Rigid gauges and F-zips, and the fundamental sheaf of gauges G_n

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Introduction

In this paper we study $D - \varphi$-gauges introduced by J.M. Fontaine and U. Jannsen, the author's advisor. These are objects of (Frobenius)-linear algebra over a perfect field k of positive characteristic. Fontaine and Jannsen define invariants for smooth projective varieties over k, which take values in gauges, by means of e.g. syntomic cohomology of a sheaf $G_n^{\cdot}$ built essentially of the sheaves $\mathcal{O}_n^{\text{cris}}$.

In the first section we study $D_1 - \varphi$-gauges over a perfect field k. A $D_1 - \varphi-$gauge over k is a graded module M of finite type over the graded ring $k[f, v]/(fv)$ (with f in degree 1 and v in degree -1) together with an Frobenius-semi-linear isomorphism $\varphi : M^{\infty} \xrightarrow{\sim} M^{-\infty}$. Fontaine defined the subcategory of rigid gauges to be those $D_1 - \varphi$-gauges with $\text{im} v = \ker f$, $\text{im} f = \ker v$ and $\ker(f, v) = 0$.
We study the structure and morphisms of rigid $D_1 - \varphi$-gauges. The underlying D_1-module of a rigid gauge is isomorphic to $\bigoplus_{k=1}^{d} D_1(m_k)$ with some numbers m_k. The morphisms of $D_1 - \varphi$-gauges can be described by matrices over k which satisfy some Frobenius-linear equations. The composition of morphisms is in general not given by matrix multiplication, but we give an explicit description of the matrix of a composition of two morphisms.
An F-zip over k is a finite-dimensional k-vectorspace, with an ascending and a descending filtration with semi-linearily isomorphic subquotients (see A.2). There is a functor from rigid $D_1 - \varphi$-gauges to F-zips over k, due to Fontaine, by sending M to $M^{-\infty}$. The filtrations are defined by the images of v_r^{∞} resp φf_r^{∞} (With e.g. v_r^{∞} the map $M^r \to M^{-\infty}$ induced by v). We construct a functor in the opposite direction by mapping an F-zip $(M, C^{\bullet}, D_{\bullet}, \varphi)$ to $(\oplus^{\sigma} D_r) \times_{\text{gr}^D(^{\sigma}M)} (\oplus C^r)$, which is a rigid $D_1 - \varphi$-gauge. The main result is that these functors are quasi-invers to each other, i.e. the category of F-zips over k is a full subcategory of the category of $D_1 - \varphi$-gauges. It is equivalent to the category of rigid $D_1 - \varphi$-gauges.

The second section introduces quasi-étale morphisms. A morphism of schemes is called quasi-étale (or quiet) if is locally a composition of étale morphisms and successive extractions of p.-th roots. By the latter we mean a morphism of rings $A \to A[T]/(T^p - \alpha)$ with some element α. Stability of quiet morphisms under compostition and base-change is shown. One important property of classes of morphisms we study, is the "lifting property": We say that a class τ of scheme morphisms satisfies the lifting property, if for every nil-immersion $U \to T$ and for every τ-morphism f to U, Zariski-locally there is a τ-morphism g to T, such that f is the base-change of g. It is shown that quasi-étale morphisms satisfy the lifting property.
Later we shall show that the quasi-étale cohomology of certain sheaves is equal to syntomic cohomology. This is true for example for $\mathcal{O}_n^{\text{cris}}$.

In the third section we will study different topologies and the associated cohomology for direct images of quasi-coherent crystals. After a quick review of the large crystalline site endowed with different topologies, we define three axioms on classes τ of scheme-morphisms (T1)-(T3). (T1) lists some standard properties, like flatness, stability under base-change etc., (T2) is similar to the lifting property and (T3) demands that every extraction of a p.-th root has to be in τ. If a class satisfies all three axioms we call it *p-crystalline*.

If the first two are satisfied for a class τ, we can construct a morphism of topoi from the large τ-crystalline topos to the large τ-topos $v : (X/S)_{CRIS,\tau} \to X_\tau$: The main reason is, that the lifting property together with flatness ensures, that a τ-covering can be lifted to a τ-crystalline covering of any PD-thickening.

A crystal is a special, "rigid" sheaf on the crystalline site, and it is called quasi-coherent if it is defined by quasi-coherent modules. In the following, cohomology of direct images of quasi-coherent sheaves under the morphisms v is studied. For example $\mathcal{O}_n^{\text{cris}}$ is the direct image of the crystalline structure-sheaf $\mathcal{O}_{X/W_n}$. The main result is the following comparison theorem:

If two classes τ and τ' of morphisms satisfy the three axioms, τ-cohomology of the direct image of a quasi-coherent crystal agrees with its τ'-cohomology. The proof combines the facts that higher direct images $R^q v_* F$ of a quasi-coherent crystal F vanish and that crystalline cohomology of a quasi-coherent crystal is independent of the topology on the crystalline site.

The fourth section is devoted to the study of the sheaves G_n. They are defined by Fontaine and Jannsen and are one of the central constructions. We shall give and proof a small formulaire of elementary properties of G_n.

First the sheaves $\mathcal{O}_n^{\text{cris}}$ are defined: $\mathcal{O}_n^{\text{cris}}(Y) = H^0_{\text{cris}}(Y/W_n, \mathcal{O}_{Y/W_n})$ and a relation to a divided power envelope of the pre-sheaf of Witt-vectors is given. It follows that there is a Frobenius φ on $\mathcal{O}_n^{\text{cris}}$. The image of Frobenius for $n = 1$ is determined: There is a canonical monomorphism $\mathcal{O} \hookrightarrow \mathcal{O}_1^{\text{cris}}$, and the image of Frobenius is the image of this monomorphism. Furthermore there is an epimorphism $\mathcal{O}_1^{\text{cris}} \twoheadrightarrow \mathcal{O}$. Both compositions equal the respective Frobenius.

We study the fundamental exact sequence

$$0 \to \mathcal{O}_n^{\text{cris}} \to \mathcal{O}_{m+n}^{\text{cris}} \to \mathcal{O}_m^{\text{cris}} \to 0.$$

The graded sheaf of rings G_n is defined by letting G_n^r be the cokernel of p^n-multiplication on $\hat{G}_m^r = \ker(\mathcal{O}_m^{\text{cris}} \xrightarrow{\varphi} \mathcal{O}_m^{\text{cris}} \to \mathcal{O}_r^{\text{cris}})$ for $m \geq n + r$ (This is independent of m). There are global sections f and v of respective degree 1 and -1. We show "strictness", i.e. that $(f_r^\infty, v_r^{-\infty})$ is injective and some kind of rigidity, namely that the sequences $G_n \xrightarrow{f^n} G_n \xrightarrow{v^n} G_n$ and $G_n \xrightarrow{v^n} G_n \xrightarrow{f^n} G_n$ are exact. There is a ringhomomorphism $\varphi : G_n \to \mathcal{O}_n^{\text{cris}}$, which is, on G_n^r, informally be given by "division by p^r" after Frobenius on $\mathcal{O}_m^{\text{cris}}$. The images $F_r = \text{im}\varphi_r$ in $\mathcal{O}_n^{\text{cris}}$ define an ascending filtration.

Now we consider characteristic p, i.e. $n = 1$. The kernel of Frobenius $\mathcal{J}_1^{[1]}$ is a divided power-ideal in $\mathcal{O}_1^{\text{cris}}$, and the higher powers define a descending filtration $F^r = \mathcal{J}_1^{[r]}$ on

$\mathcal{O}_1^{\mathrm{cris}}$. It is a result of Fontaine, that the subquotients of both filtrations are isomorphic, via the Cartier-isomorphism. Kernel and cokernel of $v : G_n^r \to G_n^{r-1}$ are isomorphic to F_r, while kernel and cokernel of $f_r : G_n^{r-1} \to G_n^r$ are isomorphic to F^r. The second statement follows from the first with rigidity and Cartier-morphism. There is an exact sequence

$$0 \to G_n \to G_{m+n} \to G_m \to 0.$$

We wish to compare chomology of G_n for different topologies. One possible way to do this is via p-good algebras and Čech-cohomology. A smooth k-algebra is called good, if it admits a system of parameters, and a k-algebra is p-good, if it is the quotient of a good algebra $\mathcal{A}$, by a regular sequence $(f_1^p, \ldots, f_r^p)$. Every syntomic k-scheme can be covered by p-good algebras in p-topology and every syntomic covering can be p-refined (i.e. in the topology generated by p.-th roots) to a covering consisting of p-good algebras in a very particular form, a so called p-good covering. Fontaine computed the value of the sheaves $\mathcal{O}_1^{\mathrm{cris}}$, $\mathcal{J}_1^{[r]}$ and $\mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}$ explicitely over p-good algebras.
We are then able to show with Čech-cohomology computations that cohomology of the subquotient $\mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}$ does not depend on the p-crystalline topology used. This easily implies that the cohomology of G_n is also independent of this choice.

In the fifth chapter we study relations between F-zips and different notions of gauges over a scheme X over $\mathbb{F}_p$. The correct notion should be the one of $\varphi - G_1$-crystal. A $\varphi - G_1$-module is a graded G_1-module M plus an isomorphism $\Phi : \mathcal{O}_n^{\mathrm{cris}}{}_{\varphi^\nwarrow} \otimes_{G_n} M \to \mathcal{O}_n^{\mathrm{cris}}{}_{pr^\nwarrow} \otimes_{G_n} M$. A $\varphi - G_1$-module which comes from the small Zariski-site is called a $\varphi - G_1$-crystal. If X is a field, a $\varphi - G_1$-crystal is the same as a $D_1 - \varphi$-gauge. It turned out that it is necessary to modify the notion of F-zip slightly, for details see the appendix. It seemed also, that modified F-zips are the right definition for extending the notion of F-zip to a higher level, i.e. for definig F-zips over W_n.
First we introduce the notions of $D_1 - \varphi$-gauges over X. These are graded $\mathcal{O}[f,v]/(fv)$-modules plus an isomorphism $(M^\infty)^{(p)} \to M^{-\infty}$. Again there is a notion of strictness and rigidity, given exactly as in the case of fields. If we want to compare with F-zips we have to introduce a property of locally freeness: A $D_1 - \varphi$-gauge is called locally free if all graded pieces are locally free and if kernel and cokernel of v_r and f_r are locally direct summands. There is a functor from modified F-zips to $D_1 - \varphi$-gauges which induces an equivalence of categories between modified F-zips and rigid locally free $D_1 - \varphi$-gauges. The functor and its quasi-inverse functor are essentially given as in the case of fields.
To a $D_1 - \varphi$-gauge M we can assign a $G_1 - \varphi$-module by tensoring: $G_1 \otimes_{D_1} M$ is a $D_1 - \varphi$-crystal, with the morphism $D_1 \to G_1$ which is given by $D_1^0 = \mathcal{O} \hookrightarrow \mathcal{O}_1^{\mathrm{cris}} = G_1^0$ and $f \mapsto f$, $v \mapsto v$.

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Conventions and notations

- For a set E we let $\sharp E$ be the cardinality of E.
- The n.-th unit-matrix is denoted by E_n.
- For an abelian Group A we let A_p be p-torsion: $A_p = \{a \in A | pa = 0\}$.
- All gradings are indexed with $\mathbb{Z}$. We write a graded object M as $M = \bigoplus_{r \in \mathbb{Z}} M^r$.
- For a graded object M we and $n \in \mathbb{N}$ we define the n.-th twist of M to be the graded object $M(n)$ with $M(n)^r = M^{n+r}$
- If $f : M \to N$ is a morphism of graded modules of degree n, we write f_r for $M^r \to N^{n+r}$.
- Descending filtrations are marked with an upper index, ascending filtrations with a lower index.
- Let $C^\bullet$ and $D_\bullet$ be a descending and an ascending filtration of an object M. If the subquotients exist we define

  $$\mathrm{gr}^r_C M = C_r/C_{r-1} \text{ and } \mathrm{gr}^D_r M = D^r/D^{r+1}.$$
- Monomorphisms are symbolized by $\hookrightarrow$, epimorphisms by $\twoheadrightarrow$ and for isomorphisms we use $\xrightarrow{\sim}$.
- If $f : X \to Y$ is morphism of schemes, the associated morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ will be denoted by $f^\sharp$. For an $\mathcal{O}_Y$-module M we denote $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}M$ with f^*M.
- If M is a $\mathcal{O}_X$-module over a scheme $X/\mathbb{F}_p$ we let $M^{(p)} = M \otimes_{\mathcal{O}_X \nearrow F} \mathcal{O}_X$ with the absolute Frobenius $F : x \mapsto x^p$.
- By "DP" we mean divided powers. If an ideal in a ring is furnished with divided powers, we denote by γ_n the n.-th divided power.

Contents

1 Elementary calculations on rigid modules and gauges

Let k be a perfect field of characteristic $p > 0$, let its absolute Frobenius be denoted by σ. Let $W = W(k)$ be the ring of Witt-vectors of k, and $W_n = W/p^n$ as usual. The Frobenius morphism on W and W_n will also be denoted by σ. It raises each component to the p.-th power.

For a module M over W (resp. W_n) we define $({}^\sigma M)$ to be the scalar restriction along Frobenius: Multiplication of an element m by a scalar λ is by definition $\lambda^p m$. For $n = 1$, i.e. for a k-vectorspace we have the alternative description $({}^\sigma M) \cong M \otimes_{k \nearrow_{\sigma^{-1}}} k$.
If M is W-module, we let $M^{(p)} = M \otimes_{W \nearrow_\sigma} W$ be scalar extension along Frobenius (analogously for W_n-modules).

Definition 1.0.1. A *σ-linear map* $\Phi : M \to N$ of modules over W (resp. W_n) is a W-linear (resp. W_n-linear) map $M \to ({}^\sigma N)$. Equivalently we can say that Φ is an additive map, such that for any element $m \in M$ and any scalar λ it holds $\Phi(\lambda m) = \sigma(\lambda)\Phi(m)$ $(= \lambda^p \Phi(m)$ for $n = 1)$.

Remark 1.0.2. Let $\alpha : V \to W$ be a map of k-vectorspaces with respective bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$. Let the matrix of α with respect to these bases be denoted by $A = (a_{ij})$. The matrix of ${}^\sigma\alpha : {}^\sigma V \to {}^\sigma W$ is given by

$$ {}^\sigma A := ({}^\sigma a_{ij}) := (a_{ij}^{\frac{1}{p}}). $$

Indeed the underlying map is the same, hence, if we denote scalar multiplication in ${}^\sigma W$ by $*$, we have:

$$ \begin{aligned} {}^\sigma\alpha(v_i) &= \sum_{k=1}^{m} a_{ki} w_k \\ &= \sum_{k=1}^{m} a_{ki}^{\frac{1}{p}} * w_k. \end{aligned} $$

1.1 Preliminaries

Remark 1.1.1. (i) We say that a graded module M over a graded ring R is of finite type (or finitely generated) if there are homogenous elements $m_1 \ldots, m_r$, such that their R-linear span is M. Equivalently M is of finite type if there is an epimorphism of graded R-modules

$$ \bigoplus_{k=1}^{r} R(i_k) \twoheadrightarrow M $$

where $R(i_k)$ is R with twisted grading.

The notion of gauges was introduced by Fontaine and Jannsen. They found many notions of gauges in different situations, this is the simplest:

Definition 1.1.2. (Due to Fontaine and Jannsen)
(i) Let D be the $\mathbb{Z}$-graded, commutative ring $D = W[f, v]/(fv - p)$ where f and v are variables of degree 1 and -1 respectively.
(ii) A *D-module* is a graded module over D of finite type.
(iii) For a D-module M we let

$$M^{-\infty} = \varinjlim_{k \in \mathbb{Z}, \geq} M^k \cong M/(v-1)M$$

$$M^{\infty} = \varinjlim_{k \in \mathbb{Z}, \leq} M^k \cong M/(f-1)M$$

where the transition maps in the limit are given by multiplication with v resp. f.
For the maps into the limit we write

$$v_r^{\infty} : M^r \to M^{-\infty}$$

$$f_r^{\infty} : M^r \to M^{\infty}$$

(iv) A *$D - \varphi$-module* is a D-module M with a σ-linear map $\varphi : M^{\infty} \to M^{-\infty}$.
(v) A $D - \varphi$-module is called a *$D - \varphi$-gauge* if φ is an isomorphism.

Definition 1.1.3. For a natural number n we let $D_n = D/(p^n) = W_n[f, v]/(fv - p)$. Similarily we get the notion of *D_n-modules, $D_n - \varphi$-modules* and *$D_n - \varphi$-gauges.*

Remark 1.1.4. (i) We will view a D_n-module $M = \bigoplus_r M^r$ as a diagram of W_n-modules

$$\cdots \leftrightarrows M^{r-1} \leftrightarrows M^r \leftrightarrows M^{r+1} \leftrightarrows \cdots$$

with W_n-linear maps $v_r : M^r \to M^{r-1}$ and $f_r : M^r \to M^{r+1}$, such that for every r it holds $v_r f_{r-1} = p$ and $f_r v_{r+1} = p$.
(ii) Since a D_n-module is assumed to be of finite type, multiplication by f is an isomorphism for very large degree and multiplication by v is an isomorphism for very small degree. This follows from simple calculations in graded modules:
Let M be the quotient of $D_n(i_1) \oplus \ldots \oplus D_n(i_s)$ by the submodule generated by homogenous elements $m_1, \ldots, m_r$ of respective degrees $d_1, \ldots, d_r$. Then the graded piece M^s is (for $s \geq \max\{d_1, \ldots, d_r, -i_1, \ldots, -i_s, 0\}$)

$$f^{s+i_1} W_n \oplus \ldots \oplus f^{s+i_r} W_n / (f^{s-d_1} m_1, \ldots, f^{s-d_r} m_r)$$

and multiplication by f is an isomorphism. We can treat v analogously.
(iii) Thus we can in effect represent a D_n-module by a *finite* diagram like in (i):

$$M^a \leftrightarrows M^{a+1} \leftrightarrows \cdots \leftrightarrows M^b$$

for some integers $a \leq b$.

There are many interesting subcategories in the category of D_n-modules, especially for $n = 1$, i.e. in characteristic p. We need the following notions of "strictness" and "rigidity", due to Fontaine. They are defined purely in terms of easy linear algebra, but give a subcategory which allows us to compare gauges with other constructions of algebraic geometry, especially Moonen and Wedhorns F-zips.

Definition 1.1.5. (due to Fontaine.) Let M be a D_1-module.
(i) M is called *strict* if $M^r \to M^{r-1} \oplus M^{r+1}$ is injective for all $r \in \mathbb{Z}$.
(ii) M is called *rigid* if M is strict and it holds $\mathrm{im} v = \ker f$ and $\ker f = \mathrm{im} v$ (or equivalently if $\mathrm{im} v_{r+1} = \ker f_r$ and $\mathrm{im} f_{r-1} = \ker v_r$ for all r).

1.2 Structure and morphisms

First we find, that for rigid modules the dimension of the homogenous parts does not change with varying degree.

Lemma 1.2.1. *Let M be a rigid D_1-module.*
(i) It holds $\dim M^r = \dim M^{r+1}$ for all $r \in \mathbb{Z}$.
(ii) One has $\mathrm{rk} f_r - \mathrm{rk} f_{r-1} = \mathrm{rk} v_r - \mathrm{rk} v_{r+1} \geq 0$.
(iii) Furthermore $v_r|_{\mathrm{im} v_{r+1}}$ and $f_r|_{\mathrm{im} f_{r-1}}$ are monomorphisms.

Proof. (i) Consider the exact sequence

$$M^{r-1} \overset{f_{r-1}}{\to} M^r \overset{v_r}{\to} M^{r-1}$$

which gives the exact sequence

$$0 \to \mathrm{im} f_{r-1} \to M^r \to \mathrm{im} v_r \to 0.$$

Thus we have $\mathrm{rk} f_{r-1} + \mathrm{rk} v_r = \dim M^r$ and analogously by the exact sequence

$$M^{r+1} \overset{v_{r+1}}{\to} M^r \overset{f_r}{\to} M^{r+1}$$

we get that $\mathrm{rk} f_r + \mathrm{rk} v_{r+1} = \dim M^r$ for all r.
(ii) Let $n = \dim M^r$. It is (because of the strictness)

$$\begin{aligned} n &\geq \dim\ker v_r + \dim\ker f_r \\ &= n - \mathrm{rk} v_r + n - \mathrm{rk} f_r \\ &= \mathrm{rk} f_{r-1} + \mathrm{rk} v_r - \mathrm{rk} v_r + n - \mathrm{rk} f_r \\ &= \mathrm{rk} f_{r-1} - \mathrm{rk} f_r + n \end{aligned}$$

and a similar calculation shows that this equals $\mathrm{rk} v_{r+1} - \mathrm{rk} v_r + n$.
(iii) If $v^2 m = 0$ it follows $(v(vm), f(vm)) = (0, pm) = 0$ and strictness implies $vm = 0$. □

We will now classify rigid modules and gauges. Firstly we can associate to every rigid module its type and we can define standard modules for every type:

Definition 1.2.2. (i) A *type* is a map $\mathbb{Z} \to \mathbb{N}$ with finite support.
(ii) Let M be a rigid D_1-module. The type τ of M is defined to be the following map:

$$\tau : \mathbb{Z} \to \mathbb{N} \; ; \; n \mapsto \mathrm{rk} f_n - \mathrm{rk} f_{n-1}$$

We define $h(\tau), \tau_1, \ldots, \tau_{h(\tau)}$ by:

$$\mathrm{Supp}(\tau) = \{\tau_1 < \tau_2 < \ldots < \tau_{h(\tau)}\}$$

(iii) We need some auxiliary notations:
For $M \subseteq \mathbb{Z}$ let $d(\tau, M) = \sum_{m \in M} \tau(m)$
$d(\tau) = d(\tau, \mathbb{Z})$ (the "dimension")
(iv) For a type τ let $M(\tau)$ be the rigid D_1-module with $M(\tau)^n = k^{d(\tau)}$ for all n and

$$f_n = \begin{pmatrix} E_{d(\tau,(-\infty,n])} & 0 \\ 0 & 0 \end{pmatrix} \text{and } v_{n+1} = \begin{pmatrix} 0 & 0 \\ 0 & E_{d(\tau,[n+1,\infty))} \end{pmatrix}$$

(v) If furthermore $A \in Gl_{d(\tau)}(k)$ then let $M(\tau, A)$ be $M(\tau)$ with $M^\infty \to ({}^\sigma M^{-\infty})$ given by A (in the standard base).

Remark 1.2.3. Let M be a rigid D_1-module of type τ.
(i) The dimension $d(\tau)$ is the dimension of each graded piece:

$$\begin{aligned} d = d(\tau) &= \sum_{n \in \mathbb{Z}} \tau(n) \\ &= \sum \mathrm{rk} f_n - \mathrm{rk} f_{n-1} \\ &= \mathrm{rk} f_N \\ &= \dim M^N \end{aligned}$$

for $N \gg 0$, since f_N is an isomorphism.
(ii) We have for $n \in \mathbb{Z}$

$$\begin{aligned} \tau(n) &= \mathrm{rk} f_n - \mathrm{rk} f_{n-1} \\ &= d - \dim \ker f_n - \dim \mathrm{im} f_{n-1} \\ &= d - \dim \ker f_n - \dim \ker v_n \end{aligned}$$

Since M is strict, the kernels intersect in 0 so there is a (non canonical) decomposition

$$M^n \cong \ker v_n \oplus \ker f_n \oplus V_n$$

with $\dim V_n = \tau(n)$.

(iii) Let M be a D-module. There are maps:

$$\delta_s : M^r \longrightarrow M^{r+s}$$

$$m \mapsto \begin{cases} f^s m \text{ for } s \geq 0 \\ v^s m \text{ for } s \leq 0. \end{cases}$$

Sometimes we omit δ_s in the notation, i.e. we write m for the image of m in any graded piece.

Lemma 1.2.4. *Let τ be a type. There is an isomorphism:*

$$M(\tau) \cong \bigoplus_{n \in \mathbb{Z}} D_1(-n)^{\tau(n)}.$$

Proof. clear. □

The first classification result is the following:

Proposition 1.2.5. *Every rigid $D_1 - \varphi$-module M of type τ is isomorphic to $M(\tau, A)$ for an $A \in Gl_{d(\tau)}(k)$.*

Proof. For each n there is a decomposition

$$(*) \qquad M^n = \ker(f_n) \oplus \ker(v_n) \oplus V_n,$$

where V_n is a $\tau(n)$-dimensional k-vector space. Choose a basis $b_n^1, \ldots, b_n^{\tau(n)}$ of V_n for each $n \in \mathbb{Z}$. Let $h = h(\tau) = \sharp\mathrm{Supp}(\tau)$.
The following claim will prove, that each M^n has a basis
$b_{\tau_1}^1, \ldots, b_{\tau_1}^{\tau(\tau_1)}, b_{\tau_2}^1, \ldots, b_{\tau_2}^{\tau(\tau_2)}, \ldots, b_{\tau_h}^1, \ldots, b_{\tau_h}^{\tau(\tau_h)}$ (the images of those vectors, cf. above).

Claim:
The images of $b_{\tau_1}^1, \ldots, b_{\tau_1}^{\tau(\tau_1)}, \ldots, b_{\tau_s}^1, \ldots, b_{\tau_s}^{\tau(\tau_s)}$ form a basis of $\ker v_n$, where we define $s = s(n)$ by letting $\{\tau_1, \ldots, \tau_s\}$ the part of the support of τ which is strictly smaller than n. Analogously a basis of $\ker f_n$ is given.

We prove the claim by induction: For $n \ll 0$ multiplication by v is an isomorphism, so $\ker v_n = 0$. Now let the basis of $\ker v_{n-1}$ be of the described form. Since $\ker v_{n-1} \oplus V_{n-1}$ is a complement of $\ker f_{n-1}$ by $(*)$, multiplication by f is a monomorphism on $\ker v_{n-1} \oplus V_{n-1}$. Its image is $\ker v_n$ and the claim follows.

We use this basis to identify M^n with k^d. This identification gives the desired matrices for the f's and v's. □

Remark 1.2.6. We can visualize rigid D_1-modules as diagrams. Let for example $\tau(0) = 2$, $\tau(2) = 1$ and 0 elsewhere. The picture for $M(\tau)$ is

$$\begin{array}{ccccccc} \longleftarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow \\ \longleftarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow \\ \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longrightarrow \\ & 0 & & & & 2 & \end{array}$$

The columns symbolize the graded pieces, each point stands for a standard basis vector. An arrow starting in a point means sending the basis vector to itself, an arrow terminating in a point means sending the basis vector to 0:

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ e & \mapsto & e \\ 0 & \mapsfrom & e \end{array}$$

Lemma 1.2.7. *Let $n \in \mathbb{Z}$ and let M be a D_1-module. There is a k-linear isomorphism*

$$\mathrm{Hom}(D_1(n), M) \cong M^{-n},$$

i.e. $D_1(n)$ is a "free object of rank one in degree $-n$".

