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Sandra Wieland und Harald Garcke

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SURFACTANT SPREADING ON THIN VISCOUS FILMS: NONNEGATIVE SOLUTIONS OF A COUPLED DEGENERATE SYSTEM

SANDRA WIELAND* AND HARALD GARCKE†

Abstract. We consider the Navier-Stokes system for an incompressible fluid coupled with a convection-diffusion equation for surfactant molecules on the free surface. The lubrication approximation leads to a coupled system of parabolic equations, consisting of a degenerate fourth order equation for the film height and a second order equation for the surfactant concentration. A proof based on energy estimates shows the existence of global weak solutions which in addition fulfill an integral inequality (entropy inequality) which ensures positivity properties for the solution.

Key words. Partial differential equations, degenerate parabolic equation, thin liquid film, surfactant spreading, free surface, fluid interface.

AMS subject classifications. 35K55, 35K65, 35K35, 76A20, 76D08

1. Introduction. The aim of this paper is to prove the existence of nonnegative, weak solutions of the following system of coupled nonlinear, degenerate parabolic partial differential equations

$$h_t + \left(\tfrac{1}{3}\, h^3\, h_{xxx} + \tfrac{1}{2}\, h^2\, \sigma(\Gamma)_x\right)_x = 0, \tag{1.1}$$

$$\Gamma_t + \left(\tfrac{1}{2}\, h^2\, \Gamma\, h_{xxx} + h\, \Gamma\, \sigma(\Gamma)_x\right)_x = D\, \Gamma_{xx} \tag{1.2}$$

with suitable initial and boundary conditions. The above system appears in the lubrication theory for thin films on which surfactant molecules diffuse (see [GG90], [JG92]). In (1.1)-(1.2) the function h describes the height of the film and Γ is the concentration of the surfactants. The monotone decreasing function σ models the surfactant dependent surface tension whereas the monotone behavior takes into account that the surfactant molecules lower the surface tension. The first equation follows from mass conservation and the term in brackets is the total horizontal velocity. The second equation is a convection-diffusion equation describing mass balance for the surfactants. The term in brackets in (1.2) is up to the function Γ the horizontal velocity on top of the film and hence this term accounts for transport of Γ induced by the velocity field. We will discuss these issues in more detail in Section 2.
The analysis for (1.1)-(1.2) is difficult due to the fact that the system degenerates as h tends to zero. Equation (1.1) with $\sigma = 0$ is the thin film equation

$$h_t + \left(\tfrac{1}{3} h^3\, h_{xxx}\right)_x = 0 \tag{1.3}$$

which has been studied by many authors (see [BF90], [B96], [BBD95], [BP96], [DGG98] and the references therein). Due to the fact that the equation is of fourth order and since no maximum principle is valid, it is difficult to analyze the thin film equation. For example it is not clear how to show nonnegativity of solutions. For the thin film equation (1.3) a priori estimates follow from the identity

$$\frac{d}{dt} \int_\Omega \tfrac{1}{2} h_x^2 + \int_\Omega \tfrac{1}{3} h^3\, h_{xxx}^2 = 0 \tag{1.4}$$

*Department of Mathematics, Duke University, Box 90320, Durham, NC 27708-0320, USA, (sandra@math.duke.edu).
†NWF I - Mathematik, Universität Regensburg, 93040 Regensburg, Germany (harald.garcke@mathematik.uni-regensburg.de).

which holds under appropriate boundary conditions. But also further integral estimates of which

$$\frac{d}{dt}\int_\Omega G(h) + \tfrac{1}{3}\int_\Omega h\,h_{xx}^2 = 0 \tag{1.5}$$

with $G''(h) = h^{-2}$ is the easiest example hold and give further a priori estimates (see [BF90], [BBD95], [BP96], [DGG98] for details). By now many deep results such as nonnegativity of solutions, finite speed of propagation, results on the long time behavior of solutions, waiting time behavior and regularity results have been shown by using global and local versions of the above mentioned a priori estimates (see [BF90], [BBD95], [B96], [DGG98]). The mathematical analysis of the system (1.1)-(1.2) is even more involved. Renardy ([R1-96], [R2-96], [R97]) studied this system and variants of it. He showed local existence results and studied shock profiles in certain singular perturbed variants of (1.1)-(1.2). Barrett, Garcke and Nürnberg ([BGN03]) studied and analyzed a finite element method for (1.1)-(1.2) and they present several numerical simulations showing an extreme thinning of the film due to convection resulting from surface tension gradients. We also refer to Grün, Lenz and Rumpf ([GLR02]) for numerical simulations based on a finite volume method.

In this paper we will show global existence of weak solutions to (1.1)-(1.2). Fundamental for our approach is a proper generalization of the energy identity (1.4) to the case of surfactants. This is not straightforward and therefore, we will reconsider the derivation for (1.1)-(1.2) from the full free boundary problem for the Navier-Stokes equations. In particular we will derive an energy inequality for the full problem and taking the scaling of the lubrication approximation into account we can derive an energy estimate for (1.1)-(1.2). This will be done in Section 2 and the a priori estimates will be the main ingredients in our existence analysis presented in Section 3. Generalization of the integral identities ("entropy" identities) for the thin film equation do not seem to hold for (1.1)-(1.2). But in Section 4 we will show that at least one of the "entropy" estimates still can be generalized to (1.1)-(1.2) and this will allow us to show that solutions to positive initial data can only become zero on a set of measure zero. This shows that the influence of surfactants does not lead to dead cores, i.e. sets with positive measure on which h becomes zero.

2. The models. The motion of an incompressible viscous fluid on a bounded solid substrate is governed by the Navier-Stokes equations

$$\rho_0\{\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\} = \mu\,\Delta\mathbf{u} - \nabla p, \tag{2.1}$$

$$\operatorname{div}\,\mathbf{u} = 0 \tag{2.2}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity field, p the pressure and ρ_0 the constant density. We assume that the fluid does not penetrate the substrate given by $\{(x, y, z) \in \mathbb{R}^3 \,|\, z = 0\}$ and that there is no flux across the lateral boundary. We also impose no-slip boundary conditions for the velocity, i.e.

$$\mathbf{u}_{,z=0} = 0. \tag{2.3}$$

We assume that the evolving free surface C_t is given as the graph of a smooth time-dependent height function $h = h(t, x, y)$ on a spatial domain $\Omega \subset \mathbb{R}^2$, i.e.

$$C_t = \{(x, y, z) \big|\, (x, y) \in \Omega,\, z = h(t, x, y)\}. \tag{2.4}$$

At the interface (briefly abbreviated as $\{z = h\}$) a kinematic boundary condition dictates that the normal component of the liquid velocity balances the speed of the interface (the convective time-derivative $\frac{dh}{dt}$ of h), i.e.

$$u_{3,z=h} = \frac{dh}{dt} = h_t + u_1\,h_x + u_2\,h_y. \tag{2.5}$$

In order to balance the shear stress tensor

$$T(\mathbf{u},p) := -p\,\mathrm{Id} + \mu\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$$

of the fluid we look at its normal and tangential component separately. The normal stress exhibits a jump equal to the surface tension times the curvature (later referred to as normal stress condition):

$$\boldsymbol{\nu}\cdot T\boldsymbol{\nu} = \sigma\kappa, \tag{2.6}$$

where $\boldsymbol{\nu}$ is the outer unit normal and κ is the sum of the principal curvatures of the interface at a given point on C_t.

For the description of the tangential shear stress on the interface it becomes important to clarify the role of the surface tension dependence on the surfactant concentration Γ more explicitly. As a result of the surface agency of the surfactant monolayer, the surface concentration gradient of surfactant molecules opposes the surface tension gradient. To restore the disturbed equilibrium at the surface (caused by inhomogeneously distributed surfactants), the tangential part of the surface tension gradient balances the tangential component of the shear stress $T\boldsymbol{\nu}$, i.e.

$$\boldsymbol{\tau}\cdot T\boldsymbol{\nu} = \partial_{\boldsymbol{\tau}}\,\sigma(\Gamma) \tag{2.7}$$

for all tangents $\boldsymbol{\tau}$, where $\partial_{\boldsymbol{\tau}}$ denotes the gradient in direction of the tangent vector $\boldsymbol{\tau}$. This effect is known as the (solutal) Marangoni effect.

2.1. Mass balance for the surfactant concentration. We now derive the evolution equation for the surfactant concentration by looking at the mass balance for the surfactant on subsets $C_t' \subset C_t$. More precisely let

$$C' = \bigcup_{t\in[t_1,t_2]} \{t\}\times C_t'$$

be a smooth three-dimensional surface with boundary $\partial C'$, such that $C_t' \subset \mathbb{R}^2$ are smooth two dimensional surfaces with boundary $\partial C_t'$. Let Γ be the concentration of the surfactant which is a function on C. Then the mass balance on C' reads as follows

$$\frac{d}{dt}\Big(\int_{C_t'}\Gamma\,dS^2\Big) = -\int_{\partial C_t'}\Big(\Gamma\mathbf{u}_{tan} - D\nabla_s\Gamma\Big)\cdot\mathbf{n}_{\partial C_t'}\,dS^1 + \int_{\partial C_t'}\Gamma\,v_{\partial C_t'}\,dS^1. \tag{2.8}$$

Here $\mathbf{u}_{tan}$ is the tangential component of the velocity, i.e.

$$\mathbf{u}_{tan} := \mathbf{u} - (\mathbf{u}\cdot\boldsymbol{\nu})\boldsymbol{\nu},$$

where $\boldsymbol{\nu}\in\mathbb{R}^3$ is the normal to the interface C_t'. Furthermore, $\nabla_s\Gamma$ denotes the surface gradient of Γ on C_t. The vector $\mathbf{n}_{\partial C_t'}$ is the outer unit normal to $\partial C_t'$, i.e. $\mathbf{n}_{\partial C_t'}$ lies in the tangent space to C_t' and is perpendicular to $\partial C_t'$. The quantity $v_{\partial C_t'}$ is the normal boundary velocity of $\partial C_t'$ as a subcurve of C_t', i.e. $v_{\partial C_t'}$ describes the local decrease or increase of the surface area of C_t' due to the tangential velocity of the boundary $\partial C_t'$. To compute $v_{\partial C_t'}$ at a point $x\in\partial C_t'$ let $y(s)\in\partial C_t'$ be such that $y(t)=x$. Then $y'(t)$ denotes the velocity of the curve y. Any motion tangential to $\partial C_t'$ does not lead to an increase of area, i.e. only the component

$$v_{\partial C_t'} := y'(t)\cdot\mathbf{n}_{\partial C_t'}$$

changes the surface area of C_t'.

