Renormalization and Applications of Baryon Distribution Amplitudes in QCD

Dissertation

zur Erlangung des Doktorgrades
der Naturwissenschaften (Dr. rer. nat.)
der naturwissenschaftlichen Fakultät II – Physik
der Universität Regensburg

vorgelegt von

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Mai 2009

Promotionsgesuch eingereicht am: 27 Mai 2009
Die Arbeit wurde angeleitet von: Prof. Dr. V. M. Braun
Das Kolloquium fand am 17 Juli 2009 statt.

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1 Introduction

The properties of baryons in general, and of nucleons in particular, have been in the focus of experimental and theoretical efforts since Heisenberg's realization that protons and neutrons are the "fundamental" building blocks of the nuclei. On the theory side Hideki Yukawa "gave a theory of the forces which keep the nucleons together" [1, 2], whereas Stern's experiments provided first measurements of their magnetic properties [3, 1]. The form factors of the nucleons, which describe the distribution of charge and magnetization, could be measured in Hofstadter's elastic electron-nucleon scattering experiments at SLAC in the 1950s [4]. Simultaneously, the notion of fundamental nucleons was severely questioned, when the discovery of a multitude of new heavy particles coined *hadrons*, suggested some higher organization principle. Such a principle was found when the *quarks* introduced by Gell-Mann [5] could be associated with the partons discovered in Deep Inelastic Scattering (DIS) experiments [6, 7, 8].

An early attempt to capture the inner structure of hadrons based on the idea of constituent quarks spawned the quark model [5, 9, 10] which is currently still in use. With the rise of Quantum ChromoDynamics (QCD) [11], one could hope that a consistent description of hadrons based on first principles was within reach. However, many non-trivial hadronic properties such as form factors, mass and shape are due to the inner structure of the hadrons, which is governed by low energy QCD effects that cannot be described perturbatively. Although there have been new promising developments (see e.g [12]), it is at the moment still impossible to determine these properties from first principles without relying on computational methods such as Lattice QCD. Therefore, it was necessary to develop further techniques which have a solid footing on the principles of QCD, but require some additional external input.

A first step in this direction was provided by the idea of QCD factorization. That is, separating processes in two parts: short-distance physics which can be accessed using perturbative methods and long-distance contributions which are parameterized by universal functions and describe the structure of the hadrons.

Among the more refined factorization approaches are such renowned methods as Heavy Quark Effective Theory (HQET) [13] and Soft Collinear Effective Theory (SCET) [14, 15].

A priori, the long-distance functions seem to be hopelessly complicated, as each hadron is in principle composed of infinitely many interacting partons. If each degree of freedom is equally important, this prohibits any realistic description. The central question is therefore, whether the relevant degrees of freedom can be isolated for a given physical situation. One possible option is the substitution of partonic by new, effective degrees of freedom; this philosophy is employed by e.g. Chiral Perturbation Theory (χPT) [16, 17] or AdS/QCD [18, 19, 20]. A rather different course of action is feasible for processes where only a specific subset of all the possible parton configurations contained in the full hadron wavefunction contributes.

Such a situation arises in so-called hard exclusive processes which require the partons of fast moving hadrons to be "close together", that is, the partons are at small transverse distances from each other. This kinematic situation naturally favors the configuration with the least possible number of constituents, as the probability for a tight bunch of partons to stay in immediate vicinity decreases rapidly with the number of partons. Therefore, the phenomenological description of these hard exclusive processes does not require the full information on the hadron wave function and the relevant dynamics can be condensed into so-called *distribution amplitudes* (DAs). These distribution amplitudes describe hadrons in terms of spin and longitudinal momentum configurations of constituent partons with the transversal momentum dependence already integrated out.

On the one hand, only the few Fock states with the lowest number of partons are expected to play a role in hard exclusive reactions and all other DAs can be neglected. Therefore, one obtains a drastic simplification compared to the infinite tower of states contributing to the wavefunction. On the other hand, precise measurements of exclusive processes cannot be used to access the full wavefunction, but only the first few DAs; a situation similar to deep inelastic scattering, where only one-particle probabilities (parton distributions) can be extracted.

The DAs represent the major external input for pQCD calculations of form factors [21, 22], Light-Cone Sum Rules (LCSRs) [23, 24, 25] or SCET and have to be evaluated in a separate nonperturbative calculation e.g. using SVZ Sum Rules [26, 27, 28]. The distribution amplitudes for baryons, that is hadrons whose quantum numbers can be generated by some combination of three valence quarks, are the subjects of this thesis. We study the scale dependence of higher twist DAs in some detail and show how they can be used to calculate form factors in the framework of light-cone sum rules.

This thesis is organized as follows:
In the course of the next chapter we give a short reminder on the basics of quantum chromodynamics. We introduce its lagrangian density and explain the tools essential for our analysis of the distribution amplitudes: dimensional regularization, the running coupling and the renormalization group equations. Chap. 3 is dedicated to two theoretical concepts: the *spinor formalism* which is fairly non-standard in context of QCD calculations and *conformal symmetry*, an extension of the well-known Poincaré symmetry. We show how one can use this spinor formalism to construct a complete basis of one particle light-ray operators that feature definite conformal spin and collinear twist. In Chapter 4 this basis, one of our main results, is used to formulate a novel approach for the study of the scale dependence of higher twist distribution amplitudes. Using our basis as starting point, we can find a complete classification of the baryon distribution amplitudes of twist 4. After explaining the general strategy for the calculation of the renormalization kernels which determine the scale dependence of the distribution amplitudes, we give one detailed example of how our formalism works in practice, before presenting the anomalous dimension spectra. A first application of our results is the determination of the so-called Wandzura-Wilczek contribution to the twist-4 nucleon distribution amplitudes. In Chapter 5 the light-cone sum rule formalism is briefly introduced. We discuss the peculiarities of excited states in this framework, which is the main motivation for our subsequent definition of a completely new set of distribution amplitudes – the $N^\star$ distribution amplitudes; they can be determined using lattice QCD methods. We use these DAs to calculate the electromagnetic form factors of the $N\gamma \to N^\star$ transition, which could not be obtained using LCSRs previously. A good agreement with the most recent experimental data is found. We finish with a short conclusion and an outlook on future applications and possible improvements in Chap. 6.

There was a young fellow from Trinity,
Who took the square root of infinity.
But the number of digits,
Gave him the fidgets;
He dropped Math and took up Divinity.

George Gamov

2 Setting the Scene: Quantum Chromodynamics

The concept of a new quantum number, color [29, 30], was originally proposed to avoid spin-1/2 quarks with bosonic statistics and required the invariance of hadron states, which are color-neutral, under global $SU(3)$ transformations. The promotion of the global symmetry to a local gauge symmetry marks the birth of the quantum field theory of strong interactions – quantum chromodynamics. Given the success of quantum electrodynamics which was based on the same construction principle, this step was natural and further "strengthened by the [...] ability to quantize gauge theories in a manner that was at once unitary and renormalizable" [31].

In contrast to QED, whose gauge group is abelian, QCD includes a nonlinear interaction of the gauge bosons which themselves carry color charge. This property of quantum chromodynamics is the origin of many of its nontrivial features and has until now inhibited any attempt to solve QCD. In fact no single approximate method can cover all energy scales and a multitude of approaches has been devised, each only valid in a specific region. The most famous of these approaches is perturbative QCD or pQCD, which successfully predicted the strong dynamics for very short distances to astounding accuracy, but cannot make quantitative statements on the low energy behavior of the theory – the domain of nonperturbative methods.

In this Chapter we give an elementary introduction to the basics of quantum chromodynamics. Starting with the Lagrangian of QCD, we present two different gauge fixing prescriptions: covariant and axial gauges. In Sect. 2.3 we introduce the concept of dimensional regularization. Two important features of renormalization are discussed in Sect. 2.4: the running of the coupling constant and the renormalization group equation, which are instrumental for the study of the scale dependence of baryon distribution amplitudes in Chap. 4. For a detailed account of QCD, we refer the reader to standard textbooks like [32, 33, 34, 35].

2.1 The Lagrangian of QCD

The dynamics of the color charged spin-1/2 quarks and their interaction via spin-1 vector bosons, the gluons, can be condensed into the *Lagrangian density* of QCD.

It has the following form[1] [11]

$$\mathcal{L}_{\mathrm{QCD}} = \mathcal{L}_{\mathrm{Cl}} + \mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{\mathrm{gh}} \,, \tag{2.1}$$

where the first term corresponds to the classical Lagrangian density[2]

$$\mathcal{L}_{\mathrm{Cl}} = \sum_{\substack{f=u,d,\\ s,c,b,t}} \left(\bar{q}_f^a\right)_i(x) \left(i\gamma^\mu_{ij} D^{ab}_\mu - m_f \delta_{ij}\delta^{ab}\right) \left(q^b_f\right)_j(x) - \frac{1}{4} F^A_{\mu\nu}(x) F^{A,\mu\nu}(x) \,, \tag{2.2}$$

x being a space-time four-vector. The sum in (2.2) runs over the six different quark flavors. Each quark field q_f transforms in the fundamental representation of the gauge group $SU(3)$ and carries a color $a, b = 1, \ldots, N_c = 3$ as well as a Dirac spinor index $i, j = 1, \ldots, 4$. The gluon field strength tensor $F^A_{\mu\nu}$ and the covariant derivative D_μ which incorporates the interaction of quark and gluon fields, are given by

$$D^{ab}_\mu = \partial_\mu \delta^{ab} - igA^A_\mu T^{A,ab} \,, \tag{2.3}$$

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu + \partial_\nu A^A_\mu + g f^{ABC} A^B_\mu A^C_\nu \,. \tag{2.4}$$

The gluon field A^A_μ transforms according to the adjoint representation. The color index[3] A, therefore, runs from 1 to $N_c^2 - 1 = 8$. The strength of the interaction is controlled by the strong coupling constant g.

The T^A are the generators of the $SU(3)$ and close a Lie algebra [32, 36]

$$\left[T^A, T^B\right] = i f^{ABC} T^C \,, \tag{2.5}$$

$$\mathrm{Tr}\left\{T^a\right\} = 0 \,, \tag{2.6}$$

where the coefficients f^{ABC} are the structure constants of the algebra. In (2.3) the T^A are the standard hermitian, traceless 3×3 matrices associated with the fundamental representation.

The set of Dirac 4×4 matrices γ_μ obeys the algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{2.7}$$

[1] Thoughout this thesis we work in "god-given" units, i.e., $\hbar = c = 1$ [33].

[2] We make use of Einstein's sum convention, i.e., a summation over indices that appear twice is assumed.

[3] We use the convention that capital color indices correspond to "adjoint" and lower case indices to "fundamental" fields.

where we use the definitions of Bjorken and Drell [37] for the metric

$$g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)\ . \tag{2.8}$$

For completeness, we also introduce the γ_5 matrix

$$\gamma_5 = -\frac{i}{4!}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\epsilon_{\mu\nu\rho\sigma}\ ,\quad \epsilon_{0123} = 1\ , \tag{2.9}$$

$$\{\gamma_5, \gamma^\mu\} = 0\ . \tag{2.10}$$

Note that various renowned textbooks, such as [32, 34, 35], use different sign conventions. This is a standard source of errors.

Since we are working in a gauge theory, two field configurations that are related via a gauge transformation correspond to one and the same physical state. As one has to avoid a double counting of unphysical degrees of freedom for a proper quantization of the gluon field, it is necessary to fix the gauge. This is achieved by introducing the term $\mathcal{L}_{\mathrm{gf}}$ in Eq. (2.1). There are two different families of gauge fixing terms. On the one hand, there is the class of covariant gauge fixing terms

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\zeta}\left(\partial^\mu A^A_\mu(x)\right)\left(\partial^\nu A^A_\nu(x)\right)\ , \tag{2.11}$$

which corresponds to a condition of the type $\partial^\mu A^A_\mu(x) = g(x)$ with $g(x)$ being an arbitrary scalar function. The gauge parameter ζ itself is unphysical; all physical quantities must be independent of ζ and any choice is valid. This gauge fixing procedure leads to a rather simple gluon propagator. However, in non-abelian gauge theories, such as QCD, it is then necessary to introduce a non-vanishing third term, the ghost Lagrangian $\mathcal{L}_{\mathrm{gh}}$

$$\mathcal{L}_{\mathrm{gh}} = -gf^{ABC}\bar{c}^A(x)\partial^\mu\left(A^B_\mu(x)c^C(x)\right) - \bar{c}^A(c)\partial^2 c^A(x)\ . \tag{2.12}$$

The new fields c, the Faddeev-Popov ghosts [38], are scalar fields that obey Fermi-Dirac statistics. Therefore, they cannot have any physical meaning and their only raison d'être is the cancelation of unphysical gluon polarizations.

The second class of gauge-fixing terms corresponds to the so-called axial gauges. In this case the gauge condition reads $n^\mu A^A_\mu(x) = g(x)$ and $\mathcal{L}_{\mathrm{gf}}$ takes the form

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\zeta}\left(n^\mu A^A_\mu(x)\right)\left(n^\nu A^A_\nu(x)\right)\ , \tag{2.13}$$

where n is an arbitrary four-vector[4]. This class does not require any ghost fields, but has the disadvantage that the gluon propagator takes a quite complicated form and calculations beyond one loop become tedious. In Chapter 4 this gauge (with $\zeta = 0$) is employed for the calculation of the one-loop renormalization kernels.

[4]This introduces a “preferred” direction into the Lagrangian. However, this will not cause complications in the calculations relevant for this thesis.

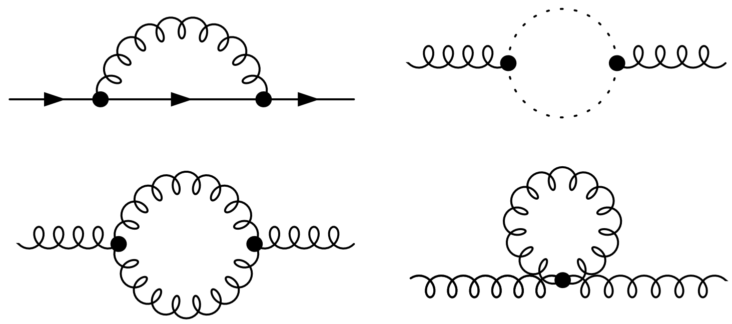


Figure 2.1: One-loop self-energy diagrams: As usual, the straight lines correspond to quarks, the curly lines to gluons and the dotted lines to ghosts.

2.2 A Few Words on Perturbation Theory

Everything one needs to know about QCD should, in principle, be encoded in the Lagrangian $\mathcal{L}_{\text{QCD}}$. However, since the gauge and matter sector in the Lagrangian are intertwined via the covariant derivative, QCD (like QED) cannot be solved analytically. In order to have some predictive power, it is necessary to simplify the problem by using some approximate method. The standard approaches include SVZ and Light-Cone Sum Rules, Lattice QCD, chiral perturbation theory, large N_c expansion and QCD perturbation theory.

The latter is based on the observation that the action S_{QCD}

$$S_{QCD} = i \int \mathrm{d}^4x\, \mathcal{L}_{QCD}(x) = i \int \mathrm{d}^4x\, \mathcal{L}_{\text{kinetic}}(x) + i \int \mathrm{d}^4x\, \mathcal{L}_{\text{int}}(x)\ , \tag{2.14}$$

can conveniently be split in two parts: the free or kinetic part $\mathcal{L}_{\text{kinetic}}$ is bilinear in the fields and the interaction part $\mathcal{L}_{\text{int}}$ contains the cubic and quartic terms. The free part does not depend on the gauge coupling g and can be solved exactly, whereas each term in $\mathcal{L}_{\text{int}}$ is at least linear in g. Under the assumption that g is small, the action can be expanded in the QCD path-integral and all Green's functions can be approximated by a series in the strong coupling

$$\alpha_s = \frac{g^2}{4\pi}\ , \tag{2.15}$$

which is the analogue of the fine structure constant α_{em} of QED.

The standard visualization of this perturbative expansion are the Feynman diagrams. Fig. 2.1 shows some $\mathcal{O}(\alpha_s)$ corrections to two-point Green's functions, i.e. propagators. The Feynman Rules (see [33]) tell us that the momentum p of the virtual particles "running" inside the loops is not restricted by the on-shell condition $p^2 = m^2$, where m is the mass of the particle, and that one has to integrate over all possible momenta. This seems unnatural for two reasons. First of all, it is known that at energies close to the Plank mass M_P our theory has to be inconsisted, as gravitational interaction becomes strong and one can not even be sure what metric should be used. Secondly, the momentum integrals turn out to be divergent in the ultra-violet region (that is for high momenta).

2.3 Dimensional Regularization

In order to deal with the divergences arising due to the loop corrections to the Green's functions in QCD, the first step is to *regularize* them. That is, introduce some new auxiliary parameter ϵ, the regulator. While in the limit of $\epsilon \to 0$ (or $\epsilon \to \infty$) the original divergence is recovered, finite values of the regulator lead to finite corrections, which simplifies handling and isolating the infinities.

There are quite a few standard choices for this procedure, the *regularization schemes*, on the market; the simplest one being Cut-Off regularization, where the loop integral is restricted to momenta smaller than some arbitrary, large scale M_{CO}. This has the disadvantage to explicitly break the Ward identities and therefore gauge invariance, see e.g. [33].

Throughout this thesis we will make use of a more sophisticated scheme: *Dimensional Regularization* (DR) [39]. In this regularization the Feyman diagrams are evaluated in $D = 4 - 2\epsilon$ space-time dimensions [40]. The singularities then arise as poles in ϵ as $\epsilon \to 0$. The result of a one-loop calculation typically takes the following form

$$\frac{A}{\epsilon} + B\,, \tag{2.16}$$

where A is the residue of the ϵ pole and B is a finite term. However, setting the number of space-time dimensions equal to D forces us to consider a Lagrangian with mass dimension D to keep the action

$$S = i \int \mathrm{d}^D x \, \mathcal{L}_{\mathrm{QCD}}$$

dimensionless. First of all, this requirement changes the canonical dimensions of the fields. The term $m_q \bar{q}_i^f q_i^f$ in Eq. (2.1) implies that the quark fields now have mass dimension $(D-1)/2$ and $F_A^{\mu\nu} F_{\mu\nu}^A$ corresponds to $\dim[A^{\mu,A}] = (D-2)/2$. Therefore, the gauge coupling g has to be modified for the term $ig\bar{q}^i \not{A}^{ij} q^j$ to have the correct dimension [41]:

$$g \to g\mu^{\epsilon}\,. \tag{2.17}$$

Here μ is a mass parameter. We see that DR also introduces an arbitrary scale.

In fact, not only the fields and couplings are modified by the change of space-time dimensions, but also the algebra for the Dirac γ matrices changes.

Especially the definition of the γ_5 matrix in D dimensions is highly non-trivial and inconsistencies may occur, if this is not treated accurately. We will adopt the so-called naive dimensional regularization scheme, which only modifies the metric $g^{\mu\nu}$ to take the changed space-time dimensions into account

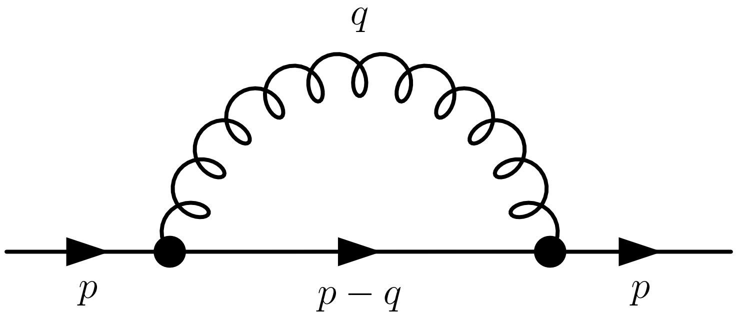
$$g_\mu{}^\mu = D\,.$$

The other relations, such as

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \quad \text{and} \quad \{\gamma_\mu, \gamma_5\} = 0$$

are left untouched. This prescription is known [40] to lead to inconsistencies when traces of γ matrices including a γ_5 are involved. However, for our purposes this simple modification[5] is sufficient and no ambiguities will appear.

A simple example: Let us consider the simplest one-loop diagram in QCD, the self-energy of a massless quark.



Using the standard techniques for the calculations of one-loop integrals and the Feynman rules given in App. A, one arrives at[6]

$$i\Sigma_{\alpha\beta} = i\,\not{p}\,C_F\,\delta_{\alpha\beta}\,g^2\frac{2(1-\epsilon)\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}}\left(\frac{\mu^2}{-p^2}\right)^{\epsilon}\beta(2-\epsilon, 1-\epsilon)\,, \tag{2.18}$$

where $C_F = \frac{N_c^2-1}{2N_c}$ is a color factor. For the details of the calculation see e.g. [33].

The pole in ϵ is hidden in the Euler-Gamma function Γ; it can be expanded around $\epsilon = 0$

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) \quad , \, \gamma_E = 0.57721\ldots\,. \tag{2.19}$$

Using the expansion for the Euler-Beta function

$$\beta(2-\epsilon, 1-\epsilon) = \frac{\Gamma(2-\epsilon)\Gamma(1-\epsilon)}{\Gamma(3-2\epsilon)} = \frac{1+2\epsilon}{2} + \mathcal{O}(\epsilon^2)\,, \tag{2.20}$$

[5] More sophisticated prescriptions are Dimensional Reduction [42] or the famous 't Hooft-Veltman Prescription [43, 44].

[6] We use the abbreviation $\not{p} = \sum_\mu p_\mu \gamma^\mu$, which is commonly called *Feynman slash*.

we obtain

$$i\Sigma_{\alpha\beta} = -i\not{p}\, C_F\, \delta_{\alpha\beta}\frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon} + \ln 4\pi - \gamma_E + 1 + \ln\left(\frac{\mu^2}{-p^2}\right)\right) . \qquad (2.21)$$

The appearance of the $\frac{1}{\epsilon}$ pole in combination with the term $\ln 4\pi - \gamma_E$ is a generic feature of DR. More important, the residue does not depend on μ directly, but, as we will see shortly, only via the strong coupling constant α_s which is scale dependent.

2.4 Renormalization in a Nutshell

Having isolated the divergences in a Laurent series in $\frac{1}{\epsilon}$, we still have to remove the divergences from the Green's functions. This is possible by introducing renormalized fields and QCD parameters (masses and coupling)

$$\begin{aligned} q_B &= \mathcal{Z}_q^{1/2} q_R \qquad & A_B^{A,\mu} &= \mathcal{Z}_3^{1/2} A_R^{A,\mu} \\ g_B &= \mathcal{Z}_g \mu^{\epsilon} g_R \qquad & m_B &= \mathcal{Z}_m m_R \; . \end{aligned} \qquad (2.22)$$

The index B indicates the unrenormalized, "bare" quantities and R the renormalized ones. The $\mathcal{Z}$ factors are the so-called renormalization constants or renormalization factors. They are divergent and have been chosen in such a way that all Green's functions are finite once expressed though renormalized quantities alone.

The renormalization constants are not uniquely defined. They have to absorb the divergences, which fixes their divergent part, but it is possible to add an arbitrary finite term which does not depend on μ or any (external) momenta. These different choices are referred to as *renormalization schemes*. For example, by dropping all finite terms from Eq. (2.21) one can determine the renormalization constant $\mathcal{Z}_q$ of the quark fields to

$$\mathcal{Z}_q^{MS} = 1 - C_F \frac{\alpha_s}{4\pi}\frac{1}{\epsilon} + \mathcal{O}(\alpha_s^2) \; . \qquad (2.23)$$

This renormalization scheme is the *Minimal Subtraction Scheme* (MS) [39]. Its name comes from the fact that only the pole is subtracted, that is the minimal amount of terms possible. However, equally well one could take advantage of this freedom in the definition of the $\mathcal{Z}$ factors and get rid of the spurious $\ln 4\pi - \gamma_E$ term. This scheme is called *Modified Minimal Subtraction* ($\overline{MS}$) [45] and one obtains

$$\mathcal{Z}_q^{\overline{MS}} = 1 - C_F \frac{\alpha_s}{4\pi}\left[\frac{1}{\epsilon} + \ln 4\pi - \gamma_E\right] + \mathcal{O}(\alpha_s^2) \; . \qquad (2.24)$$

This choice obviously is equivalent to a redefinition of the scale μ, because after replacing

$$\mu \to \mu^{\overline{MS}} = \frac{e^{\gamma_E/2}}{\sqrt{4\pi}}\mu \qquad (2.25)$$

in Eq. (2.21) $i\Sigma_{\alpha\beta}$ takes the form

$$i\Sigma_{\alpha\beta} = i\,\not{p}\, C_f\, \delta_{\alpha\beta} \frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon} + 1 + \ln\frac{\mu^2}{-p^2}\right) . \qquad (2.26)$$

The great advantage of the MS-type schemes is that the $\mathcal{Z}$ factors depend on $\mu^{\overline{MS}/MS}$ only through the coupling constant $\alpha_s^{\overline{MS}/MS}$, but never explicitly. In the following, we always adopt the $\overline{MS}$ scheme and can therefore drop the superscript $\overline{MS}$ for simplicity.

2.4.1 The β-function of QCD

From Eq. (2.22) it is obvious that the renormalized coupling constant $g_R \equiv g$ and therefore also α_s depend on the scale[7] μ. We follow [40] and define:

$$\frac{\mathrm{d}g(\mu)}{\mathrm{d}\ln\mu} = \beta\left(g(\mu), \epsilon\right) . \qquad (2.27)$$

$\beta(g, \epsilon)$, the so-called QCD β function, can then be obtained by comparing (2.27) and (2.22):

$$\begin{aligned}
\beta(g,\epsilon) &= \frac{\mathrm{d}}{\mathrm{d}\ln\mu}\left(g_B \mu^{-\epsilon}\mathcal{Z}_g^{-1}\right) = g_B \mu \frac{\mathrm{d}}{\mathrm{d}\mu}\left(\mu^{-\epsilon}\mathcal{Z}_g^{-1}\right) = \\
&= g_B\left(-\epsilon\mu^{-\epsilon}\mathcal{Z}_g^{-1} - \mu^{1-\epsilon}\mathcal{Z}_g^{-2}\frac{\mathrm{d}\mathcal{Z}_g}{\mathrm{d}\mu}\right) = -\epsilon g - g\mu\frac{1}{\mathcal{Z}_g}\frac{\mathrm{d}\mathcal{Z}_g}{\mathrm{d}\mu} \\
&=: -\epsilon g + \beta(g) , \qquad (2.28)
\end{aligned}$$

where we identified

$$\beta(g) := -g\mu\frac{1}{\mathcal{Z}_g}\frac{\mathrm{d}\mathcal{Z}_g}{\mathrm{d}\mu} . \qquad (2.29)$$

Now recall that the renormalization constants do not depend on μ directly in the $\overline{MS}$ scheme. Therefore, $\mathcal{Z}_g$ can be expanded in a Laurent series [41]

$$\mathcal{Z}_g = 1 + \sum_{n=1}^{\infty}\frac{\mathcal{Z}_{g,n}(g)}{\epsilon^n}$$

with μ-independent coefficients. As

$$\beta(g) = -g\mu\frac{1}{\mathcal{Z}_g}\frac{\mathrm{d}\mathcal{Z}_g}{\mathrm{d}\mu} \overset{2.27}{=} -g\frac{1}{\mathcal{Z}_g}\frac{\mathrm{d}\mathcal{Z}_g}{\mathrm{d}g}\cdot\beta(g,\epsilon) , \qquad (2.30)$$

inserting this expansion yields [41]:

$$\left(1 + \frac{\mathcal{Z}_{g,1}}{\epsilon} + \frac{\mathcal{Z}_{g,2}}{\epsilon^2} + \ldots\right)\beta(g) = -g\frac{1}{\epsilon}\beta(g,\epsilon)\left(\frac{\mathrm{d}\mathcal{Z}_{g,1}}{\mathrm{d}g} + \frac{1}{\epsilon}\frac{\mathrm{d}\mathcal{Z}_{g,2}}{\mathrm{d}g}\ldots\right) . \qquad (2.31)$$

[7] The bare coupling g_B is, of course, independent of the scale μ.

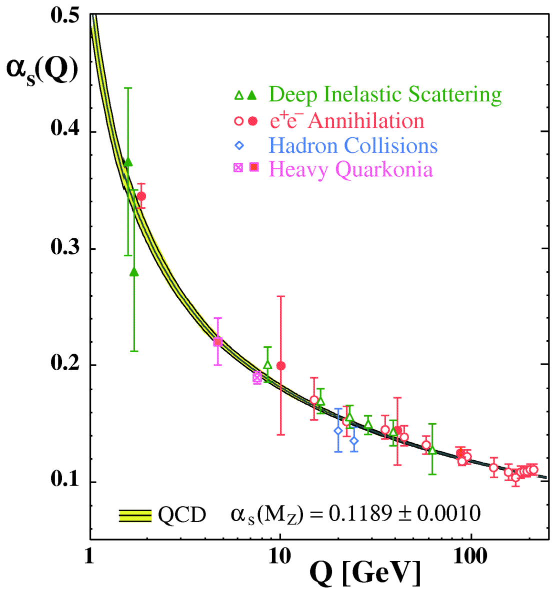


Figure 2.2: The strong coupling α_s in the $\overline{MS}$ scheme at the scale Q. For this plot the four-loop result for the β-function and the initial condition $\alpha_s(M_Z) = 0.1189 \pm 0.0010$ were utilized. The figure is taken from [46].

Since $\beta(g)$ is finite, one can compare the terms of $\mathcal{O}(\epsilon^0)$ on the left- and right-hand side of (2.31) using $\frac{1}{\epsilon}\beta(g,\epsilon) = -g + \mathcal{O}(1/\epsilon)$. One obtains

$$\beta(g) = g^2 \frac{\mathrm{d}\mathcal{Z}_{g,1}}{\mathrm{d}g} = 2g^3 \frac{\mathrm{d}\mathcal{Z}_{g,1}}{\mathrm{d}g^2} \; . \tag{2.32}$$

This implies that the QCD β-function $\beta(g)$ can be obtained from the residue of the $\frac{1}{\epsilon}$-pole alone. $\mathcal{Z}_{g,1}$ itself can be written as a perturbative expansion in the coupling constant g. A detailed calculation of the leading order term can be found in [47] and we will quote the renowned result for the running coupling [48, 49]

$$\mu \frac{\mathrm{d}g(\mu)}{\mathrm{d}\mu} = -\frac{g^3}{16\pi^2}\left[\frac{11}{3}N_c - \frac{2}{3}N_f\right] = -\frac{g^3}{16\pi^2} b_0 \tag{2.33}$$

or equivalently

$$\mu \frac{\mathrm{d}\alpha_s(\mu)}{\mathrm{d}\mu} = -b_0 \frac{\alpha_s^2}{2\pi} \; , \tag{2.34}$$

where N_f is the number of active flavors at the scale μ, i.e. all quark flavors with a mass smaller than μ. The differential equation (2.34) can be solved and the result has the form

$$\alpha_s(\mu) = \frac{4\pi}{b_o \ln\left(\frac{\mu^2}{\Lambda^2_{QCD}}\right)} . \tag{2.35}$$

Λ_{QCD} is a scheme dependent mass scale; it can be determined by "measuring" α_s at some scale μ', i.e. matching theory prediction and experiment at this scale.

As $b_0 > 0$ for $N_f < 17$, we see that for large scales the QCD coupling gets small and the perturbative ansatz works perfectly. In fact, α_s goes to zero for $\mu \to \infty$ and the quarks can be considered as quasi-free non-interacting particles. This behavior has been coined **asymptotic freedom** [49] and is illustrated in Fig. 2.2. On the other hand, it seems that approaching the scale Λ_{QCD} from above causes the coupling to "explode", a behavior sometimes called *infrared slavery*, as (2.35) has a pole at $\mu = \Lambda_{QCD}$. This argument, however, is flawed. Eq. (2.35) was obtained in perturbation theory and therefore, cannot hold in a region where perturbative QCD breaks down. Still, the growth of the coupling at low energies or, equivalently, large distances indicates a peculiar behavior exhibited by QCD: quarks appear only in bound states, the hadrons, and no free quarks have been (experimentally) observed. This phenomenon is called **confinement**.

2.4.2 Renormalization Group Equations

Equation (2.27) is one example of a so-called *renormalization group equation* (RGE). This class of equations describes the dependence of renormalized quantities on the renormalization scale μ. For example, the RGE for the mass will take the form [40]

$$\mu \frac{\mathrm{d}m(\mu)}{\mathrm{d}\mu} = -\gamma_m(g) m(\mu) \quad \text{with} \quad \gamma_m(g) = \frac{\mu}{\mathcal{Z}_m} \frac{\mathrm{d}\mathcal{Z}_m}{\mathrm{d}\mu} \; . \tag{2.36}$$

The renormalization group function γ_m is called quark mass *anomalous dimension*. Using the same line of argumentation as previously for the β-function, one easily finds that γ_m can be obtained from the residue of the corresponding $\mathcal{Z}_m$ factor via

$$\gamma_m = -2g^2 \frac{\mathrm{dRes}\,(\mathcal{Z}_m)}{\mathrm{d}g^2} = -2\alpha_s \frac{\mathrm{dRes}\,(\mathcal{Z}_m)}{\mathrm{d}\alpha_s} \; . \tag{2.37}$$

This formula is true not only for masses but for all renormalized quantities in the $\overline{MS}$ scheme. The anomalous dimensions can generically be expanded in a series in α_s

$$\gamma_m(\alpha_s) = \gamma_m^{(0)} \frac{\alpha_s}{2\pi} + \gamma_m^{(1)} \left(\frac{\alpha_s}{2\pi}\right)^2 + \ldots \; .$$

In leading order in α_s the RGE (2.36) can be solved by separation of variables. One obtains

$$m(\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_i)}\right)^{\frac{\gamma_m^{(0)}}{b_0}} m(\mu_i) \; , \tag{2.38}$$

where μ_i, $m(\mu_i)$ are the initial conditions. It should be noted that the RGE given in (2.36) can trivially be rewritten in the "standard" form

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma_O\right] O^{\text{ren}}(\mu) = 0 \tag{2.39}$$

for a generic renormalized local operator $O^{\text{ren}}(\mu, \alpha_s(\mu))$ with anomalous dimension γ_O. An operator that follows such a RGE is called *multiplicatively renormalizable*, as its dependence on the scale μ can be expressed via a single multiplicative factor, see (2.38).

In general, the situation is more complicated. Due to renormalization an operator can be affected by admixtures of operators with the same quantum numbers. If a set of operators $\{O_i\}$, $i = 1, \ldots, n$ is closed under renormalization, that is each operator in the set only receives admixtures due to operators also in $\{O_i\}$, the RGEs have the form

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma_{ij}\right] \vec{O}_j^{\text{ren}}(\mu) = 0 \ , \tag{2.40}$$

where γ is the $n \times n$ matrix of anomalous dimensions and $\vec{O}$ a vector consisting of the operators O_i. By diagonalizing the matrix γ, Eq. (2.40) can be reduced to n decoupled differential equations of the form (2.39).

What to keep in mind

The subtleties and challenges of the renormalization procedure are numerous and we refer the reader to standard textbooks, such as [50, 51], for further reading. For our purposes this small detour will be sufficient. Apart from Eqs. (2.39), (2.38) and (2.40) it is useful to keep this convenient property of our renormalization scheme in mind:

> *In the $\overline{MS}$ scheme the leading order anomalous dimension (of an operator) is equal to the negative of the double residue of the corresponding renormalization constant.*

"Shut up and calculate!"

