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Motivic Galois Theory for Motives of Level ≤ 1

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Abstract. Let $\mathcal{T}$ be a Tannakian category over a field k of characteristic 0 and let $\pi(\mathcal{T})$ be its fundamental group. First we prove that there is a bijection between the Tannakian subcategories of $\mathcal{T}$ and the normal affine group sub-$\mathcal{T}$-schemes of $\pi(\mathcal{T})$. Then we apply this result to the Tannakian category $\mathcal{T}_1(k)$ generated by motives of level ≤ 1 defined over k, whose fundamental group is called the motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ of motives of level ≤ 1. We find four short exact sequences of affine group sub-$\mathcal{T}_1(k)$-schemes of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$, correlated one to each other by inclusions and projections. Moreover, given a 1-motive M, we compute explicitly the biggest Tannakian subcategory of the one generated by M, whose fundamental group is commutative.

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Introduction.

Let k be a field of characteristic 0 and let $\overline{k}$ be its algebraic closure. Let $\mathcal{T}$ be a Tannakian category over k. The tensor product of $\mathcal{T}$ allows us to define the notion of Hopf algebras in the category Ind$\mathcal{T}$ of Ind-objects of $\mathcal{T}$. The category of affine group $\mathcal{T}$-schemes is the opposite of the category of Hopf algebras in Ind$\mathcal{T}$. The fundamental group $\pi(\mathcal{T})$ of $\mathcal{T}$ is the affine group $\mathcal{T}$-scheme $\mathrm{Sp}(\Lambda)$, whose Hopf algebra Λ is endowed for each object X of $\mathcal{T}$ with a morphism $X \longrightarrow \Lambda \otimes X$ functorial in X, and is universal for these properties. Those morphisms $\{X \longrightarrow \Lambda \otimes X\}_{X \in \mathcal{T}}$ define *an action of the fundamental group* $\pi(\mathcal{T})$ *on each object of* $\mathcal{T}$. For each Tannakian subcategory $\mathcal{T}'$ of $\mathcal{T}$, let $H_{\mathcal{T}}(\mathcal{T}')$ be the kernel of the faithfully flat morphism of group $\mathcal{T}$-schemes $I : \pi(\mathcal{T}) \longrightarrow i\pi(\mathcal{T}')$ corresponding to the inclusion functor $i : \mathcal{T}' \longrightarrow \mathcal{T}$. In particular we have the short exact sequence of group $\pi(\mathcal{T})$-schemes

$$0 \longrightarrow H_{\mathcal{T}}(\mathcal{T}') \longrightarrow \pi(\mathcal{T}) \longrightarrow i\pi(\mathcal{T}') \longrightarrow 0. \tag{0.1}$$

In [6] 6.6, Deligne proves that the Tannakian category $\mathcal{T}'$ is equivalent, as tensor category, to the subcategory of $\mathcal{T}$ generated by those objects on which the action of $\pi(\mathcal{T})$ induces a trivial action of $H_{\mathcal{T}}(\mathcal{T}')$. In particular, this implies that the fundamental group $\pi(\mathcal{T}')$ of $\mathcal{T}'$ is isomorphic to the group $\mathcal{T}$-scheme $\pi(\mathcal{T})/H_{\mathcal{T}}(\mathcal{T}')$. The group $\pi(\mathcal{T})$-scheme $H_{\mathcal{T}}(\mathcal{T}')$ characterizes the Tannakian subcategory $\mathcal{T}'$ modulo $\otimes$-equivalence. In fact we have the following result (theorem 1.6): *there is bijection between the Tannakian subcategories of $\mathcal{T}$ and the normal affine group sub-$\mathcal{T}$-schemes of $\pi(\mathcal{T})$, which associates*

- to each Tannakian subcategory $\mathcal{T}'$ of $\mathcal{T}$, the kernel $H_{\mathcal{T}}(\mathcal{T}')$ of the morphism of $\mathcal{T}$-schemes $I : \pi(\mathcal{T}) \longrightarrow i\pi(\mathcal{T}')$ corresponding to the inclusion $i : \mathcal{T}' \longrightarrow \mathcal{T}$;

- to each normal affine group sub-$\mathcal{T}$-scheme H of $\pi(\mathcal{T})$, the Tannakian subcategory $\mathcal{T}(H)$ of objects of $\mathcal{T}$ on which the action of $\pi(\mathcal{T})$ induces a trivial action of H.

Hence we obtain a clear dictionary between Tannakian subcategories of $\mathcal{T}$ and normal affine group sub-$\mathcal{T}$-schemes of the fundamental group $\pi(\mathcal{T})$ of $\mathcal{T}$. Pierre Deligne has pointed out to the author that we can see this bijection as a "reformulation" of [10] II 4.3.2 b) and g). The proof of theorem 1.6 is based on [7] theorem 8.17.

We apply this dictionary to the Tannakian category $\mathcal{T}_1(k)$ generated by motives of level ≤ 1 defined over k (in an appropriate category of mixed realizations). We want to make precise that in this article we restrict to motives of level ≤ 1 because we are interested in motivic (and hence geometric) results and until now we know concretely only motives of level ≤ 1. The fundamental group of $\mathcal{T}_1(k)$ is called the *motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ of motives of level ≤ 1.*

For each fibre functor ω of $\mathcal{T}_1(k)$ over a k-scheme S, $\omega\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ is the affine group S-scheme $\underline{\mathrm{Aut}}^{\otimes}_S(\omega)$ which represents the functor which associates to each S-scheme T, $u : T \longrightarrow S$, the group of automorphisms of $\otimes$-functors of the functor $u^*\omega$. In particular, for each embedding $\sigma : k \longrightarrow \mathbb{C}$, the fibre functor ω_σ "Hodge realization" furnishes the $\mathbb{Q}$-algebraic group

$$\omega_\sigma\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) = \mathrm{Spec}\,\big(\omega_\sigma(\Lambda)\big) = \underline{\mathrm{Aut}}^{\otimes}_{\mathbb{Q}}(\omega_\sigma) \tag{0.2}$$

which is the *Hodge realization of the motivic Galois group of $\mathcal{T}_1(k)$.*

The weight filtration W_* of motives of level ≤ 1 induces an increasing filtration W_* of 3 steps in the motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$. Each of these 3 steps can be reconstructed as intersection of the group sub-$\mathcal{T}_1(k)$-schemes $H_{\mathcal{T}_1(k)}(\mathrm{Gr}_i^W\mathcal{T}_1(k))$ (for $i = 0, -1, -2$) and $H_{\mathcal{T}_1(k)}(\mathrm{W}_{-1}\mathcal{T}_1(k))$ of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$. The explicit computation of these normal sub-$\mathcal{T}_1(k)$-schemes will provide, according to (0.1), four exact short sequences of group sub-$\mathcal{T}_1(k)$-schemes of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ (theorem 2.11). These short exact sequences are correlated one to each other by inclusions and projections, and they also involve the filtration W_* of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$. One of these short exact sequences is

$$(0.3) \qquad 0 \longrightarrow \mathrm{Res}_{\overline{k}/k}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(\overline{k})) \longrightarrow \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \stackrel{\pi}{\longrightarrow} \mathrm{Gal}(\overline{k}/k) \longrightarrow 0.$$

If τ is an element of $\mathrm{Gal}(\overline{k}/k)$, then $\pi^{-1}(\tau)$ is $\underline{\mathrm{Hom}}^{\otimes}(\mathrm{Id}, \tau \circ \mathrm{Id})$, where Id and $\tau \circ \mathrm{Id}$ have to be regarded as functors on $\mathcal{T}_1(\overline{k})$ (corollary 2.13). By (0.2), the short exact sequence (0.3) is the geometrical origin (i.e. the motivic version) of the short exact sequence of $\mathbb{Q}$-algebraic groups

$$(0.4) \qquad 0 \longrightarrow \underline{\mathrm{Aut}}^{\otimes}_{\mathbb{Q}}(\omega_{\overline{\sigma}\,|\mathcal{T}_1(\overline{k})}) \longrightarrow \underline{\mathrm{Aut}}^{\otimes}_{\mathbb{Q}}(\omega_{\sigma}) \longrightarrow \mathrm{Gal}(\overline{k}/k) \longrightarrow 0$$

where $\overline{\sigma} : \overline{k} \longrightarrow \mathbb{C}$ is the embedding of $\overline{k}$ in $\mathbb{C}$ which extends $\sigma : k \longrightarrow \mathbb{C}$. This last sequence is the restriction to motives of level ≤ 1 of the sequence found by P. Deligne and U. Jannsen in [5] II 6.23 and [8] 3.7 respectively. Moreover the equality $\pi^{-1}(\tau) = \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Id}, \tau \circ \mathrm{Id})$ is the motivic version of the one found by P. Deligne and U. Jannsen (loc. cit.).

Let M be a 1-motive defined over k. The motivic Galois group $\mathcal{G}_{\mathrm{mot}}(M)$ of M is the fundamental group of the Tannakian subcategory $\langle M \rangle^{\otimes}$ of $\mathcal{T}_1(k)$ generated by M. In [2], we compute the unipotent radical $W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M))$ of the Lie algebra of $\mathcal{G}_{\mathrm{mot}}(M)$: it is the semi-abelian variety defined by the adjoint action of the graduated $\mathrm{Gr}^W_*(W_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M))$ on itself. This result allows us to compute the derived group of the unipotent radical $W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M))$ (proposition 3.3). Applying then theorem 1.6, we construct explicitly the biggest Tannakian subcategory of $\langle M \rangle^{\otimes}$ which has a commutative motivic Galois group (theorem 3.8).

