Z. Phys. A – Atoms and Nuclei 318, 205–213 (1984)

Zeitschrift für Physik A Atoms and Nuclei

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The Nuclear Surface as a Collective Variable

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Received June 20, 1984

In heavy nuclei where the thickness of the diffused edge is relatively small, a certain sharp effective surface can be defined which characterizes the shape of the nucleus, and it can be considered as a collective dynamic variable. It is shown that the problem of fluid dynamics can be simplified by reducing it to simple linearized equations for the dynamics in the nuclear interior and boundary conditions set at the effective dynamic sharp surface of the density distribution. These conditions are derived from the fluid dynamical equations. Transitional densities obtained from this simple model are compared with the numerical solution of fluid dynamical equations.

1. Introduction

Many theories of complex nuclei exploit the energy-density approximation expressing the total energy of the system as

$$E=\int d^3r\mathscr{E}(\rho,\quad \rho). \tag{1}$$

Here, ρ is the average particle density which, in dynamical problems, is a function of time. The properties of $\mathscr{E}(\rho)$ are such that in heavier nuclei the density distribution is characterized by a relatively sharp edge. Dynamic theories which exploit expansions in $\delta\rho(\mathbf{r}t)$ fail if the amplitude of the surface distortion is of the order of or greater than the surface diffuseness. Only inside the nuclear volume can such approximations be validated. The solution can be helped if the nuclear surface itself is introduced as a dynamical variable.

It has been shown, indeed, in [1] that for a realistic $\mathscr{E}(\rho)$ the static problem can be reduced to isolated problems of finding the shape of the effective nuclear surface and determining the density distributions in the inside region and across the surface region. Finding the density distribution can be avoided altogether in certain cases by using phenomenological parameters of the nuclear droplet model. It will be shown below that the same is also true for fluid dynamics which can be reduced to simplified equations which determine dynamics of the nuclear surface and of the volume density distribution. It can facilitate developing a unified model [2, 3] which combines macroscopic features with the single-particle motion.

2. Statics

In the static case, the energy functional (1) is usually considered in connection with the problem of equilibrium density distribution which gives extremum of (1) under the subsidiary condition of particle number conservation,

$$A=\int d^3r\rho(\mathbf{r}). \tag{2}$$

(The difference between the density distributions of protons and neutrons is neglected here). Another constraint fixes a certain deformation parameter

$$Q=\int d^3r\rho(\mathbf{r})\,q(\mathbf{r}). \tag{3}$$

The function $q(\mathbf{r})$ here can be a multipole moment, or it can be chosen in such a way that Q determines

the distance between the centers of mass of the two halves of the stretched nucleus [4]. In this section, which essentially repeats the contents of [1], some arguments will be presented which will be needed in the treatment of the dynamical case and which were omitted in the original publication.

Approximate solution to the problem can be found by exploiting explicitly the property of low compressibility of the nuclear matter. In the nuclear volume, the density differs little from its equilibrium value $\bar{\rho}$ for infinite matter which equals approximately $3A/4\pi R^3$. The density gradient is large near the edge of the density distribution, where the density drops sharply from its central value to zero in relatively thin surface layer the width a of which is small as $A^{-1/3}$ relative to the nuclear size. The relevant Lagrange equation is

$$\frac{\delta \mathscr{E}}{\delta \rho} - \lambda + \lambda_a q(\mathbf{r}) = 0, \tag{4}$$

and a reasonably accurate solution to it can be found by developing approximations to (4) separately in the nuclear interior and in the nuclear edge [1], see also Appendix. Required unique solution can be determined by matching the two approximations. Quantities λ and λ_Q in (4) are two Lagrange multipliers corresponding to constraints (2) and (3).

In the nuclear interior one uses the expansion

$$\mathscr{E}(\rho) = \rho \bar{E} \cdot (1 + (K/18\bar{E})(1 - \rho/\bar{\rho})^2 + \ldots), \tag{5}$$

where the compressibility modulus $K, \bar{E} < 0$ and $\bar{\rho}$ are parameters relevant to infinite nuclear matter. Of them, $\bar{\rho}$ and $\bar{E}$ are known directly from empirical data. Inside the nucleus, the derivatives of ρ in (A7) may be neglected and by means of (5) one gets the interior density as

$$\rho_{\text{in}} = \bar{\rho} \cdot \left(1 - \frac{9}{K} \Lambda(\mathbf{r})\right). \tag{6}$$

Here,

$$\Lambda = \lambda_A + \lambda_Q q(\mathbf{r}), \tag{7}$$

where λ_A is the finite-size correction to the particle separation energy $\mu = -\bar{E}$ for infinite matter.

To resolve the problem near the nuclear edge we introduce the effective surface ($\mathscr{E}.\mathscr{S}.$) of the density distribution, determined as geometrical locus of points of maximal gradient of density $\rho(\mathbf{r})$. The derivative of ρ in direction perpendicular to the surface is in the surface region large as R/a relative to other derivatives and here it is convenient to introduce a surface-based coordinate system by choosing one of the coordinates (ξ) directed perpendicular to the surface ($\xi = +\infty$ at large distances and $\xi = 0$ at the surface). The so far undetermined shape of the surface is defined formally by a certain shape function, for example, as $y = Y(z)$ in cylindrical coordinates, should axially symmetrical shapes be considered. The transformation relationship to new coordinates are readyly obtained, see Appendix.

