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ON A CAHN-HILLIARD MODEL FOR PHASE SEPARATION WITH ELASTIC MISFIT

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ABSTRACT. Existence of solutions to a Cahn-Hilliard system taking elastic stresses into account is shown. The analysis is hindered by a logarithmic singularity and a quadratic term in the displacement gradient. This makes careful a priori estimates for both terms necessary. In particular, a global version of L^p-estimates for gradients of nonlinear elliptic systems is shown in order to control the logarithmic singularity.

1. INTRODUCTION

The Cahn-Hilliard model describes phase separation and coarsening in alloys (see [6, 22] and the references cited therein). The Cahn-Hilliard equation is a fourth order diffusion equation for the concentrations of the alloy components and the evolution is driven by diffusion potentials (the chemical potentials) which are given as the first variation of a Ginzburg-Landau energy. The Ginzburg-Landau energy is non-convex with several local minima leading to the occurence of different phases (see e.g. [6]).

In many alloys the phase separation process is drastically influenced by elastic interactions (see [9, 12]) which are not taken into account in the standard Cahn-Hilliard equation. Therefore, Cahn and Larché [20] and later Onuki [25] included elastic effects into the original model. In this modified model the displacement of the lattice is a further unknown field for which an elliptic system has to be solved which is coupled to the diffusion equations. It is the goal of this paper to mathematically analyze the resulting system of equations. The mathematical difficulty lies in the fact that equations contain terms that are singular with

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respect to the alloy concentrations and that the strain tensor enters the diffusion equations quadratically. Therefore, it will turn out that it is necessary to show a higher integrability result for the gradient of the displacement. To obtain this result we use a perturbation method due to Giaquinta and Modica [15]. In particular we need to show higher integrability up to the boundary - something which is not covered by the original results of Giaquinta and Modica.

Let us now derive the governing equations. We assume that the alloy consists of N components. The concentration of component k we denote by c_k $(k = 1, ..., N)$ and therefore, the vector $\mathbf{c} = (c_k)_{k=1,...,N}$ has to fulfil the constraint $\sum_{k=1}^{N} c_k = 1$, i.e. $\mathbf{c}$ lies in the affine hyperplane

$$\Sigma := \{\mathbf{c}' = (c'_k)_{k=1,...N} \in \mathrm{I\!R}^N \mid \sum_{k=1}^{N} c'_k = 1\,\}.$$

Since only non-negative values for the c_k are physically meaningful we also introduce the Gibbs simplex

$$\mathbf{G} := \{\mathbf{c}' \in \mathrm{I\!R}^N \mid \sum_{i=1}^{N} c'_k = 1 \text{ and } c'_k \geq 0 \text{ for } k = 1, ..., N\}\,.$$

To describe elastic effects we define the displacement field $\mathbf{u}(\mathbf{x})$, i.e. a material point $\mathbf{x}$ in the undeformed body will be at the point $\mathbf{x}+\mathbf{u}(\mathbf{x})$ after deformation. Since in phase separation processes the displacement gradient usually is small, we consider an approximative theory based on the linearised strain tensor

$$\mathcal{E}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^t\right).$$

A generalised Ginzburg–Landau free energy taking elastic effects into account is of the form

$$E(\mathbf{c}, \mathbf{u}) = \int_\Omega \left\{\frac{\gamma}{2}|\nabla\mathbf{c}|^2 + \Psi(\mathbf{c}) + W(\mathbf{c}, \mathcal{E}(\mathbf{u})) + W^*(\mathcal{E}(\mathbf{u}))\right\} \tag{1}$$

where $\Omega \subset \mathrm{I\!R}^n$, $n \in \mathrm{I\!N}$, is a bounded domain with Lipschitz boundary. The first term in the energy is the gradient part with a small parameter $\gamma > 0$ and this term penalizes rapid spatial variations in the concentrations. The second summand Ψ is the homogeneous free energy density at zero stress and a free energy density Ψ derived from a mean–field theory is the sum of a logarithmic entropy term and a pairwise interaction term and has the form

$$\Psi(\mathbf{c}) = \theta \sum_{k=1}^{N} c_k \ln c_k + \tfrac{1}{2}\mathbf{c} \cdot \mathbf{A}\mathbf{c}, \tag{2}$$

where $\theta \in \mathbb{R}^+$ is the absolute temperature and the entries of the matrix $\mathbf{A} = (A_{kl})_{k,l=1,...,N}$ are the constant parameters that describe pairwise interactions between the components. For simplicity we have rescaled the absolute temperature such that the Boltzmann constant k_B is equal to one. We assume that $\mathbf{A}$ is symmetric. A typical example is $\mathbf{A} = \mathcal{X}(\mathbf{e}\mathbf{e}^t - Id)$ which means that the interactions between all components have the same magnitude $\mathcal{X} \in \mathbb{R}^+$. Let us point out that phase separation only occurs if $\mathbf{A}$ has negative eigenvalues which implies that $\mathbf{c} \longmapsto \mathbf{c} \cdot \mathbf{A}\mathbf{c}$ is not positive definite. Then the second term in (2) is non–convex while the first one is convex. For small temperatures θ the homogeneous free energy Ψ then has more than one local minima, leading to the occurrence of different phases. Chemical energies of the form (2) have been studied for example by De Fontaine [5], Hoyt [16] and Elliott and Luckhaus [7]. The first two summands in the total free energy E are classical contributions to a Ginzburg–Landau free energy. Energies consisting of two terms of this form go back to van der Waals [26]. In the theory of phase separation in alloys they have been introduced by Cahn and Hilliard [4].

The last two terms in the free energy take elastic effects into account. The term $W(\mathbf{c}, \mathcal{E})$ is the elastic free energy density and a typical form is

$$W(\mathbf{c}, \mathcal{E}) = \frac{1}{2} \left(\mathcal{E} - \mathcal{E}^\star(\mathbf{c})\right) : \mathcal{C}(\mathbf{c}) \left(\mathcal{E} - \mathcal{E}^\star(\mathbf{c})\right). \tag{3}$$

where the : –product between two tensors $\mathbf{A}, \mathbf{B}$ is defined to be $\mathbf{A} : \mathbf{B} := \operatorname{tr}(\mathbf{A}^t\mathbf{B})$. Here, $\mathcal{C}(\mathbf{c})$ is the concentration dependent elasticity tensor mapping symmetric tensors in $\mathbb{R}^{n\times n}$ into itself. We require $\mathcal{C}(\mathbf{c})$ to be symmetric and positive definite. The quantity $\mathcal{E}^\star(\mathbf{c})$ is the symmetric stress free strain (or eigenstrain) at concentration $\mathbf{c}$. This is the value the strain tensor attains if the material were uniform with concentration $\mathbf{c}$ and unstressed. If the vector $\mathbf{c}$ is equal to one of the standard cartesian basis vectors $\mathbf{e}_1, ..., \mathbf{e}_N$ then the system is equal to a pure component. In this case $\mathcal{E}^\star(\mathbf{e}_k)$ is the value of the strain tensor if the material consists only of component k and is unstressed. The function $\mathcal{E}^\star$ is a suitable extension to all of Σ. Usually a linear extension of the form

$$\mathcal{E}^\star(\mathbf{c}) = \sum_{k=1}^{N} c_k \mathcal{E}^\star(\mathbf{e}_k)$$

is assumed (Vegard's law). The elastic energy density (3) is the standard choice which goes back to the early work of Eshelby [8] and Khachaturyan [17] (see also [21, 9]). However, we will obtain results for more general densities.

The remaining term $W^\star(\mathcal{E})$ represents energy effects due to externally applied forces. For simplicity, we assume that

$$W^*(\mathcal{E}') := -\mathcal{E}' : \mathcal{S}^*$$

for a constant externally applied stress tensor S^* (see e.g. [9]).

To describe evolution phenomena in the system, we consider mass diffusion for the individual components leading to diffusion equations for the concentrations. Mechanical equilibrium is attained on a much faster time scale than diffusion takes place. Therefore, we will assume a quasi–static equilibrium for $\mathbf{u}$, i.e. for all times

$$\nabla \cdot \mathcal{S} = 0,$$

where

$$\mathcal{S} = W_{,\mathcal{E}}(\mathbf{c}, \mathcal{E}(\mathbf{u}))$$

is the stress tensor. We remark that the solution of the elastic system in general depends on time since $\mathbf{c}$ in general is time dependent. It is always assumed that W depends on its second argument only through its symmetric part, i.e. $W(\mathbf{c}', \mathcal{E}') = W(\mathbf{c}', (\mathcal{E}')^t)$. This implies that $\mathcal{S} = W_{,\mathcal{E}}(\mathbf{c}', \mathcal{E}')$ is symmetric.

The diffusion equations for the concentrations c_k $(k = 1, ..., N)$ are based on mass balances for the individual components. To define the mass balance we need to introduce chemical potentials μ_k which are defined as the variational derivative of the total free energy E with respect to c_k, i.e.

$$\mu_k = -\gamma \Delta c_k + \Psi_{,c_k}(\mathbf{c}) + W_{,c_k}(\mathbf{c}, \mathbf{u}).$$

Now Onsager's postulate [23, 24, 18] says that each thermodynamic flux is linearly related to every thermodynamic force. Since in our case the thermodynamic forces are the negative chemical potential gradients, we obtain the phenomenological equations (see Kirkaldy and Young [18], p. 137)

$$\mathbf{J}_k = -\sum_{l=1}^{N} L_{kl} \nabla \mu_l \tag{4}$$

with a constant matrix $\mathbf{L} = (L_{kl})_{k=1,...,N; l=1,...,N} \in \mathbb{R}^{N \times N}$. The Onsager reciprocity law (see [18], p. 137, and [23, 24]) states that the matrix $\mathbf{L}$ has to be symmetric, which we assume in the following. Having defined the flux, the diffusion equations follow from the balance of mass as

$$\partial_t c_k = -\nabla \cdot \mathbf{J}_k. \tag{5}$$

To ensure that the diffusion equations (5) are consistent with the constraint $\sum_{k=1}^{N} c_k = 1$ we require that the fluxes have to fulfil a linear dependency of the form

$$\sum_{k=1}^{N} \mathbf{J}_k = 0. \tag{6}$$

Since the identities (4) and (6) have to hold for all possible chemical potentials we have to impose

$$\sum_{l=1}^{N} L_{kl} = 0. \tag{7}$$

This property of the mobility matrix $\mathbf{L} = (L_{kl})_{k=1,\ldots,N;l=1,\ldots,N}$ we assume from now on. As a consequence the diffusion equations (5) become

$$\begin{aligned} \partial_t c_k &= \nabla \cdot \left(\sum_{l=1}^{N} L_{kl} \nabla \mu_l \right) \\ &= \sum_{l=1}^{N} L_{kl} \Delta \frac{1}{N} \sum_{m=1}^{N} (\mu_l - \mu_m) . \end{aligned}$$

Hence, the diffusion equations can be expressed via the chemical potential differences $(\mu_l - \mu_k)$. In particular, the evolution can be described via the vector of generalised chemical potential differences

$$\begin{aligned} \mathbf{w} &= \frac{1}{N} \left(\sum_{m=1}^{N} (\mu_l - \mu_m) \right)_{l=1,\ldots,N} \\ &= \mathbf{P} \boldsymbol{\mu} \end{aligned}$$

where $\mathbf{P}$ is the euclidian projection of $\mathrm{I\!R}^N$ onto

$$T\Sigma = \{ \mathbf{d}' = (d'_k)_{k=1,\ldots N} \in \mathrm{I\!R}^N \mid \sum_{k=1}^{N} d'_k = 0 \}$$

which is the tangent space to Σ. A simple computation yields that $\mathbf{w}$ is the variational derivative of E when one takes the constraint $\sum_{k=1}^{N} c_k = 1$ into account. In fact, introducing $\mathbf{e} = (1, ..., 1)$, we get $\mathbf{w} = \boldsymbol{\mu} - \frac{1}{N} (\boldsymbol{\mu} \cdot \mathbf{e}) \mathbf{e}$ where the second term is the Lagrange multiplier associated to the constraint $\sum_{k=1}^{N} c_k = 1$.