N. David Mermin
'What's Wrong with this Pillow?'

3 Technical Background

As mentioned briefly in the Introduction, two more advanced concepts are to be presented in this chapter; the first one being the so-called spinor formalism. It is based on the early observation by Weyl [52] and van der Waerden [53] that the Dirac equation [54] for 4-spinors of massless fermions can be rewritten in terms of two separate differential equations for two-component spinors, the Weyl spinors. Over the course of Sect. 3.1 we will show that working with Weyl spinors as basic objects of the theory allows for a simple classification of the transformation properties of generic tensors with respect to the Lorentz group. While this formalism is utilized frequently in supersymmetric theories, where the spinor nature of the supersymmetric generators makes this a natural choice, it rarely sees use in QCD or QED, although its merits have been pointed out frequently [55, 56].

We begin in Sect. 3.1 with van der Waerden's idea that the transformation of chiral and antichiral Weyl spinors can conveniently be indicated by employing dotted and undotted indices. After explaining how to include Lorentz indices in this spinor notation, we give a short summary of translation rules and relations useful for working in this formalism. The following section deals with gauge fields and we show that, analogously to the 4-spinor, the field strength tensor can be decomposed into two components transforming according to irreducible representations of the Lorentz group. In Sect. 3.1.3 we discuss how to project arbitrary tensor operators onto definite spin, which turns out to be straightforward in spinor notation and is, in fact, the main reason why we work with it. Note that only recently a detailed review [56] on these two-component spinor techniques was published[1].

The second part of Chap. 3 is dedicated to the concept of conformal symmetry and its application to multi-particle light-ray operators (the study of the

[1] [56] uses different sign conventions in its definitions and therefore one has to be careful not to miss a sign. The authors of [56] provide alternative versions with different conventions on their website [57] .

renormalization properties of such operators is the aim of Chapter 4). The symmetry itself has been known for more than a century and its early applications included complicated problems in electrostatics. Moreover, the study of the conformal properties of two-dimensional field theories is a large area of research due to connections with string theory. In 4-dimensional field theories, conformal symmetry has generally been treated rather stepmotherly, as an existing conformal symmetry is usually broken at quantum level. We explain this in more detail in Section 3.2. A notable exception are the so-called super-conformal $\mathcal{N} = 4$ Yang-Mills theories, which feature a vanishing β-function. They are the basis of the famous AdS/CFT conjecture [12] – one the most active fields in mathematical physics [58].

As conformal symmetry is not among the standard tools of QCD, see [59] for the current state-of-the-art, we give a short introduction to the structure of the conformal algebra in Sect. 3.2.1 and show how the generators look like in spinor notation. After restricting ourselves to the so-called $SL(2,\mathbb{R})$ subgroup, which corresponds to the projective Moebius transformations on a light-ray, we construct a basis of one-particle operators with "good" conformal properties[2] in Sect. 3.2.3. While our explicit derivation does not take into account issues related to the fact that QCD is a gauge theory, this basis can be generalized to full QCD with the tools introduced in this chapter. This construction strategy, along with the new, complete one-particle basis, see Eq. (3.63), represents the main result of this chapter and one of the central novelties of this thesis.

3.1 Spinor Formalism

In the case of free massless spin-1/2 fermion fields q the Dirac equation [54] assumes the form

$$p^{\mu}\,(\gamma_{\mu})_{ij}\,q_j(p) = 0, \tag{3.1}$$

where p is the momentum of the fermion, i, j are spinor indices and μ is a Lorentz index. It is possible to decouple the equation for the upper two components of the bispinor from the equation for the lower two components by a specific choice for the γ matrix basis. This is the so-called Weyl representation, see e.g. [33], which, of course, respects the usual commutation relations

$$\{\gamma_{\mu}, \gamma_{\nu}\} = 2g_{\mu\nu} \quad \text{for } \mu, \nu = 0, 1, 2, 3\,. \tag{3.2}$$

The fact that such a separation is possible is equivalent to the statement that the Dirac equation preserves the chirality of massless fermions.

[2]Each operator has well-defined collinear twist, helicity and conformal spin.

Following [53], it is then convenient to introduce the following notation for the four-dimensional Dirac bispinor:

$$q = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\beta}} \end{pmatrix}, \qquad \bar{q} = (\chi^\beta, \bar{\psi}_{\dot{\alpha}}). \tag{3.3}$$

ψ corresponds to the chiral, $\bar{\chi}$ to the anti-chiral Weyl spinor. The somewhat peculiar notation with dotted and undotted indices has a distinct advantage [53]:
The irreducible representations of the Lorentz group are labeled by two spins $(s, \bar{s})$. The chiral spinor ψ_α transforms according to $(1/2, 0)$, the anti-chiral spinor $\bar{\chi}^{\dot{\alpha}}$ according to $(0, 1/2)$. Hence, it is very simple to read off the transformation properties in this notation. It is now obvious that the standard Dirac spinor does not transform according to an irreducible representation of the Lorentz group; it rather transforms as $(1/2, 0) \oplus (0, 1/2)$.

However, in order for the separation of dotted and undotted indices, i.e. chiral and anti-chiral fields, to be useful for the classification of the transformation properties of a generic operator, it is necessary to convert also all Lorentz indices into spinor indices. This can be achieved in the following way, see also [60]:

- take an arbitrary covariant four-vector x^μ
- then define

  $$x_{\alpha\dot{\alpha}} := x_\mu (\sigma^\mu)_{\alpha\dot{\alpha}} \tag{3.4}$$

  where

  $$\sigma^\mu = (\mathbb{1}, \vec{\sigma}) \tag{3.5}$$

  and $\vec{\sigma}$ are the usual Pauli matrices
- the 2×2 matrix $x_{\alpha\dot{\alpha}}$ then contains the full information on the vector x_μ and has the transformation properties under Lorentz transformations as indicated by its indices.

One can see that this procedure actually does what it claims by observing that each covariant four-vector x^μ can be mapped to a hermitian 2×2 matrix x

$$x = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \equiv x_\mu \sigma^\mu. \tag{3.6}$$

It can easily be checked that $\det x = x_\mu x^\mu = x^2$ and that a Lorentz transformation $x'_\mu = \Lambda_\mu{}^\nu x_\nu$ corresponds to a rotation of the form $x' = M x M^\dagger$, where $M \in SL(2, \mathbb{C})$. The homomorphism $\Lambda \to M$ must then define a two-dimensional (spinor) representation of the Lorentz group $u' = Mu$ [60]. At first glance, it is actually possible to find four homomorphisms $\Lambda \rightarrow M, M^*, M^{-1,T}$ and $M^{-1\dagger}$, each of which could define a different representation and the corresponding spinors are usually denoted as $u_\alpha, \bar{u}_{\dot{\alpha}}, u^\alpha$ and $\bar{u}^{\dot{\alpha}}$, respectively, i.e. $u'_\alpha = M_\alpha{}^\beta u_\beta$, $\bar{u}'_{\dot{\alpha}} = M^{*\dot{\beta}}_{\dot{\alpha}} \bar{u}_{\dot{\beta}}$ etc. However, not all these representations are independent, as the Lorentz group has only two non-equivalent spinor representations $(1/2, 0)$ and $(0, 1/2)$. One finds that

$$i\sigma_2 M = M^{-1,T} i\sigma_2 \qquad \text{and} \qquad i\sigma_2 M^* = M^{-1,\dagger} i\sigma_2 . \tag{3.7}$$

The transformation properties indicated by the notation of Eq. (3.4) are therefore indeed realized.

3.1.1 Working with the Spinor Formalism

We have seen that it is possible to map a Lorentz vector x^μ to an $SL(2,\mathbb{C})$ matrix $x_{\alpha\dot{\alpha}}$ which transforms under Lorentz transformations as a tensor product of two Weyl spinors. While it is obvious that this simplifies the identification of the transformation properties, it is not yet clear how to work with this notation in practice.

Writing the operator $i\sigma_2$ which intertwines the equivalent representations u_α and u^α as well as $\bar{u}^{\dot{\alpha}}$ and $\bar{u}_{\dot{\alpha}}$ in explicit matrix form

$$i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{3.8}$$

it is easy to see that $i\sigma_2$ is equal to the two dimensional Levi-Civita tensor ϵ. By defining

$$\epsilon_{12} = \epsilon^{12} = -\epsilon_{\dot{1}\dot{2}} = -\epsilon^{\dot{1}\dot{2}} = 1 \tag{3.9}$$

we end up with the following rule for raising and lowering the spinor indices

$$u^\alpha = \epsilon^{\alpha\beta} u_\beta , \qquad u_\alpha = u^\beta \epsilon_{\beta\alpha} , \qquad \bar{u}^{\dot{\alpha}} = \bar{u}_{\dot{\beta}} \epsilon^{\dot{\beta}\dot{\alpha}} , \qquad \bar{u}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{u}^{\dot{\beta}} . \tag{3.10}$$

The definition (3.9) is not unique and an equivalent choice can be found in [56]. Note that $\epsilon_\alpha{}^\beta = -\epsilon^\beta{}_\alpha = \delta^\beta_\alpha$ and $\epsilon^{\dot{\alpha}}{}_{\dot{\beta}} = -\epsilon_{\dot{\beta}}{}^{\dot{\alpha}} = \delta^{\dot{\alpha}}_{\dot{\beta}}$. Due to the trivial identity

$$\epsilon_{ab}\epsilon_{cd} = \epsilon_{ac}\epsilon_{bd} - \epsilon_{ad}\epsilon_{bc} , \qquad a, d, b, c \in 1, 2 , \tag{3.11}$$

the Fierz transformations for Weyl spinors take the simple form

$$(u_1 u_2)(v_1 v_2) = (u_1 v_1)(u_2 v_2) - (u_1 v_2)(u_2 v_1) . \tag{3.12}$$

For products of two spinors one has to keep in mind that dotted and undotted indices "do not talk to each other". That means the product $u\bar{v} \stackrel{?}{=} u_\alpha \bar{v}^{\dot{\alpha}}$ is ill defined as the two spinors have different transformation properties and their indices can therefore never be contracted, even though the notation $_\alpha$ and $^{\dot\alpha}$ may bear some similarity. When the spinor indices are not displayed explicitly, it is usually assumed that undotted indices are contracted "up-down"

$$(uv) = u^\alpha v_\alpha = -u_\alpha v^\alpha \,, \tag{3.13}$$

whereas dotted indices are contracted "down-up"

$$(\bar u \bar v) = \bar u_{\dot\alpha} \bar v^{\dot\alpha} = -\bar u^{\dot\alpha} \bar v_{\dot\alpha} \,, \tag{3.14}$$

which is consistent with Eq. (3.10). Note that mixing up these conventions is a standard source for sign errors.

As mentioned above, the $SL(2,\mathbb{C})$ matrix M is not unique and one can in principle map the vector x to $x_{\alpha\dot\alpha} = x_\mu(\sigma^\mu)_{\alpha\dot\alpha}$ or to $\bar x^{\dot\alpha\alpha} = x^\mu(\bar\sigma_\mu)^{\dot\alpha\alpha}$, where $(\bar\sigma^\mu)^{\dot\alpha\beta} = (\mathbb{1}, -\vec\sigma) = (\sigma^\mu)^{\beta\dot\alpha}$. It turns out that introducing both, σ and $\bar\sigma$, is rather convenient, as one can easily express the Lorentz invariant scalar product

$$a_\mu b^\mu = \frac{1}{2} a_{\alpha\dot\alpha} \bar b^{\dot\alpha\alpha} = \frac{1}{2} \bar a^{\dot\alpha\alpha} b_{\alpha\dot\alpha} \tag{3.15}$$

as well as the Dirac matrices and the charge conjugation matrix $\mathcal{C}$

$$\gamma^\mu = \begin{pmatrix} 0 & [\sigma^\mu]_{\alpha\dot\beta} \\ [\bar\sigma^\mu]^{\dot\alpha\beta} & 0 \end{pmatrix} \,, \qquad \not{a} = \begin{pmatrix} 0 & a_{\alpha\dot\beta} \\ \bar a^{\dot\alpha\beta} & 0 \end{pmatrix} \tag{3.16}$$

$$\sigma^{\mu\nu} = \begin{pmatrix} [\sigma^{\mu\nu}]_\alpha{}^\beta & 0 \\ 0 & [\bar\sigma^{\mu\nu}]^{\dot\alpha}{}_{\dot\beta} \end{pmatrix} \,, \quad \gamma_5 = \begin{pmatrix} -\delta_\alpha^\beta & 0 \\ 0 & \delta^{\dot\alpha}_{\dot\beta} \end{pmatrix} \,, \quad \mathcal{C} = \begin{pmatrix} -\epsilon_{\alpha\beta} & 0 \\ 0 & -\epsilon^{\dot\alpha\dot\beta} \end{pmatrix} \tag{3.17}$$

in terms of these two matrices. Here

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{i}{2}[\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu]_\alpha{}^\beta \,, \qquad (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = \frac{i}{2}[\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu]^{\dot\alpha}{}_{\dot\beta} \,. \tag{3.18}$$

There also exist two useful identities involving the σ_μ matrices that come handy for the calculation of Feynman diagrams:

$$\sigma^\mu_{\alpha\dot\alpha}\,(\bar\sigma^\nu)^{\dot\alpha\alpha} = 2g^{\mu\nu} \,, \qquad \sigma^\mu_{\alpha\dot\alpha}\,\bar\sigma^{\dot\beta\beta}_\mu = 2\delta_\alpha^\beta\,\delta_{\dot\alpha}^{\dot\beta} \,. \tag{3.19}$$

3.1.2 Gluon Fields and Equations of Motion

In addition to the quark fields, represented by the Weyl spinors ψ and $\bar\chi$, cp. (3.3), we need an equivalent expression for the gluon fields. The gluon field strength tensor[3] $F_{\mu\nu}$ transforms as $(1,0) \oplus (0,1)$. Therefore, one should find a decomposition into two new objects that transform as $(1,0)$ and $(0,1)$, respectively. Let us consider

$$F_{\alpha\beta,\dot\alpha\dot\beta} = \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}F_{\mu\nu}. \tag{3.20}$$

[3] We will for the moment neglect color indices; they are not relevant in what follows.

Contracting (3.20) with $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot\alpha\dot\beta}$ gives zero because of (3.19) as $F_{\mu\nu}$ is antisymmetric under the exchange $\mu \leftrightarrow \nu$. Symmetrization of $F_{\alpha\beta,\dot\alpha\dot\beta}$ with respect to $\alpha\beta$ and $\dot\alpha\dot\beta$ gives obviously also zero. That means that one is able to define

$$F_{\alpha\beta,\dot\alpha\dot\beta} = 2\left(\epsilon_{\dot\alpha\dot\beta} f_{\alpha\beta} - \epsilon_{\alpha\beta}\bar f_{\dot\alpha\dot\beta}\right), \tag{3.21}$$

where $f_{\alpha\beta}$ is a chiral and $\bar f_{\dot\alpha\dot\beta}$ is an anti-chiral tensor. The factor 2 is included for convenience. Both $f_{\alpha\beta}$ and $\bar f_{\dot\alpha\dot\beta}$ are symmetric and thus transform as $(1,0)$ and $(0,1)$, respectively. They can be expressed as

$$f_{\alpha\beta} = \frac{i}{4}\sigma^{\mu\nu}_{\alpha\beta}F_{\mu\nu} \qquad \text{and} \qquad \bar f_{\dot\alpha\dot\beta} = -\frac{i}{4}\bar\sigma^{\mu\nu}_{\dot\alpha\dot\beta}F_{\mu\nu}\,. \tag{3.22}$$

Hence, the gluon field strength tensor F and the dual tensor $\tilde F$,

$$F^{\mu\nu} = \frac{i}{2}\left(\sigma^{\mu\nu}_{\alpha\beta}f^{\alpha\beta} - \bar\sigma^{\mu\nu}_{\dot\alpha\dot\beta}\bar f^{\dot\alpha\dot\beta}\right) \quad \text{and} \quad \widetilde F^{\mu\nu} = \frac{1}{2}\left(\sigma^{\mu\nu}_{\alpha\beta}f^{\alpha\beta} + \bar\sigma^{\mu\nu}_{\dot\alpha\dot\beta}\bar f^{\dot\alpha\dot\beta}\right), \tag{3.23}$$

are completely determined by f and $\bar f$.

The equations of motion for quark and gluon fields also can be translated into the spinor language. Let $D_\mu = \partial_\mu - igA_\mu$ be the usual covariant derivative. The Dirac equations for the quark fields read

$$\bar D^{\dot\alpha\alpha}\psi_\alpha(x) = 0\,, \qquad D_{\alpha\dot\alpha}\bar\chi^{\dot\alpha}(x) = 0\,. \tag{3.24}$$

The equations of motion for the fields $f, \bar f$ are given by

$$\bar D_{\dot\beta}{}^{\alpha,AB} f^B_{\alpha\beta} = g\left(\bar\psi^a_{\dot\beta}T^A_{ab}\psi^b_\beta + \chi^a_\beta T^A_{ab}\bar\chi^b_{\dot\beta}\right),\; D^{AB}_{\beta\dot\alpha}\bar f^{\dot\alpha,B}{}_{\dot\beta} = g\left(\bar\psi^a_{\dot\beta}T^A_{ab}\psi^b_\beta + \chi^a_\beta T^A_{ab}\bar\chi^b_{\dot\beta}\right), \tag{3.25}$$

where $A, B = 1,\ldots,8$ and $a, b = 1,\ldots,3$ are color indices for the adjoint and fundamental representation[4].

3.1.3 General Tensors

With the translation rules described in the previous section, every tensor $T_{\mu_1\ldots\mu_n}$ given in vector representation can be linked to a tensor $T_{\alpha_1\ldots\alpha_n,\dot\beta_1\ldots\dot\beta_{\bar n}}$ in spinor represention via

$$T_{\alpha_1\ldots\alpha_n,\dot\beta_1\ldots\dot\beta_n} = \sigma^{\mu_1}_{\alpha_1\dot\beta_1}\ldots\sigma^{\mu_n}_{\alpha_n\dot\beta_n}T_{\mu_1\ldots\mu_n}\,. \tag{3.26}$$

For applications in QCD it is often necessary to project an operator onto given spin or, equivalently, twist. This is usually done by symmetrizing and antisymmetrizing the (Lorentz) indices. In spinor representation this is much simpler,

[4]Note that the color structure as well as a factor $\pm i$ is missing in our original publication [62].

since one does not have to distinguish between vector and spinor indices. The leading twist part of the tensor (3.26) is given by

$$T_{\{\alpha_1\ldots\alpha_n\},\{\dot\beta_1\ldots\dot\beta_n\}},$$

where $\{\ldots\}$ denotes the symmetrization with respect to the included indices.

Let us introduce an auxiliary spinor ξ. Symmetrization is then conveniently achieved by contraction of all open spinor indices with ξ. In our example:

$$T_\xi = \xi^{\alpha_1}\ldots\xi^{\alpha_n}\,T_{\alpha_1\ldots\alpha_n,\dot\beta_1\ldots\dot\beta_{\bar n}}\,\bar\xi^{\dot\beta_1}\ldots\bar\xi^{\dot\beta_{\bar n}}\,. \qquad (3.27)$$

For the fundamental fields we define the abbreviations

$$\psi_\xi = (\xi\psi) = \xi^\alpha\psi_\alpha \qquad\qquad f_\xi = \xi^\alpha\xi^\beta f_{\alpha\beta}\,,$$
$$\bar\chi_\xi = (\bar\chi\bar\xi) = \bar\chi_{\dot\alpha}\bar\xi^{\dot\alpha} \qquad\qquad \bar f_\xi = \bar f_{\dot\alpha\dot\beta}\bar\xi^{\dot\alpha}\bar\xi^{\dot\beta}\,. \qquad (3.28)$$

If the tensor $T_{\alpha_1\ldots\alpha_n,\dot\beta_1\ldots\dot\beta_n}$ had been symmetric from the very beginning, T_ξ would contain the same information as the tensor itself. In order to restore the full tensor, we have to introduce derivatives with respect to the spinor components ξ^α and $\bar\xi^{\dot\alpha}$:

$$\partial_\beta\xi^\alpha = \frac{\partial}{\partial\xi^\beta}\xi^\alpha = \epsilon_\beta{}^\alpha = \delta^\alpha_\beta\,, \qquad \bar\partial^{\dot\beta}\bar\xi_{\dot\alpha} = \frac{\partial}{\partial\bar\xi_{\dot\beta}}\bar\xi_{\dot\alpha} = \epsilon^{\dot\beta}{}_{\dot\alpha} = \delta^{\dot\beta}_{\dot\alpha}\,. \qquad (3.29)$$

Here one has to keep in mind that the raising and lowering rules for the derivatives

$$\frac{\partial}{\partial\xi^\beta} = \epsilon_{\beta\alpha}\frac{\partial}{\partial\xi_\alpha}\,, \qquad\qquad \frac{\partial}{\partial\bar\xi_{\dot\beta}} = \epsilon^{\dot\beta\dot\alpha}\frac{\partial}{\partial\bar\xi^{\dot\alpha}} \qquad (3.30)$$

differ from the rules for the spinors themselves, compare (3.10). It is now straightforward to restore the symmetric tensor from T_ξ. One obtains:

$$T_{\alpha_1\ldots\alpha_n,\dot\beta_1\ldots\dot\beta_{\bar n}} = \frac{(-1)^{\bar n}}{n!\,\bar n!}\frac{\partial}{\partial\xi^{\alpha_1}}\cdots\frac{\partial}{\partial\xi^{\alpha_n}}\frac{\partial}{\partial\bar\xi^{\dot\beta_1}}\cdots\frac{\partial}{\partial\bar\xi^{\dot\beta_{\bar n}}}T_\xi\,. \qquad (3.31)$$

3.2 Conformal Symmetry

The Poincaré group is the fundamental symmetry group of space-time. It is an extension of the Lorentz group and also includes translations. In fact a possible definition of an elementary particle following Wigner's classification [60] is:
An elementary particle corresponds to a nonnegative energy, irreducible representation of the Poincaré group.
Invariance under Poincaré transformations is one of the requirements for any meaningful quantum field theory. If the Lagrangian of such a theory does not contain any intrinsic scale (such as a mass), it possesses an additional classical symmetry: the so-called **dilatation** symmetry which corresponds to scale

transformations. The massless QCD Lagrangian does not only enjoy Poincaré and dilatation symmetry, but is also invariant under **special conformal transformations**, a combination of translations and space-time inversions. This additional symmetry of the Lagrangian is not obvious and can only be checked by explicit calculation. The group encompassing Poincaré transformations, dilatations and special conformal transformations is the so-called **conformal group**. It is generated by the maximal non-trivial extension of the Poincaré algebra.

Since we are working with massless quarks, the Lagrangian of QCD enjoys conformal symmetry at classical level. However, this symmetry is broken by quantum effects. The corresponding anomaly is the conformal anomaly: the classically conserved dilatation current $J^\mu_{\rm D}$,

$$[\partial_\mu J^\mu_{\rm D}(x)]_{\rm classical} = 0$$

receives quantum corrections resulting in [59]

$$\partial_\mu J^\mu_{\rm D}(x) = -\frac{\beta(\alpha_s)\alpha_s}{8\pi} F^A_{\mu\nu}(x) F^{\mu\nu}_A(x) \; . \tag{3.32}$$

We see that the non-vanishing QCD β-function, see Chap. 2,

$$\beta(\alpha_s) = -b_0 \alpha_s^2 + \mathcal{O}(\alpha_s^3) \tag{3.33}$$

is responsible for the breaking of scale invariance. Actually, this is a quite intuitive statement, as we have already seen that the quantum effects require a renormalization procedure that forces us to introduce some scale μ into our theory. This automatically breaks scale invariance and the β-function can be seen as a measure for the strength of the theory's dependence on this scale.

However, there are two situations, where conformal symmetry is still present in QCD. First of all, for $\alpha_s \to 0$, i.e. for $\mu \to \infty$, the right-hand side of Eq. (3.32) vanishes due to the running of the coupling and conformal symmetry is restored in this limit. The second case arises if one stays strictly "$\mathcal{O}(\alpha_s^1)$"; that is, one does not take into account corrections of the order α_s^2 in the calculations. To this accuracy the conformal anomaly vanishes as the QCD β-function, cp. (3.33), does not contribute in order α_s^1.

As we want to study the one-loop renormalization of baryon distribution amplitudes in Chap. 4, conformal symmetry stays intact and is the key to solving the renormalization group equations. The reason for this is that the RGEs must, to our accuracy, respect conformal symmetry; we will see that this simplifies the calculation. In the following, we discuss the features of conformal symmetry relevant to our objective. For a detailed review we refer the reader to [59].

3.2.1 The Generators of the Conformal Group

The algebra of the full conformal group involves the 10 standard generators of the Poincaré group (4 translations $\mathbf{P}_\mu$, 6 Lorentz boosts and spatial rotations

$\mathbf{M}_{\mu\nu}$) as well as the dilatation $\mathbf{D}$ and four special conformal transformations $\mathbf{K}_\mu$. The algebra has the following form [61]

$$
\begin{aligned}
&[\mathbf{D},\mathbf{K}_\mu]=i\mathbf{K}_\mu, && [\mathbf{K}_\mu,\mathbf{P}_\nu]=-2i(g_{\mu\nu}\mathbf{D}+\mathbf{M}_{\mu\nu}), && [\mathbf{D},\mathbf{M}_{\mu\nu}]=0,\\
&[\mathbf{D},\mathbf{P}_\mu]=-i\mathbf{P}_\mu, && [\mathbf{K}_\rho,\mathbf{M}_{\mu\nu}]=i(g_{\rho\mu}\mathbf{K}_\nu-g_{\rho\nu}\mathbf{K}_\mu), && [\mathbf{K}_\mu,\mathbf{K}_\nu]=0,\\
&[\mathbf{P}_\mu,\mathbf{P}_\nu]=0, && [\mathbf{M}_{\mu\nu},\mathbf{P}_\rho]=g_{\mu\rho}\mathbf{P}_\nu-g_{\nu\rho}\mathbf{P}_\mu,\\
&[\mathbf{M}_{\mu\nu},\mathbf{M}_{\rho\sigma}]=g_{\mu\rho}\mathbf{M}_{\nu\sigma}-g_{\mu\sigma}\mathbf{M}_{\nu\rho}-g_{\nu\rho}\mathbf{M}_{\mu\sigma}+g_{\nu\sigma}\mathbf{M}_{\mu\rho}\,. \qquad (3.34)
\end{aligned}
$$

It can be translated into spinor notation in the usual way. More important than the algebra is the action of the generators on the quantum fields in the spinor representation. To this end, let us define the following abbreviations $\Phi_\xi=\{\psi_\xi,\chi_\xi,f_\xi\}$ and $\bar{\Phi}_\xi=\{\bar{\psi}_\xi,\bar{\chi}_\xi,\bar{f}_\xi\}$, where we used the notation introduced in Sect. 3.1.3. The action on Φ then takes the form

$$i[\mathbf{P}_{\alpha\dot\alpha},\Phi(x)]=\partial_{\alpha\dot\alpha}\Phi(x)\;\equiv\;iP_{\alpha\dot\alpha}\Phi(x)\,, \qquad (3.35a)$$

$$i[\mathbf{D},\Phi(x)]=\frac{1}{2}\left(x_{\alpha\dot\alpha}\partial^{\alpha\dot\alpha}+2t+\xi^\alpha\frac{\partial}{\partial\xi^\alpha}+\bar\xi_{\dot\alpha}\frac{\partial}{\partial\bar\xi_{\dot\alpha}}\right)\Phi(x)\;\equiv\;iD\,\Phi(x)\,, \qquad (3.35b)$$

$$i[\mathbf{M}_{\alpha\beta},\Phi(x)]=\frac{1}{4}\left(x_{\alpha\dot\gamma}\partial_\beta{}^{\dot\gamma}+x_{\beta\dot\gamma}\partial_\alpha{}^{\dot\gamma}-2\xi_\alpha\frac{\partial}{\partial\xi^\beta}-2\xi_\beta\frac{\partial}{\partial\xi^\alpha}\right)\Phi(x)\equiv iM_{\alpha\beta}\Phi(x), \qquad (3.35c)$$

$$i[\overline{\mathbf{M}}_{\dot\alpha\dot\beta},\Phi(x)]=\frac{1}{4}\left(x_{\gamma\dot\alpha}\partial^\gamma{}_{\dot\beta}+x_{\gamma\dot\beta}\partial^\gamma{}_{\dot\alpha}-2\bar\xi_{\dot\alpha}\frac{\partial}{\partial\bar\xi^{\dot\beta}}-2\bar\xi_{\dot\beta}\frac{\partial}{\partial\bar\xi^{\dot\alpha}}\right)\Phi(x)\equiv i\overline{M}_{\dot\alpha\dot\beta}\Phi(x), \qquad (3.35d)$$

$$
\begin{aligned}
i[\mathbf{K}_{\alpha\dot\alpha},\Phi(x)]=&\left(x_{\alpha\dot\gamma}x_{\gamma\dot\alpha}\,\partial^{\gamma\dot\gamma}+2tx_{\alpha\dot\alpha}+2\xi_\alpha\bar{x}_{\dot\alpha}{}^{\beta}\frac{\partial}{\partial\xi^\beta}+2\bar\xi_{\dot\alpha}x_{\alpha\dot\beta}\frac{\partial}{\partial\bar\xi_{\dot\beta}}\right)\Phi(x)\\
&\equiv iK_{\alpha\dot\alpha}\Phi(x)\,. \qquad (3.35e)
\end{aligned}
$$

As usual $\partial_{\alpha\dot\alpha}=\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$ and t is the *geometric twist* of the field Φ. It is defined as $t=l^{can}-s-\bar{s}$, where l^{can} is the canonical scaling dimension of the field and $(s,\bar{s})$ its Lorentz spin. Our notation reflects the necessity to separate between the generators' action on quantum fields – denoted by boldface letters – and the corresponding differential operators acting on the coordinates, for which we use normal fonts.

Since we will be working with fast moving hadrons, one can consider them to be a tight bunch of partons on a light-ray. It is therefore convenient to introduce two light-like vectors

$$n_{\alpha\dot\alpha},\;\; n^2=0 \qquad \text{and} \qquad \tilde{n}_{\alpha\dot\alpha},\;\; \tilde{n}^2=0 \qquad \text{with}\;\; (n\cdot\tilde{n})=\frac{1}{2}\,. \qquad (3.36)$$

As $\det n=n^2=0$, it is possible to write a light-like vector as a tensor product of two spinors

$$n_{\alpha\dot\alpha}=\lambda_\alpha\bar\lambda_{\dot\alpha} \qquad\qquad \tilde{n}_{\alpha\dot\alpha}=\mu_\alpha\bar\mu_{\dot\alpha}\,. \qquad (3.37)$$

This is easy to understand: the vanishing of the determinant is synonymous to a non-trivial kernel of the matrix n. Hence, there exists a two component "vector" $\bar{v}^{\dot\alpha}$ so that $n_{\alpha\dot\alpha}\bar{v}^{\dot\alpha} = 0$. The construction (3.37) takes this into account, as one can take $\lambda \perp v$. The normalization $(n \cdot \tilde{n}) = 1/2$ corresponds to $(\mu\lambda) = -(\lambda\mu) = 1$. As we have some freedom in choosing the two light-rays, it is convenient to take

$$\lambda^\alpha = (1,0)\,, \qquad \lambda_\alpha = (0,1)\,,$$
$$\mu^\alpha = (0,1)\,, \qquad \mu_\alpha = (-1,0)\,. \tag{3.38}$$

With this choice the derivatives in n and $\tilde{n}$ direction become

$$\partial^{2\dot2} = 2(n\cdot\partial)\,, \qquad \partial^{1\dot1} = 2(\tilde{n}\cdot\partial) \tag{3.39}$$

and the derivatives in the perpendicular plane are just $\partial^{1\dot2}$ and $\partial^{2\dot1}$.

3.2.2 The $SL(2,\mathbb{R})$ Subgroup

It was already noted that fields living on the light-ray n will play a central role in our analysis. The coordinates of such fields can readily be described by a single real number z:

$$\Phi(x) \to \Phi(zn) \equiv \Phi(z). \tag{3.40}$$

A generic non-local light-ray operator

$$\begin{aligned}\mathcal{O}(z_1,z_2,..,z_k) :&= \Phi_1(z_1)\Phi_2(z_2)\cdot\ldots\cdot\Phi_k(z_k)\\ &= \prod_{i=1}^{k}\left[\sum_{n_i}\frac{z_i^{n_i}}{2^{n_i}n_i!}(\partial^{2\dot2})^{n_i}\Phi_i(0)\right]\end{aligned} \tag{3.41}$$

can be written as a sum over local operators of the same *collinear twist*[5] as $\mathcal{O}$ itself. This is obvious as each derivative $\partial^{2\dot2}$ adds one unit of dimension but also, as $\partial^{2\dot2} \sim n\cdot\partial$, one unit of spin projection onto the light-ray. So each polynomial in z_i uniquely defines one local operator with the same twist as $\mathcal{O}$.

By restricting the fields to the light-ray we also reduce the symmetry of our problem. Instead of the full conformal group only the subgroup on the ray, $SL(2,\mathbb{R})$, is needed. It corresponds to the Moebius transformations on the line $x = zn$:

$$z \to \frac{az+b}{cz+d}\,, \qquad ab-cd = 1\,, \qquad a,b,c,d \in \mathbb{R}\,. \tag{3.42}$$

Hence, there are just three generators which are usually labeled as S_+, S_- and S_0 [59]. S_+ and S_- are related to special conformal transformations and translations, respectively. S_0 is given as a linear combination of dilatation and Lorentz

[5] *Collinear* or *light-cone twist* is defined as canonical dimension minus spin projection on the light-cone.

rotation. For our purpose only the explicit expressions

$$S_+ = \frac{1}{2} x_{2\dot\gamma} x_{\gamma\dot 2} \partial^{\gamma\dot\gamma} + x_{2\dot 2}\left(t + \xi^\beta \frac{\partial}{\partial \xi^\beta} + \bar\xi^{\dot\beta}\frac{\partial}{\partial \bar\xi^{\dot\beta}}\right) - x_{\beta\dot 2}\xi^\beta \frac{\partial}{\partial \xi^2} - x_{2\dot\beta}\bar\xi^{\dot\beta}\frac{\partial}{\partial \bar\xi^{\dot 2}}\,, \tag{3.43a}$$

$$S_- = -\frac{1}{2}\partial^{2\dot 2}\,, \tag{3.43b}$$

$$S_0 = \frac{1}{2}\left[x_{2\dot 2}\partial^{2\dot 2} + \frac{1}{2}\left(x_{2\dot 1}\partial^{2\dot 1} + x_{1\dot 2}\partial^{1\dot 2}\right) + t + \xi^1\frac{\partial}{\partial \xi^1} + \bar\xi^{\dot 1}\frac{\partial}{\partial \bar\xi^{\dot 1}}\right] \tag{3.43c}$$

are necessary. The commutation relations assume the simple form

$$[S_+, S_-] = 2S_0\,, \qquad [S_0, S_\pm] = \pm S_\pm\,. \tag{3.44}$$

3.2.3 Construction of the Conformal Basis

For the renormalization group equations to be manifestly conformally invariant, we have two fundamental prerequisites:

- We need an operator basis with "good" conformal properties. That is, a complete set of one-particle light-ray operators which transform according to an irreducible representation of the $SL(2,\mathbb{R})$ group. These operators will serve as building blocks for multi-parton operators (operators for DIS, baryon operators, etc.).
- In order to find and classify these "good" fields, it is necessary to have a set of good quantum numbers that uniquely determines each one-particle operator. Hence, additional quantum operators commuting with the generators $S_\pm$ and S_0 are required.