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In this paper k is a field of characteristic 0 embeddable in $\mathbb{C}$ and $\overline{k}$ is its algebraic closure.

1. Motivic Galois theory.

1.1. Let $\mathcal{T}$ be a Tannakian category over k, i.e. a tensor category over k which possesses a fibre functor over a nonempty k-scheme. A **Tannakian subcategory of** $\mathcal{T}$ is a strictly full subcategory $\mathcal{T}'$ of $\mathcal{T}$ which is closed under the formation of subquotients, direct sums, tensor products and duals. Note that $\mathcal{T}'$ is endowed with the restriction to $\mathcal{T}'$ of the fibre functor of $\mathcal{T}$.

The tensor product of $\mathcal{T}$ extends to a tensor product in the category Ind$\mathcal{T}$ of Ind-objects of $\mathcal{T}$. A **commutative ring** in Ind$\mathcal{T}$ is an object A of Ind$\mathcal{T}$ together with a commutative associative product $A \otimes A \longrightarrow A$ admitting an identity $1_{\mathcal{T}} \longrightarrow A$, where $1_{\mathcal{T}}$ is the unit object of $\mathcal{T}$. In Ind$\mathcal{T}$ we can define in the usual way the notion of morphism of commutative rings, the notion of Hopf algebra, ... The **category of affine $\mathcal{T}$-schemes** is the opposite of the category of commutative rings in Ind$\mathcal{T}$. An **affine group $\mathcal{T}$-scheme** is a group object in the category of affine $\mathcal{T}$-schemes. We denote $\mathrm{Sp}(A)$ the affine group $\mathcal{T}$-scheme defined by the Hopf algebra A. An **action** of an affine group $\mathcal{T}$-scheme $\mathrm{Sp}(A)$ on an object X of $\mathcal{T}$ is a morphism $X \longrightarrow X \otimes A$ satisfying the usual axioms for a A-comodule.

1.2. REMARKS:

(1) Since every Tannakian category over k contains, as Tannakian subcategory, the category of finite-dimensional k-vector spaces, *each affine k-scheme defines an affine $\mathcal{T}$-scheme* (cf. [6] 5.6).

(2) Let $\mathcal{T}$ be the Tannakian category of representations of an affine group scheme G over k: $\mathcal{T} = \mathrm{Rep}_k(G)$. In this case, affine $\mathcal{T}$-schemes are affine k-schemes endowed with an action of G. The inclusion of affine k-schemes in the category of affine $\mathcal{T}$-schemes as described in 1.2 (1), is realized adding the trivial action of G (cf. [6] 5.8).

1.3. The **fundamental group** $\pi(\mathcal{T})$ of a Tannakian category $\mathcal{T}$ is the affine group $\mathcal{T}$-scheme $\mathrm{Sp}(\Lambda)$, whose Hopf algebra Λ is endowed for each object X of $\mathcal{T}$ with a morphism

$$\lambda_X : X^{\vee} \otimes X \longrightarrow \Lambda \qquad (1.3.1)$$

functorial in X, and is universal for these properties. The existence of the fundamental group $\pi(\mathcal{T})$ is proved in [7] 8.4, 8.10, 8.11 (iii). The morphisms (1.3.1), which can be rewritten on the form $X \longrightarrow X \otimes \Lambda$, define *an action of the fundamental group $\pi(\mathcal{T})$ on each object of $\mathcal{T}$*. In particular, the morphism $\Lambda \longrightarrow \Lambda \otimes \Lambda$ represents the action of $\pi(\mathcal{T})$ on itself by inner automorphisms (cf. [6] 6.1).

1.4. EXAMPLES:

(1) From the main theorem on neutral Tannakian categories, we know that the Tannakian category $\mathrm{Vec}(k)$ of finite dimensional k-vector spaces is equivalent to the category of finite-dimensional k-representations of $\mathrm{Spec}\,(k)$. In this case, affine $\mathcal{T}$-schemes are affine k-schemes and $\pi(\mathrm{Vec}(k))$ is $\mathrm{Spec}\,(k)$.

(2) Let $\mathcal{T}$ be the Tannakian category of k-representations of an affine group scheme G over k: $\mathcal{T} = \mathrm{Rep}_k(G)$. From [6] 6.3, the fundamental group $\pi(\mathcal{T})$ of $\mathcal{T}$ is the affine group k-scheme G endowed with its action on itself by inner automorphisms.

1.5. By [7] 6.4, to any exact and k-linear $\otimes$-functor $u : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ between Tannakian categories over k, corresponds a morphism of affine group $\mathcal{T}_2$-schemes

$$(1.5.1) \qquad U : \pi(\mathcal{T}_2) \longrightarrow u\pi(\mathcal{T}_1).$$

For each object X_1 of $\mathcal{T}_1$, the action (1.3.1) of $\pi(\mathcal{T}_1)$ on X_1 induces an action of $u\pi(\mathcal{T}_1)$ on $u(X_1)$. Via (1.5.1) this last action induces the action (1.3.1) of $\pi(\mathcal{T}_2)$ on the object $u(X_1)$ of $\mathcal{T}_2$.

Exactly as in theorem [10] II 4.3.2 (g) we have the following dictionary between the functor u and the morphism U:

(1) U is faithfully flat (i.e. flat and surjective) if and only if u is fully faithful and every subobject of $u(X_1)$ for X_1 an object of $\mathcal{T}_1$, is isomorphic to the image of a subobject of X_1.

(2) U is a closed immersion if and only if every object of $\mathcal{T}_2$ is isomorphic to a subquotient of an object of the form $u(X_1)$, for X_1 an object of $\mathcal{T}_1$.

We can now state the dictionary between Tannakian subcategories of $\mathcal{T}$ and normal affine group sub-$\mathcal{T}$-schemes of the fundamental group $\pi(\mathcal{T})$ of $\mathcal{T}$:

1.6. **Theorem**

Let $\mathcal{T}$ be a Tannakian category over k, with fundamental group $\pi(\mathcal{T})$. There is a bijection between the Tannakian subcategories of $\mathcal{T}$ and the normal affine group sub-$\mathcal{T}$-schemes of $\pi(\mathcal{T})$, which associates

- to each Tannakian subcategory $\mathcal{T}'$ of $\mathcal{T}$, the kernel $H_{\mathcal{T}}(\mathcal{T}')$ of the morphism of affine group $\mathcal{T}$-schemes $I : \pi(\mathcal{T}) \longrightarrow i\pi(\mathcal{T}')$ corresponding to the inclusion functor $i : \mathcal{T}' \longrightarrow \mathcal{T}$. In particular, we can identify the fundamental group $\pi(\mathcal{T}')$ of $\mathcal{T}'$ with the affine group $\mathcal{T}$-scheme $\pi(\mathcal{T})/H_{\mathcal{T}}(\mathcal{T}')$.

- to each normal affine group sub-$\mathcal{T}$-scheme H of $\pi(\mathcal{T})$, the Tannakian subcategory $\mathcal{T}(H)$ of objects of $\mathcal{T}$ on which the action (1.3.1) of $\pi(\mathcal{T})$ induces a trivial action of H.

PROOF: Let $i : \mathcal{T}' \longrightarrow \mathcal{T}$ be the inclusion functor of a Tannakian subcategory $\mathcal{T}'$ of $\mathcal{T}$. Denote by $H_{\mathcal{T}}(\mathcal{T}') = \ker\big(\pi(\mathcal{T}) \xrightarrow{I} i\pi(\mathcal{T}')\big)$ the kernel of the morphism $I : \pi(\mathcal{T}) \longrightarrow i\pi(\mathcal{T}')$ corresponding via 1.5 to the inclusion functor i. In particular, we have the exact sequence of affine group $\mathcal{T}$-schemes

$$0 \longrightarrow H_{\mathcal{T}}(\mathcal{T}') \longrightarrow \pi(\mathcal{T}) \longrightarrow i\pi(\mathcal{T}') \longrightarrow 0.$$

According to theorem 8.17 of [7], the inclusion functor $i : \mathcal{T}' \longrightarrow \mathcal{T}$ identifies $\mathcal{T}'$ with the Tannakian subcategory of objects of $\mathcal{T}$ on which the action (1.3.1) of $\pi(\mathcal{T})$ induces a trivial action of $H_{\mathcal{T}}(\mathcal{T}')$. In particular, we get that $\pi(\mathcal{T}') \cong \pi(\mathcal{T})/H_{\mathcal{T}}(\mathcal{T}')$.

The injectivity and the surjectivity are trivial.

1.7. Lemma

Let $\mathcal{T}$ be a Tannakian category over k, with fundamental group $\pi(\mathcal{T})$.

(i) If $\mathcal{T}_1, \mathcal{T}_2$ are two Tannakian subcategories of $\mathcal{T}$ such that $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $H_{\mathcal{T}}(\mathcal{T}_1) \supseteq H_{\mathcal{T}}(\mathcal{T}_2)$.