Let us briefly discuss what the two terms on the right hand side of (2.8) describe. The first integral describes changes of mass due to convective transport by the velocity $\mathbf{u}$ and diffusional transport where $D > 0$ is the diffusion coefficient. The second integral takes into account the change of surfactant on C'_t due to the fact that the surface C'_t increases or decreases. Our goal is now to derive a pointwise identity for the surfactant concentration from the balance law (2.8). To reformulate (2.8) we need the transport theorem (for a proof see the Appendix)

$$\frac{d}{dt}\Big(\int_{C'_t} \Gamma\, dS^2\Big) = \int_{C'_t} \Big[\partial_{(1,\mathbf{v}_{\boldsymbol{\nu}})}\Gamma - \Gamma(\mathbf{v}_{\boldsymbol{\nu}} \cdot \boldsymbol{\kappa}_{\boldsymbol{\nu}})\Big] dS^2 + \int_{\partial C'_t} \Gamma\, v_{\partial C'_t}\, dS^1. \tag{2.9}$$

In the above $\partial_{(1,\mathbf{v}_{\boldsymbol{\nu}})}\Gamma$ denotes the partial derivative of Γ in the direction $(1, \mathbf{v}_{\boldsymbol{\nu}})$, where $(1, \mathbf{v}_{\boldsymbol{\nu}})$ at a time t is the unique vector lying in the tangent space to C' and which is perpendicular to all $(0, \boldsymbol{\tau})$ where $\boldsymbol{\tau}$ is a tangent vector to C'_t. The quantity $\mathbf{v}_{\boldsymbol{\nu}}$ is the normal velocity vector and $\partial_{(1,\mathbf{v}_{\boldsymbol{\nu}})}\Gamma$ is the normal time derivative in the notation of Gurtin [Gur93]. Furthermore $\boldsymbol{\kappa}_{\boldsymbol{\nu}}$ is the mean curvature vector, i.e. $\boldsymbol{\kappa}_{\boldsymbol{\nu}}$ has the direction of the normal $\boldsymbol{\nu}$ and its length is equal to the sum of the principal curvatures. Combining (2.8) and (2.9) and using the Gauss theorem for the first integral on the right hand side of (2.8) we obtain

$$0 = \int_{C'_t} \Big[\partial_{(1,\mathbf{v}_{\boldsymbol{\nu}})}\Gamma - \Gamma(\mathbf{v}_{\boldsymbol{\nu}} \cdot \boldsymbol{\kappa}_{\boldsymbol{\nu}}) + \nabla_s \cdot (\Gamma\, \mathbf{u}_{tan} - D\nabla_s\Gamma)\Big] dS^2. \tag{2.10}$$

Since C'_t is arbitrary we obtain

$$\partial_{(1,\mathbf{v}_{\boldsymbol{\nu}})}\Gamma - \Gamma(\mathbf{v}_{\boldsymbol{\nu}} \cdot \boldsymbol{\kappa}_{\boldsymbol{\nu}}) + \nabla_s \cdot (\Gamma \mathbf{u}_{tan} - D\nabla_s\Gamma) = 0 \tag{2.11}$$

pointwise on C'. A simple computation shows

$$\mathbf{v}_{\boldsymbol{\nu}} = \frac{h_t}{1 + |\nabla h|^2}(-\nabla h, 1) = \frac{h_t}{\sqrt{1 + |\nabla h|^2}}\boldsymbol{\nu},$$

where the normal $\boldsymbol{\nu}$ is given by

$$\boldsymbol{\nu} = \frac{1}{\sqrt{1 + |\nabla h|^2}}(-\nabla h, 1).$$

Hence we obtain from (2.5) that $\mathbf{u} \cdot \boldsymbol{\nu} = \mathbf{v}_{\boldsymbol{\nu}} \cdot \boldsymbol{\nu}$ and therefore

$$\mathbf{v}_{\boldsymbol{\nu}} \cdot \boldsymbol{\kappa}_{\boldsymbol{\nu}} = \mathbf{u} \cdot \boldsymbol{\kappa}_{\boldsymbol{\nu}} = (\mathbf{u} \cdot \boldsymbol{\nu})\kappa.$$

As a result (2.11) can be rewritten as

$$\partial_{(1,\mathbf{v}_{\boldsymbol{\nu}})}\Gamma - \Gamma(\mathbf{u} \cdot \boldsymbol{\nu})\kappa + \nabla_s \cdot (\Gamma \mathbf{u}_{tan} - D\,\nabla_s\Gamma) = 0. \tag{2.12}$$

Using $\mathbf{u} = \mathbf{u}_{tan} + (\mathbf{u} \cdot \boldsymbol{\nu})\boldsymbol{\nu}$ and the fact that $\nabla_s(\Gamma \mathbf{u} \cdot \boldsymbol{\nu}) \cdot \boldsymbol{\nu} = 0$ we obtain using the following sign convention for the curvature $\kappa = -\nabla_s \cdot \boldsymbol{\nu}$ the identity

$$\nabla_s \cdot (\Gamma \mathbf{u}) = \nabla_s \cdot (\Gamma \mathbf{u}_{tan}) - \Gamma(\mathbf{u} \cdot \boldsymbol{\nu})\kappa$$

and finally (2.12) takes the form

$$\partial_{(1,\mathbf{v}_{\boldsymbol{\nu}})}\Gamma + \nabla_s \cdot (\Gamma \mathbf{u}) = D\,\Delta_s\Gamma. \tag{2.13}$$

2.2. Energy for the free surface problem. Assuming constant temperature, a fundamental equation of chemical thermodynamics relates the concentration dependent surface tension σ to the free energy g and the chemical potential g' (see [V00]), where both functions depend on the surfactant concentration Γ:

$$\sigma(\Gamma) = g(\Gamma) - \Gamma\, g'(\Gamma). \tag{2.14}$$

With convexity of the free energy this relation implies a monotone decrease of surface tension for nonnegative concentration, which is consistent with the surface agency of the surfactant molecules we described before.
The total energy decomposes into the kinetic energy of the flow and the free energy of the surface:

$$E(t) = \int\limits_{\Omega_t} \frac{\rho_0}{2}\mathbf{u}^2(t,\cdot)\, dV^3 + \int\limits_{C_t} g(\Gamma(t,\cdot))\, dS^2, \tag{2.15}$$

where $\Omega_t := \{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in \Omega,\ 0 < z < h(t,x,y)\}$ is the domain occupied by the liquid. A careful computation (see Appendix 5.2) reveals the dissipation of the total energy:

$$\frac{d}{dt}E(t) = -\int\limits_{\Omega_t} \nabla\mathbf{u} : T\, dV^3 - D\int\limits_{C_t} g''(\Gamma)\, |\nabla_s\Gamma|^2 dS^2. \tag{2.16}$$

Here the imposed boundary conditions (2.3), (2.5), (2.6) and (2.7), a no-slip boundary condition at the lateral boundary and a 90° angle condition at points where the free surface intersects the lateral boundary have been used. The importance of (2.16) for the analysis lies in the fact that with integration over a bounded time interval $[0,T]$ a priori estimates for $\mathbf{u}, \Gamma$ and their spatial gradients result.

2.3. Lubrication approximation. As our subsequent analysis will be restricted to two spatial dimensions (x,z) we state the following derivation in the two dimensional setting and introduce the variable $\mathbf{u} = (u,w)$ where u represents the horizontal velocity in the direction of x and w the vertical velocity in the direction of z.
Scalings appropriate for a lubrication approximation of surfactant driven thin films (see ([GG90]) are

$$\hat{x} = \tfrac{1}{L}\, x, \qquad \hat{z} = \tfrac{1}{H}\, z, \qquad \hat{t} = \tfrac{\epsilon U}{H}\, t,$$

$$\hat{u} = \tfrac{1}{U}\, u, \quad \hat{w} = \tfrac{1}{\epsilon U}\, w, \quad \hat{h} = \tfrac{1}{H}\, h, \quad \hat{\Gamma} = \tfrac{1}{\Gamma_m}\, \Gamma,$$

where L represents the typical horizontal length scale, H the typical film height, U the typical horizontal velocity and Γ_m the critical surfactant concentration. Considering thin films we assume that the parameter

$$\epsilon = \frac{H}{L}$$

is small. We also need to scale the pressure and the surface energy density and in this context the scaling

$$\hat{p} = \frac{H}{S}\, p \quad \text{und} \quad \hat{\sigma} = \frac{1}{S}\big(\sigma - \sigma_m\big) \tag{2.17}$$

has been chosen (see [GG90], [JG92]). Here S is the spreading coefficient, i.e. the surface tension difference at the surfactant monolayer edge and σ_m the surface tension of the saturated surface. Defining the typical horizontal velocity as

$$U = \frac{\epsilon S}{\mu}$$

with μ being the dynamic viscosity of the fluid, ensures that the Marangoni force at the free surface remains as a dominant force in the tangential stress boundary condition.

Lubrication theory now gives

$$p(t,x,z) = -\mathcal{S}\, h_{xx}(t,x), \tag{2.18}$$

$$u(t,x,z) = z\, u_z(t,x,h(t,x)) + p_x(t,x,z)\big(\frac{1}{2}z^2 - h(t,x)\, z\big) \tag{2.19}$$

where

$$u_z(t,x,h(t,x)) = \sigma(\Gamma(t,x))_x$$

and $\mathcal{S}$ denotes the rescaled capillary constant. The incompressibility of the flow and the evolution equation for Γ imply to leading order

$$h_t = -\partial_x\Big(\int_0^{h(t,x)} u(t,x,z)\, dz\Big), \tag{2.20}$$

$$\Gamma_t = -\partial_x\Big(\Gamma(t,x)\, u(t,x,h(t,x))\Big) + D\Gamma_{xx} \tag{2.21}$$

(see [GG90] for details). Using the representation of u in (2.19) we get the system (1.1)-(1.2) which we are going to analyze in the following. Furthermore we take the scaling of the lubrication approximation to approximate the energy inequality (2.16) which gives

$$\frac{d}{dt}\int_\Omega \Big(\frac{\mathcal{S}}{2}h_x^2 + g(\Gamma)\Big) + \int_\Omega \int_0^{h(t,x)} u_z^2 + \mathcal{D}\int_\Omega g''(\Gamma)\,\Gamma_x^2 = 0, \tag{2.22}$$

where $\mathcal{D}$ denotes the rescaled diffusion constant. Making use of the representation (2.19) we obtain

$$\frac{d}{dt}\int_\Omega \Big(\frac{\mathcal{S}}{2}h_x^2 + g(\Gamma)\Big) + \int_\Omega h\, \Big\{(\sigma(\Gamma)_x)^2 + \mathcal{S}\, h\, h_{xxx}\,\sigma(\Gamma)_x + \frac{1}{3}h^2\, |\mathcal{S}h_{xxx}|^2\Big\} \tag{2.23}$$

$$+\mathcal{D}\int_\Omega g''(\Gamma)\,\Gamma_x^2 = 0. \tag{2.24}$$

We remark that the second term in brackets is positive since the middle term can be absorbed by the first and the third term using Young's inequality. As a result the above identity gives a priori estimates for h and Γ. These estimates will be crucial for the analysis presented in the following sections.

We observe that through lubrication approximation the complexity of the free boundary problem is reduced - compensated by higher order spatial derivatives for h.

3. Existence of Weak Solutions. We are interested in solutions of the system of nonlinear partial differential equations we derived in Section 2.3 namely

$$h_t + \partial_x\Big(a_2(h)\,\sigma(\Gamma)_x + a_3(h)\, h_{xxx}\Big) = 0 \qquad \text{in } \Omega_T, \tag{3.1}$$

$$\Gamma_t + \partial_x\Big(\Gamma\, a_1(h)\,\sigma(\Gamma)_x + \Gamma\, a_2(h)\, h_{xxx}\Big) = \mathcal{D}\,\Gamma_{xx} \quad \text{in } \Omega_T \tag{3.2}$$

where we impose the no-flux and initial boundary conditions

$$h_x = h_{xxx} = \Gamma_x = 0 \quad \text{on } (0,T)\times\partial\Omega \tag{3.3}$$

$$h(0,\cdot) = h_0 \quad \text{in } \Omega \tag{3.4}$$

$$\Gamma(0,\cdot) = \Gamma_0 \quad \text{in } \Omega. \tag{3.5}$$

For later use we introduce the coefficient functions a_i for $i = 1, 2, 3$, where $a_i(s) = \frac{1}{i}s^i$. It turns out that solutions of appropriate regularity exist in a weak sense. This result is formulated in the following theorem. Before we state the result we need to formulate certain assumptions on the coefficients. In detail we assume:

(A1) The functions $a_i : \mathbb{R} \to \mathbb{R}_0^+$ are continuous for $i = 1, 2, 3$.