While the first point is non-trivial, there are indeed two operators to be found that commute with the $SL(2,\mathbb{R})$ generators[6]:

$$\begin{aligned} E =& i\left(D + M_{21} + \bar M_{\dot 1\dot 2}\right) = \\ =& x_{1\dot 1}\partial^{1\dot 1} + \frac{1}{2}\left(x_{2\dot 1}\partial^{2\dot 1} + x_{1\dot 2}\partial^{1\dot 2} + 2t\right) + \xi^2\frac{\partial}{\partial \xi^2} + \bar\xi^{\dot 2}\frac{\partial}{\partial \bar\xi^{\dot 2}}\,, \end{aligned} \tag{3.45}$$

$$H = i(\bar M_{\dot 1\dot 2} - M_{21}) = \frac{1}{2}\left(x_{2\dot 1}\partial^{2\dot 1} - x_{1\dot 2}\partial^{1\dot 2} + \xi^1\frac{\partial}{\partial \xi^1} - \xi^2\frac{\partial}{\partial \xi^2} - \bar\xi^{\dot 1}\frac{\partial}{\partial \bar\xi^{\dot 1}} + \bar\xi^{\dot 2}\frac{\partial}{\partial \bar\xi^{\dot 2}}\right). \tag{3.46}$$

E is exactly the operator measuring the *collinear* twist. This can be seen in the following way: the dilatation operator iD in (3.45) gives the scaling dimension (cf. (3.35)) of a field, whereas iM_{21} and $i\overline{M}_{\dot 1\dot 2}$ count the difference between dotted or undotted 1 and 2 spinor indices, which gives exactly *minus* the spin projection on the light-ray. So E determines the difference of dimension and spin

[6] This can be verified by explicit calculation of the commutator.

	ψ_+	ψ_-	$\bar{\chi}_+$	$\bar{\chi}_-$	f_{++}	f_{+-}	f_{--}
j	1	1/2	1	1/2	3/2	1	1/2
E	1	2	1	2	1	2	3
H	1/2	−1/2	−1/2	1/2	1	0	−1

Table 3.1: The $SL(2,\mathbb{R})$ spin, twist and helicity of the fundamental fields. The table is taken from [62].

projection of a quantum field; this is by definition just *collinear* twist. At this point there is no straightforward interpretation of H which we will suggestively call “helicity operator”. This nomenclature will become clear shortly.

A light-ray operator with definite collinear twist E transforms according to an irreducible representation of the $SL(2,\mathbb{R})$ group with *conformal spin* [59]

$$j = l^{\text{can}} - E/2\,. \tag{3.47}$$

For such fields the $SL(2,\mathbb{R})$ generators acquire their simple canonical form [59]

$$S_+ = z^2\partial_z + 2jz\,, \qquad S_0 = z\partial_z + j\,, \qquad S_- = -\partial_z\,. \tag{3.48}$$

Here all derivatives have lost their spinor indices and act on the light-ray coordinate z.

In particular the upper and lower component of the chiral quark field, ψ_1 and ψ_2, have conformal spin $j = 1$ and $j = 1/2$. Since the two fields correspond to the projection on the ‘plus’ and ‘minus’ light-cone coordinate, they are usually labeled by ψ_+ and ψ_-. This coincidently agrees with the sign of their light-cone projected spin, providing a useful mnemonic. Thus

$$\psi(z) = \lambda\,\psi_-(z) - \mu\,\psi_+(z)\,, \tag{3.49}$$

where

$$\begin{aligned} \psi_+(z) &= \lambda^\alpha\psi_\alpha(z) \equiv \psi_1(z)\,, \quad [E\psi_+](z) = \psi_+(z)\,, \quad [H\psi_+](z) = \frac{1}{2}\psi_+(z)\,, \\ \psi_-(z) &= \mu^\alpha\psi_\alpha(z) \equiv \psi_2(z)\,, \quad [E\psi_-](z) = 2\psi_-(z)\,, \quad [H\psi_-](z) = -\frac{1}{2}\psi_-(z)\,. \end{aligned} \tag{3.50}$$

Note that the eigenvalues $\pm\frac{1}{2}$ of H correspond to the helicity of the fields ψ_+ and ψ_-. This is not true for fields with bad transformation properties with respect to the collinear subgroup. A decomposition following (3.49) can be performed for each field. We define

$$\bar{\chi}_+ = \bar{\chi}_{\dot\alpha}\bar{\lambda}^{\dot\alpha}\,, \qquad \bar{\chi}_- = \bar{\chi}_{\dot\alpha}\bar{\mu}^{\dot\alpha} \tag{3.51}$$

for antichiral quark fields[7] and

$$f_{++}(z) = \lambda^\alpha \lambda^\beta f_{\alpha\beta}(z)\,, \quad f_{+-}(z) = \lambda^\alpha \mu^\beta f_{\alpha\beta}(z)\,, \quad f_{--}(z) = \mu^\alpha \mu^\beta f_{\alpha\beta}(z)\,. \tag{3.52}$$

for self-dual gluon fields. The projections of the anti self-dual gluon fields $\bar{f}$ are given by $\bar{f}_{\pm\pm} = (f_{\pm\pm})^*$. Table 3.1 summarizes the quantum numbers of the "good" fields. This set of fields is our starting point for the construction of a one-particle light-ray operator basis.

In what follows, we restrict ourselves to the case of massless fermions alone, i.e. we ignore all issues concerning gauge invariance. After we finish our construction for the fermions, we sketch the strategy, how our result can be generalized to full QCD.

One of the problems that has to be addressed is the appearance of so-called *non-quasipartonic* operators. A nonlocal operator

$$\mathcal{O} = \Phi(z_1)\Phi(z_2)\ldots\Phi(z_N) \tag{3.53}$$

is called *quasipartonic* if the number of fields N is equal to the light-cone twist E. One can easily read off, see Table 3.1, that each "plus" field adds exactly one unit of twist. So every quasipartonic operator consists only of "plus" fields. Every "minus" field increases the twist of the operator further, making it non-quasipartonic.

The quasipartonic operators have been studied in great detail in the literature and it was understood that the renormalization of such operators only requires so-called 2-to-2 kernels (see Chapter 4). This simplifies the treatment of such operators. One of the reasons for this is that operators of given twist cannot mix with operators of different twist and operators with N fields will not mix with operators with less than N fields[8]. As quasipartonic operators have by definition the minimal possible twist for any given number of fields, the set is closed under renormalization.

This is obviously no longer true for non-quasipartonic operators. They can not only mix with operators corresponding to higher Fock states, but also with operators containing the derivatives

$$\partial^{-\dot{-}} = \partial^{1\dot{1}}, \qquad \partial^{+\dot{-}} = \partial^{2\dot{1}} \qquad \text{and} \quad \partial^{-\dot{+}} = \partial^{1\dot{2}}\,. \tag{3.54}$$

Let us now consider the action of the generators (3.35) on the chiral quark operator with a derivative, i.e.

$$[\partial^{1\dot{1}}\psi_\pm](z),\ [\partial^{1\dot{2}}\psi_\pm](z),\ [\partial^{2\dot{1}}\psi_\pm](z)\ \text{and}\ [\partial^{2\dot{2}}\psi_\pm](z).$$

[7]For simplicity we neglect the dots on the + and − symbols, as they can unambiguously be restored from the corresponding field.

[8]The reverse is not true, operators **can** mix with operators with a higher number of fields.

If $S_\pm$ and S_0 do not assume their canonical form, the operators do not have definite conformal spin and will, if included in our operator basis, veil the conformal invariance of the RGEs.

The generator S_- trivially commutes with all derivatives, so there is no source for complications. Let us have a closer look at the action of S_0. Here one must again distinguish between the quantum operator $\mathbf{S_0}$ and the differential operator S_0. It follows that

$$\begin{aligned}
\left[\mathbf{S_0}, [\partial^{1\dot{1}}\psi_+](z)\right] &= \left(\partial^{1\dot{1}}[S_0\psi_+](x)\right)_{x=zn} = (z\partial_z + 1)\,[\partial^{1\dot{1}}\psi_+](z)\,,\\
\left[\mathbf{S_0}, [\partial^{1\dot{1}}\psi_-](z)\right] &= \left(\partial^{1\dot{1}}[S_0\psi_-](x)\right)_{x=zn} = \left(z\partial_z + \frac{1}{2}\right)[\partial^{1\dot{1}}\psi_-](z)\,, \qquad (3.55)
\end{aligned}$$

where one has to take into account that $\partial^{\alpha\dot{\alpha}}x_{\beta\dot{\beta}} = 2\delta^\alpha_\beta\delta^{\dot{\alpha}}_{\dot{\beta}}$ and $x_{2\dot{2}} = z$. Analogously one gets:

$$\begin{aligned}
S_0[\partial^{1\dot{2}}\psi_+](z) &= \left(z\partial_z + \frac{3}{2}\right)[\partial^{1\dot{2}}\psi_+](z) \quad & S_0[\partial^{2\dot{1}}\psi_+](z) &= \left(z\partial_z + \frac{3}{2}\right)[\partial^{2\dot{1}}\psi_+](z)\\
S_0[\partial^{1\dot{2}}\psi_-](z) &= (z\partial_z + 1)\,[\partial^{1\dot{2}}\psi_-](z) & S_0[\partial^{2\dot{1}}\psi_-](z) &= (z\partial_z + 1)\,[\partial^{2\dot{1}}\psi_-](z)\\
S_0[\partial^{2\dot{2}}\psi_+](z) &= (z\partial_z + 2)\,[\partial^{2\dot{2}}\psi_+](z) & S_0[\partial^{2\dot{2}}\psi_-](z) &= \left(z\partial_z + \frac{3}{2}\right)[\partial^{2\dot{2}}\psi_-](z).
\end{aligned} \qquad (3.56)$$

One sees that the canonical form is always acquired. However, this is not the case for S_+. Repeating the steps above we obtain:

$$\begin{aligned}
S_+[\partial^{2\dot{2}}\psi_-](z) &= (z^2\partial_z + 4z)[\partial^{2\dot{2}}\psi_-](z)\,,\\
S_+[\partial^{2\dot{1}}\psi_+](z) &= (z^2\partial_z + 3z)[\partial^{2\dot{1}}\psi_+](z)\,,\\
S_+[\partial^{1\dot{1}}\psi_+](z) &= (z^2\partial_z + 2z)[\partial^{1\dot{1}}\psi_+](z)\,,\\
S_+[\partial^{1\dot{2}}\psi_+](z) &= (z^2\partial_z + 3z)[\partial^{1\dot{2}}\psi_+](z) - 2\psi_-(z)
\end{aligned} \qquad (3.57)$$

and

$$\begin{aligned}
S_+[\partial^{2\dot{2}}\psi_-](z) &= (z^2\partial_z + 3z)[\partial^{2\dot{1}}\psi_-](z)\,,\\
S_+[\partial^{2\dot{1}}\psi_-](z) &= (z^2\partial_z + 2z)[\partial^{2\dot{1}}\psi_-](z)\,,\\
S_+[\partial^{1\dot{1}}\psi_-](z) &= (z^2\partial_z + z)[\partial^{1\dot{1}}\psi_-](z)\,,\\
S_+[\partial^{1\dot{2}}\psi_-](z) &= (z^2\partial_z + 2z)[\partial^{1\dot{2}}\psi_-](z)\,.
\end{aligned} \qquad (3.58)$$

Obviously S_+ deviates from the standard form only for $[\partial^{1\dot{2}}\psi_+](z)$. One can use the equations of motion to circumvent this. The Dirac equation, Eq. (3.24), connects transversal, plus and minus derivatives:

$$\partial^{1\dot{2}}\psi_+ = -\partial^{2\dot{2}}\psi_-\,, \qquad\qquad \partial^{2\dot{1}}\psi_- = -\partial^{1\dot{1}}\psi_+\,. \qquad (3.59)$$

The "bad" term $[\partial^{1\dot{2}}\psi_+](z)$ can be removed from each expression by replacing it with $[-\partial^{2\dot{2}}\psi_-](z)$, which has good transformation properties. The second equation, $\partial^{2\dot{1}}\psi_- = -\partial^{1\dot{1}}\psi_+$, relates two objects with favorable transformation properties. One is free to keep any one of the two in the operator basis. Eliminating $\partial^{2\dot{1}}\psi_-$ in favor of $\partial^{1\dot{1}}\psi_+$ is slightly more advantageous, as the basis will be more symmetric with respect to the appearance of ψ_+ and ψ_-. As $\partial^{2\dot{2}} \sim \frac{\partial}{\partial z}$, it cannot give rise to new operators and must be removed from the basis. We then end up with four independent operators with one derivative:

$$\partial^{1\dot{1}}\psi_+,\ \partial^{1\dot{1}}\psi_-, \partial^{2\dot{1}}\psi_+ \ \text{and}\ \partial^{1\dot{2}}\psi_- . \tag{3.60}$$

Since there are also operators with more than just one derivative, it might seem necessary to repeat the analysis presented above ad infinitum. However, as the four operators in (3.60) transform according to irreducible representations of the collinear subgroup, they can take the place of ψ_+ and ψ_- in (3.55)-(3.59) if one adjusts the expressions for the changed conformal spin. This allows us, for an arbitrary number of derivatives, to find a set of fields which belong to a conformal spin-j representation of the $SL(2,\mathbb{R})$ group

$$\begin{aligned}\psi_+^{(j,m)}(z) =& [(\partial^{2\dot{1}})^{2j-2}(\partial^{1\dot{1}})^{2m}\psi_+](z)\,,\\ \psi_-^{(j,m)}(z) =& [(\partial^{1\dot{2}})^{2j-1}(\partial^{1\dot{1}})^{2m}\psi_-](z)\end{aligned} \tag{3.61}$$

by removing all unwanted combinations of derivatives with help of (3.55).

All previous observations do not take the (self) interacting gauge fields of QCD into account. In order to move to a true gauge theory, the following adjustments are necessary:

- All derivatives must be replaced by covariant ones

  $$\partial \to D = \partial - igA\,.$$

  Thus, the conformal transformation properties of the gauge field A are needed. The commutation of two covariant derivatives gives an additional field, a field strength tensor. However, we can drop terms proportional to commutators and treat the derivatives as commuting ones, since we are only interested in a one-particle basis.

- The (anti-)self-dual gluon fields $\bar{f}$ and f have to be included

- Each light-ray field must be connected to a gauge link

  $$\Phi(z) \to [0,z]\Phi(z)\,,$$

  where

  $$[0,z] = \text{Pexp}\left[-\frac{1}{2}igz\int_0^1 du\, A^{2\dot{2}}(uz)\right] \tag{3.62}$$

  is the path-ordered exponent along the light-ray in the appropriate representation of the gauge group (adjoint for the gluon fields, fundamental for quark fields).

While the calculations are more extensive, everything turns out to work analogously. Therefore, we just quote the final results in the next section.

3.2.4 The Conformal One-Particle Operator Basis

Our complete basis of one-particle light-ray operators for chiral quark and self-dual gluon fields is made up of the following fields [62]:

$$\begin{aligned}
\psi_+^{(j,m)}(z) &= \left(D^{2\dot{1}}\right)^{2j-2}\left(D^{1\dot{1}}\right)^{2m}\psi_1(z)\,,\\
\psi_-^{(j,m)}(z) &= \left(D^{1\dot{2}}\right)^{2j-1}\left(D^{1\dot{1}}\right)^{2m}\psi_2(z)\,,\\
\bar{\chi}_+^{(j,m)}(z) &= \left(D^{1\dot{2}}\right)^{2j-2}\left(D^{1\dot{1}}\right)^{2m}\bar{\chi}_{\dot{1}}(z)\,,\\
\bar{\chi}_-^{(j,m)}(z) &= \left(D^{2\dot{1}}\right)^{2j-1}\left(D^{1\dot{1}}\right)^{2m}\bar{\chi}_{\dot{2}}(z)\,,\\
f_{++}^{(j,m)}(z) &= \left(D^{2\dot{1}}\right)^{2j-3}\left(D^{1\dot{1}}\right)^{2m}f_{11}(z)\,,\\
f_{--}^{(j,m)}(z) &= \left(D^{1\dot{2}}\right)^{2j-1}\left(D^{1\dot{1}}\right)^{2m}f_{22}(z)\,,\\
f_{+-}^{(1,m)}(z) &= \left(D^{1\dot{1}}\right)^{2m}f_{12}(z)\,.
\end{aligned} \tag{3.63}$$

A field carrying the superscript j transforms according to the representation T^j of the $SL(2,\mathbb{R})$ group. Note that a reordering of the covariant derivatives in (3.63) does not affect the transformation properties. Collinear twist E and helicity H take the following values:

$$\begin{aligned}
E\,\psi_\pm^{(j,m)} &= (2j+4m\mp 1)\psi_\pm^{(j,m)}\,, & E\,\bar{\chi}_\pm^{(j,m)} &= (2j+4m\mp 1)\bar{\chi}_\pm^{(j,m)}\,,\\
E\,f_{\pm\pm}^{(j,m)} &= (2j+4m\mp 2)f_{\pm\pm}^{(j,m)}\,, & E\,f_{+-}^{(1,m)} &= (2+4m)f_{+-}^{(1,m)}\,,
\end{aligned} \tag{3.64}$$

$$\begin{aligned}
H\,\psi_\pm^{(j,m)} &= \pm\left(2j-1\mp\frac{1}{2}\right)\psi_\pm^{(j,m)}\,, & H\,\bar{\chi}_\pm^{(j,m)} &= \mp\left(2j-1\mp\frac{1}{2}\right)\psi_\pm^{(j,m)}\,,\\
H\,f_{\pm,\pm}^{(j,m)} &= \pm\left(2j-1\mp 1\right)f_{\pm,\pm}^{(j,m)}\,, & H\,f_{+-}^{(1,m)} &= 0\,.
\end{aligned} \tag{3.65}$$

The basis for the anti-self-dual gluon field is then defined by $\bar{f} = f^*$. Note that we do not display the gauge links explicitly, but they are, as usual, always implied. It is, however, possible to drop them by making use of a special gauge, like light-cone gauge $n\cdot A = A^{2\dot{2}} = 0$ or Fock-Schwinger Fixed-Point gauge $x_\mu A^\mu(x) = 0$. We will make use of this in the actual calculations, see Chap. 4.3.4.

Finally, by taking a color-singlet product of the fields defined in (3.63), $\Phi^{j,m} = \{\psi_\pm^{(j,m)},\ldots,f_{+-}^{(j=1,m)}\}$, and their antichiral counterparts $\bar{\Phi}^{j,m}$ at different light-ray positions $z_1,\ldots,z_N$, we obtain a complete basis of gauge-invariant N-particle operators [62]

$$\mathcal{O}(z_1,\ldots,z_N) = \Phi^{j_1,m_1}(z_1)\ldots\Phi^{j_N,m_N}(z_N)\,. \tag{3.66}$$

Each operator transforms according to the representation $T^{j_1}\otimes\ldots\otimes T^{j_N}$ of the collinear conformal group $SL(2,\mathbb{R})$ and has twist $E = E_1+\ldots+E_N$.

The basis (3.63) represents one of the main results of this thesis. It can be used to construct generic multi-particle operators of any twist with good conformal properties by following the construction principle of Eq. (3.66). In the next chapter, we use it to define an operator basis for baryon operators of twist 4. Other possible applications include e.g. higher-twist operators for deep inelastic lepton-baryon scattering.

"Thou, nature, art my goddess; to thy laws my services are bound ..."

"King Lear" – W. Shakespeare

4 Renormalization of Baryon Distribution Amplitudes

The baryon distribution amplitudes of leading collinear twist have been studied in great detail over the last 30 years [63, 64, 71, 21] and a complete understanding of the scale dependence of corresponding matrix elements was reached [61]. However, beyond leading twist much less is known.

The full set of twist-4 three-quark distribution amplitudes for nucleons was defined for the first time in [65]. They have been successfully used to determine various form factors of the nucleon, see for example [66, 67], and are instrumental for perturbative studies of processes involving an helicity flip – such as the electromagnetic Pauli form factor [68].

Until recently only the first few parameters corresponding to the next-to-leading order in conformal spin could be evaluated nonperturbatively using QCD sum rules [64, 65]. As the anomalous dimensions for twist-4 operators of lowest dimension have been known for quite some time [72], a full analysis of the scale dependence was not yet necessary. The possibility to determine the matrix elements of higher dimensional operators using lattice QCD was suggested in [73, 74]; this will require a deeper understanding of the renormalization properties of the twist-4 operators.

Furthermore, starting with twist 4 a new phenomenon occurs: the leading Fock state operators (containing three quark fields) can mix with four-particle operators. This has not been studied yet, and if strong mixing were to be found the standard arguments why higher Fock states can be neglected in most calculations would be weakened.

The aim of this Chapter is to develop a general framework for the study of the scale dependence of higher twist baryon distribution amplitudes, which are matrix elements of higher twist baryon operators. We use the approach of [61], which made heavy use of conformal symmetry, as starting point and employ the operator basis developed in Chap. 3 to extend it to higher twists. While our approach is valid for arbitrary twist, we focus on the special case of twist-4, as it is most relevant for QCD phenomenology.

In Sect. 4.1 we introduce the most common baryon DAs: the nucleon distribution amplitudes. After an illustration of the physical meaning of a distribution amplitude, we give an abstract definition in terms of matrix elements of non-local operators. We show, using our operator basis, that there are three quasipartonic and three non-quasipartonic nucleon DAs of twist-4.

In order to determine the scale dependence of the amplitudes, we need to construct a general non-local operator basis. This is done for generic flavor structures in Sect. 4.2. As QCD perturbation theory preserves the chirality of massless quarks, we distinguish between chiral operators and operators of mixed chirality. For completeness, we also give the relations of our operator basis to the nucleon distribution amplitudes defined in the previous section.

Sect. 4.3 is the central part of this chapter. We introduce a Schrödinger equation-like renormalization group equation for the set of non-local operators, and explain our strategy for its solution. The implications of conformal symmetry for the renormalization kernels are addressed in Sect. 4.3.1, where we also give an explicit example how the functional form of the kernels is restricted by symmetry. After presenting the final expressions for the Hamiltonians, which are the main results of this chapter, a detailed exemplary calculation for one of the previously unknown 2-to-3 kernels is shown.

The invariance of the RGE under the full conformal group provides connections among kernels of different twist. In Sect. 4.4 this is used to devise an additional non-trivial check of our results.

The spectra of anomalous dimensions are obtained numerically and gathered in Sect. 4.5. They are among of our central results and allow statements on the strength of the mixing between three- and four-particle operators. In addition to that, we find that the chiral quark sector is in fact integrable; a feature already found for the twist-3 baryon operators [61].

Before concluding this chapter with a short summary, we show the application of our results to the case of nucleon distribution amplitudes (Sect. 4.6). We see that using the multiplicatively renormalizable operators allows for an identification of the so-called Wandzura-Wilczek contributions.

4.1 Nucleon Distribution Amplitudes

A nucleon state $|N(p)\rangle$ with momentum p is a superposition of all Fock states with quantum numbers matching the nucleon's, schematically

$$|N(p)\rangle = |qqq\rangle\langle qqq|N\rangle + |qqqF\rangle\langle qqqF|N\rangle + |qqq\bar{q}q\rangle\langle qqq\bar{q}q|N\rangle + \dots\,, \tag{4.1}$$

where $|qqq\rangle$ denotes the leading Fock state consisting of nothing but the three valence quarks, $|qqqF\rangle$ contains an additional gluon, $|qqq\bar{q}q\rangle$ an additional sea quark pair and so on. In order to get a hold on this multitude of states and to keep the familiar picture of a multi-particle wave function, that was successful in the description of e.g. the positronium atom [69], the leading state can be associated with an analogous Bethe-Salpeter wave function [70]

$$\Psi_{BS}(x_i, k_{i,\perp}) = \langle u(x_1, k_{1\perp}) u(x_2, k_{2\perp}) d(x_3, k_{3\perp}) | N(p) \rangle \,, \tag{4.2}$$

x_i being the longitudinal momentum fraction carried by the ith quark and $k_{i\perp}$ its transverse momentum. Hence $0 \leq x_i \leq 1$ and $x_1 + x_2 + x_3 = 1$. For illustration let us consider the prime example for a hard exclusive reaction involving a nucleon [22]. A proton observed in an infinite momentum frame (i.e. $p \to \infty$) is struck by a hard photon coming from a direction orthogonal to the light-cone. The initial proton can then be perceived as a bunch of parallel moving partons which have only relatively small transverse momenta. This picture comes very natural since a large transverse momentum would result in an "unstable" proton as one constituent would detach itself from the other partons. After absorbing the hard photon, its momentum must be distributed among the partons for the proton to stay intact. The final state will then again consist of a tight bundle moving along a light-ray. In this case the object, which captures the relevant internal dynamic has been coined leading-twist *(nucleon) distribution amplitude* [22]. It can be defined in terms of the Bethe-Salpeter wave function (4.2) with the transverse degrees of freedom integrated out. That is

$$\Phi_3(x_i, \mu) = Z(\mu) \int^{|k_\perp| < \mu} \mathrm{d}^3 k_{i,\perp} \Psi_{BS}(x_i, k_{i,\perp}) \,, \tag{4.3}$$

where $Z(\mu)$ is the product of the renormalization factors of the three quark fields.

While this definition allows some intuitive picture of the physical meaning of the distribution amplitude, it is not suited for actual applications, as the Bethe-Salpeter wave function is virtually unknown. Therefore, we advocate a more abstract definition of the distribution amplitudes in terms of matrix elements of non-local light-ray operators.

4.1.1 The Leading Twist Distribution Amplitude

Let us define the projection onto chiral quark fields following [65]

$$q^{\uparrow(\downarrow)} = \frac{1 \pm \gamma_5}{2} q \,. \tag{4.4}$$

$q^{\uparrow/\downarrow}$ then correspond to the Weyl spinors $\bar{\chi}^{\dot{\alpha}}$ and ψ_α. The leading twist-three nucleon distribution amplitude $\Phi_3(x_i, \mu)$ is given by the matrix element:

$$\langle 0 | \epsilon^{ijk} (u_i^\downarrow(z_1 n)\, C \not{n} u_j^\uparrow(z_2 n))\, \not{n} d_k^\downarrow(z_3 n) | N(p) \rangle =$$

$$= -\frac{1}{2} f_N(pn)\ \not{n} N^{\downarrow}(p) \int \mathcal{D}x\, e^{-ipn\sum_i x_i z_i}\, \Phi_3(x_i,\mu)\,, \tag{4.5}$$

where

$$\int \mathcal{D}x = \int_0^1 \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3\, \delta(1 - x_1 - x_2 - x_3) \tag{4.6}$$

and i, j, k are color indices. Note that this definition differs from the standard one [65] by reversal of all spin projections; Φ_3 is not affected by this in any way. It is possible to translate Eq. (4.5) into spinor notation; this is in principle trivial, as one can use the relations given in (3.16). However, as we want to keep the auxiliary spinors μ and λ dimensionless, we have to carry out a simple redefinition of n and $\tilde{n}$ which is proportional to the nucleon momentum p:

$$n = m_N^{-1} \lambda \otimes \bar{\lambda}, \qquad \tilde{n} = m_N\, \mu \otimes \bar{\mu}\,,$$

where m_N is the nucleon mass. The calculation itself has been relayed to Appendix B.1, and we just quote the result:

$$\langle 0|\epsilon^{ijk} \psi_+^{u,i}(z_1) \bar{\chi}_+^{u,j}(z_2) \psi_+^{d,k}(z_3)|N(p)\rangle = \frac{-1}{2} m_N (pn)\, N_+^{\downarrow} \int \mathcal{D}x\, e^{-i(Pn)\sum x_i z_i}\, \Phi_3(x) \tag{4.7}$$

where the first superscript, u or d, denotes the quark flavor. In this form it is fairly easy to see that the twist of the operator is actually three, since it consists of three chiral "plus" fields. Furthermore, we see that the leading twist operator is part of the operator basis derived in the previous section. It is obvious that there cannot be a second, independent twist-3 operator with the quantum numbers of the nucleon. The only candidate

$$\epsilon^{ijk} \psi_+^{u,i}(z_1) \psi_+^{u,j}(z_2) \psi_+^{d,k}(z_3)$$

contains only quark fields with helicity $+1/2$ and therefore corresponds to the Δ baryon.

4.1.2 Next-to-Leading Twist DAs

There are all in all six independent twist-4 nucleon distribution amplitudes. Three of them, Φ_4, Ψ_4 and Ξ_4, belong to the leading three-quark Fock state and have been studied quite extensively in [65, 66]. In spinor representation their definitions take the form

$$\begin{aligned}
\langle 0|\epsilon^{ijk} \psi_+^{u,i}(z_1) \bar{\chi}_+^{u,j}(z_2) \psi_-^{d,k}(z_3)|P\rangle &= -\frac{1}{4} (\mu\lambda)\, m_N N_+^{\uparrow} \int \mathcal{D}x\, e^{-i(pn)\sum x_i z_i}\, \Phi_4(x),\\
\langle 0|\epsilon^{ijk}\, \bar{\chi}_+^{u,i}(z_1)\, \psi_-^{u,j}(z_2)\, \psi_+^{d,k}(z_3)\,|P\rangle &= -\frac{1}{4} (\mu\lambda)\, m_N N_+^{\uparrow} \int \mathcal{D}x\, e^{-i(pn)\sum x_i z_i}\, \Psi_4(x),\\
\langle 0|\epsilon^{ijk} \psi_-^{u,i}(z_1) \psi_+^{u,j}(z_2) \psi_+^{d,k}(z_3)|P\rangle &= -\frac{1}{4} (\mu\lambda)\, m_N N_+^{\downarrow} \int \mathcal{D}x\, e^{-i(pn)\sum x_i z_i}\, \Xi_4(x).
\end{aligned} \tag{4.8}$$

The spinor product $\mu\lambda = 1$ has not been carried out, as this guarantees that the number of spinors on the right- and left-hand side of the equation is the same. All three DAs arise as matrix elements of non-quasipartonic operators.

The remaining three four-particle distribution amplitudes have not been studied before and were defined in our work [62] for the first time. Each involves an additional gluon field compared to (4.8):

$$\begin{aligned}
&\langle 0|ig\epsilon^{ijk}\psi_+^{u,i}(z_1)\bar{\chi}_+^{u,j}(z_2)[\bar{f}_{++}(z_4)\psi_+^d(z_3)]^k|P\rangle = \\
&\qquad = \frac{1}{4}m_N(pn)^2 N_+^\uparrow \int \mathcal{D}x\, e^{-i(pn)\sum x_i z_i}\, \Phi_4^g(x)\,, \\
&\langle 0|ig\epsilon^{ijk}\,\bar{\chi}_+^{u,i}(z_1)\,[\bar{f}_{++}(z_4)\psi_+^u(z_2)]^j\,\psi_+^{d,k}(z_3)\,|P\rangle = \\
&\qquad = \frac{1}{4}m_N(pn)^2 N_+^\uparrow \int \mathcal{D}x\, e^{-i(pn)\sum x_i z_i}\, \Psi_4^g(x)\,, \\
&\langle 0|ig\epsilon^{ijk}[\bar{f}_{++}(z_4)\psi_+^u(z_1)]^i\,\psi_+^{u,j}(z_2)\,\psi_+^{d,k}(z_3)|P\rangle \\
&\qquad = \frac{1}{4}m_N(pn)^2 N_+^\downarrow \int \mathcal{D}x\, e^{-i(pn)\sum x_i z_i}\, \Xi_4^g(x)\,.
\end{aligned} \tag{4.9}$$

Note that the integration measure now has to ensure that the sum of the momentum fractions of all four partons is equal to one, hence

$$\int \mathcal{D}x = \int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4\,\delta(1-\sum_{i=1}^4 x_i)\,. \tag{4.10}$$

If even higher twist DAs are considered the number of independent amplitudes grows dramatically. Already at twist five the three known three-quark DAs [65] will be complemented by matrix elements involving the Fock states: $|qqqF\rangle$, $|qqqFF\rangle$ and $|qqq\bar{q}q\rangle$. At this level the mere classification of all independent distribution amplitudes would be a non-trivial task.

4.2 The Complete Twist-4 Operator Basis

As we have seen in the previous section, there are six twist-4 nucleon distribution amplitudes. Two of them, Ξ_4 and Ξ_4^g, involve only chiral quark fields, whereas Φ_4, Ψ_4, Φ_4^g and Ψ_4^g feature of both chiral and antichiral quarks. It is well known that chirality is preserved in QCD perturbation theory [56]; therefore, the two sets of distribution amplitudes cannot mix under renormalization and we can treat them separately right from the start. In the following we construct an operator basis for each case; pure chiral and mixed chirality operators that is. The operators are built from the good one-particle light-ray operators found in Eq. (3.63).

4.2.1 Chiral Operators

The two chiral nucleon distribution amplitudes Ξ_4 and Ξ_4^g have the quantum numbers $E = 4$ and $H = +1/2$. All operators of the basis are, therefore, required to share these quantum numbers, otherwise operator mixing is not possible.

The chiral three-quark distribution amplitude Ξ_4 is related to matrix elements of the operators

$$\begin{aligned} Q_1(z_1,z_2,z_3) &= \epsilon^{ijk}\psi_-^{a,i}(z_1)\,\psi_+^{b,j}(z_2)\,\psi_+^{c,k}(z_3)\,,\\ Q_2(z_1,z_2,z_3) &= \epsilon^{ijk}\psi_+^{a,i}(z_1)\,\psi_-^{b,j}(z_2)\,\psi_+^{c,k}(z_3)\,,\\ Q_3(z_1,z_2,z_3) &= \epsilon^{ijk}\psi_+^{a,i}(z_1)\,\psi_+^{b,j}(z_2)\,\psi_-^{c,k}(z_3)\,. \end{aligned} \tag{4.11}$$

i,j,k are color and a,b,c are flavor indices[1]. For the nucleon one would have to set two flavor indices to *up* and one to *down* type flavor. It is, however, convenient to consider the general case, as the nucleon flavor structure can always be restored. The corresponding chiral quasipartonic operators are

$$\begin{aligned} G_1(z_1,z_2,z_3,z_4) &= ig\epsilon^{ijk}(\mu\lambda)\,[\bar f_{++}(z_4)\psi_+^{a}(z_1)]^i\,\psi_+^{b,j}(z_2)\,\psi_+^{c,k}(z_3)\,,\\ G_2(z_1,z_2,z_3,z_4) &= ig\epsilon^{ijk}(\mu\lambda)\,\psi_+^{a,i}(z_1)\,[\bar f_{++}(z_4)\psi_+^{b}(z_2)]^j\,\psi_+^{c,k}(z_3)\,,\\ G_3(z_1,z_2,z_3,z_4) &= ig\epsilon^{ijk}(\mu\lambda)\,\psi_+^{a,i}(z_1)\,\psi_+^{b,j}(z_2)\,[\bar f_{++}(z_4)\psi_+^{c}(z_3)]^k\,, \end{aligned} \tag{4.12}$$

where the factor $\mu\lambda$ is useful for translating the expressions back to the normal Dirac notation since $F_{+,\mu\bar\lambda} = -(\mu\lambda)\bar f_{++}$.

It turns out that the three operators in (4.12) are not independent because

$$G_1(z_1,z_2,z_3,z_4) + G_2(z_1,z_2,z_3,z_4) + G_3(z_1,z_2,z_3,z_4) = 0\,. \tag{4.13}$$

This identity is a direct consequence of gauge invariance. Consider the operator

$$\mathbb{O} := \epsilon^{ijk}(\mu\lambda)\,\psi_+^{a,i}(z_1)\,\psi_+^{b,j}(z_2)\,\psi_+^{c,k}(z_3).$$

Performing an infinitesimal global gauge transformation

$$\psi_+^h \to [e^{igT^A\epsilon}\psi_+]^h = \psi_+^h + igT^{A,hl}\psi_+^l\cdot\epsilon + \mathcal{O}(\epsilon^2)\,,$$

one gets

$$\begin{aligned} \mathbb{O} \to \mathbb{O} + ig\Big(T^{A,il}\psi_+^l(z_1)\psi_+^j(z_2)\psi_+^k(z_3) &+ \psi_+^i(z_1)T^{A,jl}\psi_+^l(z_2)\,\psi_+^l(z_3)\\ &+\psi_+^i(z_1)\psi_+^j(z_2)\,T^{A,kl}\psi_+^l(z_3)\Big)\cdot\epsilon + \mathcal{O}(\epsilon^2)\,. \end{aligned} \tag{4.14}$$

Multiplying the sum in the brackets with $\bar f_{++}^A$ gives the left-hand side of (4.13). However, $\mathbb{O}$ is gauge invariant; any term in Eq. (4.14) proportional to ϵ must vanish identically, which proves (4.13).