(ii) If H_1, H_2 are two normal subgroups of $\pi(\mathcal{T})$ such that $H_1 \subseteq H_2$, then $\mathcal{T}(H_1) \supseteq \mathcal{T}(H_2)$. In particular we have the projection

$$\pi(\mathcal{T}(H_1)) = \pi(\mathcal{T})/H_1 \longrightarrow \pi(\mathcal{T}(H_2)) = \pi(\mathcal{T})/H_2.$$

1.8. Let ω be a fibre functor of the Tannakian category $\mathcal{T}$ over a k-scheme S, namely an exact k-linear $\otimes$-functor from $\mathcal{T}$ to the category of quasi-coherent sheaves over S. It defines a $\otimes$-functor, denoted again ω, from Ind$\mathcal{T}$ to the category of quasi-coherent sheaves over S. If $\pi(\mathcal{T}) = \mathrm{Sp}(\Lambda)$ we define

$$\omega(\pi(\mathcal{T})) = \mathrm{Spec}(\omega(\Lambda)). \tag{1.8.1}$$

According to [7] (8.13.1), the spectrum $\mathrm{Spec}(\omega(\Lambda))$ is the affine group S-scheme $\underline{\mathrm{Aut}}_S^{\otimes}(\omega)$ which represents the functor which associates to each S-scheme T, $u : T \longrightarrow S$, the group of automorphisms of $\otimes$-functors of the functor

$$\begin{aligned} \omega_T : \mathcal{T} &\longrightarrow \{\text{locally free scheaves of finite rank over } T\} \\ X &\longmapsto u^*\omega(X). \end{aligned}$$

From the formalism of [6] 5.11, we have the following dictionary:

- to give the group $\mathcal{T}$-scheme $\pi(\mathcal{T}) = \mathrm{Sp}(\Lambda)$ is the same thing as to give, for each fibre functor ω over a k-scheme S, the group S-scheme $\underline{\mathrm{Aut}}_S^{\otimes}(\omega)$, in a functorial way with respect to ω and in a compatible way with respect to the base changes $S' \longrightarrow S$.

- let $u : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ be a k-linear $\otimes$-functor between Tannakian categories over k. To give the corresponding morphism $U : \pi(\mathcal{T}_2) \longrightarrow u\pi(\mathcal{T}_1)$ of group $\mathcal{T}_2$-schemes, is the same thing as to give, for each fibre functor ω of $\mathcal{T}_2$ over a k-scheme S, a morphism of group S-schemes $\underline{\mathrm{Aut}}_S^{\otimes}(\omega) \longrightarrow \underline{\mathrm{Aut}}_S^{\otimes}(\omega \circ u)$, in a functorial way with respect to ω.

1.9. **Lemma**
Let $u_1 : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ and $u_2 : \mathcal{T}_2 \longrightarrow \mathcal{T}_3$ be two exact and k-linear $\otimes$-functors between Tannakian categories over k. Denote by $U_1 : \pi(\mathcal{T}_2) \longrightarrow u_1\pi(\mathcal{T}_1)$ and $U_2 : \pi(\mathcal{T}_3) \longrightarrow u_2\pi(\mathcal{T}_2)$ the morphisms of affine group $\mathcal{T}_2$-schemes and $\mathcal{T}_3$-schemes defined respectively by u_1 and u_2. Then the morphism of affine group $\mathcal{T}_3$-schemes corresponding to $u_2 \circ u_1$ is

$$U = u_2U_1 \circ U_2 : \pi(\mathcal{T}_3) \longrightarrow u_2\pi(\mathcal{T}_2) \longrightarrow u_2u_1\pi(\mathcal{T}_1).$$

Moreover,

(i) if $u_2 \circ u_1 \equiv 1_{\mathcal{T}_3}$ then $U : \pi(\mathcal{T}_3) \longrightarrow \mathrm{Sp}\,(1_{\mathcal{T}_3})$,

(ii) if $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_3$ and $u_2 \circ u_1 = id$, then $U = id$.

PROOF: The morphism of group $\mathcal{T}_2$-schemes $U_1 : \pi(\mathcal{T}_2) \longrightarrow u_1\pi(\mathcal{T}_1)$ furnishes a morphism of group $\mathcal{T}_3$-schemes $u_2U_1 : u_2\pi(\mathcal{T}_2) \longrightarrow u_2u_1\pi(\mathcal{T}_2)$ which corresponds to the following system of morphisms: for each fibre functor ω of $\mathcal{T}_3$ over a k-scheme S, we have a morphism of group S-schemes

$$\underline{\mathrm{Aut}}^{\otimes}_S(\omega \circ u_2) \longrightarrow \underline{\mathrm{Aut}}^{\otimes}_S\big((\omega \circ u_2) \circ u_1\big). \tag{1.9.1}$$

Denote by $U : \pi(\mathcal{T}_3) \longrightarrow u_2u_1\pi(\mathcal{T}_1)$ the morphism of group $\mathcal{T}_3$-schemes corresponding to the functor $u_2 \circ u_1 : \mathcal{T}_1 \longrightarrow \mathcal{T}_3$. To have the morphism U (resp. U_2) of $\mathcal{T}_3$-schemes is the same thing as to have, for each fibre functor ω of $\mathcal{T}_3$ over a k-scheme S, a morphism of group S-schemes

$$\begin{array}{rl} & \underline{\mathrm{Aut}}^{\otimes}_S(\omega) \longrightarrow \underline{\mathrm{Aut}}^{\otimes}_S\big(\omega \circ (u_2 \circ u_1)\big) \\ (\text{resp.} & \underline{\mathrm{Aut}}^{\otimes}_S(\omega) \longrightarrow \underline{\mathrm{Aut}}^{\otimes}_S\big(\omega \circ u_2\big) \quad) \end{array} \tag{1.9.2}$$

Hence, according to (1.9.1) we observe that $U = u_2U_1 \circ U_2$. The remaining assertions are clear from (1.9.2): in particular, if $u_2 \circ u_1 \equiv 1_{\mathcal{T}_3}$, we have that $\underline{\mathrm{Aut}}^{\otimes}_S\big(\omega_{|\langle 1_{\mathcal{T}_3}\rangle^{\otimes}}\big) = \mathrm{Spec}\,(k)$ for each fibre functor ω of $\mathcal{T}_3$ over a k-scheme S.

2. Motivic Galois groups.

2.1. Let $MR(k)$ be the category of mixed realizations (for absolute Hodge cycles) over k defined by U. Jannsen in [8] I 2.1. The **Tannakian category of Artin motives $\mathcal{T}_0(k)$ over k** is the Tannakian subcategory of $MR(k)$ generated by pure realizations of 0-dimensional varietes. *$\mathcal{T}_0(k)$ is a neutral Tannakian category over $\mathbb{Q}$ with fibre functors $\{\omega_\sigma\}_{\sigma:k\longrightarrow\mathbb{C}}$ "the Hodge realizations".* Through each Hodge realization, $\mathcal{T}_0(k)$ is equivalent to the category of finite-dimensional $\mathbb{Q}$-representations of $\mathrm{Gal}(\overline{k}/k)$, i.e.

$$\begin{array}{rcl} \mathcal{T}_0(k) & \cong & \mathrm{Rep}_{\mathbb{Q}}(\mathrm{Gal}(\overline{k}/k)) \\ X & \longmapsto & \mathbb{Q}^{X(\overline{k})}. \end{array} \tag{2.1.1}$$

Here and in the following we regard $\mathrm{Gal}(\overline{k}/k)$ as a constant, pro-finite affine group scheme over $\mathbb{Q}$. By (2.1.1) it is clear that $\mathcal{T}_0(\overline{k})$ is equivalent to the Tannakian category of finite-dimensional $\mathbb{Q}$-vector spaces.

A **1-motive** M over k consists of

(a) a group scheme X over k, which is locally for the étale topology, a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module,

(b) a semi-abelian variety G defined over k, i.e. an extention of an abelian variety A by a torus $Y(1)$, which cocharacter group Y,

(c) a morphism $u : X \longrightarrow G$ of group schemes over k. 1-motives are mixed motives of level ≤ 1: the weight filtration W_* on $M = [X \stackrel{u}{\longrightarrow} G]$ is $\mathrm{W}_i(M) = M$

for each $i \geq 0$, $\mathrm{W}_{-1}(M) = G$,$\mathrm{W}_{-2}(M) = Y(1)$, $\mathrm{W}_j(M) = 0$ for each $j \leq -3$. If we denote $\mathrm{Gr}_n^{\mathrm{W}} = \mathrm{W}_n/\mathrm{W}_{n-1}$, we have $\mathrm{Gr}_0^{\mathrm{W}}(M) = X$, $\mathrm{Gr}_{-1}^{\mathrm{W}}(M) = A$ and $\mathrm{Gr}_{-2}^{\mathrm{W}}(M) = Y(1)$. The **Tannakian category $\mathcal{T}_1(k)$ of 1-motives over** k is the Tannakian subcategory of $MR(k)$ generated by mixed realizations of 1-motives: *$\mathcal{T}_1(k)$ is a neutral Tannakian category over $\mathbb{Q}$ with fibre functors $\{\omega_\sigma\}_{\sigma:k\longrightarrow\mathbb{C}}$ "the Hodge realizations".* For each object M of $\mathcal{T}_1(k)$, we denote by $M^\vee = \underline{\mathrm{Hom}}(M, \mathbb{Z}(0))$ its dual. The Cartier dual of an object M of $\mathcal{T}_1(k)$ is the object $M^* = M^\vee \otimes \mathbb{Z}(1)$ of $\mathcal{T}_1(k)$. We will denote by $W_{-1}\mathcal{T}_1(k)$ (resp. $\mathrm{Gr}_0^W \mathcal{T}_1(k)$, ...) the Tannakian subcategory of $\mathcal{T}_1(k)$ generated by all $W_{-1}M$ (resp. $\mathrm{Gr}_0^W M$, ...) with M a 1-motive.

Since the category $MR(k)$ of mixed realizations is $\mathbb{Q}$-linear, in the following we work with iso-1-motives (called just 1-motives below).

If a Tannakian category $\mathcal{T}$ is generated by motives, the fundamental group $\pi(\mathcal{T})$ is called the **motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathcal{T})$ of $\mathcal{T}$**.