The leading approximation for $\rho(\mathbf{r})$ in the edge region is obtained by retaining terms of zeroth order in R/a in the Lagrange equation transformed to new coordinates. It is reduced then to ordinary differential equation and its solution,

$$\rho_{\text{edge}} = \bar{\rho} w_0^2(\xi), \tag{8}$$

see (A15), determines the shape of the density distribution near the edge of the nucleus. Within this approximation it is the same as in the case of semi-infinite matter [5]. It satisfies asymptotic conditions

$$\rho_{\text{edge}} = \rho'_{\text{edge}} = 0 \tag{9}$$

at $\xi = \infty$ and approaches unity as an exponent at $\xi = -\infty$. (The primes mark ξ-derivatives here). The location of $\rho_{\text{edge}}(\xi)$ relative to the surface is determined by requiring that the density gradient is maximal at $\xi = 0$ and it has the transition width characterized by a small dimensionless parameter γ of the order of a/R, see (A3). The quantity $\rho_{\text{edge}}(\xi)$ gives the density distribution in the direction perpendicular to the surface. In principle, its accuracy is sufficient to obtain the surface tension constant [1, 5]

$$\sigma = \tfrac{1}{2} |\bar{E}| \bar{\rho}^{-1} R^{-1} \int_{-\infty}^{+\infty} \rho_{\text{edge}}^{-1} (\rho'_{\text{edge}})^2 d\xi. \tag{10}$$

Expressed in dimensionless quantities used in Appendix, σ is small quantity of the order of γ. Through (10), the surface tension constant is related to microscopic properties of the energy functional [5]. This microscopic definition is, however, hardly needed because, as a phenomenological constant, σ is known better than any Skyrme force. It should also be noted that properties of ρ_{edge}, as determined in (8, 9), are so specific that simple parametrization of this quantity in the form of Fermi-distribution-like function with empirical value of the diffuseness parameter can well be used, unless for some special reasons more detailed behavior of $\rho(\mathbf{r})$ in the edge region is required, see also in Sect. 4.

This leading approximation does not solve yet the problem of finding the equilibrium density distribution and to meet this end terms of the order of γ must be kept in the equilibrium density equation. Multiplying (A7) by w' one obtains that within this

improved accuracy the following relationship must hold,

$$(\gamma^2 w'^2 - w^2 \varepsilon(w^2))' - 2\Lambda w w' + 2\gamma^2 (\ln g^{1/2})' w'^2 = 0. \quad (11)$$

The another coordinate η of the $(\xi\eta)$-coordinate system enters only as a parameter through $\Lambda(\mathbf{r})$ and $g^{1/2}$. Here, $w = \rho^{1/2}$ and g are defined in Appendix, and demensionless units are used. Solution to (11) has now correct asymptotic behavior, namely, it turns into (6) at $\xi = -\infty$ where the derivatives of w are small. In this region

$$\varepsilon(\rho_{\text{in}}) = 9\Lambda^2/2K. \quad (12)$$

Now, (11) is integrated over ξ between $\xi = +\infty$ outside the nucleus and some value $\xi_{\text{in}} < 0$ which is on the inside at a distance larger than γ from the surface. The integral of the first term in (11) can be neglected because after the integration it involves quantities w' and $\varepsilon(w^2)$ taken at ξ_{in} where they are of higher order of smallness. The integrands in the second and third integrals are sharply peaked at the surface. The relatively smooth factors can, therefore, be taken out off the integrals at $\xi = 0$ and approximation $\rho = \rho_{\text{edge}}(\xi)$ can be used in deriving these integrals without violating the accuracy. Taking (A12) and (A17) into account, it is concluded then that the following relationship must hold at the effective surface

$$2\sigma H = -(\bar{\rho}\Lambda)_{\mathscr{E}.\mathscr{S}.}. \quad (13)$$

Here,

$$H = \tfrac{1}{2}(R_1^{-1} + R_2^{-1}) \quad (14)$$

is local curvature of the surface. According to (13), Λ is of the order of γ, too. Relationship (13) involves first and second derivatives of the surface shape function $Y(z)$, see (A10), and is, therefore, second-order differential equation which determines the equilibrium shape of the nucleus distorted by the field of external force $\Lambda(\mathbf{r})$. Expression (6) and function $w_0(\xi)$ complete definition of density distribution in the entire space. When the Coulomb force is included, the equation determining the profile function becomes an integral-differential one. This does not hinder its solution, however, [4]. Note, that the Lagrange multiplier λ_A is related to the scale factor R through (7, 13, 14), in this indirect way securing conservation of the particle number. The quantity in l.h.s. of (13) is interpreted as pressure component acting at and normal to the surface and related to the surface curvature. From (13, 4, 7) it can be concluded that the variational derivative of the energy density taken at the surface and derived for the equilibrium density distribution can also be related to the surface tension, namely,

$$\left(\frac{\delta\mathscr{E}}{\delta\rho}\right)_{\mathscr{E}.\mathscr{S}.} = \bar{E} + 2\frac{\sigma}{\bar{\rho}} H. \quad (15)$$