[1]For simplicity we usually do not display the flavor indices explicitly and assume that the first quark carries flavor a, the second flavor b and so on.

4.2.2 Operators of Mixed Chirality

Analogously, one can define the operator basis for the mixed chirality operators. The distribution amplitudes Φ_4, Ψ_4, Φ_4^g and Ψ_4^g also have collinear twist $E = 4$, but helicity $H = -1/2$ as opposed to $+1/2$ for the chiral amplitudes. Again one can find three independent operators matching these quantum numbers:

$$\begin{aligned}
\mathcal{Q}_1(z_1, z_2, z_3) =&\epsilon^{ijk}\, \psi_-^{a,i}(z_1)\, \psi_+^{b,j}(z_2)\, \bar{\chi}_+^{c,k}(z_3)\,, \\
\mathcal{Q}_2(z_1, z_2, z_3) =&\epsilon^{ijk}\, \psi_+^{a,i}(z_1)\, \psi_-^{b,j}(z_2)\, \bar{\chi}_+^{c,k}(z_3)\,, \\
\mathcal{Q}_3(z_1, z_2, z_3) =&\frac{1}{2}\epsilon^{ijk}\, \psi_+^{a,i}(z_1)\, \psi_+^{b,j}(z_2)\, [\bar{\chi}_+^{3/2}]^{c,k}(z_3)\,,
\end{aligned} \tag{4.15}$$

where $\bar{\chi}_+^{3/2} \equiv \bar{\chi}_+^{(3/2,0)} = -(\mu D\bar{\lambda})\bar{\chi}_+ \equiv -D_{\mu\bar{\lambda}}\bar{\chi}_+$, cp. Eq. (3.63). Note that the naive choice for the third operator

$$\hat{\mathcal{Q}}_3(z_1, z_2, z_3) = \epsilon^{ijk}\, \psi_+^{a,i}(z_1)\, \psi_+^{b,j}(z_2)\, \bar{\chi}_-^{c,k}(z_3)$$

has the wrong helicity ($H = 3/2$). This can be read off Table 3.1, since the helicity of $\hat{\mathcal{Q}}_3$ is the sum of the helicities of the involved one-particle operators. For the four particle case there again exist three operators

$$\begin{aligned}
\mathcal{G}_1(z_1, z_2, z_3, z_4) =&ig\epsilon^{ijk}\, (\mu\lambda)\, [\bar{f}_{++}(z_4)\psi_+^a(z_1)]^i\, \psi_+^{b,j}(z_2)\, \bar{\chi}_+^{c,k}(z_3)\,, \\
\mathcal{G}_2(z_1, z_2, z_3, z_4) =&ig\epsilon^{ijk}\, (\mu\lambda)\, \psi_+^{a,i}(z_1)\, [\bar{f}_{++}(z_4)\psi_+^b(z_2)]^j\, \bar{\chi}_+^{c,k}(z_3)\,, \\
\mathcal{G}_3(z_1, z_2, z_3, z_4) =&ig\epsilon^{ijk}\, (\mu\lambda)\, \psi_+^{a,i}(z_1)\, \psi_+^{b,j}(z_2)\, [\bar{f}_{++}(z_4)\bar{\chi}_+^c(z_3)]^k\,.
\end{aligned} \tag{4.16}$$

The same argument as before guarantees the identity

$$\mathcal{G}_1(z_1, z_2, z_3, z_4) + \mathcal{G}_2(z_1, z_2, z_3, z_4) + \mathcal{G}_3(z_1, z_2, z_3, z_4) = 0\,, \tag{4.17}$$

so that the operators $\mathcal{G}_i$ are not independent. Therefore, there are only two independent chiral distribution amplitudes with one gluon field, Ψ_4^g and Φ_4^g, instead of three.

4.2.3 Nucleon Matrix Elements

The matrix elements of the operators between vacuum and nucleon state exhibit additional symmetries. The nucleon has isospin 1/2 and this property is reflected in the matrix elements. Furthermore, the identity of quark flavors, two u quarks in case of the proton and two d quarks for the neutron, generates an additional symmetry.

The matrix elements of the chiral operators Q_i and G_i between vacuum and proton can be defined, see also [62], as

$$\begin{aligned}
\phi_i(z_1, z_2, z_3) &= \langle 0|Q_i(z_1, z_2, z_3)|P\rangle\,, \\
\phi_i^g(z_1, z_2, z_3, z_4) &= \langle 0|G_i(z_1, z_2, z_3, z_4)|P\rangle\,,
\end{aligned} \tag{4.18}$$

where the first two quarks are of u-type flavor, that is $a = u$ and $b = u$ in Eqs. (4.11) and (4.12). The identity of the u quarks leads to

$$\phi_1(z_1, z_2, z_3) = \phi_2(z_2, z_1, z_3)\,, \tag{4.19}$$

whereas the isospin condition enforces the relation[2]

$$\phi_3(z_2, z_3, z_1) = -\phi_1(z_1, z_2, z_3) - \phi_1(z_1, z_3, z_2)\,. \tag{4.20}$$

For the four particle operators one obtains:

$$\begin{aligned}\phi_2^g(z_1, z_2, z_3, z_4) &= \phi_1^g(z_2, z_1, z_3, z_4)\,,\\ \phi_3^g(z_2, z_3, z_1, z_4) &= -\phi_1^g(z_1, z_2, z_3, z_4) - \phi_1^g(z_1, z_3, z_2, z_4)\,.\end{aligned} \tag{4.21}$$

The matrix elements of the mixed chirality operators feature similar relations, if the quarks of same chirality also have the same flavor. However, the two distribution amplitudes Φ_4 and Ψ_4 are directly related to the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ with flavors $a = c = u$ and $b = d$; the u quarks have different chirality. The matrix element corresponding to the operator $\mathcal{Q}_3$,

$$\begin{aligned}&\frac{1}{2}\langle 0|\epsilon^{ijk}\psi_+^{u,i}(z_1)[\bar{\chi}_+^{-3/2}]^{u,j}(z_2)\psi_+^{d,k}(z_3)|P\rangle = \\ &\qquad = \frac{i}{4}(\mu\lambda)(pn)m_N N_+^{\uparrow}\int \mathcal{D}x\, e^{-i(pn)\sum x_i z_i}\, D_4(x)\,,\end{aligned} \tag{4.22}$$

did not appear in the set of distributions amplitudes (4.8). The reason for this is that the distribution amplitude D_4 is not independent and can be expressed in terms of matrix elements of other twist-4 operators. To show this, let us define the matrix elements of mixed chirality as

$$\begin{aligned}\varphi_k(z_1, z_2, z_3) =& \langle 0|\mathcal{Q}_k(z_1, z_2, z_3)|P\rangle\,,\\ \varphi_k^g(z_1, z_2, z_3, z_4) =& \langle 0|\mathcal{G}_k(z_1, z_2, z_3, z_4)|P\rangle\,.\end{aligned} \tag{4.23}$$

One can show that the relation[3]

$$\begin{aligned}\varphi_3(z_1, z_2, z_3) =& \frac{\partial}{\partial z_1}\varphi_1(z_1, z_2, z_3) + \frac{\partial}{\partial z_2}\varphi_2(z_1, z_2, z_3)\\ &- \frac{1}{2}\int_0^1 d\tau\left(z_{13}\,\varphi_1^g(z_1, z_2, z_3, z_{13}^\tau) + z_{23}\,\varphi_2^g(z_1, z_2, z_3, z_{23}^\tau)\right)\end{aligned} \tag{4.24}$$

holds. Here we used the notation

$$z_{ik} = z_i - z_k\,, \qquad \bar{\tau} = 1 - \tau\,, \qquad z_{ik}^\tau = \bar{\tau} z_i + \tau z_k\,. \tag{4.25}$$

The proof of (4.24) is straightforward but lengthy. It can be found in App. B.2.

[2] Isospin relations for nucleon matrix elements were studied in great detail in [104].
[3] Note that in [62] an incorrect momentum space representation of this relation is presented.

4.3 RGE and Renormalization Kernels

With an operator basis for chiral operators as well as for operators of mixed chirality firmly established, renormalization and mixing can be discussed. For a set of renormalized non-local operators $\{O\}$ the renormalization group equation takes a form similar to the one discussed in Chapter 2, Eq. (2.40):

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+\frac{\alpha_s}{2\pi}\mathbb{H}\right)\{O\}(z_1,\ldots,n_N)=0\,, \tag{4.26}$$

where $\mathbb{H}$, the renormalization kernel, is now an integral operator which acts on the coordinates of the non-local operators [59]. The functional form of this RGE is often referred to as Schrödinger equation-like: the first two terms can be thought of as an equivalent to the time-derivative in the time-dependent Schrödinger equation and the renormalization kernel $\mathbb{H}$ plays the role of the Hamiltonian[4]. As $\mathbb{H}$ only acts on the coordinates and not on μ or g, one can solve the "time-independent" equation

$$\mathbb{H}\,O_i(z_1,\ldots,n_N)=E_i\,O_i(z_1,\ldots,n_N) \tag{4.27}$$

separately. Here the eigenvalues E_i correspond, up to the trivial factor $\frac{\alpha_s}{2\pi}$, to the 1-loop anomalous dimensions and the eigenfunctions to multiplicatively renormalizable operators.

For our baryon operators the Hamiltonian $\mathbb{H}$ will take the form of a six-by-six matrix, since the operator basis consists of three three-quark operators, see Eq. (4.11) for the chiral case or Eq. (4.15) otherwise, and three four-particle operators, Eq. (4.12) and Eq. (4.16), respectively. The determination of these matrix elements is the main task of this section.

$\mathbb{H}$ can be cast into the form

$$\mathbb{H}=\begin{pmatrix}\mathbb{H}_q & \mathbb{H}_{qg}\\ 0 & \mathbb{H}_g\end{pmatrix} \tag{4.28}$$

by ordering the basis, e.g. for the chiral case the "vector" has the form

$$\mathbb{O}^{\text{chiral}}(\vec{z})=\begin{pmatrix}Q_1(z_1,\ldots,z_3)\\ Q_2(z_1,\ldots,z_3)\\ Q_3(z_1,\ldots,z_3)\\ G_1(z_1,\ldots,z_4)\\ G_2(z_1,\ldots,z_4)\\ G_3(z_1,\ldots,z_4)\end{pmatrix}\,. \tag{4.29}$$

The 3×3 submatrices $\mathbb{H}_q$ and $\mathbb{H}_g$ describe the mixing of the three and four particle operators among themselves and are, therefore, given by 2-to-2 kernels:

[4] We will use 'renormalization kernel' and 'Hamiltonian' synonymously.

the integral operators, which are the matrix elements of $\mathbb{H}_q$ and $\mathbb{H}_g$ map only two-particle operators to two-particle operators.

To illustrate this let us schematically consider the matrix element $(\mathbb{H}_q)_{11}$ for the chiral case. From Eqs. (4.28) and (4.29) follows that $(\mathbb{H}_q)_{11}$ maps $Q_1(z_1, z_2, z_3)$ onto itself. Thus, it has to be a sum of four terms: three 2-to-2 integral operators corresponding to one-loop diagrams involving the three combinations of quark pairs, (12), (13) and (23), as well as a trivial (constant) term stemming from the quark field renormalization due to self-energy diagrams. More complicated structures – such as kernels involving all three quark fields at once – can only arise from Feynman diagrams featuring more than one loop and are not important for the $\mathcal{O}(\alpha_s)$-RGEs.

As we have already seen in Sect. 4.2, the three gluonic operators are not independent and the operators G_3 and $\mathcal{G}_3$ have to be removed from the basis. Replacing G_3 by $-G_1 - G_2$, the last row and the last column of $\mathbb{H}$ can be eliminated and one ends up with a 5×5 matrix, $\widetilde{\mathbb{H}}$. The relation between $\mathbb{H}$ and $\widetilde{\mathbb{H}}$ is obvious:

$$\begin{aligned} [\widetilde{\mathbb{H}}_q]_{ik} &= [\mathbb{H}_q]_{ik}\,, & i,k &= 1,2,3 \\ [\widetilde{\mathbb{H}}_g]_{ik} &= [\mathbb{H}_g]_{ik} - [\mathbb{H}_g]_{i3}\,, & i,k &= 1,2 \\ [\widetilde{\mathbb{H}}_{qg}]_{ik} &= [\mathbb{H}_{qg}]_{ik} - [\mathbb{H}_{qg}]_{i3}\,, & i = 1,2,3, \quad k &= 1,2\,. \end{aligned} \tag{4.30}$$

It was already mentioned that any non-local operator can be written as a sum over local operators (e.g. by performing a formal Taylor expansion, see Eq. (3.41)):

$$\mathbb{O}^{\text{non-local}}(z_1, \dots, z_n) = \sum_{N,p} \Psi_{N,p}(z_1, \dots, z_n)\mathbb{O}^{\text{local}}_{N,p}\,. \tag{4.31}$$

N counts the number of (covariant) derivatives in the expansion, p labels the different possibilities to apply N derivatives to n fields and $\Psi_{N,p}(z_1, \dots, z_n)$ is a homogeneous polynomial of degree N. A renormalization kernel $\mathbb{H}$ will only "see" the polynomials, as it acts on the coordinates. An operator $\mathbb{O}^{\text{local}}_{N,p}$ satisfying the RGE

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma_{\mathbb{O}_{N,p}}\right)\mathbb{O}^{\text{local}}_{N,p} = 0$$

corresponds to a vector of polynomials $\Psi_{N,p}(z_1, \dots, z_n)$, which is an eigenfunction of the Hamilton operator $\mathbb{H}$

$$[\mathbb{H}\Psi_{N,q}](z_1, \dots, z_n) = E_{N,p}\Psi_{N,q}(z_1, \dots, z_n)\,, \tag{4.32}$$

with $\gamma_{\mathbb{O}_{N,p}} \sim E_{N,p}$. Therefore, we only have to consider the action of the kernel on the space of homogeneous polynomials. These have the form

$$\Psi^i_{N,q}(z_1, z_2, z_3) = \sum_{\substack{k_1,\dots,k_3 \\ k_1+k_2+k_3=N}} \psi^{(i)N,q}_{k_1\,k_2\,k_3}\, z_1^{k_1}\, z_2^{k_2}\, z_3^{k_3} \tag{4.33}$$

for the quark operators, $i = 1, 2, 3$, and

$$\Psi^i_{N,q}(z_1, z_2, z_3, z_4) = \sum_{\substack{k_1,\ldots,k_4 \\ k_1+\ldots+k_4=N-2}} \psi^{(i)N,q}_{k_1\,k_2\,k_3\,k_4}\, z_1^{k_1}\, z_2^{k_2}\, z_3^{k_3}\, z_4^{k_4} \tag{4.34}$$

for the quark-gluon operators, $i = 1, 2$. The sum in (4.34) contains $N-2$ instead of N as the additional gluon field has a canonical dimension of 2.

However, even though we can determine the eigenfunctions of (4.32) once we know the functional form of the Hamiltonian, it is not immediately clear how the multiplicatively renormalizable operators look like. They are linear combinations of local operators that can be represented as [59]

$$\begin{aligned}\mathcal{O}_{\mathrm{mult}} = & \sum_{i=1}^{3} \left[\widetilde{\mathcal{P}}^i_{N,q}(\partial_1, \partial_2, \partial_3)\mathbb{O}_i(z_1, z_2, z_3)\right]_{z_i=0} + \\ & + \sum_{i=4}^{5} \left[\widetilde{\mathcal{P}}^i_{N,q}(\partial_1, \partial_2, \partial_3, \partial_4)\mathbb{O}_i(z_1, z_2, z_3, z_4)\right]_{z_i=0} ,\end{aligned} \tag{4.35}$$

where the $\mathbb{O}_i$ were defined in Eq. (4.29) and the $\widetilde{\mathcal{P}}^i_{N,q}(x_1, x_2, x_3(, x_4))$ are homogeneous polynomials. These polynomials are affiliated with the polynomials in coordinate space $\Psi^i_{N,q}(\vec{z})$, but *not* the same. Their relation is similar to the relation of a function to its Fourier transformed: we can say that $\widetilde{\mathcal{P}}^i_{N,q}(\vec{x})$ is the momentum space representation of $\Psi^i_{N,q}(\vec{z})$ and the x_i correspond to momentum fractions.

In [61, 59] the translation rules from one representation to another were derived. They amount to the substitution

$$z_i^n \longrightarrow \frac{x_i^n}{\Gamma(n+2j)} , \tag{4.36}$$

where j is the conformal spin of the field with coordinate z_i. Note that conformal symmetry demands that the polynomials $\widetilde{\mathcal{P}}$ fulfill an orthogonality relation

$$\int \mathcal{D}x\; x_1^{2j_1-1}\, x_2^{2j_2-1}\, x_3^{2j_3-1}\; \widetilde{\mathcal{P}}^{j_1j_2j_3}_l(x_1, x_2, x_3)\, \widetilde{\mathcal{P}}^{j_1j_2j_3}_k(x_1, x_2, x_3) = \mathcal{N}\delta_{lk} , \tag{4.37}$$

where $\mathcal{N}$ is some normalization constant and the integral measure is defined in (4.6), see [59] for the derivation of (4.36) and (4.37).

4.3.1 Conformal Symmetry and Evolution Kernels

The renormalization kernels are yet to be determined. Conformal symmetry provides a tool to classify and constrain the possible functional forms. To understand how this works, we want to consider the simplest example: a 2-to-2 kernel $K^{1\,1/2}_{1/2\,1}$ mapping conformal spins $j_1 = 1/2$ and $j_2 = 1$ to $i_1 = 1$ and

$i_2 = 1/2$. This type of kernel occurs for example as part of $\mathbb{H}_q$. Unfortunately, the analysis of the transformation properties of the kernel turns out to be quite involved and we need a certain mathematical armamentarium.

Therefore, we first give the final result and present a detailed, slightly heuristic derivation in a separate section (see below):

$$[K^{1\,1/2}_{1/2\,1}\varphi](z_1, z_2) = \int_0^1 \mathrm{d}\alpha \int_0^1 \mathrm{d}\beta \; \frac{1}{\alpha} \; \kappa\left(\frac{\alpha\beta}{\bar{\alpha}\bar{\beta}}\right) \varphi(z_{12}^{\alpha}, z_{21}^{\beta}) \,, \tag{4.38}$$

where $\frac{\alpha\beta}{\bar{\alpha}\bar{\beta}}$ is the so-called conformally invariant ratio and κ is an arbitrary function that must not produce any poles throughout the integration region. This limits the functional form of $\kappa(x)$. In the end only a finite amount of choices, which typically contain δ-functions or Heaviside-Θ-functions, are allowed. Each of the possible functions κ generates one elementary renormalization kernel.

For arbitrary conformal spins, see [75, 61], as well as for the mapping $(1, 1, 3/2) \to (1, 1/2)$ [62], which is necessary for the quark-gluon block in (4.28), analogous expressions can be found. The multitude of possible kernels is collected in App. C.

These conformal "elementary kernels" provide us with a powerful check of any calculation of the full Hamiltonian $\mathbb{H}$, as one must be able to rewrite all kernels in terms of linear combinations of the elementary kernels. In Sect. 4.4 we will develop an additional check that is able to verify the coefficients in the linear combinations.

How to obtain $[K^{1\,1/2}_{1/2\,1}\varphi](z_1, z_2)$

Let us now derive the kernel $K^{1\,1/2}_{1/2\,1}$. To this end, it is convenient to introduce an $SL(2,\mathbb{R})$ invariant scalar product. This can be achieved by observing that the $SU(1,1)$ group and the $SL(2,\mathbb{R})$ group actually have the same algebra, and the generators have the same form. The $SU(1,1)$ invariant scalar product is known [76] and reads:

$$\langle f_1, f_2\rangle_j = \int_{|z|<1} D_j z\, \overline{f_1(z)}\, f_2(z) \,, \qquad D_j z = \frac{2j-1}{\pi}(1-|z|^2)^{2j-2} d^2 z \,, \tag{4.39}$$

where j is the conformal spin and the functions f_1 and f_2 are polynomials in the complex variable z, $\overline{f(z)} = (f(z))^*$. Note that the integration is performed over the complex unit disc and includes the weight function

$$\frac{2j-1}{\pi}(1-|z|^2)^{2j-2} \,.$$

In mathematics this scalar product is associated with *weighted Bergman spaces* which have been studied in great detail, see [77] for an excellent introduction.

The generalization to the case of a function of more than one variable is straightforward and one obtains

$$\langle f_1, f_2 \rangle = \left(\prod_{k=1}^{n} \int_{|z_k|<1} D_{j_k}\, z_k \right) \overline{f_1(z_1, \ldots, z_n)}\, f_2(z_1, \ldots, z_n)\,. \tag{4.40}$$

Now consider the case where we have a single field $\Phi(z)$ which has conformal spin j. As usual, it can be identified with a polynomial $p(z)$. The kernel $K^j(z, w)$ with the property

$$p(z) = \int_{|w|<1} D_j w\; K^j(z, w) p(w) \tag{4.41}$$

is called *reproducing kernel*: $K^j(z, w)$ maps the represention T^j onto itself.

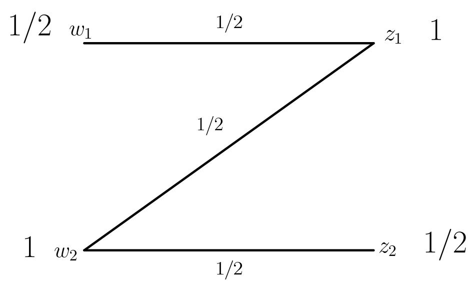
The reproducing kernel (RK) can be obtained via explicit calculation [77] and has the form

$$K^j(z, w) = \frac{1}{(1 - z\bar{w})^{2j}}\ . \tag{4.42}$$

It turns out that the RK can be used as starting point for the construction of more complicated (multi-particle) kernels; this brings us back to our original problem: the kernel mapping $T^{1/2} \otimes T^1$ onto $T^1 \otimes T^{1/2}$. $K^{1\,1/2}_{1/2\,1}(z_1, z_2, w_1, w_2)$ must fulfill the requirement

$$\phi^{1,1/2}(z_1, z_2) = \int D_1 w_2 D_{1/2} w_1\; K^{1\,1/2}_{1/2\,1}(z_1, z_2, w_1, w_2) \phi^{1/2,1}(w_1, w_2)\ , \tag{4.43}$$

where ϕ^{j_1,j_2} are polynomials corresponding to the operators with conformal spins (j_1, j_2). Obviously, in total half a unit of conformal spin has to be "transported" from the second particle to the first one. Heuristicly, this can be visualized in diagrammatic form via



Each line carries half a unit of spin. On the left-hand side are two fields, one with conformal spin 1/2 at coordinate w_1 and one with spin 1 at coordinate w_2. On the right-hand side $z_1 \,\hat{=}\, j{=}1$ and $z_2 \,\hat{=}\, j{=}1/2$. We can construct the full kernel with the correct conformal properties from this diagram by treating each line as a reproducing kernel with conformal spin 1/2. One gets

$$\int D_1 w_2\, D_{1/2} w_1\, \frac{1}{(1 - z_1\bar{w}_1)^1} \frac{1}{(1 - z_2\bar{w}_2)^1} \frac{1}{(1 - z_1\bar{w}_2)^1} \times$$

$$\times\,\widetilde{\kappa}\left(\frac{(1-z_1\bar{w}_1)(1-z_2\bar{w}_2)}{(1-z_2\bar{w}_1)(1-z_1\bar{w}_2)}\right)\phi^{1/2,1}(w_1,w_2)\,, \qquad (4.44)$$

where $\widetilde{\kappa}$ is an arbitrary function of its argument

$$\frac{(1-z_1\bar{w}_1)(1-z_2\bar{w}_2)}{(1-z_2\bar{w}_1)(1-z_1\bar{w}_2)}\,,$$

which is invariant under $SL(2,\mathbb{R})$ transformations [75]. Therefore, $\tilde{\kappa}$ is neither constrained by conformal symmetry nor does it affect the transformation properties of Eq. (4.44).

Replacing $\widetilde{\kappa}$ in (4.44) by its Mellin transformed

$$\widetilde{\kappa}(x)=\int_{c-i\infty}^{c+i\infty}\mathrm{d}j\;\; x^j F(j) \qquad (4.45)$$

one obtains

$$\int\limits_{c-i\infty}^{c+i\infty}\mathrm{d}j\int D_1w_2D_{1/2}w_1\,\frac{1}{(1-z_1\bar{w}_1)^{1-j}}\frac{1}{(1-z_2\bar{w}_2)^{1-j}}\frac{1}{(1-z_1\bar{w}_2)^{1+j}}\frac{1}{(1-z_2\bar{w}_1)^{j}}$$
$$\times\,F(j)\;\phi^{1/2,1}(w_1,w_2)\;. \qquad (4.46)$$

By introducing two Feynman parameters α and β we can combine the denominators with $\bar{w}_1$ and $\bar{w}_2$

$$\int\limits_0^1\mathrm{d}\alpha\int\limits_0^1\mathrm{d}\beta\int\limits_{c-i\infty}^{c+i\infty}\mathrm{d}j\int D_1w_2D_{1/2}w_1\,\frac{\alpha^{j-1}\bar{\alpha}^{-j}}{(1-z_{12}^{\alpha}\bar{w}_1)^1}\frac{\beta^j\bar{\beta}^{-j}}{(1-z_{21}^{\beta}\bar{w}_2)^2}\times$$
$$\times\,F(j)\phi^{1/2,1}(w_1,w_2)\;=$$
$$=\int\limits_0^1\mathrm{d}\alpha\int\limits_0^1\mathrm{d}\beta\left[\int\limits_{c-i\infty}^{c+i\infty}\mathrm{d}j\left(\frac{\alpha\beta}{\bar{\alpha}\bar{\beta}}\right)^j F(j)\right]\cdot\frac{1}{\alpha}$$
$$\times\int D_1w_2D_{1/2}w_1\frac{1}{(1-z_{12}^{\alpha}\bar{w}_1)^1}\frac{1}{(1-z_{21}^{\beta}\bar{w}_2)^2}\phi^{1/2,1}(w_1,w_2) \qquad (4.47)$$

The whole second line amounts to the reproducing kernel for $\phi^{1/2,1}(w_1,w_2)$ with the correct conformal spins. We obtain

$$\int\limits_0^1\mathrm{d}\alpha\int\limits_0^1\mathrm{d}\beta\left[\int\limits_{c-i\infty}^{c+i\infty}\mathrm{d}j\left(\frac{\alpha\beta}{\bar{\alpha}\bar{\beta}}\right)^j F(j)\right]\cdot\frac{1}{\alpha}\cdot\phi^{1/2,1}(z_{12}^{\alpha},z_{21}^{\beta})\;. \qquad (4.48)$$

As a final step one has to identify the factor in square brackets with $\widetilde{\kappa}\left(\frac{\alpha\beta}{\bar{\alpha}\bar{\beta}}\right)$ via the Mellin transformation (4.45). The result coincides with the kernel $[K^{1\,1/2}_{1/2\,1}\varphi](z_1,z_2)$ given in (4.38).

4.3.2 Renormalization Kernels I : Chiral Operators

Finally, everything is ready for the calculation of the renormalization kernels. Instead of presenting the full calculation, we opt to quote the final results following [62] and show the details for the calculation of the simplest chiral 2-to-3 kernel, see Sect.4.3.4. In fact the 2-to-2 kernels (for both quark and gluon block, $\mathbb{H}_q$ and $\mathbb{H}_g$, respectively) have been known for quite some time and can be found in the literature [61, 78, 79, 80, 81]. Further, it turns out that explicitly keeping factors of N_c stemming from color structures is advantageous for organizing the terms. Of course, all expressions only make sense for $N_c = 3$ as the operator basis is not gauge invariant for a different number of colors.

The kernels for the chiral operators are generally simpler than their mixed chirality counterparts. This is natural, as two quark fields differ only by color and flavor indices which do not play a major role in the computation of the kernels. Let us denote the chiral Hamiltonian by the superscript $\psi\psi\psi$, that is $\mathbb{H}^{\psi\psi\psi}$. The quark block then takes the form:

$$\mathbb{H}_q^{\psi\psi\psi} = \left(1 + \frac{1}{N_c}\right) \begin{pmatrix} \mathrm{H} & \mathcal{H}_{12}^e & \mathcal{H}_{13}^e \\ \mathcal{H}_{21}^e & \mathrm{H} & \mathcal{H}_{23}^e \\ \mathcal{H}_{31}^e & \mathcal{H}_{32}^e & \mathrm{H} \end{pmatrix}, \tag{4.49}$$

where

$$\mathrm{H} = \mathcal{H}_{12}^v + \mathcal{H}_{23}^v + \mathcal{H}_{31}^v - \frac{1}{2}. \tag{4.50}$$

and the expressions for the calligraphic two-particle Hamiltonians can be found in App. C.

The renormalization of the chiral four-particle operators and their mixing among themselves is described by the gluon block $\mathbb{H}_g$. The reduced Hamiltonian $\widetilde{\mathbb{H}}_g$ can be restored via Eqs. (4.30). $\mathbb{H}_g$ can conveniently be written as

$$\mathbb{H}_g^{\psi\psi\psi\bar{f}} = N_c\,\mathbb{H}_g^{(1)} + \mathbb{H}_g^{(0)} + \frac{1}{N_c}\,\mathbb{H}_g^{(-1)} + \frac{21}{2}. \tag{4.51}$$

The off-diagonal matrix elements assume a rather simple form

$$\begin{aligned} [\mathbb{H}_g^{(0)}]_{ik} &= \mathcal{H}_{ik}^v - \mathcal{H}_{k4}^v + 2\,\mathcal{H}_{k4}^+ - \frac{1}{2}, \\ [\mathbb{H}_g^{(-1)}]_{ik} &= -\,2\,\mathcal{H}_{k4}^-, \end{aligned} \tag{4.52}$$

where $i \neq k$ and $i, k = 1, 2, 3$. The diagonal part of the matrices in Eq. (4.51) is more involved:

$$[\mathbb{H}_g^{(1)}]_{kk} = \mathcal{H}_{k4}^v - 2\,\mathcal{H}_{k4}^+, \qquad [\mathbb{H}_g^{(-1)}]_{kk} = \mathcal{H}_{12}^v + \mathcal{H}_{23}^v + \mathcal{H}_{31}^v - 2\,\mathcal{H}_{k4}^-,$$
$$[\mathbb{H}_g^{(0)}]_{kk} = \mathcal{H}_{k+1,k-1}^v + \mathcal{H}_{k+1,4}^v + \mathcal{H}_{k-1,4}^v +$$

$$- 2\left(\mathcal{H}^{+}_{k+1,4} + \mathcal{H}^{+}_{k-1,4} + \mathcal{H}^{-}_{k+1,4} + \mathcal{H}^{-}_{k-1,4}\right) . \quad (4.53)$$

Here again k is equal to $1, 2$ or 3 and the indices on the right-hand side of the equations are cyclic, i.e. the subscript $k+1 = 4$ is identified with 1 and $k-1 = 0$ with 3.

The quark-gluon mixing block is found to be

$$\mathbb{H}^{\psi\psi\psi}_{qg} = -\frac{1}{2}\mathcal{H}^{\psi\psi\psi}_{qg} \quad (4.54)$$

with

$$\begin{aligned}
[\mathcal{H}^{\psi\psi\psi}_{qg}]_{kk} =& \frac{1}{N_c}\left(\mathcal{V}^{(1)}_{k,k+1,(4)} + \mathcal{V}^{(1)}_{k,k-1,(4)}\right) - \mathcal{V}^{(2)}_{k,k+1,(4)} - \mathcal{V}^{(2)}_{k,k-1,(4)} ,\\
[\mathcal{H}^{\psi\psi\psi}qg]_{ik} =& \mathcal{V}^{(1)}_{ik(4)} + \mathcal{V}^{(2)}_{ik(4)} ,
\end{aligned} \quad (4.55)$$

where the subscripts are again cyclic. We will derive these kernels in some detail in Sect. 4.3.4.

4.3.3 Renormalization Kernels II : Operators of Mixed Chirality

Following the presentation in the previous section, we just quote the final results for the three 3×3 blocks of Hamiltonian $\mathbb{H}^{\psi\psi\bar{\chi}}$ for the operators of mixed chirality (4.15), (4.16).