2.2. EXAMPLES:

(1) The motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathbb{Z}(0))$ of the unit object $\mathbb{Z}(0)$ of $\mathcal{T}_1(k)$ is the affine group $\langle\mathbb{Z}(0)\rangle^\otimes$-scheme $\mathrm{Sp}(\mathbb{Z}(0))$. For each fibre functor "Hodge realization" ω_σ, we have that $\omega_\sigma\big(\mathcal{G}_{\mathrm{mot}}(\mathbb{Z}(0))\big) := \mathrm{Spec}\big(\omega_\sigma(\mathbb{Z}(0))\big) = \mathrm{Spec}(\mathbb{Q})$, which is the Mumford-Tate group of $\mathrm{T}_\sigma(\mathbb{Z}(0))$.

(2) Let $\langle\mathbb{Z}(1)\rangle^\otimes$ be the neutral Tannakian category over $\mathbb{Q}$ defined by the k-torus $\mathbb{Z}(1)$. The motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathbb{Z}(1))$ of the torus $\mathbb{Z}(1)$ is the affine group $\langle\mathbb{Z}(1)\rangle^\otimes$-scheme $\mathbb{G}_m$ defined by the $\mathbb{Q}$-scheme $\mathbb{G}_{m/\mathbb{Q}}$ (cf. remark 1.2 (1)). For each fibre functor "Hodge realization" ω_σ, we have that $\omega_\sigma\big(\mathbb{G}_m\big) = \mathbb{G}_{m/\mathbb{Q}}$, which is the Mumford-Tate group of $\mathrm{T}_\sigma(\mathbb{Z}(1))$.

(3) If k is algebraically closed, the motivic Galois group of motives of CM-type over k is the Serre group (cf. [9] 4.8).

2.3. **Lemma-Definition**

The motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_0(k))$ of $\mathcal{T}_0(k)$ is the affine group $\mathbb{Q}$-scheme $\mathrm{Gal}(\overline{k}/k)$ *endowed with its action on itself by inner automorphisms. We denote it by $\mathcal{GAL}(\overline{k}/k)$. In particular, for any fibre functor ω over* $\mathrm{Spec}\,(\mathbb{Q})$ *of $\mathcal{T}_0(k)$, the affine group scheme $\omega(\mathcal{GAL}(\overline{k}/k)) = \underline{\mathrm{Aut}}^\otimes_{\mathrm{Spec}\,(\mathbb{Q})}(\omega)$ is canonically isomorphic to* $\mathrm{Gal}(\overline{k}/k)$

PROOF: Since $\mathcal{T}_0(k) \cong \mathrm{Rep}_{\mathbb{Q}}(\mathrm{Gal}(\overline{k}/k))$, this lemma is an immediate consequence of remark 1.4 (2).

2.4. REMARK: In the category of affine group $\mathcal{T}_0(k)$-schemes, there are two $\mathcal{T}_0(k)$-schemes defined by the Galois group $\mathrm{Gal}(\overline{k}/k)$:

- the affine group $\mathcal{T}_0(k)$-scheme $\mathcal{GAL}(\overline{k}/k)$ which is the affine group $\mathbb{Q}$-scheme $\mathrm{Gal}(\overline{k}/k)$ endowed with its action on itself by inner automorphisms. It is the fundamental group of the Tannakian category $\mathcal{T}_0(k)$ of Artin motives.

- the affine group $\mathcal{T}_0(k)$-scheme $\mathrm{Gal}(\overline{k}/k)$ which is the affine group $\mathbb{Q}$-scheme $\mathrm{Gal}(\overline{k}/k)$ endowed with the trivial action of $\mathrm{Gal}(\overline{k}/k)$ (cf. 1.2 (2)). It is a $\mathbb{Q}$-scheme viewed as a $\mathcal{T}_0(k)$-scheme.

2.5. **Lemma**

(i) The Tannakian subcategory $\mathcal{T}_0(k)$ of $MR(k)$ is equivalent (as tensor category) to the Tannakian subcategory $\mathrm{Gr}_0^W \mathcal{T}_1(k)$.

(ii) We have the following anti-equivalence of tensor categories

$$\mathcal{T}_0(k) \otimes \langle \mathbb{Z}(1) \rangle^{\otimes} \longrightarrow \mathrm{Gr}_{-2}^W \mathcal{T}_1(k)$$

which is defined on the generators by $X \otimes \mathbb{Z}(1) \longmapsto X^{\vee}(1)$.

PROOF: (i) It is a consequence of (2.1.1).
(ii) According to (i), we can view an object X of $\mathcal{T}_0(k)$ as the character group of a torus T defined over k. The dual $X^{\vee}$ of X in the Tannakian category $MR(k)$, can be identified with the cocharacter group of T which can be written, according to our notation, as $X^{\vee}(1)$. The anti-equivalence between the category of character groups and the category of cocharacter groups furnishes the desired anti-equivalence.

2.6. **Corollary**

(i) $\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_0^W \mathcal{T}_1(k)) = \mathcal{GAL}(\overline{k}/k)$,

(ii) $\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_{-2}^W \mathcal{T}_1(k)) = i_1 \mathcal{GAL}(\overline{k}/k) \times i_2 \mathbb{G}_m$,

where $i_1 : \mathcal{T}_0(k) = \mathcal{T}_0(k) \otimes \mathrm{Vec}(\mathbb{Q}) \longrightarrow \mathcal{T}_0(k) \otimes \langle \mathbb{Z}(1) \rangle^{\otimes}$ *and* $i_2 : \langle \mathbb{Z}(1) \rangle^{\otimes} = \mathrm{Vec}(\mathbb{Q}) \otimes \langle \mathbb{Z}(1) \rangle^{\otimes} \longrightarrow \mathcal{T}_0(k) \otimes \langle \mathbb{Z}(1) \rangle^{\otimes}$ *identify respectively* $\mathcal{T}_0(k)$ *and* $\langle \mathbb{Z}(1) \rangle^{\otimes}$ *with full subcategories of* $\mathcal{T}_0(k) \otimes \langle \mathbb{Z}(1) \rangle^{\otimes}$ *. (In the following, we will avoid the symbols* i_1 *and* i_2*).*

PROOF:
(i) Consequence of 2.5 (*i*) and 2.3.
(ii) From 2.5 (*ii*) and [9] (2.40.5), we have that

$$\mathcal{G}_{\mathrm{mot}}\big(\mathrm{Gr}_{-2}^W \mathcal{T}_1(k)\big) = \mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_0(k) \otimes \langle \mathbb{Z}(1) \rangle^{\otimes}\big) = i_1 \mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_0(k)\big) \times i_2 \mathbb{G}_m.$$

2.7. REMARKS:

(1) The motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_0(\overline{k}))$ is the affine group $\mathcal{T}_0(\overline{k})$-scheme $\mathrm{Sp}\,(1_{\mathcal{T}_0(\overline{k})})$ defined by the $\mathbb{Q}$-scheme $\mathrm{Spec}\,(\mathbb{Q})$ (cf. (2.1.1) with $k = \overline{k}$ and 1.4 (2)).

(2) Since the category $\mathrm{Gr}_{-2}^W \mathcal{T}_1(\overline{k})$ is equivalent to the Tannakian category generated by the torus $\mathbb{Z}(1)$, the motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_{-2}^W \mathcal{T}_1(\overline{k}))$ is $\mathbb{G}_m$.

2.8. The weight filtration W_* on objects of $\mathcal{T}_1(k)$ induces an increasing filtration, always denoted by W_*, on the motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ of $\mathcal{T}_1(k)$ (cf. [10] Chapitre IV §2). We describe this filtration through the action (1.3.1) of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ on the generators of $\mathcal{T}_1(k)$. For each 1-motive M over k,

$W_0(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))) = \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$

$W_{-1}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))) = \{g \in \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \mid (g - id)M \subseteq W_{-1}(M),$
$(g - id)W_{-1}(M) \subseteq W_{-2}(M), (g - id)W_{-2}(M) = 0\},$

$W_{-2}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))) = \{g \in \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \mid (g - id)M \subseteq W_{-2}(M),$
$(g - id)W_{-1}(M) = 0\},$

$W_{-3}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))) = 0.$

According to the motivic analogue of [3] §2.2, $\mathrm{Gr}_0^W(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)))$ is an reductive group sub-$\mathcal{T}_1(k)$-scheme of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ and $W_{-1}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)))$ is the unipotent radical of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$. Now we will prove that this filtration W_* of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ can be reconstruct from the following group sub-$\mathcal{T}_1(k)$-schemes of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$: $H_{\mathcal{T}_1(k)}(\mathrm{Gr}_i^W \mathcal{T}_1(k))$ (for $i = -1, -2$), and $H_{\mathcal{T}_1(k)}(W_{-1}\mathcal{T}_1(k))$. These group sub-$\mathcal{T}_1(k)$-schemes are the motivic generalization of the algebraic $\mathbb{Q}$-groups introduced in [1] §2. Because of the commutativity of the inclusion morphisms, we have the inclusions

$$H_{\mathcal{T}_1(k)}(W_{-1}\mathcal{T}_1(k)) \subseteq H_{\mathcal{T}_1(k)}(\mathrm{Gr}_{-1}^W \mathcal{T}_1(k)) \cap H_{\mathcal{T}_1(k)}(\mathrm{Gr}_{-2}^W \mathcal{T}_1(k)),$$
$$H_{\mathcal{T}_1(k)}(W_0/W_{-2}\mathcal{T}_1(k)) \subseteq H_{\mathcal{T}_1(k)}(\mathrm{Gr}_0^W \mathcal{T}_1(k)) \cap H_{\mathcal{T}_1(k)}(\mathrm{Gr}_{-1}^W \mathcal{T}_1(k)).$$

By lemma 2.3, the Tannakian category $\mathrm{Gr}_0^W \mathcal{T}_1(k)$ of Artin motives is a Tannakian subcategory of $\mathrm{Gr}_{-2}^W \mathcal{T}_1(k)$ and therefore according to 1.7 we have that

$$H_{\mathcal{T}_1(k)}(\mathrm{Gr}_0^W \mathcal{T}_1(k)) \supseteq H_{\mathcal{T}_1(k)}(\mathrm{Gr}_{-2}^W \mathcal{T}_1(k)). \tag{2.8.1}$$

Moreover the Cartier duality furnishes the anti-equivalence of tensor categories

$$\begin{aligned} W_0/W_{-2}\mathcal{T}_1(k) &\longrightarrow W_{-1}\mathcal{T}_1(k) \\ M &\longmapsto M^*. \end{aligned} \tag{2.8.2}$$

2.9. **Lemma**

(i) $W_{-1}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))) = \cap_{i=-1,-2} H_{\mathcal{T}_1(k)}(\mathrm{Gr}_i^W \mathcal{T}_1(k))$,

(ii) $W_{-2}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))) = H_{\mathcal{T}_1(k)}(W_{-1}\mathcal{T}_1(k)) = H_{\mathcal{T}_1(k)}(W_0/W_{-2}\mathcal{T}_1(k))$.