When the distorting force is absent ($\lambda_Q = 0$), Eq. (13) yields spherical shape with $H = \text{const}$. In this case, the inside density (6) can be explicitly obtained. By means of (12) it is determined as

$$\rho_{\text{in}} = \bar{\rho} \cdot (1 + 24\pi R^2 \sigma / KA). \quad (16)$$

The total energy (1) is

$$E = E_V + |\bar{E}| \int d^3 r (\rho\varepsilon(\rho) + \tfrac{1}{4}\gamma^2 R^2 (\nabla\rho)^2/\rho), \quad (17)$$

where

$$E_V = \bar{E} A$$

is volume energy. In the integral term in (17) more important contribution comes from the edge region where the integrand has a sharp maximum. Substituting there $\rho = \rho_{\text{edge}}(\xi)$, the integral is expressed as essentially surface integral, which equals surface energy E_s with the surface tension constant σ to which both terms contribute equally. The volume part of the integral in (17) is due to the deviation of ρ from its infinite matter value $\bar{\rho}$ and it contains the dependence of total energy on the compressibility modulus K. By means of (6, 7) this is obtained equal to

$$\frac{K}{18\bar{\rho}} \int_V d^3 r (\rho - \bar{\rho})^2 = 2E_s^2/AK \quad (18)$$

Parameter K appears, thus, only in terms proportional to $A^{1/3}$ in the mass formula to which many other factors contribute. Droplet terms of this order of magnitude in the mass formulas cannot be determined uniquely from mass fits, which are usually performed over a limited number of nuclei, because small changes in leading terms have far grater effects. Neither can this parameter be obtained from Skyrme forces fitted to nuclear masses and radii.
Generalization to the case of a two-component was presented in [7] and droplet dynamics is considered in the next section.

3. Surface Dynamics

Dynamics of the particle density $\rho(\mathbf{r}t)$ and velocity distribution $\mathbf{u}(\mathbf{r}t)$ is determined by the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\mathbf{u}) = 0, \quad (19)$$

and Euler equation

$$\frac{D\mathbf{u}}{Dt}+\frac{1}{m}\nabla\left(\frac{\delta\mathscr{E}}{\delta\rho}\right)=0, \tag{20}$$

where

$$\frac{D\mathbf{u}}{Dt}=\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\nabla)\mathbf{u}. \tag{21}$$

Multiplying (20) by $m\rho\mathbf{u}$, integrating the result over the entire space and using (19), it is obtained by means of the Gauss theorem that

$$\frac{\partial}{\partial t}\left(\int d^3r\frac{m}{2}\rho u^2+\int d^3r\mathscr{E}(\rho)\right)=0. \tag{22}$$

The total energy

$$E=\int d^3r\left(\frac{m}{2}\rho u^2+\mathscr{E}(\rho)\right) \tag{23}$$

is, thus, conserved. The system of Eqs. (19, 20) is usually considered in the limit of small velocity when the non-liner term in (21) can be omitted. It will be assumed also that the flow is irrotational and

$$\mathbf{u}=\nabla\chi. \tag{24}$$

Equations (19, 21) become then,

$$\frac{\partial}{\partial t}\rho(\mathbf{r}t)+\nabla\rho\nabla\chi+\rho\Delta\chi=0, \tag{25}$$

$$\frac{\delta\mathscr{E}}{\delta\rho}=-m\frac{\partial\chi}{\partial t}+\lambda. \tag{26}$$

To solve these equations we use the notion of dynamic effective sharp surface ($\mathscr{D.E.S.}$). As in the static case, the effective surface is defined as locus of points of maximal density gradient and is a function of time now. The shape of the effective surface is determined by a certain profile function, such as, for example, $y=Y(zt)$ in cylindrical coordinates. Partial derivatives with respect to spacial coordinates are transformed to new $(\xi\eta)$-coordinates by means of (A8,9) in the same way as in the static case. The surface dynamics must be considered, however, in the transformation of partial time derivatives. One has for arbitrary function $F(\mathbf{r}t)$,

$$\frac{\partial}{\partial t}F(\mathbf{r}t)=\frac{\partial}{\partial t}F(\xi\eta t)+\dot{\xi}\frac{\partial}{\partial\xi}F(\xi\eta t)+\dot{\eta}\frac{\partial}{\partial\eta}F(\xi\eta t) \tag{27}$$

where $F(\xi\eta t)$ is the same quantity expressed in surface-based coordinates,

$$\dot{\xi}=-\frac{1}{\tau}\frac{\partial}{\partial t}Y(\eta t), \tag{28}$$

$$\dot{\eta}=-\frac{1}{\tau^2}\left(1+\frac{\xi}{R_2}\right)^{-1}\left(\frac{\partial Y}{\partial\eta}\frac{\partial Y}{\partial z}-\frac{\xi}{\tau}\frac{\partial^2 Y}{\partial\eta\,\partial t}\right) \tag{29}$$

are velocities of time-variations of ξ and η for the fixed spacial point $\mathbf{r}$. Taken with the inverse sign, $\dot{\xi}$ is the same as the velocity u_s of normal displacement of the surface,

$$u_s=-\dot{\xi}(\eta t). \tag{30}$$

Transformed to $(\xi\eta)$-coordinates, the continuity equation is

$$\frac{\partial}{\partial t}\rho(\xi\eta t)+\left(\dot{\xi}+\frac{\partial\chi}{\partial\xi}\right)\frac{\partial\rho}{\partial\xi}+\left(\dot{\eta}+\frac{1}{g_\eta}\frac{\partial\chi}{\partial\eta}\right)\frac{\partial\rho}{\partial\eta}+\rho\Delta\chi=0. \tag{31}$$