Proof. Clear. $\square$

The Hom-Sets of rigid modules and gauges can be computed explicitely: Morphisms of D_1-modules can be described as matrices. The matrices of the k-linear maps between the graded pieces are given by elimination of certain matrix-entries. Compability with φ gives conditions on the involved matrices. First the following easy statement:

Lemma 1.2.8. *Let τ and σ be types.*
(i) There is a k-linear isomorphism $\mathrm{Hom}_{D_1-Mod}(M(\tau), M(\sigma)) \cong M(d(\tau) \times d(\sigma), k)$.
(ii) Let $C = (c_{ij}) \in M(d(\tau) \times d(\sigma), k)$ and let φ_C denote the corresponding morphism. Let C^n denote the matrix of φ_C in degree n (w.r.t. the standard base). Then the j.-th column of C^n is given by

$$\begin{pmatrix} c_{1j} \\ \vdots \\ c_{d(\sigma,(-\infty,\tau_{t(j)}]),j} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ for } \tau_{t(j)} < n \quad \begin{pmatrix} c_{1j} \\ \vdots \\ \\ \vdots \\ c_{d(\sigma)j} \end{pmatrix} \text{ for } \tau_{t(j)} = n \quad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c_{d(\sigma,[\tau_{t(j)},\infty)),j} \\ \vdots \\ c_{d(\sigma)j} \end{pmatrix} \text{ for } \tau_{t(j)} > n$$

where we let $t(m) = i$ if $\sum_{j=1}^{i-1} \tau(\tau_j) < m \leq \sum_{j=1}^{i} \tau(\tau_j)$ (See the following remark).
(iii) Let $A \in M_{d(\tau)}(k)$ and $B \in M_{d(\sigma)}(k)$. Then

$$\mathrm{Hom}(M(\tau, A), M(\sigma, B)) = \{C \in M(d(\tau) \times d(\sigma), k) | ({}^{\sigma}C)^{-\infty} A = BC^{\infty}\}.$$

Proof. (i) This is clear because $M(\tau)$ is a direct sum of $D_1(a)$ for some integers a, and $D_1(a)$ is "free". We will give a direct construction:

Let $M = M(\tau)$, $N = M(\sigma)$. To a matrix C we assign a morphism as follows:
Consider $M^n = \ker(f_n) \oplus \ker(v_n) \oplus V_n$ like above and choose a basis for V_n which consists of standard basis vectors. We define the image of $e_n \in V_{\tau_{t(n)}}$ by the n.-th column of C. This completely determines a morphism $M \to N$ because the basis of each M^i consists of images of basis vectors of the V_n.
(ii) This is easy to see (the hardest part is writing it down) because the f's and v's kill coordinates just in the way described above.
(iii) clear. □

Remark 1.2.9. We use t to parametrize the basis of $M(\tau)^n$: The m.-th basis vector of $M(\tau)^n$ is the image of the m.-th standard basis vector in $M(\tau)^{\tau_{t(m)}}$. This means t has values

$$\underbrace{1,\ldots,1}_{\tau(\tau_1)-times},\ \underbrace{2,\ldots,2}_{\tau(\tau_2)-times},\ldots,\ \underbrace{h,\ldots,h}_{\tau(\tau_h)-times}$$

Example 1.2.10. Let $\tau(0) = 1, \tau(1) = 2, \tau(2) = 3, \tau(3) = 1$ and 0 elsewhere.
Let $\alpha : M(\tau) \to M(\tau)$ be given by $A \in M(7 \times 7, k)$. The matrix of α in second degree, i.e. of α^2, arises from A be eliminating the 0-marked entries:

$$X = \left(\begin{array}{c|cc|ccc|c} * & * & * & * & * & * & 0 \\ \hline 0 & * & * & * & * & * & 0 \\ 0 & * & * & * & * & * & 0 \\ \hline 0 & 0 & 0 & * & * & * & 0 \\ 0 & 0 & 0 & * & * & * & 0 \\ 0 & 0 & 0 & * & * & * & 0 \\ \hline 0 & 0 & 0 & * & * & * & * \end{array}\right).$$

The matrices of $\alpha^{-\infty}$ (resp. α^{∞}) arise from A be eliminating all entries in blocks right (resp. left) of the diagonal blocks.

Example 1.2.11. (i) Let τ, σ be one-dimensional types and $\epsilon, \lambda \in k^{\times}$. Then

$$\operatorname{Hom}(M(\tau,\epsilon), M(\sigma,\lambda)) = \begin{cases} 0 & \text{if } \tau_1 \neq \sigma_1 \\ \{\gamma \in k | \gamma^p = \gamma(\epsilon/\lambda)^p\} & \text{if } \tau_1 = \sigma_1 \end{cases}$$

because ${}^{\sigma}\gamma\epsilon = \lambda\gamma \Leftrightarrow \gamma^{\frac{1}{p}}\epsilon = \lambda\gamma \Leftrightarrow \gamma\epsilon^p = \lambda^p\gamma^p$.
(ii) In particular $\operatorname{End}(M(\tau,\epsilon)) = \{\gamma \in k | \gamma^p = \gamma\} = \mathbb{F}_p$ and $\operatorname{Aut}(M(\tau,\epsilon)) = \mathbb{F}_p^{\times}$.

Remark 1.2.12. (i) One can see easily that for a matrix $X \in M_n(k)$ and a type τ the determinant of X_i in $M(\tau)$ is given for all i (!) by

$$\prod_{n=1}^{h(\tau)} \det X(\tau, n)$$

where $X(\tau, n) = (X_{ij} | d(\tau, (-\infty, \tau_n - 1] < i, j \leq d(\tau, (-\infty, \tau_n]))$.
(N.B. This only depends on the partition of $d(\tau)$ induced by τ). So a matrix yields an invertible morphism if its "τ-determinant" is nonzero, its determinant can vanish very well:
(ii) Example: Let $\tau(0) = \tau(1) = 1$ and let τ be zero elsewhere. The matrix $X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ with $\det X = 0$ gives an invertible endomorphism

$$\alpha_X : M(\tau) \rightarrow M(\tau)$$

where α_X^n corresponds to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ for $n \leq 0$ resp. $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ for $n \geq 1$.
(iii) The composition of two endomorphisms is in general not given by the product of the corresponding matrices, but one has to multiply the matrices on every level (in fact multiplication in degree $-\infty$ and ∞ suffices). For example: The matrix corresponding to $\alpha_X \circ \alpha_X$ is $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

This defines different ring-structures on $M_n(k)$:

Definition 1.2.13. (i) Let τ be a type. Define a ring-structure on $M_{d(\tau)}(k)$ by structure transport along the isomorphism of abelian groups:

$$\mathrm{End}_{D_1-Mod}(M(\tau)) \xrightarrow{\cong} M_{d(\tau)}(k)$$

Denote this multiplication by $A \cdot^\tau B$.
(ii) To fix some notation: For $M \in M_{d(\tau)}(k)$ and $a \leq b \in \{1, \ldots, h(\tau)\}$ let $M^{a,b}$ be the $d \times d$ matrix $(M_{ij})_{l<i,j<g}$ for
$d = d(\tau, \{\tau_a, \ldots, \tau_b\})$ and $l = d(\tau, \{\tau_1, \ldots, \tau_{a-1}\})$ and $g = d(\tau, \{\tau_{b+1}, \ldots, \tau_{h(\tau)}\})$

Lemma 1.2.14. *Let τ be a type and let $A, B \in M_{d(\tau)}(k)$. Then*

$$(A \cdot^\tau B)_{ij} = (A^{a,b} B^{a,b})_{i-l, j-l}$$

for $l = d(\tau, \{\tau_1, \ldots, \tau_{a-1}\})$ and $a = \inf(t(i), t(j))$, $b = \sup(t(i), t(j))$.

Proof. Assume $i \leq j$, so that $a = t(i)$ and $b = t(j)$. Let $x = (A \cdot^\tau B)_{ij}$. According to Lemma 1.2.8 we get $x = (A^\infty B^\infty)_{ij}$. The i.-th line of A^∞ has l leading zeros, the j.-th column has $g = d(\tau, \{\tau_{b+1}, \ldots, \tau_{h(\tau)}\})$ zeros at its bottom. Thus multiplying this vectors gives the same as multiplying the $(i-l)$.-th line of $A^{a,b}$ with the $(j-l)$.-th column of $B^{a,b}$. □

Remark 1.2.15. (i) Let us make clear the previous statements. First we subdivide the matrices according to the partition associated to τ, for example:

$$X = \left(\begin{array}{c|cc|ccc|c} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \odot & \odot & \odot & \odot & \odot & \cdot \\ \cdot & \odot & \odot & \odot & \odot & \odot & * \\ \hline \cdot & \odot & \odot & \odot & \odot & \odot & \cdot \\ \cdot & \odot & \odot & \odot & \odot & \odot & \cdot \\ \cdot & \odot & \odot & \odot & \odot & \odot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right)$$

Here the part marked with $\odot$ is $X^{2,3}$.
For computing one specific entry of a multiplication $A \cdot^\tau B = M$ one chooses a and b like above. Then $X^{a,b}$ is the smallest quadratic submatrix containing the considered entry and containing complete blocks on the diagonal (For $X = A, B$ or M). For example multiply $A^{2,4}$ by $B^{2,4}$ to determine the entry marked with $*$.
(ii) These ringstructures obviously depend only on the partition associated to τ.

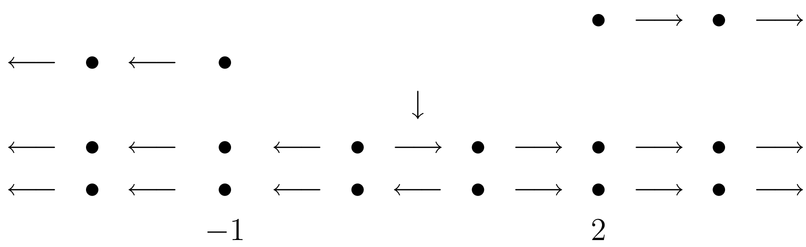
Remark 1.2.16. (i) The categories of D_n-modules, $D_n - \varphi$-modules and $D_n - \varphi$-gauges are abelian categories. This is almost clear, we only have to check that there are well-defined φ's on kernel and cokernel of a map. Let $\alpha : M \to N$ be a morphism of $D_n - \varphi$-modules and let $m \in (M^\infty)^{(p)}$. Then $(\alpha^\infty)^{(p)}(m) = 0 \Rightarrow \varphi \circ (\alpha^\infty)^{(p)}(m) = 0 \Leftrightarrow \alpha^{-\infty} \circ \varphi(m) = 0$, so that φ on M restricts to a well defined map $\varphi : (\ker \alpha^\infty)^{(p)} \to \ker \alpha^{-\infty}$. If M and N are gauges, the first implication $\Rightarrow$ is an equivalence and $\ker \alpha$ is a gauge. By a dual argument the cokernel of a map of $D_n - \varphi$-modules (resp. $D_n - \varphi$-gauges) is a $D_n - \varphi$-module (resp. $D_n - \varphi$-gauge).
(ii) Thereof it follows that a morphism of $D_n - \varphi$-gauges is a monomorphism (resp. epimorphism) if and only if it is set-theoretically injective (resp. surjective).

Example 1.2.17. (A 2-dimensional rigid $D_1 - \varphi -$gauge without nontrivial rigid subobjects)
Now let $\tau(0) = \tau(1) = 1$ and 0 else and let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let σ be a type of dimension 1 and $\epsilon \in k^\times$.
We assume a monomorphism: $\iota : M(\sigma, \epsilon) \to M(\tau, A)$ given by $\begin{pmatrix} x \\ y \end{pmatrix}$. Then $\mathrm{Supp}(\sigma) \subseteq \{0, 1\}$, else $\iota^\infty = 0$ or $\iota^{-\infty} = 0$. We can assume $\sigma(0) = 1$, the case $\sigma(1) = 1$ is treated analogously.
So we have $\iota^\infty = \begin{pmatrix} x \\ 0 \end{pmatrix}$ and $\iota^{-\infty} = \begin{pmatrix} x \\ y \end{pmatrix}$.
Now $A\iota^\infty = ({}^\sigma\iota^{-\infty})\epsilon \Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} {}^\sigma x\epsilon \\ {}^\sigma y\epsilon \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} {}^\sigma x\epsilon \\ {}^\sigma y\epsilon \end{pmatrix}$. This implies $x = y = 0$ which is a contradiction to injectivity.

The rigid $D_1 - \varphi$-gauge $M(\tau, A)$ has nontrivial non-rigid subobjects. Consider for example the $D_1 - \varphi$-gauge $M = D_1/(f)(1) \oplus D_1/(v)(-2)$ with φ defined by the matrix (1). There is a monomorphism

$$D_1/(f)(1) \oplus D_1/(v)(-2) \hookrightarrow M(\tau, A)$$

given by $D_1/(f) \ni 1 \mapsto e_2 \in M(\tau, A)^{-1}$ and $D_1/(v) \ni 1 \mapsto e_1 \in M(\tau, A)^2$. The picture is the following:



1.3 Connections with F-zips

We want to establish an equivalence of categories between the category of rigid $D_1 - \varphi$-gauges and the category of F-zips over a field k, introduced by Moonen and Wedhorn in their paper [MW](See appendix).

Lemma 1.3.1. *There is a functor (due to Fontaine)*

$$F : (\textit{rigid } D_1 - \varphi - \textit{gauges }) \longrightarrow (F - \textit{zips over } k)$$

defined as follows:

For a rigid $D_1 - \varphi-$gauge $M = (M^r, \varphi)$ let $F(M)$ be the $F-$zip $(F(M), C^\bullet, D_\bullet, \overline{\varphi_r})$ with

- $F(M) = M^{-\infty}$
- $C^r = \mathrm{im}(v_r^\infty)$
- $D_r = \mathrm{im}(\varphi \circ f_r^\infty)$ *and*
- $\overline{\varphi_r}$ *given by the diagram with exact lines*

$$\begin{array}{ccccccc} M^{r-1} \oplus M^{r+1} & \xrightarrow{(f,v)} & M^r & \xrightarrow{pr v_r^\infty} & \mathrm{gr}_C^r F(M) & \longrightarrow & 0 \\ \downarrow id & & \downarrow id & & \downarrow \overline{\varphi_r} & & \\ M^{r-1} \oplus M^{r+1} & \xrightarrow{(f,v)} & M^r & \xrightarrow{pr \varphi f_r^\infty} & \mathrm{gr}_r^D({}^\sigma F(M)) & \longrightarrow & 0 \end{array}$$

Explicitly $\overline{\varphi_r}$ is given as follows: For $x \in gr^r M$ we choose a preimage $\tilde{x}$ under v_r^∞ which is well defined modulo $\ker v_r^\infty$. Then $\overline{\varphi_r}(x)$ equals $pr \circ \varphi \circ f_r^\infty(\tilde{x})$ which is well defined because $\ker v_r^\infty = \ker v_r = \mathrm{im} f_{r-1}$ maps to 0 under $pr \circ \varphi \circ f_r^\infty$.

For a morphism g of $D_1 - \varphi-$gauges we let $Fg = g^{-\infty}$.

Proof. The lines form complexes by definition and since $fv = vf = p = 0$. By rigidity we know that $\ker v_r^\infty = \ker v_r$ ($\Leftrightarrow v|_{\mathrm{im}v}$ is a monomorphism). So for $m \in M^r$ mapping to 0 there exists m' with

$$v_r^\infty(m) = v_{r+1}^\infty(m')$$

which implies

$$v_{r+1}(m') - m \in \ker v_r^\infty = \ker v_r = \mathrm{im} f_{r-1}$$

and $m \in \mathrm{im} v_{r+1} + \mathrm{im} f_{r-1}$. Exactness of the second line is shown analogously.

Obviously $\overline{\varphi}_r$ is an isomorphism. So $F(M)$ is indeed an F-zip.

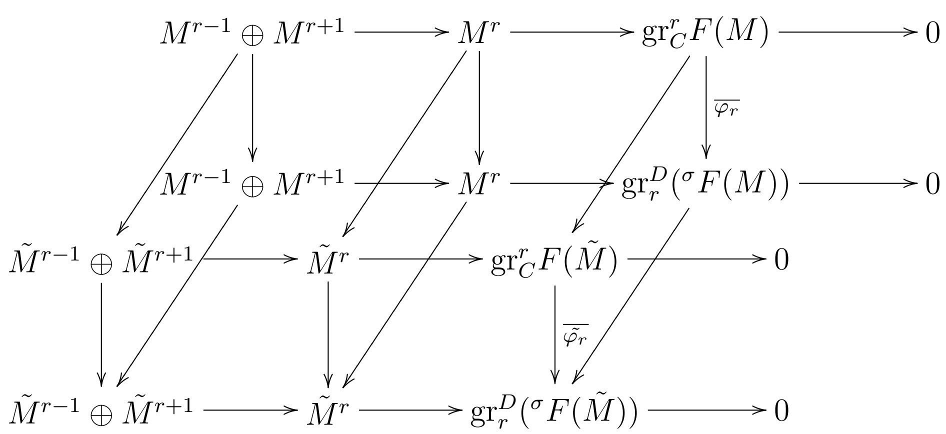
Finally we have to show that a morphism of gauges induces indeed a morphism of F-zips: Let g be a morphism $M \to \tilde{M}$ (with $M = (M, f, v, \varphi)$ and $\tilde{M} = (\tilde{M}, \tilde{f}, \tilde{v}, \tilde{\varphi})$) then

$$\begin{aligned} Fg(Fil^r F(M)) &= g^{-\infty} v_r^\infty(M^r) \\ &= \tilde{v}_r^\infty g^r(M^r) \\ &\subseteq \tilde{v}_r^\infty(\tilde{M}^r) \\ &= Fil^r F(\tilde{M}) \end{aligned}$$

and analogously

$$\begin{aligned} ({}^\sigma Fg)(Fil_r^\sigma F(M)) &= ({}^\sigma g^{-\infty})\varphi f_r^\infty(M^r) \\ &= \tilde{\varphi} g^\infty f_r^\infty(M^r) \\ &= \tilde{\varphi}\tilde{f}_r^\infty g^r(M^r) \\ &\subseteq \tilde{\varphi}\tilde{f}_r^\infty(\tilde{M}^r) \\ &= Fil_r^\sigma F(\tilde{M}). \end{aligned}$$

Thus Fg respects both filtrations. Furthermore we have the commutative diagram with exact lines:



which shows us that Fg is compatible with $\overline{\varphi}_r$ and $\overline{\tilde{\varphi}}_r$. □

Remark 1.3.2. The type of a rigid $D_1-\varphi$-gauge M coincides with the type of the associated F-zip $F(M)$:
Recall $\mathrm{Fil}^r F(M) =$ image of v_r^∞, so $\dim \mathrm{Fil}^r F(M) = \mathrm{rk} v_r^\infty = \mathrm{rk} v_r$. The type of $F(M)$ in the sense of F-zips takes value $\dim \mathrm{Fil}^r F(M) - \dim \mathrm{Fil}^{r+1} F(M)$ at r (see appendix), and the type of M in the sense of gauges takes value $\mathrm{rk} v_r - \mathrm{rk} v_{r+1}$.

Example 1.3.3. Let $\tau(0) = \tau(1) = 1$ and zero elsewhere. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $M = M(\tau, A)$. Consider $(k^2, C^\bullet, D_\bullet, \overline{\varphi}) = FM$. We have

$$C^0 = k^2, C^1 =< e_2 >, C^2 = 0$$

and

$$({}^\sigma D_{-1}) = 0, ({}^\sigma D_0) =< Ae_1 >=< e_2 >, ({}^\sigma D_1) = k^2.$$

It is easy to see that the Hom-sets are the same:

$$\begin{aligned}
\mathrm{End}(M(\tau, A)) &= \left\{ X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k) \middle| ({}^\sigma X^{-\infty})A = AX^\infty \right\} \\
&= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k) \middle| \begin{pmatrix} ({}^\sigma a) & 0 \\ ({}^\sigma c) & ({}^\sigma d) \end{pmatrix} A = A \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k) \middle| \begin{pmatrix} 0 & ({}^\sigma a) \\ ({}^\sigma d) & ({}^\sigma c) \end{pmatrix} = \begin{pmatrix} 0 & d \\ a & b \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k) \middle| ({}^\sigma a) = d, ({}^\sigma d) = a, ({}^\sigma c) = b \right\}
\end{aligned}$$

The endomorphisms of the image of M are matrices which have to respect the filtrations (here: the subspace spanned by the second standard basis vectors) and which have to be compatible with the isomorphism of the associated graded spaces. These conditions give the following description:
$\mathrm{End}(FM(\tau, A)) = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in M_2(k) \middle| ({}^\sigma a) = d, ({}^\sigma d) = a \right\}.$
The functor $F : \mathrm{End}(M(\tau, A)) \to \mathrm{End}(FM(\tau, A))$ maps $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$.

We will now construct the quasi-inverse functor.

Proposition 1.3.4. *There is a functor*

$$G : (F - \textit{zips over } k) \longrightarrow (\textit{rigid } D_1 - \varphi - \textit{gauges })$$

Explicitely G is defined by the following construction:

Given a F-zip $M = (M, C^\bullet, D_\bullet, \overline{\varphi}_\bullet)$ over k let:

$$GM^r = \ker({}^\sigma D_r \oplus C^r \twoheadrightarrow \mathrm{gr}_r^D({}^\sigma M) | (x, y) \mapsto x - \overline{\varphi}_r(y))$$

$$f_r = ((^\sigma D_r) \oplus C^r \xrightarrow{(\iota,0)} (^\sigma D_{r+1}) \oplus C^{r+1}) \text{ restricted to } GM^r$$

$$v_r = ((^\sigma D_r) \oplus C^r \xrightarrow{(0,\iota)} (^\sigma D_{r-1}) \oplus C^{r-1}) \text{ restricted to } GM^r$$

$$\varphi : GM^\infty = (^\sigma D_\infty) = (^\sigma M) \xrightarrow{id} (^\sigma M) = (^\sigma C^{-\infty}) = (^\sigma GM^{-\infty})$$

where ι stands for inclusions, and $C_\infty = C_s$ for $s \gg 0$, $D^{-\infty} = D^s$ for $s \ll 0$. If $\alpha : M \to \tilde{M}$ is a morphism of F-zips we define $G\alpha$ in degree r by the commutative diagram with exact lines:

$$\begin{array}{ccccccc}
GM^r & \longrightarrow & (^\sigma D_r) \oplus C^r & \longrightarrow & \mathrm{gr}_r^D(^\sigma M) & \longrightarrow & 0 \\
\downarrow{\scriptstyle G\alpha^r} & & \downarrow{\scriptstyle (^\sigma\alpha,\alpha)} & & \downarrow & & \\
G\tilde{M}^r & \longrightarrow & (^\sigma \tilde{D}_r) \oplus \tilde{C}^r & \longrightarrow & \mathrm{gr}_r^{\tilde{D}}(^\sigma \tilde{M}) & \longrightarrow & 0
\end{array}$$

Proof. First we show that GM is a gauge:

- The f_r are well defined: The image of $(\iota x, 0)$ in $gr^\sigma_{r+1}M$ is $x - 0 = x$ which is 0 for $x \in (^\sigma C_r)$. The same argument shows that v_r is well defined.

- GM is a strict gauge:

  $$(f_r, v_r) : GM^r \to GM^{r+1} \oplus GM^{r-1}$$

  is restriction of

  $$(^\sigma D_r) \oplus C^r \hookrightarrow (^\sigma D_{r+1}) \oplus C^{r-1} \hookrightarrow (^\sigma D_{r+1}) \oplus C^{r+1} \oplus (^\sigma D_{r-1}) \oplus C^{r-1}$$

  with

  $$(c, d) \mapsto (c, d) \text{ and } (c, d) \mapsto (c, 0, 0, d)$$

  which is injective, so (f_r, v_r) is injective itself.

- For rigidity consider for example:

  $$(^\sigma D_{r+1}) \oplus C^{r+1} \overset{v_{r+1}}{\to} (^\sigma D_r) \oplus C^r \overset{f_r}{\to} (^\sigma D_{r+1}) \oplus C^{r+1}$$

  Let $m = (x, y) \in GM^r$ with $f_r(m) = 0$, which means $m = (0, y)$ for some y. Since $m \in GM^r$ it follows $\overline{\varphi}_r(y) = 0$. This is equivalent to $y = 0 \in \mathrm{gr}_r^D M \Leftrightarrow y \in D^{r+1}$, which is equivalent to $m \in \mathrm{im} v$, by surjectivity of $(^\sigma D_{r+1}) \to \mathrm{gr}_{r+1}^D(^\sigma M)$.

For functoriality we note that the diagram in the statement is obviously commutative. Compability of $G\alpha$ with f and v amounts to stating that kernels are functorial (in the proof of 1.3.1 we used a dual argument). Compability of $G\alpha$ with φ is immediate. □

Remark 1.3.5. Let M be an F-zip like in the proposition. The W_1-modules $C = \bigoplus_{r\in\mathbb{Z}} C^r$ and $D = \bigoplus_{r\in\mathbb{Z}} D_r$ are modules over $W_1[f,v]/(fv)$ if we let f-multiplication be induced by the inclusions and v-multiplication be 0 on D (respectively v-multiplication induced by the inclusions and f-multiplication 0 on CD). We view the W_1-module $\mathrm{gr}^D({}^\sigma M)$ as $W_1[f,v]/(fv)$-module with f and v operating as 0.

There are canonical morphisms of graded $W_1[f,v]/(fv)$-modules

$$({}^\sigma D) \to \mathrm{gr}^D({}^\sigma M) \text{ with } x \mapsto prx$$

$$C \to \mathrm{gr}^D({}^\sigma M) \text{ with } x \mapsto \overline{\varphi}_r \circ prx \text{ in degree } r.$$

With these morphisms we can describe $G(M)$ as a fibered product

$$G(M) = ({}^\sigma D) \times_{\mathrm{gr}^D({}^\sigma M)} C.$$

The main theorem of this section is the following:

Theorem 1.3.6. *Let k be an arbitrary perfect field of caracteristic p. The categories of rigid $D_1 - \varphi -$modules and F-zips over k are equivalent. The equivalence is given by the functors F and G which are quasiinverse to each other.*

Proof. (i) We show that $FG \cong id$. Let $M = (M, C_\bullet, D^\bullet, \overline{\varphi}_\bullet)$ be a F-zip and $N = (N, \tilde{C}_\bullet, \tilde{D}^\bullet, \overline{\psi}_\bullet) = FGM$.
1) It is $N = (GM)^{-\infty} = D_{-\infty} = M$.
2) Use the snake lemma on

$$\begin{array}{ccccccccc}
0 & \longrightarrow & GM^r & \longrightarrow & ({}^\sigma D_r) \oplus C^r & \longrightarrow & \mathrm{gr}^D_r({}^\sigma M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle v_r^\infty} & & \downarrow{\scriptstyle (0,\iota)} & & \downarrow & & \\
0 & \longrightarrow & D_{-\infty} & \longrightarrow & D_{-\infty} & \longrightarrow & 0 & &
\end{array}$$

and get the exact sequence

$$({}^\sigma D_r) \to \mathrm{gr}^D_r({}^\sigma M) \to GM^{-\infty}/\mathrm{im} v_r^\infty \to GM^{-\infty}/C^r \to 0.$$

The first map is surjective, so the second one is 0 and we see that $\tilde{C}^r = \mathrm{im} v_r^\infty$ is equal to C^r.
3) Analogously we see $\tilde{D}_r \cong D_r$.
4) For checking that $\overline{\psi}_r = \overline{\varphi}_r$, first note that a general element of GM^r is of form $(\overline{\varphi}_r(y), y)$ (with $\overline{\varphi}_r(y)$ actually a lift of $\overline{\varphi}_r(y)$ under ${}^\sigma D_r \to \mathrm{gr}^D_r({}^\sigma M)$). Consider the following

diagram

$$
\begin{array}{ccccccc}
\overline{\varphi}_r(y) & & ({}^\sigma D_\infty) & \longrightarrow & ({}^\sigma D_\infty) & \longrightarrow & 0 \\
\uparrow & & \uparrow{\scriptstyle f_r^\infty} & & \uparrow{\scriptstyle (\iota,0)} & & \uparrow \\
(\overline{\varphi}_r(y), y) & & GM^r & \longrightarrow & ({}^\sigma D_r) \oplus C^r & \longrightarrow & \mathrm{gr}_r^D({}^\sigma M) \\
\downarrow & & \downarrow{\scriptstyle v_r^\infty} & & \downarrow{\scriptstyle (0,\iota)} & & \downarrow \\
y & & D^{-\infty} & \longrightarrow & C^{-\infty} & \longrightarrow & 0
\end{array}
$$

Recall that $\overline{\psi}_r$ is defined by applying φf_r^∞ to a preimage under v_r^∞. Note that in the image of G we have always $\varphi = id$, so

$$\overline{\psi}_r(y) = \overline{\varphi}_r(y).$$

(ii) We have to check that $GF \cong id$. Let $M = (M, f, v, \varphi)$ be a rigid $D_1 - \varphi$-gauge and let $FM = (M^{-\infty}, C^\bullet, D_\bullet, \overline{\varphi}_\bullet)$.
1) The map $(\varphi f_r^\infty, v_r^\infty) : M^r \to GFM^r$ is a canonical isomorphism, since the construction of $\overline{\varphi}_r$ implies that

$$0 \to M^r \overset{(\varphi f_r^\infty, v_r^\infty)}{\longrightarrow} ({}^\sigma D_r) \oplus C^r \overset{(x,y) \mapsto x - \overline{\varphi}_r(y)}{\longrightarrow} gr_r^D({}^\sigma M^{-\infty}) \to 0$$

is exact:

Composition is 0 by definition of $\overline{\varphi}_r$.
Injectivity follows by strictness.
Let $(x, y) \in {}^\sigma D_r \oplus C^r$ with $x - \overline{\varphi}_r(y) = 0$. Choose a preimage $\tilde{y}$ with $v_r^\infty(\tilde{y}) = y$. We get

$$
\begin{aligned}
x - \overline{\varphi}_r(y) = 0 \quad & \Rightarrow \quad x = \overline{\varphi}_r(v_r^\infty(\tilde{y})) \\
& \Rightarrow \quad x = \varphi f_r^\infty(\tilde{y}) \\
& \Rightarrow \quad (x, y) = (\varphi f_r^\infty, v_r^\infty)(\tilde{y})
\end{aligned}
$$

2) Consider the commutative diagram

$$
\begin{array}{ccc}
M^r & \overset{f}{\longrightarrow} & M^{r+1} \\
{\scriptstyle (\varphi f_r^\infty, v_r^\infty)} \downarrow & & \downarrow {\scriptstyle (\varphi f_{r+1}^\infty, v_{r+1})} \\
GFM^r & \overset{(\iota,0)}{\longrightarrow} & GFM^{r+1}
\end{array}
$$

which guarantees the compatibility with f.
3)Analogously we show compatibility with v.