The quark block is given by [62]

$$\mathbb{H}^{\psi\psi\bar{\chi}}_{q} = \left(1 + \frac{1}{N_c}\right)\mathcal{H}^{\psi\psi\bar{\chi}}_{q} \quad (4.56)$$

where[5]

$$\mathcal{H}^{\psi\psi\bar{\chi}}_{q} = \begin{pmatrix} \mathrm{H} + \mathcal{H}^{d}_{13} - \mathcal{H}^{+}_{23} & \mathcal{H}^{e}_{12} & z_{13}\mathcal{H}^{+}_{13} \\ \mathcal{H}^{e}_{21} & \mathrm{H} + \mathcal{H}^{d}_{23} - \mathcal{H}^{+}_{13} & z_{23}\mathcal{H}^{+}_{23} \\ z^{-1}_{13}(\mathbb{1} - 2\mathcal{H}^{d}_{13}) & z^{-1}_{23}(\mathbb{1} - 2\mathcal{H}^{d}_{23}) & \mathrm{H} - 2(\mathcal{H}^{+}_{13} + \mathcal{H}^{+}_{23}) + 3 \end{pmatrix} . \quad (4.57)$$

The gluon block $\mathbb{H}^{\psi\psi\bar{\chi}\bar{f}}_{g}$ can again be split in three terms with different powers of N_c and a constant:

$$\mathbb{H}^{\psi\psi\bar{\chi}\bar{f}}_{g} = N_c\,\mathbb{H}^{\psi\psi\bar{\chi}\bar{f},(1)}_{g} + \mathbb{H}^{\psi\psi\chi\bar{f},(0)}_{g} + \frac{1}{N_c}\,\mathbb{H}^{\psi\psi\bar{\chi}\bar{f},(-1)}_{g} + \frac{21}{2} . \quad (4.58)$$

The matrix elements are given by

$$[\mathbb{H}^{\psi\psi\bar{\chi}\bar{f},(-1)}_{g}]_{kk} = \mathcal{H}^{v}_{12} + \mathcal{H}^{v}_{13} + \mathcal{H}^{v}_{23} - \mathcal{H}^{+}_{13} - \mathcal{H}^{+}_{23} - 2(1 - \delta_{k,3})\mathcal{H}^{-}_{k4} + \delta_{k3}P_{34}\mathcal{H}^{e}_{43},$$

[5] Recall $z_{ij} = z_i - z_j$, cf. (4.25).

$$[\mathbb{H}_g^{\psi\psi\bar{\chi}\bar{f},(-1)}]_{12} = -2\mathcal{H}_{24}^{-}\,, \qquad [\mathbb{H}_g^{\psi\psi\bar{\chi}\bar{f},(-1)}]_{21} = -2\mathcal{H}_{14}^{-}\,,$$
$$[\mathbb{H}_g^{\psi\psi\bar{\chi}\bar{f},(-1)}]_{j3} = P_{34}\mathcal{H}_{43}^{e}\,, \qquad [\mathbb{H}_g^{\psi\psi\bar{\chi}\bar{f},(-1)}]_{3j} = -2\mathcal{H}_{14}^{-} \tag{4.59}$$

and

$$\begin{aligned}
[\mathbb{H}_g^{\psi\psi\bar{\chi}\bar{f},(0)}]_{11} =& \mathcal{H}_{23}^{v} + \mathcal{H}_{24}^{v} + \mathcal{H}_{34}^{v} - \mathcal{H}_{23}^{+} - 2\mathcal{H}_{24}^{+} - 2\mathcal{H}_{24}^{-} + P_{34}\mathcal{H}_{43}^{e}\,,\\
[\mathbb{H}_g^{\psi\psi\bar{\chi}\bar{f},(0)}]_{22} =& \mathcal{H}_{13}^{v} + \mathcal{H}_{14}^{v} + \mathcal{H}_{34}^{v} - \mathcal{H}_{13}^{+} - 2\mathcal{H}_{14}^{+} - 2\mathcal{H}_{14}^{-} + P_{34}\mathcal{H}_{43}^{e}\,,\\
[\mathbb{H}_g^{\psi\psi\bar{\chi}\bar{f},(0)}]_{33} =& \mathcal{H}_{12}^{v} + \mathcal{H}_{14}^{v} + \mathcal{H}_{24}^{v} - 2(\mathcal{H}_{14}^{+} + \mathcal{H}_{24}^{+} + \mathcal{H}_{14}^{-} + \mathcal{H}_{24}^{-})\,,\\
[\mathbb{H}_g^{\psi\psi\bar{\chi}\bar{f},(0)}]_{12} =& \mathcal{H}_{12}^{v} - \mathcal{H}_{24}^{v} + 2\mathcal{H}_{24}^{+} - \frac{1}{2}\,,\\
[\mathbb{H}_g^{\psi\psi\bar{\chi}\bar{f},(0)}]_{21} =& \mathcal{H}_{21}^{v} - \mathcal{H}_{14}^{v} + 2\mathcal{H}_{14}^{+} - \frac{1}{2}\,,\\
[\mathbb{H}_g^{\psi\psi\bar{\chi}\bar{f},(0)}]_{j3} =& \mathcal{H}_{j3}^{v} - \mathcal{H}_{34}^{v} - \ \mathcal{H}_{j3}^{+} - \frac{1}{2}\,,\\
[\mathbb{H}_g^{\psi\psi\bar{\chi}\bar{f},(0)}]_{3j} =& \mathcal{H}_{3j}^{v} - \mathcal{H}_{j4}^{v} - \mathcal{H}_{j3}^{+} + 2\mathcal{H}_{j4}^{+} - \frac{1}{2}\,,
\end{aligned} \tag{4.60}$$

where $k = 1,2,3$, $j = 1,2$ and $\delta_{kk'}$ is the usual Kronecker symbol. The operator P_{ij} exchanges the ith and jth argument, e.g. the action of P_{34} on some function φ is given as

$$P_{34}\varphi(z_1,z_2,z_3,z_4) = \varphi(z_1,z_2,z_4,z_3).$$

For the last remaining submatrix, the quark-gluon block $\mathbb{H}^{\psi\psi\bar{\chi}\bar{f}}$, we find [62]

$$\mathbb{H}_{qg}^{\psi\psi\bar{\chi}\bar{f}} = -\frac{1}{2}\mathcal{H}_{qg}^{\text{mixed}}\,, \tag{4.61}$$

where

$$\begin{aligned}
\left[\mathcal{H}_{qg}^{\text{mixed}}\right]_{jk} =& \left[\mathcal{H}_{qg}^{\psi\psi\psi}\right]_{jk} + [\Delta\mathcal{H}_{qg}]_{jk}\ , \qquad j,k = 1,2\\
[\mathcal{H}_{qg}^{\text{mixed}}]_{3k} =& \frac{2}{z_{k3}}\left(\mathcal{V}_{k3(4)}^{(b)} - \frac{1}{3}\mathcal{V}_{k3(4)}^{(a)} - \frac{1}{2}\mathcal{V}_{k3(4)}^{(3)} + \frac{1}{2}\mathcal{V}_{k3(4)}^{(4)}\right)\,, \qquad k = 1,2\\
[\mathcal{H}_{qg}^{\text{mixed}}]_{33} =& -2\sum_{j=1}^{2}\frac{1}{z_{j3}}\left(\mathcal{V}_{j3(4)}^{(a)} - \frac{1}{3}\mathcal{V}_{j3(4)}^{(b)} + \frac{4}{3}\mathcal{V}_{j3(4)}^{(c)} + \frac{1}{6}\mathcal{V}_{j3(4)}^{(3)} + \frac{1}{2}\mathcal{V}_{j3(4)}^{(4)}\right)\,.
\end{aligned} \tag{4.62}$$

$\mathcal{H}_{qg}^{\psi\psi\psi}$ is given in Eq. (4.55) and

$$\begin{aligned}
[\Delta\mathcal{H}_{qg}]_{12} =& [\Delta\mathcal{H}_{qg}]_{21} = 0\,,\\
[\Delta\mathcal{H}_{qg}]_{jj} =& \frac{1}{3}\mathcal{V}_{j3(4)}^{(a)} - \mathcal{V}_{j3(4)}^{(b)}\,,\\
[\Delta\mathcal{H}_{qg}]_{j3} =& \mathcal{V}_{j3(4)}^{(a)} - \frac{1}{3}\mathcal{V}_{j3(4)}^{(b)} + \frac{4}{3}\mathcal{V}_{j3(4)}^{(c)}\,,
\end{aligned} \tag{4.63}$$

for $j = 1,2$.

4.3.4 Explicit Example: The Chiral 2-to-3 Kernel

[6]In principle, the calculation of the kernels given in Sect. 4.3.2 and 4.3.3 is straightforward. However, it is technically challenging if one is not used to the formalism. We will try to shed some light on the "tricks" necessary to simplify this kind of calculation using the chiral $\psi_+\bar{f}_{++}\psi_+ \rightarrow \psi_-\psi_+$ kernels as an instructive example. While it does not represent the most complicated kernel, it already requires some effort to work out. We abstain from relaying this calculation to a separate appendix for the following reason: there is, to our knowledge, no explicit calculation for this kind of kernel to be found anywhere in the literature.

Starting point is the operator, cf. (4.11),

$$Q_1(a_1, a_2, a_3) = \epsilon^{abc}\psi_-^a(a_1)\psi_+^b(a_2)\psi_+^c(a_3)\ , \tag{4.64}$$

where a, b, c are color indices and the flavor indices have been neglected for simplicity. We want to study the mixing of Q_1 with the three chiral quasipartonic operators G_1, G_2 and G_3 defined in (4.12). At one-loop level there are two 2-to-3 kernels mapping the four particle operators G_i to the operator Q_1, which can play a role:

- $\psi_-^a(a_1)\psi_+^b(a_2)$ can mix with $[\bar{f}_{++}\psi_+]^a\,\psi_+^b$ and $\psi_+^a\,[\bar{f}_{++}\psi_+]^b$
- $\psi_-^a(a_1)\psi_+^c(a_3)$ can mix with $[\bar{f}_{++}\psi_+]^a\,\psi_+^c$ and $\psi_+^a\,[\bar{f}_{++}\psi_+]^c$

The third combination of two quarks, $\psi_+^b(a_1)\psi_+^c(a_3)$, cannot produce an additional gluon field $\bar{f}_{++}$.

It is obvious that we do not need to consider both 2-to-3 kernels since the result for $\psi_-^a(a_1)\psi_+^c(a_3)$ can be restored from the one for $\psi_-^a(a_1)\psi_+^b(a_2)$ by replacing $a_2 \leftrightarrow a_3$. We define the Hamiltonian corresponding to the 2-to-3 kernel involving the ith and jth quark in Q_1 as

$$\mathbb{H}_{qg}^{(ij)}(a_1, a_2, a_3) := \left[\mathbb{H}_{qg}^{(ij)}\right]_{11} G_1 + \left[\mathbb{H}_{qg}^{(ij)}\right]_{12} G_2 + \left[\mathbb{H}_{qg}^{(ij)}\right]_{13} G_3\ . \tag{4.65}$$

These kernels are then related to the matrix elements of the quark-gluon mixing block, see (4.28), by

$$\left[\mathbb{H}_{qg}^{\text{chiral}}\right]_{11} = \left[\mathbb{H}_{qg}^{(12)}\right]_{11} + \left[\mathbb{H}_{qg}^{(13)}\right]_{11} \tag{4.66}$$

$$\left[\mathbb{H}_{qg}^{\text{chiral}}\right]_{12} = \left[\mathbb{H}_{qg}^{(12)}\right]_{12} \tag{4.67}$$

and so on.

[6]Readers, who do not wish to follow through the whole rather lengthy calculation, can skip this section; the rest of this thesis does not rely on concept or equations introduced therein.

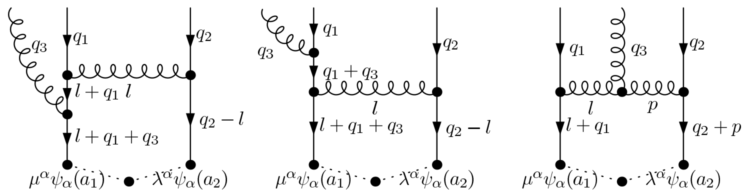


Figure 4.1: The three Feynman diagrams relevant for the chiral 2-to-3 kernels. The "mirror diagrams" where the gluon is emitted from the "plus" quark are omitted.

General Considerations

Before we start with the calculation, there are some issues that need to by discussed. The operator Q_1 is a gauge invariant object. Therefore, we can freely choose any gauge and the results are independent of our choice; the complexity of the calculation, however, is not. There are two standard choices in the literature: Fock-Schwinger Fixed-Point gauge [82, 83], which was used for a similar calculation in [81] and light-cone gauge $(nA)(x) = 0$ [84]. We opt for the latter, since this will allow us to set all path-ordered exponents equal to $\mathbb{1}$. In spinor notation this gauge corresponds to

$$n_{\alpha\dot{\alpha}}\bar{A}^{\dot{\alpha}\alpha}(x) = 0 \qquad \Leftrightarrow \qquad \lambda_\alpha\bar{\lambda}_{\dot{\alpha}}\bar{A}^{\dot{\alpha}\alpha} = 0$$
$$\Leftrightarrow \qquad A_{++} = 0 \,. \tag{4.68}$$

However, we have to pay a price for this simplification. The gluon propagator in D dimensions assumes the form [84]

$$\overline{A^A(x)A}^B(0) = \int \frac{\mathrm{d}^D k}{(2\pi)^D i} \frac{(g_{\mu\nu} - \frac{n_\mu k_\nu + n_\nu k_\mu}{nk})\delta^{AB}}{k^2} e^{-ikx} \,, \tag{4.69}$$

which is more complicated than in Feynman gauge. $A, B = 1, .., 8$ are color indices of the adjoint representation.

Next, one has to determine the Feynman diagrams that will contribute to the kernel mapping the operator $\bar{f}_{++}\psi_+\psi_+$ to $\psi_-\psi_+$. In order to keep the calculation manageable, we will only consider the three diagrams given in Fig. 4.1; as the third quark, which is necessary to have a gauge invariant operator, is always a pure spectator at one-loop level, we omitted it for simplicity. Note that there are diagrams, where the gluon is emitted from the "plus" quark. Although this is not obvious, these diagrams give no relevant contribution[7] and we can ignore them.

We also need to make a comment concerning the second diagram in Fig. 4.1. It seems to be unnecessary as it is not irreducible and should not play a role

[7] If this were not the case, one could generate diagrams giving rise to $\bar{f}_{++}\psi_+\psi_+ \to \psi_+\psi_+$; this is forbidden due to different twists.

for renormalization. In fact, this type of diagram – sometimes called "equation of motion" diagram – is the diagrammatical visualization of terms that arise in the $\psi_-\psi_+ \to \psi_-\psi_+$ kernels and require the use of the Dirac equation, i.e. the expression $(\bar\partial^{\dot\alpha\alpha}\psi_\alpha)\psi_+$ is zero up to terms involving three fields

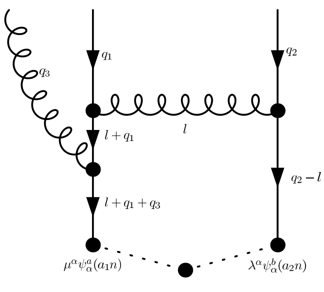
$$\bar\partial^{\dot\alpha\alpha}\psi_\alpha = \bar D^{\dot\alpha\alpha}\psi_\alpha + ig\bar A^{\dot\alpha\alpha}\psi_\alpha = 0 + ig\bar A^{\dot\alpha\alpha}\psi_\alpha \;.$$

If one only wants to access the 2-to-2 kernels, one can neglect terms with three fields altogether, but then they have to be included in the 2-to-3 kernels. As we do not wish to recalculate the known 2-to-2 kernels, but are interested in the more challenging 2-to-3 kernels, the term arising due to the equations of motion can be restored by including the second diagram in Fig. 4.1 and keeping only the terms where the propagator with momentum $q_1 + q_3$ is canceled.

The calculation itself is carried out in dimensional regularization with $D = 4 - 2\epsilon$. To avoid any ambiguities in the definition of spinors in D dimensions we generally only change to the spinor formalism at the very end of the calculation of the one-loop integrals, i.e. after Feynman parameter integrals have been introduced, the momentum integral taken and the divergence isolated in a $\frac{1}{\epsilon}$-pole. As we are interested in the anomalous dimensions, the only terms relevant for our analysis are logarithmically divergent in the ultra-violet region. Thus we can drop all ultra-violet finite terms and consider two terms to be equal, if they only differ by a finite term. This simplifies the bookkeeping drastically, e.g. the replacement $g \to g\mu^{-\epsilon}$ will only modify the finite part.

As a further simplification, we do not write the third quark (which plays the role of a passive spectator) nor the Levi-Civita tensor ϵ^{abc} explicitly.

The First Diagram



We start with the diagram on the left. For completeness we indicate the gauge links by dotted lines. Even though it is a bit more tedious, it is convenient to start in coordinate space before the relevant Wick contractions are carried out. The risk to lose an overall sign right from the start is diminished that way.

One obtains:

$$(ig)^3\mu\psi^a(a_1n)\lambda\psi^b(a_2n)\int \mathrm{d}^D y\int \mathrm{d}^D z\int \mathrm{d}^D w\; \bar q^{e'}(y)\not{A}^{e'e}(y)q^e(y)$$

$$\cdot\, \bar q^{f'}(z)\not{A}^{f'f}(z)q^f(z)\cdot \bar q^{g'}(w)\not{A}^{g'g}(w)q^g(w) \overset{Wick}{\to}$$

$$\to -ig^3\int \mathrm{d}^D y\;\mathrm{d}^D y\;\mathrm{d}^D z\int \frac{\mathrm{d}^D(p,l,q,k)}{(2\pi)^{4D}}\times$$

$$\times\frac{\mu\not{p}\not{A}^{ae}(q_3)\not{q}\gamma_\mu\psi^f(q_1)\cdot\left[-g^{\mu\nu}+\frac{n^\mu l^\nu+n^\nu l^\mu}{nl}\right]\cdot\lambda\not{k}\gamma_\nu\psi^g(q_2)}{p^2k^2l^2q^2}\,T^{A,ef}T^{A,bg}\times$$

$$\times\, \mathrm{e}^{(-ip(a_1 n - y) - iq(y-z) - ik(a_2 x - w) - il(z-w) - iq_1 z - iq_2 w - iq_3 y)} = \dots \tag{4.70}$$

After performing the three integrations in coordinate space, one is left with three delta functions

$$(2\pi)^{3D}\delta^D(q+q_3-p)\delta^D(q_2-k-l)\delta^D(l+q_3-q)\ ,$$

which allow us to remove all but one momentum integration. We get:

$$\begin{aligned}\dots = -ig^3 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\mu(\not{l}+\not{q}_1+\not{q}_3)\not{A}^B(q_3)(\not{l}+\not{q}_1)\gamma_\mu \psi^f(q_1)\cdot \lambda(\not{q}_2-\not{l})\gamma_\nu \psi^g(q_2)}{(l+q_1+q_3)^2(l+q_1)^2(q_2-l)^2 l^2}\cdot \\ \cdot (T^{B,ae}T^{A,ef}T^{A,bg}) \left[-g^{\mu\nu} + \frac{n^\mu l^\nu + n^\nu l^\mu}{nl}\right] \cdot e^{-i(l+q_1+q_3)a_1 n - i(q_2-l)a_2 n}\ .\end{aligned} \tag{4.71}$$

Let us introduce the abbreviations

$$\mathcal{N}_1 = \frac{1}{(l+q_1+q_3)^2(l+q_1)^2(q_2-l)^2 l^2} \quad \text{and} \quad \mathcal{C}_1 = T^{B,ae}T^{A,ef}T^{A,bg}\ .$$

We consider the three terms arising due to the three summands in the gluon propagator separately.

$$\begin{aligned} I_1 :=& ig^3 \mathcal{C}_1 \int \frac{\mathrm{d}^D l}{(2\pi)^D}\mathcal{N}_1 \cdot \mu(\not{l}+\not{q}_1+\not{q}_3)\not{A}^B(q_3)(\not{l}+\not{q}_1)\gamma_\mu\psi^f(q_1)\cdot \\ &\cdot \lambda(\not{q}_2-\not{l})\gamma^\mu\psi^g(q_2)\cdot e^{-i(l+q_1+q_3)a_1 n - i(q_2-l)a_2 n} \end{aligned} \tag{4.72}$$

$$\begin{aligned} I_2 :=& -ig^3 \mathcal{C}_1 \int \frac{\mathrm{d}^D l}{(2\pi)^D}\mathcal{N}_1 \frac{1}{ln}\mu(\not{l}+\not{q}_1+\not{q}_3)\not{A}^B(q_3)(\not{l}+\not{q}_1)\not{n}\psi^f(q_1)\cdot \\ &\cdot \lambda(\not{q}_2-\not{l})\not{l}\psi^g(q_2) e^{-i(l+q_1+q_3)a_1 n - i(q_2-l)a_2 n} \end{aligned} \tag{4.73}$$

$$\begin{aligned} I_3 :=& -ig^3 \mathcal{C}_1 \int \frac{\mathrm{d}^D l}{(2\pi)^D}\mathcal{N}_1 \frac{1}{ln}\mu(\not{l}+\not{q}_1+\not{q}_3)\not{A}^B(q_3)(\not{l}+\not{q}_1)\not{l}\psi^f(q_1)\cdot \\ &\cdot \lambda(\not{q}_2-\not{l})\not{n}\psi^g(q_2) e^{-i(l+q_1+q_3)a_1 n - i(q_2-l)a_2 n} \end{aligned} \tag{4.74}$$

We shall start with I_1. Going over to spinor notation using the rules given in Sect. 3.1 one obtains:

$$\begin{aligned} I_1 =& ig^3 \int \frac{\mathrm{d}^D l}{(2\pi)^D}\mathcal{N}_1 \cdot \mu^\alpha (l+q_1+q_3)_{\alpha\dot\beta}\bar{A}^{B,\dot\beta\gamma}(l+q_1)_{\gamma\dot\delta}(\bar\sigma^\mu)^{\dot\delta\rho}\psi^f_\rho(q_1)\cdot \\ &\cdot \lambda^\sigma(q_2+l)_{\sigma\dot\tau}(\bar\sigma_\mu)^{\dot\tau\zeta}\psi^g_\zeta(q_2)\mathcal{C}_1\cdot e^{-i(l+q_1+q_3)a_1 n - i(q_2-l)a_2 n} = \\ =& -2ig^3 \int \frac{\mathrm{d}^D l}{(2\pi)^D}\mathcal{N}_1 \cdot \mu^\alpha (l+q_1+q_3)_{\alpha\dot\beta}\bar{A}^{B,\dot\beta\gamma}(l+q_1)_{\gamma\dot\tau}\psi^f_\rho(q_1)\cdot \\ &\cdot \lambda^\sigma(q_2+l)_{\sigma\dot\tau}\psi^g_\rho(q_2)\mathcal{C}_1\cdot e^{-i(l+q_1+q_3)a_1 x_1 - i(q_2-l)a_2 x_2} \end{aligned} \tag{4.75}$$

Here the identity

$$(\bar\sigma^\mu)^{\dot\delta\rho}(\bar\sigma_\mu)^{\dot\tau\zeta} = -2\epsilon^{\rho\zeta}\epsilon^{\dot\delta\dot\tau}$$

was employed, and the formal replacement $n \to x_1$ and $n \to x_2$ was implemented. This replacement allows the removal of the factors $\mu^\alpha(l+q_1+q_3)_{\alpha\dot\beta}$ and $\lambda^\sigma(q_2+l)_{\sigma\dot\tau}$.

To this end let us for the moment treat x_1 and x_2 as independent objects and put $x_1 = \mu\otimes\bar\mu$ as well as $x_1 = \lambda\otimes\bar\lambda$. Then one can replace $\mu^\alpha(l+q_1+q_3)_{\alpha\dot\beta}$ and $\lambda^\sigma(q_2+l)_{\sigma\dot\tau}$ by derivatives acting on the exponential function:

$$\begin{aligned}&\mu^\alpha(l+q_1+q_3)_{\alpha\dot\beta}\cdot\lambda^\sigma(q_2+l)_{\sigma\dot\tau}\cdot e^{-i(l+q_1+q_3)a_1x_1-i(q_2-l)a_2x_2} = \\ &= -\frac{1}{a_1a_2}\frac{\partial}{\partial\bar\mu_{\dot\beta}}\frac{\partial}{\partial\bar\lambda_{\dot\tau}}e^{-i(l+q_1+q_3)a_1x_1-i(q_2-l)a_2x_2}\ .\end{aligned} \tag{4.76}$$

After taking the derivatives we have to replace x_1 and x_2 again by n. Then I_1 takes the form

$$\begin{aligned}I_1 =&\frac{2ig^3}{a_1a_2}\frac{\partial}{\partial\bar\mu_{\dot\beta}}\frac{\partial}{\partial\bar\lambda_{\dot\tau}}\int\frac{\mathrm{d}^Dl}{(2\pi)^D}\mathcal{N}_1\mathcal{C}_1 \\ &\bar A^{B,\dot\beta\gamma}(l+q_1)_{\gamma\dot\tau}\psi^f_\rho(q_1)\cdot\lambda^\sigma(q_2+l)_{\sigma\dot\tau}\psi^g_\rho(q_2)e^{-i(l+q_1+q_3)a_1x_1-i(q_2-l)a_2x_2}\ .\end{aligned} \tag{4.77}$$

The denominator in $\mathcal{N}_1$ can be combined using the standard Feynman parameter trick. Shifting the integration variable l one obtains

$$\begin{aligned}I_1 =&\frac{2ig^3}{a_1a_2}\mathcal{C}_1\frac{\partial}{\partial\bar\mu_{\dot\beta}}\frac{\partial}{\partial\bar\lambda_{\dot\tau}}\int\frac{\mathrm{d}^Dl}{(2\pi)^D}\int_0^1\mathrm{d}\alpha\int_0^{\bar\alpha}\mathrm{d}\beta\int_0^{1-\beta-\alpha}\mathrm{d}\gamma\times \\ &\times\frac{\Gamma(4)\bar A^{B,\dot\beta\gamma}(q_3)(l+q_1-\alpha q_1-\alpha q_3-\beta q_1-\gamma q_2)^{\dot\tau}_\gamma\psi^f_\rho(q_1)\psi^{\rho,g}(q_2)}{[l^2+M^2]^4}\times \\ &\times\exp\left(-i(l+(1-\alpha-\beta)q_1+(1-\alpha)q_3+\gamma q_2)a_1x_1\right)\times \\ &\times\exp\left(-i(q_2-l+\alpha(q_1+q_2)+\beta q_1-\gamma q_2)a_2x_2\right)\ ,\end{aligned} \tag{4.78}$$

where M^2 depends only on the momenta q_i and the Feynman parameters α, β and γ. $\Gamma(z)$ is the usual Gamma function. Performing the loop integral in $D = 4-2\epsilon$ dimensions, M^2 does not affect the residue of the $\frac{1}{\epsilon}$ pole, thus its functional dependence on Feynman parameters and momenta is unimportant. In fact a logarithmic divergence can only appear after the exponent has been expanded to third order in l, as four powers of l are required in the numerator. Therefore, one easily finds that (up to finite terms)

$$\begin{aligned}I_1\ \sim&\frac{\partial}{\partial\bar\mu_{\dot\beta}}\frac{\partial}{\partial\bar\lambda_{\dot\tau}}\bar A^{B,\dot\beta}((1-\alpha)a_1x_1+\alpha a_2x_2)\cdot(a_1x_1-a_2x_2)^2(a_1x_1-a_2x_2)^{\dot\tau}_\gamma\times \\ &\times\psi^f_\rho((1-\alpha-\beta-\gamma)a_1x_1+(\alpha+\beta+\gamma)a_2x_2)\times \\ &\times\psi^{\rho,g}((1-\gamma)a_2x_2+\gamma a_1x_1)\ .\end{aligned} \tag{4.79}$$

However, the right hand side of (4.79) vanishes after carrying out the derivatives and resubstituting $x_i \to n = \bar\lambda\otimes\lambda$. This is obvious since

$$nA = 0,\ \ n^2 = 0,\ \ \lambda^\alpha\lambda_\alpha = 0,\ \ \lambda^\alpha n_{\alpha\dot\beta} = \bar\lambda^{\dot\beta}n_{\alpha\dot\beta} = 0$$

and there are effectively "too many" λ spinors present.

$$\Longrightarrow I_1 = 0 + \text{finite terms} \tag{4.80}$$

Now we turn to the calculation of the second term, I_2. Making use of the Dirac equation $\not{q}_2\psi(q_2) = 0$, it is possible to replace $\lambda(\not{q}_2 - \not{l})\not{l}\psi^g(q_2)$ by $-(q_2 - l)^2\lambda\psi^g(q_2)$[8]. This factor cancels one of the propagators in the denominator $\mathcal{N}_1$. Hence

$$\begin{aligned} I_2 =& ig^3\mathcal{C}_1 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{(ln)(l+q_1+q_3)^2 l^2 (l+q_2)^2} \times \\ &\times \mu(\not{l} + \not{q}_1 + \not{q}_3)\not{A}^B(q_3)(\not{l} + \not{q}_1)\not{n}\psi^f(q_1) \cdot \lambda\psi^g(q_2) \times \\ &\times e^{-i(l+q_1+q_3)a_1 n - i(q_2-l)a_2 n} \ . \end{aligned} \tag{4.81}$$

In order to remove the factor ln from the denominator we make use of the identity

$$\left(-i(a_1 - a_2) \int_0^1 \mathrm{d}s\, e^{-is(a_1-a_2)ln} \right) + \frac{1}{ln} = \frac{1}{ln} e^{-i(a_1-a_2)ln} \ . \tag{4.82}$$

Note that the second term on the left-hand side does not induce any ultra-violet divergences in Eq. (4.81); we can safely neglect it. We are left with:

$$\begin{aligned} I_2 =& (a_1 - a_2) g^3 \mathcal{C}_1 \int_0^1 \mathrm{d}s \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{(l+q_1+q_3)^2 l^2 (l+q_2)^2} \times \\ &\times \mu(\not{l} + \not{q}_1 + \not{q}_3)\not{A}^B(q_3)(\not{l} + \not{q}_1)\not{n}\psi^f(q_1) \cdot \lambda\psi^g(q_2) \times \\ &\times e^{-i(s\cdot l+q_1+q_3)a_1 n - i(q_2 - s\cdot l)a_2 n} \ . \end{aligned} \tag{4.83}$$

Going over to Feynman parameter integrals and shifting the momentum integration yields

$$\begin{aligned} I_2 =& (a_1 - a_2) g^3 \mathcal{C}_1 \int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}\alpha \int_0^{\bar{\alpha}} \mathrm{d}\beta \int \frac{\mathrm{d}^D l}{(2\pi)^D} \times \\ &\times \frac{\Gamma(3)\mu(\not{l} + (\bar{\alpha}-\beta)\not{q}_1 + \bar{\alpha}\not{q}_3)\not{A}^B(q_3)(\not{l} + (\bar{\alpha}-\beta)\not{q}_1 - \alpha q_3)\not{n}\psi^f(q_1)}{[l^2 - M^2]^3} \times \\ &\times \lambda\psi^g(q_2) \exp\left(-is(a_1-a_2)(ln) - i(1 - s\alpha - s\beta)a_1(q_1 n)\right. \\ &\qquad - i(1-s\alpha)a_1(q_3 n) - ia_2(q_2 n) \\ &\qquad \left. -is(\alpha+\beta)a_2(q_1 n) - is\alpha a_2(q_3 n)\right) \ , \end{aligned} \tag{4.84}$$

where the functional dependence of M^2 on Feynman parameters and momenta is again irrelevant for our purpose. It is necessary to take the integral over the momentum l. One only needs to consider the expansion

$$e^{-i(a_1-a_2)(ln)} = 1 - i(a_1 - a_2)(ln) + \mathcal{O}\left((ln)^2\right)$$

[8] In fact, there a term containing an additional A field also arises. However, it leads to a five particle operator; thus one can neglect it.

up to and including the term linear in ln. All further terms do not produce a logarithmic divergence or vanish exactly because they are proportional to n^2 or A^{++}. To keep the expressions manageable it is convenient to define

$$\Lambda(\epsilon) := \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\Gamma(2)}{[l^2 + M^2]} - \text{finite terms}\,. \tag{4.85}$$

Then one obtains up to ultra-violet finite terms:

$$\begin{aligned} I_2 = &- g^3 \Lambda(\epsilon) \mathcal{C}_1 \int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}\alpha \int_0^{\bar{\alpha}} \mathrm{d}\beta \;\times \\ &\times \Big\{ (a_1 - a_2) \cdot \mu \not{A}^B(a_{12}^{s\alpha}) \not{n} \psi^f\Big(a_{12}^{(s\alpha+s\beta)}\Big) \cdot \lambda \psi^g(a_2) \\ &\quad + (a_1 - a_2)^2 \cdot \Big[(1-\alpha-\beta) s (q_1 n) \mu \not{n} \not{A}^B(a_{12}^{s\alpha})\, \psi^f\Big(a_{12}^{(s\alpha+s\beta)}\Big) \\ &\qquad - s\alpha (q_3 n) \mu \not{n} \not{A}^B(a_{12}^{s\alpha})\, \psi^f\Big(a_{12}^{(s\alpha+s\beta)}\Big) \Big] \cdot \lambda \psi^g(a_2) \Big\}\,, \end{aligned} \tag{4.86}$$

where $a_{12}^x = (1-x)a_1 n + x a_2 n$. The expression (4.86) requires some rewriting to remove the factors $q_i n$. The following relations prove to be useful for this task:

- Recall that $\psi^f\Big(a_{12}^{(s\alpha+s\beta)}\Big) = \psi^f(q_1)\, e^{-i(s\alpha+s\beta)q_1 n}$ which implies

$$(q_1 n) \cdot \psi^f\Big(a_{12}^{(s\alpha+s\beta)}\Big) = \frac{1}{i(a_1 - a_2)s} \frac{\partial}{\partial\beta} \psi^f\Big(a_{12}^{(s\alpha+s\beta)}\Big)$$

- Analogously one obtains

$$(q_3 n) \cdot A^{B,\nu}(\ldots) = i\, n^\mu \left[\partial_\mu A^{B,\nu}\right](\ldots)$$

- Integration by parts yields the relation

$$\begin{aligned} &\int_0^{1-\alpha} \mathrm{d}\beta\; (1-\alpha-\beta) s \frac{\partial}{\partial\beta} \psi^f\left(a_{12}^{(s\alpha+s\beta)}\right) = \\ &\qquad = -s(1-\alpha)\psi^f\left(a_{12}^{s\alpha}\right) + \int_0^{1-\alpha} \mathrm{d}\beta\; s\psi^f\left(a_{12}^{(s\alpha+s\beta)}\right) \end{aligned}$$

Finally, we go over to spinor notation. I_2 then simplifies to

$$\begin{aligned} I_2 = &- g^3 \Lambda(\epsilon) \mathcal{C}_1 \Big[2 \int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}\alpha\; (a_2 - a_1)(1-\alpha) A_{-+}^B(a_{12}^{s\alpha}) \cdot \psi_+^f(a_{12}^{s\alpha}) \cdot \psi_+^g(a_2) \\ &+ \int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}\alpha \int_0^{\bar{\alpha}} \mathrm{d}\beta\; s\alpha (a_{12})^2 [D_{++} A_{-+}^B]\,(a_{12}^{s\alpha}) \cdot \psi_+^f(a_{12}^{(s\alpha+s\beta)}) \cdot \psi_+^g(a_2) \Big]\,. \end{aligned} \tag{4.87}$$

In the first line of (4.87) we perform the substitutions

$$v := s\alpha \;\;\Rightarrow\;\; \int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}\alpha \ldots \to \int_0^1 \mathrm{d}\alpha \int_0^\alpha \mathrm{d}v\, \frac{1}{\alpha} \ldots \to \int_0^1 \mathrm{d}v \int_v^1 \mathrm{d}\alpha\, \frac{1}{\alpha} \ldots\,,$$

whereas substituting

$$v := s\alpha \ \text{ and } \ u := s(\alpha+\beta)$$

in the second line leads to

$$\int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}\alpha \int_0^{1-\alpha} \mathrm{d}\beta \ \ldots \to \int_0^1 \mathrm{d}s \int_0^s \mathrm{d}v \int_v^s \mathrm{d}u \ \frac{1}{s^2} \ldots \to \int_0^1 \mathrm{d}v \int_v^1 \mathrm{d}u \int_u^1 \mathrm{d}s \ \frac{1}{s^2} \ldots \ .$$

The overall result for I_2 is

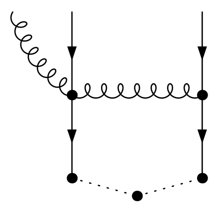
$$\begin{aligned} I_2 =& g^3 \Lambda(\epsilon) \mathcal{C}_1 \int \mathrm{d}v \ [\ln v + (1-v)](a_1 - a_2) A^B_{-+} \, (a^v_{12}) \, \psi^f_+ \, (a^v_{12}) \, \psi^g_+ \, (a_2) \\ &+ 1/2 g^3 \Lambda(\epsilon) \mathcal{C}_1 \int_0^1 \mathrm{d}v \int_v^1 \mathrm{d}u \ a^2_{12} \ \frac{v\bar{u}}{u} [D_{++} A_{-+}] \, (a^v_{12}) \, \psi^f_+ \, (a^u_{12}) \, \psi^g_+ \, (a_2) \ . \end{aligned} \tag{4.88}$$

The last remaining term from the first diagram, I_3, has the form

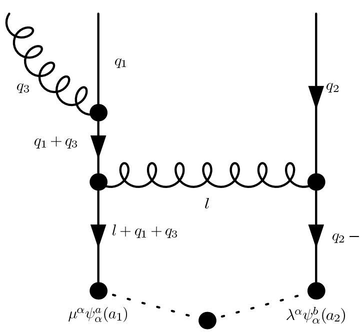
$$\begin{aligned} I_3 :=& - i g^3 \mathcal{C}_1 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \mathcal{N}_1 \frac{1}{ln} \times \\ & \times \mu(\not l + \not q_1 + \not q_3) \not A^B(q_3)(\not l + \not q_1) \not l \psi^f(q_1) \cdot \lambda(\not q_2 - \not l) \not n \psi^g(q_2) \times \\ & \times e^{-i(l+q_1+q_3)a_1 n - i(q_2 - l)a_2 n} \\ =& \, i g^3 \mathcal{C}_1 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{e^{-i(l+q_1+q_3)a_1 n - i(q_2-l)a_2 n}}{(nl)(l+q_1+q_3)^2 l^2 (l+q_1)^2} \times \\ & \times \mu(\not l + \not q_1 + \not q_3) \not A^B(q_3) \psi^f(q_1) \cdot \lambda(\not q_2 - \not l) \not n \psi^g(q_2) \ , \end{aligned} \tag{4.89}$$

where the equations of motion were used to obtain the last line.

It is advantageous not to simplify this expression any further. The structure of the numerator corresponds schematically to the (fictional) Feynman diagram depicted to the right. This structure will also appear during the calculation of the remaining two diagrams and we will see that the sum of these terms vanishes.