Inside the volume, the amplitude of density distortions is small and, according to (19), one can write

$$\frac{\partial}{\partial t}\rho_d+\bar{\rho}\Delta\chi(\mathbf{r}t)=0, \tag{32}$$

where ρ_d is the dynamic part of $\rho(\mathbf{r}t)$. Near the surface the density changes sharply, considered as a function of $\mathbf{r}$ as well as t. However, in $(\xi\eta)$-coordinates only ξ-derivative is large in this region all other derivatives retaining the same magnitudes as they were inside the volume. The second term in (31) is $1/\gamma$ times larger than any other term in this region and equality (31) can be fulfilled only if the coefficient in this term turns into zero where the density gradient is maximal. Therefore, one has at the surface

$$\frac{\partial}{\partial\xi}\chi(o\eta t)=(\mathbf{un})_{\mathscr{D.E.S.}}. \tag{33}$$

Here, $\mathbf{n}$ is the unit vector perpendicular to effective surface and velocity u_s was determined in (28, 30). Equation (33) can be considered as a boundary condition set at the effective surface, which the volume solutions should satisfy. The physical significance of (33) is apparent: Normal-to-surface velocity of displacement of the surface must be the same as the normal component of the mean velocity of the matter. Considered together with Eq. (32) for the volume region, it is equivalent to original continuity Eq. (25). In particular, it provides that

$$\frac{d}{dt}\int d^3r\,\rho(\mathbf{r}t)=0. \tag{34}$$

To prove it, we split this integral into a volume part in which the integration is performed over the finite volume limited by $\mathscr{D.E.S.}$, and narrow surface region where the density derivative is large. Near the sur-

face

$$\frac{\partial \rho(\mathbf{r}t)}{\partial t} \approx \dot{\xi} \frac{\partial \rho(\xi)}{\partial \xi}, \tag{35}$$

so, that the second part of the integral can be expressed as surface integral over $\mathscr{D.E.S.}$ In the volume integration smooth quantities determined for the nuclear volume can be used and by means of continuity Eq. (32) and Gauss theorem this integral can also be expressed as a surface integral. It can be shown then that the two surface integrals cancel each other, securing (34). Note, that in case of small amplitude multipole vibrations Eq. (34) is trivial because for $l=2,3$ the volume and surface components of (34) turn into zero independently after integration over the angles. For monopole vibration ($l=0$) (34) is effectively equivalent to boundary condition (33), see also in [3].

To derive the other boundary condition one can exploit formal analogy between Euler Eq. (26) and Eq. (4) for static equilibrium density. Arbitrary constant in (26) is equal to Lagrange parameter λ, so, that hydrodynamical Euler equation would turn into Lagrange equation for equilibrium density distribution in the static case when $\chi=0$. Comparing (4) and (26) one notes that the static distortion force in r.h.s. of (4) is replaced with a dynamic quantity in (26). It makes it possible to use directly relationships of Sect. 2 replacing in them $\Lambda(\mathbf{r})$ by a dynamic quantity $m\partial\chi/\partial t-\lambda$. From (6) it is obtained, then, that in the volume region

$$\rho_d(\mathbf{r}t) = -\frac{\bar{\rho}}{v^2}\frac{\partial}{\partial t}\chi(\mathbf{r}t), \tag{36}$$

where

$$v=(K/9m)^{1/2} \tag{37}$$

is the so-called first sound velocity. It can be proved in the same way that (13) is valid also in dynamic case, if Λ is replaced by $m\partial\chi/\partial t-\lambda$, as above. By means of (13) it can be written as

$$\left(m\bar{\rho}\frac{\partial\chi}{\partial t}\right)_{\mathscr{D.E.S.}} = -2\,\sigma(H-H_{\text{stat}}) \equiv -P_s. \tag{38}$$

Another form of this relationship is

$$\left(\frac{K}{9}\rho_d\right)_{\mathscr{DES}} = P_s. \tag{39}$$

Here, P_s is pressure excess due to dynamic distortion of surface. (Only the leading terms were kept in above relationships.) Equation (38) or (39) can be considered as equation of motion of $\mathscr{D.E.S.}$ coupled to volume distortions of the density. Indeed, when solution of (32, 36) with boundary condition (33) is substituted, (38) or (39) turn into partial derivative differential equation for profile function $Y(zt)$. The energy expression (22) can also be transformed by means of (39). It can be shown that in first order in γ

$$\frac{\partial}{\partial t}\left\{\int_V d^3r\left(\frac{m}{2}\rho u^2+\rho\bar{E}(1-\varepsilon(\rho))\right)\right\}$$
$$= -\oint dS(\boldsymbol{n}\boldsymbol{u})\frac{mu^2}{2}\rho - \oint dS\,P_s u_s. \tag{40}$$

Here, in l.h.s. the integration is performed over volume restricted by $\mathscr{D.E.S.}$ and involves smooth quantities defined for the nuclear volume. The integral represents variation of the volume energy. In r.h.s., one term is the flow of kinetic energy through the surface and the other is work performed by surface tension forces per unit time.