4) Compatibility with φ is essentially trivial: After passage to the limit we have a commutative diagram

$$\begin{array}{ccc} M^{\infty} & \xrightarrow{\varphi} & ({}^{\sigma}M^{-\infty}) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle id} \\ GFM^{\infty} & \xrightarrow{id=\varphi_{GFM}} & ({}^{\sigma}GFM^{-\infty}) \end{array}$$

□

Remark 1.3.7. Moonen and Wedhorn mentioned that F-zips do not form an abelian category thus rigid $D_1 - \varphi$-modules do neither by the theorem. Let us make clear one point where this fails:
Let $\tau(0) = 2$ and $\tilde{\tau}(-1) = \tilde{\tau}(1) = 1$ and zero elsewhere. Let A be the matrix permuting two basis vectors. Consider the kernel of

$$\alpha : M(\tau, A) \to M(\tilde{\tau}, A)$$

defined at $M(\tau, A)^0 \to M(\tilde{\tau}, A)^0$ by the identity matrix:

$$\begin{array}{ccccccc} & & \longleftarrow & \bullet & \longrightarrow & & \\ & & \longleftarrow & \bullet & \longrightarrow & & \\ & & & \downarrow & & & \\ \longleftarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow \\ \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longrightarrow \end{array}$$

This is indeed a morphism of gauges because

$$\begin{aligned} ({}^{\sigma}\alpha^{-\infty})A &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} A\alpha^{\infty} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Obviously $\dim\ker\alpha^0 = 2$ and $\dim\ker\alpha^{-1} = \ker\alpha^1 = 1$ which shows us that $\ker\alpha$ cannot be rigid.

2 Quasi-étale morphisms

We define quasi-étale (or quiet) scheme-morphisms introduced by Fontaine and Jannsen and show stability under composition and base-change. Furthermore we introduce a lifting property which will be important to study quasi-coherent quiet crystals.
Let p be a fixed prime number. All consideres schemes are over $\mathbb{F}_p$.

2.1 Definition and Properties

Definition 2.1.1. (i) Let $f : Y \to X$ be a morphism of schemes. We say that f is *quasi-étale* (or *quiet*), if locally f can be written as a composition of an étale morphism after a p-th root.
(ii) A morphism of schemes is a *p-th root* (or an *extraction of a p-th root* or a *p-morphism*) if it is locally a composition of morphisms of the form $\mathrm{Spec}(A[x]/(x^p - \alpha)) \to \mathrm{Spec}(A)$ for $\alpha \in A$.

Here are some simple properties of p-th roots:

Remark 2.1.2. (i) Because p-th roots are locally of finite type and flat, hence open morphisms and because open immersions are étale, one can choose the p-th root in the definition of a quasi-étale morphism to be surjective.
(ii) The fibers of a p-th root f over a point $x = \mathrm{Spec}(k)$ are locally of the form $\mathrm{Spec}(R)$ where $R \to K$ is a nilpotent thickening of the residue field extension of a point over x, which is a purely inseparable field extension of k. Especially f is radicial.
One can see this immediately by induction: Assume $A \to B$ a p-th root with such a fiber R and $B \to B[x]/(x^p - \beta)$ a morphism. Let $\mathrm{Spec}(\tilde{K})$ be the point over $\mathrm{Spec}(K)$. Consider the diagram

$$\begin{array}{ccccc} R[x]/(x^p - \beta) & \longrightarrow & K[x]/(x^p - \overline{\beta}) & \xrightarrow{\pi} & \tilde{K} \\ \uparrow & & \uparrow & & \\ R & \longrightarrow & K & & \end{array}$$

where the square is cocartesian and π is surjective (resp. an isomorphism) if in K a p-th root of $\overline{\beta}$ exists (resp. does not exist). Then $R[x]/(x^p - \beta) \to \tilde{K}$ has the required form.
(iii) Obviously a p-th root is integral.

Lemma 2.1.3. *The property "quasi-étale" is stable under composition and base change.*

Proof. The base change of an étale morphism is étale, so we have to show that the base change of a p-th root is again a p-th root. This is evident.
For stability under composition consider $f \circ g'$ where $f : Y \to X$ is a faithfully flat p-th root and $g' : Z \to Y$ is étale. We know that f is integral, surjective and radicial and thus

with [SGA4.2] VIII 1.1 one can find a cartesian square

$$\begin{array}{ccc} Z & \xrightarrow{f'} & E \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X \end{array}$$

where g is étale . Then f' is a p-th root. Because p-th roots resp. étale morphisms are stable under composition, this implies the lemma. □

Lemma 2.1.4. *Quasi-étale morphisms are syntomic.*

Proof. This is clear (see Appendix A.1). □

2.2 Lifting property

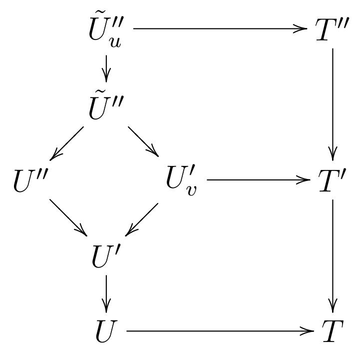
The following lifting property is the main point which allows us to use quiet morphisms for our purposes. It enables us to refine a quiet covering to a covering which can be lifted infinitesimally, to be more precise: Given an infinitesimal thickening, any quiet covering of the source can be refined to a covering which can be lifted along the thickening.
Thus the topology τ generated by classes of morphisms satisfying this lifting property (in fact a weaker form suffices) has good properties on infinitesimal sites. Especially it gives a connection from the infinitesimal τ-site to the large τ-site. If the classes are additionally assumed to be flat, they actually behave well with respect to the crystalline site.

Proposition 2.2.1. *(Quasi-étale lifting property) Let $f' : U' \to U$ be quasietale and $U \to T$ be a (closed) nilimmersion. Then every point u of U' possesses a neighborhood U'_u (for Zariski-topology) such that we ca find a cartesian square with a quasi-étale T-scheme T':*

$$\begin{array}{ccc} U'_u & \longrightarrow & T' \\ \downarrow & & \\ U' & & \downarrow \\ \downarrow & & \\ U & \longrightarrow & T. \end{array}$$

Proof. It is enough to treat the cases that f' is either étale or a p-th root:

Assume $f : U' \overset{f_2}{\to} U'' \overset{f_1}{\to} U$ and the lifting property is satisfied by f_1 and f_2. Let $u \in U'$ and $v = f_2(u) \in U''$. The following diagram implies that the lifting property holds also for f:



The three squares are cartesian. First T' and U'_v are defined by the lifting property of f_1, then $\tilde{U}''$ is defined as fibered product and $\tilde{U}_{u''}, T''$ are defined by lifting property of f_2.

If f' is étale this is a weaker statement than topological invariance of étale morphisms [EGA4.4] 18.1.2.
So let f' be a p-th root. Because the problem is local in T we can assume that $T = \mathrm{Spec}(A)$ is affine, so $U = \mathrm{Spec}(B)$ with $A \twoheadrightarrow B$. Furthermore we can, by the argument from above, reduce to the case that $U' = \mathrm{Spec}(B[t]/(t^p - \beta))$ for a $\beta \in B$, but then our problem is solved by $\mathrm{Spec}(A[t]/(t^p - \alpha))$ for a preimage α of β. □

Remark 2.2.2. The lifing property holds for other classes, too: It is also true for the class of syntomic morphisms (see [Be2] Lemme 1.1.9), for étale morphisms and for the class of extractions of p-th roots.

Remark 2.2.3. Syntomic morphisms were used by Fontaine and Messing in their paper [FM] to give means of calculating crystalline cohomology. They identified crystalline cohomology of a smooth scheme over a field k with syntomic cohomology of some special sheaf $\mathcal{O}_n^{\mathrm{cris}}$ on the syntomic site over k:

$$H^n_{\mathrm{cris}}(X/W_n, \mathcal{O}_{X/W_n}) \cong H^n_{syn}(X, \mathcal{O}_n^{\mathrm{cris}})$$

But the class of syntomic morphisms is very large, so the natural question arose whether one could replace the class "syntomic" with some smaller classes. One possible solution are quasi-étale morphisms but the even smaller class of p-th roots is also a possible substitute.

3 Cohomology of quasi-coherent crystals

3.1 Topologies

We work consistently over large sites and topoi. Here we recall the basic definitions of the large crystalline site with different topologies as introduced in [Be2].

Definition 3.1.1. (i) Let (S, I, γ) be a PD-scheme and X an S-scheme to which γ extends. Let $CRIS(X/S)$ or $CRIS(X/S, I, \gamma)$ denote the category with objects

$$(i : U \hookrightarrow T, \delta) = (U, T, \delta)$$

where U is an X-scheme, i a closed immersion into an S-scheme T and δ is a PD-structure on the ideal $J \subseteq \mathcal{O}_T$ of i, which is compatible with γ. Such an i is called a *PD-thickening of U*.
A morphism in $CRIS(X/S)$ is a commutative diagram

$$\begin{array}{ccc} T' & \xrightarrow{v} & T \\ \uparrow & & \uparrow \\ U' & \xrightarrow{u} & U \end{array}$$

where v is a PD-morphism. A morphism is called *cartesian* if its diagram is cartesian.
(ii) We say that a morphism (u, v) has a particular property (e.g. being an open immersion, syntomic etc.) if $v : T' \to T$ has the corresponding property.

We use the procedure of constructing topologies in [SGA3.1] exp. IV 6.2. In order to use the good properties of topologies built in this way, e.g. a criterion for presheaves to be sheaves ([SGA3.1] exp.IV 6.2.3), we have to check some technical conditions, but these are almost all trivial.

Definition 3.1.2. Let τ be a class of flat, locally of finite type scheme morphisms which is stable under composition and base-change. Assume that τ contains all open immersions.
(i) Let $\tau(X)$ be the big τ-site, and X_τ the corresponding topos.
(ii) Let $CRIS(X/S)_\tau$ be the category $CRIS(X/S)$ endowed with the topology generated by surjective families of open immersions and surjective τ-morphisms of affine objects. Let $(X/S)_{CRIS,\tau}$ be the corresponding topos.
(iii) Let $\mathcal{O}_{X/S}$ be the presheaf of rings

$$(U, T) \mapsto \mathcal{O}_T$$

which in fact is a sheaf (by flatness of τ). Is is called the *structural sheaf* of $(X/S)_{CRIS,\tau}$. We use $\mathcal{O}_{X/S}$ to give $(X/S)_{CRIS,\tau}$ the structure of a ringed topos.

Our aim is to construct a morphism of topoi

$$v_\tau : (X/S)_{CRIS,\tau} \to X_\tau$$

for different classes of morphisms τ.
In their paper [FM] Fontaine and Messing implicitely use this morphism; the existence of v_τ, as we will construct it, implies that the presheaf $\mathcal{O}_n^{\text{cris}}$ is a sheaf for syntomic topology. We will give axioms for classes τ that guarantee the existence of v_τ. The key point is the property that that coverings should admit a refinement which can be lifted to a crystalline covering. This is closely related to the lifting property of the previous section, in fact it follows from the lifting property.

Definition 3.1.3. Let τ be a class of morphisms of schemes. Consider the following properties:
(T1) (Stability, flatness and open immersions) The class τ is stable under base change and composition and every τ-morphism is flat and locally of finite presentation. Furthermore every open immersion is τ.
(T2) (Refinement) The class τ admits τ-crystalline refinements (see next definition).
(T3) (p-morphisms) Every p-th root morphism is τ.
If τ satisfies (T1),(T2) and (T3) we call τ a *p-crystalline* class.

Definition 3.1.4. Let τ be a class of scheme-morphisms. We say that τ *admits crystalline refinements* if for every PD-thickening $U \to T$ and every surjective family of τ-morphisms $\{U_i \to U\}$ there exists a refinement $\{V_j \to U\}$ with $\epsilon : J \to I$ and PD-thickenings $V_j \to T_j$ with T-schemes T_j such that the diagrams

$$\begin{array}{ccc} V_j & \longrightarrow & T_j \\ \downarrow & & \\ U_{\epsilon(j)} & & \downarrow f_j \\ \downarrow & & \\ U & \longrightarrow & T \end{array}$$

are cartesian with PD-morphisms $f_j \in \tau$.

We get the already announced relation to the lifting property by the following lemma.

Lemma 3.1.5. *Consider*
(T2') (Lifting property) Let $f' : U' \to U$ be a τ-morphism and $U \to T$ a closed nilimmersion. Then every point u of U' possesses a Zariski-open neighborhood U'_u s.t. there exists a cartesian square

$$\begin{array}{ccc} U'_u & \longrightarrow & T' \\ \downarrow & & \\ U' & & \downarrow \\ \downarrow & & \\ U & \longrightarrow & T \end{array}$$

with $T' \to T$ a τ-morphism. Then ((T2') and (T1)) $\Rightarrow$ (T2).

Proof. Note that divided powers extend along flat morphisms ([Be] I 2.7.4). □

Remark 3.1.6. The lemma shows us that the following classes of morphisms satisfy (T2), i.e. admit crystalline refinements:

- syntomic
- quasi-étale
- p.-th roots
- étale
- Zariski-morphisms

since they satisfy the lifting property (see 2.2.1). Zariski- and étale coverings can be extendend to crystalline coverings even without refining because of their topological invariance and flatness (We call morphisms *Zariski-morphisms* if they are locally open immersions).

Remark 3.1.7. The axioms play different roles:
(i) Condition (T1) implies that toplogies on X-schemes resp. on $CRIS(X/S)$ are of a specific form described in [SGA3.1] IV Proposition 6.2.1 (p.239):
In notation of loc. cit. we let $C = CRIS(X/S)$ (resp. the category of X-schemes), C' the full subcategory of affine objects of $CRIS(X/S)$ (resp. affine X-schemes), P the set of open coverings and P' the set of finite τ-coverings. The axioms (P1), (P2), (P3), (a) and (b) of loc. cit. are immediately verified using Lemma 3.1.8 below, for (c) we notice that τ-morphisms are open and affine schemes are quasi-compact.
(ii) Assuming (T1) and (T2) guarantees that there is a morphism from the τ-crystalline site into the big τ-site.
(iii) We shall see that assuming all three axioms (T1)-(T3) for a class τ implies that higher direct images under v_τ of a quasi-coherent crystal M on $(X/S)_{CRIS,\tau}$ vanish:

$$R^q(v_\tau)_*M = 0 \text{ for } q > 0.$$

Lemma 3.1.8. *Let τ be a class with (T1). Let $(U', T') \to (U, T)$ be a τ-morphism and $(U_1, T_1) \to (U, T)$ be a morphism. Then the fibered product $(U', T') \times_{(U,T)} (U_1, T_1)$ in $CRIS(X/S)_\tau$ exists and is canonically isomorphic to $(U' \times_U U_1, T' \times_T T_1)$.*

Proof. See [Be2] 1.1.2. The proof works in our situation because τ-morphisms are flat. □

Remark 3.1.9. (i) We need the following simple technical statement: If τ is a class which admits τ-crystalline refinements then obviously every surjective family which can be refined by a τ-family, admits a τ-crystalline refinement. Especially every surjective family of *source-locally* τ-morphisms (i.e morphisms $f : X \to Y$ such that there exists an open covering (U_i) of X with τ-morphisms $f|_{U_i}$) admits a τ-crystalline refinenment.
(ii) For all cases which we consider here, a morphism is a τ-morphism if and only if it is source-locally a τ-morphism. This is obvious since the defining properties for τ are always local properties.

3.2 A morphism of topoi

This subsection contains the technical main part of this section.
Concerning sites and topoi we use notations and conventions of [SGA4.1], but we write $P(\cdot)$ for the category of presheaves (in [SGA4.1] denoted by $\hat{\cdot}$) and $S(\cdot)$ for the category of sheaves (in [SGA4.1] denoted by $\tilde{\cdot}$) on some site.
Recall that a morphism of topoi is a pair of adjoint functors, "direct image" and "pull-back", with exact "pull-back"-functor.

Theorem 3.2.1. *Let τ be a class which is flat, locally of finite presentation and stable under composition and base-extension, i.e. τ satisfies (T1). Assume that τ admits crystalline refinements ((T2)). Then there is morphism of topoi*

$$v_\tau : (X/S)_{CRIS,\tau} \to X_\tau$$

We denote v_τ also by v if the topology is fixed.

Proof. The proof is based on [K].
By [SGA4.1] IV Proposition 4.9.4 a continuous([SGA4.1] III.1) and left exact functor from the site $\tau(X)$ to the topos $(X/S)_{CRIS,\tau}$ gives a morphism of topoi

$$(X/S)_{CRIS,\tau} \to X_\tau.$$

Let

$$V : \tau(X) \to (X/S)_{CRIS,\tau}$$

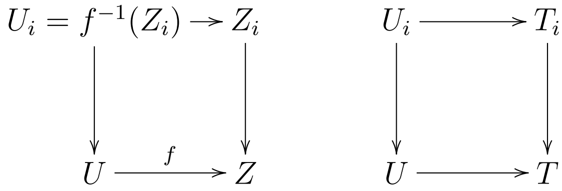
be defined by $Z \mapsto ((U,T) \mapsto \mathrm{Hom}_X(U,Z))$. Then V is a well-defined, continuous and left-exact functor.
(i) <u>V is well-defined</u>:
This is Lemma 3.2.2.
(ii) <u>V is continuous:</u>
Consider the functor $W : P((X/S)_{CRIS,\tau}) \to P(\tau(X))$ given by $F \mapsto F \circ V$. We have to show that W maps sheaves into sheaves, i.e.

$$W : S((X/S)_{CRIS,\tau}) \cong (X/S)_{CRIS,\tau} \to S(X_\tau) = X_\tau,$$

with P denoting presheaves and S denoting sheaves. Note that the category of sheaves on a topos is equivalent to this topos by representing sheaves (See [SGA4.1] IV Corollaire 1.2.1). Let F be a τ-crystalline sheaf.
If Z is an X-scheme, then $WF(Z) = F(V(Z)) = \mathrm{Hom}_{(X/S)_{CRIS,\tau}}(VZ, F)$ (The value of the sheaf F at the sheaf VZ). By [SGA3.1] IV 6.2.3 we have to check sheaf-condition for certain types of coverings:
<u>Part 1:</u> Let $\{Z_i \to Z\}$ be a Zariski-covering of X-schemes and let

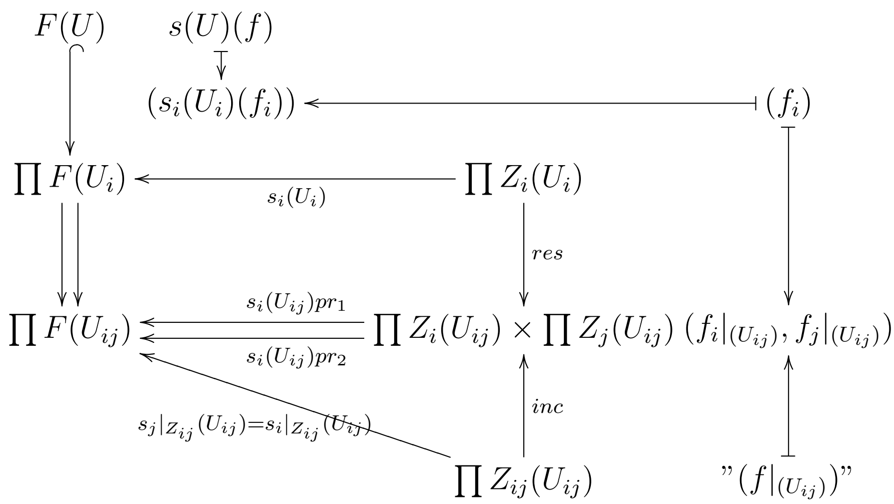
$$(s_i) \in \ker(\prod \mathrm{Hom}(VZ_i, F) \rightrightarrows \prod \mathrm{Hom}(VZ_{ij}, F))$$

with $Z_{ij} = Z_i \times_Z Z_j = Z_i \cap Z_j$. We have to show that there is exactly one $s \in \mathrm{Hom}(VZ, F)$ mapping to (s_i). Let $U = (U, T)$ be in $CRIS(X/S)_\tau$ and $f \in VZ(U) = \mathrm{Hom}_X(U, Z)$. Let $U_i = f^{-1}(Z_i)$ and $U_i = (U_i, T_i)$ defined by requiring the following squares to be cartesian:

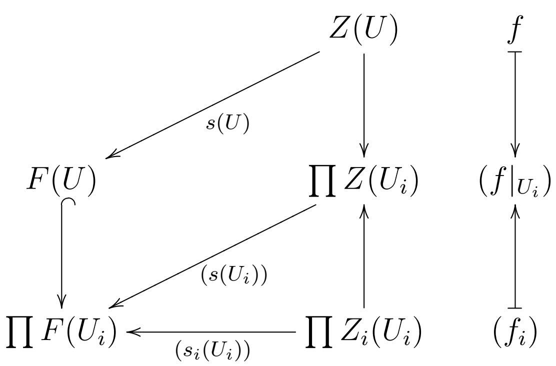


and $T_i \to T$ being an open immersion. In the right diagram the closed immersions have PD-ideals and the open immersions are PD-morphisms (See remark 3.1.6).
Let $f_i = f|_{U_i} : U_i \to Z_i$ be in $Z_i(U_i)$.Then $s(U)(f)$ can be constructed by the following commutative diagram:

(∗)

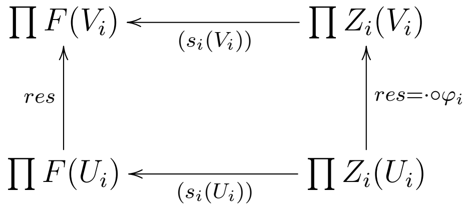


The uniqueness follows, under the hypothesis that s maps to s_i, from the following commutative diagram (for every fixed f separately):



Furthermore we have to show s defines a morphism of sheaves. Let $\varphi : V \to U$ (in the crystalline site). Then $s(V)(f|_V)$ is determined by the image of $f_i|_{V_i} = f_i\varphi_i : V_i \to Z_i$

with $V_i = \varphi^{-1}(U_i) = (f\varphi)^{-1}(Z_i)$. Now the following diagram is commutative because each s_i is a morphism of sheaves.



Finally we show that s maps to s_i for $i \in I$. Let $U = (U, T)$ be in $CRIS(X/S)_\tau$ and fix $k \in I$. For $f_k \in VZ_k(U) = \mathrm{Hom}_X(U, Z_k)$ it is by definition $s|_{Z_k}(U)(f_k) = s(U)(\iota \circ f_k)$ (for $\iota : Z_k \hookrightarrow Z$). Note that in diagram $(*)$ it is $U_k = U$ and thus $s(U)(\iota f_k) = s_k(U_k)(f_k)$.

<u>Part 2:</u>

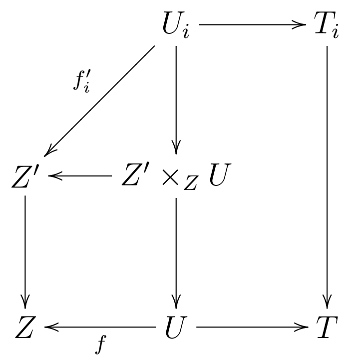
To check sheaf-condition for a finite τ-covering of affine schemes it suffices to check it for a faithfully flat source-locally τ-morphism, since by the first part the presheaf WF maps sums to products and the sequence for a finite covering $\{\mathrm{Spec}B_i \to \mathrm{Spec}A\}$ can be identified with the one for $\{\mathrm{Spec}(\prod B_i) \twoheadrightarrow \mathrm{Spec}A\}$ (It is $WF(\mathrm{Spec}(\prod B_i)) \cong WF(\coprod \mathrm{Spec}(B_i)) \cong \prod WF(\mathrm{Spec}B_i))$.

Now let $Z' \to Z$ be a surjective source-locally τ-morphism with affine $Z' = \mathrm{Spec}B$ and $Z = \mathrm{Spec}A$. Let

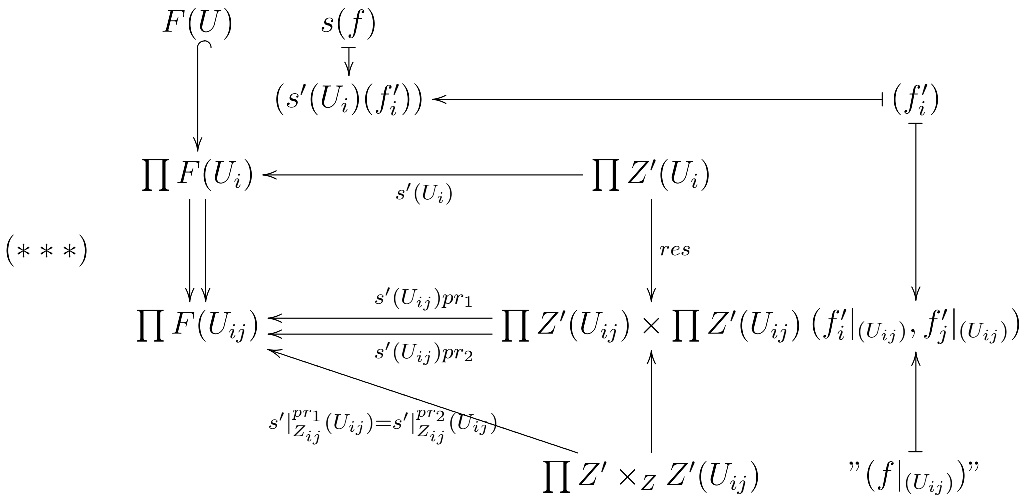
$$s' \in \ker(\mathrm{Hom}(VZ', F) \rightrightarrows \mathrm{Hom}(V(Z' \times_Z Z'), F))$$

We have to find exactly one $s \in \mathrm{Hom}(VZ, F)$ which maps to s', so let $U = (U, T) \in CRIS(X/S)_\tau$ and $f \in VZ(U) = \mathrm{Hom}_X(U, Z)$. The surjective family $\{Z' \times_Z U \to U\}$ can be refined to a τ-crystalline covering $\{U_i = (U_i, T_i) \to U\}$ by (T2) and remark 3.1.9. Define f'_i by the following diagram

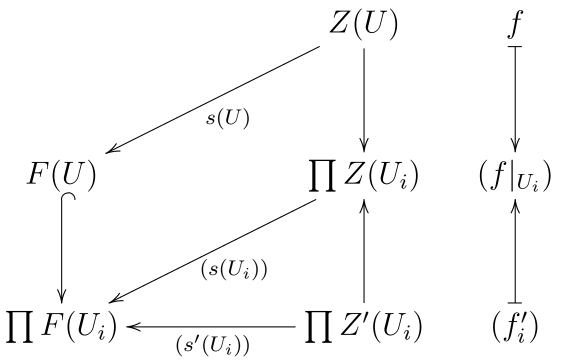
$(**)$



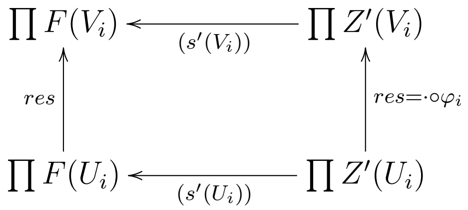
Let $U_{ij} = (U_i \times_U U_j, T_i \times_T T_j)$. Similar to part 1 we define $s(f)$ by the commutative diagram



Because of the following commutative diagram this is the only possible choice for the image of f, if we suppose s maps to s':



Again we have to show s defines a morphism of sheaves. Let $\varphi : (V, S) \to (U, T)$ (in the crystalline site). First note that $V_i = (V_i, S_i) = (V \times_U U_i, S \times_T T_i)$ is a τ-crystalline refinement of $\{V \times_Z Z' \to V\}$.
Then $s(V)(f|_V)$ is determined by the image of $(f|_V)_i = f'_i\varphi_i : V_i \to U_i \to Z'$ with φ_i the basechange of φ to U_i. Now the following diagram is commutative because s' is a morphism of sheaves.