Altogether we have to solve the system of equations

$$\begin{aligned} \partial_t \mathbf{c} &= \mathbf{L} \Delta \mathbf{w}, && (8) \\ \mathbf{w} &= \mathbf{P} \left(-\gamma \Delta \mathbf{c} + \Psi_{,\mathbf{c}}(\mathbf{c}) + W_{,\mathbf{c}}(\mathbf{c}, \mathcal{E}(\mathbf{u})) \right), && (9) \\ \nabla \cdot \mathcal{S} &= 0, && (10) \\ \mathcal{S} &= W_{,\mathcal{E}}(\mathbf{c}, \mathcal{E}(\mathbf{u})) && (11) \end{aligned}$$

on $\Omega_T := \Omega \times (0,T)$ (for $T > 0$ arbitrary).

If only two components are present, one can use the constraint $c_1 + c_2 = 1$ to reduce the system for the concentrations to a single equation. In this case the equations (8)–(11) were first stated by Larché and Cahn [20] for $\gamma = 0$ and for nonzero γ by Onuki [25].

As boundary conditions we impose no–flux conditions for the $\mathbf{J}_k$ and the natural boundary conditions which one obtains from variations of the energy functional with respect to $\mathbf{c}$ and $\mathbf{u}$. Therefore, we require

$$
\begin{aligned}
\mathbf{L}\nabla\mathbf{w}\cdot\mathbf{n} &= 0, &(12)\\
\nabla\mathbf{c}\cdot\mathbf{n} &= 0, &(13)\\
\mathcal{S}\cdot\mathbf{n} &= \mathcal{S}^*\cdot\mathbf{n}, &(14)
\end{aligned}
$$

where $\mathbf{n}$ is the outer unit normal to $\partial\Omega$. In addition, we impose initial conditions for $\mathbf{c}$, i.e.

$$\mathbf{c}(\mathbf{x},0) = \mathbf{c}^0(\mathbf{x}) \tag{15}$$

for a given function $\mathbf{c}^0$ with $\mathbf{c}^0(\mathbf{x}) \in \Sigma$ for all $\mathbf{x} \in \Omega$. Since we assume that the mechanical equilibrium is obtained instantaneously no initial conditions for $\mathbf{u}$ are needed. We remark that the no–flux boundary condition implies that the total mass of the solution to (8)–(15) is a conserved quantity, i.e. for all $t > 0$

$$\int_\Omega \mathbf{c}(\mathbf{x},t)d\mathbf{x} = \int_\Omega \mathbf{c}^0(\mathbf{x})d\mathbf{x}.$$

Other boundary conditions are possible. For example we could impose Dirichlet conditions for $\mathbf{u}$ on parts of the boundary $\partial\Omega$. This means to prescribe the deformation on parts of the boundary and hence a unique $\mathbf{u}$ could be determined. The boundary condition (14) on the other hand prescribes $\mathbf{u}$ only up to infinitesimal rigid displacements (i.e. translations and infinitesimal rotations). This is typical for problems in elasticity that are based on a linearised strain tensor $\mathcal{E}$. The non–uniqueness in $\mathbf{u}$ will have no effect on the evolution of $\mathbf{c}$ since only $\mathcal{E}(\mathbf{u})$ enters the equation for $\mathbf{w}$. Therefore we define the space of infinitesimal rigid displacements

$$
\begin{aligned}
X_{ird} := \{\mathbf{u} \in H^1(\omega, \mathbf{R}^n) \mid \ &\text{there exist } \mathbf{b} \in \mathbf{R}^n \quad \text{and a skew symmetric}\\
&\mathbf{A} \in \mathbf{R}^{n\times n} \text{ such that } \mathbf{u}(\mathbf{x}) = \mathbf{b} + \mathbf{A}\mathbf{x}\}
\end{aligned}
$$

and the space of functions perpendicular to X_{ird}

$$X_1 := \{\mathbf{u} \in H^1(\Omega, \mathbb{R}^n) \mid (\mathbf{u},\mathbf{v})_{H^1} = 0 \quad \text{for all} \quad \mathbf{v} \in X_{ird}\} = X_{ird}^\perp.$$

We remark that a homogeneous free energy of the form (2) implies that $\Psi_{,\mathbf{c}}$, a term which enters the equation for the chemical potential differences $\mathbf{w}$, becomes singular if one of the c_k $(k = 1, ..., N)$ tends to zero. In the literature the logarithmic term often is approximated by a polynomial leading to a smooth free energy density Ψ as studied for example in [11]. But a smooth Ψ does not guarantee that the concentrations remain non–negative. On the other hand it will turn out that the singular term, due to the presence of the logarithmic contribution to the homogenous free energy density, prevents the c_k from attaining negative values. It is the goal of this paper to study the evolution equations which result from a homogeneous free energy density of the form (2). As already mentioned the system contains singular terms and the analysis is much more complicated than the one in previous articles [11, 3]. In particular in [3] the system (8)–(11) is studied with a scalar equation for the concentration c, a γ that depends on c and an additional viscosity term that regularises the system.

The case of the Cahn–Hilliard system with logarithmic free energy but without elasticity has been already studied by Elliott and Luckhaus [7]. They proved an existence and uniqueness result. For their analysis it was crucial to derive a priori estimates by differentiating the equation for the chemical potential differences $\mathbf{w}$ (see equation (9)) with respect to time. Since in general the solutions of the elastic Cahn–Hilliard system will not be smooth enough to allow differentiating with respect to time, we had to develop a new method. Let us mention one main difficulty and the technique to overcome this difficulty. The strain tensor enters the equation of the chemical potential differences quadratically. Hence, we need to establish a higher integrability result for the strain. We will apply and extend the technique of Giaquinta and Modica[15] (see also [14]), who derived for solutions to elliptic systems a higher integrability result. This is the base to show that $\ln c_i$, and hence $\Psi_{,\mathbf{c}}$ lie in $L^q(\Omega_T)$ (for some $q > 1$). In conclusion we can derive that the chemical potentials are well defined. In particular, we can also show that the concentrations c_i are positive almost everywhere.

Finally, we remark that a basic ingredient in the existence proof are a priori estimates which stem from the Lyapunov property of the free energy (1). These follow formally from the identity

$$\frac{d}{dt}\int_\Omega \Big\{\frac{\gamma}{2}|\nabla \mathbf{c}|^2 + \Psi(\mathbf{c}) + W(\mathbf{c},\mathcal{E}) + W^*(\mathcal{E})\Big\} + \sum_{k,l=1}^{N}\int_\Omega L_{kl}\nabla w_k \nabla w_l = 0$$

and the fact that $\mathbf{L}$ is positive definite.

2. The main result

In this section we formulate the main existence and uniqueness results. We need the following assumptions.

Assumptions:

(A1) $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary,
(A2) $\gamma > 0$,
(A3) the homogeneous free energy density Ψ is of the form (2) with $\theta > 0$ and a symmetric $\mathbf{A} \in \mathbb{R}^{N \times N}$,
(A4) for the elastic energy density $W \in C^1(\mathbb{R}^N \times \mathbb{R}^{n \times n}, \mathbb{R})$ we assume
  (A4.1) $W(\mathbf{c}', \mathcal{E}')$ only depends on the symmetric part of $\mathcal{E}' \in \mathbb{R}^{n\times n}$, i.e. $W(\mathbf{c}', \mathcal{E}') = W(\mathbf{c}', (\mathcal{E}')^t)$ for all $\mathbf{c}' \in \mathbb{R}^N$ and $\mathcal{E}' \in \mathbb{R}^{n\times n}$,
  (A4.2) $W_{,\mathcal{E}}(\mathbf{c}', .)$ is strongly monotone uniformly in $\mathbf{c}'$, i.e. there exists a $c_1 > 0$ such that for all symmetric $\mathcal{E}_1', \mathcal{E}_2' \in \mathbb{R}^{n\times n}$

$$(W_{,\mathcal{E}}(\mathbf{c}', \mathcal{E}_2') - W_{,\mathcal{E}}(\mathbf{c}', \mathcal{E}_1')) : (\mathcal{E}_2' - \mathcal{E}_1') \geq c_1 |\mathcal{E}_2' - \mathcal{E}_1'|^2,$$

  (A4.3) there exists a constant $C_2 > 0$ such that for all $\mathbf{c}' \in \Sigma$ and all symmetric $\mathcal{E}' \in \mathbb{R}^{n\times n}$
$$|W(\mathbf{c}', \mathcal{E}')| \leq C_2(|\mathcal{E}'|^2 + |\mathbf{c}'|^2 + 1),$$
$$|W_{,\mathbf{c}}(\mathbf{c}', \mathcal{E}')| \leq C_2(|\mathcal{E}'|^2 + |\mathbf{c}'|^2 + 1),$$
$$|W_{,\mathcal{E}}(\mathbf{c}', \mathcal{E}')| \leq C_2(|\mathcal{E}'| + |\mathbf{c}'| + 1),$$
(A5) the energy density of the applied forces is assumed to be of the form $W^*(\mathcal{E}') = -\mathcal{E}' : \mathcal{S}^*$ with a constant symmetric tensor $\mathcal{S}^*$,
(A6) the mobility matrix $\mathbf{L} = (L_{kl})_{k=1,\dots,N; l=1,\dots,N}$ is assumed to be
  (A6.1) symmetric,
  (A6.2) to fulfil $\sum_{l=1}^N L_{kl} = 0$,
  (A6.3) to be positive definite on $T\Sigma$,
(A7) the initial data $\mathbf{c}^0 \in H^1(\Omega, \mathbb{R}^N)$ are assumed to fulfil $\mathbf{c}^0 \in \mathbf{G}$ almost everywhere and

$$\fint_\Omega c_k^0 > 0 \qquad \text{for} \quad k = 1, \dots, N.$$

Let us comment on the stated assumptions. The assumptions on (A2) and (A6) guarantee that the system (8), (9) defines a semi-linear parabolic system of fourth order in the variable $\mathbf{c}$. The assumption (A4.2) ensures that the elastic part of the equation defines a quasi–linear elliptic system in $\mathbf{u}$ and in addition it follows from (A4.2) that there exist positive constants c_3 and C_3 such that

$$W(\mathbf{c}', \mathcal{E}') \geq c_3 |\mathcal{E}'|^2 - C_3 \left(|\mathbf{c}'|^2 + 1\right)$$

for all $\mathbf{c}' \in \Sigma$ and all symmetric $\mathcal{E}' \in \mathrm{I\!R}^{n\times n}$. The assumption on the mean value of $\mathbf{c}$ does not give any practical limitation. A zero mean value for one component, i.e. $⨍_\Omega c_k^0 = 0$, together with $\mathbf{c}^0 \in \mathbf{G}$ would imply that $c_k = 0$ almost everywhere. Therefore, component k does not appear at all, which means that we can reduce the system to a $N-1$ system containing all components beside the k–th component.

The main result of this paper is an existence result for the elastic Cahn–Hilliard system with a logarithmic free energy density (2).