(A2) For $A : \mathbb{R} \to Mat_{2,2}(\mathbb{R}_0^+)$ there exist $d_1, d_3 \in C(\mathbb{R}_0^+)$ with $d_i(s) = 0 <=> s = 0$ such that

$$A : s \mapsto \begin{pmatrix} a_3(s) & a_2(s) \\ a_2(s) & a_1(s) \end{pmatrix}$$

has the property

$$\xi^T A(s)\, \xi \geq d_3(s)\, \xi_1^2 + d_1(s)\, \xi_2^2$$

for all $\xi = (\xi_1, \xi_2)^T \in \mathbb{R}^2$.

(A3) There exist $k, l > 0$ such that for all $s \in \mathbb{R}$

$$a_2(s) \leq C\, s^k \sqrt{d_1(s)} \text{ and } a_3(s) \leq C\, s^k \sqrt{d_3(s)},$$
$$a_2(s) \leq C\, s^l \sqrt{d_3(s)} \text{ and } a_1(s) \leq C\, s^l \sqrt{d_1(s)}.$$

(A4) The function g lies in $C^{2,1}_{loc}(\mathbb{R})$.

(A5) There exists a $c_g > 0$ such that $g''(s) \geq c_g$ for all $s > 0$.

(A6) There exists a $C_g > 0$ such that $g''(s) \leq C_g\, s^r$ for all $s > 0$ and with $r \in (0, 2)$.

In the following $\langle \cdot, \cdot \rangle$ denotes the dual product between a linear functional and a point in the corresponding normed space.

THEOREM 3.1. *Let the initial data fulfill $h_0 \geq 0$, $h_0 \in H^{1,2}(\Omega)$, $\Gamma_0 \in L^2(\Omega)$ and $g'(\Gamma_0) \in L^2(\Omega)$. Assume also that (A1)-(A6) hold. Then there exists a weak solution (h, Γ) of problem (3.1)-(3.5) such that*

$$\int_0^T \langle h_t, \zeta \rangle - \int_{\Omega_T} a_2(h)\, \sigma(\Gamma)_x\, \zeta_x - \int_{\Omega_T \setminus \Omega_T^0} \mathcal{S} a_3(h)\, h_{xxx}\, \zeta_x = 0, \tag{3.6}$$

$$\int_0^T \langle \Gamma_t, \zeta \rangle - \int_{\Omega_T} \Gamma\, a_1(h)\, \sigma(\Gamma)_x\, \zeta_x - \int_{\Omega_T \setminus \Omega_T^0} \mathcal{S} \Gamma\, a_2(h)\, h_{xxx}\, \zeta_x + \mathcal{D} \int_{\Omega_T} \Gamma_x\, \zeta_x = 0 \tag{3.7}$$

for all $\zeta \in L^3\big(0, T; H^{2,2}(\Omega)\big)$ with $\zeta_x = 0$ on $(0, T) \times \partial\Omega$ and

$$\Omega_T^0 := \{(t, x) \in \Omega_T;\ h(t, x) = 0\}.$$

The solutions h and Γ have the following regularity properties

$$h \in L^2\big(0, T; H^{1,2}(\Omega)\big) \cap H^{1,2}\big(0, T; (H^{1,2}(\Omega))^*\big) \cap C^{\frac{1}{8}, \frac{1}{2}}(\overline{\Omega_T}),$$

$$\Gamma \in L^\infty\big(0, T; L^2(\Omega)\big) \cap L^2\big(0, T; H^{1,2}(\Omega)\big) \cap H^{1, \frac{3}{2}}\big(0, T; (H^{1,3}(\Omega))^*\big) \cap L^6(\Omega_T).$$

Furthermore $h \in L^2\big(0, T; H^{3,2}(\Omega \setminus \{h > \delta\}\big)$ for all $\delta > 0$ and in addition it holds $h \geq 0$ almost everywhere.

To prove the above theorem we are following a 2-step approach: First we **regularize the degeneracy** which is apparent for $h = 0$. For this purpose we approximate the equation for h by a family of nondegenerate equations

$$h_t^\delta + \partial_x \left(a_2(h^\delta)\, \sigma(\Gamma^\delta)_x + [a_3(h^\delta) + \delta]\, \mathcal{S} h_{xxx}^\delta \right) = 0$$

where $\delta > 0$. The surfactant concentration Γ^δ is still subject to the nondegenerate equation

$$\Gamma_t^\delta + \partial_x \left(\Gamma^\delta\, a_1(h^\delta)\, \sigma(\Gamma^\delta)_x + \Gamma^\delta\, a_2(h^\delta)\, \mathcal{S} h_{xxx}^\delta \right) = \mathcal{D}\, \Gamma_{xx}^\delta$$

as we assume $\mathcal{D} \neq 0$. On the boundary $(0,T) \times \partial\Omega$ we impose again no-flux conditions that are

$$h_x^\delta = h_{xxx}^\delta = \Gamma_x^\delta = 0 \tag{3.8}$$

and for $t = 0$ we require the same initial data as for the degenerate problem

$$\begin{aligned} h^\delta(0,\cdot) &= h_0 \quad \text{in } \Omega, \\ \Gamma^\delta(0,\cdot) &= \Gamma_0 \quad \text{in } \Omega. \end{aligned}$$

In a second step we use a **Galerkin approximation** which transforms the system of partial differential equations into a system of ordinary differential equations. As basis functions for the finite dimensional space E_k we select the eigenfunctions ϕ_0, ϕ_1, $\phi_2, \ldots$ of $-\partial_{xx}$ with zero Neumann boundary conditions belonging to eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$. The usual approach would be to make an ansatz for h and Γ in $E_n = \operatorname{span}(\phi_0, \ldots, \phi_n)$ but in this case it turned out that it is necessary to reformulate the equation for Γ in terms of the chemical potential $g'(\Gamma)$ and to set up a Galerkin-Ansatz for $g'(\Gamma)$ instead or Γ. With the help of this transformation we are able to establish the necessary a priori estimates to prove global existence of Galerkin solutions.

Since $g' \in C^1(\mathbb{R})$ and $g'' > 0$ there exists $W := (g')^{-1}$ with

$$(W' \circ g')(s) = \frac{1}{g''(s)},$$

so that we can easily transform (3.2) into an equation for $v := g'(\Gamma)$ using that $\sigma(\Gamma)_x = -W(v)\, v_x$:

$$\partial_t\, W(v) - \partial_x \left(W^2(v)\, a_1(h)\, v_x \right) + \partial_x \left(W(v)\, a_2(h)\, \mathcal{S} h_{xxx} \right) = \mathcal{D}\, W(v)_{xx}. \tag{3.9}$$

Due to the approximating properties of the eigenfunctions ϕ_k there exist sequences $(\beta_{kn})_{n\in\mathbb{N}}$ and $(\gamma_{kn})_{n\in\mathbb{N}}$ for $h_0 \in H^{1,2}(\Omega)$ and $g'(\Gamma_0) \in L^2(\Omega)$ such that

$$\begin{aligned} h_{0n} &= \sum_{k=0}^{n} \beta_{kn}\phi_k \quad \text{with} \quad h_{0n} \to h_0 \quad \text{in} \quad H^{1,2}(\Omega), \\ v_{0n} &= \sum_{k=0}^{n} \gamma_{kn}\phi_k \quad \text{with} \quad v_{0n} \to v_0 := g'(\Gamma_0) \quad \text{in} \quad L^2(\Omega). \end{aligned}$$

Furthermore we make a Galerkin-ansatz for $h^\delta(t)$ and $v^\delta(t) = g'(\Gamma^\delta(t))$ of the form

$$\begin{aligned} h_n^\delta(t) &= \sum_{k=0}^{n} b_{kn}^\delta(t)\phi_k \quad \text{for all } t \in]0,T[\quad \text{with } h_n^\delta(0) = h_{0n}, \\ v_n^\delta(t) &= \sum_{k=0}^{n} c_{kn}^\delta(t)\phi_k \quad \text{for all } t \in]0,T[\quad \text{with } v_n^\delta(0) = v_{0n}, \end{aligned}$$

where according to (3.1) and (3.9) the functions $b^\delta_{kn}(t)$ and $c^\delta_{kn}(t)$ are subject to the following **Galerkin** equations where $(\cdot,\cdot)$ denotes the L^2-scalar product:

$$\frac{d}{dt}\Big(h^\delta_n(t),\phi_j\Big)+\Big(a_2(h^\delta_n(t))\,W(v^\delta_n(t))v^\delta_{n,x}(t),\phi_j'\Big) \qquad (3.10)$$

$$-\Big(a_3(h^\delta_n(t))\,\mathcal{S}h^\delta_{n,xxx}(t)+\delta\mathcal{S}h^\delta_{n,xxx}(t),\phi_j'\Big)=0,$$

$$\frac{d}{dt}\Big(W(v^\delta_n(t)),\phi_j\Big)+\Big(W^2(v^\delta_n(t))\,a_1(h^\delta_n(t))\,v^\delta_{n,x}(t),\phi_j'\Big) \qquad (3.11)$$

$$-\Big(W(v^\delta_n(t))\,a_2(h^\delta_n(t))\mathcal{S}h^\delta_{n,xxx}(t)-\mathcal{D}\,W'(v^\delta_n(t))v^\delta_{n,x}(t),\phi_j'\Big)=0.$$

Proof. The **first step** to take is to prove the existence of local solutions of the Galerkin equations (3.10)- (3.11).

In the **second step** we derive a priori estimates which allow us to extend the local solutions established in the first step towards global solutions of the Galerkin equations.

As the Galerkin equations are an approximation to the nondegenerate system of partial differential equations we establish in the **third step** the convergence of the Galerkin method.

Generalizing ideas of Bernis and Friedman [BF90] we can show in the final **fourth step** that solutions of the nondegenerate system converge to solutions of the degenerate system.