Finally it is to show that the restriction of s equals s'. Let $U = (U, T)$ be in $CRIS(X/S)_\tau$ and let $f' \in VZ'(U) = \mathrm{Hom}_X(U, Z')$. Let f'_i be associated to $z \circ f'$ (with $z : Z' \to Z$)

according to $(**)$, then $f_i' = f'|_{U_i}$. It is:

$$\begin{aligned}(s|_{Z'}(U)(f'))|_{U_i} &= (s(U)(z \circ f'))|_{U_i} && \text{by definition}\\ &= (s'(U_i)(f_i')) && \text{by diagram } (***)\\ &= (s'(U)(f'))|_{U_i} && \text{since } f_i' = f'|_{U_i}\end{aligned}$$

and we conclude that $s|_{Z'} = s'$.

(iii) V is left exact:

By [SGA4.1] I 2.4.2. it is enough to show that V maps the final object $X \in \tau(X)$ to the final object $e \in (X/S)_{CRIS,\tau}$, which is trivial as $VX((U,T)) = \mathrm{Hom}_X(U,X)$, and that V commutes with fiber-products:

We have $V(Z_1 \times_Z Z_2)((U,T)) = \mathrm{Hom}_X(U, Z_1 \times_Z Z_2) = \mathrm{Hom}(U,Z_1) \times_{\mathrm{Hom}(U,Z)} \mathrm{Hom}(U,Z_2) = VZ_1((U,T)) \times_{VZ((U,T))} VZ_2((U,T)) = VZ_1 \times_{VZ} VZ_2((U,T))$. Note that the fiber product of sheaves is given by the set-theoretic fiberproduct over each object. □

Lemma 3.2.2. *Let τ be a class satisfying (T1) and let V be as in the theorem, Z an X-scheme. Then VZ is a τ-crystalline sheaf.*

Proof. If $\{(U_i,T_i) \to (U,T)\}$ is a covering in τ-topology then $\{U_i \to U\}$ is an $fppf$-covering and the following isomorphic (cf. 3.1.8) sequences

$$VZ(U,T) \to \prod VZ(U_i,T_i) \rightrightarrows \prod VZ((U_i,T_i) \times_{(U,T)} (U_j,T_j))$$

$$\mathrm{Hom}_X(U,Z) \to \prod \mathrm{Hom}_X(U_i,Z) \rightrightarrows \prod \mathrm{Hom}_X(U_{ij},Z)$$

are exact since representable presheaves are $fppf$-sheaves.

□

Remark 3.2.3. We wish to make clear the precise nature of the direct image functor for the morphism of topoi in the theorem. If f is a left-exact, continuous functor from a site C (where finite projective limits are assumed to be representable) into a topos E, the associated morphism of topoi g is given by

$$g_* : F \mapsto F \circ f$$

and g^* a left adjoint to g_*. For details describing the "pull-back"-functor see [SGA4.1] IV 4.9.

So in our situation $v_* : F \mapsto \mathrm{Hom}_{(X/S)_{CRIS,\tau}}(V(\cdot), F)$ is the functor we called W.

3.3 Crystals

We restrict our attention now to what Grothendieck called *special sheaves* on the crystalline site, but which is better known as *crystal*. These are in a sense rigid presheaves which in fact are sheaves.

Remark 3.3.1. Let τ be a class satisfying (T1), and let (U, T) be an element of the site $CRIS(X/S)$. Every presheaf F on $CRIS(X/S)$ defines by restriction a Zariski-presheaf on T:

$$T \supseteq T' \mapsto F(U \times_T T', T')$$

We denote this presheaf by F_T.

Definition 3.3.2. Let (X/S) be as in 3.1.1. A $\mathcal{O}_{X/S}$-module presheaf F is called a *crystal* if:
(i) it is a Zariski-sheaf and
(ii) for any morphism $(g, f) : (U', T') \to (U, T)$ the induced morphism $(f^* F_T) \to F'_T$ is an isomorphism of $\mathcal{O}_{T'}$-modules (Here f^* denotes pullback of module-sheaves, i.e. $f^* F_T = f^{-1} F_T \otimes_{f^{-1}\mathcal{O}_T} \mathcal{O}_{T'}$).
A crystal is said to be *quasi-coherent* if every F_T is quasi-coherent.

Lemma 3.3.3. *A quasi-coherent crystal is a sheaf for any topology coming from a class τ satisfying (T1).*

Proof. Again using [SGA3.1] IV 6.2.3 it suffices to show the exactness of the sequence to a covering $\{(g, f) : (\mathrm{Spec}R, \mathrm{Spec}B) \to (\mathrm{Spec}S, \mathrm{Spec}A)\}$ with f source-locally τ and surjective, hence faithfully flat (cf. the proof of the theorem). If $F_{\mathrm{Spec}A} = \tilde{M}$ for an A-module M then $F_{\mathrm{Spec}B} \cong f^* F_{\mathrm{Spec}A} \cong f^* \tilde{M} \cong M \tilde{\otimes}_A B$ and this sequence is

$$M \to M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B$$

which is well known to be exact (See e.g. [SGA1] VIII 1.7.) □

Proposition 3.3.4. *Let τ and τ' satisfy (T1) with $\tau \subseteq \tau'$. Then there is a canonical morphism of coarsening topology*

$$\alpha : (X/S)_{CRIS,\tau'} \to (X/S)_{CRIS,\tau}$$

and one has $R^q \alpha_* F = 0$ for $q > 0$ if F is a quasi-coherent crystal.

Proof. See [Be2] 1.1.19. for a similar statement. The proof is exactly the same. □

Quasi-coherent crystals have τ-crystalline cohomology independent of τ for a very general choice of the class τ:

Corollary 3.3.5. *Let τ and τ' satisfy (T1). Then for a quasi-coherent crystal F it holds:*

$$H^n((X/S)_{CRIS,\tau}, F) \cong H^n((X/S)_{CRIS,\tau'}, F)$$

Proof. Both topologies are finer than Zariski-topology. Consider the two morphisms of coarsening topology:

$$(X/S)_{CRIS,\tau} \to (X/S)_{CRIS,zar}$$

and

$$(X/S)_{CRIS,\tau'} \to (X/S)_{CRIS,zar}.$$

Using the Leray spectral sequence both terms are isomorphic to Zariski-crystalline cohomology of F. □

Remark 3.3.6. Let $\tau \subseteq \tau'$ satisfy (T1). Obviously there is a morphism of coarsening topology

$$\beta : X_{\tau'} \to X_\tau.$$

The direct-image functor is the canonical inclusion. Pullback is given be assigning to every τ-sheaf its τ'-sheafification.

3.4 Direct images of crystals and their cohomology

This subsection contains the main results of this section: The comparison of cohomology of direct images of coherent crystals for p-crystalline classes τ for a k-scheme X and $S = W_n(k)$ with a perfect field k.
We study the following commutative diagram of topoi with τ and τ' satisfying (T1) and (T2) and $\tau \subseteq \tau'$.

$$\begin{array}{ccc} (X/S)_{CRIS,\tau} & \xrightarrow{v_\tau} & X_\tau \\ \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} \\ (X/S)_{CRIS,\tau'} & \xrightarrow{v_{\tau'}} & X_{\tau'} \end{array}$$

We will see that quasi-coherent crystals are acyclic for direct images of all the morphisms if we assume in addition (T3).

Remark 3.4.1. Let E be a topos, X an object of E and F an abelian sheaf. Then $H^q(X, F)$ is defined as $R^q\Gamma_X F$ for $\Gamma_X : M \mapsto \mathrm{Hom}_E(X, M)$ (Γ_X a functor on abelian sheaves). See [SGA4.2] V 2.1.

Proposition 3.4.2. *Let τ be a p-crystalline class, X a scheme over k and $S = \mathrm{Spec}W_n(k)$ for a perfect field k. Let F be a quasi-coherent crystal on $(X/S)_{CRIS,\tau}$. Then one has for $q > 0$*

$$R^q v_* F = 0.$$

Proof. The proof is essentially the same as of [Ba] Proposition 1.17. (which deals with the case of syntomic topology):
Recall that $R^q v_* F$ is the sheaf associated to the presheaf

$$X_\tau \ni U \mapsto H^q(v^*U, F)$$

(See [SGA4.2] V 5.1). Under the identifications $S(X_\tau) \cong X_\tau \cong S(\tau(X))$ this corresponds to the sheaf associated to the presheaf

$$\tau(X) \ni U \mapsto F^q(U) := H^q(v^*U, F).$$

(U stands for the sheaf represented by U and v^*U sends (Y,T) to $\mathrm{Hom}_X(Y,U)$).

By Lemma 3.4.4 it is $F^q(U) \cong H^q((U/S)_{CRIS,\tau}, F)$.
It is sufficient to check

$$(*) \qquad (F^q)^\sharp(U) = 0$$

for affine $U \in \tau(X)$. This statement is implied by $\varinjlim_{\mathfrak{U}} H^0(\mathfrak{U}, F^q) = 0$ where the limit runs over all coverings of U and H^0 denotes 0. Čech-cohomology:

$$H^0(\mathfrak{U}, F^q) = \ker(\prod_{U_i \in \mathfrak{U}} F^q(U_i) \rightrightarrows \prod_{U_i, U_j \in \mathfrak{U}} F^q(U_i \times_U U_j))$$

Thus it suffices to show: For every affine $V \in \tau(X)$ there exists a family of faithfully flat $\tau-$morphisms $\{V_k \to V\}$ with $\varinjlim F^q(V_k) = 0$:
Assume the U_i affine, let $u_i \in F^q(U_i)$. Choose U_i' (a member of the family for U_i), such that u_i maps to $0 \in F^q(U_i')$. Then $\{U_i'\}$ is a refinement of $\{U_i\}$, where (u_i) vanishes.
For Spec$A$ an affine X-scheme define inductively

$$A_0 = A \text{ and } A_n = A_{n-1}[x_a; a \in A_{n-1}]/(x_a^p - a)$$

Let $\hat{A}$ be the perfect envelope $\varinjlim A_n$. Then $\hat{A} = \varinjlim B_i$ for some surjective p.-th roots Spec$B_i \to A$ which are τ by (T3). Now we have

$$\varinjlim F^q(\mathrm{Spec}(B_i)) = \varinjlim H^q((\mathrm{Spec}B_i/W_n)_\tau, F) \cong \varinjlim H^q((\mathrm{Spec}B_i/W_n)_{CRIS,ét}, F)$$

by corollary 3.3.5, which by [Ka] 2.4.3 in turn is isomorphic to

$$H^q((\mathrm{Spec}\hat{A}/W_n)_{CRIS,ét}, F) \cong H^q((\mathrm{Spec}\hat{A}/W_n)_{CRIS,zar}, F).$$

The proposition follows from the following one since coherent modules have no higher cohomology on affine schemes. □

Proposition 3.4.3. *Let A be a k-algebra with surjective Frobenius morphism, k a perfect field of characteristic $p > 0$. Then there exists a W_n-divided-power thickening (U,T) in $CRIS(\mathrm{Spec}A/W_n)_{Zar}$ with affine T such that for any abelian zariski-crystalline sheaf F and each $q \in \mathbb{N}$ it holds:*

$$H^q((\mathrm{Spec}A/W_n)_{zar}, F) \cong H^q(T, F_T)$$

for F_T the induced abelian Zariski-sheaf on T.

Proof. [Ba] 1.16. ☐

To complete the proof: First recall the definition of the global section functor for presheaves F on a site C: $\Gamma(F) = \mathrm{Hom}(e, F) = \varprojlim_{U \in C} F(U)$, where e is the final object in the associated category of sheaves (the constant sheaf with value $\{*\}$). If in C there is a final object Y, it obviously holds $\Gamma(F) = F(Y)$.

Lemma 3.4.4. *Let Z be an X-scheme and τ a class with (T1). Then the restriction functor $res : (X/S)_{CRIS,\tau} \to (Z/S)_{CRIS,\tau}$ is exact (on abelian sheaves). Furthermore it holds: $\Gamma_{(Z/S)} \circ res \cong \mathrm{Hom}_{(X/S)}(v^*Z, \cdot)$ (Again Z stands for the sheaf represented by Z and v^*Z maps (U,T) to $\mathrm{Hom}_X(U,Z)$). We conclude that*

$$H^q((Z/S)_{CRIS,\tau}, F) \cong H^q(v^*Z, F)$$

for any abelian sheaf F on $(X/S)_{CRIS,\tau}$.

Proof. Exactness is immediate. Let e be the final object in the topos $(Z/S)_{CRIS,\tau}$. Then we define a map

$$\alpha : \Gamma(F) = \mathrm{Hom}_{(Z/S)}(e, F) \to \mathrm{Hom}_{(X/S)}(v^*Z, F)$$

by sending $(f_{(U,T)})$ to the morphism of sheaves which sends $\varphi \in v^*Z((U',T'))$ for $(U',T') \in CRIS(X/S)$ to $f_{(U',T')}$ (Here (U',T') is viewed as an object of $CRIS(Z/S)$ via $\varphi \in v^*Z((U',T')) = \mathrm{Hom}_X(U',Z)$).
The inverse map

$$\beta : \mathrm{Hom}_{(X/S)}(v^*Z, F) \to \mathrm{Hom}_{(Z/S)}(e, F)$$

is given by $\psi \mapsto (s_{(U,T)})$ with the global section $s_{(U,T)}$ defined as follows:
For $(U,T) \in CRIS(Z/S)$ the structural map $U \to Z$ is an Element in $v^*Z((U,T)) = \mathrm{Hom}_X(U,Z)$. Define $s_{(U,T)} = \psi(U,Z)(U \to Z) \in F(U,T)$.

This is indeed the inverse map:
Let $f = (f_{(U,T)})$ be a global section of F. Let $s_{(U,T)}$ be associated to $\alpha(f)$. Then

$$\begin{aligned} \beta\alpha(f)_{(U,T)} &= s_{(U,T)} \\ &= \alpha(f)(U,T)(U \to Z) \\ &= f_{(U,T)}. \end{aligned}$$

Let $\psi \in \mathrm{Hom}_{(X/S)}(v^*Z, F)$ and let $(s_{(U,T)})_{(U,T)} = \beta(\psi)$. Then we compute:

$$\begin{aligned} \alpha\beta(\psi) &= (\mathrm{Hom}_X(U',Z) \ni \varphi \mapsto s_{(U',T')}) \\ &= (\mathrm{Hom}_X(U',Z) \ni \varphi \mapsto \psi(U',T')(U' \to Z) = \psi(\varphi)) \\ &= \psi. \end{aligned}$$

☐

Proposition 3.4.5. *Let τ and τ' satisfy (T1) and (T2) with $\tau \subseteq \tau'$. Then diagram of morphisms of topoi*

$$\begin{array}{ccc} (X/S)_{CRIS,\tau} & \xrightarrow{v_\tau} & X_\tau \\ \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} \\ (X/S)_{CRIS,\tau'} & \xrightarrow{v_{\tau'}} & X_{\tau'} \end{array}$$

is commutative where β is defined by coarsening topology.

Proof. Again by [SGA4.1] IV proposition 4.9.4 it is enough to show that $v_{\tau'}^*\beta^*$ and $\alpha^* v_\tau^*$ restricted to $\tau(X)$ are isomorphic. It is $\beta^* Z = Z$ since representable presheaves are sheaves on both X_τ and $X_{\tau'}$ by (T1). Furthermore v^*Z is in both cases given by the presheaf

$$VZ : (U, T) \mapsto \mathrm{Hom}_X(U, Z),$$

which is actually a sheaf for τ and τ' (see Lemma 3.2.2). Now note that inverse image morphisms of coarsening topology morphisms map a sheaf for the coarse topology to the associated sheaf in the fine topology. □

Remark 3.4.6. Assume all τ satisfy (T1). By Lemma 3.3.3 we know that for a quasi-coherent crystal F the underlying sheaf on $(X/S)_{CRIS,\tau}$ does not dependend on τ. By proposition 3.3.4 we know furthermore that its cohomology is independent of τ.
The proposition now tells us that the underlying sheaf of the direct image of F does not depend on τ (for all τ satisfying (T1) and (T2)). Proposition 3.4.2 implies that its cohomology is independent of τ (assuming (T3) fulfilled) under the assumptions of the proposition. We summarize in the main theorem of this section:

Theorem 3.4.7. *Let τ and τ' satisfy (T1), (T2) and (T3) and let X be a scheme over k and $S = \mathrm{Spec} W_n(k)$ for a perfect field k. Then one has for a quasi-coherent crystal F:*

$$H^q(X_\tau, v_*F) \cong H^q(X_{\tau'}, v_*F)$$

In particular, if we let $F = \mathcal{O}_{X/S}$:

$$H^q(X_\tau, \mathcal{O}_n^{cris}) \cong H^q(X_{\tau'}, \mathcal{O}_n^{cris}).$$

Proof. We can assume that τ is the class of p.-th roots; this is the smallest possible class. We have a commutative diagram of morphisms of topoi

$$\begin{array}{ccc} (X/S)_{CRIS,\tau} & \xrightarrow{v_\tau} & X_\tau \\ \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} \\ (X/S)_{CRIS,\tau'} & \xrightarrow{v_{\tau'}} & X_{\tau'} \end{array}$$

Using Leray spectral sequence for v_τ and $v_{\tau'}$, recalling $R^q(v_\tau)_*F = 0$ for $q > 0$ by 3.4.2 and the preceding remark, we get isomorphisms:

$$H^q((X/S)_{CRIS,\tau}, F) \cong H^q(X_\tau, v_*F)$$

and

$$H^q((X/S)_{CRIS,\tau'}, F) \cong H^q(X_{\tau'}, v_*F)$$

Combining this with the isomorphism

$$H^q((X/S)_{CRIS,\tau'}, F) \cong H^q((X/S)_{CRIS,\tau}, F)$$

of the remark, the claim follows. □

Theorem 3.4.8. *Let $\tau \subseteq \tau'$ both satisfy (T1), (T2) and (T3). Then one has with (X/S) like in 3.4.7 and a quasi-coherent crystal F:*

$$R^q\beta_*(v_*F) = 0$$

for $q > 0$ and $\beta : X_{\tau'} \to X_\tau$ like above.

Proof. Let $G = v_*F$. We know that $R^q\beta_*G$ is the τ-sheaf associated to the presheaf $U \mapsto H^q_{\tau'}(U, G)$, so it is enough to show that this associated sheaf takes zero-values on affine X-schemes.
We have $H^q_{\tau'}(U, G) \cong H^q(U_{\tau'}, G) \cong H^q((U/S)_{CRIS,\tau'}, F)$ by the previous results. This is $F^q(U)$ in notation of the proof of 3.4.2. There it was shown that $(F^q)^\sharp(U) = 0$ for affine U. □

4 Topologies and the sheaves G_n

Let X be a scheme of finite type over a perfect field k of positive characteristic p. Let W denote the ring of Witt-vectors and let W_n denote the truncated ring of Witt-vectors.

4.1 The ring $\mathcal{O}_n^{\mathrm{cris}}$

Here we give the main definitions of this section. This subsection also contains outlines of some proofs Fontaine sketched in his conversation with Jannsen.

Recall section 3 that there is a morphism of topoi

$$v : (X/W_n(k))_{CRIS,syn} \to X_{syn}$$

Definition 4.1.1. (i) Define the sheaf of rings in X_{syn}

$$\mathcal{O}_n^{\mathrm{cris}} = v_* \mathcal{O}_{X/W_n}.$$

(ii) Let $\mathcal{O}_0^{\mathrm{cris}} = 0$.

Obviously $\mathcal{O}_n^{\mathrm{cris}}$ is a sheaf in X_τ for any class of morphisms τ contained in the class of syntomic morphisms. By definition we have for $Y \to X$

$$\mathcal{O}_n^{\mathrm{cris}}(Y) = H^0((Y/W_n)_{CRIS,syn}, \mathcal{O}_{Y/W_n})$$

This is the original definition (to be found in [FM]). There is another description of $\mathcal{O}_n^{\mathrm{cris}}$ which is very useful for computations:

Remark 4.1.2. One knows (see [FM]), that

$$\tilde{W}_n^{DP} \cong \mathcal{O}_n^{\mathrm{cris}}.$$

The presheaf W_n maps a k-algebra A to $W_n(A)$ and the presheaf W_n^{DP} is the divided power envelope of W_n with respect to the ideal $I = \{(a_0, \dots, a_n) | a_0^{p^n} = 0\}$, compatible with standard divided powers on pW_n.
The isomorphism is constructed as follows:
There is a canonical map:

$$W_n \to \mathcal{O}_n^{\mathrm{cris}}$$

given over a k-algebra A by

$$(a_0, \dots, a_{n-1}) \mapsto (\hat{a}_0^{p^n} + p\hat{a}_1^{p^{n-1}} + \dots + p^{n-1}\hat{a}_{n-1}^p)_{(U,T)} \in \varprojlim_{(U,T)} \Gamma(T, \mathcal{O}_T).$$

Recall that the limit is taken over all A-schemes U with W_n-thickenings $U \hookrightarrow T$. Fix a component $(U,T) = (\mathrm{Spec} A \overset{f}{\leftarrow} U \overset{i}{\rightarrow} T)$. Now let $\hat{a}_i$ be a preimage (under $i^\sharp$) of $f^\sharp(a_i)$, then $p^i \hat{a}_i^{p^{n-i}}$ does not depend on the choice of the lifting:

Two preimages $\hat{a}_i, \hat{b}_i$ of a_i differ by an element in the divided power ideal $\ker f^\sharp$, say $\hat{a}_i = \hat{b}_i + \alpha$. Then

$$p^i \hat{a}_i^{p^{n-i}} = p^i \hat{a}_i^{p^{n-i}} + p^i \sum_{i>0} \binom{p^{n-i}}{i} \alpha^i \hat{b}_i^{(p^{n-i}-i)}.$$

Since α has divided powers we can write $p^i \binom{p^{n-i}}{i} \alpha^i = p^i \binom{p^{n-i}}{i} i!\gamma_i(\alpha)$. This equals 0 because the p-valuation of the coefficient is $i + n - i - v_p(i) + v_p(i!) \geq n$ and $\Gamma(T, \mathcal{O}_T)$ is annihilated by p^n (see A.3.4).
Since the image of I under this map is contained in $p\mathcal{O}_n^{\mathrm{cris}}$ we can extend the map to morphism

$$W_n^{DP} \to \mathcal{O}_n^{\mathrm{cris}}.$$

Syntomic sheafification gives the desired map

$$\tilde{W}_n^{DP} \to \mathcal{O}_n^{\mathrm{cris}}.$$

Fontaine and Messing claim that this is an isomorphism, see [FM] 1.4.

There is a Frobenius φ on $\mathcal{O}_n^{\mathrm{cris}}$ which is compatible with the canonical projections

$$v : \mathcal{O}_n^{\mathrm{cris}} \to \mathcal{O}_{n-1}^{\mathrm{cris}}.$$

It is induced by Frobenius on W_n :

$$F : W_n \to W_n$$

$$(a_0, \ldots, a_{n-1}) \mapsto (a_0^p, \ldots, a_{n-1}^p)$$

It is clear that F extends to the divided power envelope.

Lemma 4.1.3. *(Computation of $\mathcal{O}_n^{\mathrm{cris}}(\mathrm{Spec} k)$) Let k be a perfect field of positive characteristic p. Then one has:*

$$\mathcal{O}_n^{\mathrm{cris}}(\mathrm{Spec}(k)) \cong W_n^{DP}(k) \cong W_n(k)$$

whith notation from above.

Proof. By definition, $W_n^{DP}(k)$ is the divided power envelope of $W_n(k)$ along the ideal $\{(a_0, \ldots, a_{n-1}) | a_0^{p^n} = 0\}$, compatible with the canonical ones on $pW_n(k)$. By definition and since k is perfect (and reduced),

$$\begin{aligned} pW_n &= \{p(b_0, \ldots, b_{n-1}) | b_i \in k\} \\ &= \{(0, b_0^p, \ldots, b_{n-2}^p) | b_i \in k\} \\ &= \{(a_0, \ldots, a_{n-1}) | a_i \in k, a_0 = 0\} \\ &= \{(a_0, \ldots, a_{n-1}) | a_i \in k, a_0^{p^n} = 0\} \end{aligned}$$

So the ideals coincide and forming the divided power envelope is trivial, which shows the second isomorphism. The first is in [FM]1.4 (Again because k is perfect). □

Definition 4.1.4. Let $\mathcal{O}$ be the structural sheaf, i.e. $\mathcal{O}(Y) = \Gamma(Y, \mathcal{O}_Y)$ for $Y \to X$.

Lemma/Definition 4.1.5. *(due to Fontaine)*
(i) There is an epimorphism of p-sheaves (topology consisting of p.-th roots)

$$\mathcal{O}_n^{\mathrm{cris}} \to \mathcal{O}$$

given in terms of Witt-vectors by $(a_0, \ldots, a_{n-1}) \mapsto a_0^{p^n}$ (cf. [FM] II.2.1). The morphism is surjective for any topology finer than topology of p.-th roots.
(ii) The kernel of the map $\mathcal{O}_1^{\mathrm{cris}} \to \mathcal{O}$, denoted $\mathcal{J}_1^{[1]}$, is a PD-ideal. Its r.-th divided powers are denoted by $\mathcal{J}_1^{[r]}$.

Proof. (i) The morphism $W_n \to \mathcal{O}$ with $(a_0, \ldots, a_{n-1}) \mapsto a_0^{p^n}$ maps $I \mapsto 0$, so forming DP-envelope and taking the associated syntomic sheaf we get

$$\begin{array}{ccccc} W_n & \longrightarrow & W_n^{DP} & \longrightarrow & \mathcal{O}_n^{\mathrm{cris}} \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{O} & & \end{array}$$

Hence, to show surjectivity it is enough to show that for any section $a \in \mathcal{O}(\mathrm{Spec}A)$, with a k-algebra A, there exists a p-covering $f : A \to B$, such that the restriction $a|_B = f(a)$ has a preimage in $W_n(B)$.
This is obvious: Take $B = A[t]/(t^{p^n} - a)$, then $(t, 0, \ldots, 0)$ is the desired preimage.
(ii) The kernel of $W_n \to \mathcal{O}$ is the ideal used un the divided power envelope. □

Lemma 4.1.6. *In characteristic p, ie. if $n = 1$, there is a monomorphism*

$$\rho : \mathcal{O} \hookrightarrow \mathcal{O}_1^{\mathrm{cris}}$$

(due to Fontaine).