Theorem 2.1. (Existence) *Assume (A1)-(A7). Then there exists a triple* $(\mathbf{c}, \mathbf{w}, \mathbf{u}) \in L^2(0,T; H^1(\Omega, \mathrm{I\!R}^N)) \times L^2(0,T; H^1(\Omega, \mathrm{I\!R}^N)) \times L^2(0,T; X_1)$ *with* $c \in \Sigma$ *a.e. and* $\mathbf{P}\Psi_{,\mathbf{c}}(\mathbf{c}) \in L^1(\Omega_T)$ *which solves the elastic Cahn–Hilliard system in the following sense:*
(i)

$$-\int_{\Omega_T} \partial_t \boldsymbol{\xi} \cdot (\mathbf{c} - \mathbf{c}^0) + \int_{\Omega_T} \mathbf{L}\nabla\mathbf{w} : \nabla\boldsymbol{\xi} = 0 \tag{16}$$

for all $\boldsymbol{\xi} \in L^2(0,T; H^1(\Omega, \mathrm{I\!R}^N))$ *with* $\partial_t\boldsymbol{\xi} \in L^2(\Omega_T)$ *and* $\boldsymbol{\xi}(T) = 0$,
(ii)

$$\int_{\Omega_T} \mathbf{w}\cdot\boldsymbol{\zeta} = \int_{\Omega_T} \{\gamma\nabla\mathbf{c} : \nabla\boldsymbol{\zeta} + \mathbf{P}\Psi_{,\mathbf{c}}(\mathbf{c})\cdot\boldsymbol{\zeta} + \mathbf{P}W_{,\mathbf{c}}(\mathbf{c}, \mathcal{E}(\mathbf{u}))\cdot\boldsymbol{\zeta}\} \tag{17}$$

for all $\boldsymbol{\zeta} \in L^2(0,T; H^1(\Omega, \mathrm{I\!R}^N)) \cap L^\infty(\Omega_T, \mathrm{I\!R}^N)$, *and*
(iii)

$$\int_{\Omega_T} W_{,\mathcal{E}}(\mathbf{c}, \mathcal{E}(\mathbf{u})) : \nabla\boldsymbol{\eta} = \int_{\Omega_T} \mathcal{S}^* : \nabla\boldsymbol{\eta} \tag{18}$$

for all $\boldsymbol{\eta} \in L^2(0,T; H^1(\Omega, \mathrm{I\!R}^n))$.

In addition the solution has the following properties:
(i) $\mathbf{c} \in C^{0,\frac{1}{4}}([0,T]; L^2(\Omega))$,
(ii) $\partial_t\mathbf{c} \in L^2\left(0,T; (H^1(\Omega))^*\right)$,
(iii) there exists a $p > 2$ *such that* $\mathbf{u} \in L^\infty\left(0,T; W^{1,p}(\Omega, \mathrm{I\!R}^n)\right)$,
(iv) there exists a $q > 1$ *such that for* $k \in \{1, ..., N\}$

$$\ln c_k \in L^q(\Omega_T).$$

In particular, $c_k > 0$ *almost everywhere.*

We can prove a uniqueness theorem in the case of homogeneous linear elasticity and under the assumption that the stress free strain varies linearly with the concentration, i.e.

$$\mathcal{E}^\star(\mathbf{c}) = \sum_{k=1}^N c_k \mathcal{E}_k^\star, \tag{19}$$

where the $\mathcal{E}_k^\star = \mathcal{E}^\star(\mathbf{e}_k)$ are the stress free strains in the case that the material were uniformly equal to component k. Altogether the elastic part of the free energy has the form

$$W(\mathbf{c},\mathcal{E}) = \frac{1}{2}\left(\mathcal{E} - \mathcal{E}^\star(\mathbf{c})\right) : \mathcal{C}\left(\mathcal{E} - \mathcal{E}^\star(\mathbf{c})\right) \tag{20}$$

with a constant positive definite tensor $\mathcal{C}$ which is assumed to fulfil the usual symmetry conditions of linear elasticity.

Theorem 2.2. (Uniqueness) *In addition to the assumptions of Theorem 2.1 we assume that W has the form (19), (20).*

Then there exists a unique solution of the elastic Cahn–Hilliard system with logarithmic free energy in the sense of Theorem 2.1.

The uniqueness theorem can be shown in exactly the same way as in the case of a smooth homogeneous free energy Ψ and we therefore omit the proof (see the proof of Theorem 4.1 in [11]).

3. A regularised problem

Our goal is to approximate the singular system by a system with smooth free energies such that the results of [11] can be applied.

First of all we assume that the elastic free energy density W fulfils

(A4.4) $W_{,\mathbf{c}}(\mathbf{c}',\mathcal{E}') = 0$ for all $\mathbf{c}' \in \mathbb{R}^N$ with $|\mathbf{c}'| > 2$ and all $\mathcal{E}' \in \mathbb{R}^{n\times n}$.

This assumption is without loss of generality, because the solution turns out to lie on the Gibbs simplex and therefore has modulus less than two.

Furthermore, for given $\delta > 0$ we replace Ψ by the C^2–function

$$\Psi^\delta(\mathbf{c}') = \theta \sum_{k=1}^{N} \psi^\delta(c_k') + \tfrac{1}{2}\mathbf{c}' \cdot \mathbf{A}\mathbf{c}' \tag{21}$$

with

$$\psi^\delta(d) := \begin{cases} d\ln d & \text{for} \quad d \geq \delta\,, \\ \left(d\ln\delta - \frac{\delta}{2} + \frac{d^2}{2\delta}\right) & \text{for} \quad d < \delta. \end{cases} \tag{22}$$

For later use we define

$$\Psi^{1,\delta}(\mathbf{c}') = \theta \sum_{k=1}^{N} \psi^\delta(c_k') \qquad \text{and} \quad \Psi^2(\mathbf{c}') = \tfrac{1}{2}\mathbf{c}' \cdot \mathbf{A}\mathbf{c}'.$$

The same regularisation has been used by Elliott and Luckhaus [7] in their existence proof for the Cahn–Hilliard system without elasticity. The following lemma (for a proof see Elliott and Luckhaus [7]) states that Ψ^δ is uniformly bounded from below on Σ.

Lemma 3.1. *There exist a $\delta_0 > 0$ and a $K > 0$ such that for all $\delta \in (0, \delta_0)$*

$$\Psi^\delta(\mathbf{c}') \geq -K \qquad \text{for all} \quad \mathbf{c}' \in \Sigma.$$

The following lemma states an existence result for the regularised problem and collects a priori estimates and compactness results, which can be obtained similar as in [11] and we therefore only sketch the proof.

Lemma 3.2. *Suppose the homogeneous free energy density is of the form (21).*

(a) For all $\delta \in (0, \delta_0)$ there exists a weak solution $(\mathbf{c}^\delta, \mathbf{w}^\delta, \mathbf{u}^\delta)$ of the elastic Cahn–Hilliard system in the sense specified in Theorem 2.1.

(b) Moreover, there exists a constant $C > 0$ such that for all $\delta \in (0, \delta_0)$

$$\sup_{t\in[0,T]} \left\{ \|\mathbf{c}^\delta(t)\|_{H^1(\Omega)} + \|\mathbf{u}^\delta(t)\|_{H^1(\Omega)} \right\} \leq C,$$

$$\sup_{t\in[0,T]} \int_\Omega \Psi^{1,\delta}(\mathbf{c}^\delta(t)) + \|\nabla \mathbf{w}^\delta\|_{L^2(\Omega_T)} \leq C$$

and

$$\|\mathbf{c}^\delta(t_2) - \mathbf{c}^\delta(t_1)\|_{L^2(\Omega)} \leq C|t_2 - t_1|^{\frac{1}{4}}$$

for all $t_1, t_2 \in [0, T]$.

(c) Furthermore, one can extract a subsequence $(\mathbf{c}^\delta)_{\delta\in\mathcal{R}}$, where $\mathcal{R} \subset (0, \delta_0)$ is a countable set with zero as the only cluster point, such that

$$\begin{array}{llll} (i) & \mathbf{c}^\delta & \to & \mathbf{c} \text{ in } C^{0,\alpha}([0,T]; L^2(\Omega)) \quad \text{for all } \alpha \in (0, \tfrac{1}{4}), \\ (ii) & \mathbf{c}^\delta & \to & \mathbf{c} \text{ almost everywhere}, \\ (iii) & \mathbf{c}^\delta & \to & \mathbf{c} \text{ weak-* in } L^\infty(0,T; H^1(\Omega)), \\ (iv) & \mathbf{u}^\delta & \to & \mathbf{u} \text{ in } L^2(0,T; H^1(\Omega)), \end{array}$$

as $\delta \in \mathcal{R}$ tends to zero.

Proof. The regularised problem fulfils the assumptions of Theorem 3.1 in [11], for all $\delta \in (0, \delta_0)$. To show this, one makes use of Lemma 3.1. Hence, a weak solution of the regularised problem exists. The a priori estimates in (b) follow from the Lyapunov property of the energy E and embedding theorems. In this context we refer to the Lemmas 3.3 and 3.4 in [11] which give the estimates in (b) by convergence and lower semi-continuity properties. To show that the constant on the right hand side does not depend on δ, one has to check that $E^\delta(\mathbf{c}^0, \mathbf{u}^0)$ does not depend on δ (see the proof of Lemma 3.3 in [11]). This is implied by the facts that the initial data $\mathbf{c}^0$ lie in $H^1(\Omega, \mathbb{R}^N)$ and only attain values on the Gibbs simplex. The convergence properties in (c) follow as in the proofs of the Lemmas 3.4 and 3.5 in [11]. $\square$

What remains to be done? It is our goal to show compactness for the chemical potential differences $\{\mathbf{w}^\delta\}_{\delta\in(0,\delta_0)}$. We already established a uniform estimate for $\{\nabla\mathbf{w}^\delta\}_{\delta\in(0,\delta_0)}$, i.e. it is enough to control the spatial mean values of $\{\mathbf{w}^\delta\}_{\delta\in(0,\delta_0)}$ to get a uniform bound in $L^2(0,T;H^1(\Omega,\mathrm{I\!R}^N))$. This will be our first step. Thereafter, it is possible to show the existence of a subsequence of $\{\mathbf{w}^\delta\}_{\delta\in(0,\delta_0)}$ which converges weakly in $L^2(0,T;H^1(\Omega,\mathrm{I\!R}^N))$ to a limit $\mathbf{w}$. Then it remains to prove that the following equation holds in a weak sense

$$\mathbf{w} = \mathbf{P}\left(-\gamma\Delta\mathbf{u} + \Psi_{,\mathbf{c}}(\mathbf{c}) + W_{,\mathbf{c}}(\mathbf{c},\mathcal{E}(\mathbf{u}))\right), \tag{23}$$

where

$$\Psi_{,\mathbf{c}}(\mathbf{c}) = \theta\left(\ln c_k + 1\right)_{k=1,\dots,N} + \mathbf{A}\mathbf{c}.$$

The problem is that $\ln c_k$ might be singular. Our goal is to establish a uniform estimate for $\left(\psi^\delta\right)'(c_k^\delta)$ in $L^q(\Omega_T)$ for some $q>1$. We remind the reader that ψ^δ is an approximation of $\psi(d) = d\ln d$. To show the L^q–bound we first derive the integrability of $\mathcal{E}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^t)$ in $L^p(\Omega_T)$ for some $p>2$. This implies that $W_{,\mathbf{c}}$ lies in $L^{\frac{p}{2}}(\Omega_T)$ which allows to multiply the equations in (23) by an appropriate power of $\left(\psi^\delta\right)'(c_k^\delta)$, leading to uniform L^q–bounds (for some $q>1$) for $\left(\psi^\delta\right)'(c_k^\delta)$.