3.1. Local existence of solutions to the Galerkin system. To make use of standard theory for systems of ordinary equations we have to work out the structure of the equations (3.10)-(3.11). Since we can write

$$\frac{d}{dt}(W(v^\delta_n(t)),\phi_j)=\Big[B(c^\delta_n(t))\,\frac{d}{dt}c^\delta_n(t)\Big]_j$$

where $B(c^\delta_n(t))=\big[B_{jk}(c^\delta_n(t))\big]_{j,k}$ is the matrix

$$B_{jk}(c^\delta_n(t)):=\int_\Omega W'(v^\delta_n(t,x))\phi_k(x)\phi_j(x)\,dx$$

and $c^\delta_n(t)$ is the vector $(c^\delta_{nk}(t))_{0\le k\le n}$, we simply have to make sure that $B(c^\delta_n(t))$ is symmetric and positive definite and therefore invertible. For $\xi=(\xi_0,...,\xi_n)\in\mathbb{R}^{n+1}$ we obtain

$$\sum_{j,k}\xi_jB_{jk}(c^\delta_n(t))\xi_k=\int_\Omega W'(v^\delta_n(t,x))\big(\sum_j\xi_j\phi_j(x)\big)^2\,dx.$$

Since $W'>0$ and the eigenfunctions ϕ_j are linear independent we obtain that $B(c^\delta_n(t))$ is positive definite and we can multiply equation (3.11) with $B^{-1}(c^\delta_n(t))$. As a result we have to solve a system of first-order ordinary differential equations of the following type: for given initial data $\beta_n,\,\gamma_n\in\mathbb{R}^{n+1}$ and for any value of $\delta>0$ we look for functions $b^\delta_n,\,c^\delta_n:[0,T]\to\mathbb{R}^{n+1}$ which satisfy the equations

$$\frac{d}{dt}\left(b^\delta_n,c^\delta_n\right)(t)=F\big(t,(b^\delta_n(t),c^\delta_n(t))\big)^T, \qquad (3.12)$$

$$\left(b^\delta_n,c^\delta_n\right)(0)=\left(\beta_n,\gamma_n\right) \qquad (3.13)$$

where

$$F:[0,T]\times\mathbb{R}^{2(n+1)}\to\mathbb{R}^{2(n+1)}$$

$$(t,y_1,y_2)\mapsto\begin{bmatrix}(f_1(y_1,y_2),\Phi')\\(f_2(y_1,y_2),B^{-1}(c^\delta_n(t))\,\Phi')\end{bmatrix}$$

with $\Phi = (\phi_0, ..., \phi_n)$ and

$$f_2(y_1, y_2) := -W^2(y_2 \cdot \Phi)\, a_1(y_1 \cdot \Phi)\, y_2 \cdot \Phi' + \mathcal{S} W(y_2 \cdot \Phi)\, a_2(y_1 \cdot \Phi)\, y_1 \cdot \Phi''' - \mathcal{D}\, W'(y_2 \cdot \Phi)\, y_2 \cdot \Phi',$$
$$f_1(y_1, y_2) := -a_2\big(y_1 \cdot \Phi\big)\, W(y_2 \cdot \Phi)\, y_2 \cdot \Phi' + \mathcal{S}\, a_3\big(y_1 \cdot \Phi\big)\, y_1 \cdot \Phi''' + \delta \mathcal{S}\, y_1 \cdot \Phi'''.$$

By assumptions (A1)-(A3) the right hand side F satisfies a local Lipschitz condition with respect to $\boldsymbol{y}$. Therefore by the Picard-Lindelöf theorem a unique local solution of the initial value problem (3.12),(3.13) exists.

3.2. Global existence of solutions for the Galerkin system. In this section we will use a priori estimates in order to extend the local solution to a global solution. Since ϕ_k'' is a multiple of ϕ_k we can plug in $p_n^\delta(t) = -\mathcal{S} h_{n,xx}^\delta(t)$ as a test function in (3.10) and $v_n^\delta(t)$ as test function in (3.11) and get:

$$\mathcal{S}\Big(\frac{d}{dt} h_{n,x}^\delta(t), h_{n,x}^\delta(t)\Big) - \Big(a_2(h_n^\delta(t))\, W(v_n^\delta(t)) v_{n,x}^\delta(t), \mathcal{S} h_{n,xxx}^\delta(t)\Big) \qquad (3.14)$$
$$+ \Big(a_3(h_n^\delta(t))\, \mathcal{S} h_{n,xxx}^\delta(t) + \delta \mathcal{S} h_{n,xxx}^\delta(t), \mathcal{S} h_{n,xxx}^\delta(t)\Big) = 0,$$
$$\Big(\frac{d}{dt} W(v_n^\delta(t)), v_n^\delta(t)\Big) + \Big(W^2(v_n^\delta(t))\, a_1(h_n^\delta(t))\, v_{n,x}^\delta(t), v_{n,x}^\delta(t)\Big) \qquad (3.15)$$
$$- \Big(W(v_n^\delta(t))\, a_2(h_n^\delta(t)) \mathcal{S} h_{n,xxx}^\delta(t) - \mathcal{D}\, W'(v_n^\delta(t)) v_{n,x}^\delta(t), v_{n,x}^\delta(t)\Big) = 0.$$

Defining

$$\Gamma_n^\delta := (g')^{-1}(v_n^\delta)$$

we can recalculate

$$W'(v_n^\delta)\, v_{n,t}^\delta\, v_n^\delta = \Gamma_{n,t}^\delta\, g'(\Gamma_n^\delta) = \partial_t\, g(\Gamma_n^\delta),$$
$$W(v_n^\delta)\, v_{n,x}^\delta = -\sigma(\Gamma_n^\delta)_x \qquad \text{and} \qquad W'(v_n^\delta)\, |v_{n,x}^\delta|^2 = g''(\Gamma_n^\delta)\, |\Gamma_{n,x}^\delta|^2.$$

Using these formulas we receive by adding (3.14) and (3.15):

$$\frac{d}{dt}\left\{ \frac{\mathcal{S}}{2} \left\| h_{n,x}^\delta(t) \right\|_{L^2(\Omega)}^2 + \left\| g(\Gamma_n^\delta(t)) \right\|_{L^1(\Omega)} \right\} \qquad (3.16)$$
$$+ \int_\Omega \Big\{ a_3(h_n^\delta(t))\, \big|\mathcal{S} h_{n,xxx}^\delta(t)\big|^2 + 2 a_2(h_n^\delta(t))\, \mathcal{S} h_{n,xxx}^\delta(t)\, \sigma(\Gamma_n^\delta(t))_x$$
$$+ a_1(h_n^\delta(t)) \big|\sigma(\Gamma_n^\delta(t))_x\big|^2 \Big\} + \delta \left\| \mathcal{S} h_{n,xxx}^\delta(t) \right\|_{L^2(\Omega)}^2 + \mathcal{D} \left\| \sqrt{g''(\Gamma_n^\delta(t))}\, \Gamma_{n,x}^\delta(t) \right\|_{L^2(\Omega)}^2 = 0.$$

Assumption (A2) guarantees that the integral in the second line of (3.16) is nonnegative, so that by integration of (3.16) over $[0,T]$ we get for any given $0 < T < \infty$:

$$\frac{\mathcal{S}}{2} \sum_{k=1}^{n} \big(b_{kn}^\delta(T)\big)^2 \lambda_k + \int_\Omega g(W(c_n^\delta(T) \cdot \Phi)) \leq C(\beta_n, \gamma_n). \qquad (3.17)$$

Taking $\phi_0 = 1$ in (3.10) gives in addition that b_{0n}^δ is bounded and we can deduce that b_n^δ resp. c_n^δ are a priori bounded and therefore extentable. As a conclusion we have shown that the Galerkin equations have global solutions

$$h_n^\delta,\ \Gamma_n^\delta \in C^1\big(0, T; C^\infty(\Omega)\big).$$

3.3. Convergence of the Galerkin method. Let $\zeta \in L^3\big(0,T;H^{2,2}(\Omega)\big)$ be arbitrarily chosen with $\zeta_x = 0$ on $(0,T)\times\partial\Omega$. Then there exist functions $\zeta_n(t,\cdot) = P_n\zeta(t,\cdot) \in \operatorname{span}\{\phi_0,...,\phi_n\}$ such that for $n\to\infty$: $\zeta_n(t,\cdot)\to\zeta(t,\cdot)$ in $H^{1,3}(\Omega)$ for almost all $t\in[0,T]$. Using the convergence theorem of Lebesgue we finally get

$$\zeta_n \to \zeta \;\text{ in }\; L^3\big(0,T;H^{1,3}(\Omega)\big) \;\text{ for }\; n\to\infty.$$

Plugging ζ_n into the Galerkin-equations (3.10)-(3.11) we have to show that in the following weak formulation we can pass to the limit for $n\to\infty$:

$$\int_0^T \langle h^\delta_{n,t}(t),\zeta_n(t)\rangle - \int_{\Omega_T} a_2(h^\delta_n)\,\sigma(\Gamma^\delta_n)_x\,\zeta_{n,x} - \int_{\Omega_T} a_3(h^\delta_n)\,\mathcal{S}h^\delta_{n,xxx}\zeta_{n,x} - \delta\int_{\Omega_T}\mathcal{S}h^\delta_{n,xxx}\zeta_{n,x} = 0, \tag{3.18}$$

$$\int_0^T \langle \Gamma^\delta_{n,t}(t),\zeta_n(t)\rangle - \int_{\Omega_T} \Gamma^\delta_n\, a_1(h^\delta_n)\,\sigma(\Gamma^\delta_n)_x\,\zeta_{n,x} - \int_{\Omega_T} \Gamma^\delta_n\, a_2(h^\delta_n)\,\mathcal{S}h^\delta_{n,xxx}\zeta_{n,x} + \mathcal{D}\int_{\Omega_T}\Gamma^\delta_{n,x}\zeta_{n,x} = 0. \tag{3.19}$$

To ensure convergence we have to establish appropriate convergence properties for the integrands involved. Exploiting (3.16) together with assumption (A2) and the convergence properties of the initial data (L^2-convergence of $\big(h^\delta_{n,x}(0)\big)_{n\in\mathbb{N}} \to h_{0,x}$ and L^1-convergence of $\big(g(\Gamma^\delta_n(0))\big)_{n\in\mathbb{N}} \to g(\Gamma_0)$) we can deduce the following:

$$\Big(h^\delta_{n,x}\Big)_{n\in\mathbb{N}} \quad \text{is uniformly bounded in } L^\infty(0,T;L^2(\Omega)), \tag{3.20}$$

$$\Big(g(\Gamma^\delta_n)\Big)_{n\in\mathbb{N}} \quad \text{is uniformly bounded in } L^\infty(0,T;L^1(\Omega)), \tag{3.21}$$

$$\Big((g''(\Gamma^\delta_n))^{\frac12}\,\Gamma^\delta_{n,x}\Big)_{n\in\mathbb{N}} \quad \text{is uniformly bounded in } L^2(0,T;L^2(\Omega)), \tag{3.22}$$

$$\Big((d_3(h^\delta_n))^{\frac12}\,h^\delta_{n,xxx}\Big)_{n\in\mathbb{N}} \quad \text{is uniformly bounded in } L^2(0,T;L^2(\Omega)), \tag{3.23}$$

$$\Big((d_1(h^\delta_n))^{\frac12}\,\sigma(\Gamma^\delta_n)_x\Big)_{n\in\mathbb{N}} \quad \text{is uniformly bounded in } L^2(0,T;L^2(\Omega)), \tag{3.24}$$

$$\sqrt{\delta}\Big(h^\delta_{n,xxx}\Big)_{n\in\mathbb{N}} \quad \text{is uniformly bounded in } L^2(0,T;L^2(\Omega)). \tag{3.25}$$

Using (3.20) and the Sobolev embedding theorem we can improve the result for h^δ_n as follows:

$$\exists\, C>0 \quad \forall\, n\in\mathbb{N} \quad \forall\,\delta>0 \quad \operatorname*{ess\,sup}_{t\in[0,T]} \|h^\delta_n(t)\|_{L^\infty(\Omega)} \le C.$$

Using (3.21)-(3.22) together with (A4)-(A5) we are able to establish uniform bounds for Γ^δ_n with respect to n and δ as follows:

$$(\Gamma^\delta_n)_{n\in\mathbb{N}} \quad \text{is uniformly bounded in } L^\infty\big(0,T;L^2(\Omega)\big),$$
$$(\Gamma^\delta_n)_{n\in\mathbb{N}} \quad \text{is uniformly bounded in } L^2\big(0,T;H^{1,2}(\Omega)\big).$$

Applying an embedding theorem for parabolic function spaces (see e.g. [DiB93]) these results can be combined to:

$$(\Gamma^\delta_n)_{n\in\mathbb{N}} \quad \text{is uniformly bounded in } L^4(0,T;L^\infty(\Omega))\cap L^6(\Omega_T).$$