Proof. Let U be over X and let $x \in \mathcal{O}(U)$. Then there exists a p-covering $\{V \to U\}$ such that $x|_V$ posesses a preimage $y \in \mathcal{O}_1^{\mathrm{cris}}(V)$ by the previous lemma. Let I be the kernel of $\mathcal{O}_1^{\mathrm{cris}} \to \mathcal{O}$. Consider the commutative diagram with exact rows and columns, given by sheaf-condition of I, $\mathcal{O}_1^{\text{cris}}$ respectively $\mathcal{O}$,

$$\begin{array}{ccccc}
I(V \times_U V) & \longrightarrow & \mathcal{O}_1^{\text{cris}}(V \times_U V) & \longrightarrow & \mathcal{O}(V \times_U V) \\
\uparrow & & \uparrow & & \uparrow \\
I(V) & \longrightarrow & \mathcal{O}_1^{\text{cris}}(V) & \longrightarrow & \mathcal{O}(V) \\
\uparrow & & \uparrow & & \uparrow \\
I(U) & \longrightarrow & \mathcal{O}_1^{\text{cris}}(U) & \longrightarrow & \mathcal{O}(U) \\
\uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & 0
\end{array}$$

The element y is mapped to 0 in $\mathcal{O}(V \times_U V)$, so $y' :=$ (image of y in $\mathcal{O}_1^{\text{cris}}(V \times_U V)$) lies in $I(V \times_U V)$. Since I is a PD-ideal $y'^p = p!\gamma_p(y') = 0$, so $y^p \in \mathcal{O}_1^{\text{cris}}(U)$. We let $\rho(x) = y^p$. Two preimages of $x|_V$ differ by an element $d \in I(V)$, for which we have again $d^p = 0$, showing that the definition is independent of this choice.
Let V' be another covering. Then both V and V' refine the covering $V \coprod V'$ so we can without loss of generality assume that $V' \to V$. In this case it is clear that ρ does not depend on the choice of the covering: If y is a preimage of $x|_V$, then $y|_{V'}$ is a preimage of $x|_{V'}$.
Injectivity is proven in [FJ] Proposition 3.3.2 on p.13: One can show injectivity over p-good algebras (for a definition see 4.3), and this suffices. □

Lemma 4.1.7. *(i) The composition*

$$\mathcal{O} \hookrightarrow \mathcal{O}_1^{\text{cris}} \twoheadrightarrow \mathcal{O}$$

equals Frobenius on $\mathcal{O}$.
(ii) The composition

$$\mathcal{O}_1^{\text{cris}} \twoheadrightarrow \mathcal{O} \hookrightarrow \mathcal{O}_1^{\text{cris}}$$

equals Frobenius φ on $\mathcal{O}_1^{\text{cris}}$.

Proof. (i) Let $x \in \mathcal{O}(U)$ and $\rho(x) = y^p$ be the image of x in $\mathcal{O}_1^{\text{cris}}$ like in the proof of the previous proposition. The image of y^p in $\mathcal{O}(V)$ equals $x|_V^p$ and since the restriction map $\mathcal{O}(U) \to \mathcal{O}(V)$ is injective, the image of $\rho(x) = y^p$ in $\mathcal{O}(U)$ is x^p.
(ii) Let $z \in \mathcal{O}_1^{\text{cris}}(U)$, and let x be the image of z in $\mathcal{O}(U)$. In the construction of ρ we can choose $V = U$ and $\rho(z)$ is defined as p.-th power of a preimage of x, hence $\rho(z) = z^p$. □

Lemma 4.1.8. *The sequence*

$$0 \to \ker \varphi \to \mathcal{O}_1^{\mathrm{cris}} \to \mathcal{O} \to 0$$

is exact. In other words $\mathcal{J}_1^{[1]}$ is isomorphic to the kernel of Frobenius on $\mathcal{O}_1^{\mathrm{cris}}$.

Proof. This follows immediately from the factorization of φ in 4.1.7(ii). □

Remark 4.1.9. (i) We recall [FM] that for natural numbers m and n there is an exact sequence of τ-sheaves

$$0 \to \mathcal{O}_n^{\mathrm{cris}} \xrightarrow{\pi} \mathcal{O}_{n+m}^{\mathrm{cris}} \xrightarrow{\nu} \mathcal{O}_m^{\mathrm{cris}} \to 0$$

with the canonical projection ν defined as divided power envelope and sheafification of

$$W_{n+m} \to W_m$$

$$(a_0, \dots, a_{n+m-1}) \mapsto (a_0^{p^n}, \dots, a_{m-1}^{p^n})$$

The map π is defined by the diagram

$$\begin{array}{ccc} \mathcal{O}_n^{\mathrm{cris}} & \xrightarrow{\pi} & \mathcal{O}_{n+m}^{\mathrm{cris}} \\ & \nwarrow \quad \nearrow{\scriptstyle p^m} & \\ & \mathcal{O}_{n+m}^{\mathrm{cris}} & \end{array}$$

which means locally $\pi x = p^m \hat{x}$ for any lift $\hat{x}$ of x in $\mathcal{O}_{n+m}^{\mathrm{cris}}$. Sometimes we will denote π by

$$\pi^m : \mathcal{O}_n^{\mathrm{cris}} \to \mathcal{O}_{m+n}^{\mathrm{cris}}$$

to avoid any confusion. We note that the images of p^m and π coincide: We can identify $\mathcal{O}_n^{\mathrm{cris}}$ with the image of p^m-multiplication in $\mathcal{O}_{n+m}^{\mathrm{cris}}$.
(ii)This sequence identifies $\mathcal{O}_n^{\mathrm{cris}}$ to the kernel of p^n-multiplication in $\mathcal{O}_{m+n}^{\mathrm{cris}}$ also: Multiplication by p^n factors

$$\mathcal{O}_{m+n}^{\mathrm{cris}} \xrightarrow{\nu} \mathcal{O}_m^{\mathrm{cris}} \xrightarrow{\pi^n} \mathcal{O}_{m+n}^{\mathrm{cris}}$$

with π^n the monomorphism of the sequence with m and n switched. So for a local section x we have $p^n x = 0 \Rightarrow \nu x = 0 \Rightarrow x \in \mathrm{im}\pi^m = \mathrm{im} p^m$ locally $\Rightarrow p^n x = 0$.
(iii) We can identity $\mathcal{O}_n^{\mathrm{cris}}$ with the cokernel of p^n-multiplicationin $\mathcal{O}_{n+m}^{\mathrm{cris}}$, by switching m and n above: $\mathrm{im}\tilde{\pi} = \mathrm{im}p^n$, which means $\mathcal{O}_n^{\mathrm{cris}} \cong \mathrm{coker}\pi = \mathrm{coker}p^n$. Together we have the exact sequence

$$0 \to \mathcal{O}_n^{\mathrm{cris}} \xrightarrow{\pi} \mathcal{O}_{n+m}^{\mathrm{cris}} \xrightarrow{p^n} \mathcal{O}_{n+m}^{\mathrm{cris}} \xrightarrow{v} \mathcal{O}_n^{\mathrm{cris}} \to 0$$

(iv) The maps π and ν commute with Frobenius φ as one easily checks.

Remark 4.1.10. For calculations the following fact is very useful:

We can "divide by p" in $\mathcal{O}_n^{\mathrm{cris}}$ in the following sense: Let $x, y \in \mathcal{O}_n^{\mathrm{cris}}$

$$px = py \Rightarrow x = y \text{ in } \mathcal{O}_{n-1}^{\mathrm{cris}}.$$

Indeed, with $\nu : \mathcal{O}_n^{\mathrm{cris}} \to \mathcal{O}_{n-1}^{\mathrm{cris}}$, it holds $\pi(\nu x) = px = py = \pi(\nu y)$. The injectivity of π implies $\nu x = \nu y$.

Notation 4.1.11. (i) For an integer z let $\underline{z} = \max(z, 0)$. Then it is $z + z' \leq \underline{z + z'} \leq \underline{z} + \underline{z'}$.
(ii) For integers z, z' let $\underline{(z, z')} = \underline{z} + \underline{z'} - \underline{z + z'}$. One verifies easily that

$$\underline{(z, z')} = \begin{cases} \min(|z|, |z'|) & \text{if } z \text{ and } z' \text{ have different signs} \\ 0 & \text{else.} \end{cases}$$

In particular $\underline{(z, z')} = \underline{(-z, -z')}$.

Remark 4.1.12. Let $r \in \mathbb{N}$ and let s be such that $r + s \geq 0$. The morphism $\mathcal{O}_r^{\mathrm{cris}} \to \mathcal{O}_{r+s}^{\mathrm{cris}}$ denotes π or ν if $s \geq 0$ resp. $s \leq 0$. It is defined in both cases by the diagram

$$\begin{array}{ccc} \mathcal{O}_r^{\mathrm{cris}} & \longrightarrow & \mathcal{O}_{r+s}^{\mathrm{cris}} \\ \uparrow & & \uparrow \\ \mathcal{O}_{r+\underline{s}}^{\mathrm{cris}} & \xrightarrow{p^{\underline{s}}} & \mathcal{O}_{r+\underline{s}}^{\mathrm{cris}} \end{array}$$

as one easily checks.

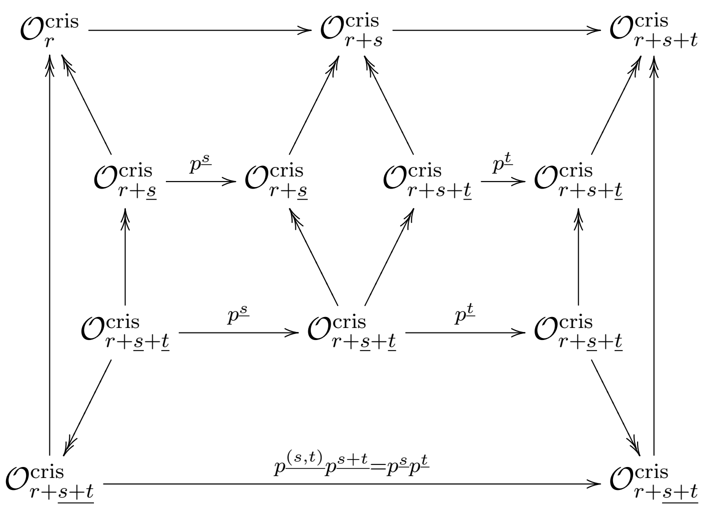
The next two lemmas deal with interaction of π and ν.

Lemma 4.1.13. *Let $r \in \mathbb{N}$, and s, t such that $r + s, r + t, r + s + t \geq 0$. The composition*

$$\mathcal{O}_r^{\mathrm{cris}} \to \mathcal{O}_{r+s}^{\mathrm{cris}} \to \mathcal{O}_{r+s+t}^{\mathrm{cris}}$$

equals the morphism $\mathcal{O}_r^{\mathrm{cris}} \to \mathcal{O}_{r+s+t}^{\mathrm{cris}}$ followed by multiplication with $p^{\underline{(s,t)}}$.

Proof. This follows completely from the commutative diagram, where all "vertical" morphisms are the canonical epimorphisms



□

Lemma 4.1.14. *Let $s \geq 0$.*
(i) The composition

$$\mathcal{O}_r^{\mathrm{cris}} \to \mathcal{O}_{r+s}^{\mathrm{cris}} \to \mathcal{O}_r^{\mathrm{cris}}$$

equals multiplication with p^s.
(ii) The composition

$$\mathcal{O}_{r+s}^{\mathrm{cris}} \to \mathcal{O}_r^{\mathrm{cris}} \to \mathcal{O}_{r+s}^{\mathrm{cris}}$$

equals multiplication with p^s.
(iii) Let $r \in \mathbb{N}$, and s, t such that $r+s, r+t, r+s+t \geq 0$. The diagram

$$\begin{array}{ccc} \mathcal{O}_r^{\mathrm{cris}} & \longrightarrow & \mathcal{O}_{r+s}^{\mathrm{cris}} \\ \downarrow & & \downarrow \\ \mathcal{O}_{r+t}^{\mathrm{cris}} & \longrightarrow & \mathcal{O}_{r+s+t}^{\mathrm{cris}} \end{array}$$

commutes.

Proof. This follows immediately from the previous lemma. It can also easily be shown by using the definitions. □

4.2 The fundamental gauges G_n

We will construct a sheaf of gauges on the big τ-site for some p-crystalline class τ. This construction is due to Fontaine and Jannsen, we prove the technical statements. So let τ be a p-crystalline class.

Let $m, n \in \mathbb{Z}$. We will use exact sequences

$$(*) \qquad 0 \to \mathcal{O}_{\underline{n}}^{\mathrm{cris}} \xrightarrow{\pi} \mathcal{O}_{\underline{n}+\underline{m}}^{\mathrm{cris}} \xrightarrow{\nu} \mathcal{O}_{\underline{m}}^{\mathrm{cris}} \to 0$$

Definition 4.2.1. Let $\hat{G}_m^r$ be the kernel of

$$\mathcal{O}_m^{\mathrm{cris}} \xrightarrow{\varphi} \mathcal{O}_m^{\mathrm{cris}} \to \mathcal{O}_{\underline{r}}^{\mathrm{cris}}$$

for $r \in \mathbb{Z}$ and $m \in \mathbb{N}$ with $r \leq m$. The sequence $(*)$ implies that a section $x \in \mathcal{O}_m^{\mathrm{cris}}$ is in $\hat{G}_m^r$ if and only if locally $\varphi x = p^r y$ for $y \in \mathcal{O}_m^{\mathrm{cris}}$.
Let $G_n = \oplus G_n^r$ be the sheaf of graded abelian groups with

$$G_n^r = \hat{G}_m^r / p^n \hat{G}_m^r$$

for $m \geq \max(n, n+r)$.

Remark 4.2.2. For $r \le 0$ it is by definition $\mathcal{O}^{\mathrm{cris}}_{\underline{r}} = 0$ and hence $\hat{G}^r_m = \mathcal{O}^{\mathrm{cris}}_m$. It follows

$$G^r_n \cong \mathcal{O}^{\mathrm{cris}}_n$$

for $r \le 0$.

Lemma 4.2.3. *The definition is independent of* m*: Fix* $m \ge \max(n, n+r)$ *and let*

$$K^r_n = \hat{G}^r_{m+1}/p^n\hat{G}^r_{m+1}$$

and

$$H^r_n = \hat{G}^r_m/p^n\hat{G}^r_m.$$

Then $K^r_n \cong H^r_n$.

Proof. Consider the commutative diagram with exact rows and columns (coming from snake lemma applied to the right two exact columns)

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{O}^{\mathrm{cris}}_1 & \longrightarrow & \mathcal{O}^{\mathrm{cris}}_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow \\
0 & \longrightarrow & \hat{G}^r_{m+1} & \longrightarrow & \mathcal{O}^{\mathrm{cris}}_{m+1} & \xrightarrow{\nu\varphi} & \mathcal{O}^{\mathrm{cris}}_r \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \hat{G}^r_m & \longrightarrow & \mathcal{O}^{\mathrm{cris}}_m & \xrightarrow{\nu\varphi} & \mathcal{O}^{\mathrm{cris}}_r \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

Considering cokernels of p^n-multiplication of the first column we get an exact sequence

$$\mathcal{O}^{\mathrm{cris}}_1 \xrightarrow{\pi} K^r_n \to H^r_n \to 0.$$

The map π is zero: Consider the diagram (the defining diagram of π)

$$\begin{array}{ccccc}
\mathcal{O}^{\mathrm{cris}}_1 & & \xrightarrow{\quad\pi\quad} & & \hat{G}^r_{m+1} \\
\uparrow & & & {\scriptstyle p^n}\nearrow & \downarrow \\
\mathcal{O}^{\mathrm{cris}}_{m+1} & \xrightarrow{p^{m-n}} & \hat{G}^r_{m+1} & \xrightarrow{p^n} & \mathcal{O}^{\mathrm{cris}}_{m+1}
\end{array}$$

which is commutative by the next lemma (with $s = 1$).

□

Lemma 4.2.4. *Let $m \geq n + r$ and $s \geq 0$. The map $p^m : \mathcal{O}^{\text{cris}}_{m+s} \to \mathcal{O}^{\text{cris}}_{m+s}$ factors in*

$$\mathcal{O}^{\text{cris}}_{m+s} \overset{p^{m-n}}{\to} \hat{G}^r_{m+s} \overset{p^n}{\to} \hat{G}^r_{m+s} \hookrightarrow \mathcal{O}^{\text{cris}}_{m+s}$$

Proof. Obviously $p^{m-n}\mathcal{O}^{\text{cris}}_{m+s}$ is mapped to 0 in $\mathcal{O}^{\text{cris}}_r$ since $m - n \geq r$. □

Lemma 4.2.5. *The graded module G_n is a graded ring with multiplication defined by the following commutative diagram:*

$$\begin{array}{ccc} G^r_n \times G^s_n & \longrightarrow & G^{r+s}_n \\ \uparrow & & \uparrow \\ \hat{G}^r_m \times \hat{G}^s_m & \longrightarrow & \hat{G}^{r+s}_m \\ \downarrow & & \downarrow \\ \mathcal{O}^{\text{cris}}_m \times \mathcal{O}^{\text{cris}}_m & \xrightarrow{mult} & \mathcal{O}^{\text{cris}}_m \end{array}$$

for $m \geq \max(\underline{n} + \underline{r} + \underline{s})$.

Proof. Firstly we have to show that multiplication on $\mathcal{O}^{\text{cris}}_m$ restricts to $\hat{G}$. Let $x \in \hat{G}^r_m$ and $x' \in \hat{G}^s_m$ so that locally $\varphi x = p^{\underline{r}} y$ and $\varphi x' = p^{\underline{s}} y'$ for $y, y' \in \mathcal{O}^{\text{cris}}_r$. Then it is $\varphi(xx') = p^{\underline{r}+\underline{s}} yy'$ which means $xx' \in \hat{G}^{\underline{r}+\underline{s}}_m \subseteq \hat{G}^{\underline{r+s}}_m \subseteq \hat{G}^{r+s}_m$.
It is clear that multiplication is well defined after dividing out the image of p^n-multiplication.
□

Definition 4.2.6. There are two distinguished global sections of G_n:
(i) Let f be $p \in G^1_n$ (Note that $p \in \hat{G}^1_m$ for all m).
(ii) Let v be $1 \in G^{-1}_n = \mathcal{O}^{\text{cris}}_n$.

Remark 4.2.7. (i) Multiplication by f is given via

$$\hat{G}^r_m \xrightarrow{\cdot p} \hat{G}^{r+1}_m$$

(multiplication by p). Indeed, if $\varphi x = p^r y$, then $\varphi p x = p^{r+1} y$.
(ii) Multiplication by v is given via the canonical inclusions

$$\hat{G}^r_m \to \hat{G}^{r-1}_m.$$

(iii) By the above remarks G_n is a graded $\mathcal{O}^{\text{cris}}_n[f, v]/(fv - p)$-module.

Remark 4.2.8. Let k be a perfect field. We compute $G_n(k)$. It is

$$\mathcal{O}^{\text{cris}}_n(\text{Spec}k) \cong W^{DP}_n(k) \cong W_n(k),$$

(See 4.1.3). In particular $\varphi(\text{Spec}k)$ is an isomorphism. So

$$\hat{G}^r_m(\text{Spec}k) = \ker(\nu\varphi) = \ker(\varphi\nu) = \ker \nu = \text{im}\pi \cong W_{m-\underline{r}}(k)$$

It follows that

$$G^r_n(\text{Spec}k) \cong W_n(k)$$

for all $r \in \mathbb{Z}$.
Explicitely $\psi : W_n(k) \to G_n^r(\mathrm{Spec} k)$ is given by

$$W_n(k) \overset{pr}{\twoheadleftarrow} W_{m-\underline{r}}(k) \overset{\pi}{\to} \hat{G}_m^r(\mathrm{Spec} k) \to G_n^r(\mathrm{Spec} k)$$

with the canonical projection pr.
($pr : W_m \to W_n$ is the map we use to identify W_n with $W_m/p^n W_m$. It maps $(a_0, \ldots, a_{m-1})$ to $(a_0, \ldots, a_{n-1})$. This map should not be confused with $\nu : W_m \to W_n; (a_0, \ldots, a_n) \mapsto (a_0^{p^{m-n}}, \ldots, a_{n-1}^{p^{m-n}})$. One Obviously has $\nu = F^{m-n} pr$. Using pr guarantees that the isomorphism is independent of the auxiliary index m).

By definition of π we can also write

$$W_n(k) \overset{pr}{\twoheadleftarrow} W_{m-\underline{r}} \overset{\nu}{\twoheadleftarrow} W_m(k) \overset{p^{\underline{r}}}{\to} \hat{G}_m^r(\mathrm{Spec} k) \to G_n^r(\mathrm{Spec} k).$$

So the isomorphism maps $a = (a_0, \ldots, a_{n-1})$ to $p^{\underline{r}}(\sqrt[p^{\underline{r}}]{a_0}, \ldots, \sqrt[p^{\underline{r}}]{a_{n-1}}, 0, \ldots, 0)$ in $G_n^r(\mathrm{Spec} k) \twoheadleftarrow \hat{G}_m^r(\mathrm{Spec} k) \subseteq W_m(k)$.
The map

$$\psi : \bigoplus W_n \xrightarrow{\sim} G_n$$

is an isomorphism of W_n-modules. We now examine the ring-structure on $\bigoplus W_n$ induced by ψ. Denote this multiplication by $*$.

Let $a \in W_n(k) \cong G_n^r$ and $b \in W_n(k) \cong G_n^s$ and choose m large enough. Then a, b correspond to $\psi(a) = p^{\underline{r}}\hat{a}, \psi(b) = p^{\underline{s}}\hat{b}$ for liftings $\hat{a}, \hat{b}$ under $pr \circ v$ and

$$\psi(a)\psi(b) = p^{\underline{r+s}}\hat{a}\hat{b} = p^{\underline{(r,s)}}p^{\underline{r+s}}\hat{a}\hat{b} = p^{\underline{(r,s)}}\psi(ab) = \psi(p^{\underline{(r,s)}}ab)$$

It follows that $a_r * b_s = p^{\underline{(r,s)}}ab$ which shows that

$$W_n(k)[f, v]/(fv - p) \xrightarrow{\sim} G_n(k)$$

with $f \mapsto 1$ in degree 1 and $v \mapsto 1$ in degree -1 is an isomorphism of W_n-algebras. Indeed it is obviously

$$W_n[f, v]/(fv - p) \xrightarrow{\sim} \bigoplus W_n$$

with $f \mapsto 1$ in degree 1 and $v \mapsto 1$ in degree -1 an isomorphism of W_n-modules. The preimage of a_r (resp. b_s) is $a_r f^{\underline{r}} v^{\underline{-r}}$ (resp. $b_s f^{\underline{s}} v^{\underline{-s}}$), and it holds

$$\begin{aligned} a_r f^{\underline{r}} v^{\underline{-r}} b_s f^{\underline{s}} v^{\underline{-s}} &= a_r b_s f^{\underline{r}+\underline{s}} v^{\underline{-r}+\underline{-s}} \\ &= a_r b_s f^{\underline{r+s}} v^{\underline{-r-s}} f^{\underline{(r,s)}} v^{\underline{(-r,-s)}} \\ &= a_r b_s f^{\underline{r+s}} v^{\underline{-r-s}} p^{\underline{(r,s)}} \end{aligned}$$

which is the preimage of $a_r * b_s$. This shows compatibility with multiplication.

Proposition 4.2.9. *(Strictness and rigidity) Let $n \in \mathbb{N}$.*
(i) The map

$$(f_r^\infty, v_r^\infty) : G_n^r \longrightarrow G_n^\infty \oplus G_n^{-\infty}$$

is a monomorphism for all r.
(ii) The sequences

$$G_n \xrightarrow{f^n} G_n \xrightarrow{v^n} G_n$$

and

$$G_n \xrightarrow{v^n} G_n \xrightarrow{f^n} G_n$$

are exact.

Proof. Always choose m large enough.
(i) Let $x \in \hat{G}_m^r$, such that $v^s x = 0$ and $f^s x = 0$, where without loss of generality $s \geq n$. This implies that

$$x = p^n y$$

locally and that $p^s x = p^n z$ where $\varphi z = p^{r+s} t$ locally. It follows that

$$p^n z = p^s x = p^{n+s} y$$

and hence

$$p^{n+s} \varphi(y) = p^{n+s+r} t$$

locally. So in $\mathcal{O}_{m-n-s}^{\mathrm{cris}}$ it holds $\varphi(y) = p^r t$, i.e. $x = 0 \in G_n^r$.

(ii) It is clear that the sequences are complexes since all modules are annihilated by p^n and $f^n v^n = v^n f^n = p^n$.
So let $x \in \hat{G}_m^r$, such that $v^n x = 0$ in G_n^{r+n}. This means that $x = p^n y$ with $\varphi(y) = p^{\underline{r-n}} z$ locally. This is equivalent to saying that x is the image of y under f^n.
Now let $x \in \hat{G}_m^r$ with $f^n x = 0$. It follows that $p^n x = p^n y$ with $y \in \hat{G}_m^{r+n}$, i.e. $\varphi y = p^{\underline{r+n}} z$ for a z. This implies $x = y$ in $\hat{G}_{m-n}^r$, so x is the image of y under v^n. □

Corollary 4.2.10. G_1 *is rigid.*

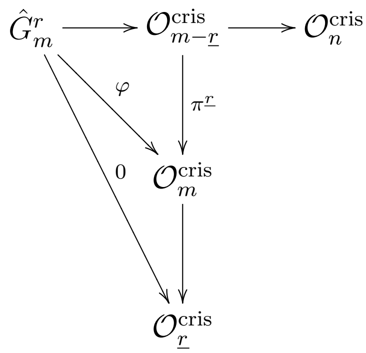
Proof. Clear. □

Definition 4.2.11. (Due to Fontaine) There is a (non-graded) surjective ringhomomorphism (see 4.2.14)

$$\varphi : G_n \to \mathcal{O}_n^{\mathrm{cris}}$$

constructed as follows:
Let $m \geq n + r$ and $r \in \mathbb{Z}$. The composition $\hat{G}_m^r \xrightarrow{\varphi} \mathcal{O}_m^{\mathrm{cris}} \xrightarrow{\nu} \mathcal{O}_{\underline{r}}^{\mathrm{cris}}$ is 0 by definition, so the

commutative diagram



gives a map $\hat{G}^r_m \to \mathcal{O}^{\mathrm{cris}}_{m-r} \to \mathcal{O}^{\mathrm{cris}}_n$. Let φ_r be defined by this map followed by multiplication with $p^{\underline{-r}}$:

$$\varphi_r : \hat{G}^r_m \to \mathcal{O}^{\mathrm{cris}}_{m-r} \to \mathcal{O}^{\mathrm{cris}}_n \overset{p^{\underline{-r}}}{\to} \mathcal{O}^{\mathrm{cris}}_n$$

Remark 4.2.12. Let $r \in \mathbb{Z}$. For $x \in \hat{G}^r_m$, so that $\varphi x = p^r y$ with $y \in \mathcal{O}^{\mathrm{cris}}_m$, it holds

$$\varphi_r(x) = p^{\underline{-r}}(\text{image of } y \text{ in } \mathcal{O}^{\mathrm{cris}}_n)$$

since $p^r y = \pi(y')$, where y' is the image of y in $\mathcal{O}^{\mathrm{cris}}_{m-\underline{r}}$ and by definition $\varphi_r(x) = p^{\underline{-r}}(\text{image of } y' \text{ in } \mathcal{O}^{\mathrm{cris}}_n)$.

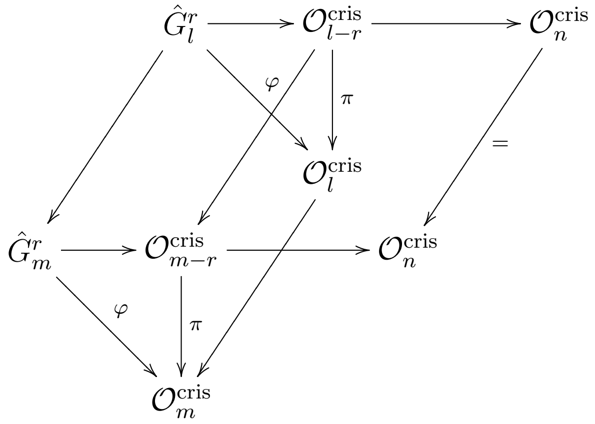
Lemma 4.2.13. *φ is well-defined.*

Proof. Let $r \geq 0$. For $r < 0$ the lemma is clear.
1) Let $x \in G^r_n$ be the image of $y_1, y_2 \in \hat{G}^r_m$, i.e. $y_1 = y_2 + p^n g$ with $g \in \hat{G}^r_m$. Let $\varphi(y_2) = p^r y_2'$. We get

$$\begin{aligned} \varphi(y_1) &= \varphi(y_2 + p^n g) \\ &= \varphi(y_2) + p^n \varphi g \\ &= p^r y_2' + p^r p^n g' \end{aligned}$$

for some element $g' \in \mathcal{O}^{\mathrm{cris}}_m$. But in $\mathcal{O}^{\mathrm{cris}}_n$ it holds $y_2' = y_2' + p^n g'$, so $\varphi_r(x)$ does not depend on the choices of y_1, y_2.