The uniform estimates for $\left(\psi^\delta\right)'(c_k^\delta)$ together with the almost everywhere convergence of $\{\mathbf{c}^\delta\}_{\delta\in(0,\delta_0)}$ yields the convergence of $\left(\psi^\delta\right)'(c_k^\delta)$ to $\ln c_k + 1$ in $L^1(\Omega_T)$. This is enough to pass to the limit in the equation for $\mathbf{w}^\delta$ and to show that $c_k > 0$ almost everywhere for $k=1,\dots,N$.

As pointed out above we first have to derive a uniform bound on $\{\mathbf{w}^\delta\}_{\delta\in(0,\delta_0)}$.

Lemma 3.3. *(i) There exists a constant $C>0$ independent of δ such that for all $\delta\in(0,\delta_0)$*

$$\int_0^T \left(\fint_\Omega \mathbf{P}\Psi^{1,\delta}_{,\mathbf{c}}(\mathbf{c}^\delta)\right)^2(t)\,dt < C$$

and

$$\|\mathbf{w}^\delta\|_{L^2(0,T;H^1(\Omega))} < C.$$

(ii) There exists a subsequence $(\mathbf{w}^\delta)_{\delta\in\mathcal{R}}$ where $\mathcal{R}\subset(0,\delta_0)$ is a countable set with zero as the only cluster point such that

$$\mathbf{w}^\delta \to \mathbf{w} \text{ weakly in } L^2(0,T;H^1(\Omega)).$$

PROOF. We define

$$\mathbf{w}_{\mathbf{0}}^{\delta} = \mathbf{w}^{\delta} - \boldsymbol{\lambda}^{\delta}$$

with

$$\boldsymbol{\lambda}^{\delta} = \fint_{\Omega} \mathbf{w}^{\delta} = \fint_{\Omega} \left(\mathbf{P}\Psi^{\delta}_{,\mathbf{c}}(\mathbf{c}^{\delta}) + \mathbf{P}W_{,\mathbf{c}}(\mathbf{c}^{\delta}, \mathcal{E}(\mathbf{u}^{\delta})) \right).$$

To derive a bound on the Lagrange multipliers $\boldsymbol{\lambda}^{\delta}$ we generalise an idea of Barrett and Blowey [2]. Since $\mathbf{w}^{\delta}$ fulfils (17) with homogeneous free energy density Ψ^{δ} we have:

$$\int_{\Omega} (\mathbf{w}_{\mathbf{0}}^{\delta} + \boldsymbol{\lambda}^{\delta}) \cdot \boldsymbol{\zeta} = \qquad (24)$$
$$\int_{\Omega} \left\{ \gamma \nabla \mathbf{c}^{\delta} : \nabla \boldsymbol{\zeta} + \mathbf{P}\Psi^{\delta}_{,\mathbf{c}}(\mathbf{c}^{\delta}) \cdot \boldsymbol{\zeta} + \mathbf{P}W_{,\mathbf{c}}(\mathbf{c}^{\delta}, \mathcal{E}(\mathbf{u}^{\delta})) \cdot \boldsymbol{\zeta} \right\}$$

for all $\boldsymbol{\zeta} \in H^1(\Omega, \mathbb{R}^N) \cap L^{\infty}(\Omega, \mathbb{R}^N)$ and for almost all $t \in (0, T)$.

For all elements $\mathbf{k}$ lying on the Gibbs simplex $\mathbf{G}$ we obtain by using the convexity of $\Psi^{1,\delta}$ and the fact that $\mathbf{k} - \mathbf{c}^{\delta} \in T\Sigma$ almost everywhere

$$\begin{aligned} \int_{\Omega} \Psi^{1,\delta}(\mathbf{k}) &\geq \int_{\Omega} \Psi^{1,\delta}(\mathbf{c}^{\delta}) + \int_{\Omega} \Psi^{1,\delta}_{,\mathbf{c}}(\mathbf{c}^{\delta}) \cdot (\mathbf{k} - \mathbf{c}^{\delta}) \\ &= \int_{\Omega} \Psi^{1,\delta}(\mathbf{c}^{\delta}) + \int_{\Omega} \mathbf{P}\Psi^{1,\delta}_{,\mathbf{c}}(\mathbf{c}^{\delta}) \cdot (\mathbf{k} - \mathbf{c}^{\delta}). \qquad (25) \end{aligned}$$

Since W fulfils $W_{,\mathbf{c}}(\mathbf{c}', \mathcal{E}') = 0$ for $|\mathbf{c}'| \geq 2$, we can also choose $\boldsymbol{\zeta} = \mathbf{k} - \mathbf{c}^{\delta}$ as a test function in (24) for almost all $t \in (0, T)$. Taking the resulting expression and using inequality (25) we conclude

$$\begin{aligned} \int_{\Omega} \Psi^{1,\delta}(\mathbf{k}) \geq & \int_{\Omega} \Psi^{1,\delta}(\mathbf{c}^{\delta}) - \int_{\Omega} \mathbf{P}\Psi^{2}_{,\mathbf{c}}(\mathbf{c}) \cdot (\mathbf{k} - \mathbf{c}^{\delta}) \\ & - \int_{\Omega} \mathbf{P}W_{,\mathbf{c}} \cdot (\mathbf{k} - \mathbf{c}^{\delta}) + \int_{\Omega} \gamma \nabla \mathbf{c}^{\delta} : \nabla \mathbf{c}^{\delta} \\ & + \int_{\Omega} \mathbf{w}_{\mathbf{0}}^{\delta} \cdot (\mathbf{k} - \mathbf{c}^{\delta}) + \int_{\Omega} \boldsymbol{\lambda}^{\delta} \cdot (\mathbf{k} - \mathbf{c}^{\delta}) \end{aligned}$$

for almost all $t \in (0, T)$. We want to use the above inequality to establish an estimate of the term

$$\int_{\Omega} \boldsymbol{\lambda}^{\delta} \cdot (\mathbf{k} - \mathbf{c}^{\delta}).$$

Using

$$|\mathbf{P}W_{,\mathbf{c}}(\mathbf{c}', \mathcal{E}') \cdot (\mathbf{k} - \mathbf{c}')| \leq \begin{cases} C_2(|\mathcal{E}'|^2 + 1) & \text{if } |\mathbf{c}'| < 2, \\ 0 & \text{if } |\mathbf{c}'| \geq 2, \end{cases}$$

Lemma 3.1, the a priori estimates of Lemma 3.2 and Poincaré's inequality for functions with mean value zero, we obtain for almost all $t \in (0,T)$

$$\begin{aligned}\int_\Omega \boldsymbol{\lambda}^\delta \cdot (\mathbf{k} - \mathbf{c}^\delta) \quad &\leq \quad C\Big(1 + \|\nabla \mathbf{w}^\delta(t)\|_{L^2(\Omega)} \left(1 + \|\mathbf{c}^\delta(t)\|_{L^2(\Omega)}\right) + \\ &\quad + \|\mathbf{c}^\delta(t)\|^2_{L^2(\Omega)} + \|\nabla \mathbf{u}^\delta\|^2_{L^2(\Omega)}\Big). \qquad (26)\end{aligned}$$

Assumption (A7$'$) and the fact that $\int_\Omega \mathbf{c}^\delta(t)$ is constant in time ensures the existence of a $\rho > 0$ such that for all $k \in \{1, ..., N\}$ and all $t \in (0,T]$

$$\rho < \fint_\Omega c_k^\delta(t) < 1 - \rho.$$

Choosing

$$\mathbf{k} = \fint_\Omega \mathbf{c}^\delta(t) + \rho \, \mathrm{sign}(\lambda_k^\delta - \lambda_l^\delta)\,(\mathbf{e}_k - \mathbf{e}_l) \in \mathbf{G}$$

in (26) gives

$$|\lambda_k^\delta - \lambda_l^\delta|(t) \leq \frac{C}{\rho|\Omega|}\left(1 + \|\nabla \mathbf{w}^\delta(t)\|_{L^2(\Omega)}\right).$$

Integrating $|\lambda_k^\delta - \lambda_l^\delta|^2(t)$ from 0 to T and using the identity $\boldsymbol{\lambda}^\delta = \frac{1}{N}\left(\sum_{l=1}^N (\lambda_k^\delta - \lambda_l^\delta)\right)_{k=1,...,N}$ leads to

$$\int_0^T |\boldsymbol{\lambda}^\delta|^2(t)\,dt \leq C.$$

This, together with the growth condition for W and the a priori estimates of Lemma 3.2, gives an estimate for the spatial mean values of $\mathbf{w}^\delta$ in $L^2(0,T)$. Hence, the Poincaré inequality yields the second inequality in (i). The second hypothesis then follows from a compactness argument. □

4. Higher integrability for the strain tensor

In this subsection we use a perturbation argument to show that the deformation gradient has the following integrability property:
There exists a $p > 2$ such that for almost all $t \in [0,T]$ we have $\nabla \mathbf{u}(t) \in L^p(\Omega)$.

Lemma 4.1. (Higher integrability: interior estimates) *Suppose that $\mathbf{c} \in L^\sigma(\Omega, \mathbb{R}^N)$, $\sigma > 2$ and that (A4) and (A5) hold.*

Then there exists a $p \in (2, \sigma]$, independent of $\mathbf{c}$, such that for all $\mathbf{u} \in H^1(\Omega, \mathbb{R}^n)$ which fulfil for all $\boldsymbol{\eta} \in H^1(\Omega, \mathbb{R}^n)$ the identity

$$\int_\Omega W_{,\mathcal{E}}(\mathbf{c}, \mathcal{E}(\mathbf{u})) : \nabla \boldsymbol{\eta} = \int_\Omega \mathcal{S}^* : \nabla \boldsymbol{\eta} \tag{27}$$

the integrability property

$$\nabla \mathbf{u} \in L^p_{loc}(\Omega, \mathbb{R}^{n \times n})$$

holds. In particular, for all $\Omega' \subset\subset \Omega$ it holds

$$\|\nabla \mathbf{u}\|_{L^p(\Omega', \mathbb{R}^{n\times n})} \le C \left(\|\nabla \mathbf{u}\|_{L^2(\Omega, \mathbb{R}^{n\times n})} + \|\mathbf{c}\|_{L^p(\Omega, \mathbb{R}^n)} + 1 \right)$$

where $C = C(\Omega, \Omega', C_2, c_2, c_1, \mathcal{S}^*, n, \sigma, p)$ is independent of $\mathbf{c}$.

Proof. The proof is based on a Caccioppoli inequality, a reverse Hölder inequality and a perturbation argument due to Gehring [13] and Giaquinta and Modica [15]. This technique is well known for elliptic systems. In our case additional difficulties arise in the derivation of the Caccioppoli inequality because the direct estimates only control $\mathcal{E}(\mathbf{u})$ rather than $\nabla \mathbf{u}$. Therefore, we present the derivation of the Caccioppoli inequality in detail.

Let $\mathbf{x}_0 \in \Omega$ and $R > 0$ be such that

$$Q_{2R}(\mathbf{x}_0) := \{ x \in \mathbb{R}^n \,|\, |x_i - x_{0i}| < 2R \} \subset \Omega.$$

Then we define a cutoff function $\zeta \in C_0^\infty(\Omega)$ with the properties

i) $\zeta = 0$ in $\Omega \setminus Q_{2R}(\mathbf{x}_0)$,

ii) $0 \le \zeta \le 1$ in Ω and $\zeta = 1$ in $Q_R(\mathbf{x}_0)$,

iii) $|\nabla \zeta| \le \frac{2}{R}$.