Finally using (3.23)-(3.24), assumption (A3) and the previous statements allow us to prove that

$$\mathcal{I}_n^\delta := a_2(h_n^\delta)\,\sigma(\Gamma_n^\delta)_x + a_3(h_n^\delta)\,\mathcal{S}h_{n,xxx}^\delta + \delta\mathcal{S}h_{n,xxx}^\delta \qquad \text{is bounded uniformly in } L^2(\Omega_T),$$
$$\mathcal{J}_n^\delta := \Gamma_n^\delta\, a_1(h_n^\delta)\,\sigma(\Gamma_n^\delta)_x + \Gamma_n^\delta\, a_2(h_n^\delta)\,\mathcal{S}h_{n,xxx}^\delta - \mathcal{D}\,\Gamma_{n,x}^\delta \qquad \text{is bounded uniformly in } L^{\frac{3}{2}}(\Omega_T)$$

and therefore we obtain

$$(h_{n,t}^\delta)_{n\in\mathbb{N}} \text{ is uniformly bounded in } L^2(0,T;(H^{1,2}(\Omega))^*),$$
$$(\Gamma_{n,t}^\delta)_{n\in\mathbb{N}} \text{ is uniformly bounded in } L^{\frac{3}{2}}(0,T;(H^{1,3}(\Omega))^*).$$

To demonstrate the convergence of (3.18)-(3.19) we list the following: Since $(h_n^\delta)_{n\in\mathbb{N}}$ is uniformly bounded in $C^{\frac{1}{8},\frac{1}{2}}(\overline{\Omega_T})$ (Hölder-continuity of $h_n^\delta(t,x)$ with resp. to $t\in[0,T]$ and $x\in\Omega$, see [BF90]) we conclude that $(h_n^\delta)_{n\in\mathbb{N}}$ converges uniformly to h^δ for $n\to\infty$. This implies together with the reflexivity of $L^2(0,T;(H^{1,2}(\Omega))^*)$ that $\partial_t h_n^\delta \overset{*}{\rightharpoonup} \partial_t h^\delta$ in $L^2\big(0,T;(H^{1,2}(\Omega))^*\big)$. By Poincare's Lemma we can prove that the boundedness of $(h_{n,xxx}^\delta)_{n\in\mathbb{N}}$ in $L^2(\Omega_T)$ implies for all δ the convergence $h_n^\delta \rightharpoonup h^\delta$ in $L^2(0,T;H^{3,2}(\Omega))$.

The convergence results for $(\Gamma_n^\delta)_{n\in\mathbb{N}}$ are not as good as for $(h_n^\delta)_{n\in\mathbb{N}}$. Using the compactness lemma of Aubin-Lions (see [LIO69]) we get strong convergence of $\Gamma_n^\delta\to\Gamma^\delta$ in $L^2(0,T;L^q(\Omega))$ for all $q\in[1,\infty)$. Besides we have the weak-convergence of $(\partial_t\,\Gamma_n^\delta)_{n\in\mathbb{N}}$ to $\partial_t\,\Gamma^\delta$ in $L^{\frac{3}{2}}(0,T;(H^{1,3}(\Omega))^*)$ and the strong convergence of $\Gamma_n^\delta\to\Gamma^\delta$ in $L^q(\Omega_T)$ for all $q\in[2,6)$ using the boundedness of $(\Gamma_n^\delta)_{n\in\mathbb{N}}$ in $L^6(\Omega_T)$ together with an interpolation estimate for L^p norms (see [E98]).

In order to get a convergence result for $g'(\Gamma_n^\delta)$ we make use of (A4) and (A6): By Lebesgue's theorem and the strong convergence of $(\Gamma_n^\delta)_{n\in\mathbb{N}}$ in $L^q(\Omega_T)$ we first conclude the convergence of $(g''(\Gamma_n^\delta))_{n\in\mathbb{N}}$ to $g''(\Gamma^\delta)$ in $L^3(\Omega_T)$. By an interpolation estimate for L^p-norms this can be extended to $g''(\Gamma_n^\delta)\to g''(\Gamma^\delta)$ in $L^q(\Omega_T)$ for all $q\in[3,\frac{6}{r})$, $r\in(0,2)$. Combining this with the weak convergence $\Gamma_{n,x}^\delta\rightharpoonup\Gamma_x^\delta$ in $L^2(\Omega_T)$ we therefore get $(g''(\Gamma_n^\delta))^{\frac{1}{2}}\Gamma_{n,x}^\delta \rightharpoonup (g''(\Gamma^\delta))^{\frac{1}{2}}\Gamma_x^\delta$ in $L^2(\Omega_T)$ for $n\to\infty$. Since $\sigma(\Gamma_n^\delta)_x$ decomposes into $\sigma(\Gamma_n^\delta)_x = -\Gamma_n^\delta(g''(\Gamma_n^\delta))^{\frac{1}{2}}\cdot(g''(\Gamma_n^\delta))^{\frac{1}{2}}\Gamma_{n,x}^\delta$ we can prove that $\sigma(\Gamma_n^\delta)_x \rightharpoonup \sigma(\Gamma^\delta)_x$ in $L^s(\Omega_T)$ for $s\in[\frac{2}{3},\frac{6}{4+r})$.

Applying these convergence results to (3.18)- (3.19) we get that the Galerkin-solutions $(h_n^\delta,\Gamma_n^\delta)$ converge for any fixed $\delta>0$ to a weak solution (h^δ,Γ^δ) of the nondegenerate problem

$$\int_0^T \langle h_t^\delta,\phi\rangle - \int_{\Omega_T} a_2(h^\delta)\,\sigma(\Gamma^\delta)_x\phi_x - \int_{\Omega_T} a_3(h^\delta)\,\mathcal{S}h_{xxx}^\delta\phi_x - \delta\int_{\Omega_T}\mathcal{S}h_{xxx}^\delta\phi_x = 0, \tag{3.26}$$

$$\int_0^T \langle \Gamma_t^\delta,\phi\rangle - \int_{\Omega_T} \Gamma^\delta a_1(h^\delta)\,\sigma(\Gamma^\delta)_x\phi_x - \mathcal{S}\int_{\Omega_T}\Gamma^\delta a_2(h^\delta)\,h_{xxx}^\delta\phi_x + \mathcal{D}\int_{\Omega_T}\Gamma_x^\delta\phi_x = 0. \tag{3.27}$$

3.4. Existence of weak solutions of the degenerate problem. When we take the limit $\delta\to 0$ we lose control over h_{xxx}^δ in $L^2(\Omega_t)$. Therefore similar as in [BF90] we introduce the sets $\Omega_T\backslash\Omega_T^0$ with

$$\Omega_T^0 := \{(t,x)\in\Omega_T;\ h(t,x)=0\}$$

on which convergence of the terms involving third derivatives of h^δ holds since $h^\delta_{xxx} \rightharpoonup h_{xxx}$ in $L^2(\Omega_T \backslash \Omega_T^\eta)$ with $\Omega_T^\eta := \{(t,x) \in \Omega_T;\ |h(t,x)| \leq \eta\}$ and since

$$\int_{\Omega_T^\eta} a_3(h^\delta)\mathcal{S}h^\delta_{xxx}\phi_x \leq C\|h^\delta\|^k_{L^\infty(\Omega_T^\eta)}\,\|d_3(h^\delta)^{\frac{1}{2}}\mathcal{S}h^\delta_{xxx}\|_{L^2(\Omega_T)}\,\|\phi_x\|_{L^2(\Omega_T)} \leq C\eta^k, \tag{3.28}$$

$$\int_{\Omega_T^\eta} \Gamma^\delta a_2(h^\delta)\mathcal{S}h^\delta_{xxx}\phi_x \leq C\|h^\delta\|^l_{L^\infty(\Omega_T^\eta)}\|\Gamma^\delta d_3(h^\delta)^{\frac{1}{2}}\mathcal{S}h^\delta_{xxx}\|_{L^{\frac{3}{2}}(\Omega_T)}\,\|\phi_x\|_{L^3(\Omega_T)} \leq C\eta^l, \tag{3.29}$$

where we made use of (A3) and the bounds (3.20)-(3.23), which are uniform with respect to δ.

Those terms in (3.26)-(3.27) in which h^δ_{xxx} does not occur are not affected and we can pass to the limit for $\delta \to 0$ in the same way as in Section 3.3. Since $\delta\|h^\delta_{xxx}\|^2_{L^2(\Omega_T)}$ is uniformly bounded (see (3.25)) we conclude furthermore

$$\delta \int_{\Omega_T} h^\delta_{xxx}\,\phi_x \leq \delta\|h^\delta_{xxx}\|_{L^2(\Omega_T)} \cdot \|\phi_x\|_{L^2(\Omega_T)} \to 0 \quad \text{for} \quad \delta \to 0.$$

In the remaining terms involving h^δ_{xxx} we can pass to the limit as in [BF90] using the estimates (3.28) and (3.29). Taking $h_- := min\{h, 0\}$ as a test function in (3.6) gives $h \geq 0$ almost everywhere (see [Yin92] or the discussion in [BGN03]).
□

4. Nonnegativity. In this section a rigorous proof for the existence of nonnegative solutions is given. For positive initial data we will generate strictly positive solutions of an approximation problem. In analogy to the single thin film equation we are looking for a functional $G(h)$ such that we can derive further estimates from the identity

$$\frac{d}{dt}\int_\Omega G(h) = \int_\Omega G'(h)h_t = \frac{1}{2}\int_\Omega G''(h)h_x\,h^2\sigma(\Gamma)_x + \frac{\mathcal{S}}{3}\int_\Omega G''(h)h_x\,h^3 h_{xxx}. \tag{4.1}$$

Making the ansatz $G''(h) = h^\alpha$ for any $\alpha > 0$ a formal computation leads to

$$\frac{d}{dt}\int_\Omega G(h) = \frac{1}{2}\int_\Omega h^{2+\alpha}\,h_x\,\sigma(\Gamma)_x - \frac{\mathcal{S}\,(3+\alpha)}{3}\int_\Omega h^{2+\alpha}\,h_x^2\,h_{xx} - \frac{\mathcal{S}}{3}\int_\Omega h^{3+\alpha}h^2_{xx},$$

which suggests to take $\alpha = -2$. For this choice the first term on the right hand side is bounded which follows from the a priori estimates we derived in Section 3 and the second term vanishes due to the Neumann boundary condition for h. After integration over $[0,T]$ we therefore receive formally the entropy equation:

$$\int_\Omega G(h(T,\cdot)) + \frac{\mathcal{S}}{3}\int_{\Omega_T} h\,h^2_{xx} = \frac{1}{2}\int_{\Omega_T} h_x\,\sigma(\Gamma)_x + \int_\Omega G(h(0,\cdot)).$$

4.1. Approximation by positive solutions. Starting with the boundary problem

$$
\begin{aligned}
h_t + \partial_x \Big[a_2(h)\, \sigma(\Gamma)_x + a_3(h)\, \mathcal{S} h_{xxx} \Big] &= 0 \quad \text{in } \Omega_T, \\
\Gamma_t + \partial_x \Big[\Gamma\, a_1(h)\, \sigma(\Gamma)_x + \Gamma\, a_2(h)\, \mathcal{S} h_{xxx} - \mathcal{D}\, \Gamma_x \Big] &= 0 \quad \text{in } \Omega_T, \\
h_x = h_{xxx} = \Gamma_x &= 0 \quad \text{on } (0,T) \times \partial\Omega, \\
h(0,\cdot) &= h_0 \quad \text{in } \Omega, \\
\Gamma(0,\cdot) &= \Gamma_0 \quad \text{in } \Omega
\end{aligned}
$$

we will follow an idea by [BF90] and regularize the coefficient functions $a_i(s)$ and lift the initial data h_0 such that the entropy equation can be derived rigorously and the existence proof for weak solutions can be imitated. Both requirements are fulfilled by choosing the regularization

$$
a_i^\epsilon(s) = \frac{s^{n+i}}{s^n + \epsilon s^3} \quad \text{for} \quad i \in \{1,2,3\}. \tag{4.2}
$$