2) Let $x \in G^r_n$ be the image of $y \in \hat{G}^r_l$ and $z \in \hat{G}^r_m$, where we can assume $l \geq r$. Furthermore we can assume that $\hat{G}^r_l \to \hat{G}^r_m$ maps $y \mapsto z$ by 1). The commutative (see 4.1.14) diagram



implies the lemma. □

Lemma 4.2.14. *The map φ from above is a ringhomomorphism.*

Proof. Let $x \in \hat{G}_m^r, y \in \hat{G}_m^s$, so that locally there exist $x', y' \in \mathcal{O}_m^{\text{cris}}$ with $\varphi x = p^{\underline{r}} x'$ and $\varphi y = p^{\underline{s}} y'$. So

$$\begin{aligned} \varphi_r(x)\varphi_s(y) &= p^{\underline{-r}} x' p^{\underline{-s}} y' \\ &= p^{\underline{-r-s}} p^{\underline{(-r,-s)}} x' y' \\ &= p^{\underline{-(r+s)}} p^{\underline{(r,s)}} x' y' \\ &= \varphi_{r+s}(xy) \end{aligned}$$

since $\varphi(xy) = p^{\underline{r}+\underline{s}} x'y' = p^{\underline{r+s}}(p^{\underline{(r,s)}} x'y')$. □

Remark 4.2.15. (i) This map may be seen as "φ followed by division by p^r" (an inverse of π). Indeed it is by definition

$$\hat{G}_m^r = \{x \in \mathcal{O}_m^{\text{cris}} | \varphi(x) = \pi^{\underline{r}}(y) \text{ for a (unique) } y \in \mathcal{O}_{m-r}^{\text{cris}}\}$$

and $\pi^{\underline{r}}(y) = p^{\underline{r}}\hat{y}$ for a lifting $\hat{y}$ of y in $\mathcal{O}_m^{\text{cris}}$. Thus for an element in G_n^r represented by $x \in \hat{G}_m^r$ one has

$$\varphi_r(x) = \pi^{-1}(\varphi(x))$$

(ii) For a perfect field k and $n = 1$ this reads as

$$\varphi : k[f, v]/(fv) \to k$$

which is the absolute frobenius on k and maps f to 1 and v to 0. We can write φ as composition

$$\varphi : k[f, v]/(fv) \xrightarrow{\varphi} k[f, v]/(fv, f-1) \xrightarrow{\sim} k$$

Lemma/Definition 4.2.16. *(i) The map φ is compatible with f-multiplication, i.e. for $g \in G_n$ it holds:*

$$\varphi(fg) = \varphi(g)$$

(ii) The image of $\varphi_r : G_n^r \to \mathcal{O}_n^{\text{cris}}$ is denoted by $F_r\mathcal{O}_n^{\text{cris}}$. These images define an ascending filtration on $\mathcal{O}_n^{\text{cris}}$.

Proof. (i) is clear since φ is multiplicative and $\varphi(f) = 1$.
(ii) By the first part φ_r factors over φ_{r+1}, so the image of φ_r is contained in the image of φ_{r+1}. □

We now restrict attention to the case mod p, i.e. in the following we assume $n = 1$.

Lemma 4.2.17. *(i) The kernel of* φ_r *equals* $\ker\varphi$ *(With* φ *coming from* φ *on* $\mathcal{O}^{\text{cris}}_{r+1}$ *if we identify* G^r_1 *as quotient of* $\hat{G}^r_{r+1} \subseteq \mathcal{O}^{\text{cris}}_{r+1}$*). This is also equal to the kernel of* $f : G^r_1 \to G^{r+1}_1$*.*
(ii) There is an exact sequence

$$0 \to \ker\varphi \to \hat{G}^r_{r+1} \xrightarrow{\varphi_r} F_r\mathcal{O}^{\text{cris}}_1 \to 0$$

Proof. (i) The kernel of φ on $\mathcal{O}^{\text{cris}}_{r+1}$ is contained in $\hat{G}^r_{r+1}$. Let $x \in \hat{G}^r_{r+1}$, which means by definition $\varphi x = p^r y = \pi(y)$, and $y \in \mathcal{O}^{\text{cris}}_1$ is $\varphi_r(x)$. Since π is a monomorphism, $\varphi_r(x) = y = 0 \Leftrightarrow \varphi(x) = 0$. For the second part of (i) let $x \in G^r_1$ be represented by $x \in \hat{G}^r_{r+2}$. Then $\varphi_r(x) = 0 \Leftrightarrow \varphi(x) = 0 \in \mathcal{O}^{\text{cris}}_{r+1} \Leftrightarrow x \in \hat{G}^{r+1}_{r+2}$. This is equivalent to $px = 0 \in G^{r+1}_1$: In general it holds for $g \in \hat{G}^r_m$ and $m \geq n + r$, that $p^n g = 0 \in G^r_n \Leftrightarrow g \in \hat{G}^r_m$. If $p^n g = p^n y$ with $y \in \hat{G}^r_m$, it follows that $x = y$ in $\mathcal{O}^{\text{cris}}_{m-n}$, and since $m - n \geq r$, we get that $\varphi(g) = \varphi(y) = 0 \in \mathcal{O}^{\text{cris}}_r$, i.e. $g \in \hat{G}^r_m$.
(ii) clear. □

Proposition 4.2.18. *(Cartier isomorphism) There is an exact sequence*

$$0 \to \mathcal{J}^{[r+1]}_1 \to \mathcal{J}^{[r]}_1 \to F_r\mathcal{O}^{\text{cris}}_1 / F_{r-1}\mathcal{O}^{\text{cris}}_1 \to 0$$

Proof. By Fontaine, see [FJ] 3.3.4. on p.14. □

The following two sequences are claimed to be exact by Fontaine. We give the proof here:

Lemma 4.2.19. *There is an exact sequence*

$$0 \to F_r\mathcal{O}^{\text{cris}}_1 \to G^{r+1}_1 \xrightarrow{v} G^r_1 \xrightarrow{\varphi_r} F_r\mathcal{O}^{\text{cris}}_1 \to 0.$$

Proof. There is an exact sequence:

$$0 \to \hat{G}^{r+1}_{r+2} \to \hat{G}^r_{r+2} \xrightarrow{\varphi_r} F_r\mathcal{O}^{\text{cris}}_1 \to 0$$

To see exactness in the middle, let $x \in \hat{G}^r_{r+2}$, so that $\varphi(x) = p^r\hat{y} = \pi(y)$ for some $\hat{y}, y$, with $\pi : \mathcal{O}^{\text{cris}}_2 \to \mathcal{O}^{\text{cris}}_{r+2}$. The image of x under φ_r is by 4.2.12 the image of y in $\mathcal{O}^{\text{cris}}_1$. This image is 0 if and only if $y = py'$ for some y', and this happens if and only if $\hat{y} = p\tilde{y}$ for some $\tilde{y}$. At last $\hat{y}$ is divisible by p if and only if $\varphi(x) = p^{r+1}\tilde{y} \Leftrightarrow x \in \hat{G}^{r+1}_{r+2}$.

We apply the snake-lemma to p-multiplication of this sequence and get the exact sequence

$$(\hat{G}^r_{r+2})_p \to F_r\mathcal{O}^{\text{cris}}_1 \to G^{r+1}_1 \to G^r_1 \to F_r\mathcal{O}^{\text{cris}}_1 \to 0.$$

The first map is 0:
Let $x \in \hat{G}^r_{r+2}$ with $px = 0$, which implies $x = \pi^{r+1}y$ for some y. So we can write $\varphi(x) = \pi^r\pi^1\varphi(y)$. By definition $\varphi_r(x)$ is the image of $\pi^1\varphi(y)$ in $\mathcal{O}^{\text{cris}}_1$, which is 0. □

Remark 4.2.20. To prove the next lemma we need to know exactly how $F_r\mathcal{O}_1^{\text{cris}}$ is identified with $\ker v_{r+1}$. Let $r \geq 0$:
We have the commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & & & F_r\mathcal{O}_1^{\text{cris}} & \dashrightarrow & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \hat{G}_{r+2}^{r+1} & \longrightarrow & \hat{G}_{r+2}^{r} & \xrightarrow{\varphi_r} & F_r\mathcal{O}_1^{\text{cris}} & \longrightarrow & 0 \\
 & & \downarrow p & & \downarrow p & & \downarrow p=0 & & \\
0 & \longrightarrow & \hat{G}_{r+2}^{r+1} & \longrightarrow & \hat{G}_{r+2}^{r} & \xrightarrow{\varphi_r} & F_r\mathcal{O}_1^{\text{cris}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & \dashrightarrow & G_1^{r+1} & \xrightarrow{v_{r+1}} & G_1^{r} & \longrightarrow & F_r\mathcal{O}_1^{\text{cris}} & \longrightarrow & 0.
\end{array}$$

Let $x \in \ker v_{r+1} \subseteq G_1^{r+1}$ be given by $x \in \hat{G}_{r+2}^{r+1}$. Let $x = py$ with $y \in \hat{G}_{r+2}^{r}$, so that there is a z with $\varphi y = p^r z$. The image of x in $F_r\mathcal{O}_1^{\text{cris}}$ is by defintion $\varphi_r(y) = z$.
But $\varphi x = \varphi(py) = p^{r+1}z$, so the definition of φ_{r+1} yields $\varphi_{r+1}(x) = z$, i.e. $x \mapsto \varphi_{r+1}(x)$ in $F_r\mathcal{O}_1^{\text{cris}}$. Informally "$\varphi_{r+1}(x) = \frac{\varphi x}{p^{r+1}} = \frac{p\varphi y}{p^{r+1}} = \frac{\varphi y}{p^r} = \varphi_r(y)$".

Lemma 4.2.21. *There is an analogous exact sequence*

$$0 \to F^s\mathcal{O}_1^{\text{cris}} \to G_1^{s-1} \xrightarrow{f} G_1^s \to F^s\mathcal{O}_1^{\text{cris}} \to 0.$$

Proof. Let $F_s = F_s\mathcal{O}_1^{\text{cris}}$ and $F^s = F^s\mathcal{O}_1^{\text{cris}}$. We proceed by induction.
By definition $F^r\mathcal{O}_1^{\text{cris}} = \mathcal{O}_1^{\text{cris}}$ for $r \leq 0$, so assume that the sequence is exact for all $r \leq s$ and $s \geq 0$. We can apply the snake lemma to multiplication by v of

$$0 \to \ker f \to G_1^r \to \text{im} f \to 0.$$

This gives us the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & F_{r-1} & \longrightarrow & F_r & \dashrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker f_r & \longrightarrow & G_1^r & \xrightarrow{f} & \text{im} f_r & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow v & & \downarrow v=0 & & \\
0 & \longrightarrow & F^r & \longrightarrow & G_1^{r-1} & \xrightarrow{f} & \text{im} f_{r-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \varphi_{r-1} & & \downarrow \sim & & \\
 & \dashrightarrow & C & \longrightarrow & F_{r-1} & \xrightarrow[\sim]{\overline{f}} & F_{r-1} & \longrightarrow & 0.
\end{array}$$

First, the middle column is exact by 4.2.19. By rigitiy of G_1, it follows that $v|_{\text{im} f_r} = 0$ and with 4.2.19 we get $\text{im} f_r = \ker v_{r+1} \cong F_r$ and hence exactness of the right column.

By the preceding remark $\mathrm{im} f_{r-1}$ is identified with F_{r-1} via φ_r, hence $\varphi_r f = \varphi_{r-1}$ implies that $\overline{f}$ is an isomorphism.
It follows that $C \cong F_r/F_{r-1}$ and $\ker f_r \cong F^{r+1}$ by the Cartier-isomorphism 4.2.18.

Multiplication by v and snake-lemma give also the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_r & \xrightarrow{\sim} & F_r & \longrightarrow & N & \dashrightarrow & \\
 & & \downarrow{\scriptstyle \sim} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{im} f_r & \longrightarrow & G_1^{r+1} & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle v=0} & & {\scriptstyle v}\downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{im} f_{r-1} & \longrightarrow & G_1^r & \longrightarrow & F^r & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sim} & & \downarrow & & \downarrow & & \\
 & \dashrightarrow & F_{r-1} & \longrightarrow & F_r & \longrightarrow & D & \longrightarrow & 0.
\end{array}$$

Again, by 4.2.19, the middle column is exact. By rigidity, $v|_{\mathrm{im} f} = 0$ and with 4.2.19, $\mathrm{im} f_r = \ker v_{r+1} \cong F_r$. Together with the injectivity of $F_{r-1} \to F_r$ we get $N = 0$. This implies that $D \cong F_r/F_{r-1}$ and hence $C \cong F^{r+1}$ by Cartier-isomorphism. □

Lemma 4.2.22. *Let $m, n > 0$. There is an exact sequence of graded modules*

$$0 \to G_n \xrightarrow{\overline{\pi}} G_{m+n} \xrightarrow{\nu} G_m \to 0.$$

Proof. Let $\overline{\pi}$ defined as quotient of the restriction of $\pi^m : \mathcal{O}_{n+r}^{\mathrm{cris}} \to \mathcal{O}_{n+m+r}^{\mathrm{cris}}$ to $\hat{G}_{n+r}^r$, which has image in $\hat{G}_{m+n+r}^r$. Let ν be defined by the identity on $\hat{G}_{m+n+r}^r$.
Surjectivitiy of ν is immediate.
The following considerations are locally:
To check injectivity of $\overline{\pi}$, let $x \in \hat{G}_{n+r}^r$ such that $\pi(x) = p^m \hat{x}$ is divideable by p^{m+n} in $\hat{G}_{m+n}^r$, say $\pi(x) = p^{m+n} z$, where $\varphi z = 0 \in \mathcal{O}_r^{\mathrm{cris}}$. This means $\pi(x) = \pi(p^n z)$ and the injectivity of π implies that $x = p^n z = 0$ in G_n^r.
Let $x \in G_n^r \twoheadleftarrow \hat{G}_{n+r}^r$, then the image of x in G_m^r is $p^m \hat{x} = 0$, where $\hat{x}$ is a lifting of x in $\hat{G}_{m+n+r}^r$ (First choose a lifting in $\mathcal{O}_{m+n+r}^{\mathrm{cris}}$, then it is clear that it is in $\hat{G}_{m+n+r}^r$). So the sequence is a complex.
Finally let $x \in \hat{G}_{m+n+r}^r$ such that $\nu(x) = 0 \Leftrightarrow x = p^m y$ for $y \in \hat{G}_{m+n+r}^r$. Obviously $x = \pi(y')$ where y' is the image of y in $\hat{G}_{n+r}^r$. □

4.3 p-good algebras and coverings

In this subsection we introduce a class of k-algebras, which are very useful for our considerations. In sites contained in the syntomic site, with topologies finer than p-topology, there are enough p-good-algebras. More exactly: Every syntomic k-algebra can be p-covered with p-good algebras. The sections of $\mathcal{O}_1^{\mathrm{cris}}$, $\mathcal{J}_1^{[r]}$ and $\mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}$ can be computed explicitely over p-good algebras.

Definition 4.3.1. Let A be a k-algebra. We denote by $A^{p^{-1}}$ the k-algebra A with structural morphism $k \to A \overset{x \mapsto x^p}{\to} A$.

Definition 4.3.2. (Fontaine) (i) A smooth k-algebra $\mathcal{A}$ is called *good* if it admits a *system of parameters*, i.e. if there are elements $t_1, \dots, t_d \in \mathcal{A}$ such that $dt_1, \dots, dt_d$ form a $\mathcal{A}$-basis of $\Omega^1_{\mathcal{A}/k}$.
(ii)A k-algebra A is called *p-good* if there is a smooth and good k-algebra $\mathcal{A}$, a regular sequence $u_1, \dots, u_n$ and an isomorphism of k-algebras

$$A \cong \mathcal{A}/(u_1^p, \dots, u_n^p).$$

Lemma 4.3.3. *Let $\mathcal{A}$ be a smooth k-algebra. Then the Frobenius-morphism*

$$\mathcal{A} \to \mathcal{A}; x \mapsto x^p$$

is a faithfully flat p-th root-morphism.

Proof. (Based on an idea of Fontaine) Every smooth k-algebra is Zariski-locally good, so we can assume that $\mathcal{A}$ is good. Let $t_1, \dots, t_d$ be a system of parameters for $\mathcal{A}$. By [Ty] III.Thm 1 the elements $t_1, \dots, t_d$ form a p-basis of $\mathcal{A}$, this means $\{t_1^{s_1} \cdots t_d^{s_d} | 0 \leq s_i < p\}$ is linearly independent over $k[\mathcal{A}^p] = \mathcal{A}^p$ and

$$\mathcal{A} = k[\mathcal{A}^p, t_1, \dots, t_d] = \mathcal{A}^p[t_1, \dots, t_d],$$

where $\mathcal{A}^p$ denotes p-th powers of elements of $\mathcal{A}$. Let

$$\alpha : \mathcal{A}[T_1, \dots, T_d]/(T_1^p - t_1, \dots, T_d^p - t_d) \to \mathcal{A}$$

be given by $x \mapsto x^p$ on $\mathcal{A}$ and $T_i \mapsto t_i$, which is obviously well-defined. Let $I = \{s = (s_1, \dots, s_d) | 0 \leq s_i < p\}$. With $t_1, \dots, t_d$ being a p-basis, every $x \in \mathcal{A}$ can be written uniquely as

$$x = \sum_{s \in I} a_s^p t_1^{s_1} \cdots t_d^{s_d}.$$

Let

$$\beta(x) := \sum_{s=(s_1,\dots,s_d)} a_s T_1^{s_1} \cdots T_d^{s_d} \in \mathcal{A}[T_1, \dots, T_d]/(T_1^p - t_1, \dots, T_d^p - t_d).$$

One sees immediately that β is an inverse for α and that the following triangle commutes

$$\begin{array}{ccc} \mathcal{A}[T_1, \dots, T_d]/(T_1^p - t_1, \dots, T_d^p - t_d) & \xrightarrow[\sim]{\alpha} & \mathcal{A} \\ & \nwarrow \qquad \nearrow {\scriptstyle x \mapsto x^p} & \\ & \mathcal{A} & \end{array}$$

which completes the proof. ∎

Lemma 4.3.4. *Let A be a syntomic k-algebra. Then there is a p-covering of A by p-good k-algebras.*

Proof. Zariski-locally A is given as $\mathcal{A}/(f_1, \dots, f_n)$ where $\mathcal{A}$ is a smooth and good k-algebra and $f_1, \dots, f_n$ is a regular sequence. Let $\mathcal{B}$ be $\mathcal{A}^{p^{-1}}$. The square

$$\begin{array}{ccc} \mathcal{A}/(f_1, \dots, f_n) & \longrightarrow & \mathcal{B}/(f_1^p, \dots, f_n^p) \\ \uparrow & & \uparrow \\ \mathcal{A} & \xrightarrow{x \mapsto x^p} & \mathcal{B} \end{array}$$

is cartesian. So the top morphism is a faithfully flat p.-th root since $\mathcal{A} \to \mathcal{B}$ is, by 4.3.3. The structural morphism of $\mathcal{B}$ is Frobenius on k, which is an isomorphism, followed by the smooth structural morphism of $\mathcal{A}$, so $\mathcal{B}$ is smooth. The sequence $f_1^p, \dots, f_n^p$ is regular in $\mathcal{B}$ by flatness ([EGA4.1] 0.15.2.5) and the lemma follows. □

We are interested in computing Čech-cohomoloy of $\mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}$. We can do this only for very special coverings, but this will suffice to give an isomorphism of cohomology of $\mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}$ for all p-crystalline topologies τ.

Lemma 4.3.5. *Let A be a p-good algebra, identified with $\mathcal{A}/(t_1^p, \dots, t_n^p)$ like in the definition. Let*

$$B_0 = A[Y_1, \dots, Y_r]/(v_1, \dots, v_m)$$

be a syntomic faithfully flat A-algebra and assume that $v_1, \dots, v_m$ form a regular sequence in $A[Y_1, \dots, Y_r]$.
There is a commutative triangle

$$\begin{array}{ccc} A = \mathcal{A}/(t_1^p, \dots, t_n^p) & \longrightarrow & \mathcal{A}[Y_1, \dots, Y_r]/(t_1^p, \dots, t_n^p, v_1, \dots, v_m) = B_0 \\ & \searrow^{\alpha} \quad \swarrow_{\beta} & \\ & \mathcal{B}[Y_1, \dots, Y_r]/(u_1^p, \dots, u_n^p, v_1^p, \dots, v_m^p) = B & \end{array}$$

with the following properties:
(i) $\mathcal{B}$ is the smooth k-Algebra $\mathcal{A}^{p^{-1}}$.
(ii) $(u_1^p, \dots, u_n^p, v_1^p, \dots, v_m^p)$ is a regular sequence in $\mathcal{B}[Y_1, \dots, Y_r]$.
(iii) β is a faithfully flat p-th root-morphism.
(iv) $\alpha(t_i) = u_i$.
In other words, there is a p-th root-refinement of $A \to B_0$ by a p-good algebra B.

Proof. Define $\mathcal{B} = \mathcal{A}^{p^{-1}}$ and $\hat{\beta} : \mathcal{A}[Y_1, \dots, Y_r] \to \mathcal{B}[Y_1, \dots, Y_r]$ by $x \mapsto x^p$. By 4.3.3 the morphism $\hat{\beta}$ is a faithfully flat p-th root. Let $u_i = \beta(t_i) = t_i^p$, then by flatness of $\hat{\beta}$, the sequence $(u_1^p, \dots, u_n^p, v_1^p, \dots, v_m^p)$ is regular ([EGA4.1] 0.15.2.5).

The commutative square

$$\begin{array}{ccc} \mathcal{B}[Y_1, \dots, Y_r] & \longrightarrow & \mathcal{B}[Y_1, \dots, Y_r]/(u_1^p, \dots, u_n^p, v_1^p, \dots, v_m^p) \\ \uparrow \hat{\beta} & & \uparrow \beta \\ \mathcal{A}[Y_1, \dots, Y_r] & \longrightarrow & A[Y_1, \dots, Y_r]/(t_1^p, \dots, t_n^p, v_1, \dots, v_m) \end{array}$$

is cartesian, so β is a faithfully flat p-th root. □

Definition 4.3.6. We call a covering $A \to B$ of a p-good algebra like in the previous lemma a *p-good covering.*

Lemma 4.3.7. *Let A be a p-good algebra, identified with $\mathcal{A}/(t_1^p, \dots, t_n^p)$. Let*

$$\mathfrak{U} = \{U_j \to \mathrm{Spec}A\}_{j \in J}$$

be a syntomic covering. Then

$\mathfrak{U}$ possesses a refinement in p-topology by a finite covering

$$\mathfrak{V} = \{A \to B_i\}_{i \in I}$$

of the following form:

$$A = \mathcal{A}/(t_1^p, \dots, t_n^p) \to B_i = \mathcal{B}_i[X_1, \dots, X_{s_j}]/(u_1^p, \dots, u_n^p, v_{i,1}^p, \dots, v_{i,m_i}^p)$$

where $\mathcal{B}_i = \mathcal{A}_{f_i}^{p^{-1}}$ for an element $f_i \in \mathcal{A}$ and $u_1^p, \dots, u_n^p, v_{i,1}^p, \dots, v_{i,m_i}^p$ is a regular sequence in $\mathcal{B}_i[X_1, \dots, X_{s_i}]$ and such that $t_i \mapsto u_i$.
Furthermore, for $i_0, \dots, i_n \in I$, the tensor product $B_{i_0} \otimes \dots \otimes B_{i_n}$ is also p-good.

Proof. There is a covering of $\mathrm{Spec}A$ by principal open subsets $\mathrm{Spec}A_{f_i}$ such that $\mathfrak{U}$ can be refined by a covering of the form $\{A \to A_{f_i}[X_1, \dots, X_{s_i}]/(v_{i,1}, \dots, v_{i,m_i})\}$ with a regular sequence $v_{i,1}, \dots, v_{i,m_i}$ by considering the local definition of syntomic morphisms. Since $\mathrm{Spec}A$ is quasi-compact and syntomic morphisms are open we can assume I finite and applying 4.3.5 implies the lemma.
The last statement is clear, for details see the proof of 4.4.5. □

4.4 The comparison theorem for G_n

In this subsection we shall show as main result that cohomology of G_n is independent of the choice of a p-crystalline topology. Using homological algebra and some exact sequences of subsection 4.2, we can easily see that it suffices to show this independence for $\mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}$. This follows from the exactness of the Čech-sequence for p-good coverings, which we shall prove with explicit computations.

Definition 4.4.1. We say that an abelian syntomic sheaf F is *p-good* if the natural morphism

$$H(X_{syn}, F) \to H(X_\tau, F)$$

is an isomorphism for all p-crystalline topologies τ which are coarser than syntomic topology.

Remark 4.4.2. (i) If in an exact sequence (starting and ending with 0) all terms except one are known to be p-good, then the remaining one also has to be p-good.
(ii) In section 3 we have shown that $\mathcal{O}_n^{\mathrm{cris}}$ is p-good.
(iii) Sheaves arising from coherent modules on X_{zar} are p-good, see e.g. [M]III.3.7. (We say that a sheaf F comes from the coherent module M if $F(Y \xrightarrow{f} X) = f^*M(Y) = (\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}M)(Y))$. In particular the structural sheaf $\mathcal{O}$ is p-good.

Lemma 4.4.3. *If we assume $\mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}$ to be p-good, then G_1^r is p-good.*

Proof. (1) The sheaves $\mathcal{O}$ and $\mathcal{O}_n^{\mathrm{cris}}$ are p-good as noted above. The exact sequence

$$0 \to \mathcal{J}_1^{[1]} \to \mathcal{O}_1^{\mathrm{cris}} \to \mathcal{O} \to 0$$

implies that $\mathcal{J}_1^{[1]}$ is p-good.

(2) The image of $\varphi : \mathcal{O}_1^{\mathrm{cris}} \to \mathcal{O}_1^{\mathrm{cris}}$ is $\mathcal{O}$ by 4.1.7. The definition of $\hat{G}_m^1$

$$0 \to \hat{G}_m^1 \to \mathcal{O}_m^{\mathrm{cris}} \xrightarrow{\varphi v} \mathrm{im}\varphi \to 0$$

implies that $\hat{G}_m^1$ is p-good. Consider the monomorphism of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & \mathcal{O}_n^{\mathrm{cris}} & \xrightarrow{\gamma} & (\mathcal{O}_1^{\mathrm{cris}})_{p^n} \\ & & \downarrow & & \downarrow \pi & & \downarrow \\ 0 & \longrightarrow & \hat{G}_m^1 & \longrightarrow & \mathcal{O}_m^{\mathrm{cris}} & \xrightarrow{\varphi v} & \mathcal{O}_1^{\mathrm{cris}} \end{array}$$

where the first row is p^n-torsion of the second one. With $m \geq 1+n$ we have $\varphi v \pi = 0$ so the injectivity of the vertical morphisms implies that $\gamma = 0$. Hence $K \cong \mathcal{O}_n^{\mathrm{cris}}$ is p-good. The exact sequence

$$0 \to K \to \hat{G}_m^1 \xrightarrow{p^n} \hat{G}_m^1 \to G_n^1 \to 0$$

shows that G^1_n is p-good.