PROOF: This result follows from the definition of the filtration W_* of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ and from (2.8.1) and (2.8.2).

2.10. Before stating the main theorem of this paragraph, we need some more notation. Consider the base extension functor

$$\begin{aligned} E : \mathcal{T}_1(k) &\longrightarrow \mathcal{T}_1(\overline{k}) \\ M &\longmapsto M \otimes_k \overline{k}. \end{aligned} \tag{2.10.1}$$

According to (2.1.1) for $k=\overline{k}$, the images through E of the objects of $\mathcal{T}_0(k)$ are in the Tannakian subcategory generated by the unit object $1_{\mathcal{T}_1(\overline{k})}$ of $\mathcal{T}_1(\overline{k})$ and they generate it. Moreover if M is an object of $\mathcal{T}_1(\overline{k})$, it can be written as a subquotient of $M' \otimes_k \overline{k}$ for some object M' of $\mathcal{T}_1(k)$: in fact, for M' we can take the restriction of scalars $\mathrm{Res}_{k'/k} M_0$ with M_0 a model of M over a finite extension k' of k.

2.11. **Theorem**
We have the following diagram of affine group $\mathcal{T}_1(k)$-schemes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Res}_{\overline{k}/k}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(\overline{k})) & \longrightarrow & \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \longrightarrow & \mathcal{GAL}(\overline{k}/k) & \longrightarrow & 0 \\
 & & \uparrow & & || & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Res}_{\overline{k}/k} H_{\mathcal{T}_1(\overline{k})}(\langle \mathbb{Z}(1)\rangle^{\otimes}) & \longrightarrow & \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \longrightarrow & \mathcal{GAL}(\overline{k}/k)\times \mathbb{G}_m & \longrightarrow & 0 \\
 & & \uparrow & & || & & \uparrow & & \\
0 & \longrightarrow & W_{-1}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \longrightarrow & \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \longrightarrow & \mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W \mathcal{T}_1(k)) & \longrightarrow & 0 \\
 & & \uparrow & & || & & \uparrow & & \\
0 & \longrightarrow & W_{-2}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \longrightarrow & \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \longrightarrow & \mathcal{G}_{\mathrm{mot}}(W_{-1}\mathcal{T}_1(k)) & \longrightarrow & 0
\end{array}$$

where all horizontal short sequences are exact and where the vertical arrows on the left are inclusions and those on the right are surjections.

PROOF: We will prove the exactness of these four horizontal short sequences applying theorem 1.6 to the following Tannakian subcategories of $\mathcal{T}_1(k)$: $\mathcal{T}_0(k)$, $\widetilde{W}\mathcal{T}_1(k), \mathrm{Gr}_{-2}^W \mathcal{T}_1(k)$ and $\mathrm{Gr}_*^W \mathcal{T}_1(k)$. From 2.10 and 1.5 we have the closed immersion $e : \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(\overline{k})) \hookrightarrow E\, \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$. Moreover by [8] 2.20 (e), the group $\mathcal{T}_1(k)$-scheme $\mathrm{Res}_{\overline{k}/k}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(\overline{k}))$ is a sub-$\mathcal{T}_1(k)$-scheme of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$. As observed in 2.10, the objects of $\mathcal{T}_0(k)$ are exactly those objects of $\mathcal{T}_1(k)$ on which the sub-$\mathcal{T}_1(k)$-scheme $\mathrm{Res}_{\overline{k}/k}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(\overline{k}))$ of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ acts trivially. Hence by 1.6, we get the first exact sequence.
As observed in 2.7 (2), $\mathrm{Gr}_{-2}^W \mathcal{T}_1(\overline{k})$ is equivalent as tensor category to the Tannakian subcategory $\langle \mathbb{Z}(1)\rangle^{\otimes}$ of $\mathcal{T}_1(\overline{k})$ generated by the k-torus $\mathbb{Z}(1)$. Hence the objects of $\mathrm{Gr}_{-2}^W \mathcal{T}_1(k)$ are exactly those objects of $\mathcal{T}_1(k)$ on which, after extension of scalars, the sub-$\mathcal{T}_1(\overline{k})$-scheme $H_{\mathcal{T}_1(\overline{k})}(\langle \mathbb{Z}(1)\rangle^{\otimes})$ of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(\overline{k}))$ acts trivially. Applying 1.6, we obtain the second exact sequence.
For the third sequence, remark that the objects of $\mathrm{Gr}_*^W \mathcal{T}_1(k)$ are exactly those objects of $\mathcal{T}_1(k)$ on which the unipotent radical $W_{-1}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ acts trivially, and therefore the group $\mathcal{T}_1(k)$-scheme characterizing $\mathrm{Gr}_*^W \mathcal{T}_1(k)$ is $H_{\mathcal{T}_1(k)}(\mathrm{Gr}_*^W \mathcal{T}_1(k)) = W_{-1}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$.
The fourth exact sequence is a consequence of lemma 2.9 (ii).
Finally, in order to prove that the left vertical arrows are inclusions and that the right vertical arrows are surjections, it is enough to apply lemma 1.7.

2.12. As corollary, we get the motivic version of [5] II 6.23 (a), (c) and [8] 4.7 (c), (e). But before stating this corollary, we recall some facts:

If $F_1, F_2 : \mathcal{T}_1(\overline{k}) \longrightarrow \mathcal{T}_1(\overline{k})$ are two functors, we define $\underline{\mathrm{Hom}}^{\otimes}(F_1, F_2)$ to be the functor which associates to each $\mathcal{T}_1(\overline{k})$-scheme $\mathrm{Sp}(B)$, the set of morphisms of $\otimes$-functors from $(F_1)_{\mathrm{Sp}(B)} : X \longmapsto F_1(X) \otimes B$ to $(F_2)_{\mathrm{Sp}(B)} : X \longmapsto F_2(X) \otimes B$ ($(F_1)_{\mathrm{Sp}(B)}$ and $(F_2)_{\mathrm{Sp}(B)}$ are $\otimes$-functors from $\mathcal{T}_1(\overline{k})$ to the category of modules over $\mathrm{Sp}(B)$).

Moreover, each element τ of $\mathrm{Gal}(\overline{k}/k)$ defines a functor $\tau : \mathcal{T}_1(\overline{k}) \longrightarrow \mathcal{T}_1(\overline{k})$ in the following way: since as observed in 2.10, the category $\mathcal{T}_1(\overline{k})$ is generated by motives of the form $E(M)$ with $M \in \mathcal{T}_1(k)$, it is enough to define $\tau E(M)$. We put $\tau E(M) = M \otimes_k \tau \overline{k}$.

Consider the inclusions of Tannakian categories $\mathcal{T}_0(k) \xrightarrow{I_0} \mathrm{Gr}_*^W \mathcal{T}_1(k) \xrightarrow{I} \mathcal{T}_1(k)$. From 1.5, we obtain the faithfully flat morphisms of group $\mathcal{T}_1(k)$-schemes

$$\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \xrightarrow{i} I\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W \mathcal{T}_1(k)) \xrightarrow{Ii_0} \mathcal{I}_0 \mathcal{GAL}(\overline{k}/k)$$

where $Ii_0 : I\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W \mathcal{T}_1(k)) \longrightarrow \mathcal{I}_0 \mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_0^W \mathcal{T}_1(k))$ is defined by the morphism $i_0 : \mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W \mathcal{T}_1(k)) \longrightarrow I_0 \mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_0^W \mathcal{T}_1(k))$ corresponding to the inclusion I_0. Denote by $\iota_0 : \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \longrightarrow \mathcal{I}_0 \mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_0^W \mathcal{T}_1(k))$ the faithfully flat morphism corresponding to the inclusion $\mathcal{I}_0 : \mathrm{Gr}_0^W \mathcal{T}_1(k) \longrightarrow \mathcal{T}_1(k)$. In particular, by 1.9 we have that $\iota_0 = Ii_0 \circ i$.

By 1.5, the functor "take the graded" $\mathrm{Gr}_*^W : \mathcal{T}_1(k) \longrightarrow \mathrm{Gr}_*^W \mathcal{T}_1(k)$ corresponds to the closed immersions of affine group $\mathrm{Gr}_*^W \mathcal{T}_1(k)$-schemes

$$\mathrm{gr}_*^W : \mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W \mathcal{T}_1(k)) \longrightarrow \mathrm{Gr}_*^W \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)),$$

which identifies the motivic Galois group of $\mathrm{Gr}_*^W \mathcal{T}_1(k)$ with the quotient Gr_0^W of the motivic Galois group of $\mathcal{T}_1(k)$.