We see that for any fixed $\epsilon > 0$, $i \in \{1,2,3\}$ and $n > 3$:

$$
\frac{a_i^\epsilon(s)}{s^{n+i-3}} = \mathcal{O}(1) \qquad \text{for} \quad s \to 0 \quad \text{and} \quad \frac{s^n}{s^n + \epsilon s^3} = \mathcal{O}(1) \qquad \text{for} \quad s \to \infty.
$$

As a consequence the matrix $A^\epsilon(s) := \frac{s^n}{s^n + \epsilon\, s^3} A(s)$ of the regularized coefficients fulfills the requirements of Theorem 3.1 as long as $A(s) = \begin{pmatrix} a_3(s) & a_2(s) \\ a_2(s) & a_1(s) \end{pmatrix}$ does. Furthermore we lift the initial data for h with ϵ^θ, $0 < \theta < \frac{1}{n-3}$, so that they become strictly positive and formulate the approximation problem P^ϵ as follows:

$$
\begin{aligned}
h_t^\epsilon + \partial_x \Big[a_2^\epsilon(h^\epsilon)\, \sigma(\Gamma^\epsilon)_x + a_3^\epsilon(h^\epsilon)\, \mathcal{S} h_{xxx}^\epsilon \Big] &= 0 && \text{in } \Omega_T, && (4.3)\\
\Gamma_t^\epsilon + \partial_x \Big[\Gamma^\epsilon\, a_1^\epsilon(h^\epsilon)\, \sigma(\Gamma^\epsilon)_x + \Gamma^\epsilon\, a_2^\epsilon(h^\epsilon)\, \mathcal{S} h_{xxx}^\epsilon \Big] &= \mathcal{D}\, \Gamma_{xx}^\epsilon && \text{in } \Omega_T, && (4.4)\\
h_x^\epsilon = h_{xxx}^\epsilon = \Gamma_x^\epsilon &= 0 && \text{on } (0,T) \times \partial\Omega, && (4.5)\\
h^\epsilon(0,\cdot) &= h_0 + \epsilon^\theta && \text{in } \Omega, && (4.6)\\
\Gamma^\epsilon(0,\cdot) &= \Gamma_0 && \text{in } \Omega. && (4.7)
\end{aligned}
$$

For this system we can state the following

THEOREM 4.1. **(Existence of positive approximative solutions).** *Let the assumptions of Theorem 3.1 hold and assume in addition that a_1^ϵ, a_2^ϵ, a_3^ϵ are given by (4.2) with $n \geq 5$. Then there exist for all $\epsilon > 0$ functions $(h^\epsilon, \Gamma^\epsilon)$ with $h^\epsilon > 0$ in Ω_T and*

$$
\begin{aligned}
h^\epsilon &\in H^{1,2}\big(0,T; (H^{1,2}(\Omega))^*\big) \cap L^2\big(0,T; H^{1,2}(\Omega_T) \cap H^{3,2}(\Omega_T)\big) \cap C^{\frac{1}{8},\frac{1}{2}}(\overline{\Omega_T}), \\
\Gamma^\epsilon &\in H^{1,\frac{3}{2}}\big(0,T; (H^{1,3}(\Omega))^*\big) \cap L^\infty\big(0,T; L^2(\Omega)\big) \cap L^2\big(0,T; H^{1,2}(\Omega)\big),
\end{aligned}
$$

such that for all $\zeta \in L^3(0,T; H^{1,3})$ equations (4.3)-(4.4) are fulfilled in the following weak sense

$$
\int_0^T \langle h_t^\epsilon(t), \zeta(t) \rangle\, dt - \int_{\Omega_T} a_2^\epsilon(h^\epsilon)\, \sigma(\Gamma^\epsilon)_x\, \zeta_x - \mathcal{S} \int_{\Omega_T} a_3^\epsilon(h^\epsilon)\, h_{xxx}^\epsilon \zeta_x = 0, \tag{4.8}
$$

$$
\int_0^T \langle \Gamma_t^\epsilon(t), \zeta(t) \rangle\, dt - \int_{\Omega_T} \Gamma^\epsilon a_1^\epsilon(h^\epsilon)\, \sigma(\Gamma^\epsilon)_x\, \zeta_x - \mathcal{S} \int_{\Omega_T} \Gamma^\epsilon a_2^\epsilon(h^\epsilon)\, h_{xxx}^\epsilon \zeta_x = -\mathcal{D} \int_{\Omega_T} \Gamma_x^\epsilon \zeta_x. \tag{4.9}
$$

Proof. For the existence part we can imitate the prove of existence of solutions to (3.1)-(3.2). We receive functions $(h^\epsilon, \Gamma^\epsilon)$ such that for all $\zeta \in L^3(0,T;H^{1,3})$:

$$\int_0^T \langle h_t^\epsilon(t), \zeta(t)\rangle\, dt - \int_{\Omega_T} a_2^\epsilon(h^\epsilon)^*\, \sigma(\Gamma^\epsilon)_x\, \zeta_x - \int_{\Omega_T\setminus\Omega_T^{\epsilon,0}} a_3^\epsilon(h^\epsilon)^*\, \mathcal{S} h_{xxx}^\epsilon \zeta_x = 0,$$

$$\int_0^T \langle \Gamma_t^\epsilon(t), \zeta(t)\rangle\, dt - \int_{\Omega_T} \Gamma^\epsilon a_1^\epsilon(h^\epsilon)^*\, \sigma(\Gamma^\epsilon)_x\, \zeta_x - \int_{\Omega_T\setminus\Omega_T^{\epsilon,0}} \Gamma^\epsilon a_2^\epsilon(h^\epsilon)^*\, \mathcal{S} h_{xxx}^\epsilon \zeta_x = -\mathcal{D}\int_{\Omega_T} \Gamma_x^\epsilon \zeta_x$$

with $\Omega_T^{\epsilon,0} := \{(t,x) \in \Omega_T;\ h^\epsilon(t,x) = 0\}$ and $a_i^\epsilon(s)^* := a_i^\epsilon(s)\chi_{\{s>0\}}$. We already know that $h^\epsilon \geq 0$ for all $t \in [0,T]$ and almost all $x \in \Omega$. We now want to show that the set $\Omega_T^{\epsilon,0}$ is empty. Let g_ϵ and G_ϵ be defined as follows:

$$g_\epsilon(s) := -\Big(\frac{1}{s} + \frac{\epsilon\, s^{2-n}}{n-2}\Big),$$

$$G_\epsilon(s) := \Big(\log\frac{1}{s} + \frac{\epsilon s^{3-n}}{(n-3)(n-2)}\Big).$$

Then $G_\epsilon'(s) = g_\epsilon(s)$ and

$$g_\epsilon'(s) = G_\epsilon''(s) = \frac{s}{a_3^\epsilon(s)} = \frac{s^{n+1} + \epsilon s^4}{s^{n+3}}.$$

Since $(h^\epsilon)_{\epsilon>0}$ is uniformly bounded in $C^0(\overline{\Omega_T})$ there exists an $A > 0$ such that $max\, h^\epsilon \leq A$ for all ϵ and we get

$$g_\epsilon(s) = -\int_s^A \frac{r}{a_3^\epsilon(r)}\, dr \leq 0 \quad \text{und} \quad G_\epsilon(s) = -\int_s^A g_\epsilon(r)\, dr \geq 0. \tag{4.10}$$

Since h_0 is strictly positive and since h^ϵ is continuous we conclude that there exists a time t^* such that h^ϵ is strictly positive on $[0,t^*]$. On this time interval the system (4.8)-(4.9) is strictly parabolic. Therefore parabolic regularity implies that h is smooth on this time interval. Hence the following computations are justified. Choosing $g_\epsilon(h^\epsilon)$ as test function for (4.3) leads to

$$\int_\Omega G_\epsilon(h^\epsilon(t^*,\cdot)) - \mathcal{S}\int_{\Omega_{t^*}} h^\epsilon h_x^\epsilon h_{xxx}^\epsilon = \int_\Omega G_\epsilon(h^\epsilon(0,\cdot)) + \int_{\Omega_{t^*}} \frac{h^\epsilon\, a_2^\epsilon(h^\epsilon)}{a_3^\epsilon(h^\epsilon)} h_x^\epsilon\, \sigma(\Gamma^\epsilon)_x.$$

Using the boundary condition (4.5) the second term reduces to

$$-\int_{\Omega_{t^*}} h^\epsilon h_x^\epsilon h_{xxx}^\epsilon = \underbrace{\frac{1}{3}\int_{\Omega_{t^*}} \partial_x\,(h_x^\epsilon)^3}_{=0} + \int_{\Omega_{t^*}} h^\epsilon (h_{xx}^\epsilon)^2$$

and we get

$$\int_\Omega G_\epsilon(h^\epsilon(t^*,\cdot)) + \int_{\Omega_{t^*}} h^\epsilon (h_{xx}^\epsilon)^2 = \int_\Omega G_\epsilon(h_0^\epsilon) + \int_{\Omega_{t^*}} h_x^\epsilon\, \sigma(\Gamma^\epsilon)_x. \tag{4.11}$$

Since $h_x^\epsilon\,\sigma(\Gamma^\epsilon)_x$ is uniformly bounded in $L^1(\Omega_T)$ we have to check the first term on the right hand side. By definition we know that

$$\int_\Omega G_\epsilon(h_0^\epsilon) = \int_\Omega \frac{\epsilon(h_0^\epsilon)^{3-n}}{(n-3)(n-2)} + \int_\Omega G_0(h_0^\epsilon).$$

From this point we see the boundedness of the expression since $0 < \theta < \frac{1}{n-3}$ and therefore $\epsilon(h_0^\epsilon)^{3-n} = \epsilon h_0^{3-n} + \epsilon^{1+\theta(3-n)} \to 0$ for $\epsilon \to 0$ and since pointwise in Ω

$$-\log h_0^\epsilon = G_0(h_0^\epsilon) = \log \frac{1}{h_0 + \epsilon^\theta} \to \log \frac{1}{h_0} \quad \text{for } \epsilon \to 0.$$

From the above we hence obtain that for all $t \in [0, t^*]$

$$\int_\Omega G_\epsilon(h^\epsilon(t,\cdot)) \le C. \tag{4.12}$$

Let's assume $t_* < T$ and let $t_0 \in (t_*, T]$ be the first time such that $h^\epsilon(t_0, x_0) = 0$ for $x_0 \in \Omega$. Let $(t_n)_{n\in\mathbb{N}}$ be the sequence $t_n \nearrow t_0$ for $n \to \infty$. We then conclude

$$h^\epsilon(t_n,\cdot) \to h^\epsilon(t_0,\cdot) \quad \text{uniformly in } \Omega \text{ for } \quad n \to \infty. \tag{4.13}$$

To prove that the entropy is still bounded in t_0 we apply Fatou's lemma. Since $G_\epsilon(h^\epsilon(t_n,\cdot))$ is bounded from below uniformly with respect to n we get

$$\int_\Omega \liminf_{n\to\infty} G_\epsilon(h^\epsilon(t_n,\cdot)) \le \liminf_{n\to\infty} \int_\Omega G_\epsilon(h^\epsilon(t_n,\cdot)) \le C. \tag{4.14}$$

This leads to the following contradiction: Using the Hölder-Regularity of h^ϵ we conclude that for all $x \in \Omega$

$$h^\epsilon(t_0, x) \le C|x - x_0|^{\frac{1}{2}}$$

which gives for $n > 3$

$$\int_\Omega [h^\epsilon(t_0,x)]^{3-n} \ge C \int_\Omega |x - x_0|^{\frac{3-n}{2}}. \tag{4.15}$$

The right hand side in (4.15) is unbounded since $\frac{n-3}{2} \ge 1$ for $n \ge 5$ and this contradicts the facts (4.13) and (4.14). Hence we can conclude that h^ϵ are strictly positive for all times $t \in [0, T]$. ☐

4.2. Convergence of the regularized problem. In this subsection we show that initial data with a positive height possess solutions to (1.1)-(1.2) which do not form dead cores, i.e. regimes with zero height cannot have positive measure.