(3) The Cartier-isomorphism 4.2.18

$$0 \to \mathcal{J}_1^{[r+1]} \to \mathcal{J}_1^{[r]} \to F_r\mathcal{O}_1^{\text{cris}}/F_{r-1}\mathcal{O}_1^{\text{cris}} \to 0$$

together with the assumption implies that $F_r\mathcal{O}_1^{\text{cris}}/F_{r-1}\mathcal{O}_1^{\text{cris}}$ is p-good. By induction over r it follows that $F_r\mathcal{O}_1^{\text{cris}}$ is p-good: $F_0\mathcal{O}_1^{\text{cris}}$ is the image of $\varphi_0 : G^0_1 \to \mathcal{O}_1^{\text{cris}}$, but φ_0 is the same as $\varphi : \mathcal{O}_1^{\text{cris}} \to \mathcal{O}_1^{\text{cris}}$.

(4) The exactness (4.2.19) of

$$0 \to F_r\mathcal{O}_1^{\text{cris}} \to G_1^{r+1} \xrightarrow{v} G_1^r \to F_r\mathcal{O}_1^{\text{cris}} \to 0$$

implies with an induction over r that G^r_1 is p-good. □

The rest of the subsection is devoted to showing that $\mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}$ is p-good, with help of p-good coverings.

Lemma 4.4.4. *Let A be a p-good algebra, identified with $\mathcal{A}/(u_1^p, \ldots, u_d^p)$ like in the definition. Then $\mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}$ is a free A-module with basis $\{\gamma_{m_1}(u_1)\cdots\gamma_{m_d}(u_d) | \sum m_i = r\}$.*

Proof. By Fontaine, see [FJ] Prop. 3.3.1. on p.11. □

Proposition 4.4.5. *Let*

$$\{A = \mathcal{A}/(t_1^p, \ldots, t_n^p) \to B_i = \mathcal{B}_i[X_1, \ldots, X_{s_i}]/(u_1^p, \ldots, u_n^p, v_{i,1}^p, \ldots, v_{i,m_i}^p)\}_{i \in I}$$

be a covering like in 4.3.7. Then the Čech-Complex

$$0 \to \mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}(A) \to \prod \mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}(B_i) \to \prod J_1^{[r]}/\mathcal{J}_1^{[r+1]}(B_i \otimes_A B_j) \to \ldots$$

is exact.

Proof. Let $\eta = (i_0, \ldots, i_l) \in I^{l+1}$ and $B_\eta = B_{i_0} \otimes_A \ldots \otimes_A B_{i_l}$. Let $\mathcal{B}_\eta = \mathcal{B}_{i_0} \otimes_A \ldots \otimes_A \mathcal{B}_{i_l}$ and $S = S_\eta = s_{i_0} + \ldots + s_{i_l}$ and $M = M_\eta = n + m_{i_0} + \ldots + m_{i_l}$. Let $\iota_* : B_{i_*} \to B_\eta$ be the canonical inclusions. Define

$$\begin{aligned}
Y_1 &= \iota_0(X_1) \\
&\vdots \\
Y_{s_{i_0}} &= \iota_0(X_{s_{i_0}}) \\
Y_{s_{i_0}+1} &= \iota_1(X_1) \\
&\vdots \\
Y_{s_{i_0}+s_{i_1}} &= \iota_1(X_{s_{i_1}}) \\
&\vdots \\
Y_{S_\eta} &= \iota_l(X_{s_{i_l}})
\end{aligned}$$

and define

$$\begin{aligned} w_1 &= t_1 \\ &\vdots \\ w_n &= t_n \\ w_{n+1} &= \iota_0(v_{i_0,1}) \\ &\vdots \\ w_{n+m_{i_0}} &= \iota_0(v_{i_0,m_{i_0}}) \\ w_{n+m_{i_0}+1} &= \iota_1(v_{i_1,1}) \\ &\vdots \\ w_{n+m_{i_0}+m_{i_0}} &= \iota_1(v_{i_1,m_{i_1}}) \\ &\vdots \\ w_{M_\eta} &= \iota_l(v_{i_l,m_{i_l}}) \end{aligned}$$

Then we have

$$B_\eta \cong \mathcal{B}_\eta[Y_1, \dots, Y_{S_\eta}]/(w_1^p, \dots, w_{M_\eta}^p)$$

By the lemma above

$$\mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}(B_\eta) \cong \bigoplus_{\sum m_j = r} B_\eta \gamma_{m_1}(w_1) \cdots \gamma_{m_M}(w_M)$$

The key observation is now that this module is isomorphic to the r-th graded piece of

$$B_{i_0}[Z_1, \dots, Z_{n+m_{i_0}}] \otimes_{A[Z_1,\dots,Z_n]} \cdots \otimes_{A[Z_1,\dots,Z_n]} B_{i_l}[Z_1, \dots, Z_{n+m_{i_l}}]$$

which is seen easily by counting the bases:
It holds $B_{i_0}[Z_1, \dots, Z_{n+m_{i_0}}] \otimes_{A[Z_1,\dots,Z_n]} \cdots \otimes_{A[Z_1,\dots,Z_n]} B_{i_l}[Z_1, \dots, Z_{n+m_{i_l}}] \cong B_\eta[W_1, \dots, Z_{M_\eta}]$, which follows from the easy formula: $R_1[Z, X] \otimes_{R[Z]} R_2[Z, Y] \cong R_1 \otimes_R R_2[Z, X, Y]$ for R-algebras R_1 and R_2. The isomorphism maps $\gamma_{m_1}(w_1) \cdots \gamma_{m_M}(w_M)$ to $W_1^{m_1} \cdots W_M^{m_M}$.

Note that this isomorphism is compatible with the inclusions of $\otimes$-product, and hence with Čech-complexes: Let, for simplicity of notation, $\theta = (i_j, \dots, i_n, j)$ and consider the inclusion of the first n factors $B_\eta \to B_\theta$:
The image of w_i is w_i for all i, so the image of $\gamma_{m_1}(w_1) \cdots \gamma_{m_M}(w_M)$ under $\mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}(B_\eta) \to \mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}(B_\theta)$ is $\gamma_{m_1}(w_1) \cdots \gamma_{m_M}(w_{M_\eta}) \gamma_0(w_{m_\eta+1}) \cdots \gamma_0(w_{m_\theta}) = \gamma_{m_1}(w_1) \cdots \gamma_{m_M}(w_{M_\eta})$.
The image of W_i under $B_\eta[W_1, \dots, W_{M_\eta}] \to B_\theta[W_1, \dots, W_{M_\theta}]$ is W_i, so $W_1^{m_1} \cdots W_M^{m_M}$ is mapped to $W_1^{m_1} \cdots W_M^{m_M}$ under $B_\eta[W_1, \dots, W_{M_\eta}]^r \to B_\theta[W_1, \dots, W_{M_\theta}]^r$.

So it is enough to show exactness of the graded complex

$$\begin{aligned} 0 \to A[Z_1, \dots, Z_n] &\to \prod B_i[Z_1, \dots, Z_{n+m_i}] \to \\ &\to \prod B_i[Z_1, \dots, Z_{n+m_i}] \otimes_{A[Z_1,\dots,Z_n]} B_j[Z_1, \dots, Z_{n+m_j}] \to \cdots \end{aligned}$$

This complex is the Čech-complex of $\mathcal{O}$ with respect to the covering $\{A[Z_1,\dots,Z_n] \to B_i[Z_1,\dots,Z_{n+m_i}]\}$ and by 4.4.7 it is isomorphic to the Čech-complex of $\mathcal{O}$ with respect to the faithfully flat covering $\{A[Z_1,\dots,Z_n] \to B\}$ with $B = \prod B_i[Z_1,\dots,Z_{n+m_i}]$. This sequence is well known to be exact ([M] I.2.18). □

Lemma 4.4.6. *Let P be a sheaf on the site $(SYN/k)_{syn}$ and let τ be a p-crystalline topology.*

Assume $\check{H}^q(\mathfrak{U}/V,P) = 0$ for all p-good V and for all coverings $\mathfrak{U}$ in some cofinal system (with respect to τ) for all $q > 0$.
Then it follows that the associated sheaf in p-topology of the presheaf $U \mapsto H^q_\tau(U,P)$ is 0 for all $q > 0$.

Proof. The proof is based on proposition III.2.12 in [M]. Denote with $\underline{H}^q(P)$ as usual the presheaf $U \mapsto H^q(U,P)$ and by $\underline{\check{H}}^q(P)$ the presheaf $U \mapsto \check{H}^q(U,P)$ (in both cases cohomology for the p-crystalline topology τ).
Let $\check{H}^q(\mathfrak{U}/V,P) = 0$ for all p-good V and for all coverings $\mathfrak{U}$ in some cofinal system. We show:

(1) $H^q(V,P) = 0$ for p-good V and $q > 0$:
The assumption implies that $\underline{\check{H}}^q(P)$ takes 0 as value on p-good schemes. Since first Čech-cohomology-group and the first cohomology group of a sheaf agree, this implies that $\underline{H}^1(P)$ is 0 on p-good schemes. We proceed by induction. Let us assume that $\underline{H}^p(P)$ is 0 on p-good schemes for all p with $1 \le p < q$.
Then $\underline{\check{H}}^p(\underline{H}^{q-p}(P))$ is 0 on p-good schemes:
We have $q - p < q$ and so $\underline{H}^{q-p}(P)$ is 0 on p-good schemes. This implies for V a p-good scheme: $\underline{\check{H}}^p(\underline{H}^{q-p}(P))(V) = \varinjlim_{\mathfrak{V}} \check{H}^p(\mathfrak{V}/V, \underline{H}^{q-p}(P)) = 0$, because every covering $\mathfrak{V}$ of V can be refined by a covering $\mathfrak{V}'$, which consists of p-good schemes and is such that for $V_1,\dots,V_n \in \mathfrak{V}'$, also $V1_1 \otimes_V \dots \otimes_V V_n$ is a p-good algebra (see Lemma 4.3.7).
Furthermore $\underline{\check{H}}^q(\underline{H}^0(P)) = \underline{\check{H}}^q(P)$ is 0 on p-good schemes since P is a sheaf and $\underline{\check{H}}^0(\underline{H}^q(P))$ is 0 on p-good schemes by [M] III.2.9.
Thus we have showed that $\underline{\check{H}}^m(\underline{H}^n(P))$ takes 0-values on p-good schemes for all $m+n = q$. Using Čech spectral sequence the induction is completed.

(2) Because every p-covering of a syntomic scheme possesses a p-refinement consisting of p-good schemes, (1) implies $\check{H}^0_p(V, \underline{H}^q(P)) = 0$ for syntomic k-schemes schemes V (where $\check{H}^0_p$ denotes Čech-cohomology with respect to p-coverings). From this, it follows that the associated sheaf is 0. □

Lemma 4.4.7. *Let $\mathfrak{U} = \{A \to A_i\}_{i\in I}$ be a faithfully flat covering of a k-algebra A and let $\mathcal{F}$ be a sheaf on some site over k, where coverings are finer than Zariski-coverings. Assume that I is finite. Then the Čech-complex of $\mathcal{F}$ with respect to $\mathfrak{U}$ is equal to the Čech-complex of $\mathcal{F}$ with respect to $\{A \to \prod A_i\}$.*

Proof. Using that $\mathcal{F}$ is a sheaf, in particular $\mathcal{F}(\prod B_j) = \prod \mathcal{F}(B_j)$ for a finite family of rings (B_j), we calculate :

$$\begin{aligned}\prod_{(i_o,\dots,i_r)} \mathcal{F}(A_{i_0} \otimes \ldots \otimes A_{i_r}) &= \mathcal{F}(\prod_{(i_o,\dots,i_r)} A_{i_0} \otimes \ldots \otimes A_{i_r}) \\ &= \mathcal{F}(\prod A_i \otimes \ldots \otimes \prod A_i)\end{aligned}$$

using that finite sums and products of modules coincide. One easily sees that the maps in the Čech-complexes coincide. □

We do not use the following lemma but it gives a different approach to prove the main theorem of this subsection. The proof would get more technical, while one had to prove 4.4.5 only for a covering which consists of one p-good algebra.

Lemma 4.4.8. *Let A be a p-good algebra and let $\{\mathrm{Spec}A_{f_i} \subseteq \mathrm{Spec}A\}$ be a Zariski-covering of $\mathrm{Spec}A$ with localizations of A by elements f_i. Then the Čech-complex*

$$(*) \quad 0 \to \mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}(A) \to \prod\nolimits_i \mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}(A_i) \to \prod_{i,j} \mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}(A_i \otimes_A A_j) \to \ldots$$

is exact.

Proof. We can write $A = \mathcal{A}/(t_1^p, \ldots, t_n^p)$ with smooth and good $\mathcal{A}$ and a regular sequence t_i. It follows for $f \in A$, that $A_f = \mathcal{A}_f/(t_1^p, \ldots, t_n^p)$, with $\mathcal{A}_f$ smooth and good and (t_i) regular. This implies

$$A_{f_{i_1}} \otimes_A \ldots \otimes_A A_{f_{i_s}} = \mathcal{A}_{f_{i_1}\cdots f_{i_s}}/(t_1^p, \ldots, t_n^p).$$

We abbreviate $A_{i_1 \ldots i_s} = A_{f_{i_1}} \otimes_A \ldots \otimes_A A_{f_{i_s}}$. The description for $\mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}$ over p-good algebras gives

$$\mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}(A_{i_1 \ldots i_s}) = \bigoplus_{(m_1,\dots,m_n),\sum m_i = r} A_{i_1 \ldots i_s} \cong A_{i_1 \ldots i_s}^d$$

where d is the cardinality of $\{(m_1, \ldots, m_n) \in \mathbb{N}^n | \sum m_i = r\}$.
It follows, that the Čech-complex $(*)$ is the same the as the Čech-complex of $\mathcal{O}^d_{\mathrm{Spec}A}$ over $\mathrm{Spec}A$ with respect to the covering $\{\mathrm{Spec}A_{f_i} \subseteq \mathrm{Spec}A\}$. This is well known to be exact, see e.g. [Li] 5.2.2 Lemma 2.17. □

Theorem 4.4.9. *The sheaves G_1^r are p-good.*

Proof. We have to show, by the reduction above (4.4.3), that $P := \mathcal{J}_1^{[r]}/\mathcal{J}_1^{[r+1]}$ is p-good. This follows from the previous lemmas:
Let $\alpha : \mathrm{Spec}k_\tau \to \mathrm{Spec}k_p$. Then $R^q\alpha_*P$ is the p-sheafification of $U \mapsto H^q(U, P)$. The computation of Čech-cohomology above (4.4.5 shows that $\check{H}^q(\mathfrak{U}/V, P) = 0$ for all p-good V and for all coverings $\mathfrak{U}$ like in 4.3.7, which form a cofinal system of coverings, for all $q > 0$). Then lemma 4.4.6 proves the theorem. □

Corollary 4.4.10. *The sheaves G_n are p-good for all n.*

Proof. This follows inductively with 4.2.22. □

5 φ-G_n-modules and crystals

5.1 Definition

The following generalizes the notion of $D_n - \varphi$-gauges for a larger basis and is due to Fontaine. Let X be a scheme over $\mathbb{F}_p$.

Definition 5.1.1. (φ-G-modules modulo p^n or φ-G_n-modules, due to Fontaine)
Let E be a site contained in $SYN(X)$ and let G_n be the sheaf over E defined above.
(i) A *pre-φ-G_n-module* over E is a graded G_n-Module M equipped with an additive map

$$\varphi : M \to M/(v-1)M$$

such that $\varphi(\lambda m) = \varphi(\lambda)\varphi(m)$ for $\lambda \in G_n^0 = \mathcal{O}_n^{\text{cris}}$. (Note that $\varphi(\lambda)$ is the image of λ under the map φ from 4.2.11(i).) This is the same as to give an $\mathcal{O}_n^{\text{cris}}$-linear map

$$\Phi : \mathcal{O}_n^{\text{cris}}{}_{\varphi\nwarrow} \otimes_{G_n} M \to \mathcal{O}_n^{\text{cris}}{}_{pr\nwarrow} \otimes_{G_n} M$$

with the projection $pr : G_n \to G_n/(v-1)G_n \cong \mathcal{O}_n^{\text{cris}}$, by setting $\Phi(\lambda \otimes m) := \Phi_\varphi(\lambda \otimes m) := \lambda\varphi(m)$.
(ii) A *$\varphi - G_n$-module* (or a *$\varphi - G$-module modulo p^n*) is a pre-$\varphi - G_n$-module (M, Φ) such that Φ is an isomorphism.
(iii) A morphism of a $\varphi - G_n$-module is a graded morphism $\alpha : M \to N$ of G_n-modules compatible with φ, i.e the diagram

$$\begin{array}{ccc} M & \xrightarrow{\varphi_M} & \mathcal{O}_1^{\text{cris}} \otimes_{G_1} M \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle id\otimes\alpha} \\ N & \xrightarrow{\varphi_N} & \mathcal{O}_1^{\text{cris}} \otimes_{G_1} N \end{array}$$

is commutative.

Remark 5.1.2. (i) Indeed to give φ is equivalent to give Φ:
Given Φ like above we let $\varphi_\phi(m) = \Phi(1 \otimes m)$. One immediately sees that one recovers φ from Φ_φ and Φ from φ_Φ.
(ii) One immediately checks that a graded morphism of G_n-modules is compatible with φ if and only if it is compatible with Φ.

Remark 5.1.3. For a perfect field k and $n = 1$ over k it holds:

$$\begin{aligned} \mathcal{O}_1^{\text{cris}}{}_{\varphi\nwarrow} \otimes_{G_n} M &= k_{\varphi\nwarrow} \otimes_{k[f,v]/(fv)} M \\ &= k[f,v]((fv, f-1)_{\varphi\nwarrow} \otimes_{k[f,v]/(fv)} M \\ &= k[f,v]/(fv)_{\varphi\nwarrow} \otimes_{k[f,v]/(fv)} M/(f-1)M \\ &= (M/(f-1)M)^{(p)} \end{aligned}$$

and

$$\begin{aligned}\mathcal{O}_{1\ pr\nwarrow}^{\text{cris}} \otimes_{G_n} M &= k_{pr\nwarrow} \otimes_{k[f,v]/(fv)} M \\ &= k[f,v]((fv, v-1) \otimes_{k[f,v]/(fv)} M \\ &= k[f,v]/(fv) \otimes_{k[f,v]/(fv)} M/(v-1)M \\ &= M/(v-1)M,\end{aligned}$$

so we have

$$\Phi : (M/(f-1)M)^{(p)} \to M/(v-1)M.$$

It follows that the sections over k of a pre-$G_1 - \varphi$-module (resp. of a $G_1 - \varphi$-module) form a $D_1 - \varphi$-module (resp. a $D_1 - \varphi$-gauge).

Remark 5.1.4. Let R be a graded Ring and let M, N be graded R-modules. Then there is an isomorphism

$$M \otimes_R N \cong \bigoplus_r (\bigoplus_{m+n=r} M^m \otimes_{R^0} N^n / U_r)$$

where U_r is the subgroup generated by elements

$$rm \otimes n - m \otimes rn$$

with $r \in R^{r'}, m \in M^{m'}, n \in N^{n'}$ and $r' + m' + n' = r$. This induces a grading on $M \otimes_R N$.

Lemma 5.1.5. *There is a pair of adjoint funtors*

$$(j_*, j^*) = j : \textit{pre- } \varphi - G_n\textit{-modules over } X_\tau \to \textit{pre- } \varphi - G_n\textit{-modules over } X_{zar}$$

where X_{zar} is the small Zariski-site of a scheme X of characteristic p with

$$j_* : M \mapsto M \textit{ restricted to } X_{zar}$$

$$j^* : N \mapsto j^{-1} N \otimes_{j^{-1} G_{n,zar}} G_{n,\tau}$$

where j^{-1} is the inverse image fuctor of the morphism of topoi $X_\tau \to X_{zar}$ (this maps $F \in X_{zar}$ to the associated sheaf of $U \mapsto \varinjlim_V F(V)$ where V runs over Zariski-open neighborhoods of the image of U in X).

Proof. By Fontaine ([Fo] p.3). □

Definition 5.1.6. A *pre-φ-G_n-crystal* M is a pre-$\varphi - G_n$-module of finite type such that $j^* j_* M \to M$ is an isomorphism. We call M a *$\varphi - G_n$-crystal* if it is moreover a $\varphi - G_n$-module (i.e. Φ is an isomorphism).

Remark 5.1.7. Obviously a pre-φ-G_n-crystal is determined by its restriction to the small Zariski-site. So by remark 5.1.3 the datum of a φ-G_n-crystal over a perfect field k is equivalent to give a $D_n - \varphi$-gauge over k.

Remark 5.1.8. Let M be a $G_{n,zar}$-module of finite type with ad : $M \to j_* j^* M$ an isomorphism. Then j^*ad : $j^* j_* j^* M \to j^* M$ is also an isomorphism, so $N = j^* M$ is a pre-φ-G_n-crystal.

5.2 Connections with F-Zips II

Let X be a scheme over $\mathbb{F}_p$. We will construct a functor from modified F-Zips over X to $\varphi - G_1$-crystals over X.

Definition 5.2.1. (i) Let D_1 be the sheaf of graded algebras $\mathcal{O}[f,v]/(fv)$ where the degree of f is defined to be 1 and the degree of v is -1.
(ii) Let $D_1^{\mathrm{cris}} = \mathcal{O}_1^{\mathrm{cris}}[f,v]/(fv)$ be graded again with f in degree 1 and v in degree -1.

Remark 5.2.2. The "limit terms" of D_1^{cris} are isomorphic to $\mathcal{O}_1^{\mathrm{cris}}$, consider for example

$$\begin{aligned} D_1^{-\infty} &\overset{dfn}{=} D_1/(v-1)D_1 \\ &= \mathcal{O}_1^{\mathrm{cris}}[f,v]/(fv, v-1) \\ &= \mathcal{O}_1^{\mathrm{cris}}[f,v]/(f, v-1) \\ &\cong \mathcal{O}_1^{\mathrm{cris}} \end{aligned}$$

The limit terms of D_1 are obviously isomorphic to $\mathcal{O}$.
(ii) There are obvious graded ring-homomorphisms by sending f to f and v to v

$$D_1 \to D_1^{\mathrm{cris}} \to G_1.$$

The first one is induced by the morphism $\mathcal{O} \to \mathcal{O}_1^{\mathrm{cris}}$, for the second one note that, by definition, $G_1^0 = \mathcal{O}_1^{\mathrm{cris}}$.

We wish to assign a $G_1 - \varphi$-crystal to any modified F-zip over X. We can use a similar construction as in the case of an F-zip over a field, which gives us a $D_1 - \varphi$-gauge. There is an notion of rigidity for $D_1 - \varphi$-gauges and the category of rigid (and locally free) $D_1 - \varphi$-gauges turns out to be equivalent to the category of modified F-zips.

Definition 5.2.3. We consider sheaves on the small Zariski-site of X.
(i) A D_1*-module* is a graded module over D_1 of finite type.
(ii) A D_1*−*φ*-module* is a D_1 module M with a Frobenius semi-linear map $\varphi : M^\infty \to M^{-\infty}$.
(iii) A $D_1 - \varphi$*-gauge* is a $D_1 - \varphi$ module M, such that the induced $\mathcal{O}$-linear map $\varphi : (M^\infty)^{(p)} \to M^{-\infty}$ is an isomorphism.

We wish to characterize those $D_1 - \varphi$-gauges which correspond to F-zips. In order to do this we need the following properties of $D_1 - \varphi$-modules:

Definition 5.2.4. (i) A D_1-module M is called *strict* if for all r the maps

$$(v_r^\infty, f_r^\infty) : M^r \to M^{-\infty} \oplus M^\infty$$

are injective.
(ii) A $D_1 - \varphi$-module M is *rigid* if it is strict and if

$$M \xrightarrow{f} M \xrightarrow{v} M$$

$$M \xrightarrow{v} M \xrightarrow{f} M$$

are exact.
(iii) A $D_1 - \varphi$-module M is called *locally free* if all M^r are locally free $\mathcal{O}$-modules (necessarily of finite type) and if both kernel and image of

$$v_r : M^r \to M^{r-1}$$

and

$$f_r : M^r \to M^{r+1}$$

are locally direct summands for all r.

Construction 5.2.5. Let $M = \underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet)$ be a modified F-Zip over X. We wish to assign a $D_1 - \varphi$-gauge to M.

(1) Define the graded D_1-modules

$$C = \bigoplus_{r \in \mathbb{Z}} C^r$$

with trivial f-multiplication and v-multiplication induced by the inclusions of the descending filtration,

$$D = \bigoplus_{r \in \mathbb{Z}} D_r$$

with f-multiplication induced by the inclusions of the ascending filtration and with trivial v-multiplication.

$$grM = \bigoplus_{r \in \mathbb{Z}} gr_r^D M$$

with both trivial f- and v-multiplication.
By definition the morphisms

$$p_1 : C \to grM$$

with $(c_r) \mapsto (\varphi_r \circ pr)$ and

$$p_2 : D \to grM$$

given by the canonical projections are graded D_1-module-homomorphisms.

(2) Let $\tilde{M} = C \times_{grM} D$ be the fibered product of graded D_1-modules with respect to the maps p_i from above. This is easily to be computed as

$$\tilde{M}^r = \ker((\varphi_r \circ pr, -pr) : C^r \oplus D_r \to gr_r^D M)$$

Obivously $fv = 0 = vf$.

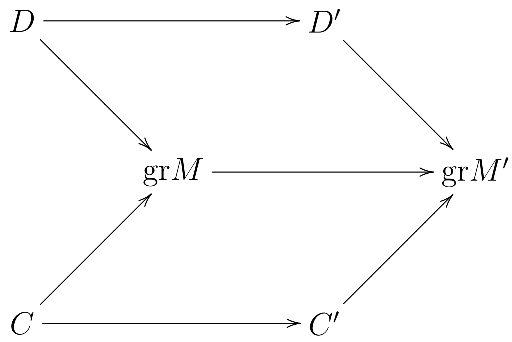
(3) We have to define $\varphi : (\tilde{M}^{\infty})^{(p)} \to \tilde{M}^{-\infty}$. We compute both sides:

$$\begin{aligned}(\tilde{M}^{\infty})^{(p)} &= ((C \times_{\mathrm{gr}M} D)^{\infty})^{(p)} \\ &\cong (C^{\infty} \times_{\mathrm{gr}M^{\infty}} D^{\infty})^{(p)} \\ &\cong (0 \times_0 M)^{(p)} \\ &\cong M^{(p)}\end{aligned}$$

$$\begin{aligned}\tilde{M}^{-\infty} &= (C \times_{\mathrm{gr}M} D)^{-\infty} \\ &\cong C^{-\infty} \times_{\mathrm{gr}M^{-\infty}} D^{-\infty} \\ &\cong M^{(p)} \times_0 0 \\ &\cong M^{(p)}\end{aligned}$$

We let φ be defined by the identity and the canonical isomorphisms above.

(4) Let $\alpha : (M, C^{\bullet}, D_{\bullet}, \varphi_{\bullet}) \to (M', C'^{\bullet}, D'_{\bullet}, \varphi'_{\bullet})$ be a morphism of F-zips. With α being compatible with the filtrations and φ resp. φ', there are induced maps $\alpha : C \to C'$, $\alpha : D \to D$ and $\alpha : \mathrm{gr}M \to \mathrm{gr}M'$, which are compatible with the maps p_i resp. p'_i. In other words α gives rise to a commutative diagram of graded D_1-modules



which induces a graded D_1-linear morphism

$$\tilde{\alpha} : C \times_{\mathrm{gr}M} D \to C' \times_{\mathrm{gr}M'} D'.$$

It is easy to see that this map is compatible with φ as defined in (3).

Proposition 5.2.6. *The assignment* $(M, \alpha) \mapsto (\tilde{M}, \tilde{\alpha})$ *defines a functor*

$$D : (\textit{modified } F - \textit{zips over } X) \longrightarrow (D_1 - \varphi - \textit{gauges over } X)$$

Its essential images consists of locally free rigid gauges.