2.13. **Corollary**
(i) We have the following diagram of affine group $\mathcal{T}_1(k)$-schemes in which all the short sequences are exact:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & W_{-1}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & & & & \\
 & & & & \downarrow & & & & \\
0 & \rightarrow & \mathrm{Res}_{\overline{k}/k}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(\overline{k})) & \rightarrow & \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \stackrel{\iota_0}{\rightarrow} & \mathcal{I}_0\mathcal{GAL}(\overline{k}/k) & \rightarrow & 0 \\
 & & \downarrow & & i \downarrow & & \| & & \\
0 & \rightarrow & I\mathrm{Res}_{\overline{k}/k}\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W\mathcal{T}_1(\overline{k})) & \rightarrow & I\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W\mathcal{T}_1(k)) & \stackrel{Ii_0}{\rightarrow} & II_0\mathcal{GAL}(\overline{k}/k) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

(ii) The morphism $I\mathrm{gr}_*^W : I\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W \mathcal{T}_1(k)) \longrightarrow \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ *of affine group $\mathcal{T}_1(k)$-schemes is a section of i.*
(iii) For any $\tau \in \mathcal{GAL}(\overline{k}/k)$, $\iota_0^{-1}(\tau) = \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Id}, \tau \circ \mathrm{Id})$, *regarding* Id *and* $\tau \circ \mathrm{Id}$ *as functors on* $\mathcal{T}_1(\overline{k})$. *In an analogous way,* $Ii_0^{-1}(\tau) = \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Id}, \tau \circ \mathrm{Id})$, *regarding* Id *and* $\mathrm{Id} \circ \tau$ *as functors on* $\mathrm{Gr}_*^W \mathcal{T}_1(\overline{k})$.

PROOF: (i) We have only to prove the exactness of the second horizontal short sequence. In order to do this, we apply theorem 1.6 to the Tannakian category $\mathrm{Gr}_0^W \mathcal{T}_1(k)$ viewed this time as subcategory of $\mathrm{Gr}_*^W \mathcal{T}_1(k)$: in fact by (2.10.2), Artin motives are exactly the kernel of the base extension functor and therefore $H_{\mathrm{Gr}_*^W \mathcal{T}_1(k)}(\mathrm{Gr}_0^W \mathcal{T}_1(k)) = \mathrm{Res}_{\overline{k}/k} \mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W \mathcal{T}_1(\overline{k}))$.
(ii) Since

$$\mathrm{Gr}_*^W \circ I = id : \mathrm{Gr}_*^W \mathcal{T}_1(k) \stackrel{I}{\longrightarrow} \mathcal{T}_1(k) \stackrel{\mathrm{Gr}_*^W}{\longrightarrow} \mathrm{Gr}_*^W \mathcal{T}_1(k),$$

from 1.9 we have that

$$\mathrm{Gr}_*^W i \circ \mathrm{gr}_*^W = id : \mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W \mathcal{T}_1(k)) \stackrel{\mathrm{gr}_*^W}{\longrightarrow} \mathrm{Gr}_*^W \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \stackrel{\mathrm{Gr}_*^W i}{\longrightarrow} \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$$

and therefore $i \circ I \mathrm{gr}_*^W = id$.
(iii) By [7] 8.11, the fundamental group $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ represents the functor $\underline{\mathrm{Aut}}^{\otimes}(\mathrm{Id})$ which associates to each $\mathcal{T}_1(k)$-scheme $\mathrm{Sp}(B)$ the group of automorphisms of $\otimes$-functors of the functor

$$\begin{aligned} \mathrm{Id}_{\mathrm{Sp}(B)} : \mathcal{T}_1(k) &\longrightarrow \{\text{modules over } \mathrm{Sp}(B)\} \\ X &\longmapsto X \otimes B. \end{aligned}$$

Hence if g is an element of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))(\mathrm{Sp}\, B) = \underline{\mathrm{Aut}}^{\otimes}(\mathrm{Id})(\mathrm{Sp}\, B)$, for each pair of objects M and N of $\mathcal{T}_1(k)$ and for each morphism $f : M \longrightarrow N$ of $\mathcal{T}_1(k)$, we have the commutative diagram

$$\begin{array}{ccc} M \otimes B & \stackrel{g_M}{\longrightarrow} & M \otimes B \\ f \otimes id_B \downarrow & & \downarrow f \otimes id_B \\ N \otimes B & \stackrel{g_N}{\longrightarrow} & N \otimes B. \end{array}$$

Let M and N be two objects of $\mathcal{T}_1(k)$. Since $\mathrm{Hom}_{\mathcal{T}_1(\overline{k})}(E(M), E(N))$ is an object of $\mathrm{Rep}_{\mathbb{Q}}(\mathrm{Gal}(\overline{k}/k))$, it can be regarded as an Artin motive over k. Moreover, the elements of $\mathrm{Hom}_{\mathcal{T}_1(k)}(M, N)$ are exactly the elements of $\mathrm{Hom}_{\mathcal{T}_1(\overline{k})}(E(M), E(N))$ which are invariant under the action of $\mathrm{Gal}(\overline{k}/k)$.
Let g be an element of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))(\mathrm{Sp}\, B) = \underline{\mathrm{Aut}}^{\otimes}(\mathrm{Id})(\mathrm{Sp}\, B)$, and let $\iota_0(g) = \tau \in \mathrm{Gal}(\overline{k}/k)$. This means that g acts via τ on $\mathrm{Hom}_{\mathcal{T}_1(\overline{k})}(E(M), E(N))$. Then for any morphism $h : E(M) \longrightarrow E(N)$ of $\mathcal{T}_1(\overline{k})$, we have the commutative diagram

$$\begin{array}{ccc} E(M) \otimes B & \stackrel{E(g_M)}{\longrightarrow} & E(M) \otimes B \\ h \otimes id_B \downarrow & & \downarrow \tau h \otimes id_B \\ E(N) \otimes B & \stackrel{E(g_N)}{\longrightarrow} & E(N) \otimes B. \end{array} \quad (2.13.1)$$

Since M and N are defined over k, $E(M)$ and $E(N)$ are respectively isomorphic to $\tau E(M)$ and $\tau E(N)$ and therefore the upper line of (2.13.1) defines a morphism

$$E(M) \otimes B \longrightarrow \tau E(M) \otimes B$$

which is functorial in $E(M)$ and B, and which is compatible with tensor products. Moreover we have already observed in 2.10 that the Tannakian category $\mathcal{T}_1(\overline{k})$ is generated by motives of the form $E(M)$ with $M \in \mathcal{T}_1(k)$. We can then conclude that g defines an element of $\underline{\mathrm{Hom}}^{\otimes}(\mathrm{Id}, \tau \circ \mathrm{Id})$, regarding Id and $\tau \circ \mathrm{Id}$ as functors on $\mathcal{T}_1(\overline{k})$.

3. Case of a 1-motive.

3.1. Almost all results of section 2 are true, if we restrict to the Tannakian subcategory of $\mathcal{T}_1(k)$ generated by a 1-motive M. However it is not true that the Tannakian category generated by $M/W_{-2}M$ is equivalent to the Tannakian category generated by $W_{-1}M$. Hence the lemma 2.9 must be modified in the following way

$$\begin{aligned} W_{-1}(\mathcal{G}_{\mathrm{mot}}(M)) &= \cap_{i=-1,-2} H_{\mathcal{T}_1(k)}(\mathrm{Gr}_i^W M), \\ W_{-2}(\mathcal{G}_{\mathrm{mot}}(M)) &= H_{\mathcal{T}_1(k)}(M/W_{-2}M) \cap H_{\mathcal{T}_1(k)}(W_{-1}M). \end{aligned} \tag{3.1.1}$$

3.2. In order to construct the biggest Tannakian subcategory of the one generated by M, whose motivic Galois group in commutative, we need a more symmetric description of M: according [4] (10.2.14), to have M is equivalent to have the $(X, Y^{\vee}, A, A^*, v, v^*, \psi)$ where

- X and $Y^{\vee}$ are two group k-schemes, which are locally for the étale topology, constant group schemes defined by a finitely generated free $\mathbb{Z}$-module;
- A and A^* are two abelian varieties defined over k, dual to each other;
- $v : X \longrightarrow A$ and $v^* : Y^{\vee} \longrightarrow A^*$ are two morphisms of group schemes over k; and
- ψ is a trivialization of the pull-back $(v, v^*)^* \mathcal{P}_A$ by (v, v^*) of the Poincaré biextension $\mathcal{P}_A$ of (A, A^*).