Inside the nuclear volume (32, 36) can be replaced with equivalent wave equations for the dynamic part $\rho_d(\mathbf{r}t)$ of total density,

$$\frac{\partial^2}{\partial t^2}\rho_d - v^2\Delta\rho_d = 0 \tag{41}$$

and analogous equation for velocity potential $\chi(\mathbf{r}t)$. Solutions to this equation are generally expressed as sums of harmonic functions. So, in spherical coordinates

$$\rho_d(\mathbf{r}t) = \bar{\rho}\,\Sigma_{ln}a_{nl}(t)j_l(k_{nl}r)\,Y_{l0}(\cos\vartheta), \tag{42}$$

$$\chi(\mathbf{r}t) = \Sigma_{ln}(k_{nl})^{-2}\dot{a}_{nl}(t)j_l(k_{nl}r)\,Y_{l0}(\cos\vartheta). \tag{43}$$

Here, $j_l(x)$ are spherical Bessel functions. Amplitudes a_{nl} are periodic functions of time with frequencies

$$\omega_{nl} = vk_{nl} \tag{44}$$

and k_{nl} are found from conditions (33, 39) which couple volume distortions to surface motion.

Important particular case is small amplitude vibrations around spherical shape. One has, then, approximately,

$$\dot{\xi} = -\dot{R}(\vartheta t) = -R\,\Sigma_l\dot{\alpha}_l\,Y_{l0}(\cos\vartheta) \tag{45}$$

and

$$H-R^{-1} = \frac{1}{2R}\Sigma_l(l-1)(l+2)\alpha_l(t)\,Y_{l0}(\cos\vartheta), \tag{46}$$

where

$$R(\vartheta t) = R+\delta R(\vartheta t) = R(1+\Sigma_l\alpha_l\,Y_{l0}(\cos\vartheta)) \tag{47}$$

Quantities in both sides of (33, 39) are linear in amplitudes α_l, so that distortion of the surface itself can be neglected there. (Of course, it does not change significance of these relationships as corre-

sponding to dynamic surface.) Condition of periodicity selects only one term in each sum (42, 43). Boundary conditions (33, 39) relate amplitudes of surface and volume vibrations,

$$\alpha_l = \frac{KA^{1/3}}{3(l-1)(l+2)b_{\text{surf}}} a_{nl} j_l(k_{nl}R) = a_{nl} \cdot (k_{nl}R)^{-1} j_l'(k_{nl}R) \tag{48}$$

and $b_{\text{surf}} = 17\text{–}19$ MeV is the coefficient in surface energy term of the mass formula. Here, j_l' is the derivative of j_l with respect to its argument. Substitution of (48) into the other boundary condition leads to characteristic equation.

$$j_l'(x) = \frac{A^{1/3} Kx}{3(l-1)(l+2)b_{\text{surf}}} j_l(x), \tag{49}$$

which determines $x = k_{nl}R$ and then the frequency ω_{nl} through (44). Noting that the coefficient in the r.h.s. of (49) is a large quantity, accurate approximate analytical solutions can be obtained. For $l \geqq 2$ there exist roots $k_{0l}R \ll 1$ ($n=0$, $l \geqq 2$) which correspond to so-called surface, or long wave-length mode of density oscillations. Other roots are approximately found as

$$x_{nl} = x_{nl}^{(0)} + 3b_{\text{surf}}(l-1)(l+2)/(x_{nl}^{(0)} KA^{1/3}), \tag{50}$$

where $x_{nl}^{(0)}$ is the nth root of the Bessel function $j_l(x)$. Without the correction, Eq. (50) would correspond to condition of fixed spherical surface of the droplet, and the correction is, apparently, due to surface dynamics. These results are essentially the same as the ones described by Bohr and Mottelson in their solution of the problem of multipole density oscillations of a liquid droplet with a dynamic sharp surface [8]. Our derivation shows that such a solution takes into account the diffuseness of the nuclear surface in the first order in aH. Widely spread criticism of the so-called sharp surface approximation is, therefore, groundless, and one should not be surprized by close agreement of result of numerical solutions of hydrodynamical equations [9] with the simple model suggested by Bohr-Mottelson. To this it must be added that the characteristic equation (49), as well as (6.676) of [8] are valid in the monopole case, too, in spite of reservations stated in [8]. The misunderstanding was caused by the fact that coefficients of characteristic equation were expressed in [8] in terms of quantities which become meaningless for $l=0$, although such were not used in the derivations.

For the monopole case ($l=0$), the frequency relationship (44) can be written, by means of (50), in the form of standard equation for the frequency of sound oscillations,

$$\omega = (K_A/9m)^{1/2}(x_{nl}^{(0)}/R),$$

in which K_A is an effective incompressibility constant. Including the effect of surface displacement

$$K_A = K\{1 + 6b_{\text{surf}}(l-1)(l+2)/(x_{nl}^{(0)})^2 KA^{1/3}\}. \tag{51}$$

It has the same structure as suggested by several authors. The correction is significantly smaller than what is required to explain the A-dependence of the monopole frequencies. The effective value of K_A may contain corrections which originate from other sources.