Proof. It is clear that the assignment is well defined and functorial.
Furthermore $\tilde{M}$ is strict because the map $(f_r^{\infty}, v_r^{\infty}) : (C \times_{\mathrm{gr}M} D)^r \to M^{(p)} \oplus M$ sends (c, d) to (c, d). To show rigidity consider for example:

$$\tilde{M}^r \xrightarrow{f} \tilde{M}^{r+1} \xrightarrow{v} \tilde{M}$$

which is by definition isomorphic to

$$C^r \times_{\mathrm{gr}M^r} D_r \xrightarrow{(c,d)\mapsto(c,0)} C^{r+1} \times_{\mathrm{gr}M^{r+1}} D_{r+1} \xrightarrow{(c,d)\mapsto(0,d)} C^r \times_{\mathrm{gr}M^r} D_r.$$

Obviously composition is 0. Let $(c,d) \in C^{r+1} \times_{\mathrm{gr}M^{r+1}} D_{r+1}$ mapping to 0, which means $d = 0$. This implies that the image of c in $\mathrm{gr}M$ is 0, so c comes from C^r. The surjectivity of $D_r \to \mathrm{gr}M^r$ now gives a preimage of (c,d).

Finally we have to show that $\tilde{M}$ is locally free:
The exact sequence

$$0 \to \tilde{M}^r \to C^r \oplus D_r \to \mathrm{gr}^D_r M \to 0$$

implies that $\tilde{M}^r$ is locally free (the kernel of a epimorphism of locally free sheaves is again locally free). Consider the commutative diagram, arising from an application of the snake-lemma to the last two lines:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker f_r & \longrightarrow & C^r & \longrightarrow & \mathrm{gr}^D_r M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{M}^r & \longrightarrow & C^r \oplus D_r & \xrightarrow{(\varphi_r pr, -pr)} & \mathrm{gr}^D_r M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_r} & & \downarrow{\scriptstyle (0,\iota)} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \tilde{M}^{r+1} & \longrightarrow & C^{r+1} \oplus D_{r+1} & \longrightarrow & \mathrm{gr}^D_{r+1} M & \longrightarrow & 0
\end{array}$$

(Note that $C^r \to \mathrm{gr}^D_r M$ is surjective). It follows that $\ker f_r$ is locally free. Since $C^\bullet$ is a filtration whose terms are locally direct summands, locally, there exist sections

$$s : \mathrm{gr}^r_C(M^{(p)}) \to C^r$$

of pr. Let $U \subseteq X$ be open and such that s exists. Then over U the diagram

$$\begin{array}{ccc}
C^r & \longrightarrow & \mathrm{gr}^D_r M \\
\uparrow{\scriptstyle (id, -s(\varphi_r)^{-1}pr)} & & \uparrow{\scriptstyle =} \\
C^r \oplus D_r & \longrightarrow & \mathrm{gr}^D_r M
\end{array}$$

is commutative:

$$\varphi_r pr(c - s(\varphi_r)^{-1} prd) = \varphi_r prc - prd$$

for $(c,d) \in C^r \oplus D_r$. So $(id, -s\varphi_r^{-1}pr)$ induces a splitting of the monomorphism $\ker f_r \to \tilde{M}^r$ which shows that it is locally a direct summand.
A similar argument can be applied to $\mathrm{im} f_r$, but with precisely the same argument we can show that $\ker v_r$ is locally a direct summand. But now rigidity gives us, that $\mathrm{im} f_r$ and $\mathrm{im} v_r$ are locally direct summands. □

Proposition 5.2.7. *There is a functor*

$$G : (\text{ locally free rigid } D_1 - \varphi - \text{gauges over } X) \longrightarrow (\text{modified } F - \text{zips over } X)$$

given as follows:

To a locally free rigid D_1−φ-module N we assign a modified F-zip $G(N) = (G(N), C^\bullet, D_\bullet, \varphi_\bullet)$ by

$$G(N) := N^\infty$$
$$C^r := \mathrm{im}((\varphi)^{-1} \circ v_r^\infty) \subseteq (N^\infty)^{(p)}$$
$$D_r := \mathrm{im} f_r^\infty \subseteq N^\infty.$$

Let φ_r be given by the diagram with exact lines:

$$\begin{array}{ccccccc}
N^{r-1} \oplus N^{r+1} & \longrightarrow & N^r & \xrightarrow{pr\circ(\varphi)^{-1}\circ v_r^\infty} & \mathrm{gr}_C^r(GN^{(p)}) & \longrightarrow & 0 \\
\downarrow = & & \downarrow = & & \downarrow \varphi_r & & \\
N^{r-1} \oplus N^{r+1} & \longrightarrow & N^r & \xrightarrow{pr f_r^\infty} & \mathrm{gr}_r^D GN & \longrightarrow & 0
\end{array}$$

Proof. It is clear that the filtrations are locally direct summands. It follows similarily as in the proof of 1.3.1 that (the lines are exact and) $G(N)$ is an F-zip over X:

The lines are complexes by the relation $fv = vf = 0$ and by rigidity. Let (locally) $n \in N^r$ with $pr(\varphi)^{-1}v_r^\infty n = 0$. So it exists n' with $(\varphi^{-1})v_{r+1}^\infty n' = (\varphi^{-1})v_r^\infty n$ which means

$$v_{r+1}^\infty n' = v_r^\infty n$$

and hence

$$vn' - n \in \ker v_r^\infty = \ker v_r = \mathrm{im} f_{r-1}.$$

We conclude that $n \in \mathrm{im} v_{r+1} + \mathrm{im} f_{r-1}$. It follows analogously that the second line is exact. We conclude that $G(N)$ is an F-zip.

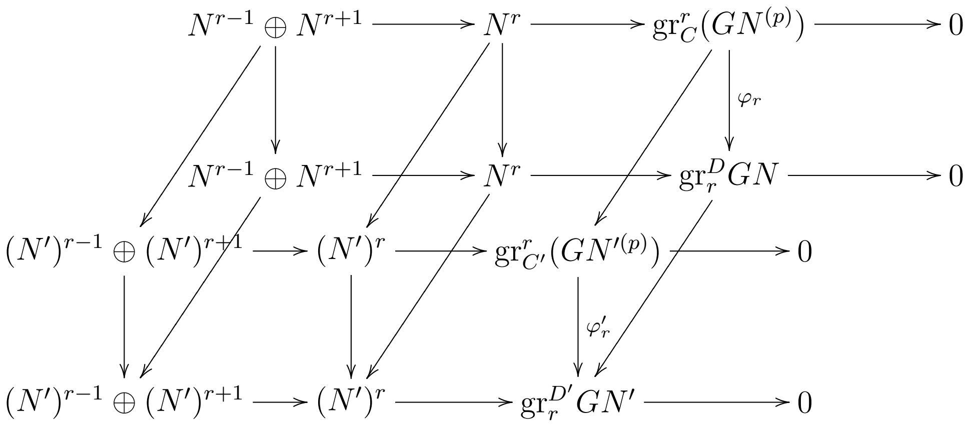
We have to show that a morphism of gauges induces indeed a morphism of F-zips: Let g be a morphism $N \to N'$ (with $N = (N, \varphi)$ and $N' = (N', \varphi')$) and let $\tilde{g}$ be the associated map $g^\infty : (\tilde{N}, C^\bullet, D_\bullet, \varphi_\bullet) \to (\tilde{N}', C'^\bullet, D'_\bullet, \varphi'_\bullet)$. Then it holds

$$\begin{aligned}
\tilde{g}^{(p)}(C^r) &= (g^\infty)^{(p)}(\varphi^{-1}v_r^\infty(N^r)) \\
&= (\varphi')^{-1}g^{-\infty}v_r^\infty(N^r) \\
&= (\varphi')^{-1}v_r^\infty g^r(N^r) \\
&\subseteq C'^r
\end{aligned}$$

and analogously

$$\begin{aligned}\tilde{g}(D_r) &= g^\infty(f_r^\infty(N^r) \\ &= f_r^\infty g^r(N^r) \\ &\subseteq D'_r\end{aligned}$$

Thus $\tilde{g}$ respects both filtrations. Furthermore we have the commutative diagram with exact lines:



which shows us that $\tilde{g}$ is compatible with φ_r and φ'_r. □

Proposition 5.2.8. *The functor D induces an equivalence of categories*

$$D : (\textit{modified } F - \textit{zips over } X) \longrightarrow (\textit{ locally free rigid } D_1 - \varphi - \textit{gauges over } X)$$

A quasi-inverse is given by G.

Proof. Let $M = (M, C^\bullet, D_\bullet, \varphi_\bullet)$ be a modified F-zip and let $M' = (M', C'^\bullet, D'_\bullet, \varphi'_\bullet) = G(\tilde{M})$. We write $\tilde{M} = (\tilde{M}, \varphi)$.

1) By definition $M' = \tilde{M}^\infty = (C \times_{\mathrm{gr}M} D)^\infty \cong D_\infty = M$.

2) The filtration step C'^r is by definition the image of

$$C^r \times_{\mathrm{gr}_r M} D_r = \tilde{M}^r \overset{\varphi^{-1}v_r^\infty}{\longrightarrow} (\tilde{M}^\infty)^{(p)} \cong M^{(p)}.$$

This map is given by $(c, d) \mapsto c$ by definition of φ (In the image of $\tilde{\cdot}$ the isomorphism φ "is" the identity). But with the morphism $D_r \to \mathrm{gr}_r M$ being surjective, the image $\mathrm{im}(pr_1 : C^r \times_{\mathrm{gr}_r M} D_r \to \tilde{M}^{(p)}) = C'^r$ is equal to C^r.

3) Analogously we see $D'_r \cong D_r$.

4) Let us check that $\varphi'_r = \varphi_r$. The diagram with exact lines

$$\begin{array}{ccccc} \tilde{M}^r & \xrightarrow{pr\circ(\varphi)^{-1}\circ v_r^\infty} & \mathrm{gr}^r_{C'}(M'^{(p)}) & \longrightarrow & 0 \\ \Big\downarrow{=} & & \Big\downarrow{\varphi_r} & & \\ \tilde{M}^r & \xrightarrow{pr f_r^\infty} & \mathrm{gr}_r^{D'} M' & \longrightarrow & 0 \end{array}$$

commutes, if we identify M with M' by the canonical isomorphism from above: For $m = (c, d) \in \tilde{M}^r = C^r \times_{\mathrm{gr}_r M} D_r$ we have

$$\varphi_r pr \varphi^{-1} v_r^\infty(c, d) = \varphi_r c$$

and

$$pr f_r^\infty(c, d) = d.$$

Since $(c, d) \in C^r \times_{\mathrm{gr}_r M} D^r$, it follows that $\varphi_r c = d$ in $\mathrm{gr}_r M = \mathrm{gr}_r^{D'} M'$ and hence commutativity. Surjectivity and the definition of φ'_r implies that $\varphi'_r = \varphi_r$.

Now let $N = (N, \varphi)$ be a locally free rigid $D_1 - \varphi$-gauge. We write $M = (M, C^\bullet, D_\bullet, \varphi_\bullet) = G(N)$ and $N' = (N', \varphi') = \tilde{G(N)}$.

1) The map

$$(\varphi^{-1} v_r^\infty, f_r^\infty) : N^r \to N'^r = C^r \times_{\mathrm{gr} N} D_r = \mathrm{im}(\varphi^{-1} v_r^\infty) \times_{\mathrm{gr}_r M} \mathrm{im}(f_r^\infty)$$

is well defined by the definition of φ_r: Recall that the morphisms in the fibered product $C^r \to \mathrm{gr}_r M \leftarrow D_r$ are given by $c \mapsto \varphi_r c; d \leftarrow\!\shortmid d$ and φ_r maps an element x to $f_r^\infty y$ where y is such that $\varphi^{-1} v_r^\infty y = x$. It is injective by rigidity and surjective, since for $(x, z) \in \mathrm{im}(\varphi^{-1} v_r^\infty) \times_{\mathrm{gr}_r M} \mathrm{im}(f_r^\infty)$ there is again by definition of φ_r a $y \in N^r$ such that $x = \varphi^{-1} v_r^\infty y$ and $z = f_r^\infty x$. So N and N' are isomorphic $\mathcal{O}$-modules.

2) Consider the commutative diagram

$$\begin{array}{ccc} N^r & \xrightarrow{f} & N^{r+1} \\ \Big\downarrow{(\varphi^{-1} v_r^\infty, f_r^\infty)} & & \Big\downarrow{(\varphi^{-1} v_{r+1}^\infty, f_{r+1}^\infty)} \\ N'^r & \xrightarrow{(0, \iota)} & N'^{r+1} \end{array}$$

which guarantees the compatibility with f.

3)Analogously we show compatibility with v.

4) At last we have to show compatibility with φ. From 1) we get that the pieces of degree ∞ (resp $-\infty$) are identified via the isomorphisms

$$id : (N'^\infty)^{(p)} = (0 \times_{\mathrm{gr} N} N^\infty)^{(p)} \cong (N^\infty)^{(p)}$$

and

$$\varphi : (N'^{-\infty}) = ((N^{\infty})^{(p)} \times_{\mathrm{gr}N} 0) \cong (N^{\infty})^{(p)} \to N^{-\infty}.$$

So the diagram

$$\begin{array}{ccc}
(N'^{\infty})^{(p)} & \xrightarrow{\varphi'=id} & N'^{-\infty} \\
\Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle \varphi} \\
(N^{\infty})^{(p)} & \xrightarrow{\varphi} & N^{-\infty}
\end{array}$$

commutes.

□

The next step is to construct a functor from $D_1 - \varphi$-gauges to $G_1 - \varphi$-crystals. This is given simply be tensor product:

Construction 5.2.9. Let M be a $D_1 - \varphi$-gauge over X.

(1) Let $G(M) = G_1 \otimes_{j^{-1}D_1} j^{-1}M \cong G_1 \otimes_{j^{-1}G_1^{\mathrm{zar}}} j^{-1}(G_1^{\mathrm{zar}} \otimes_{D_1} M) \cong j^*(G_1^{\mathrm{zar}} \otimes_{D_1} M)$. This is a graded G_1-module and it holds (if we suppress "j^{-1}"):

$$\begin{array}{rcl}
j^* j_* G(M) & = & G_1 \otimes_{G_1^{\mathrm{zar}}} j_*(G_1 \otimes_{D_1} M) \\
& \cong & G_1 \otimes_{G_1^{zar}} G_1^{\mathrm{zar}} \otimes_{D_1} M \\
& \cong & G(M)
\end{array}$$

(2) We have to construct an $\mathcal{O}_1^{\mathrm{cris}}$-linear morphism

$$\mathcal{O}_1^{\mathrm{cris}}{}_{\varphi\nwarrow} \otimes_{G_1} G(M) \to \mathcal{O}_1^{\mathrm{cris}}{}_{pr\nwarrow} \otimes_{G_1} G(M)$$

For the definition we consider both domain and codomain (We will suppress "j^{-1}"):

$$\begin{array}{rcll}
\mathcal{O}_1^{\mathrm{cris}}{}_{\varphi\nwarrow} \otimes_{G_1} G(M) & \cong & \mathcal{O}_1^{\mathrm{cris}}{}_{\varphi\nwarrow} \otimes_{G_1} G_1 \otimes_{D_1} M & \\
& \cong & \mathcal{O}_1^{\mathrm{cris}}{}_{\varphi\nwarrow} \otimes_{D_1} M & \\
& \cong & D_1^{\mathrm{cris}}/(f-1)D_1^{\mathrm{cris}}{}_{\varphi\nwarrow} \otimes_{D_1} M & (1) \\
& \cong & D_1^{\mathrm{cris}}/(f-1)D_1^{\mathrm{cris}} \otimes_{D_1} D_1{}_{\varphi\nwarrow} \otimes_{D_1} M & (2) \\
& \cong & D_1^{\mathrm{cris}} \otimes_{D_1} (M/(f-1)M)^{(p)} & \\
& \cong & D_1^{\mathrm{cris}} \otimes_{D_1} (M^{\infty})^{(p)} &
\end{array}$$

Isomorphism (1) holds since the diagram

$$\begin{array}{ccccc}
D_1 & & \longrightarrow & & G_1 \\
\Big\downarrow{\scriptstyle \varphi} & & & & \Big\downarrow{\scriptstyle \varphi} \\
D_1^{\mathrm{cris}} & \longrightarrow & D_1^{\mathrm{cris}}/(f-1) & \xrightarrow{\sim} & \mathcal{O}_1^{\mathrm{cris}}
\end{array}$$

commutes: Both possible ways are given by $\mathcal{O} \to \mathcal{O}_1^{\mathrm{cris}} \xrightarrow{\varphi} \mathcal{O}_1^{\mathrm{cris}}$ and the condition $f \mapsto 1$ and $v \mapsto 0$.
Isomorphism (2) follows from the commutativity of

$$\begin{array}{ccc} D_1 & \xrightarrow{\varphi} & D_1 \\ \downarrow & & \downarrow \\ D_1^{\mathrm{cris}} & \xrightarrow{\varphi} & D_1^{\mathrm{cris}} \end{array}$$

Now consider the codomain

$$\begin{aligned} {\mathcal{O}_1^{\mathrm{cris}}}_{pr\nwarrow} \otimes_{G_1} G(M) &\cong {\mathcal{O}_1^{\mathrm{cris}}}_{pr\nwarrow} \otimes_{G_1} G_1 \otimes_{D_1} M \\ &\cong {\mathcal{O}_1^{\mathrm{cris}}}_{pr\nwarrow} \otimes_{D_1} M \\ &\cong D_1^{\mathrm{cris}}/(v-1){D_1^{\mathrm{cris}}}_{pr\nwarrow} \otimes_{D_1} M \qquad (3) \\ &\cong D_1^{\mathrm{cris}} \otimes_{D_1} M/(v-1)M \\ &\cong D_1^{\mathrm{cris}} \otimes_{D_1} M^{-\infty} \end{aligned}$$

Isomorphism (3) holds since the diagram

$$\begin{array}{ccc} D_1 & \longrightarrow & G_1 \\ \downarrow & & \downarrow{\scriptstyle pr} \\ D_1^{\mathrm{cris}}/(v-1) & \xrightarrow{\sim} & \mathcal{O}_1^{\mathrm{cris}} \end{array}$$

commutes: Both possible ways are given by $\mathcal{O} \to \mathcal{O}_1^{\mathrm{cris}}$ and the condition $f \mapsto 0$ and $v \mapsto 1$.

Define

$$\Phi : {\mathcal{O}_1^{\mathrm{cris}}}_{\varphi\nwarrow} \otimes_{G_1} G(M) \to {\mathcal{O}_1^{\mathrm{cris}}}_{pr\nwarrow} \otimes_{G_1} G(M)$$

by requiring the diagram

$$\begin{array}{ccc} {\mathcal{O}_1^{\mathrm{cris}}}_{\varphi\nwarrow} \otimes_{G_1} G(M) & \longrightarrow & {\mathcal{O}_1^{\mathrm{cris}}}_{pr\nwarrow} \otimes_{G_1} G(M) \\ \downarrow{\scriptstyle \sim} & & \uparrow{\scriptstyle \sim} \\ D_1^{\mathrm{cris}} \otimes_{D_1} (M^{\infty})^{(p)} & \xrightarrow{id\otimes\varphi} & D_1^{\mathrm{cris}} \otimes_{D_1} M^{-\infty} \end{array}$$

to be commutative. This obviously gives an $\mathcal{O}_1^{\mathrm{cris}}$-linear isomorphism.
Explicitely one computes for $\lambda \otimes g \otimes m \in {\mathcal{O}_1^{\mathrm{cris}}}_{\varphi\nwarrow} \otimes_{G_1} G_1 \otimes_{D_1} M$

$$\lambda \otimes g \otimes m \mapsto \lambda\varphi(g) \otimes 1 \otimes \varphi(m) \in {\mathcal{O}_1^{\mathrm{cris}}}_{pr\nwarrow} \otimes G_1 \otimes_{D_1} M$$

where $\varphi(g)$ denotes the image of g under $G_1 \to \mathcal{O}_1^{\mathrm{cris}}$ and $\varphi(m)$ is the preimage of m_1, where m_1 is the image of $m \otimes 1$ under $M^{(p)} \to (M/(f-1)M)^{(p)} \xrightarrow{\varphi} M/(v-1)M$ (note that $\varphi(m)$ is not well defined per se, but it is well defined in the tensor product).

The associated φ-semilinear morphism

$$\varphi : G(M) \to \mathcal{O}_1^{\text{cris}}{}_{pr\nwarrow} \otimes_{G_1} G(M)$$

maps $g \otimes m \mapsto \varphi(g) \otimes 1 \otimes \varphi(m)$, with the φ's defined as above.

(3) A morphism of $D_1 - \varphi$-modules $\alpha : M \to N$ gives a morphism of the associated G_1-modules. We must show compatibility with φ. This amounts to showing commutativity of the following diagram

$$\begin{array}{ccc} G_1 \otimes_{D_1} M & \xrightarrow{\varphi_{G(M)}} & \mathcal{O}_1^{\text{cris}}{}_{pr\nwarrow} \otimes_{G_1} G_1 \otimes_{D_1} M \\ \downarrow{\scriptstyle id\otimes\alpha} & & \downarrow{\scriptstyle id\otimes id\otimes\alpha} \\ G_1 \otimes_{D_1} N & \xrightarrow{\varphi_{G(N)}} & \mathcal{O}_1^{\text{cris}}{}_{pr\nwarrow} \otimes_{G_1} G_1 \otimes_{D_1} N \end{array}$$

Commutativity follows since because for $g \otimes m \in G_1 \otimes_{D_1} M$ it holds

$$\varphi(g) \otimes 1 \otimes \varphi_N(\alpha(m)) = \varphi(g) \otimes 1 \otimes \alpha(\varphi_M(m)).$$

Proposition 5.2.10. *The assignment $M \mapsto G_1 \otimes_{j^{-1}D_1} j^{-1}M$ defines a functor*

$$G : (D_1 - \varphi - \textit{gauges over } X) \to (G_1 - \varphi - \textit{crystals over } X)$$

Proof. This is clear, since all steps of the construction above are functorial. □

A Appendix

A.1 Syntomic morphisms

Definition A.1.1. (i) A morphism of schemes f is called *locally of complete intersection* if f factors locally as a smooth morphism after a regular immersion.
(i) A morphism of schemes is called *syntomic* if it is flat and locally of complete intersection.

Remark A.1.2. (i) A syntomic morphism $fX \to Y$ can locally be written as a regular immersion in an affine n-space $\mathbb{A}^n_U$ over an open subset U of Y.
(ii) The class of morphisms of locally complete intersection is stable under base-change only for flat morphisms, but the class of syntomic morphisms is stable in general.

Lemma A.1.3. *(i) The composition of two syntomic morphisms is syntomic.*
(ii) The basechange of a syntomic morphism is syntomic.

Proof. (i) A composition $\mathbb{A}^n_S \to S \xrightarrow{\iota} T$ of a regular immersion after an affine n-space can easily be written as $\mathbb{A}^n_S \xrightarrow{\iota'} \mathbb{A}^n_T \to T$, where ι' is regular since affine n-space is flat.
(ii) See [EGA4.4] 19.3.9. □

Proposition A.1.4. *Syntomic morphisms satisfy the lifting property of 3.1.5.*

Proof. See [Be2] Lemme 1.1.9. □

A.2 F-zips

One aim of this paper is to find connections between (different notions of) gauges and so called F-zips, invented by Ben Moonen and Torsten Wedhorn.We give the basic facts about F-zips. The source is their article [MW].

Let S be a scheme of characteristic p. An F-zip over S is a locally free $\mathcal{O}_S$-module with two filtrations, whose graded quotients are semi-linearily isomorphic. Moonen and Wedhorn gave a complete classification of F-zips over algebraically closed fields, so by our theorem 1.3.6 we have a classification for rigid gauges over algebraically closed fields, which is quite explicit.

Remark A.2.1. If we speak of a filtration of a locally free $\mathcal{O}_S$-module, we always mean a filtration, such that the inclusions are locally direct summands. Hence the subquotients are locally free $\mathcal{O}_S$-modules, too.

Definition A.2.2. Let S be scheme over $\mathbb{F}_p$. An *F-zip over S* is a tuple $\underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet)$ with
(i) a locally free $\mathcal{O}_S$-module of finite rank M,
(ii) a descending filtration $C^\bullet$ on M,

(iii) an ascending filtration $D_\bullet$ on M,
(iv) a family of $\mathcal{O}_S$-linear isomorphisms

$$\varphi_n : (\mathrm{gr}^n_C M)^{(p)} \xrightarrow{\sim} \mathrm{gr}^D_n M$$

for $n \in \mathbb{Z}$.

Moonen and Wedhorn use so called "types" to classify their F-zips. The type of a filtration C of a locally free $\mathcal{O}_S$-module of finite rank M is the function

$$\tau^s_C : \mathbb{Z} \to \mathbb{N}$$

which assigns to each n the rank of the graded piece of degree n at a fixed point s. This function obviously has finite support. The mapping $s \mapsto \tau^s_C$ is locally constant, so for connected S there is a single function $\tau_C : \mathbb{Z} \to \mathbb{N}$ with finite support.
Each type τ defines an ordered partition $(N_1, \ldots, N_r)$ of $N = \sum_{n\in\mathbb{Z}} \tau(n)$ (Hence N is the rank of M). Let $S_{N_1} \times \ldots \times S_{N_r}$ be a subgroup of the symmetric group S_N in the usual way.

Theorem A.2.3. *Let S be the spectrum of an algebraically closed field. Then there is bijection*

$$\left\{ \begin{array}{c} \textit{isomorphy classes of} \\ F\textit{-zips of type } \tau \textit{ over } S \end{array} \right\} \xrightarrow{\sim} S_{N_1} \times \ldots \times S_{N_r} \backslash S_N$$

Proof. See [MW] Theorem 1 and Theorem (4.4). They explicitely construct very simple F-zips over $\mathbb{F}_p$, with their filtrations given by standard basis-vectors. Changing bases to S, these give representatives of the isomorphy classes. □

While this work was written, it appeared, that another notion of F-zip would be more suitable to work with. The definition is essentialy the same, but one has the descending filtration on $M^{(p)}$. We will call these objects modified F-zips:

Definition A.2.4. A *modified F-zip* is a is a tuple $\underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet)$ with
(i) a locally free $\mathcal{O}_S$-module of finite rank M,
(ii) a descending filtration $C^\bullet$ on $M^{(p)}$,
(iii) an ascending filtration $D_\bullet$ on M,
(iv) a family of $\mathcal{O}_S$-linear isomorphisms

$$\varphi_n : \mathrm{gr}^n_C(M^{(p)}) \xrightarrow{\sim} \mathrm{gr}^D_n M$$

for $n \in \mathbb{Z}$.

Obviously every F-zip gives rise to a modified F-zip, but not vice versa in general. But if S is the spectrum of a perfect ring the two notions coincide.

A.3 Some p-valuations

Lemma A.3.1. *Let $n = a_0 + a_1 p + \ldots + a_r p^r$ be the p-adic expansion of a natural number. Then $\nu_p(n!) = \frac{n-(a_0+\ldots+a_r)}{p-1}$.*

Lemma A.3.2. *Let $a, b \in \mathbb{Z}$ with p-adic expansions $a = a_o + pa_1 + \ldots + p^r a_r$ and $b = b_o + pb_1 + \ldots + p^s b_s$. Define ϵ_i inductively by $\epsilon_{-1} = 0$ and $p\epsilon_n + s_n = a_n + b_n + \epsilon_{n-1}$ with $0 \leq s_n < p$. Then $v_p(\binom{a+b}{a}) = \sum_i \epsilon_i$.*

Proof. See e.g. [Ku] p.116. □

Remark A.3.3. The numbers ϵ_i are the "carry digits" arising from p-adic addition.

Corollary A.3.4. *Let $r \in \mathbb{Z}$ and $i \leq p^r$. Then $v_p(\binom{p^r}{i}) = r - v_p(i)$.*

Proof. Assume $i < p^r$. Consider the p-adic expansions of i and $p^r - i$. Let $v = v_p(i)$, then one has $i = a_v p^v + a_{v+1} p^{v+1} + \ldots$ and $p^r - i = b_v p^v + b_{v+1} p^{v+1} + \ldots$ with $a_v + b_v = p$ and $a_n + b_n = p - 1$ for $v + 1 \leq n \leq r - 1$ and $a_r + b_r = 0$. Hence $\epsilon_v = \ldots = \epsilon_{r-1} = 1$ and $\epsilon_0 = \ldots = \epsilon_{v-1} = 0$. □

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