In [2], it is proved that the unipotent radical of the Lie algebra of $\mathcal{G}_{\mathrm{mot}}(M)$ is the semi-abelian variety defined by the adjoint action of the Lie algebra $(\mathrm{Gr}_*^W(W_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)), [\,,\,])$ on itself. The abelian variety B and the torus $Z(1)$ underlying this semi-abelian variety can be computed explicitly. We recall here briefly their construction:
The motive $E = W_{-1}(\underline{\mathrm{End}}(\mathrm{Gr}_*^W M))$ is a split 1-motive, whose non trivial components are the abelian variety $A \otimes X^{\vee} + A^* \otimes Y$ and the torus $X^{\vee} \otimes Y(1)$, and which is endowed of a Lie crochet $[\,,\,]$. According to corollary [2] 2.7, this Lie crochet corresponds to a $\Sigma - X^{\vee} \otimes Y(1)$-torsor $\mathcal{B}$ living over $A \otimes X^{\vee} + A^* \otimes Y$. As proved in [2] 3.3, the 1-motives $\mathrm{Gr}_*^W M$ and $\mathrm{Gr}_*^W M^{\vee}$ are Lie $(E, [\,,\,])-$modules. In particular, E acts on the components $\mathrm{Gr}_0^W M$ and $\mathrm{Gr}_0^W M^{\vee}$ through the projections:

$$(3.2.1)\qquad \begin{aligned}\alpha :&(X^\vee \otimes A)\otimes X \longrightarrow A\\ \beta :&(A^*\otimes Y)\otimes Y^\vee \longrightarrow A^*\\ \gamma :&(X^\vee \otimes Y(1))\otimes X \longrightarrow Y(1)\end{aligned}$$

Denote by $b=(b_1,b_2)$ the k-rational point $b=(b_1,b_2)$ of the abelian variety $A\otimes X^\vee + A^*\otimes Y$ defining the morphisms $v: X\longrightarrow A$ and $v^*: Y^\vee \longrightarrow A^*$. Let B be the smallest abelian sub-variety of $X^\vee\otimes A + A^*\otimes Y$ containing the point $b=(b_1,b_2)\in X^\vee\otimes A(k)\times A^*\otimes Y(k)$. The restriction $i^*\mathcal{B}$ of the $\Sigma - X^\vee\otimes Y(1)$-torsor $\mathcal{B}$ by the inclusion $i: B\longrightarrow X^\vee\otimes A\times A^*\otimes Y$ is a $\Sigma - X^\vee\otimes Y(1)$-torsor over B. Denote by Z_1 the smallest $\mathrm{Gal}(\overline{k}/k)$-module of $X^\vee\otimes Y$ such that the torus $Z_1(1)$, that it defines, contains the image of the Lie crochet $[\, ,\,]: B\otimes B\longrightarrow X^\vee\otimes Y(1)$. The direct image $p_*i^*\mathcal{B}$ of the $\Sigma - X^\vee\otimes Y(1)$-torsor $i^*\mathcal{B}$ by the projection $p: X^\vee\otimes Y(1)\longrightarrow (X^\vee\otimes Y/Z_1)(1)$ is a trivial $\Sigma-(X^\vee\otimes Y/Z_1)(1)$-torsor over B. We denote by $\pi: p_*i^*\mathcal{B}\longrightarrow (X^\vee\otimes Y/Z_1)(1)$ the canonical projection. By [2] 3.6, the morphism $u: X\longrightarrow G$ defines a point $\widetilde{b}$ in the fibre of $\mathcal{B}$ over b. We denote again by $\widetilde{b}$ the points of $i^*\mathcal{B}$ and of $p_*i^*\mathcal{B}$ over the point b of B. Let Z be the smallest sub-$\mathrm{Gal}(\overline{k}/k)$-module of $X^\vee\otimes Y$, containing Z_1 and such that the sub-torus $(Z/Z_1)(1)$ of $(X^\vee\otimes Y/Z_1)(1)$ contains $\pi(\widetilde{b})$. If we put $Z_2=Z/Z_1$, we have that $Z(1)=Z_1(1)\times Z_2(1)$.
With these notation, the unipotent radical $W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M))$ of the Lie algebra of $\mathcal{G}_{\mathrm{mot}}(M)$ is the extension of the abelian variety B by the torus $Z(1)$ defined by the adjoint action of $(B+Z(1),[\, ,\,])$ on itself.

3.3. **Proposition**
The derived group of the unipotent radical $W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M))$ *of the Lie algebra of* $\mathcal{G}_{\mathrm{mot}}(M)$ *is the torus* $Z_1(1)$.

PROOF: Let $g_1=(P,Q,\vec{v})$ and $g_2=(R,S,\vec{w})$ be two elements of $B+Z(1)$, with $P,R\in B\cap X^\vee\otimes A(k)$, $Q,S\in B\cap A^*\otimes Y(k)$, and $\vec{v},\vec{w}\in Z(1)(k)$. We have

$$(3.3.1)\qquad g_1\circ g_2=(P+R,Q+S,\vec{v}+\vec{w}+\Upsilon(P,Q,R,S)$$

where Υ is a $\mathrm{Gal}(\overline{k}/k)$-equivariant homomorphism from $(X^\vee\otimes A+A^*\otimes Y)(\overline{k})\otimes(X^\vee\otimes A+A^*\otimes Y)(\overline{k})$ to $X^\vee\otimes Y(\overline{k})$. In order to determine Υ, we have to understand how $g_1\circ g_2$ acts on the 1-motive M. The 1-motives $M/W_{-2}M$ and $W_{-1}M$ generate Tannakian subcategories of $\langle M/W_{-2}M+W_{-1}M\rangle^{\otimes}$ and therefore by 1.5 we have the surjective morphisms

(3.3.2)
$$\begin{aligned}pr_1: W_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(\langle M/W_{-2}M+W_{-1}M\rangle^{\otimes})&\longrightarrow W_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(\langle M/W_{-2}M\rangle^{\otimes})\\ pr_2: W_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(\langle M/W_{-2}M+W_{-1}M\rangle^{\otimes})&\longrightarrow W_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(\langle W_{-1}M\rangle^{\otimes}).\end{aligned}$$

Since by [2] 3.10 $\mathrm{Gr}_{-1}^W \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)$ is the abelian variety B and $W_{-2}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)$ is the torus $Z(1)$, according to (3.1.1) we get that $W_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(\langle M/W_{-2}M + W_{-1}M\rangle^{\otimes})$ is the abelian variety B. It follows that explicitly the morphisms (3.3.2) are the projections $pr_1 : B \longrightarrow B \cap X^{\vee} \otimes A$ and $pr_2 : B \longrightarrow B \cap A^* \otimes Y$.
Let $\pi : W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)) \longrightarrow W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(\langle M/W_{-2}M + W_{-1}M\rangle^{\otimes}))$ be the surjective morphism coming from the inclusion of the Tannakian sub-category $\langle M/W_{-2}M + W_{-1}M\rangle^{\otimes}$ in $\langle M\rangle^{\otimes}$. By definition of $W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M/W_{-2}M))$ we have that

$$\big(pr_1(\pi\, g_1) - id\big)\, W_0/W_{-2}(M) \subseteq A,$$
$$\big(pr_1(\pi\, g_1) - id\big)\, A = 0.$$

Hence modulo the canonical isomorphism $\underline{\mathrm{Hom}}(X;A) \cong X^{\vee} \otimes A$ which allows us to identify $pr_1(\pi\, g_1) - id$ with $P \in B \cap X^{\vee} \otimes A(k)$, we obtain that

$$P : \mathrm{Gr}_0^W(M) \longrightarrow A. \tag{3.3.3}$$

In an analogous way, by definition of $W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(W_{-1}M))$ we observe that

$$\begin{aligned}\big(pr_2(\pi\, g_2) - id\big)\, W_{-1}M &\subseteq T,\\ \big(pr_2(\pi\, g_2) - id\big)\, T &= 0.\end{aligned} \tag{3.3.4}$$

Since the Cartier dual of $W_{-1}(M)$ is the 1-motive $M^*/W_{-2}M^*$, $pr_2(\pi\, g_2)$ acts on a contravariant way on $M^*/W_{-2}M^*$, and therefore we have that

$$\big(pr_2(\pi\, g_2)' - id\big)\, M^*/W_{-2}M^* \subseteq A^*,$$
$$\big(pr_2(\pi\, g_2)' - id\big)\, A^* = 0,$$

where the symbol $'$ denote the contravariant action. Consequently, modulo the canonical isomorphism $\underline{\mathrm{Hom}}(Y^{\vee};A^*) \cong A^* \otimes Y$ which allows us to identify $pr_2(\pi\, g_2)' - id$ with $S \in B \cap A^* \otimes Y$, we have by (3.3.4) that

$$-S : A \longrightarrow Y(1). \tag{3.3.5}$$

Again modulo the canonical isomorphism $\underline{\mathrm{Hom}}(X;Y(1)) \cong X^{\vee} \otimes Y(1)$, from (3.3.3) and (3.3.5) we get that $-[P;S] : \mathrm{Gr}_0^W(M) \stackrel{P}{\longrightarrow} A \stackrel{-S}{\longrightarrow} Y(1)$. It follows that $\Upsilon(P,Q,R,S) = -[P;S]$ and using (3.3.1) we can conclude that

$$g_1 \circ g_2 - g_2 \circ g_1 = (0, 0, -[P;S] + [R;Q])$$

which is an element of $Z_1(1)$ by definition.