Now we want to test equation (27) with

$$\boldsymbol{\eta} = \zeta^2 (\mathbf{u} - \boldsymbol{\mu}) \qquad \text{with} \quad \boldsymbol{\mu} \in \mathbb{R}^n.$$

We compute

$$\mathcal{E}(\boldsymbol{\eta}) = \zeta^2 \mathcal{E}(\mathbf{u}) + \zeta \left((\mathbf{u} - \boldsymbol{\mu}) (\nabla \zeta)^t + \nabla \zeta \, (\mathbf{u} - \boldsymbol{\mu})^t \right).$$

Due to the symmetry of $W_{,\mathcal{E}}(\mathbf{c}, \mathcal{E}(\mathbf{u}))$ we obtain

$$\int_\Omega \zeta^2 \, W_{,\mathcal{E}}(\mathbf{c}, \mathcal{E}(\mathbf{u})) : \mathcal{E}(\mathbf{u}) + 2 \int_\Omega \zeta W_{,\mathcal{E}}(\mathbf{c}, \mathcal{E}(\mathbf{u})) : \left((\mathbf{u} - \boldsymbol{\mu}) (\nabla \zeta)^t \right) =$$
$$\int_\Omega \zeta^2 \mathcal{S}^* : \mathcal{E}(\mathbf{u}) + 2 \int_\Omega \zeta \mathcal{S}^* : \left((\mathbf{u} - \boldsymbol{\mu}) (\nabla \zeta)^t \right). \tag{28}$$

Assumptions (A4.2) and (A4.3) yield

$$c_1 |\mathcal{E}(\mathbf{u})|^2 \le W_{,\mathcal{E}}(\mathbf{c}, \mathcal{E}(\mathbf{u})) : \mathcal{E}(\mathbf{u}) + C_2 \left(|\mathbf{c}| + 1 \right) |\mathcal{E}(\mathbf{u})|$$

and

$$|W_{,\mathcal{E}}(\mathbf{c}, \mathcal{E}(\mathbf{u})) : \left((\mathbf{u} - \boldsymbol{\mu}) (\nabla \zeta)^t \right)| \le C_2 \left(|\mathcal{E}(\mathbf{u}) + |\mathbf{c}| + 1 \right) |\mathbf{u} - \boldsymbol{\mu}| \tfrac{2}{R}\,.$$

Since $\mathcal{S}^*$ is a constant tensor and using Young's inequality we can deduce from (28) the existence of a constant $C > 0$ depending on c_1, C_2 and $|\mathcal{S}^*|$ such that

$$
\begin{aligned}
c_1 \int_\Omega \zeta^2 |\mathcal{E}(\mathbf{u})|^2 \leq \\
&\leq C \int_\Omega \zeta^2 \left(|\mathbf{c}|^2 + 1\right) + C \int_\Omega \zeta \left(|\mathcal{E}(\mathbf{u})| + |\mathbf{c}| + 1\right) |\mathbf{u} - \boldsymbol{\mu}| \tfrac{2}{R} \\
&\leq C \int_\Omega \zeta^2 \left(|\mathbf{c}|^2 + 1\right) + \tfrac{C}{R^2} \int_{Q_{2R}(\mathbf{x}_0)} |\mathbf{u} - \boldsymbol{\mu}|^2.
\end{aligned}
$$

Employing

$$
\mathcal{E}(\zeta(\mathbf{u} - \boldsymbol{\mu})) = \zeta \mathcal{E}(\mathbf{u}) + \tfrac{1}{2} \left((\mathbf{u} - \boldsymbol{\mu})(\nabla \zeta)^t + \nabla \zeta (\mathbf{u} - \boldsymbol{\mu})^t\right)
$$

we obtain

$$
\int_\Omega |\mathcal{E}\left(\zeta(\mathbf{u} - \boldsymbol{\mu})\right)|^2 \leq 2 \left(\int_\Omega \zeta^2 |\mathcal{E}(\mathbf{u})|^2 + \int_\Omega |\mathbf{u} - \boldsymbol{\mu}|^2 |\nabla \zeta|^2\right).
$$

Now we can apply Korn's inequality for functions with boundary value zero to conclude

$$
\int_\Omega |\nabla\left(\zeta(\mathbf{u} - \boldsymbol{\mu})\right)|^2 \leq C \int_\Omega \zeta^2 \left(|\mathbf{c}|^2 + 1\right) + \tfrac{C}{R^2} \int_{Q_{2R}(\mathbf{x}_0)} |\mathbf{u} - \boldsymbol{\mu}|^2. \quad (29)
$$

Since

$$
\nabla\left(\zeta(\mathbf{u} - \boldsymbol{\mu})\right) = \zeta \nabla \mathbf{u} + (\mathbf{u} - \boldsymbol{\mu})(\nabla \zeta)^t
$$

we derive from (29) that

$$
\fint_{Q_R(\mathbf{x}_0)} |\nabla \mathbf{u}|^2 \leq C \fint_{Q_{2R}(\mathbf{x}_0)} \left(|\mathbf{c}|^2 + 1\right) + \tfrac{C}{R^2} \fint_{Q_{2R}(\mathbf{x}_0)} |\mathbf{u} - \boldsymbol{\mu}|^2.
$$

Now we choose $\boldsymbol{\mu} = \fint_{Q_{2R}(\mathbf{x}_0)} \mathbf{u}$ and use the Poincaré–Sobolev inequality (see Theorem 7.2) to conclude

$$
\fint_{Q_R(\mathbf{x}_0)} |\nabla \mathbf{u}|^2 \leq C \fint_{Q_{2R}(\mathbf{x}_0)} \left(|\mathbf{c}|^2 + 1\right) + C \left(\fint_{Q_{2R}(\mathbf{x}_0)} |\nabla \mathbf{u}|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{n}}.
$$

Finally, Proposition 7.1 with $g = |\nabla \mathbf{u}|^{\frac{2n}{n+2}}$, $q = \frac{n+2}{n}$ and $f = C\left(|\mathbf{c}|^2 + 1\right)^{\frac{n}{n+2}}$ and a covering argument leads to the assertion. $\square$

Theorem 4.1. (Higher integrability) *Suppose that* $\mathbf{c} \in L^\sigma(\Omega, \mathbb{R}^N)$, $\sigma > 2$ *and that (A4) and (A5) hold.*

Then there exists a $p \in (2, \sigma]$, *independent of* $\mathbf{c}$, *such that for all* $\mathbf{u} \in H^1(\Omega, \mathbb{R}^n)$, *which fulfil for all* $\boldsymbol{\eta} \in H^1(\Omega, \mathbb{R}^n)$ *the identity*

$$
\int_\Omega W_{,\mathcal{E}}(\mathbf{c}, \mathcal{E}(\mathbf{u})) : \nabla \boldsymbol{\eta} = \int_\Omega \mathcal{S}^* : \nabla \boldsymbol{\eta} \quad (30)
$$

the integrability property

$$\nabla \mathbf{u} \in L^p(\Omega, \mathrm{I\!R}^{n\times n})$$

holds. In particular,

$$\|\nabla \mathbf{u}\|_{L^p(\Omega,\mathrm{I\!R}^{n\times n})} \leq C\left(\|\nabla \mathbf{u}\|_{L^2(\Omega,\mathrm{I\!R}^{n\times n})} + \|\mathbf{c}\|_{L^p(\Omega,\mathrm{I\!R}^n)} + 1\right)$$

where $C = C(\Omega, C_2, c_2, c_1, \mathcal{S}^*, n, \sigma, p)$ *is independent of* $\mathbf{c}$.

PROOF. The integrability in the interior follows from Lemma 4.1. Hence, it remains to show the higher integrability at the boundary. Since Ω has a Lipschitz boundary, there exist for all $\mathbf{x}_0 \in \partial\Omega$ a Lipschitz function $h : \mathrm{I\!R}^{n-1} \to \mathrm{I\!R}$ such that – upon relabelling and reorientation of the interface if necessary – the boundary $\partial\Omega$ locally around $\mathbf{x}_0$ is the graph of h. In addition, h can be chosen such that Ω locally lies on one side of the graph. To state this property precisely we define the sets

$$\begin{aligned} Q &:= \{\mathbf{y} \in \mathrm{I\!R}^n \,|\, |y_i| < R_0 \quad \text{for} \quad i = 1, ..., n\}, \\ Q^+ &:= \{\mathbf{y} \in Q \,|\, y_n > 0\}, \\ Q^- &:= \{\mathbf{y} \in Q \,|\, y_n < 0\}, \\ Q^0 &:= \{\mathbf{y} \in Q \,|\, y_n = 0\} \end{aligned}$$

and the transformation

$$\begin{aligned} \boldsymbol{\tau} : Q &\longrightarrow \mathrm{I\!R}^n, \\ \mathbf{y} &\longmapsto \boldsymbol{\tau}(\mathbf{y}) := (y_1, ..., y_{n-1}, h(y_1, ..., y_{n-1}) + y_n). \end{aligned}$$

Then we require that there exists a $R_0 > 0$ such that

$$\begin{aligned} \boldsymbol{\tau}\left(Q^+\right) &\subset \Omega, \\ \boldsymbol{\tau}\left(Q^-\right) &\subset \mathrm{I\!R}^n \setminus \bar{\Omega}. \end{aligned}$$

In what follows, we assume that $\mathbf{x}_0$, h and R_0 are chosen such that the above requirements hold. Now we define

$$\mathbf{v} : Q^+ \to \mathrm{I\!R}^n \qquad \text{and} \quad \mathbf{d} : Q^+ \to \mathrm{I\!R}^n$$

via

$$\mathbf{v} = \mathbf{u} \circ \boldsymbol{\tau} \qquad \text{and} \quad \mathbf{d} = \mathbf{c} \circ \boldsymbol{\tau}.$$

In addition, we set for all $\mathbf{y} \in Q$

$$g(\mathbf{y}) = \begin{cases} |\nabla \mathbf{v}|^{\frac{2n}{n+2}}(\mathbf{y}) & \text{if } \mathbf{y} \in Q^+, \\ 0 & \text{if } \mathbf{y} \in Q \setminus Q^+ . \end{cases}$$

Our goal is to apply Proposition 7.1 for the function g. This then shows higher integrability of $\nabla \mathbf{v}$ and by transformation also for $\nabla \mathbf{u}$.

Claim: There are constants $b, C > 0$ such that for all $\mathbf{y}_0 \in Q$ and all $R > 0$ with $2R < \operatorname{dist}(\mathbf{y}_0, \partial Q)$

$$\fint_{Q_R(\mathbf{y}_0)} g^q\, dy \le b \left(\fint_{Q_{2R}(\mathbf{y}_0)} g\, dy \right)^q + \fint_{Q_{2R}(\mathbf{y}_0)} f^q\, dy \tag{31}$$

where $q = \frac{n+2}{n}$ and $f = C\left(|\mathbf{d}|^2 + 1\right)^{\frac{n}{n+2}}$.

To prove the claim we choose a $\mathbf{y}_0 \in Q$ and a $R < \frac{1}{2}\operatorname{dist}(\mathbf{y}_0, \partial Q)$. Then there are three possibilities:

Case 1. $Q_{\frac{3}{2}R}(\mathbf{y}_0) \cap Q^+ = \emptyset$.

The left hand side in (31) in this case is zero and hence the inequality holds.

Case 2. $Q_{\frac{3}{2}R}(\mathbf{y}_0) \cap Q^- = \emptyset$.