The Equation-of-Motion Diagram



Next, we consider the second diagram in Fig.4.1; a scaled up version is also shown to the left. Recall that we may only take terms into account, where the line between emitted and exchanged gluon is "contracted to a point", i.e. the denominator of the quark propagator $\frac{\not q_1 + \not q_3}{(q_1+q_3)^2}$ is canceled. Any term that keeps it to the very end can be neglected.

One obtains:

$$- i g^3 \mu \psi^a(a_1) \cdot \lambda \psi^b(a_2) \int \mathrm{d}^D y \int \mathrm{d}^D z \int \mathrm{d}^D w \ \bar{q}^{e'}(y) \not A^{e'e}(y) q^e(y) \cdot$$

$$\cdot \bar{q}^{f'}(z)\not{A}^{f'f}(z)q^g(z) \cdot \bar{q}^{f'}(w)\not{A}^{f'f}(w)q^g(w) \overset{Wick}{\rightarrow}$$

$$\rightarrow -ig^3 \int \frac{\mathrm{d}^D l}{(2\pi)^D} e^{-i(l+q_1+q_3)a_1 n - i(q_2-l)a_2 n} \times$$

$$\times \frac{\mu(\not{l}+\not{q}_1+\not{q}_3)\gamma_\mu(\not{q}_1+\not{q}_3)\not{A}^B(q_3)\psi^f(q_2)\cdot\lambda(\not{q}_2-\not{l})\gamma_\nu\psi^g(q_3)}{l^2(q_1+q_3)^2(q_2-l)^2(l+q_1+q_3)^2} \cdot$$

$$\cdot \left[-g^{\mu\nu} + \frac{n^\mu l^\nu + n^\nu l^\mu}{ln}\right] \cdot T^{A,af'}T^{A,bg}T^{B,f'f} \quad (4.90)$$

Let us define, in analogy to our treatment of the first diagram, the three functions J_1, J_2 and J_3, each of which corresponds to one of the terms arising due to the gluon propagator (4.69).

$$J_1 := +ig^3 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \mathcal{N}_2\, \mathcal{C}_2\, e^{-i(l+q_1+q_3)a_1 n - i(q_2-l)a_2 n} \times$$
$$\times\, \mu(\not{l}+\not{q}_1+\not{q}_3)\gamma_\mu(\not{q}_1+\not{q}_3)\not{A}^B(q_3)\psi^f(q_2)\cdot\lambda(\not{q}_2-\not{l})\gamma^\mu\psi^g(q_3) \quad (4.91)$$

$$J_2 := -ig^3 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{ln} \mathcal{N}_2\, \mathcal{C}_2\, e^{-i(l+q_1+q_3)a_1 n - i(q_2-l)a_2 n} \times$$
$$\times\, \mu(\not{l}+\not{q}_1+\not{q}_3)\not{n}(\not{q}_1+\not{q}_3)\not{A}^B(q_3)\psi^f(q_2)\cdot\lambda(\not{q}_2-\not{l})\not{l}\psi^g(q_3) \quad (4.92)$$

$$J_3 := -ig^3 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{ln} \mathcal{N}_2\, \mathcal{C}_2\, e^{-i(l+q_1+q_3)a_1 n - i(q_2-l)a_2 n} \times$$
$$\times\, \mu(\not{l}+\not{q}_1+\not{q}_3)\not{l}(\not{q}_1+\not{q}_3)\not{A}^B(q_3)\psi^f(q_2)\cdot\lambda(\not{q}_2-\not{l})\not{n}\psi^g(q_3)\,, \quad (4.93)$$

where

$$\mathcal{N}_2 := \frac{1}{l^2(q_1+q_3)^2(q_2-l)^2(l+q_1+q_3)^2} \quad \text{and} \quad \mathcal{C}_2 := T^{A,af'}T^{A,bg}T^{B,f'f}\ .$$

We can neglect J_1 right from the start, as the only way to get rid of the factor $(q_1+q_3)^2$ in the denominator is combining the two $(\not{q}_1+\not{q}_2)$ in the numerator. Then we are left with

$$J_1 \sim \frac{\mu\gamma^\mu\not{A}(q_3)\psi^f(q_1)\cdot\lambda(\not{q}_2-\not{l})\gamma_\mu\psi^g(q_2)}{l^2(q_2-l)^2(l+q_1+q_3)^2} e^{-i(a_1-a_2)(ln)} \quad (4.94)$$

which can only generate logarithmically divergent terms proportional to $\lambda_\alpha n^{\alpha\dot{\beta}}$; this is equal to zero. Therefore, J_1 does not contribute to the renormalization kernels:

$$\Longrightarrow J_1 = 0\ +\ \text{finite terms.} \quad (4.95)$$

The same argument cannot be applied to the remaining expressions J_2 and J_3. Let us consider J_2 first.

$$J_2 = -ig^3 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{ln} \mathcal{N}_2\, \mathcal{C}_2\, e^{-i(l+q_1+q_3)a_1 n - i(q_2-l)a_2 n} \times$$

$$\times\, \mu(\not{l} + \not{q}_1 + \not{q}_3)\not{n}(\not{q}_1 + \not{q}_3)\not{A}^B(q_3)\psi^f(q_2)\cdot$$
$$\cdot\, \lambda(\not{q}_2 - \not{l})(\not{l} - \not{q}_2 + \not{q}_2)\psi^g(q_3) =$$
$$\overset{\not{q}_2\psi(q_2)=0}{=}\; ig^3 \int \frac{\mathrm{d}^D l}{(2\pi)^D}\, \mathcal{C}_2\, e^{-i(l+q_1+q_3)a_1 n - i(q_2-l)a_2 n} \times$$
$$\times\, \frac{\mu(\not{l} + \not{q}_1 + \not{q}_3)\not{n}(\not{q}_1 + \not{q}_3)\not{A}^B(q_3)\psi^f(q_2)\cdot\lambda\psi^g(q_3)}{(q_1+q_3)^2 l^2 (l+q_1+q_3)^2}$$
$$\overset{(4.82)}{=}\; ig^3 \int \frac{\mathrm{d}^D l}{(2\pi)^D}\, \mathcal{C}_2\, e^{-i(q_1+q_3)a_1 n - i a_2 q_2 n}\left[\frac{1}{i}(a_1-a_2)\int_0^1 \mathrm{d}s\, e^{-is(a_1-a_2)ln} + \frac{1}{ln}\right]$$
$$\times\, \frac{\mu(\not{l} + \not{q}_1 + \not{q}_3)\not{n}(\not{q}_1 + \not{q}_3)\not{A}^B(q_3)\psi^f(q_2)\cdot\lambda\psi^g(q_3)}{(q_1+q_3)^2 l^2 (l+q_1+q_3)^2} \tag{4.96}$$

At first glance, the red colored expressions are able to create a logarithmic divergence. For this to happen, the $\not{l}$ in the numerator has to be kept, i.e. after the Feynman trick and shift of the momentum integral one has to drop all quadratic terms in the momenta q_i in the numerator. But this means that the factor $(q_1+q_3)^2$ must survive in the denominator to the very end and the whole term can be disregarded anyway. Therefore, we can safely drop $\frac{1}{ln}$ in Eq. (4.96). Introducing Feynman parameter integrals, the integral over l can be taken and up to finite terms J_2 takes the form

$$J_2 = -g^3\Lambda(\epsilon)\mathcal{C}_2(a_1-a_2)\int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}\alpha\,\bar{\alpha}\mu\not{n}\not{A}^B(a_{12}^{s\alpha})\psi^f(a_{12}^{s\alpha})\cdot\lambda\psi^g(a_2) =$$
$$\overset{v:=s\alpha}{=}\; g^3\Lambda(\epsilon)\mathcal{C}_2(a_1-a_2)\int_0^1 \mathrm{d}\alpha \int_0^\alpha \mathrm{d}v\,\frac{\bar{\alpha}}{\alpha}\mu\not{A}^B(a_{12}^v)\not{n}\psi^f(a_{12}^v)\cdot\lambda\psi^g(a_2)$$
$$\overset{\int\, v\leftrightarrow\alpha}{=}\; g^3\Lambda(\epsilon)\mathcal{C}_2(a_1-a_2)\int_0^1 \mathrm{d}v \int_v^1 \mathrm{d}\alpha\,\frac{\bar{\alpha}}{\alpha}\mu\not{A}^B(a_{12}^v)\not{n}\psi^f(a_{12}^v)\cdot\lambda\psi^g(a_2) \tag{4.97}$$

Now the integral over α is trivial and after changing to spinor notation we get

$$J_2 = -g^3\Lambda(\epsilon)\mathcal{C}_2(a_1-a_2)\int_0^1 \mathrm{d}v[\ln v - \bar{v}]A^B_{-+}(a_{12}^v)\psi_+^f(a_{12}^v)\cdot\psi_+^g(a_2)\,. \tag{4.98}$$

Note that the result for J_2 is not gauge invariant. We will see that gauge invariance is restored only in the sum over all three diagrams .

Next, we come to the third term J_3.

$$J_3 = -ig^3\mathcal{C}_2 \int \frac{\mathrm{d}^D l}{(2\pi)^D}\frac{1}{ln}\mathcal{N}_2\,\mathcal{C}_2\, e^{-i(l+q_1+q_3)a_1 n - i(q_2-l)a_2 n}\times$$
$$\times\, \mu(\not{l} + \not{q}_1 + \not{q}_3)(\not{l} + \not{q}_1 + \not{q}_3 - \not{q}_1 - \not{q}_3)\cdot$$
$$\cdot\,(\not{q}_1 + \not{q}_3)\not{A}^B(q_3)\psi^f(q_2)\cdot\lambda(\not{q}_2 - \not{l})\not{n}\psi^g(q_3) =$$
$$= -ig^3\mathcal{C}_2 \int \frac{\mathrm{d}^D l}{(2\pi)^D}\,\mathcal{C}_2\, e^{-i(l+q_1+q_3)a_1 n - i(q_2-l)a_2 n}\times$$
$$\times\left[\frac{\mu(\not{q}_1 + \not{q}_3)\not{A}^B(q_3)\psi^f(q_2)\cdot\lambda(\not{q}_2 - \not{l})\not{n}\psi^g(q_2)}{(ln)(q_1+q_3)^2\, l^2\,(q_2-l)^2}\right.$$

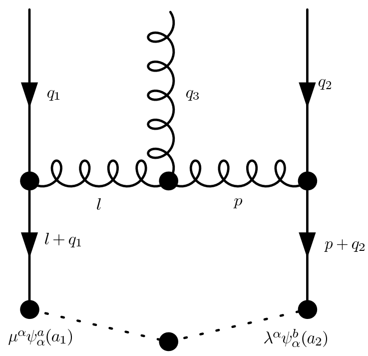
$$+\frac{\mu(\not{q}_1+\not{q}_3)\not{A}^B(q_3)\psi^f(q_2)\cdot\lambda(\not{q}_2-\not{l})\not{n}\psi^g(q_2)}{(ln)(l+q_1+q_3)^2\; l^2\;(q_2-l)^2}\Bigg]\,. \tag{4.99}$$

The first term in the square brackets can be neglected as it will keep the quark line with momentum q_1+q_3. We obtain

$$J_3 = ig^3\mathcal{C}_2\int\frac{\mathrm{d}^D l}{(2\pi)^D}\,\mathcal{C}_2\; e^{-i(l+q_1+q_3)a_1 n - i(q_2-l)a_2 n}\times$$
$$\times\left[\frac{\mu(\not{q}_1+\not{q}_3)\not{A}^B(q_3)\psi^f(q_2)\cdot\lambda(\not{q}_2-\not{l})\not{n}\psi^g(q_2)}{(ln)(l+q_1+q_3)^2\; l^2\;(q_2-l)^2}\right]\,. \tag{4.100}$$

Note that J_3 and I_3 differ only by an overall sign and by color structure. In fact, one could combine both terms using the commutation relations for the generators of $SU(3)$. In anticipation that the last Feynman diagram in Fig.4.1 will also produce a similar term, we refrain from pursuing this issue any further.

The 3-Gluon-Vertex Diagram



Here $p = q_3 - l$ and we have to keep in mind that the momentum flow of the internal gluon lines has, compared to the three-gluon vertex given in App. A, the wrong direction. This leads to some sign changes. For the moment, it is advantageous to treat l and p as independent. We start with the expression, where all Wick contractions have already been carried out.

$$g^3\int\frac{\mathrm{d}^D l}{(2\pi)^D}\, f^{ABC}T^{A,af}T^{B,bg}e^{-(l+q_1)a_1 n - i(p+q_2)a_2 n}\times$$
$$\times\frac{\mu(\not{l}+\not{q}_1)\gamma_\mu\psi^f(q_1)\cdot\lambda(\not{p}+\not{q}_2)\gamma_\nu\psi^g(q_2)\cdot A^C_\gamma(q_3)}{l^2\;p^2\;(l+q_2)^2(p+q_2)^2}\times$$
$$\times\left[g^{\alpha\beta}(p-l)^\gamma - g^{\beta\gamma}(p+q_3)^\alpha + g^{\gamma\alpha}(l+q_3)^\beta\right]\times$$
$$\times\left[-g^\mu{}_\alpha+\frac{l^\mu n_\alpha + n^\mu l_\alpha}{ln}\right]\cdot\left[-g^\nu{}_\beta+\frac{p^\nu n_\beta + n^\nu p_\beta}{pn}\right] \tag{4.101}$$

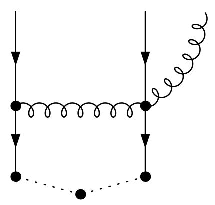
As the diagram contains two gluon propagators, in addition to the three-gluon vertex, quite a number of terms will appear. It is useful to note that any term proportional to $\mu\psi(q)$ must vanish, because it gives rise to a chiral "minus" field ψ_- which cannot appear in a baryon operator of twist 4 with a gluon field. Keeping only the logarithmically divergent terms and making use of the gauge condition $n\cdot A = 0$ and the fact that n is light-like, $n^2 = 0$ as well as $\lambda\not{n} = 0$, it is a pure matter of patience to arrive at the expression

$$g^3 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \, f^{ABD} T^{A,af} T^{B,bg} \, e^{-(l+q_1)a_1 n - i(p+q_2)a_2 n} \times$$
$$\times \Bigg[\frac{\mu(\not{l} + \not{q}_1)\not{n}\psi^f(q_1) \cdot \lambda(\not{q}_3 + \not{q}_2 - \not{l})\not{A}^C(q_3)\psi^g(q_2)}{(nl)\, l^2 \, (l+q_1)^2 (q_3+q_2+l)^2}$$
$$- \frac{\mu(\not{l} + \not{q}_1)\not{A}^C(q_3)\psi^f(q_1) \cdot \lambda(\not{q}_3 + \not{q}_2 - \not{l})\psi^g(q_2)}{(pn)(q_2-l)^2 \, (l+q_1)^2 \, (q_3+q_2+l)^2}$$
$$+ 2(q_3 n)\frac{\mu(\not{l} + \not{q}_1)\not{A}^C(q_3)\psi^f(q_1) \cdot \lambda\psi^g(q_2)}{(pn)(q_2-l)^2 \, (l+q_1)^2 \, l^2}$$
$$- \frac{2\mu(\not{l} + \not{q}_1)\not{n}\psi^f(q_1) \cdot \lambda\psi^g(q_2) \cdot (l \cdot A^C(q_3))}{(pn)(q_2-l)^2 \, l^2; (l+q_1)^2} \Bigg] . \quad (4.102)$$

Let K_1, K_2, K_3, K_4 denote the 1st, 2nd, 3rd and 4th term in the sum, respectively. Further we introduce the abbreviation

$$\mathcal{C}_3 = i f^{ABC} T^{A,af} T^{B,bg} \, .$$

The structure of the numerator of K_1 corresponds to the fictional Feynman diagram shown on the right. The gluon is emitted from the right quark line, this means that K_1 belongs to the group of diagrams coined “mirror diagrams”. The sum of all these diagrams vanishes and we do not consider them here.



Replacing p by $q_3 - l$, the second term K_2 can be written as

$$K_2 = ig^3 \mathcal{C}_3 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \, e^{-(l+q_1)a_1 n - i(q_3+q_2-l)a_2 n} \times$$
$$\times \frac{\mu(\not{l} + \not{q}_1)\not{A}^C(q_3)\psi^f(q_1) \cdot \lambda(\not{q}_3 + \not{q}_2 - \not{l})\psi^g(q_2)}{(q_3-l)n \, (q_3-l)^2 \, (l+q_1)^2 \, (q_3+q_2-l)^2} =$$
$$\overset{l \to l+q_3}{=} -ig^3 \mathcal{C}_3 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \, e^{-(l+q_1+q_3)a_1 n - i(q_2-l)a_2 n} \times$$
$$\times \frac{\mu(\not{l} + \not{q}_1 + \not{q}_3)\not{A}^C(q_3)\psi^f(q_1) \cdot \lambda(\not{q}_2 - \not{l})\psi^g(q_2)}{(ln)\, l^2 \, (l+q_1+q_3)^2 \, (q_2-l)^2} \quad (4.103)$$

Now observe that after the shift in the momentum K_2 differs from I_3, see Eq. (4.89), and J_3, see Eq. (4.100), only in color structure. Hence, one should consider the sum of the three terms:

$$I_3 + J_3 + K_2 = ig^3 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{e^{-i(l+q_1+q_3)a_1 n - i(q_2-l)a_2 n}}{(nl)(l+q_1+q_3)^2 l^2 (l+q_1)^2} \times$$
$$\times \, \mu(\not{l} + \not{q}_1 + \not{q}_3)\not{A}^B(q_3)\psi^f(q_1) \cdot \lambda(\not{q}_2 - \not{l})\not{n}\psi^g(q_2) \times$$
$$\times \, (\mathcal{C}_1 - \mathcal{C}_2 + \mathcal{C}_3) \quad (4.104)$$

Inserting the explicit expressions for the color factors we see that

$$\begin{aligned} I_3 + J_3 + K_2 \sim& (\mathcal{C}_1 - \mathcal{C}_2 + \mathcal{C}_3) = \\ &= \left[\left((T^A T^B)^{af} - (T^B T^A)^{af} \right) T^{A,bg} - i f^{ACB} T^{A,af} T^{C,bg} \right] = \\ &= i f^{ABC} T^{C,af} T^{A,bg} - i f^{ACB} T^{A,af} T^{C,bg} = 0 \,. \end{aligned} \tag{4.105}$$

With the help of Eq. (4.82) one can remove the factors $\frac{1}{pn}$ in K_3 and K_4. Introducing Feyman parameter integrals K_3 can be cast into the form

$$\begin{aligned} K_3 =& - i g^3 \Lambda(\epsilon) \mathcal{C}_3 (a_1 - a_2)^2 \int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}\alpha \int_0^{1-\alpha} \mathrm{d}\beta \; s \cdot \\ & \qquad \cdot (q_3 n) \mu \not{n} \not{A}^C\!\left(a_{12}^{(s\alpha + s\beta)}\right) \psi^f \, (a_{12}^{s\alpha}) \cdot \lambda \psi^g (a_2) = \\ =& g^3 \Lambda(\epsilon) \mathcal{C}_3 (a_1 - a_2) \int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}\alpha \int_0^{1-\alpha} \mathrm{d}\beta \; s \cdot \\ & \qquad \cdot \frac{\partial}{\partial \beta} \mu \not{A}^C\!\left(a_{12}^{(s\alpha+s\beta)}\right) \not{n} \psi^f \, (a_{12}^{s\alpha}) \cdot \lambda \psi^g (a_2) \,. \end{aligned} \tag{4.106}$$

Substituting $u := s\alpha$, $v := s(\alpha+\beta)$ and going over to spinor notation we obtain:

$$K_3 = g^3 \Lambda(\epsilon) \mathcal{C}_3 (a_1 - a_2) \int_0^1 \mathrm{d}u \int_u^1 \mathrm{d}s \frac{1}{s} \int_u^s \mathrm{d}v \; \frac{\partial}{\partial v} \mu A^C_{-+}(a_{12}^v) \; \psi_+^f \, (a_{12}^u) \; \psi_+^g (a_2) \,. \tag{4.107}$$

It is possible to simplify this expression further, as the integral over v can be taken directly; this is, however, not necessary, as one will see below.

The last remaining term, K_4, can be treated completely analogously to K_3. We just quote the result for the divergent part:

$$K_4 = - g^3 \Lambda(\epsilon) \mathcal{C}_3 \int_0^1 \mathrm{d}u \int_u^1 \mathrm{d}v \; \frac{1-v}{v} A^C_{-+}(a_{12}^v) \; \psi_+^f \, (a_{12}^u) \; \psi_+^g (a_2) \,. \tag{4.108}$$

The calculation is not yet finished. The remaining terms, i.e. K_3, K_4, J_2 as well as I_2, do not have a functional form permitted by conformal symmetry. If it is not possible to cast the sum of these term in a form corresponding to the kernels (C.2)-(C.14), the calculation must have been faulty. This is one of the most powerful checks we have at our disposal. It is useful to collect all four terms in one expression corresponding to the sum of all diagrams in Fig.4.1.

$$\begin{aligned} & I_2 + J_2 + K_3 + K_4 = \\ =& \frac{1}{2} g^3 \Lambda(\epsilon) \mathcal{C}_1 \int_0^1 \mathrm{d}v \int_v^1 \mathrm{d}u (a_1 - a_2)^2 \frac{v(1-u)}{u} [\bar{D}_{++} A_{-+}] \, (a_{12}^v) \, \psi_+^f \, (a_{12}^u) \, \psi_+^g \, (a_2) \\ & + g^3 \Lambda(\epsilon) \mathcal{C}_1 (a_1 - a_2) \int_0^1 \; \mathrm{d}v \; [\ln v + (1-v)] A^B_{-+} \, (a_{12}^v) \, \psi_+^f \, (a_{12}^v) \, \psi_+^g \, (a_2) \\ & - g^3 \Lambda(\epsilon) \mathcal{C}_2 (a_1 - a_2) \int_0^1 \mathrm{d}v [\ln v + (1-v)] A^B_{-+} (a_{12}^v) \psi_+^f (a_{12}^v) \cdot \psi_+^g (a_2) \end{aligned}$$

$$
\begin{aligned}
&- g^3\Lambda(\epsilon)\mathcal{C}_3 \int_0^1 \mathrm{d}u \int_u^1 \mathrm{d}v \, \frac{1-v}{v} A^C_{-+}(a^v_{12}) \; \psi^f_+(a^u_{12}) \; \psi^g_+(a_2) \\
&+ g^3\Lambda(\epsilon)\mathcal{C}_3(a_1-a_2) \int_0^1 \mathrm{d}u \int_u^1 \mathrm{d}s \frac{1}{s} \int_u^s \mathrm{d}v \, \frac{\partial}{\partial v} A^C_{-+}(a^v_{12}) \; \psi^f_+(a^u_{12}) \; \psi^g_+(a_2)
\end{aligned} \tag{4.109}
$$

The second, third and fourth line can be combined using the relation for the color factors $\mathcal{C}_1 - \mathcal{C}_2 + \mathcal{C}_3 = 0$, see Eq. (4.105), the identity

$$
-\ln v - (1-v) = \int_v^1 \mathrm{d}u \frac{1-u}{u}
$$

and

$$
(A_{-+}(a^v_{12}) - A_{-+}(a^u_{12})) = \int_u^v \mathrm{d}\alpha \frac{\partial}{\partial\alpha} A_{-+}(a^\alpha_{12}) \, .
$$

We arrive at

$$
\begin{aligned}
&\sum \text{ all three diagrams} = \\
&= \frac{1}{2} g^3\Lambda(\epsilon)\mathcal{C}_1 \int_0^1 \mathrm{d}v \int_v^1 \mathrm{d}u (a_1-a_2)^2 \frac{v(1-u)}{u} [\bar{D}_{++}A_{-+}] \, (a^v_{12}) \, \psi^f_+ \, (a^u_{12}) \, \psi^g_+ \, (a_2) \\
&\quad + g^3\Lambda(\epsilon)\mathcal{C}_3(a_1-a_2) \int_0^1 \mathrm{d}u \int_u^1 \mathrm{d}s \, \frac{1}{s} \int_u^s \mathrm{d}v \, \frac{\partial}{\partial v} A^C_{-+}(a^v_{12}) \; \psi^f_+(a^u_{12}) \; \psi^g_+(a_2) \\
&\quad - g^3\Lambda(\epsilon)\mathcal{C}_3(a_1-a_2) \int_0^1 \mathrm{d}u \int_u^1 \mathrm{d}s \frac{1-s}{s} \int_u^s \mathrm{d}v \, \frac{\partial}{\partial v} A^C_{-+}(a^v_{12}) \; \psi^f_+(a^u_{12}) \; \psi^g_+(a_2) \\
&= 1/2 g^3\Lambda(\epsilon)\mathcal{C}_1 \int_0^1 \mathrm{d}v \int_v^1 \mathrm{d}u (a_1-a_2)^2 \frac{v(1-u)}{u} [\bar{D}_{++}A_{-+}] \, (a^v_{12}) \, \psi^f_+ \, (a^u_{12}) \, \psi^g_+ \, (a_2) \\
&\quad + g^3\Lambda(\epsilon)\mathcal{C}_3(a_1-a_2) \int_0^1 \mathrm{d}u \int_u^1 \mathrm{d}s \int_u^s \mathrm{d}v \, \frac{\partial}{\partial v} A^C_{-+}(a^v_{12}) \; \psi^f_+(a^u_{12}) \; \psi^g_+(a_2)
\end{aligned} \tag{4.110}
$$

After replacing $\frac{\partial}{\partial v} A^C_{-+}(a^v_{12})$ by $-1/2(a_1-a_2)[\bar{D}_{++}A_{-+}]\,(a^v_{12})$, one can use that in light-cone gauge the relation between field strength tensor f_{++} and vector potential A_{-+} is given by

$$
\bar{D}_{++}A_{-+} = -2\bar{f}_{++} \, .
$$

Therefore

$$
\begin{aligned}
&\sum \text{ all three diagrams} = \\
&= -\frac{\alpha_s}{4\pi\epsilon}\mathcal{C}_1(a_1-a_2)^2 \int_0^1 \mathrm{d}v \int_v^1 \mathrm{d}u \frac{v(1-u)}{u} \cdot ig\bar{f}^C_{++} \, (a^v_{12}) \, \psi^f_+ \, (a^u_{12}) \, \psi^g_+ \, (a_2) \\
&\quad + \frac{\alpha_s}{4\pi\epsilon}\mathcal{C}_3(a_1-a_2)^2 \int_0^1 \mathrm{d}u \int_u^1 \mathrm{d}v \, (1-v) \cdot ig\bar{f}^C_{++}(a^v_{12}) \; \psi^f_+(a^u_{12}) \; \psi^g_+(a_2) \; = \ldots \, .
\end{aligned} \tag{4.111}
$$

The $\frac{1}{\epsilon}$-pole was extracted from $\Lambda(\epsilon)$ using

$$\Lambda(\epsilon) = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\Gamma(2)}{[l^2 + M^2]} - \text{finite terms} = \frac{i}{(4\pi)^2} \frac{1}{\epsilon} \,. \tag{4.112}$$

Now recall that we swept the overall factor $\epsilon^{abc}\psi_+(a_3)$ under the rug in the very beginning. By adding this again, we see first of all that the sum over the three diagrams is truly gauge invariant and secondly that the color factors can be determined using the standard Fierz identity for the Gell-Mann matrices. $\mathcal{C}_1$ and $\mathcal{C}_3$ take the form:

$$\mathcal{C}_1 = T^{B,ae} T^{A,ef} T^{A,bg} = \frac{1}{2}\delta^{fb} T^{B,ag} - \frac{1}{2N_c}\delta^{bg} T^{B,af} \tag{4.113}$$

$$\mathcal{C}_3 = i f^{ABC} T^{A,af} T^{A,bg} = \frac{1}{2}\delta^{ag} T^{C,bf} - \frac{1}{2}\delta^{bf} T^{C,ag} \,. \tag{4.114}$$

This yields

$$\begin{aligned} \ldots = {} & \frac{\alpha_s}{8\pi\epsilon} a_{12}^2 \int_0^1 \mathrm{d}v \int_v^1 \mathrm{d}u \frac{v\bar{u}}{u} \left[G_2(a_{12}^u, a_2, a_3, a_{21}^v) + \frac{1}{N_c} G_1(a_{12}^u, a_2, a_3, a_{21}^v) \right] \\ & + \frac{\alpha_s}{8\pi\epsilon} a_{12}^2 \int_0^1 \mathrm{d}u \int_u^1 \mathrm{d}v \; \bar{v} \left[G_2(a_{12}^v, a_2, a_3, a_{21}^u) - G_1(a_{12}^v, a_2, a_3, a_{21}^u) \right] \,, \end{aligned} \tag{4.115}$$

where we inserted the chiral basis from Eq. (4.12). Since the residue given in (4.115) is equal to $-\frac{\alpha_s}{4\pi}\mathbb{H}_{gq}^{12}$, we finally arrive at

$$\begin{aligned} \mathbb{H}_{gq}^{12} = {} & -\frac{1}{2} a_{12}^2 \int_0^1 \mathrm{d}v \int_0^{\bar{v}} \mathrm{d}u \; u \left[G_2(a_{12}^v, a_2, a_3, a_{21}^u) - G_1(a_{12}^v, a_2, a_3, a_{21}^u) \right] \\ & - \frac{1}{2} a_{12}^2 \int_0^1 \mathrm{d}v \int_{\bar{v}}^1 \mathrm{d}u \; \frac{\bar{u}\bar{v}}{v} \left[G_2(a_{12}^v, a_2, a_3, a_{21}^u) + \frac{G_1(a_{12}^v, a_2, a_3, a_{21}^u)}{N_c} \right] , \end{aligned} \tag{4.116}$$

which is just a linear combination of the kernels $\mathcal{V}_{12(4)}^{(1)}$ and $\mathcal{V}_{12(4)}^{(2)}$, see App. C. Our result has a form that is consistent with conformal symmetry: a non-trivial check for the calculation.

For completeness, we also give the result for $\mathbb{H}_{gq}^{13}$:

$$\begin{aligned} \mathbb{H}_{gq}^{13} = {} & -\frac{1}{2} a_{13}^2 \int_0^1 \mathrm{d}v \int_0^{\bar{v}} \mathrm{d}u \; u \left[G_3(a_{13}^v, a_2, a_3, a_{31}^u) - G_1(a_{13}^v, a_2, a_3, a_{31}^u) \right] \\ & - \frac{1}{2} a_{13}^2 \int_0^1 \mathrm{d}v \int_{\bar{v}}^1 \mathrm{d}u \; \frac{\bar{u}\bar{v}}{v} \left[G_3(a_{13}^v, a_2, a_3, a_{31}^u) + \frac{G_1(a_{13}^v, a_2, a_3, a_{31}^u)}{N_c} \right] . \end{aligned} \tag{4.117}$$

This follows readily by replacing $2 \leftrightarrow 3$ in all subscripts in Eq. (4.116). By combining Eq. (4.116) and Eq. (4.117) one can check that Eq. (4.54) is indeed reproduced.

4.4 From $SL(2,\mathbb{R})$ to the Full Symmetry

Until now the operators considered in this chapter were constructed from fields living on a single light-ray and the appropriate symmetry group is the $SL(2,\mathbb{R})$ subgroup. However, the one-loop renormalization group equations are, as discussed in Sect. 4.3.1, invariant under the full conformal group. This means, among other things, that the renormalization kernels which describe the one-loop renormalization factors "know" about the full group.

To make this more clear, let us introduce an (unknown) quantum operator $\boldsymbol{\Delta}$ which acts on a set of local operators $\mathbb{O}_i$ the following way:

$$[\boldsymbol{\Delta}, \mathbb{O}_k^B] = \mathcal{Z}_{kl}^{-1}\mathbb{O}_l^B \,, \tag{4.118}$$

where the $\mathbb{O}^B$ are bare operators and $\mathcal{Z}_{kl}$ is the matrix of renormalization factors. Since the RGEs are conformally invariant, Eq. (4.118) must still be true if an arbitrary generator of the conformal group, $\mathbb{G}$, acts on left and right hand side of the equation simultaneously. Therefore, $\mathbb{G}$ and $\boldsymbol{\Delta}$ commute

$$[\boldsymbol{\Delta}, \mathbb{G}] = 0 \,. \tag{4.119}$$

For Lorentz transformations this is in fact very intuitive, $\mathcal{Z}$ factors and anomalous dimensions do not care, if the whole system of fields is rotated in Minkowski space; e.g. for light-ray operators one light-like direction n^μ is transformed into a different one, $n'^\mu = \Lambda^\mu{}_\nu n^\mu$. As no direction was preferred to begin with, this cannot have physical consequences.

The constraints from requiring $SL(2,\mathbb{R})$ invariance of the renormalization kernels led to a set of elementary kernels, see App. C. It seems reasonable that the relation (4.119) might provide us with some connection among the different renormalization kernels. As we will see shortly, even a non-trivial relationship between kernels of *different* twist can be found.

The Generator $\mathbf{M}_2{}^1$

Collinear twist depends only on the dimension of an operator and the light-cone projection of its spin. Therefore, a global Lorentz rotation $\mathbf{M}_{\alpha\beta}$ or $\overline{\mathbf{M}}_{\dot\alpha\dot\beta}$, cf. Eq. (3.35), can modify the twist of an operator by changing its spin projection.

Let us consider the action of the generator $\mathbf{M}_2{}^1$ on the light-ray field $\psi_+(x) = \lambda^\alpha\psi_\alpha(z\cdot n)$ with $n^2 = 0$. We obtain:

$$\begin{aligned} i[\mathbf{M}_2{}^1, \psi_+(x)] &= \frac{1}{4}\left(x_{2\dot\gamma}\partial^{1\dot\gamma} + x^1{}_{\dot\gamma}\partial_2{}^{\dot\gamma} - 2\lambda_2\frac{\partial}{\partial\lambda_1} - 2\lambda^1\frac{\partial}{\partial\lambda^2}\right)\lambda^\alpha\psi_\alpha(x) = \\ &= \frac{1}{2}x_{2\dot 2}\partial^{1\dot 2}\psi_+(x) - \psi_-(x). \end{aligned} \tag{4.120}$$

One cannot use equations of motion to remove the derivative $\partial^{1\dot 2}$ and replace $\partial^{1\dot 2}\psi_+(x)$ by $-\partial^{2\dot 2}\psi_-(x)$, as this would require the derivatives to be covariant.

However, to ensure gauge invariance of the composite operators, the (anti-)chiral fields that are part of the one-particle basis (3.63) are always adorned with an additional path-ordered exponent $[z_0, z]$. The generators of the conformal group will act on the quantum fields $A_{2\dot{2}}$ sitting in the gauge links, see (3.62).

In analogy to the computation of the action of the generator of translations $\mathbf{P}_\mu$ on the gauge link presented in App. B.2, one finds that

$$i[\mathbf{M}_2{}^1, [z_0 \cdot n, z_1 \cdot n]] = ig\frac{1}{2}z_{10}\int_0^1 d\tau \; z_{01}^\tau [z_1, z_{01}^\tau]\bar{f}_{++}(z_{01}^\tau)[z_{01}^\tau, z_1] - \frac{ig}{2}z_1 n_{2\dot{2}} A^{1\dot{2}}(z_1 \cdot n) + \frac{ig}{2}z_0 n_{2\dot{2}} A^{1\dot{2}}(z_0 \cdot n). \quad (4.121)$$

The first term in the second line combines nicely with the standard derivative in Eq. (4.120) to form a covariant one; this allows us to use equations of motion. The second term in the same line seems like an artifact, since there is no quark field at coordinate z_0. In fact, if we consider gauge invariant three-quark operators, three such terms arise. The sum vanishes thanks to the relation (4.13).