3.4. Let $\{e_i\}_i$ and $\{f_j^*\}_j$ be basis of $X(\overline{k})$ and $Y^{\vee}(\overline{k})$ respectively. Choose a point P of $B \cap X^{\vee} \otimes A(k)$ and a point Q of $B \cap A^* \otimes Y(k)$ such that the abelian sub-variety

they generate in $X^\vee \otimes A + A^* \otimes Y$, is isogeneous to B. Denote by $\overline{v} : X(\overline{k}) \longrightarrow A(\overline{k})$ et $\overline{v}^* : Y^\vee(\overline{k}) \longrightarrow A^*(\overline{k})$ the $\mathrm{Gal}(\overline{k}/k)$-equivariant homomorphisms defined by $\overline{v}(e_i) = \alpha(P, e_i), \overline{v}^*(f_j^*) = \beta(Q, f_j^*)$. Moreover choose a point $\vec{q} = (q_1, \dots, q_{\mathrm{rg}\, Z_2})$ of $Z_2(1)(k)$ such that the points $q_1, \dots, q_{\mathrm{rg}\, Z_2}$ are multiplicative independent.
Let $\Gamma : Z(1)(\overline{k}) \otimes X \otimes Y^\vee(\overline{k}) \longrightarrow \mathbb{Z}(1)(\overline{k})$ be the $\mathrm{Gal}(\overline{k}/k)$-equivariant homomorphism obtained from the map $\gamma : (X^\vee \otimes Y(1)) \otimes X \longrightarrow Y(1)$, and denote by $\overline{\psi} : X \otimes Y^\vee(\overline{k}) \longrightarrow \mathbb{Z}(1)(\overline{k})$ the $\mathrm{Gal}(\overline{k}/k)$-equivariant homomorphism defined by

$$\overline{\psi}(e_i, f_j^*) = \Gamma([P,Q], \vec{q}, e_i, f_j^*). \tag{3.4.1}$$

3.5. **Lemma**
With the above notation, the Tannakian category $\langle M \rangle^\otimes$ is equivalent to the Tannakian category generated by the 1-motives $(e_i\mathbb{Z}, f_j^*\mathbb{Z}, A, A^*, \overline{v}_i, \overline{v}_j^*, \overline{\psi}_{i,j})$, where $\overline{v}_i, \overline{v}_j^*$ and $\overline{\psi}_{i,j}$ are the $\mathrm{Gal}(\overline{k}/k)$-equivariant homomorphisms obtained restricting respectively $\overline{v}, \overline{v}^*$ and $\overline{\psi}$ to $e_i\mathbb{Z}$ and $f_j^*\mathbb{Z}$:

$$\begin{aligned}
\overline{v}_i : e_i\mathbb{Z} &\longrightarrow A, & e_i &\longmapsto \alpha(P, e_i),\\
\overline{v}_j^* : f_j^*\mathbb{Z} &\longrightarrow A^*, & f_j^* &\longmapsto \alpha^*(Q, f_j^*),\\
\overline{\psi}_{i,j} : e_i\mathbb{Z} \times f_j^*\mathbb{Z} &\longrightarrow \mathcal{P}_{|e_i\mathbb{Z} \times f_j^*\mathbb{Z}}, & (e_i, f_j^*) &\longmapsto \Gamma([P,Q], \vec{q}, e_i, f_j^*).
\end{aligned}$$

PROOF: According to the proof of theorem 3.8 [2], the homomorphisms $\overline{v}, \overline{v}^*$ and $\overline{\psi}$ define a 1-motive $(X, Y^\vee, A, A^*, \overline{v}, \overline{v}^*, \overline{\psi})$ which generates the same Tannakian category as M. In order to conclude we apply theorem [1] 1.7 to the 1-motives $(X, Y^\vee, A, A^*, \overline{v}, \overline{v}^*, \overline{\psi})$ and $(e_i\mathbb{Z}, f_j^*\mathbb{Z}, A, A^*, \overline{v}_i, \overline{v}_j^*, \overline{\psi}_{i,j})$.

3.6. Consider the 1-motives

$$\begin{aligned}
M^{tab} &= \oplus_{i,j}\, (e_i\mathbb{Z}, f_j^*\mathbb{Z}, 0, 0, 0, 0, \overline{\psi}_{i,j}^{ab})\\
M^{a} &= \oplus_{i,j}\, (e_i\mathbb{Z}, f_j^*\mathbb{Z}, A, A^*, \overline{v}_i, \overline{v}_j^*, 0)\\
M^{nab} &= \oplus_{i,j}\, (e_i\mathbb{Z}, f_j^*\mathbb{Z}, A, A^*, \overline{v}_i, \overline{v}_j^*, \overline{\psi}_{i,j}^{nab})\\
M^{ab} &= \oplus_{i,j}\, (e_i\mathbb{Z}, f_j^*\mathbb{Z}, A, A^*, \overline{v}_i, \overline{v}_j^*, \overline{\psi}_{i,j}^{ab})
\end{aligned}$$

where

$$\begin{aligned}
\overline{\psi}_{i,j}^{nab}(e_i, f_j^*) &= \Gamma([P,Q], \vec{1}, e_i, f_j^*)\\
\overline{\psi}_{i,j}^{ab}(e_i, f_j^*) &= \Gamma(\vec{1}, \vec{q}, e_i, f_j^*)
\end{aligned} \tag{3.6.1}$$

According to lemma 3.5, the 1-motives M^{tab}, M^a, M^{ab} and M^{nab} belong to the Tannakian category $\langle M \rangle^\otimes$ generated by M.

3.7. **Lemma**

The Tannakian category generated by M is equivalent to the Tannakian category generated by the 1-motive $M^{tab} \oplus M^{nab}$. Moreover the 1-motives M^{ab} and $M^a \oplus M^{tab}$ generate the same Tannakian category.

PROOF: Since through the projection $p : X^\vee \otimes Y(1) \longrightarrow (X^\vee \otimes Y/Z_1)(1)$ the $\Sigma - (X^\vee \otimes Y)(1)$-torsor $i^*\mathcal{B}$ becomes trivial, confronting (3.4.1) and (3.6.1), we observe that to have the trivializations $\overline{\psi}^{ab}_{i,j}$ and $\overline{\psi}^{nab}_{i,j}$ is the same thing as to have the trivialization $\overline{\psi}_{i,j}$. Hence we can conclude by lemma 3.5.

Always because of the fact that the $\Sigma - (X^\vee \otimes Y/Z_1)(1)$-torsor $p_* i^*\mathcal{B}$ is trivial, we observe that the trivialization $\overline{\psi}^{ab}_{i,j}$ is independent of the abelian part of the 1-motive M, i.e. it is independent of $\overline{v}_i, \overline{v}^*_j$ Therefore, we can conclude that the Tannakian category generated by 1-motive M^{ab} is equivalent to one generated by the 1-motive $M^a \oplus M^{tab}$.

3.8. **Theorem**

The Tannakian category generated by M^{ab} is the biggest Tannakian subcategory of $\langle M \rangle^\otimes$ whose motivic Galois group is commutative. We have the following diagram of affine group $\langle M \rangle^\otimes$-schemes

$$\begin{array}{ccccccccc}
0 & \rightarrow & Z_2(1) & \rightarrow & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M) & \rightarrow & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M^{nab}) & \rightarrow & 0 \\
 & & \downarrow & & || & & \downarrow & & \\
0 & \rightarrow & Z(1) & \rightarrow & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M) & \rightarrow & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M^{a}) & \rightarrow & 0 \\
 & & \uparrow & & || & & \uparrow & & \\
0 & \rightarrow & Z_1(1) & \rightarrow & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M) & \rightarrow & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M^{ab}) & \rightarrow & 0 \\
 & & \downarrow & & || & & \downarrow & & \\
0 & \rightarrow & B+Z_1(1) & \rightarrow & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M) & \rightarrow & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M^{tab}) & \rightarrow & 0
\end{array}$$

where all horizontal short sequences are exact and where the vertical arrows on the left are inclusions and those on the right are surjections.

PROOF: By [2] 3.10 we know that $\mathrm{Gr}^W_*(W_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M))$ is the Lie algebra $(B + Z(1), [\,,\,])$. Now, from the definition of the 1-motives M^{nab}, M^{ab} and M^{tab}, and from (3.6.1) we observe that the torus $Z_2(1)$ acts trivially on M^{nab}, that the torus $Z_1(1)$ acts trivially on M^{ab} and that the split 1-motive $B + Z_1(1)$ acts trivially on M^{tab}. In other words we have that

$$\begin{aligned}
\mathrm{Lie}\, H_{\langle M \rangle^\otimes}(\langle M^{nab} \rangle^\otimes) &= Z_2(1) \\
\mathrm{Lie}\, H_{\langle M \rangle^\otimes}(\langle M^{ab} \rangle^\otimes) &= Z_1(1) \\
\mathrm{Lie}\, H_{\langle M \rangle^\otimes}(\langle M^{tab} \rangle^\otimes) &= B + Z_1(1)
\end{aligned}$$

and therefore we get the first, the third and the fourth short exact sequence.

Recall that by [2] 3.10 the Lie algebra $W_{-2}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)$ is the torus $Z(1)$. Moreover by construction, the 1-motive without toric part M^a generates the same Tannakian category as the 1-motive $W_0/W_{-2}M + W_{-1}M$. Hence thanks to (3.1.1) the obtain the second horizontal short exact sequence.

According to lemma 3.7, the 1-motives M^a and M^{tab} generate Tannakian subcategories of $\langle M^{ab} \rangle^{\otimes}$. By construction the 1-motive M^a generates a Tannakian subcategory of $\langle M^{nab} \rangle^{\otimes}$. Hence in order to prove that the left vertical arrows are inclusions and that the right vertical arrows are surjections, it is enough to apply the lemma 1.7.

The third exact sequence of the above diagram implies that the motivic Galois group of M^{ab} is isomorphic to the quotient $\mathrm{Lie}\, \mathcal{G}_{\mathrm{mot}}(M)/Z_1(1)$. But according proposition 3.3, $Z_1(1)$ is the derived group of $\mathrm{Lie}\, \mathcal{G}_{\mathrm{mot}}(M)$ and hence we can conclude that $\langle M^{ab} \rangle^{\otimes}$ is the biggest Tannakian subcategory of $\langle M \rangle^{\otimes}$ whose motivic Galois group is commutative.

REMARK: Among the non degenerate 1-motives, the 1-motive M^{nab} is the one which generates the biggest Tannakian subcategory of $\langle M \rangle^{\otimes}$, whose motivic Galois group is non commutative. (A 1-motive is said to be non degenerate if the dimension of the Lie algebra $W_{-1}\mathrm{Lie}\, \mathcal{G}_{\mathrm{mot}}(M)$ is maximal (cf. [1] 2.3)).

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