Denoting by L the Lipschitz constant of h, it holds that $\boldsymbol{\tau}(Q_R(\mathbf{y}_0))$ and $\boldsymbol{\tau}(Q_{\frac{3}{2}R}(\mathbf{y}_0))$ have a distance larger than $\frac{R}{4}\min(1, \frac{1}{L})$. Hence, we can choose a cutoff function $\zeta \in C_0^\infty(\Omega)$ with the properties

i) $\zeta = 0$ in $\Omega \setminus \boldsymbol{\tau}\left((Q_{\frac{3}{2}R}(\mathbf{y}_0)\right)$,
ii) $0 \le \zeta \le 1$ in Ω and $\zeta = 1$ in $\boldsymbol{\tau}\left(Q_R(\mathbf{y}_0)\right)$,
iii) $|\nabla\zeta| \le \frac{8}{R}\max(1, L)$.

Testing (30) with $\boldsymbol{\eta} = \zeta^2(\mathbf{u} - \boldsymbol{\mu})$ where $\boldsymbol{\mu} \in \mathbb{R}^n$ and concluding as in Lemma 4.1 we obtain

$$\int_{\boldsymbol{\tau}(Q_R(\mathbf{y}_0))} |\nabla\mathbf{u}|^2 \le C \int_{\boldsymbol{\tau}\left(Q_{\frac{3}{2}R}(\mathbf{y}_0)\right)} \left(|\mathbf{c}|^2 + 1\right) + \frac{C}{R^2} \int_{\boldsymbol{\tau}\left(Q_{\frac{3}{2}R}(\mathbf{y}_0)\right)} |\mathbf{u} - \boldsymbol{\mu}|^2.$$

Transforming the integrals leads to

$$\int_{Q_R(\mathbf{y}_0)} |\nabla\mathbf{v}|^2 \le C \int_{Q_{\frac{3}{2}R}(\mathbf{y}_0)} \left(|\mathbf{d}|^2 + 1\right) + \frac{C}{R^2} \int_{Q_{\frac{3}{2}R}(\mathbf{y}_0)} |\mathbf{v} - \boldsymbol{\mu}|^2$$

where C depends on L. Choosing $\boldsymbol{\mu} = \fint_{Q_{\frac{3}{2}R}(\mathbf{y}_0)} \mathbf{v}$ and using the Sobolev–Poincaré inequality we deduce

$$\int_{Q_R(\mathbf{y}_0)} |\nabla\mathbf{v}|^2 \le C \int_{Q_{\frac{3}{2}R}(\mathbf{y}_0)} \left(|\mathbf{d}|^2 + 1\right) + \frac{C}{R^2} \left(\int_{Q_{\frac{3}{2}R}(\mathbf{y}_0)} |\nabla\mathbf{v}|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{n}}.$$

This implies

$$\int_{Q_R(\mathbf{y}_0)} g^q\, dy \le \frac{C}{R^2} \left(\int_{Q_{2R}(\mathbf{y}_0)} g\, dy \right)^q + \int_{Q_{2R}(\mathbf{y}_0)} f^q\, dy. \tag{32}$$

Multiplying by R^{-n} now gives the result.

Case 3.

$$Q_{\frac{3}{2}R}(\mathbf{y}_0)\cap Q^+\neq\emptyset \qquad \text{and} \quad Q_{\frac{3}{2}R}(\mathbf{y}_0)\cap Q^-\neq\emptyset. \tag{33}$$

For all $\bar{R}>0$ we define

$$Q^+_{\bar{R}}(\mathbf{y}_0):=Q_{\bar{R}}(\mathbf{y}_0)\cap Q^+ \qquad \text{and} \quad Q^-_{\bar{R}}(\mathbf{y}_0):=Q_{\bar{R}}(\mathbf{y}_0)\cap Q^-\,.$$

From (33) it is seen that

$$Q_{2R}(\mathbf{y}_0)\cap Q^0\neq\emptyset.$$

Hence, $\boldsymbol{\tau}\left(Q_{2R}(\mathbf{y}_0)\right)$ intersects the boundary of Ω, i.e.

$$\boldsymbol{\tau}\left(Q_{2R}(\mathbf{y}_0)\right)\cap\partial\Omega\neq\emptyset.$$

The Lipschitz continuity of h guarantees

$$\operatorname{dist}\left\{\boldsymbol{\tau}\left(\partial Q^+_{2R}(\mathbf{y}_0)\right)\cap\Omega,\boldsymbol{\tau}\left(\partial Q^+_{R}(\mathbf{y}_0)\right)\cap\Omega\right\}\geq\frac{R}{2}\min\left(1,\frac{1}{L}\right)$$

which hence allows us to choose a cutoff function $\zeta\in C^\infty(\Omega)$ with the properties

i) $\zeta=0$ in $\Omega\setminus\boldsymbol{\tau}\left((Q_{2R}(\mathbf{y}_0)\right)$,

ii) $0\leq\zeta\leq 1$ in Ω and $\zeta=1$ in $\boldsymbol{\tau}\left(Q_R(\mathbf{y}_0)\cap\Omega\right)$,

iii) $|\nabla\zeta|\leq\frac{4}{R}\max(1,L)$.

Since $\zeta(\mathbf{u}-\boldsymbol{\mu})=0$, where $\boldsymbol{\mu}\in\mathbb{R}^n$, on an open part of $\partial\Omega$, Korn's inequality holds for $\zeta(\mathbf{u}-\boldsymbol{\mu})=0$. Therefore, by testing (30) with $\boldsymbol{\eta}=\zeta^2(\mathbf{u}-\boldsymbol{\mu})$ we can proceed as in Case 2 to obtain

$$\int_{Q^+_R(\mathbf{y}_0)}|\nabla\mathbf{v}|^2\leq\int_{Q^+_{2R}(\mathbf{y}_0)}C\left(|\mathbf{d}|^2+1\right)+\frac{C}{R^2}\int_{Q^+_{2R}(\mathbf{y}_0)}|\mathbf{v}-\boldsymbol{\mu}|^2.$$

From (33) we conclude

$$\mathcal{L}^n\left(Q^+_{2R}(\mathbf{y}_0)\right)\geq cR^n$$

and

$$diam Q^+_{2R}(\mathbf{y}_0)\leq CR.$$

With the help of the Sobolev–Poincaré inequality we deduce

$$\int_{Q^+_R(\mathbf{y}_0)}|\nabla\mathbf{v}|^2\leq$$

$$\int_{Q^+_{2R}(\mathbf{y}_0)}C\left(|\mathbf{d}|^2+1\right)+C\left(\mathcal{L}^n\left(Q^+_{2R}(\mathbf{y}_0)\right)\right)^{-\frac{2}{n}}\left(\int_{Q^+_{2R}(\mathbf{y}_0)}|\nabla\mathbf{v}|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{n}}$$

and hence

$$\int_{Q_R^+(\mathbf{y}_0)} |\nabla \mathbf{v}|^2 \le \int_{Q_{2R}^+(\mathbf{y}_0)} C\left(|\mathbf{d}|^2+1\right) + \frac{C}{R^2}\left(\int_{Q_{2R}^+(\mathbf{y}_0)} |\nabla \mathbf{v}|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{n}}.$$

If we integrate over the larger sets $Q_R(\mathbf{y}_0)$ and $Q_{2R}(\mathbf{y}_0)$ respectively we obtain (32). Hence, as before we multiply by R^{-n} and deduce (31).

Now (31) and Proposition 7.1 give the higher integrability at the boundary. The higher integrability at the boundary, Lemma 4.1 and a covering argument imply the desired conclusion. □

5. Higher integrability for the logarithmic free energy

The equation for the chemical potential differences $\mathbf{w}^\delta$ of the regularised system is

$$\mathbf{w}^\delta = -\gamma\Delta\mathbf{c}^\delta + \theta\mathbf{P}\left(\phi^\delta(c_k^\delta)\right)_{k=1,\dots,N} + \mathbf{P}\mathbf{A}\mathbf{c}^\delta + \mathbf{P}W_{,\mathbf{c}}(\mathbf{c}^\delta, \mathcal{E}(\mathbf{u}^\delta)), \quad (34)$$

where we define $\phi^\delta := \left(\psi^\delta\right)'$. The function ψ^δ is an approximation of $\psi(d) = d\ln d$ and hence $\phi^\delta = \left(\psi^\delta\right)'$ becomes singular as $\delta \to 0$. We remark that ψ^δ was chosen such that ϕ^δ is monotone (see (22)). This is crucial in order to show that $\phi^\delta(c_k^\delta)$ is uniformly bounded in $L^q(\Omega_T)$ for some $q>1$. We will achieve this by testing the weak formulation of the equation for w_k with an appropriate power of $\phi^\delta(c_k^\delta)$.

Lemma 5.1. *There exist constants $q>1$ and $C>0$ such that for all $\delta \in (0, \min(\frac{1}{N}, \delta_0))$ and all $k \in \{1, ..., N\}$*

$$||\phi^\delta(c_k^\delta)||_{L^q(\Omega_T)} \le C.$$

Proof. Let $r>0$. Then we define

$$\phi_r^\delta(d) = \phi^\delta(d)|\phi^\delta(d)|^{r-1}$$

where ϕ_r^δ is defined to be zero if $\phi^\delta(d)$ is zero, and hence ϕ_r^δ is continuous on $\mathbb{R}$. For $r \in (0,1)$ the function ϕ_r^δ is not differentiable at the zero of the function ϕ^δ. Hence, for $\rho>0$ we define a monotone C^1 function $\phi_r^{\delta,\rho}$ which equals ϕ_r^δ on $\mathbb{R}\setminus[0,1]$ and which converges to ϕ_r^δ in $C(\mathbb{R})$ as $\rho \to 0$.

The weak formulation of the equation for the chemical potential differences $\mathbf{w}^\delta$ is

$$\begin{aligned}\int_{\Omega_T} \mathbf{w}^\delta \cdot \boldsymbol{\zeta} \;&=\; \int_{\Omega_T} \Big\{\gamma\nabla\mathbf{c}^\delta : \nabla\boldsymbol{\zeta} + \theta\mathbf{P}\left(\phi^\delta(c_k^\delta)\right)_{k=1,\dots,N}\cdot\boldsymbol{\zeta} \\ &\qquad + \mathbf{P}\mathbf{A}\mathbf{c}^\delta\cdot\boldsymbol{\zeta} + \mathbf{P}W_{,\mathbf{c}}(\mathbf{c}^\delta, \mathcal{E}(\mathbf{u}^\delta))\cdot\boldsymbol{\zeta}\Big\}\end{aligned} \quad (35)$$

with $\boldsymbol{\zeta} \in L^2(0,T;H^1(\Omega,\mathbb{R}^N)) \cap L^\infty(\Omega_T,\mathbb{R}^N)$. The Sobolev embedding theorem and the estimates of Lemma 3.2 imply that $\mathbf{c}^\delta$ lies in $L^\infty(0,T;L^{\frac{2n}{n-2}}(\Omega))$ if $n \geq 3$, in $L^\infty(0,T;L^s(\Omega))$ for all $s \in [1,\infty)$ if $n=2$ and in $L^\infty(\Omega_T)$ if $n=1$. We can deduce from the higher integrability result from the last subsection (see Theorem 4.1) that $\nabla \mathbf{u}^\delta \in L^\infty(0,T;L^p(\Omega))$ (for some $p>2$). We choose p such that $p \in (2,4]$ and such that in addition $p \in \left(2, \frac{2n}{n-2}\right)$ if $n \geq 3$. Hence, $W_{,\mathbf{c}}(\mathbf{c}^\delta, \mathcal{E}(\mathbf{u}^\delta)) \in L^\infty(0,T;L^{\frac{p}{2}}(\Omega))$. This implies that also test functions $\boldsymbol{\zeta} \in L^2(0,T;H^1(\Omega,\mathbb{R}^N)) \cap L^{\frac{p}{p-2}}(\Omega_T,\mathbb{R}^N)$ are allowed in (35).