THEOREM 4.2. *Let the assumptions of Theorem 3.1 hold with $a_i(s) = \frac{1}{i}s^i$ and suppose*

$$h_0 \ge 0 \quad \textit{and} \quad \int_\Omega |\log h_0| < \infty. \tag{4.16}$$

Then there exists a solution of (1.1)-(1.2) which fulfills all properties required in Theorem 3.1 and in addition:

(i) $h \geq 0$ *a.e. in* Ω_T *and* $\mathcal{L}^1(\{x \in \Omega | \, h(t,x) = 0\}) = 0$ *for all* $t \geq 0$,
(ii) there exists a constant $0 < C < \infty$ *such that for all* $t \in [0,T]$

$$\int_\Omega |\log h(t,\cdot)| \leq C.$$

Proof. We again will make use of the energy estimates and therefore use $-\mathcal{S}\, h^\epsilon_{xx}$ as testfunction for (4.3) and $g'(\Gamma^\epsilon)$ as testfunction for (4.4). From

$$\frac{\mathcal{S}}{2}\int_\Omega (h^\epsilon_x(T,\cdot))^2 + \int_\Omega g(\Gamma^\epsilon(T,\cdot)) + \mathcal{D}\int_{\Omega_T} g''(\Gamma^\epsilon)(\Gamma^\epsilon_x)^2$$
$$+ \int_{\Omega_T} d^\epsilon_3(h^\epsilon)(\mathcal{S}h^\epsilon_{xxx})^2 + \int_{\Omega_T} d^\epsilon_1(h^\epsilon)(\sigma(\Gamma^\epsilon)_x)^2 \leq \frac{\mathcal{S}}{2}\int_\Omega (h^\epsilon_x(0,\cdot))^2 + \int_\Omega g(\Gamma^\epsilon(0,\cdot))$$

we then deduce in the same way as in the existence proof of weak solutions for (3.1)-(3.2) the necessary convergence results for $\epsilon \to 0$ which enable us to show that weak solutions of (4.3)-(4.7) converge to weak solutions of (3.1)-(3.5). For the evidence of this we take a closer look at one of the terms of interest:

$$\int_{\Omega_T} \Gamma^\epsilon\, a^\epsilon_2(h^\epsilon)\, h^\epsilon_{xxx}\, \zeta_x = \int_{\Omega_T} \Gamma^\epsilon\, \tilde{a}^\epsilon_2(h^\epsilon)\, (h^\epsilon)^{\frac{3}{2}} h^\epsilon_{xxx}\, \zeta_x,$$

where

$$\tilde{a}^\epsilon_2(h^\epsilon) = (h^\epsilon)^{-\frac{3}{2}} a^\epsilon_2(h^\epsilon) = \frac{(h^\epsilon)^n}{(h^\epsilon)^n + \epsilon (h^\epsilon)^3}\,(h^\epsilon)^{\frac{1}{2}} \quad \to h^{\frac{1}{2}} \quad \text{pointwise for} \quad \epsilon \to 0.$$

Since $(h^\epsilon)_{\epsilon>0}$ is uniformly bounded in $C^{\frac{1}{8},\frac{1}{2}}(\overline{\Omega_T})$, the sequence $(\tilde{a}^\epsilon_2(h^\epsilon)\zeta_x)_{\epsilon>0}$ converges strongly to $h^{\frac{1}{2}}\zeta_x$ in $L^3(\Omega_T)$ for $\epsilon \to 0$. Hence we conclude with the uniformly boundedness of $\Gamma^\epsilon\,(h^\epsilon)^{\frac{3}{2}} h^\epsilon_{xxx}$ in $L^{\frac{3}{2}}(\Omega_T)$ and the weak convergence of $(h^\epsilon_{xxx})_\epsilon$ in $L^2(\Omega_T \backslash \Omega^0_T)$ that

$$\lim_{\epsilon \to 0} \int_{\Omega_T} \Gamma^\epsilon\, a^\epsilon_2(h^\epsilon)\, h^\epsilon_{xxx}\, \zeta_x = \int_{\Omega_T \backslash \Omega^0_T} \Gamma\, a_2(h)\, h_{xxx}\, \zeta_x.$$

Having reinstalled weak solutions of the original problem as limit of weak solutions of the regularized problem we now like to establish the nonnegativity result using the entropy as the crucial tool. Since $h^\epsilon \to h$ uniformly and $h^\epsilon > 0$ we obtain $h \geq 0$. Now we assume that there exists $t_0 \in (0,T)$ such that

$$\mathcal{L}^1(E_{t_0}) > 0 \quad \text{for} \quad E_{t_0} := \{x \in \Omega | \, h(t_0,x) = 0\}.$$

Using the uniform convergence of h^ϵ there exists a $w(\epsilon)$ with $h_\epsilon(t_0,x) < w(\epsilon)$ for all $x \in E_{t_0}$ such that for all $x \in E_{t_0}$ and arbitrary $\eta > 0$ with $w(\epsilon) < \eta$

$$G_\epsilon(h^\epsilon(t_0,x)) \geq -\int_{w(\epsilon)}^{A} g_\epsilon(s)ds \geq -\int_\eta^A g_\epsilon(s)ds.$$

Since the last integral converges to $-\int_\eta^A g_0(s)$ for $\epsilon \to 0$ and since $-\int_\eta^A g_0(s) \geq c\,\log\frac{1}{\eta}$ we obtain

$$\limsup_{\epsilon \to 0} \int_\Omega G_\epsilon(h^\epsilon(t_0,x)) \geq c \log\frac{1}{\eta}\, \mathcal{L}(E_{t_0}) \to \infty \quad \text{for} \quad \eta \to 0$$

in contradiction to (4.12). As a conclusion the set of points where $h = 0$ is of Lebesgue measure zero which proves the first statement (i). The second statement (ii) we conclude from the pointwise convergence

$$G_\epsilon(h^\epsilon(t,x)) \to G_0(h(t,x)) \quad \text{for} \quad \epsilon \to 0$$

in combination with (4.12) and Fatou's lemma:

$$\int_\Omega \liminf_{\epsilon\to 0} G_\epsilon(h^\epsilon(t,x)) \le \liminf_{\epsilon\to 0} \int_\Omega G_\epsilon(h^\epsilon(t,x)) \le C,$$

since $G_0(s) = \log\frac{1}{s}$. □

5. Appendix.

5.1. Transport identity. In this appendix we prove the transport theorem (2.9). The proof is based on the following identity which is the Gauss' theorem applied to the vector field $\Gamma\frac{1}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2}}(1,\mathbf{v}_{\boldsymbol{\nu}})$

$$\int_{C'} \nabla_{C'}\cdot\Big(\Gamma\frac{1}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2}}(1,\mathbf{v}_{\boldsymbol{\nu}})\Big)\,dS^3 = \int_{\partial C'} \Gamma\frac{1}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2}}(1,\mathbf{v}_{\boldsymbol{\nu}})\cdot\mathbf{n}_{\partial C'}dS^2, \tag{5.1}$$

where $\mathbf{n}_{\partial C'}$ is the outer unit normal to $\partial C'$. First we compute the divergence under the integral on the left hand side. Taking an orthonormal basis $\{\boldsymbol{t}_1,\boldsymbol{t}_2,\boldsymbol{t}_3\}$ of the tangent space to C' the divergence of a vector field F is defined as

$$\nabla_{C'}\cdot F = \sum_{i=1}^3(\partial_{\boldsymbol{t}_i}F)\cdot\boldsymbol{t}_i.$$

Choosing $\boldsymbol{t}_1 = (0,\boldsymbol{\tau}_1)$, $\boldsymbol{t}_2 = (0,\boldsymbol{\tau}_2)$, $\boldsymbol{t}_3 = (1,\mathbf{v}_{\boldsymbol{\nu}})$ with $(\boldsymbol{\tau}_1,\boldsymbol{\tau}_2)$ being an orthonormal basis of the tangent space to C'_t and $\mathbf{v}_{\boldsymbol{\nu}}$ being such that $\mathbf{v}_{\boldsymbol{\nu}}\cdot\boldsymbol{\tau}_i = 0$ $(i=1,2)$ we obtain

$$\begin{aligned}\nabla_{C'}\cdot\Big(\frac{1}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2}}(1,\mathbf{v}_{\boldsymbol{\nu}})\Big) &= \sum_{i=1}^2\frac{1}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2}}\partial_{\boldsymbol{t}_i}(1,\mathbf{v}_{\boldsymbol{\nu}})\cdot\boldsymbol{t}_i\\ &\quad+\Big(\partial_{\boldsymbol{t}_3}\frac{1}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2}}\Big)(1+\mathbf{v}_{\boldsymbol{\nu}}^2)\\ &\quad+\frac{1}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2}}\Big(\partial_{\boldsymbol{t}_3}(1,\mathbf{v}_{\boldsymbol{\nu}})\Big)\cdot(1,\mathbf{v}_{\boldsymbol{\nu}})\end{aligned}$$

where we used that $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ and $\mathbf{v}_{\boldsymbol{\nu}}$ are orthonormal. A straightforward computation shows that the last two terms cancel and we obtain (using the fact $\mathbf{v}_{\boldsymbol{\nu}} = \alpha\boldsymbol{\nu}$ for the scalar normal velocity α) that

$$\begin{aligned}\nabla_{C'}\cdot\Big(\frac{1}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}^2}}}(1,\mathbf{v}_{\boldsymbol{\nu}})\Big) &= \frac{\alpha}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2}}\sum_{i=1}^2(\partial_{\boldsymbol{\tau}_i}\boldsymbol{\nu})\cdot\boldsymbol{\tau}_i\\ &= -\frac{1}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}^2}}}\mathbf{v}_{\boldsymbol{\nu}}\cdot\boldsymbol{\kappa}_{\boldsymbol{\nu}}\end{aligned}$$

where we set $\boldsymbol{\kappa}_{\boldsymbol{\nu}} = \kappa\boldsymbol{\nu}$ with $\kappa = -\sum_{i=1}^2(\partial_{\boldsymbol{\tau}_i}\boldsymbol{\nu})\cdot\boldsymbol{\tau}_i$. Altogether we obtain

$$\nabla_{C'}\cdot\Big(\Gamma\frac{1}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2}}(1,\mathbf{v}_{\boldsymbol{\nu}})\Big) = \nabla_{C'}\Gamma\cdot\frac{1}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2}}(1,\mathbf{v}_{\boldsymbol{\nu}}) - \Gamma\frac{1}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2}}\mathbf{v}_{\boldsymbol{\nu}}\cdot\boldsymbol{\kappa}_{\boldsymbol{\nu}}.$$