4. Density Dynamics

General dynamics of the density distributions is conveniently characterized in terms of dynamic component of the density often described as transitional density. In $\mathscr{D}.\mathscr{E}.\mathscr{S}.$-approximation, $\delta\rho(\mathbf{r}t)$ is represented as sum of volume and surface components which are determined, correspondingly, by equation of volume dynamics for $\rho_d(\mathbf{r}t)$, boundary conditions for surface dynamics and (A16) for density distribution across the moving surface. Within the accuracy of the $\mathscr{D}.\mathscr{E}.\mathscr{S}.$-approximation

$$\delta\rho(\mathbf{r}t) = \rho_d(\mathbf{r}t) w_0^2(\xi)|_{\mathscr{D}.\mathscr{E}.\mathscr{S}.} + \bar{\rho}(w_0^2(\xi)|_{\mathscr{D}.\mathscr{E}.\mathscr{S}.} - w_0^2(\xi)|_{\mathscr{E}.\mathscr{S}.}). \tag{52}$$

In $(\xi\eta)$-coordinate system, $w_0(\xi)$ does not depend on time but the density distribution which it represents may vary with time in accordance with the dynamics of nuclear shape. Determining $w_0^2(\xi)$ through (A16) is not a difficult problem for any given form of the energy density functional. However, simple and sufficiently accurate w_0 can be obtained with a model $w_0(\xi)$ in the form of the Fermi-distribution function with a certain diffuseness parameter a. Considering small distortions around spherical shape we write approximately

$$\xi = r - R(\vartheta t)$$

and

$$w_0^2(\xi) = 1 \Big/ \left(1 + \exp\left(\frac{r - R(\vartheta t)}{a}\right)\right), \tag{53}$$

where $R(\vartheta t)$ is radius (47) of the deformed surface. For small amplitude surface distortions one obtains then $\delta R \ll a$. One has then

$$\delta\rho(\mathbf{r}t) = \rho_d(\mathbf{r}t) \Big/ \left(1 + \exp\left(\frac{r - R(\vartheta t)}{a}\right)\right) + \frac{\bar{\rho}\,\delta R(\vartheta t)}{4a} \operatorname{ch}^{-2}\left(\frac{r-R}{2a}\right). \tag{54}$$

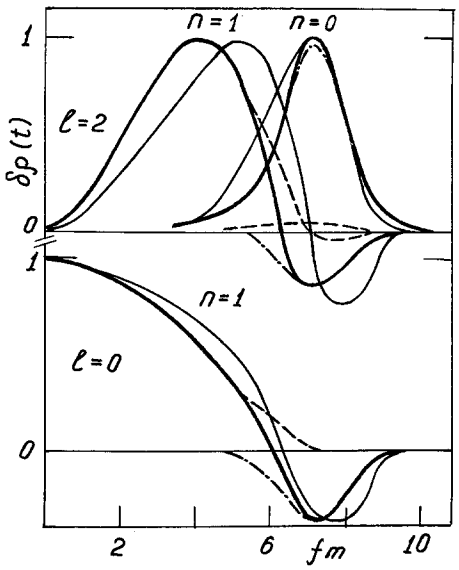

Fig. 1. Reduced transitional densities as given by (55) in the $\mathscr{D}.\mathscr{E}.\mathscr{S}.$-approximation (fat lines) and calculated numerically in [9] from linearized fluid-dynamical equations (thin lines). Radius $R=7$ Fm and $a=0.5$ Fm in (59) are the same as in [9]. Multipolarity (l) and number n of the root of characteristic equation are indicated. Curves $l=2$, $n=0$ correspond to the so-called surface mode with $kR \ll 1$. Broken and broken-dotted lines show separately contributions of volume and surface components of $\delta\rho(\mathbf{r}t)$ determined as in (55)

Taking into account relationship (48) between volume and surface amplitudes we finally find reduced transitional density

$$\frac{\delta\rho(\mathbf{r}t)}{\alpha_l}=\bar{\rho}\,Y_{l0}\left\{j_l(kr)\left(1+\exp\left(\frac{r-R}{a}\right)\right)^{-1}+\frac{R}{4a\,x_{nl}}j_l'(x_{nl})\,\mathrm{ch}^{-2}\left(\frac{r-R}{2a}\right)\right\} \tag{55}$$