We test (35) with the function

$$\boldsymbol{\zeta} = \left(\phi_r^{\delta,\rho}(c_k^\delta)\right)_{k=1,\dots N},$$

which is admissible for all $r \in (0,1]$ with $r\frac{p}{p-2} \leq 2$ (note that $\phi^\delta(d)$ is sub-linear in d). We obtain

$$\begin{aligned}
&\int_{\Omega_T} \sum_{k=1}^N w_k^\delta \cdot \phi_r^{\delta,\rho}(c_k^\delta) = \qquad\qquad (36)\\
&\quad \int_{\Omega_T} \sum_{k=1}^N \left\{ \gamma \nabla c_k^\delta \cdot \nabla \phi_r^{\delta,\rho}(c_k^\delta) + \theta \left[\phi^\delta(c_k^\delta) - \frac{1}{N}\left(\sum_{l=1}^N \phi^\delta(c_l^\delta)\right)\right] \phi_r^{\delta,\rho}(c_k^\delta) \right\}\\
&\quad + \int_{\Omega_T} \left\{ \mathbf{PAc}^\delta \cdot \left(\phi_r^{\delta,\rho}(c_k^\delta)\right)_{k=1,\dots N} + \mathbf{P}W_{,\mathbf{c}}(\mathbf{c}^\delta, \mathcal{E}(\mathbf{u}^\delta)) \cdot \left(\phi_r^{\delta,\rho}(c_k^\delta)\right)_{k=1,\dots N} \right\}.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\sum_{k=1}^N \left[\phi^\delta(c_k^\delta) - \frac{1}{N}\left(\sum_{l=1}^N \phi^\delta(c_l^\delta)\right)\right] \phi_r^{\delta,\rho}(c_k^\delta)\\
&= \frac{1}{N} \sum_{k,l=1}^N \left(\phi^\delta(c_k^\delta) - \phi^\delta(c_l^\delta)\right) \phi_r^{\delta,\rho}(c_k^\delta)\\
&= \frac{1}{N} \sum_{k<l}^N \left(\phi^\delta(c_k^\delta) - \phi^\delta(c_l^\delta)\right) \phi_r^{\delta,\rho}(c_k^\delta) + \frac{1}{N} \sum_{k>l}^N \left(\phi^\delta(c_k^\delta) - \phi^\delta(c_l^\delta)\right) \phi_r^{\delta,\rho}(c_k^\delta)\\
&= \frac{1}{N} \sum_{k<l}^N \left(\phi^\delta(c_k^\delta) - \phi^\delta(c_l^\delta)\right) \left(\phi_r^{\delta,\rho}(c_k^\delta) - \phi_r^{\delta,\rho}(c_l^\delta)\right) \geq 0
\end{aligned}$$

since both ϕ^δ and $\phi_r^{\delta,\rho}$ are monotone increasing.

Using that $\left(\phi_r^{\delta,\rho}\right)' \geq 0$ we conclude that the first term on the right hand side of (36) is non–negative. Hence, (36) implies

$$
\begin{aligned}
&\theta \int_{\Omega_T} \frac{1}{N} \sum_{k<l}^{N} \left(\phi^\delta(c_k^\delta) - \phi^\delta(c_l^\delta)\right)\left(\phi_r^{\delta,\rho}(c_k^\delta) - \phi_r^{\delta,\rho}(c_l^\delta)\right) \leq \\
\leq\ & \int_{\Omega_T} \left\{ \sum_{k=1}^{N} w_k^\delta \cdot \phi_r^{\delta,\rho}(c_k^\delta) - \right. \\
& \left. \mathbf{PA}\mathbf{c}^\delta \cdot \left(\phi_r^{\delta,\rho}(c_k^\delta)\right)_{k=1,\dots N} - \mathbf{P}W_{,\mathbf{c}}(\mathbf{c}^\delta, \mathcal{E}(\mathbf{u}^\delta)) \cdot \left(\phi_r^{\delta,\rho}(c_k^\delta)\right)_{k=1,\dots N} \right\} \\
\leq\ & C \max_{k=1,\dots,N} \|\phi_r^{\delta,\rho}(c_k^\delta)\|_{L^2(\Omega_T)} \left(\|\mathbf{w}^\delta\|_{L^2(\Omega_T)} + \|\mathbf{c}^\delta\|_{L^2(\Omega_T)}\right) \\
& + C \left(\int_{\Omega_T} |W_{,\mathbf{c}}(\mathbf{c}^\delta, \mathcal{E}(\mathbf{u}^\delta))|^{\frac{p}{2}}\right)^{\frac{2}{p}} \left(\max_{k=1,\dots,N} \int_{\Omega_T} |\phi_r^{\delta,\rho}(c_k^\delta)|^{\frac{p}{p-2}}\right)^{1-\frac{2}{p}}.
\end{aligned}
$$

Passing to the limit $\rho \searrow \mathbf{0}$ and using Theorem 4.1, the a priori estimates of Lemma 3.2 and Lemma 3.3 and the inequalities of Hölder and Young proves that there exists for all $\alpha > 0$ a constant C_α such that

$$
\begin{aligned}
\theta \int_{\Omega_T} \frac{1}{N} \sum_{k<l}^{N} \left(\phi^\delta(c_k^\delta) - \phi^\delta(c_l^\delta)\right)\left(\phi_r^\delta(c_k^\delta) - \phi_r^\delta(c_l)\right) \leq \\
\leq \alpha \left(\max_{k=1,\dots,N} \int_{\Omega_T} |\phi_r^\delta(c_k^\delta)|^{\frac{p}{p-2}}\right) + C_\alpha.
\end{aligned} \tag{37}
$$

Moreover, since $\sum_{k=1}^{N} c_k^\delta = 1$, we have

$$
\min_{k=1}^{N} c_k^\delta \leq \frac{1}{N} \leq \max_{k=1}^{N} c_k^\delta.
$$

Using this, the fact that ϕ^δ and ϕ^δ_r are monotone increasing functions and Young's inequality we deduce

$$
\begin{aligned}
&\int_{\Omega_T} \sum_{k<l}^{N} \left(\phi^\delta(c_k^\delta) - \phi^\delta(c_l^\delta)\right)\left(\phi^\delta_r(c_k^\delta) - \phi^\delta_r(c_l)\right) \geq \\
&\geq \int_{\Omega_T} \max_{k=1}^{N} |\phi^\delta(c_k^\delta) - \phi^\delta(\frac{1}{N})||\phi^\delta_r(c_k^\delta) - \phi^\delta_r(\frac{1}{N})| \\
&\geq \int_{\Omega_T} \max_{k=1}^{N} \left| |\phi^\delta(c_k^\delta)|^{r+1} - \phi^\delta(\frac{1}{N})\phi^\delta_r(c_k^\delta) - \phi^\delta(c_k^\delta)\phi^\delta_r(\frac{1}{N}) + |\phi^\delta(\frac{1}{N})|^{r+1} \right| \\
&\geq \int_{\Omega_T} \max_{k=1}^{N} \left(|\phi^\delta(c_k^\delta)|^{r+1} - |\phi^\delta(\frac{1}{N})||\phi^\delta(c_k^\delta)|^r - |\phi^\delta(c_k^\delta)||\phi^\delta(\frac{1}{N})|^r + |\phi^\delta(\frac{1}{N})|^{r+1} \right) \\
&\geq \frac{1}{2}\int_{\Omega_T} \max_{k=1}^{N} |\phi^\delta(c_k^\delta)|^{r+1} - C.
\end{aligned}
$$

If $\delta < \frac{1}{N}$ then $\phi^\delta(\frac{1}{N}) = \phi(\frac{1}{N})$, which ensures that the constant C is independ of δ. Together with (37) we have

$$
\int_{\Omega_T} \max_{k=1}^{N} |\phi^\delta(c_k^\delta)|^{r+1} \leq \frac{N}{2\theta}\alpha \left(\max_{k=1,\dots,N} \int_{\Omega_T} |\phi^\delta(c_k^\delta)|^{\frac{p}{p-2}r} \right) + C_\alpha.
$$

Setting $r = \frac{p-2}{2}$ and choosing α small enough gives the result.

□

6. Proof of the existence theorem

Proof of Theorem 2.1. We need to show that the limit $(\mathbf{c}, \mathbf{w}, \mathbf{u})$ obtained in Lemma 3.2 and Lemma 3.3 solves the elastic Cahn–Hilliard system with logarithmic free energy. To pass to the limit in the weak formulations of the equations

$$
\partial_t \mathbf{c}^\delta = \mathbf{L}\Delta \mathbf{w}^\delta \qquad \text{and} \quad \nabla \cdot \left[W_{,\mathcal{E}}(\mathbf{c}^\delta, \mathcal{E}(\mathbf{u}^\delta))\right] = 0
$$

one can apply standard arguments using the convergence properties of $(\mathbf{c}^\delta, \mathbf{w}^\delta, \mathbf{u}^\delta)$ and the growth condition on $W_{,\mathcal{E}}$ (see e.g. [11]). It remains to pass to the limit in

$$
\mathbf{w}^\delta = -\gamma\Delta\mathbf{c}^\delta + \theta\mathbf{P}\left(\phi^\delta(c_k^\delta)\right)_{k=1,\dots,N} + \mathbf{P}\mathbf{A}\mathbf{c}^\delta + \mathbf{P}W_{,\mathbf{c}}(\mathbf{c}^\delta, \mathcal{E}(\mathbf{u}^\delta)). \quad (38)
$$

Except for the term $\phi^\delta(c_k^\delta)$ one can use standard arguments (see e.g. the proof of the existence theorem in [11] for a similar context).

Let us show that $\phi^\delta(c_k^\delta)$ converges almost everywhere to $\phi(c_k)$ and that $c_k > 0$ almost everywhere. Using the convergence a.e. of c_k^δ to c_k, the Fatou lemma and Lemma 5.1 we obtain

$$\int_{\Omega_T} \liminf_{\delta\to 0} |\phi^\delta(c_k^\delta)|^q \le \liminf_{\delta\to 0} \int_{\Omega_T} |\phi^\delta(c_k^\delta)|^q \le C.$$

Next we prove that

$$\lim_{\delta\to 0} \phi^\delta(c_k^\delta) = \begin{cases} \phi(c_k) & \text{if } \lim_{\delta\to 0} c_k^\delta = c_k > 0, \\ \infty & \text{elsewhere} \end{cases} \tag{39}$$

almost everywhere. First we take $(\mathbf{x}, t) \in \Omega_T$ with $\lim_{\delta\to 0} c_k^\delta(\mathbf{x}, t) = c_k(\mathbf{x}, t) > 0$. Since $\phi^\delta(d) = \left(\psi^\delta\right)'(d) = \psi'(d) = \phi(d)$ for $d \ge \delta$ we obtain $\phi^\delta(c_k^\delta(\mathbf{x}, t)) \to \phi(c_k(\mathbf{x}, t))$. Now assume that $(\mathbf{x}, t) \in \Omega_T$ is such that $\lim_{\delta\to 0} c_k^\delta(\mathbf{x}, t) = c_k(\mathbf{x}, t) \le 0$. Then we obtain for δ small enough

$$|\phi^\delta(c_k^\delta(\mathbf{x}, t))| \ge \phi(\max\left(c_k^\delta(\mathbf{x}, t), \delta\right)).$$

The right hand side converges to ∞ as δ tends to zero which proves (39). Using (39) and Lemma 5.1, we obtain

$$c_k > 0 \qquad \text{almost everywhere,}$$

$$\int_{\Omega_T} |\phi(c_k)|^q \le C$$

and

$$\phi^\delta(c_k^\delta) \to \phi(c_k) \qquad \text{almost everywhere.}$$

Since $q > 1$ we conclude with Vitali's theorem

$$\phi^\delta(c_k^\delta) \to \phi(c_k) \qquad \text{in} \quad L^1(\Omega_T).$$

This is enough to pass to the limit in the weak formulation of (38).