Computing the surface element with the help of the above basis $\{\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3\}$ one obtains that for a function f on C'

$$\int_{C'} f dS^3 = \int_{t_1}^{t_2} \int_{C'_t} f\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2}\, dS^2\, dt.$$

Hence we obtain

$$\int_{C'} \nabla_{C'} \cdot \Big(\Gamma \frac{1}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2}}(1, \mathbf{v}_{\boldsymbol{\nu}})\Big) = \int_{t_1}^{t_2} \int_{C'_t} \Big(\partial_{(1,\mathbf{v}_{\boldsymbol{\nu}})}\Gamma - \Gamma\,(\mathbf{v}_{\boldsymbol{\nu}} \cdot \boldsymbol{\kappa}_{\boldsymbol{\nu}})\Big) dS^2\, dt. \tag{5.2}$$

It remains to compute the right hand side in (5.1). Using the identity

$$C' = \bigcup_{t\in[t_1,t_2]} \{t\} \times C'_t$$

we see that $\partial C'$ contains three parts: top, bottom and lateral boundary. For the top and similarly for the bottom (with a different sign) we obtain $\mathbf{n}_{\partial C'} = \frac{1}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2}}(1, \mathbf{v}_{\boldsymbol{\nu}})$. Hence

$$\int_{\partial C' \cap C'_{t_2}} \Gamma \frac{1}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2}}(1, \mathbf{v}_{\boldsymbol{\nu}}) \cdot \mathbf{n}_{\partial C'}\, dS^2 = \int_{C'_{t_2}} \Gamma\, dS^2 \tag{5.3}$$

and a similar formula holds for t_1 with the different sign on the right hand side. It remains to compute the integral on the lateral surface. We need to identify $\mathbf{n}_{\partial C'}$ on the lateral boundary. $\mathbf{n}_{\partial C'}$ has to be normal to $\partial C'$ and tangential to C'. Now we choose an orthogonal system $(0, \boldsymbol{\tau}_1)$, $(0, \boldsymbol{\tau}_2)$ and $(1, \mathbf{v}_{\boldsymbol{\nu}})$ such that $\boldsymbol{\tau}_2$ is tangential to $\partial C'_t$. Hence we can choose without loss of generality

$$\boldsymbol{\tau}_1 = \mathbf{n}_{\partial C'_t}.$$

Claim: $(1, \mathbf{v}_{\boldsymbol{\nu}}) + (0, v_{\partial C'_t}\, \mathbf{n}_{\partial C'_t})$ is tangential to $\partial C'$.

Proof: Assume $(t, x) \in \partial C'$. Choose a curve $(s, y(s)) \in \partial C'$ such that $(t, y(t)) = (t, x)$. Hence $(1, y'(t))$ is tangential to $\partial C'$. Since $(1, y'(t))$ lies in the tangent space to C', we have

$$(1, y'(t)) = (1, \mathbf{v}_{\boldsymbol{\nu}}) + \alpha(0, \boldsymbol{\tau}_1) + \beta(0, \boldsymbol{\tau}_2).$$

We defined $v_{\partial C'_t} = y'(t) \cdot \mathbf{n}_{\partial C'_t}$ and hence since $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = 0$ we obtain $v_{\partial C'_t} = \alpha$. This implies that

$$(1, \mathbf{v}_{\boldsymbol{\nu}}) + (0, v_{\partial C'_t}\, \mathbf{n}_{\partial C'_t})$$

is tangential to $\partial C'$.

Now we need to find numbers a and b such that

$$\mathbf{n}_{\partial C'} = a(0, \boldsymbol{\tau}_1) + b(1, \mathbf{v}_{\boldsymbol{\nu}})$$

is perpendicular to $(1, \mathbf{v}_{\boldsymbol{\nu}} + v_{\partial C'_t}\, \mathbf{n}_{\partial C'_t})$ and normal. A simple computation shows

$$b = -\frac{v_{\partial C'_t}}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2 + v_{\partial C'_t}^2}\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2}}$$

and hence

$$\frac{1}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2}}(1,\mathbf{v}_{\boldsymbol{\nu}})\cdot\mathbf{n}_{\partial C'} = -\frac{v_{\partial C'_t}}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2+v_{\partial C'_t}^2}}.$$

As a result

$$\int\limits_{\partial C'\setminus(C'_{t_2}\bigcup C'_{t_1})} \Gamma\frac{1}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2}}(1,\mathbf{v}_{\boldsymbol{\nu}})\cdot\mathbf{n}_{\partial C'}\,dS^2 = -\int\limits_{\partial C'\setminus(C'_{t_2}\bigcup C'_{t_1})} v_{\partial C'_t}\frac{\Gamma}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2+v_{\partial C'_t}^2}}\,dS^2.$$

Using $(0,\boldsymbol{\tau}_1)$ and $\mathbf{n}_{\partial C'}$ to compute the area element we obtain

$$-\int\limits_{\partial C'\setminus(C'_{t_2}\bigcup C'_{t_1})} v_{\partial C'_t}\frac{\Gamma}{\sqrt{1+\mathbf{v}_{\boldsymbol{\nu}}^2+v_{\partial C'_t}^2}}\,dS^2 = -\int\limits_{t_1}^{t_2}\int\limits_{\partial C'_t} v_{\partial C'_t}\,\Gamma dS^1\,dt. \tag{5.4}$$

Combining (5.1), (5.2), (5.3) and (5.4) gives

$$\int\limits_{C'_{t_2}}\Gamma\,dS^2 - \int\limits_{C'_{t_1}}\Gamma\,dS^2 = \int\limits_{t_1}^{t_2}\int\limits_{C'_t}\Big(\partial_{(1,\mathbf{v}_{\boldsymbol{\nu}})}\Gamma - \Gamma\,(\mathbf{v}_{\boldsymbol{\nu}}\cdot\boldsymbol{\kappa}_{\boldsymbol{\nu}})\Big)dS^2\,dt + \int\limits_{t_1}^{t_2}\int\limits_{\partial C'_t}\Gamma\,v_{\partial C'_t}dS^1 dt.$$

Differentiating with respect to t_2 now gives (2.9).

5.2. Energy identity. In this subsection we show the energy inequality (2.16). Using transport theorems (see e.g. (2.9)) we obtain

$$\begin{aligned}\frac{d}{dt}E(t) &= \int\limits_{\Omega_t}\rho_0\,\mathbf{u}\cdot\mathbf{u}_t\,dV^3 + \int\limits_{\partial\Omega_t}\frac{\rho_0}{2}\mathbf{u}^2(\mathbf{v}_{\boldsymbol{\nu}}\cdot\boldsymbol{\nu})\,dS^2\\ &\quad+\int\limits_{C_t}\Big(\partial_{(1,\mathbf{v}_{\boldsymbol{\nu}})}\,g(\Gamma) - g(\Gamma)(\mathbf{v}_{\boldsymbol{\nu}}\cdot\boldsymbol{\kappa}_{\boldsymbol{\nu}})\Big)dS^2 + \int\limits_{\partial C_t} g(\Gamma)\,v_{\partial C'_t}dS^1.\end{aligned}$$

The 90° angle condition at the outer boundary implies that the last term vanishes. Using the equations (2.1)-(2.2) for $\mathbf{u}$ and (2.11) for Γ and noticing that $\nabla\cdot T = -\nabla p + \mu\Delta\mathbf{u}$ we obtain

$$\begin{aligned}\frac{d}{dt}E(t) &= \int\limits_{\Omega_t}\Big(-\rho_0\,\mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\mathbf{u} + (\nabla\cdot T)\cdot\mathbf{u}\Big)\,dV^3 + \int\limits_{C_t}\frac{\rho_0}{2}\mathbf{u}^2(\mathbf{v}_{\boldsymbol{\nu}}\cdot\boldsymbol{\nu})dS^2\\ &\quad+\int\limits_{C_t} g'(\Gamma)\Big(\Gamma(\mathbf{v}_{\boldsymbol{\nu}}\cdot\boldsymbol{\kappa}_{\boldsymbol{\nu}}) - \nabla_s\cdot\big(\Gamma\mathbf{u}_{tan} - D\nabla_s\Gamma\big)\Big)dS^2 - \int\limits_{C_t} g(\Gamma)\,(\mathbf{v}_{\boldsymbol{\nu}}\cdot\boldsymbol{\kappa}_{\boldsymbol{\nu}})dS^2.\end{aligned}$$

Taking into account that $\nabla \cdot \mathbf{u} = 0$, $\mathbf{u} \cdot (\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{2}\mathbf{u} \cdot \nabla|\mathbf{u}|^2$ and $\mathbf{u} = \mathbf{u}_{tan} + (\mathbf{u} \cdot \boldsymbol{\nu})\boldsymbol{\nu}$ on C_t and using the boundary conditions (2.6) and (2.7) we obtain after integration by parts

$$\begin{aligned}
\frac{d}{dt}E(t) = &-\int\limits_{\Omega_t} T : \nabla \mathbf{u}\, dV^3 - \int\limits_{C_t} \frac{\rho_0}{2}\mathbf{u}^2\,(\mathbf{v}_{\boldsymbol{\nu}} - \mathbf{u}) \cdot \boldsymbol{\nu}\, dS^2 + \int\limits_{C_t} \mathbf{u}_{tan} \cdot \nabla_s\, \sigma(\Gamma) dS^2 + \int\limits_{C_t} (\mathbf{u} \cdot \boldsymbol{\nu})\sigma(\Gamma)\kappa\, dS^2 \\
&+ \int\limits_{C_t} g'(\Gamma)\Gamma(\mathbf{v}_{\boldsymbol{\nu}} \cdot \boldsymbol{\kappa}_{\boldsymbol{\nu}})\, dS^2 + g''(\Gamma)\Gamma \mathbf{u}_{tan} \cdot \nabla_s \Gamma\, dS^2 - D \int\limits_{C_t} g''(\Gamma)|\nabla_s \Gamma|^2\, dS^2 \\
&- \int\limits_{C_t} g(\Gamma)(\mathbf{v}_{\boldsymbol{\nu}} \cdot \boldsymbol{\kappa}_{\boldsymbol{\nu}})\, dS^2 \\
= &-\int\limits_{\Omega_t} T : \nabla \mathbf{u}\, dV^3 + \int\limits_{C_t} \mathbf{u}_{tan} \cdot \nabla_s \Gamma\big(\Gamma\, g''(\Gamma) - \sigma'(\Gamma)\big) dS^2 + \int\limits_{C_t} (\mathbf{u} \cdot \boldsymbol{\nu})\kappa\, \big(\sigma(\Gamma) + g'(\Gamma)\Gamma - g(\Gamma)\big) dS^2 \\
&- D \int\limits_{C_t} g''(\Gamma)|\nabla_s \Gamma|^2\, dS^2 \\
= &-\int\limits_{\Omega_t} T : \nabla \mathbf{u}\, dV^3 - D \int\limits_{C_t} g''(\Gamma)|\nabla_s \Gamma|^2\, dS^2
\end{aligned}$$

where we used (2.14) and the facts $\mathbf{v}_{\boldsymbol{\nu}} \cdot \boldsymbol{\nu} = \mathbf{u} \cdot \boldsymbol{\nu}$ and $\mathbf{v}_{\boldsymbol{\nu}} \cdot \boldsymbol{\kappa}_{\boldsymbol{\nu}} = \kappa \mathbf{u} \cdot \boldsymbol{\nu}$.

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