Combining (4.120) and (4.121) one can determine the action of $\mathbf{M}_2{}^1$ on the gauge invariant chiral operator of twist 3, $Q^{tw-3} = \psi_+\psi_+\psi_+$. For simplicity we can put $z_0 = 0$ and obtain

$$\begin{aligned} i\Big[\mathbf{M}_2{}^1, \epsilon^{abc}\,[[0,z_1]\psi_+(z_1)]^a\,[[0,z_2]\psi_+(z_2)]^b\,[[0,z_3]\psi_+(z_3)]^c\Big] = \\ \left(-z_1\frac{\partial}{\partial z_1} - 1\right) Q_1(z_1,z_2,z_3) + \frac{1}{2}\int_0^1 d\tau\; \tau z_1^2 G_1(z_1,z_2,z_3,\tau z_1) \\ + \left(-z_2\frac{\partial}{\partial z_2} - 1\right) Q_2(z_1,z_2,z_3) + \frac{1}{2}\int_0^1 d\tau\; \tau z_2^2 G_2(z_1,z_2,z_3,\tau z_2) \\ + \left(-z_3\frac{\partial}{\partial z_3} - 1\right) Q_3(z_1,z_2,z_3) - \frac{1}{2}\int_0^1 d\tau\; \tau z_3^2 [G_1+G_2](z_1,z_2,z_3,\tau z_3), \end{aligned} \quad (4.122)$$

where Q_i and G_i are part of the chiral basis (4.11) and (4.12).

We see that the operator $\mathbf{M}_2{}^1$ adds one unit of twist, thus transforming the quasipartonic twist-3 operator into a linear combination of quasipartonic and non-quasipartonic operators of twist 4. Therefore, Eq. (4.122) together with Eq. (4.119) implies some connection between the 2-to-2 kernels of twist 3, which are well understood [61], and the new 2-to-3 kernels. The final result reads:

$$\begin{aligned} i\left[\mathbf{M}_2{}^1, (\mathbb{H}^{tw-3}Q^{tw-3}(z_1,z_2,z_3))\right] = \\ = \left(-z_1\frac{\partial}{\partial z_1} - 1\right)[(\mathbb{H}_q)_{11}Q_1(z_1,z_2,z_3) + (\mathbb{H}_q)_{12}Q_2(z_1,z_2,z_3) \\ + (\mathbb{H}_q)_{13}Q_3(z_1,z_2,z_3) + (\mathbb{H}_{gq})_{11}G_1(z_1,z_2,z_3) \\ + (\mathbb{H}_{gq})_{12}G_2(z_1,z_2,z_3) + (\mathbb{H}_{gq})_{13}G_3(z_1,z_2,z_3)] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^1 \mathrm{d}\tau\ \tau z_1^2\,[(\mathbb{H}_{gq})_{11}G_1(z_1,z_2,z_3,\tau z_1)+(\mathbb{H}_{gq})_{12}G_2(z_1,z_2,z_3,\tau z_1)\\
&\qquad\qquad +(\mathbb{H}_{gq})_{13}G_3(z_1,z_2,z_3,\tau z_1)]\\
&+\left(-z_2\frac{\partial}{\partial z_2}-1\right)[(\mathbb{H}_q)_{21}Q_1(z_1,z_2,z_3)+(\mathbb{H}_q)_{22}Q_2(z_1,z_2,z_3)\\
&\qquad\qquad +(\mathbb{H}_q)_{23}Q_3(z_1,z_2,z_3)+(\mathbb{H}_{gq})_{21}G_1(z_1,z_2,z_3)\\
&\qquad\qquad +(\mathbb{H}_{gq})_{22}G_2(z_1,z_2,z_3)+(\mathbb{H}_{gq})_{23}G_3(z_1,z_2,z_3)]\\
&+\frac{1}{2}\int_0^1 \mathrm{d}\tau\ \tau z_1^2\,[(\mathbb{H}_{gq})_{21}G_1(z_1,z_2,z_3,\tau z_2)+(\mathbb{H}_{gq})_{22}G_2(z_1,z_2,z_3,\tau z_2)\\
&\qquad\qquad +(\mathbb{H}_{gq})_{23}G_3(z_1,z_2,z_3,\tau z_2)]\\
&+\left(-z_3\frac{\partial}{\partial z_3}-1\right)[(\mathbb{H}_q)_{31}Q_1(z_1,z_2,z_3)+(\mathbb{H}_q)_{22}Q_2(z_1,z_2,z_3)\\
&\qquad\qquad +(\mathbb{H}_q)_{33}Q_3(z_1,z_2,z_3)+(\mathbb{H}_{gq})_{31}G_1(z_1,z_2,z_3)\\
&\qquad\qquad +(\mathbb{H}_{gq})_{32}G_2(z_1,z_2,z_3)+(\mathbb{H}_{gq})_{33}G_3(z_1,z_2,z_3)]\\
&+\frac{1}{2}\int_0^1 \mathrm{d}\tau\ \tau z_1^2\,[(\mathbb{H}_{gq})_{31}G_1(z_1,z_2,z_3,\tau z_3)+(\mathbb{H}_{gq})_{32}G_2(z_1,z_2,z_3,\tau z_3)\\
&\qquad\qquad +(\mathbb{H}_{gq})_{33}G_3(z_1,z_2,z_3,\tau z_3)]\ ,
\end{aligned}
\tag{4.123}
$$

where $G_3 = -G_1 - G_2$. The left-hand side contains only twist-3 2-to-2 kernels, whereas the right-hand side features 2-to-2 and 2-to-3 kernels of twist 4. Since Eq. (4.123) holds for any set of polynomials $Q_i(x_1, x_2, x_3)$ and $G_i(x_1, x_2, x_3, x_4)$, it provides a powerful check for the twist-4 kernels. In fact, since all kernels can be written as a linear combination of a handful elementary kernels, the coefficients can be restored from Eq. (4.123) alone. This can be implemented in a program such as *Mathematica*[9]. The coefficients can be determined by solving a linear system of equations. We used this to check our 2-to-3 kernels and the results of Sect. 4.3.2 were verified. For the mixed chirality case analogous expressions can be constructed.

The natural question that arises is, if the full functional form of the twist-4 kernels can be restored from (4.119) and knowledge of the twist-3 kernels alone. A more careful and detailed analysis is necessary to give a definite answer. But for twist-4 operators playing a role in deep inelastic scattering processes the more complicated 2-to-3 kernels can be restored from the twist-3 kernels and twist-4 2-to-2 kernels, see [85] for details.

4.5 Anomalous Dimensions

With the Hamiltonians established, the determination of the anomalous dimensions and multiplicatively renormalizable operators corresponds to finding the

[9]Trademark of WOLFRAM Research.

eigenvalues and eigenfunctions of the operators $\mathbb{H}$. The off-diagonal block $\mathbb{H}_{qg}$ is not needed for the anomalous dimensions; it is sufficient to diagonalize quark and gluon block separately. Furthermore, since the integral kernels map polynomials of homogeneous degree in the coordinates z_i onto homogeneous polynomials, it is enough to consider the coefficient functions $\Psi^i_{N,q}$, $i = 1, \ldots, 5$, cp. Eq. (4.33).

4.5.1 Spectrum of Anomalous Dimensions I: Chiral Case

Additional Symmetries in the Chiral Sector

The chiral basis, at first glance, seems to be more "symmetric" than its mixed chirality counterpart. To put this, somewhat purely esthetical statement, on more solid ground, let us introduce the generator of cyclic permutations,

$$\mathcal{P} = P_a \otimes P_z, \tag{4.124}$$

where P_a permutes the quantum numbers of an operator and P_z the coordinates, e.g.

$$P_a \Psi^i_{N,q}(z_1, z_2, z_3) = \Psi^{i+1}_{N,q}(z_1, z_2, z_3) \tag{4.125}$$

$$P_z \Psi^i_{N,q}(z_1, z_2, z_3) = \Psi^i_{N,q}(z_3, z_1, z_2). \tag{4.126}$$

The chiral Hamiltonian $\mathbb{H}^{\psi\psi\psi}_q$ commutes with the generator $\mathcal{P}$ and we can, in principle, find simultaneous eigenfunctions of both operators. Since obviously

$$\mathcal{P}^3 = \mathbb{1}\ , \tag{4.127}$$

the eigenvalues ε of $\mathcal{P}$ are the third roots of 1, i.e. $\varepsilon \in \{1, e^{i\frac{2}{3}\pi}, e^{-i\frac{2}{3}\pi}\}$. Thus, the eigenfunctions[10] of $\mathbb{H}^{\psi\psi\psi}_q$ can be chosen to have a definite parity with respect to $\mathcal{P}$:

$$\mathcal{P}\, \Psi^{(\varepsilon)}_{N,q} = \varepsilon\, \Psi^{(\varepsilon)}_{N,q}. \tag{4.128}$$

Each eigenfunction $\Psi^{(\varepsilon)}_{N,q}$ then depends only on a single (scalar) function $\psi^{(\varepsilon)}_{N,q}$

$$\Psi^{\varepsilon}_{N,q}(z_1, z_2, z_3) = \begin{pmatrix} \varepsilon^0 & \psi^{\varepsilon}_{N,q}(z_1, z_2, z_3) \\ \varepsilon^1 & \psi^{\varepsilon}_{N,q}(z_2, z_3, z_1) \\ \varepsilon^2 & \psi^{\varepsilon}_{N,q}(z_3, z_1, z_2) \end{pmatrix}. \tag{4.129}$$

This is obvious, since $\mathcal{P}$ simultaneously permutes the coordinates as well as the rows of the vector; each eigenfunction has to be of the form given in (4.129).

The chiral quark Hamiltonian features an additional symmetry. It commutes with the operator $\mathcal{P}_{12}$ permuting the first two vector entries and the first two

[10] Recall that $\mathbb{H}^{\psi\psi\psi}_q$ is a 3×3 matrix, its eigenfunctions are three dimensional vectors.

coordinates. This can be checked by direct calculation. As

$$\begin{pmatrix} \varepsilon^0 & \psi^\varepsilon_{N,q}(z_1,z_2,z_3) \\ \varepsilon^1 & \psi^\varepsilon_{N,q}(z_2,z_3,z_1) \\ \varepsilon^2 & \psi^\varepsilon_{N,q}(z_3,z_1,z_2) \end{pmatrix} \overset{\mathcal{P}_{12}}{\rightarrow} \begin{pmatrix} \varepsilon^1 & \psi^\varepsilon_{N,q}(z_3,z_2,z_1) \\ \varepsilon^0 & \psi^\varepsilon_{N,q}(z_2,z_1,z_3) \\ \varepsilon^2 & \psi^\varepsilon_{N,q}(z_1,z_3,z_2) \end{pmatrix} \overset{\mathcal{P}}{\rightarrow} \begin{pmatrix} \varepsilon^0 & \psi^\varepsilon_{N,q}(z_3,z_2,z_1) \\ \varepsilon^2 & \psi^\varepsilon_{N,q}(z_2,z_1,z_3) \\ \varepsilon^1 & \psi^\varepsilon_{N,q}(z_1,z_3,z_2) \end{pmatrix}, \tag{4.130}$$

$[\mathcal{P}_{12}\Psi^\varepsilon_{N,q}]$ is an eigenfunction of $\mathcal{P}$ to the eigenvalue ε^2. It then follows due to

$$[\mathbb{H}^{\psi\psi\psi}_q, \mathcal{P}_{12}] = 0 \quad \text{and} \quad \left(e^{i\frac{2}{3}\pi}\right)^2 = e^{-i\frac{2}{3}\pi}$$

that the spectra of anomalous dimensions for $\varepsilon = e^{-i\frac{2}{3}\pi}$ and $\varepsilon = e^{i\frac{2}{3}\pi}$ are the same.

For the mixed chirality Hamiltonian $\mathbb{H}^{\psi\psi\bar{\chi}}_q$ no such permutation symmetry can be found and the spectrum cannot be decomposed into different sectors.

Integrability

It has been known for some ten years that the chiral twist-3 Hamiltonian possesses an integral of motion [61]. That is, there exists an operator Q that commutes with the Hamiltonian and its eigenvalues are conserved charges of the system. The Hamiltonian corresponds to a one-dimensional three-body problem and the total conformal spin as well as its projection on the light-cone already are good quantum numbers [61]. The existence of the third conserved charge Q then implies that the system is completely integrable.

It turns out that the quark part of the chiral twist-4 Hamiltonian $\mathbb{H}^{\psi\psi\psi}$ also possesses such a hidden integral of motion. To find this charge some amount of sophisticated guessing is necessary.

Let us introduce an operator [62]

$$S_{ik} = \partial_k(z_k - z_i) \equiv (\partial/\partial z_k)(z_k - z_i)\,. \tag{4.131}$$

It can be checked that this operator connects the $SL(2,\mathbb{R})$ representations $T^{j_1} \otimes T^{j_2}$ and $T^{j_2} \otimes T^{j_1}$:

$$S_{ik} T^{j_k=1/2} \otimes T^{j_i=1} = T^{j_k=1} \otimes T^{j_i=1/2} S_{ik}\,. \tag{4.132}$$

It is referred to as intertwining operator [62]. The conformal two-particle Casimir operator J_{12} for conformal spins $(1,1)$ [61] and $(1/2,1)$ then takes the form

$$J^2_{12} = S_{21}(S_{12}+1) \qquad \text{and} \qquad J^2_{12} = S_{12}S_{21} + \frac{1}{4}\,, \tag{4.133}$$

respectively. The Hamiltonian $\mathbb{H}^{\psi\psi\psi}_q$ is a three by three matrix and so must be the conserved charge. The matrix Casimir operator can be defined as follows

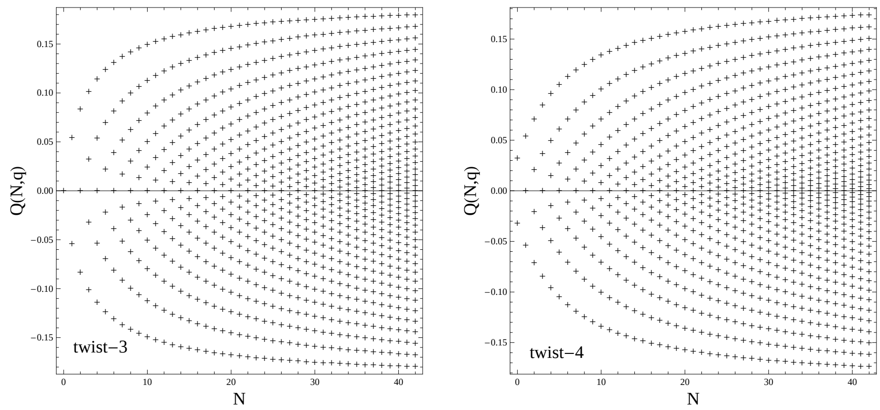


Figure 4.2: The spectrum of the conserved charge for twist-3 and twist-4 chiral quark operators. The figures are taken from [62]

[86]

$$\hat{J}_{12}^2 = \begin{pmatrix} S_{12}S_{21} & S_{12} & 0 \\ S_{21} & S_{21}S_{12} & 0 \\ 0 & 0 & S_{12}(S_{21}+1) \end{pmatrix} \tag{4.134}$$

and analogously

$$\hat{J}_{23}^2 = \begin{pmatrix} J_{23}^2 & 0 & 0 \\ 0 & J_{23}^2 + \frac{1}{4} & S_{23} \\ 0 & S_{32} & J_{23}^2 + \frac{1}{4} \end{pmatrix} \qquad \hat{J}_{23}^2 = \begin{pmatrix} J_{13}^2 + \frac{1}{4} & 0 & S_{13} \\ 0 & J_{23}^2 & 0 \\ S_{31} & 0 & J_{13}^2 + \frac{1}{4} \end{pmatrix}, \tag{4.135}$$

where J_{ik} depends on the conformal spins of the representation.

To write the operators $\hat{J}_{ik}^2$ in a compact way, one can introduce 3×3 matrices Q_{ik}^+ and Q_{ik}^-, $i < k$, which are defined by

$$[Q_{ik}^\pm]^{ik} = S_{ik}\,, \qquad [Q_{ik}^\pm]^{ki} = S_{ki} \qquad [Q_{ik}^\pm]^{ii} = [Q_{ik}^\pm]^{kk} = \frac{1}{2} \tag{4.136}$$

and

$$[Q_{ik}^+]^{jj} = \frac{1}{2} + S_{ik} \qquad [Q_{ik}^-]^{jj} = \frac{1}{2} + S_{ki}, \tag{4.137}$$

for j different from i and k, with all other matrix elements equal to zero. Then $\hat{J}_{ik}^2$ can be written as

$$\widehat{J}_{ik}^2 = \frac{1}{2}\{Q_{ik}^+, Q_{ik}^-\}\,. \tag{4.138}$$

The conserved charge should be constructed from the operators $\hat{J}_{ik}^2$ and indeed one finds that

$$\widehat{Q}_3 = \frac{i}{2}[\widehat{J}_{12}^2, \widehat{J}_{23}^2] \tag{4.139}$$

commutes with $\mathbb{H}_q^{\psi\psi\psi}$:

$$[\widehat{Q}_3, \mathbb{H}_q^{\psi\psi\psi}] = 0 \,. \tag{4.140}$$

This can be shown by calculating the commutator in the conformal basis, cp. [61, 72]. Due to Eq. (4.140) we can label any eigenfunction Ψ of $\mathbb{H}_q^{\psi\psi\psi}$ by its conserved charge q, $\widehat{Q}_3\Psi = q\Psi$. Further, one can show that

$$[\widehat{Q}_3, \mathcal{P}] = \{\widehat{Q}_3, \mathcal{P}_{12}\} = 0 \,. \tag{4.141}$$

Therefore, eigenstates with $q \neq 0$ must be degenerate, as the eigenfunctions with charge q and $-q$ correspond to the same eigenvalue. The spectrum for the operator $\widehat{Q}_3$ can be found in the right panel of Fig. 4.2. The left panel shows the spectrum of the twist-3 conserved charge for comparison [61].

The Chiral Quark Spectrum

It is known that an operator with a total derivative has the same anomalous dimensions as the corresponding operator without the total derivative. So, as far as the spectrum is concerned, these operators can be omitted. To do this, the coefficient functions $\Psi_{N,q}$ of these operators have to identified in the expansion of the non-local operators

$$\mathbb{O}(\vec{z}) = \sum_{N,q} \Psi_{N,q} \mathcal{O}_{N,q}$$

over the complete set of local operators $\mathcal{O}_{N,q}$.

This is possible using the following observation: Let us consider an expansion of a fictitious non-local operator

$$\mathbb{O}^{\text{free}}(z_1, z_2, z_3) = \sum_{N,q} \Psi_{N,q}^{\text{free}} \mathcal{O}_{N,q}^{free} \,,$$

that does *not* involve any operators with total derivatives. The action of the generator of translation along the light-ray $\mathbf{P}^{2\dot{2}} = \mathbf{P}^{++}$ on $\mathbb{O}^{\text{free}}(z_1, z_2, z_3)$ is given by

$$i \left[\mathbf{P}^{2\dot{2}}, \sum_{N,q} \Psi_{N,q}^{\text{free}} \mathcal{O}_{N,q}^{free}\right] = i \sum_{N,q} \Psi_{N,q}^{\text{free}} \left[\mathbf{P}^{2\dot{2}}, \mathcal{O}_{N,q}^{free}\right] = i \sum_{N,q} \left[P^{2\dot{2}}, \Psi_{N,q}^{\text{free}}\right] \mathcal{O}_{N,q}^{free} \,, \tag{4.142}$$

where the boldface generator acts on quantum fields and the generator in normal font acts on the coordinates, cp. (3.35). Inserting the explicit expression for P^{++}, one obtains

$$\sum_{N,q} \left[(\partial_1 + \partial_2 + \partial_3)\Psi_{N,q}^{\text{free}}(z_1, z_2, z_3)\right] \mathcal{O}_{N,q} = \sum_{N,q} \Psi_{N,q}(z_1, z_2, z_3) \left[\partial_+ \mathcal{O}_{N,q}^{\text{free}}\right]. \tag{4.143}$$

The left-hand side contains by definition only operators free of total derivatives, whereas the right-hand side is a sum over operators which explicitly contain at least one total derivative. So both sides must vanish identically, otherwise they cannot be equal to each other. Since the operators on the l.h.s. are independent, each coefficient function must be equal to zero. This provides us with a criterion to single out coefficient functions corresponding to operators without total derivatives, sometimes referred to as conformal operators:

$$(\partial_1 + \partial_2 + \partial_3)\Psi_{N,q}(\vec{z}) = 0\,. \tag{4.144}$$

It is possible to derive a second constraint by considering the generator of translations perpendicular to the light-cone $\mathbf{P}^{1\dot{2}}$. As it moves the fields away from the light-ray, this generator can – just as the generator of the Lorentz rotation ${\mathbf{M}_2}^1$ considered in Sect. 4.4 – increase the twist of an operator. In fact

$$i\,[\mathbf{P}_{1\dot{2}}, \psi_+]\,(z) = -2\partial_z\psi_-\,.$$

Thus, applying $\mathbf{P}^{1\dot{2}}$ to the chiral twist-3 operator

$$\mathbb{O}^{tw-3}(\vec{z}) = \epsilon^{ijk}\psi_+^{a,i}(z_1)\,\psi_+^{b,j}(z_2)\psi_+^{c,k}(z_3)$$

leads to

$$i[\mathbf{P}^{1\dot{2}}, \mathbb{O}^{tw-3}(\vec{z})] = -2\sum_{k=1}^{3}\frac{\partial}{\partial z_k}Q_k(\vec{z}) = -2\sum_{N,q}\left(\sum_{k=1}^{3}\frac{\partial}{\partial z_k}\Psi_{N,q}^{(k)}(\vec{z})\right)\mathbb{Q}_{N,q}\,, \tag{4.145}$$

where $Q_k(\vec{z})$ is defined in Eq. (4.11). The l.h.s. only contains operators with a total derivative, whereas the r.h.s. involves both conformal and non-conformal operators. For the equation to be fulfilled, the coefficients in front of conformal operators must vanish:

$$\partial_1\Psi_{N,q}^{(1)}(\vec{z}) + \partial_2\Psi_{N,q}^{(2)}(\vec{z}) + \partial_3\Psi_{N,q}^{(3)}(\vec{z}) = 0\,. \tag{4.146}$$

The set of shift-invariant homogeneous polynomials

$$e_{N,k}(z_1, z_2, z_3) = \frac{(z_1 - z_2)^k\,(z_1 - z_3)^{N-k}}{k!(N-k)!} \tag{4.147}$$

automatically fulfills the condition (4.143) and it is possible to calculate the Hamiltonian $\mathbb{H}_q^{\psi\psi\psi}$ in this basis, i.e.

$$\mathbb{H}_q^{\psi\psi\psi}e_{N,k} = \sum_{k'=0}^{N}(\mathbb{H}_q^{\psi\psi\psi})_{k'k}\,e_{N,k'}\,. \tag{4.148}$$

The resulting $(N+1)\times(N+1)$ matrix[11] $\mathbb{H}_q^{\psi\psi\psi}$ can be diagonalized numerically for each choice of ε [62]. This "brute-force" ansatz is the actually most

[11] Originally we had to deal with a $3(N+1)\times 3(N+1)$ matrix, but the permutation symmetry reduced the size of the matrix by a factor three, cf. (4.129).

N	$E_{N,0}$	$E_{N,1}$	$E_{N,2}$	$E_{N,3}$
0	-2^*	-	-	-
1	$-\frac{2}{3}$	$\frac{4}{3}^*$	-	-
2	$\frac{4}{3}^*$	4	-	-
3	2	$\frac{29-\sqrt{57}}{6}^*$	$\frac{29+\sqrt{57}}{6}^*$	-
4	$\frac{167-3\sqrt{481}}{30}^*$	$\frac{17}{3}$	$\frac{167+3\sqrt{481}}{30}^*$	-
5	$\frac{34}{9}$	$\frac{77}{15}^*$	$\frac{67}{9}^*$	$\frac{137}{15}$
6	3.633418^*	311/60	6.687457^*	7.724361^*

Table 4.1: Anomalous dimensions of local twist-4 chiral-quark operators with N covariant derivatives in units of $\alpha_s/(2\pi)$. The entries marked with an asterisk correspond to the operators with $\mathcal{P}$-parity $\varepsilon = e^{\pm i2\pi/3}$ and the remaining ones to $\varepsilon = 1$. The table is taken from [62].

effective option, since analytic methods such as the algebraic Bethe-Ansatz are too sophisticated for such a simple problem.

The final result for the spectrum of the chiral quark Hamiltonian for the eigenvalues with $N < 7$ is presented in Table 4.1.

All eigenvalues except the lowest one for each N are doubly degenerate. Therefore, the lowest eigenvalue must correspond to a conserved charge $\widehat{Q}_3 = 0$. The eigenvalue spectum for a larger range in N is displayed in Fig. 4.3. The upper panels show the chiral twist-4 and, for comparison, the chiral twist-3 spectrum. The smoothness of the spectra is a manifestation of the integrability of the chiral kernels. The lower two panels represent the different sectors with $\varepsilon = 1$ and $\varepsilon = e^{\frac{i2\pi}{3}}$.

The Chiral Quark-Gluon Spectrum

The chiral four-particle Hamiltonian $\mathbb{H}_g^{\psi\psi\psi\bar{f}}$ exhibits the same permutation symmetries as the three-particle Hamiltonian, and the eigenfunctions can be classified in the same way, that is by their eigenvalues $\varepsilon = 1, e^{\pm\frac{i2\pi}{3}}$ with respect to $\mathcal{P}$. An additional constraint comes from Eq. (4.13), namely

$$\sum_{i=1}^{3} \Psi_{N,q}^{(i)}(z_1, z_2, z_3, z_4) = 0\,. \tag{4.149}$$

The shift invariant polynomials

$$e_{N,k,m}(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_4)^k\,(z_2 - z_4)^m (z_3 - z_4)^{N-2-k-m}}{k!\,m!\,(N - 2 - k - m)!} \tag{4.150}$$

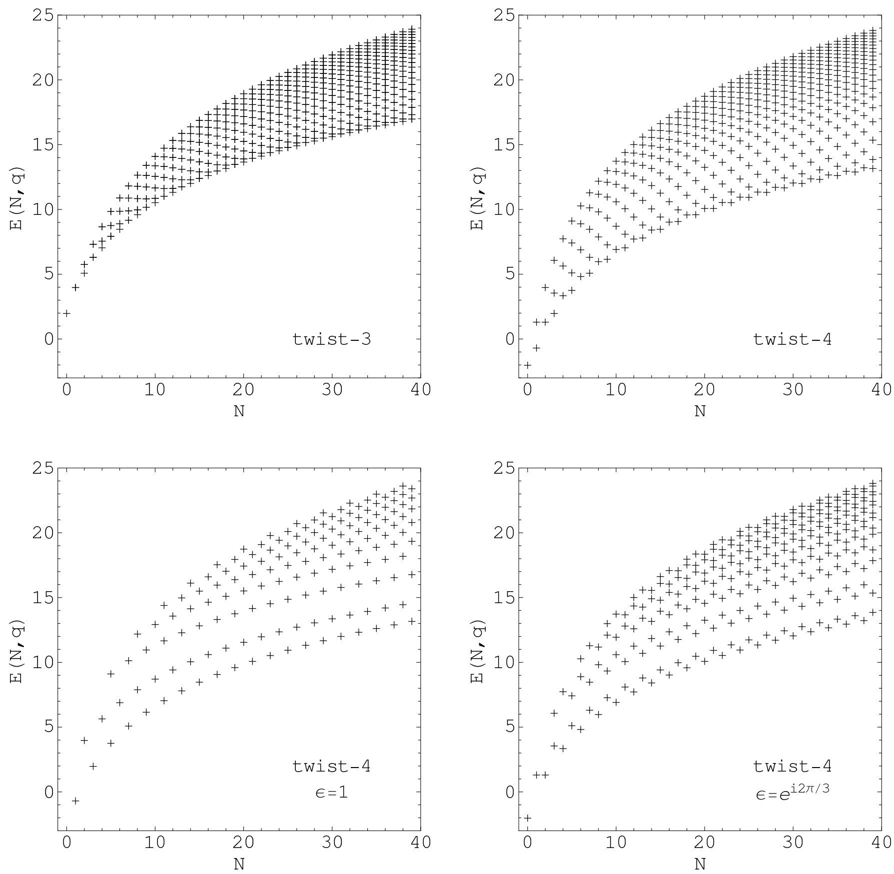


Figure 4.3: Upper panels: spectra of the chiral Hamiltonians for twist 3 and twist 4. The lower panels show the two sectors with $\varepsilon = 1$ and $\varepsilon = e^{\frac{i2\pi}{3}}$ separately. The figures are taken from [62].

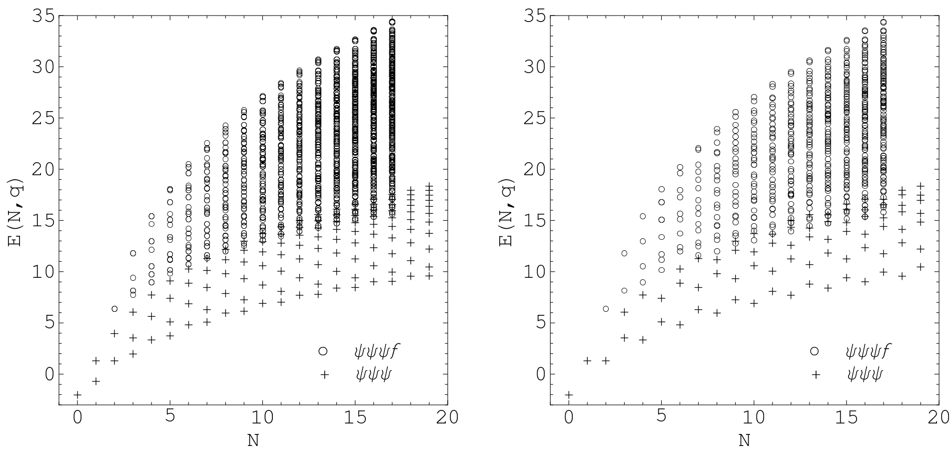


Figure 4.4: Spectra of the chiral twist-4 four-particle (open circles) and three-particle (crosses) Hamiltonians. The left panel shows the full spectrum, whereas the right panel depicts only the $\varepsilon = e^{\pm\frac{i2\pi}{3}}$ sector. The figures are taken from [62].

again provide us with a suitable basis for evaluating the Hamiltonian. Note that the degree of the polynomials in (4.150) is $N-2$. This choice is useful for the calculation of the multiplicatively renormalizable operators, as operators of different dimension do not mix and gluon operators have canonical dimension $l^{\text{can}}_{qqq\bar{f}} = l^{\text{can}}_{qqq} + 2$.

Calculating $\mathbb{H}^{\psi\psi\psi\bar{f}}$ in the basis (4.150), the resulting $N(N-1) \times N(N-1)$ matrix can be diagonalized numerically. The result is shown in Fig. 4.4. In the left panel the spectrum of anomalous dimensions for the chiral four-particle operator is indicated by the open circles. The crosses denote the twist-4 three-particle spectrum for comparison. The right panel corresponds to the $\varepsilon = e^{\pm\frac{i2\pi}{3}}$ sector alone. The two spectra start to overlap for $N > 7$ and the operator mixing between three and four particle operators is expected to become very strong at this point. For $N = 2$ and $N = 3$ the gap between the two spectra suggests a rather weak mixing, which supports the claim that the four-particle Fock states do not play a prominent role in actual applications. We can understand this qualitative claim in terms of our Schrödinger equation-like renormalization group equation picture. The anomalous dimensions depend only on the diagonal blocks, $\mathbb{H}_q$ and $\mathbb{H}_g$, but are independent of the off-diagonal block $\mathbb{H}_{qg}$, which determines the mixing of three- and four-particle operators. One can split the Hamiltonian in two pieces:

$$\mathbb{H} \to \begin{pmatrix} \mathbb{H}_q & 0 \\ 0 & \mathbb{H}_g \end{pmatrix} + \begin{pmatrix} 0 & \mathbb{H}_{gq} \\ 0 & 0 \end{pmatrix} . \tag{4.151}$$

The first summand gives the "energy eigenvalues" (anomalous dimensions) of

N	$E_{N,0}$	$E_{N,1}$	$E_{N,2}$	$E_{N,3}$	$E_{N,4}$
0	-2	-	-	-	-
1	2/9	2	-	-	-
2	$\frac{2(14-\sqrt{43})}{9}$	32/9	$\frac{2(14+\sqrt{43})}{9}$	-	-
3	$\frac{197-\sqrt{5089}}{45}$	$\frac{49-\sqrt{73}}{9}$	$\frac{49+\sqrt{73}}{9}$	$\frac{197+\sqrt{5089}}{45}$	-
4	3.706620	$\frac{589-\sqrt{11161}}{90}$	6.634936	$\frac{589+\sqrt{11161}}{90}$	7.858442

Table 4.2: Anomalous dimensions of twist-4 quark operators of mixed chirality in units of $\alpha_s/(2\pi)$; N is the total number of covariant derivatives. The table is taken from [62].

the Hamiltonian, the second term can be viewed as a perturbation. Classical Rayleigh-Schrödinger perturbation theory tells us that the correction to a (non-degenerate) state $|n\rangle$ due to a small perturbation V is given by [87]

$$|n^{\text{pert}}\rangle = |n\rangle + \sum_{p\neq n} \frac{\langle p|V|n\rangle}{E_n - E_p}|p\rangle \,, \tag{4.152}$$

where E_p is an (unperturbed) energy eigenvalue and p labels the different eigenfunctions. Therefore, even if $\mathbb{H}_{gq}$ qualifies as small, once the spectra overlap and the distance between the eigenvalues becomes tiny the mixing gets strong nonetheless[12].

4.5.2 Spectrum of Anomalous Dimensions II: Mixed Chirality Operators

The anomalous dimensions of mixed chirality operators can be obtained in a similar way. Since the permutation symmetry is absent, the spectrum cannot be split into three sectors and the numerical solution is more tedious.

For three-particle operators the constraint (4.144) is still valid but (4.146) has to be replaced by

$$\frac{\partial}{\partial z_1}\Psi^{(1)}_{N,q}(\vec{z}) + \frac{\partial}{\partial z_2}\Psi^{(2)}_{N,q}(\vec{z}) = \Psi^{(3)}_{N,q}(\vec{z})\,. \tag{4.153}$$

Using the shift-invariant polynomials one can again determine and diagonalize the Hamiltonian $\mathbb{H}^{\psi\psi\bar{\chi}}$ in this basis. The eigenvalues for $N < 5$ have been collected in Table 4.2. Fig. 4.5 shows the spectrum for operators with genuine *geometric* twist 3 on the left panel and for operators with *geometric* twist 4 on the right panel.

[12]Note that for a more quantitative statement the actual matrix elements have to evaluated nonperturbatively.

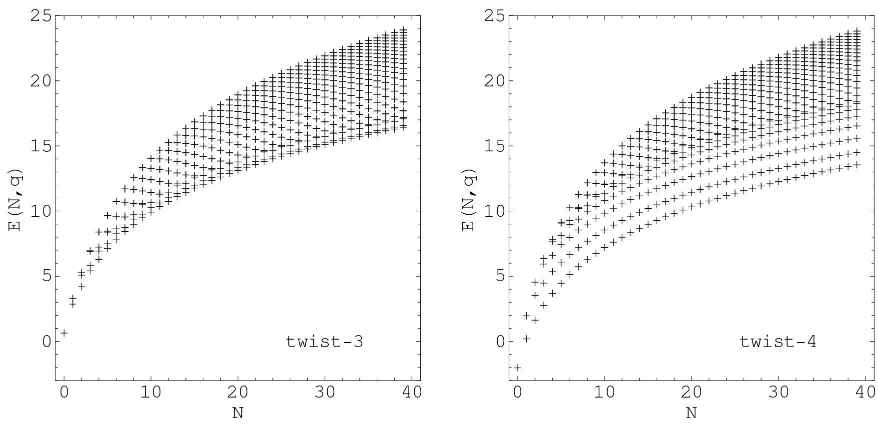


Figure 4.5: The spectrum of the Hamiltonian $\mathbb{H}_q^{\psi\psi\bar{\chi}}$. The figures are taken from [62].