where x_{nl} are roots of characteristic equation (49). Radial dependence of transitional density represented by this equation is shown in Fig. 1 (fat lines). For comparison are also presented results of numerical integration of hydrodynamical equations in [9]. Parameters $R=7$ Fm and $a=0.5$ Fm in (55) are the same as in [9]. Agreement is quite satisfactory and remaining differences can be explained as due to approximations made in derivation of (55) as well as specific properties of the energy-functional used in [9]. Multipole moments of $\delta\rho(\mathbf{r}t)$ derived for distributions (55) and such obtained in [9] differ by 10 to 15 per cent. This seems to be a rather moderate price to pay, particularly, taking into account very noticeable difference in the volume of computations. Broken and broken-dotted lines in Fig. 1 show separate contributions of volume and surface components to transitional density $\delta\rho(\mathbf{r}t)$ as given by the corresponding terms in (55). Remarcable is concentration of $\delta\rho(\mathbf{r}t)$ in the surface region for vibrations corresponding to the first root ($n=0$) of characteristic equation (49) for $l=2$. (The long wave length surface mode.) However, the surface contributions are of significance in all other cases, too. In total multipole moments of $\delta\rho(\mathbf{r}t)$, which are sums of volume and surface contributions, the dynamic surface distortions compensate almost completely the volume parts for the sound modes ($n \geqq 1$). Apparent exception is the surface mode where the surface in itself is the major contributor. In contrast, the eigen-frequencies of these modes are far less sensitive to the surface dynamics: Values of ω_{nl}, obtained from (44), (49) for $n \geqq 1$, differ very little from those obtained for fixed spherical surface and the surface mode is, again, a very important exception. It can be noted that these qualitative consulsions are very much true also for zero-sound vibration modes coupled to the dynamic surface, although, except for $l=0$, the lowest roots of the corresponding characteristic equations are not small [3].

Detailed density distributions given by (55) are not needed in derivations of physical quantities related to $\delta\rho(\mathbf{r}t)$, which are usually expressed in terms of moments of $\delta\rho$. Such can be derived by using still simpler representation for $\delta\rho$ in the form

$$\delta\rho(\mathbf{r}t)=\rho_d(\mathbf{r}t)\,\theta(r-R(\vartheta t))+\bar{\rho}\,\delta R(\vartheta t)\,\delta(r-R), \tag{56}$$

where $\theta(x)$ is the theta-function. In moments of $\delta\rho(\mathbf{r}t)$, Eq. (56) corresponds to taking into account the surface diffuseness of $\delta\rho(\mathbf{r}t)$ in first order in aH. The result is not sensitive to detailed behavior of $\delta\rho$ in the edge region. One can also note that the reduced transitional density (54) can be defined only as an expantion in the amplitude δR of the surface distortion which apparently requires that $\delta R/a \ll 1$. In practice, the inequality is not very great, particularly, for the surface contained modes. It can, however, be seen that the inclusion of higher order terms corrects the moments of $\delta\rho(\mathbf{r}t)$ by quantities of the order of $(\delta R/R)^2$ and smaller which are of no significance.

Estimates for the amplitudes of surface distortions can be obtained by expressing the resonance energies as quantities proportional to mean squared amplitudes and taking into account relationship (48) between a_l and α_l. Volume modes of 15 MeV correspond to $\delta R \approx 0.1\,a$ whereas for the surface vibrations δR and a are of the same magnitude. In these estimates hydrodynamical values for the mass coefficients were used.

5. Conclusion

Approximation of dynamical sharp effective surface ($\mathscr{D}.\mathscr{E}.\mathscr{S}.$) takes into account the diffuseness of nuclear edge in the first order in aH where a is the diffuseness parameter and H is the Gaussian curvature. It can be effectively used in solving problems of nuclear dynamics not limited by cases of small amplitude vibrations. It uses simple parametrization of total energy density in terms of quantities characteristic of infinite nuclear matter rather than effective nucleon interactions. Other quantities of importance also are simple phenomenological objects which already at the formal stage of derivations, are recognized as such and replaced with, for example, the surface tension, nuclear radius and diffuseness parameter. For such their values known from empirical data are used. The $\mathscr{D}.\mathscr{E}.\mathscr{S}.$-approach can be generalized to non-potential fluid dynamics and it can be useful also in solving the problem of quasi-particle modes coupled to dynamics of nuclear shape.

Appendix

We consider here simple expression for the energy density

$$\mathscr{E}(\rho)=\mathscr{A}(\rho)+\mathscr{B}(\rho)\frac{1}{\rho}(\nabla\rho)^2 \tag{A1}$$

where $\mathscr{A}(\rho)$ and $\mathscr{B}(\rho)$ are functions of the local density $\rho(\mathbf{r})$. The second term is small in the nuclear volume and $\mathscr{A}(\rho)$ coinsides there with approximate expression (5). Qualitative properties of equilibrium $\rho(\mathbf{r})$ are determined by characteristic structure of (A1) which becomes more apparent if it is transformed to dimensionalless quantities choosing constants $|\bar{E}|$, $\bar{\rho}$ and R as appropriate units. Equation (A1) takes then the form

$$\mathscr{E}(\rho)=|\bar{E}|\rho\left(\varepsilon(\rho)-1+\frac{1}{4}\gamma^2\frac{1}{\rho^2}(\nabla\rho)^2\right). \tag{A2}$$

Parameter

$$\gamma=(4\mathscr{B}\bar{\rho}R/|\bar{E}|)^{1/2} \tag{A3}$$

is a small quantity and

$$\varepsilon(\rho)=\mathscr{A}(\rho)/|\bar{E}|\bar{\rho}+1 \tag{A4}$$

is generally, of the order of unity, turning into zero in the nuclear volume where ρ approaches $\bar{\rho}$ and into unity at large distances from the nucleus. Lagrange equation corresponding to (A2) contains small coefficient in the term with higher derivatives which is the formal reason for the solution having a distinct edge region. As it will be seen, ratio of the diffuseness parameter a to R is of the order of γ which can, correspondingly, be estimated as

$$\gamma\approx A^{-1/3}. \tag{A5}$$

Variational equation (4) is somewhat simplified by substitution of

$$\rho=w^2(\mathbf{r}). \tag{A6}$$

It becomes

$$\gamma^2\Delta w-\tfrac{1}{2}d(w^2\varepsilon(\omega^2))/dw-\Lambda(\mathbf{r})w=0. \tag{A7}$$