□

7. Appendix

In this section we collect some known results used in the text.

Theorem 7.1. (Korn's inequality) *Let Ω be a bounded domain with Lipschitz boundary.*
i) There exists a constant $c > 0$ such that

$$\int_\Omega \mathcal{E}(\mathbf{u}) : \mathcal{E}(\mathbf{u}) \ge c\|\mathbf{u}\|_{H^1}^2$$

for all $u \in X_2 := \{\mathbf{u} \in H^1(\Omega, \mathbb{R}^n) \,|\, (\mathbf{u}, \mathbf{v})_{H^1} = 0$ for all $\mathbf{v} \in X_{ird}\} = X_{ird}^\perp$ where $X_{ird} := \{\mathbf{u} \in H^1(\Omega, \mathbb{R}^n) \,|$ there exist $\mathbf{b} \in \mathbb{R}^n$ and a skew symmetric $\mathbf{A} \in \mathbb{R}^{n\times n}$ such that $\mathbf{u}(\mathbf{x}) = \mathbf{b} + \mathbf{A}\mathbf{x}\}$.

ii) Let Γ be an open subset of the boundary $\partial\Omega$ and let $X_\Gamma := \{\mathbf{u} \in H^1(\Omega, \mathbb{R}^n) \,|\, \mathbf{u}_{|\Gamma} = 0\}$.

Then there exists a constant $c > 0$ such that

$$\int_\Omega \mathcal{E}(\mathbf{u}) : \mathcal{E}(\mathbf{u}) \geq c\|\mathbf{u}\|_{H^1}^2 \text{ for all } u \in X_\Gamma.$$

A proof can be found for example in Zeidler [27].

We needed a version of the Sobolev–Poincaré inequality for rectangles in which the dependence of the Sobolev-Poincaré constant on the diameter is specified. We demonstrate how such an estimate can be derived from the Sobolev–Poincaré inequality on the unit cube.

Theorem 7.2. (Sobolev–Poincaré inequality) *There exists a constant $C(n,p)$ such that*

$$\left(\fint_D |u - \fint_D u|^{p^*}\right)^{\frac{1}{p^*}} \leq C(n,p)(diam\, D)\left(\fint_D |\nabla u|^p\right)^{\frac{1}{p}}$$

for all rectangles $D \subset \mathbb{R}^n$ and all $u \in W^{1,p}(D)$. Here, $p \in (1,n)$, $p^* = \frac{np}{n-p}$ and $diam\, D$ is the diameter of D.

Proof. Without loss of generality we assume that

$$D = D_\mathbf{f} := \left\{ \sum_{i=1}^n x_i \mathbf{e}_i \,|0 \leq x_i \leq f_i \right\}$$

with $\mathbf{f} = (f_1, ..., f_n)$ and $0 < f_1 \leq ... \leq f_n$. All other situations can be reduced to one of these cases by a translation and a orthogonal transformation.

The Sobolev–Poincaré inequality

$$\left(\int_{D_\mathbf{e}} |v - \fint_{D_\mathbf{e}} v|^{p^*}\right)^{\frac{1}{p^*}} \leq C(n,p)\left(\int_{D_\mathbf{e}} |\nabla v|^p\right)^{\frac{1}{p}} \tag{40}$$

holds for all $v \in W^{1,p}(D_\mathbf{e})$ with a fixed constant $C(n,p)$ where $D_\mathbf{e}$ is the unit cube, i.e. $\mathbf{e} = (1, ..., 1) \in \mathbb{R}^n$. Now let $D = D_\mathbf{f}$ and $u \in W^{1,p}(D)$. Then we define

$$v(\mathbf{y}) = u(f_1 y_1, ..., f_n y_n) \qquad \text{for all } \mathbf{y} \in D_\mathbf{e}$$

and obtain

$$\nabla v(\mathbf{y}) = (f_i \partial_i u(f_1 y_1, ..., f_n y_n))_{i=1,...,n}\,. \tag{41}$$

Hence

$$|\nabla v(\mathbf{y})|^p \leq f_n^p |\nabla u(f_1 y_1, ..., f_n y_n)|^p.$$

Changing variables in (40) and using $\fint_{D_\mathbf{e}} v = \fint_{D_\mathbf{f}} u$ we obtain

$$\left(\int_D |u - \fint_D u|^{p^*}(f_1 \cdot \ldots \cdot f_n)^{-1}\right)^{\frac{1}{p^*}} \leq C(n,p) f_n \left(\int_D |\nabla u|^p (f_1 \cdot \ldots \cdot f_n)^{-1}\right)^{\frac{1}{p}}.$$

Since $\mathcal{L}^n(D) = f_1 \cdot \ldots \cdot f_n$ and since $f_n \leq diam D$ the theorem follows. □

Proposition 7.1. *Let $Q \subset \mathbb{R}^n$ be a cube, $g \in L^q_{loc}(Q)$ for a $q > 1$ and $g \geq 0$. Suppose that there exist a constant $b > 0$ and a function $f \in L^r_{loc}(Q)$ with $r > q$ and $f \geq 0$ such that*

$$\fint_{Q_R(\mathbf{x}_0)} g^q \, dx \leq b \left(\fint_{Q_{2R}(\mathbf{x}_0)} g \, dx\right)^q + \fint_{Q_{2R}(\mathbf{x}_0)} f^q \, dx$$

for each $\mathbf{x}_0 \in Q$ and all $R > 0$ with $2R < \operatorname{dist}(\mathbf{x}_0, \partial Q)$.

Then $g \in L^s_{loc}(Q)$ for $s \in [q, q+\varepsilon)$ for some $\varepsilon > 0$ and

$$\left(\fint_{Q_R(\mathbf{x}_0)} g^s \, dx\right)^{\frac{1}{s}} \leq c \left\{ \left(\fint_{Q_{2R}(\mathbf{x}_0)} g^q \, dx\right)^{\frac{1}{q}} + \left(\fint_{Q_{2R}(\mathbf{x}_0)} f^s \, dx\right)^{\frac{1}{s}} \right\}$$

for all $\mathbf{x}_0 \in Q$ and $R > 0$ such that $Q_{2R}(\mathbf{x}_0) \subset Q$. The positive constants c and ε depend on b, q, n and r.

For a proof of this proposition we refer to the book of Giaquinta [14] or the paper of Giaquinta and Modica [15].

References

[1] J.W. Barrett and J.F. Blowey, *An error bound for the finite element approximation of the Cahn–Hilliard equation with logarithmic free anergy,* Numer. Math., 72 (1995), pp. 257–287.

[2] J.W. Barrett and J.F. Blowey, *An error bound for the finite element approximation of a model for phase separation of a multi–component alloy,* IMA J. Numer. Anal., 16 (1996), pp. 257–287.

[3] E. Bonetti, P. Colli, W. Dreyer, G. Gilardi, G. Schimperna and J. Sprekels, A solid-solid phase change model accounting for mechanical effects, Physica D, 165 (2002), pp. 48–65.

[4] J. W. Cahn and J. E. Hilliard, *Free energy of a nonuniform system. I. Interfacial free energy,* Journal of Chemical Physics, Vol. 28 (1958), pp. 258–267.

[5] D. De Fontaine, *An analysis of clustering and ordering in multicomponent solid solutions–I. stability criteria,* Journal Phys. Chem. Solids, 33 (1972), pp. 287–310.

[6] C.M. Elliott, *The Cahn-Hilliard model for the kinetics of phase transitions,* in Mathematical Models for Phase Change Problems, J.F. Rodrigues (ed.), International Series of Numerical Mathematics, Vol. 88, Birkhäuser-Verlag, Basel, pp. 35–73.

[7] C. M. Elliott and S. Luckhaus *A generalised diffusion equation for phase separation of a multi–component mixture with interfacial free energy*, SFB256 University Bonn, Preprint 195 (1991).

[8] J. D. Eshelby, *Elastic inclusions and inhomogeneities,* Prog. Solid Mech., 2 (1961), pp. 89–140.

[9] P. Fratzl, O. Penrose, and J. L. Lebowitz, *Modelling of phase separation in alloys with coherent elastic misfit*, J. Stat. Physics, 95 5/6 (1999), pp. 1429–1503.

[10] H. Garcke, *On mathematical models for phase separation in elastically stressed solids*, habilitation thesis (2000).

[11] H. Garcke, On Cahn-Hilliard sytems with elasticity, Proc. Roy. Soc. Edinburgh A (to appear).

[12] H. Garcke, M. Rumpf, and U. Weikard, *The Cahn-Hilliard equation with elasticity: Finite element approximation and qualitative studies,* Interfaces and Free Boundaries, 3 (2001), pp. 101–118.

[13] F. W. Gehring, *The L^p–integrability of the partial derivatives of a quasi conformal mapping,* Acta Math., 130 (1973), pp. 265–277.

[14] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems,* Annals of Mathematics Studies, Princeton University Press, (1983).

[15] M. Giaquinta and G. Modica, *Regularity results for some classes of higher order nonlinear elliptic systems*, J. für reine u. angew. Math., 311/312 (1979), 145–169.

[16] J. J. Hoyt, *The continuum theory of nucleation in multicomponent systems,* Acta Metallurgica 38 (1990), pp. 1405–1412.

[17] A. G. Khachaturyan, *Some questions concerning the theory of phase transitions in solids,* Fiz. Tverd. Tela., 8 (1966), pp. 2709–2717. English translation in Sov. Phys. Solid. state 8 (1966), pp. 2163.

[18] J. S. Kirkaldy and D. J. Young, *Diffusion in the Condensed State,* The Institute of Metals, London (1987).

[19] F. C. Larché and J. W. Cahn, *Thermochemical equilibrium of multiphase solids under stress*, Acta Metall., 26 (1978), pp. 1579–1589.

[20] F. C. Larché and J. W. Cahn, *The effect of self–stress on diffusion in solids,* Acta Metall., 30 (1982), pp. 1835–1845.

[21] P. H. Leo, J. S. Lowengrub, and H. J. Jou, *A diffuse interface model for microstructural evolution in elastically stressed solids*, Acta Mater., 46 (1998), pp. 2113–2130.

[22] A. Novick–Cohen, *The Cahn-Hilliard equation: mathematical and modelling perspectives.* Adv. Math. Sci. Appl., 8 (1998), pp. 965–985.

[23] L. Onsager, *Reciprocal relations in irreversible processes I,* Phys. Rev. 37 (1931), pp. 405–426.

[24] L. Onsager, *Reciprocal relations in irreversible processes II,* Phys. Rev. 38 (1931), pp. 2265–2279.

[25] A. Onuki, *Ginzburg–Landau approach to elastic effects in the phase separation of solids*, J. Phys. Soc. Jpn., 58 (1989), pp. 3065–3068.

[26] J. D. van der Waals, *The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density (in Dutch),* Verhaendel. Kronik. Akad. Weten. Amsterdam, Vol. 1, (1893), Engl. translation by J.S. Rowlinson, J. Stat. Phys., 20 (1979), pp. 197–244.

[27] E. Zeidler, *Nonlinear Functional Analysis and its Applications IV*, Springer, New–York, Berlin, Heidelberg (1988).