1, i ≤ gen,
  dρgen = Join[dρgen, {Simplify[D[ρgen[m, ρ], m[i]]]}];
   i++];
For[i = 1, i ≤ gen - 1,
  dρgen = Join[dρgen, {Simplify[D[ρgen[m, ρ], ρ[i]]]}];
   i++];
```

The parameters are initialized:

```
amin = 10^-5;
c0 =.; c1 =.; c4 =.; c5 =.;
{ρ[1], ρ[2], ρ[3]} = {1, 0.1, 1};
{m[1], m[2], m[3]} = {1, 5, 20}; NC = .; ρ[gen] = ρgen[m, ρ];
```

This is algorithm A for three seas:

```
A = action[m, ρ];
act = A[[1]];
grad = Simplify[Join[A[[2]], Drop[A[[3]], -1]] + A[[3, gen]] dρgen];
crep =
  Solve[{grad[[1]] == 0, grad[[2]] == 0, grad[[3]] == 0, grad[[5]] == 0}, {c0, c1, c4, c5}][[1]];
{c0, c1, c4, c5} = {c0, c1, c4, c5} /. crep;
NClist = Solve[grad[[4]] == 0, NC];
NCfinal = {}; lNC = Length[NClist];
For[i = 1, i ≤ lNC,
  ncsol = NC /. NClist[[i]];
  If[ncsol == Re[ncsol] && ncsol > amin , NCfinal = Join[NCfinal, {ncsol}]];
  i++];
NCfinal
```

For $g = 2$ this has to be changed to

```
A = action[m, ρ];
act = A[[1]];
grad = Simplify[Join[A[[2]], Drop[A[[3]], -1]] + A[[3, gen]] dρgen];
crep = Solve[{grad[[1]] == 0, grad[[2]] == 0}, {c0, c1}][[1]];
{c0, c1} = {c0, c1} /. crep;
NClist = Solve[grad[[3]] == 0, NC];
NCfinal = {}; lNC = Length[NClist];
For[i = 1, i ≤ lNC,
  ncsol = NC /. NClist[[i]];
  If[ncsol == Re[ncsol] && ncsol > amin , NCfinal = Join[NCfinal, {ncsol}]];
  i++];
NCfinal
```

Code for section 6.2

Here we use the notations

$$\begin{aligned} \alpha &\hat{=} c_0 \\ \beta &\hat{=} c_1 \\ \gamma &\hat{=} c_4 \,. \end{aligned}$$

As before we have got some basic definitions:

```
Δ[a_, b_, c_] = a^2 + b^2 + c^2 - 2 (a b + b c + a c);
K[a_, x_, y_] = 1/a (-Sqrt[Δ[a, x^2, y^2]] (x - y) ((x + y)^2 - a) Sign[x - y] UnitStep[(x - y)^2 - a] +
      (x + y) ((x^2 - y^2)^2 - 2 a (x^2 - x y + y^2)));
Kl[a_, x_, y_] = K[a, x, y] /. UnitStep[(x - y)^2 - a] → 1;
Kr[a_, x_, y_] = K[a, x, y] /. UnitStep[(x - y)^2 - a] → 0;
dKl[a_, x_, y_] = Simplify[D[Kl[a, x, y] , y] /. Sign'[x - y] → 0];
dKr[a_, x_, y_] = D[Kr[a, x, y] , y];
Srr[m1_, m2_, m3_, q_, a_] = Simplify[4/q^3 Integrate[Kr[a, m1, m2] Kr[a, m3, q], a]];
dSrr[m1_, m2_, m3_, q_, a_] = Simplify[4/q^3 Integrate[Kr[a, m1, m2] dKr[a, m3, q], a]];
```

```
gen = 3;
μ3tot = 0; μ5tot = 0; μ3 =.; μ5 =.;
genrep = {ρ[gen + 1] → 0, m[gen + 1] → mvar};
For[i = 1, i ≤ gen + 1, i++,
  For[j = 1, j ≤ gen + 1, j++,
    μ3tot += -1/(32 π^5) ρ[i] (ρ[j] m[j]^3);
    μ5tot += 1/(256 π^5) ρ[i] ρ[j] (m[j]^5 + m[i] m[j]^4 - 2 m[i]^2 m[j]^3);
  ]];
μ3rep = μ3 → Simplify[μ3tot /. genrep]
μ5rep = μ5 → Simplify[μ5tot /. genrep]
Dμ3 = Simplify[D[μ3tot, ρ[gen + 1]] /. genrep];
Dμ5 = Simplify[D[μ5tot, ρ[gen + 1]] /. genrep];
```

```
dρμ3 = {}; dρμ5 = {};
dmμ3 = {}; dmμ5 = {}; F[μ3_, μ5_] =.;
For[i = 1, i ≤ gen, i++,
 dρμ3 = Join[dρμ3, {D[μ3 /. μ3rep, ρ[i]]}];
 dρμ5 = Join[dρμ5, {D[μ5 /. μ5rep, ρ[i]]}];
 dmμ3 = Join[dmμ3, {D[μ3 /. μ3rep, m[i]]}];
 dmμ5 = Join[dmμ5, {D[μ5 /. μ5rep, m[i]]}];
]
F[μ3_, μ5_] = α μ3 + β μ5;
DFaction[m_, ρ_, mvar_] = (D[F[μ3, μ5], μ3] Dμ3 + D[F[μ3, μ5], μ5] Dμ5) /. {μ3rep, μ5rep}
T[m_, ρ_, q_] = 1/q^3 DFaction[m, ρ, q] + γ q;