Here, Λ is defined as in (7). The first term in (A7) is small everywhere except for the edge region where some derivatives of ρ become large. Derivatives of ρ can be neglected in nuclear volume and by means of (5) one finds the internal density (6). In what follows it will be assumed that the shape of effective surface ($\mathscr{E}\mathscr{S}$) is determined by a certain profile function expressed in cylindrical coordinates as $y=Y(z)$. Coordinate ξ was defined in text, the other coordinate η can be defined as z-coordinate of the point at the surface where the perpendicular to the surface from the given point $\mathbf{r}$ crosses the surface. So defined, new coordinates $(\xi\eta)$ relate to cylindrical coordinates as

$$y=y(\xi\eta)=Y(\eta)+\frac{\xi}{\tau}, \tag{A8}$$

$$z=z(\xi\eta)=\eta-\frac{\xi}{\tau}\frac{\partial Y(\eta)}{\partial\eta}, \tag{A9}$$

where

$$\tau=\left(1+\left(\frac{\partial Y(\eta)}{\partial\eta}\right)^2\right)^{1/2}.$$

Length element

$$dl=(g_\xi d\xi^2+g_\eta d\eta^2+g_\varphi d\varphi^2)^{1/2},$$

where

$$g_\xi=1,\qquad g_\eta=(1+\xi/R_2)^2\tau^2,$$

$$g_\varphi=\left(1+\frac{\xi}{\tau}\right)^2 Y^2(\eta),$$

and R_1 and R_2 are two local curvature radii of the surface

$$R_1=\tau Y(\eta),\qquad R_2=-\tau^3/(\partial^2 Y/\partial\eta^2). \tag{A10}$$

Volume element

$$d^3r=g^{1/2}d\xi\,d\eta\,d\varphi$$

with

$$g^{1/2}=(R_1+\xi)(1+\xi/R_2). \tag{A11}$$

According to (A11), mean curvature

$$H=\frac{1}{2}(R_1^{-1}+R_2^{-1})=\frac{1}{2}\left(\frac{\partial}{\partial\xi}\ln g^{1/2}\right)_{\xi=0}. \tag{A12}$$

Laplace operator

$$\Delta_{\xi\eta}=\frac{\partial^2}{\partial\xi^2}+\frac{\partial}{\partial\xi}(\ln g^{1/2})\frac{\partial}{\partial\xi}+g^{-1/2}\frac{\partial}{\partial\eta}\left(\frac{g^{1/2}}{g_\eta}\frac{\partial}{\partial\eta}\right). \tag{A13}$$

Considering (A7) transformed to $(\xi\eta)$-coordinates, it is immediately seen that terms which contain ξ-derivatives of w are near the surface of the order of γ^0 and γ^1 whereas all other derivatives remain of the order of γ^2, as any term is inside nuclear volume. Collecting the leading terms, the following equation is obtained, approximately valid near the surface,

$$2\gamma^2 d^2 w_0(\xi)/d\xi^2-d(w_0^2\varepsilon(w_0^2))/dw_0=0. \tag{A14}$$

Solution to this equation, $w_0(\xi)$, determines the zero-order density distribution across the surface at each point along the curved surface. It is a function of ξ-coordinate alone. Multiplying (A14) by $w_0'=dw_0/d\xi$, equality (A14) can be written as

$$(\gamma^2(w_0')^2-w_0^2\varepsilon(w_0^2))'=0, \tag{A15}$$

which, together with the apparent condition, that

$$w_0(+\infty)=w_0'(+\infty)=0$$

gives $w_0(\xi)$ as solution to the well-known equation for the density distribution in a semi-infinite matter [5],

$$\gamma^2(w_0')^2-w_0^2\varepsilon(w_0^2)=0, \tag{A16}$$

see in the text. The same quantity determines also the density ditribution across the deformed – and time dependent – nuclear surface. Noting the asymptotic behavior of $\varepsilon(w^2)$ at $w\to 0$ and $w\to 1$, it can be seen that $w_0(\xi)$ turns into zero at $\xi=+\infty$ off the nuclear surface and exponentially approaches unity in the nuclear volume, with the transition width γ. It makes it possible to identify γ with dimensionless quantity a/R, as it was done in the text. Expressed in terms of dimensionless quantities the surface tension constant

$$\sigma=2\gamma^2\int_{-\infty}^{+\infty}(w_0'(\xi))^2 d\xi \tag{A17}$$

is also of the order of magnitude γ.

Better accuracy is required to determine the volume density and profile function. One possible way of solving this problem is described in Sect. 2, Eq. (11) and on.

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Note Added in Proof

Static and dynamical density distributions for realistic forms of the energy density (A1) are considered in another paper, submitted to Z. Phys. A – Atoms and Nuclei 1984.