S[m_, ρ_, q_] := (

   Module[{S, Stot, n1, n2, n3, m1, m2, m3, δa, δb},
    Stot = 0;
    For[n1 = 1, n1 ≤ gen,
     For[n2 = n1, n2 ≤ gen,
       For[n3 = 1, n3 ≤ gen,
         combi = 1;
         If[n2 > n1, combi *= 2];
         S = 0;
         m1 = m[n1]; m2 = m[n2]; m3 = m[n3];
         δa = (m2 - m1)^2; δb = (q - m3)^2;
         If[δb > δa,
          (* first case *)
          If[δa > 0,
           S += 4/q^3 NIntegrate[Kl[a, m1, m2] Kl[a, m3, q], {a, 0, δa}, AccuracyGoal → 4]];
          If[δb > 0, S += 4/q^3 NIntegrate[Kr[a, m1, m2] Kl[a, m3, q], {a,
                δa, δb}, AccuracyGoal → 4] - N[Srr[m1, m2, m3, q, δb]]],
          (* second case *)
          If[δb > 0,
           S += 4/q^3 NIntegrate[Kl[a, m1, m2] Kl[a, m3, q], {a, 0, δb}, AccuracyGoal → 4]];
          If[δa > 0,
           S += 4/q^3 NIntegrate[Kl[a, m1, m2] Kr[a, m3, q], {a, δb, δa}, AccuracyGoal → 4] -
             N[Srr[m1, m2, m3, q, δa]]]];
         Stot += combi ρ[n1] ρ[n2] ρ[n3] S;
         n3++] n2++] n1++];
    Stot]);

ST[m_, ρ_, q_] := S[m, ρ, q] + T[m, ρ, q];
```

```
dS[m_, ρ_, q_] := (
  Module[{Si, Stot, n1, n2, n3, m1, m2, m3, δa, δb},
   Stot = 0;
   For[n1 = 1, n1 ≤ gen,
    For[n2 = n1, n2 ≤ gen,
      For[n3 = 1, n3 ≤ gen,
        combi = 1;
        If[n2 > n1, combi *= 2];
        Si = 0;
        m1 = m[n1]; m2 = m[n2]; m3 = m[n3];
        δa = (m2 - m1)^2; δb = (q - m3)^2;
        If[δb > δa,
         (* first case *)
         If[δa > 0,
          Si += 4/q^3 NIntegrate[Kl[a, m1, m2] dKl[a, m3, q], {a, 0, δa}, AccuracyGoal → 4]];
         If[δb > 0,
          Si += 4/q^3 NIntegrate[Kr[a, m1, m2] dKl[a, m3, q], {a, δa, δb}, AccuracyGoal → 4] -
            N[dSrr[m1, m2, m3, q, δb]]],
         (* second case *)
         If[δb > 0,
          Si += 4/q^3 NIntegrate[Kl[a, m1, m2] dKl[a, m3, q], {a, 0, δb}, AccuracyGoal → 4]];
         If[δa > 0,
          Si += 4/q^3 NIntegrate[Kl[a, m1, m2] dKr[a, m3, q], {a, δb, δa}, AccuracyGoal → 4] -
            N[dSrr[m1, m2, m3, q, δa]]]];
        Stot += combi ρ[n1] ρ[n2] ρ[n3] Si;
        n3++] n2++] n1++];
   Stot - 3/q S[m, ρ, q]]);
```

This is the implementation of Algorithm B:

```
ϵ =.; δ =.;  α =.; β =.; γ =.; M =.; μ =.;
ϵ0 =.; δ0 =.; γ0 =.;
m[1] = μ; m[2] = 1; m[3] = M;
ρ[1] = ϵ; ρ[2] = 1;  ρ[3] = δ;

G[μ_, M_] := (Module[{},
   m[1] = μ; m[2] = 1; m[3] = M;
   ϵ =.; δ =.;  α =.; β =.;
   αβrep = Solve[{ dS[m, ρ, 1] + (D[T[m, ρ, q], q] /. q → 1) == 0,
        dS[m, ρ, M] + (D[T[m, ρ, q], q] /. q → M) == 0}, {α, β}][[1]];
   α = α /. αβrep; β = β /. αβrep;
   C1 = Simplify[dS[m, ρ, μ] + (D[T[m, ρ, q], q] /. q → μ)];
   C2 = Simplify[ST[m, ρ, μ] - ST[m, ρ, 1]];
   C3 = Simplify[ST[m, ρ, M] - ST[m, ρ, μ]];
   p1 =  Numerator[Together[C1]];
   p2 =  Numerator[Together[C2]];
   p3 =  Numerator[Together[C3]];
   ϵδrep = FindRoot[{p2, p3}, {{ϵ, ϵ0}, {δ, δ0}}];
   ϵ = ϵ /. ϵδrep; δ = δ /. ϵδrep;
   ϵ0 = ϵ; δ0 = δ;
   C1])
```

And this part of code represents Algorithm C:

```
nest[μ0start_, μ1start_, M_] := Module[{μ0 = μ0start, μ1 = μ1start}, μerr = 0.001;
  For[n = 0, n < 10, n++;
   G0 = Re[G[μ0, M]];
   G1 = Re[G[μ1, M]];
                 G0
   μn = μ0 - ───────── (μ1 - μ0);
              G1 - G0
   Gn = Re[G[μn, M]];
   If[Abs[μ0 - μn] + Abs[μ1 - μn] < μerr, Break[]];
   If[Abs[μ0 - μn] < Abs[μ1 - μn], μ1 = μn, μ0 = μn];
   Print["new interval: ", {μ0, μ1}, "   value = ", Gn];
  ];
  μ = μn]
```

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