For the quark-gluon operators the spectrum looks very similar to the chiral spectrum and it can be obtained in the same way. The main difference is that the eigenvalues are not degenerate. This is due to the missing permutation symmetry of the Hamiltonian $\mathbb{H}_g^{\psi\psi\bar{\chi}\bar{f}}$. The spectrum is shown in Fig. 4.6.

4.5.3 Multiplicatively Renormalizable Operators

While the off-diagonal blocks of the Hamiltonians $\mathbb{H}_{qg}^{\psi\psi\psi}$ and $\mathbb{H}_{qg}^{\psi\psi\bar{\chi}}$ do not play a role for the eigenvalues, they have to be taken into account if one wants to determine the eigenfunctions of $\widetilde{\mathbb{H}}$.

The eigenfunctions are, for a given N, vectors of homogeneous polynomials (in the coordinates z_i) and correspond to multiplicatively renormalizable operators. The eigenfunctions can again be found by explicit numerical diagonalization of the full Hamiltonians.

As already discussed, the multiplicatively renormalizable operators do not follow directly from the polynomials in coordinate space. The substitution rules (4.36) provide access to the dual polynomials $\widetilde{\mathcal{P}}$. These then determine the multiplicatively renormalizable operators via Eq. (4.35).

A more elegant option was used in [62]. The $SL(2,\mathbb{R})$ invariant scalar product, see Sect. 4.3.1 and [76, 88, 89], can be used to define the adjoint operator $\widetilde{\mathbb{H}}^\dagger$ and its eigenfunctions $\Psi_{N,q}^\dagger(\vec{z})$. To this end one must adjust the scalar product to take the vector nature of the eigenfunctions into account [62, 89]:

$$\langle\Psi_1|\Psi_2\rangle_5 := \sum_i^5 \langle\Psi_1^{(i)}|\Omega_{ik}\Psi_2^{(k)}\rangle\,, \tag{4.154}$$

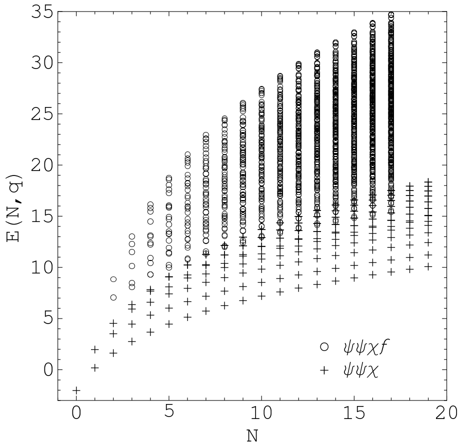


Figure 4.6: The combined spectrum of the Hamiltonians $\mathbb{H}_q^{\psi\psi\bar{\chi}}$ (crosses) and $\mathbb{H}_g^{\psi\psi\bar{\chi}\bar{f}}$ (open circles). The figure is taken from [62].

where $\langle\cdot|\cdot\rangle$ is the $SU(1,1)$ scalar product and the 5×5 matrix Ω is

$$\Omega^{\text{chiral}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \Omega^{\text{mixed}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \tag{4.155}$$

for $\widetilde{\mathbb{H}}^{\psi\psi\psi}$ and $\widetilde{\mathbb{H}}^{\psi\psi\bar{\chi}}$, respectively.

The orthogonality relation

$$\langle\Psi^\dagger_{N',q'}|\Psi_{N,q}\rangle_5 = \delta_{NN'}\delta_{qq'}\,. \tag{4.156}$$

can then be employed to project the vector of non-local operators $\mathbb{O}$, see (4.29), onto the multiplicatively renormalizable local operators $\mathbb{O}_{N,q}$. They are then given by the scalar product of $\mathbb{O}$ with the eigenfunctions of the adjoint Hamiltonian:

$$\mathbb{O}_{N,q} = \langle\Psi^\dagger_{N,q}|\mathbb{O}\rangle_5\,. \tag{4.157}$$

Both methods lead to the same result. Since the formulas are quite lengthy, we collect the multiplicatively renormalizable operators for arbitrary flavor structure in App. D.

4.6 Nucleon DAs – an Application

The nucleon distribution amplitudes of twist 3 and 4 were introduced in Sect. 4.1. The leading twist DA $\Phi_3(x_i, \mu)$ can be expanded in *momentum space* as

$$\Phi_3(x_i,\mu) = x_1 x_2 x_3 \sum_{N,q} c_{N,q}\, \phi_{N,q}(\mu)\, P_{N,q}(x_i)\,, \tag{4.158}$$

i.e. x_i are momentum fractions. The prefactor $x_1x_2x_3$ is prescribed by conformal symmetry [59]. This is the standard form used in e.g. light-cone sum rule calculations and it is necessary to transform the eigenfunctions representing multiplicatively renormalizable operators from coordinate to momentum space.

We follow [62] and use the the $SU(1,1)$ scalar product (4.40) for conformal spins $j_1 = j_2 = j_3 = 1$ and the twist-3 eigenfunctions $\Psi_{n,q}(z_1, z_2, z_3)$ defined in [61]. The polynomials $P_{N,q}$

$$P_{N,q}(x_1,x_2,x_3) = \langle e^{\sum x_i z_i}, \Psi_{n,q}(z_1,z_2,z_3)\rangle \tag{4.159}$$

are normalized to

$$c_{N,q}^{-1} = \int \mathcal{D}x\; x_1x_2x_3 |P_{N,q}|^2\,. \tag{4.160}$$

This is convenient as the coefficients $\phi_{N,q}(\mu)$ are then just given by

$$\phi_{Nq}(\mu) = \int \mathcal{D}x\, P_{N,q}(x)\, \Phi_3(x,\mu) \tag{4.161}$$

and the factors $c_{N,q}$ cancel in (4.158).

It is then only a matter of calculating the integrals to obtain the first few terms in the expansion of the twist-3 DA [62]

$$\begin{aligned}\Phi_3(x_1,x_2,x_3) = 120 x_1x_2x_3 \Big[&\phi_0^{(2/3)} + 42\,\phi_{1,0}^{(26/9)} P_{1,0}(x) + 14\,\phi_{1,1}^{(10/3)} P_{1,1}(x)\\ &+ \frac{63}{10}\,\phi_{2,0}^{(38/9)} P_{2,0}(x) + \frac{63}{2}\,\phi_{2,1}^{(46/9)} P_{2,1}(x)\\ &+ \frac{9}{5}\,\phi_{2,2}^{(16/3)} P_{2,2}(x) + \ldots\Big],\end{aligned} \tag{4.162}$$

where

$$\begin{aligned}P_{1,0}(x) =& \frac{1}{2}(x_1 - x_3)\,,\\ P_{1,1}(x) =& \frac{1}{2}(x_1 + x_3 - 2x_2)\,,\\ P_{2,0}(x) =& 3x_1^2 - 3x_1x_2 + 2x_2^2 - 6x_1x_3 - 3x_2x_3 + 3x_3^2\,,\\ P_{2,1}(x) =& (x_1 - x_3)(x_1 + x_3 - 3x_2)\,,\\ P_{2,2}(x) =& x_1^2 + x_3^2 - 12x_1x_3 + 9x_1x_2 + 9x_2x_3 - 6x_2^2\,.\end{aligned} \tag{4.163}$$

The superscript in $\phi_{N,q}^{(E_{Nq})}$ corresponds to the anomalous dimension of the coefficient [62]:

$$\phi_{N,q}^{(E_{Nq})}(\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{E_{Nq}/\beta_0} \phi_{N,q}^{(E_{Nq})}(\mu_0)\,. \tag{4.164}$$

A comparison with the standard expression for $\Phi_3(x_i, \mu)$ [65]

$$\Phi_3(x_i,\mu) = \quad 120 x_1 x_2 x_3 \left[\phi_3^0(\mu) + \phi_3^-(\mu)(x_1 - x_2) + \phi_3^+(\mu)(1 - 3x_3) + \ldots\right] \tag{4.165}$$

yields

$$\phi_3^0 = \phi_0^{(2/3)}\,, \quad \phi_3^- = 12\phi_{1,0}^{(26/9)} - \frac{7}{2}\phi_{1,1}^{(10/3)}\,, \quad \phi_3^+ = 12\phi_{1,0}^{(26/9)} + \frac{21}{2}\phi_{1,1}^{(10/3)}\,. \tag{4.166}$$

These relations have been known for quite some time [90, 91].

Twist-4 Distribution Amplitudes

Going over to the twist-4 case turns out to be more challenging. There are three reasons for this:

1. The operator basis consists of five independent light-ray operators for each chirality, so one has to work with 5-component vectors instead of a set of simple polynomials $\Psi_{N,q}(z)$.

2. The full Hamiltonian $\mathbb{H}$ is obviously *not* hermitian.

3. While the twist-4 operators corresponding to total derivatives of a twist-3 operator do not affect the spectrum of anomalous dimensions, they do play a role for the distribution amplitudes.

Each of these problems can be addressed separately. We only roughly sketch the strategy and refer to [62] for the details.

Hermicity of the Hamiltonian

The block-triangular form of the Hamiltonian already indicated that it cannot be hermitian with respect to some standard scalar product. Without this property, the eigenfunctions do not form an orthogonal system, and one cannot use an analogy of Eq. (4.159) to obtain the eigenfunctions in momentum space. However, one can use the modified $SU(1,1)$ scalar product (4.154) to determine the eigenfunctions of $\widetilde{\mathbb{H}}^\dagger$ explicitly. It is then possible to determine the momentum space expansion of the distribution amplitudes in terms of multiplicatively renormalizable operators in analogy to the twist-3 calculation, see Eq. (4.159).

Wandzura-Wilczek Contributions

The fact that the twist-4 DAs can receive sizeable contributions due to operators that are formally of collinear twist 4, but are in fact descendents of twist-3 operators, has been known for quite some time [65]. These operators cannot introduce any new nonperturbative parameters into the twist-4 distribution amplitudes, as their matrix elements can be expressed in terms of genuine twist-3 matrix elements. These contributions are usually referred to as *Wandzura-Wilczek* (WW) contributions. Obviously, the chiral twist-4 amplitude Ξ_4 is free of WW contributions, as there is no chiral twist-3 nucleon DA. For the mixed chirality DAs, it is useful to separate the Wandzura-Wilczek and the "true" twist-4 parts

$$\begin{aligned}\Phi_4(x) =& \Phi_4^{WW}(x) \ + \ \Phi_4^{tw-4}(x)\\ \Psi_4(x) =& \Psi_4^{WW}(x) \ + \ \Psi_4^{tw-4}(x)\,,\end{aligned} \qquad (4.167)$$

as the WW part can unambiguously be restored from the twist-3 DAs alone. This can then be used as a crude approximation for the full twist-4 DAs in case no information on the higher conformal spin parameters is available, see Chap. 5.2 for an application.

The separation can be achieved by using the $SL(2,\mathbb{R})$ generators S_-, S_+ and S_0. One obtains [62]

$$\Phi_4^{WW}(x) = -\sum_{N,q} \frac{c_{Nq}\,\phi_{Nq}}{(N+2)(N+3)} \left(N+2-\frac{\partial}{\partial x_3}\right) x_1 x_2 x_3\; P_{Nq}^{tw-3}(x_1,x_2,x_3)\,, \qquad (4.168)$$

$$\Psi_4^{WW}(x) = -\sum_{N,q} \frac{c_{Nq}\,\phi_{Nq}}{(N+2)(N+3)} \left(N+2-\frac{\partial}{\partial x_2}\right) x_1 x_2 x_3\; P_{Nq}^{tw-3}(x_2,x_1,x_3) \qquad (4.169)$$

for the WW contributions, where the twist-3 polynomials $P_{N,q}^{tw-3}$ and the coefficients ϕ_{Nq} are defined in (4.158) and (4.163).

The genuine twist-4 DAs Φ_4^{tw-4}, Φ_4^g, Ψ_4^{tw-4} and Ψ_4^g for $N<3$ read [62]

$$\begin{aligned}\Psi_4^{tw-4}(x,\mu) =&\ 12 x_1 x_3 \Big[\eta_{0,0}(\mu) + 4\,\eta_{1,0}(\mu)\,\mathcal{P}_{1,0}(x_2,x_3,x_1)\\ &+ \frac{20}{3}\eta_{1,1}(\mu)\mathcal{P}_{1,1}(x_2,x_3,x_1) + \left(\frac{55}{4}+\frac{25}{2\sqrt{43}}\right)\eta_{2,0}(\mu)\mathcal{P}_{2,0}(x_2,x_3,x_1)\\ &+ \frac{45}{2}\eta_{2,1}(\mu)\mathcal{P}_{2,1}(x_2,x_3,x_1) + \left(\frac{55}{4}-\frac{25}{2\sqrt{43}}\right)\eta_{2,2}(\mu)\mathcal{P}_{2,2}(x_2,x_3,x_1)\\ &+ \frac{140}{117}\eta_{2,0}^g(\mu)\,\widetilde{\mathcal{P}}_{2,0}^g(x_2,x_3,x_1) + \frac{70}{47}\eta_{2,1}^g(\mu)\,\widetilde{\mathcal{P}}_{2,1}^g(x_2,x_3,x_1)\Big]\,,\\ \Phi_4^{tw-4}(x,\mu) =& -12 x_1 x_2 \Big[\eta_{0,0}(\mu) + 4\,\eta_{1,0}(\mu)\,\mathcal{P}_{1,0}(x_3,x_1,x_2)\\ &- \frac{20}{3}\eta_{1,1}(\mu)\mathcal{P}_{1,1}(x_3,x_1,x_2) + \left(\frac{55}{4}+\frac{25}{2\sqrt{43}}\right)\eta_{2,0}(\mu)\mathcal{P}_{2,0}(x_3,x_1,x_2)\end{aligned}$$

$$
\begin{aligned}
&- \frac{45}{2}\eta_{2,1}(\mu)\mathcal{P}_{2,1}(x_3,x_1,x_2) + \left(\frac{55}{4} - \frac{25}{2\sqrt{43}}\right)\eta_{2,2}(\mu)\mathcal{P}_{2,2}(x_3,x_1,x_2) \\
&+ \frac{140}{117}\eta^g_{2,0}(\mu)\,\widetilde{\mathcal{P}}^g_{2,0}(x_3,x_1,x_2) - \frac{70}{47}\eta^g_{2,1}(\mu)\,\widetilde{\mathcal{P}}^g_{2,1}(x_3,x_1,x_2)\Big], \\
\Psi^g_4(x,\mu) =& \ \frac{1}{4}8!x_1x_2x_3x_4^2\left[\eta^g_{2,0}(\mu) + \frac{1}{3}\eta^g_{2,1}(\mu)\right], \\
\Phi^g_4(x,\mu) =& -\frac{1}{4}8!x_1x_2x_3x_4^2\left[\eta^g_{2,0}(\mu) - \frac{1}{3}\eta^g_{2,1}(\mu)\right],
\end{aligned}
\tag{4.170}
$$

where

$$
\begin{aligned}
\mathcal{P}_{1,0}(x) =& x_1 + x_2 - \frac{3}{2}x_3\,, \\
\mathcal{P}_{1,1}(x) =& x_1 - x_2 + \frac{1}{2}x_3\,, \\
\mathcal{P}_{2,1}(x) =& x_1^2 - x_2^2 - 2x_1x_3 + 3x_2x_3 - \frac{2}{3}x_3^2\,, \\
\begin{pmatrix}\mathcal{P}_{2,0}(x)\\ \mathcal{P}_{2,2}(x)\end{pmatrix} =& x_1^2 + \frac{4}{9}\left(-5 \pm \sqrt{43}\right)x_1x_2 + x_2^2 + \frac{2}{9}\left(1 \mp 2\sqrt{43}\right)x_1x_3 \\
&- \frac{1}{9}\left(17 \pm 2\sqrt{43}\right)x_2x_3 + \frac{4}{27}\left(4 \pm \sqrt{43}\right)x_3^2\,, \\
\widetilde{\mathcal{P}}^g_{2,0}(x) =& 64x_1^2 - 55x_1x_2 + \frac{11}{2}x_2^2 - 73x_1x_3 + 11x_2x_3 + \frac{17}{2}x_3^2\,, \\
\widetilde{\mathcal{P}}^g_{2,1}(x) =& 16x_1^2 - \frac{1}{3}x_2^2 - 32x_1x_3 + x_2x_3 + 5x_3^2\,.
\end{aligned}
\tag{4.171}
$$

Each coefficient $\eta(\mu)$ renormalizes multiplicatively

$$
\eta_{N,q}(\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{E_{N,q}}{b_0}} \eta_{N,q}(\mu_0)
\tag{4.172}
$$

and the anomalous dimensions can be found in Table 4.2 and in Eq. (D.9). The expressions for the WW-free chiral nucleon distribution amplitudes for $N \le 2$ are given by [62]

$$
\begin{aligned}
\Xi_4(x,\mu) =& \ 4x_2x_3\Big[\xi_{0,0}(\mu)\,\Pi_0(x) + 9\,\xi_{1,1}(\mu)\,\Pi_1(x) \\
&+ 12\,\xi_{2,0}(\mu)\,\Pi_2(x) + \frac{28}{3}\xi^g_{2,0}(\mu)\,\widetilde{\Pi}^g_2(x)\Big], \\
\Xi^g_4(x,\mu) =& \ \frac{1}{3}8!\,x_1x_2x_3x_4^2\,\xi^g_{2,0}(\mu)\Pi^g_2(x)\,,
\end{aligned}
\tag{4.173}
$$

where

$$
\begin{aligned}
\Pi_0(x) =& 1\,, \\
\Pi_1(x) =& x_1 + x_3 - \frac{3}{2}x_2\,, \\
\Pi_2(x) =& x_1^2 - 4x_1x_2 + 2x_2^2 + 2x_1x_3 - 4x_2x_3 + x_3^2\,, \\
\widetilde{\Pi}^g_2(x) =& \frac{43}{2}x_1^2 + 4x_1x_2 - 2x_2^2 - 47x_1x_3 + 4x_2x_3 + \frac{13}{2}x_3^2\,,
\end{aligned}
$$

$$\Pi_2^g(x) = 1/2 \tag{4.174}$$

and the anomalous dimensions of the coefficients $\xi_{N,q}(\mu)$ are presented in Table 4.1.

A comparison of the expressions for the three three-particle NDAs Ξ_4, Ψ_4 and Φ_4 with the original expansions in the work by Braun, Fries, Mahnke and Stein (BFMS) [65], which have been collected in App. E, yields relations between the different nonperturbative parameters.

By equating

$$\Xi_4(x_1, x_2, x_3) = \Xi_4^{\rm BFMS}(x_1, x_2, x_3) \tag{4.175}$$

and comparing the coefficients of the polynomials for $N = 1, 2$ one arrives at

$$\begin{aligned} \lambda_2 =& \xi_{0,0} \,, \\ \lambda_1 f_2^d =& \frac{4}{15}\xi_{0,0} + \frac{2}{5}\xi_{1,0} \end{aligned} \tag{4.176}$$

for the chiral amplitudes and

$$\begin{aligned} \lambda_1 =& -\eta_{00} \,, \\ \lambda_1 f_1^d =& -\frac{1}{6}\phi_{00} - \frac{3}{10}\eta_{00} - \frac{1}{5}\eta_{10} + \frac{1}{3}\eta_{11} \,, \\ \lambda_1 f_1^u =& -\frac{1}{6}\phi_{00} - \frac{1}{10}\eta_{00} - \frac{3}{5}\eta_{10} + \frac{1}{3}\eta_{11} \end{aligned} \tag{4.177}$$

for the amplitudes of mixed-chirality. We see that the twist-3 parameter ϕ_{00} appears in the relations for the twist-4 parameters f_1^d and f_1^u. This is a residue of the Wandzura-Wilczek contributions, which could not be isolated in [65].

4.7 Summary

This chapter was dedicated to the study of the scale dependence of higher-twist baryon operators. Apart from reaching a deeper theoretical understanding of higher-twist operators, our study was also fueled by recent developments in the field of nucleon distribution amplitudes [66, 67] and is therefore of relevance for phenomenology.

Since the standard techniques for the calculation of anomalous dimensions are inapt for operators of higher twist, it was necessary to adjust the framework of [61], which is based heavily on the use of conformal symmetry.

The first step was already presented in the previous chapter, where we introduced the powerful spinor formalism. It allowed us to treat spinor and Lorentz indices on the same footing. This turned out to be useful for the construction of an operators basis of one particle light-ray operators with good conformal transformation properties. This operator basis became the fundamental building block for arbitrary non-local light-ray operators and represents one of the main results of our work.

In Chapter 4 we focused on the baryon operators of twist 4, while keeping an eye on the special case of nucleon operators and their matrix elements, the nucleon distribution amplitudes. In Sect. 4.2 we devised a generic basis for non-local baryon operators of twist 4. All in all, there exist 12 independent operators; they fall into two classes: chiral operators and operators of mixed chirality. Making use of the explicit conformal properties of our basis, it was possible to restrict the functional form of the RGEs to linear combinations of a small set of elementary kernels. The subsequent determination of these renormalization kernels was presented in detail for one representative example: the previously unknown chiral 2-to-3 kernels responsible for the mixing of pure quark and quark-gluon operators. These are the main results of this chapter.

To verify our results we invented a check that is easy to implement in standard numerical programs; it relies on the invariance of the RGE under the full conformal group. This check has the potential to be extended to a completely new method for determining renormalization kernels, as it is based on connections between kernels of different twist and is not limited to one-loop order.

By diagonalizing the Hamiltonians in a specific basis, it was possible to determine the full spectrum of anomalous dimensions as well as the multiplicatively renormalizable operators. It turned out that the chiral quark sector possesses a hidden integral of motion and the system is completely integrable. As an application of our work we presented the results for the decomposition of the well-known nucleon distribution amplitudes into a linear combination of RG eigenfunctions. Especially the Eqs. (4.177) are of relevance for phenomenology, as they provide relations between twist-3 and twist-4 NDAs. In the following chapter we present a study of the $N\gamma \to N^\star$ transition form factors, which makes use of these relations.

You can't always get
what you want
But if you try sometime, yeah,
You just might find
you get what you need!

Mick Jagger & Keith Richards

5 Light-Cone Sum Rules with Baryon Distribution Amplitudes

The nickname 'exclusive processes' is used to refer to experiments where both the initial and the final state are observed. By choosing a suitable probe, for example a hard photon if one wants to study electromagnetic properties, the measured cross section can be related to the internal structure of the target. This setup is experimentally very clean as one can focus on a specific final state and ignore all other signals. A more problematic class of processes are inclusive reactions, where all possible final states have to be summed over and it is very easy to miss a particularly elusive reaction channel.

On the theory side the situation is quite the reverse. Inclusive processes can be treated with help of the optical theorem, which allows the resummation of the various final states already on parton level. On the other hand, exclusive reactions require a great deal of knowledge of the relation between constituent partons and the bound states, the hadrons. One way to include this in a calculation is the use of distribution amplitudes.

In fact, for exclusive reactions involving baryons the DAs studied in the previous chapter present the central nonperturbative input for the so-called light-cone sum rule method [23, 24, 25], a synthesis of the famous Shifman-Vhainstein-Zakharov sum rules (SVZ SRs) [26, 27, 28] and the theory of hard exclusive processes [92, 93, 21, 94]. The technique proved to be very successful in the past. Its achievements, to name a few, include the description of pion form factors [95], nucleon magnetic moments [96, 24, 97, 67] and B, D meson decay constants [98, 99]. Light-cone sum rules are especially attractive, because they do not only provide a means to consistently sum both, hard and soft contributions to exclusive processes [95], but also serve as a bridge to connect DAs and form factors.

In Sect. 5.1 we first give a brief introduction into the general philosophy of light-cone sum rules. We will see that the crucial nonperturbative objects that encode the soft contributions are matrix elements of the very same baryon operator studied in detail in Chap. 4. Furthermore, we introduce the concept of continuum subtraction and Borel transformation. The next section is dedicated to the electromagnetic form factors of the $N\gamma \to N^\star$ transition. After a short motivation we discuss the different options how the transition form factors for this process can be obtained within the LCSR approach. It turns out that one cannot use the well understood nucleon distribution amplitudes, but has to rely on distribution amplitudes for the $N^\star$. They are related to matrix elements that can be determined using Lattice QCD. We show that a proper definition of these DAs, which is one of the main results of this chapter, allows us to restore the sum rules for the $N\gamma \to N^\star$ transition directly from the sum rules for the nucleon electromagnetic form factors without an involved calculation. Our results turn out to be in good agreement with the latest Jefferson Laboratory (JLab) data on the $N^\star$ form factors. We close this chapter with a short summary.

5.1 The Philosophy of LCSRs

In a hard exclusive process, like $N\gamma \to N^\star$, a large momentum q has to be transferred to a hadronic system without "destroying" it, cf. Chap. 4.1. This can be achieved via two different mechanisms: the so-called Feynman mechanism which incorporates the soft contributions, and the hard rescattering mechanism. The rescattering relies on the possibility to redistribute the large momentum received by one parton via gluon exchanges, thus keeping the whole hadron as such intact. As each gluon exchange comes with a penalty of order $\frac{\alpha_s}{Q^2}$, this mechanism favors Fock states with a minimal number of partons at small transverse separations [100]. In the soft picture a single quark carries almost the whole momentum of the hadron. The large momentum q can be transferred to this quark without changing the relative momentum distribution in the hadron. The fast quark then recombines with the remaining partons – sometimes referred to as "soft cloud" [100]. A heuristic illustration using a meson as an example can be found in Fig. 5.1.

For generic processes it is not possible to judge a priori which one of the two mechanisms is dominant or if both have to be taken into account. While the hard part can be treated in perturbative QCD, the soft contributions require some nonperturbative approach. If both contributions are calculated with separate methods, e.g., a quark model calculation for the soft and a pQCD one for the hard part, there is the possibility to accidentally include a specific configuration in both parts, because no rigorous separation and matching procedure was or could be enforced. This so-called *double counting* of contributions can spoil the whole calculation. Along with the treatment of nonperturbative, soft contributions in general, avoiding double counting is one of the main challenges

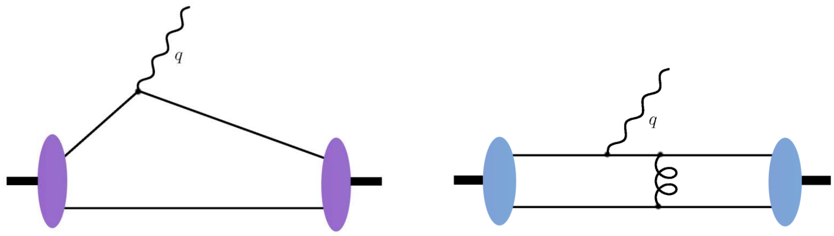


Figure 5.1: Diagrams for the determination of electromagnetic meson form factors. The straight lines represent quarks, the wiggly lines photons and the curly lines gluons. The diagram in the left panel shows schematically the function of the Feynman mechanism. A single quark with high momentum is struck by a hard photon, the fast quark then recombines with the unchanged "soft parton cloud" [100]. The right panel represents the perturbative hard rescattering mechanism.

in the theory of hard exclusive processes.

Light-cone sum rules, originally devised for the study of the weak decay $\Sigma^+ \to p\gamma$ [23], provide a means to consistently include both hard and soft parts and avoid any double counting. While the method itself is a ideological descendant of the famous SVZ sum rules, it does not make use of the local SVZ condensates [26]. By changing the expansion parameter from distance x (in coordinate space) to transverse separation x^2 of the partons [100], LCSRs allow a resummation of the SVZ operator product expansion (OPE). The local condensates ordered by their *dimension* are replaced by nonlocal matrix elements classified according to their light-cone twist E – the *distribution amplitudes*. The soft contributions are included as integrals over the end-point regions of these distribution amplitudes. Therefore, LCSRs are in a sense unique, as they express hard and soft contributions in terms of the same DAs. The nucleon distribution amplitudes introduced in Chapter 4.1 are one example for (baryon) DAs and we introduce the new $N^\star$ distribution amplitudes in Sect. 5.2.3.

Basic Example: $N\gamma \to N$

Let us now briefly sketch the paradigm of the LCSR approach using the simplest process involving baryons as an example: a nucleon absorbs a photon. This process is described by the correlation function

$$T^\mu(p,q) = \int \mathrm{d}^4x \; e^{-iqx} \langle 0 | \mathcal{T} \left\{ \eta(0) j^\mu_{\mathrm{em}}(x) \right\} | N(p) \rangle \,, \tag{5.1}$$

where $\mathcal{T}$ indicates the time-ordering of the product. The initial state $|N(p)\rangle$, a nucleon with momentum p, interacts with an electromagnetic current

$$j^\mu_{\text{em}}(x) = e_u \bar{u}^a(x)\gamma^\mu u^a(x) + e_d \bar{d}^a(x)\gamma^\mu d^a(x) , \tag{5.2}$$

where $e_u = \frac{2}{3}$ and $e_d = -\frac{1}{3}$ are the electric charges of u- and d-quark, respectively. The hadron created at space-time point x is annihilated by the current $\eta(0)$ at space-time point 0. As one is interested in the process $N\gamma \to N$, the current η must have the same quantum numbers as the nucleon, e.g., spin, isospin, charge or flavor. The correlation function then encodes the electromagnetic properties of the nucleon. Naively, it seems to be sufficient to calculate $T^\mu(p, q)$, after choosing a suitable current η, to access these properties.

However, this simple picture is not adequate. The main problem lies in the fact that it is impossible to construct a current that *only* creates or annihilates a single hadronic state. All states with the quantum numbers of the current have in principle non-vanishing overlap and can be created. So instead of a single particle the correlator T^μ describes a superposition of all transitions $N\gamma \to H$, where the hadrons H include the wanted ground state (the nucleon) as well as all resonances and the continuum. Disentangling the various contributions and separating the wanted nucleon from the "noise" (resonances and continuum) is the main task of the calculation.

Although there is no perfect current η, the choice of the current affects the quality of the LCSR prediction. In the literature one can find three standard choices: the Ioffe current [101], the Dosch current [102] and the isospin-improved Chernyak-Zhitnitsky current [64, 103]. Since the Ioffe current is known to yield the best results, we use it exclusively. For the proton it is given by

$$\eta_I(x) = \epsilon^{ijk} \left(u^i(x) \mathcal{C} \gamma^\nu u^j(x) \right) \gamma_5 \gamma_\nu d^k(x) , \tag{5.3}$$

where i, j, k are color indices and $\mathcal{C}$ is the charge conjugation matrix. The corresponding current for the neutron can be obtained using isospin relations. It is equal to (5.3) up to the exchange $d \leftrightarrow u$ and an overall minus sign.

Correlators as Sums over Hadrons

The first necessary step in order to deal with the unwanted contributions to the correlator is to explicitly introduce *all* hadronic states in (5.1). This can be achieved using the unitarity relation [104]

$$\mathbb{1} = \int \frac{\mathrm{d}^3 k}{2(2\pi)^3 k_0} \sum_{n,s} |n(k,s)\rangle\langle n(k,s)| , \tag{5.4}$$

where $k_0 = \sqrt{\vec{k}^2 - m_n^2}$ and n labels all possible orthonormal hadronic states with momentum k, mass m_n and spin s. After resolving the time-ordering one can insert (5.4) in between the two currents in Eq. (5.1) and take the momentum and coordinate integrations, see e.g. [105]. One obtains [104]

$$T^\mu(p,q) = \frac{\sum_s \langle 0|\eta_I(0)|N(p-q,s)\rangle\langle N(p-q,s)|j^\mu_{\mathrm{em}}(0)|N(p,s)\rangle}{m_N^2-(p-q)^2} + \\ + \text{resonances and continuum} \quad . \tag{5.5}$$

The first term corresponds to the wanted contribution of the nucleon and contains the nucleon mass m_N. The second term – nonchalantly labeled 'resonances and continuum' – is a highly complex object representing all the unwanted contributions. To cast Eq. (5.5) into a simpler form, let us introduce the coupling constant of the nucleon state to the Ioffe current λ_1 as

$$\frac{\lambda_1}{(2\pi)^2} N(p,s) := \langle 0|\eta_I(0)|N(p,s)\rangle \ , \tag{5.6}$$

where $N(p,s)$ is the nucleon spinor, and the electromagnetic Dirac and Pauli form factor, F_1 and F_2, via the matrix element

$$\langle N(p-q)|j^\mu_{\mathrm{em}}(0)|N(p)\rangle = \bar N(p-q)\left[\gamma^\mu F_1(Q^2) - i\frac{\sigma^{\mu\nu}q_\nu}{m_N}F_2(Q^2)\right]N(p) \ . \tag{5.7}$$

The form factors parameterize the most general Lorentz covariant form of the matrix element. Using the spin summation formula $\sum_s N(p-q,s)\bar N(p-q,s) = \not p - \not q + m_N$ one gets

$$T^\mu(p,q) = \frac{\lambda_1}{(2\pi)^2}\frac{(\not p - \not q + m_N)\left[\gamma^\mu F_1(Q^2) - i\frac{\sigma^{\mu\nu}q_\nu}{m_N}F_2(Q^2)\right]}{m_N^2-(p-q)^2}\cdot N(p) \\ + \text{resonances and continuum} \quad . \tag{5.8}$$

$F_1(Q^2)$ and $F_2(Q^2)$ are the quantities of interest, as they encode the electric and magnetic properties of the nucleon. By partitioning $T^\mu(p,q)$ into contributions of independent Lorentz structures the knowledge of the correlation function can be translated directly into knowledge of the form factors (if one can remove the continuum part).

The Correlator in QCD

Now that we expressed the correlation function via hadronic degrees of freedom, the next step is to express it in quark degrees of freedom. That is, to calculate it in the framework of QCD. To this end one inserts the explicit expressions for the currents j^μ and η_I and uses Wick's Theorem.

In leading order in QCD, i.e. without including additional gluons, we can only contract one quark-antiquark pair to form a single propagator

$$\overbrace{q^a(x)\bar q^b}(0) = \int \frac{d^4p}{(2\pi)^4 i} e^{-ipx}\delta^{ab}\frac{\not p + m}{m_q^2 - p^2} \overset{m_q\to 0}{\longrightarrow} \frac{i\not x\delta^{ab}}{2\pi^2x^4} \tag{5.9}$$

and one is left, up to Lorentz structures, with a matrix element of three quarks between vacuum and nucleon state: a distribution amplitude. Therefore, the QCD expansion of the correlator schematically has the form:

$$T^\mu(p,q) = i\int \frac{\mathrm{d}^4k}{(2\pi)^4 i}\int \mathrm{d}^4x\, e^{-i(q+k)}\frac{k^\rho}{k^2}\Gamma^{\rho\mu}_{\alpha\beta\gamma}\cdot\langle 0|u_\alpha u_\beta d_\gamma|N(p)\rangle\ , \tag{5.10}$$

where $\Gamma^{\rho\mu}_{\alpha\beta\gamma}$ is some Lorentz structure with ρ,μ being Lorentz and α,β,γ spinor indices. Note that while we only give a somewhat fuzzy expression for the QCD calculation, this part can be calculated with standard methods of perturbation theory using the general decomposition of the three quark matrix element, see App. E and [67].

Matching the two Representations

It is obvious that by simply equating Eq. (5.10) and Eq. (5.8) we do not gain much, as one cannot distinguish which terms obtained on quark level correspond to the ground state in (5.8) and which to the continuum. One idea would be to work close to $p'^2 = (p-q)^2 = m_N^2$, where it is clear that the nucleon contribution dominates. However, in this region the picture of free moving quarks also cannot be a good approximation for an (almost on-shell) hadron. To solve this conflict one can use the analytic properties of the correlation function (5.10) with respect to the variable $(p-q)^2$. Let us promote $s := (p-q)^2$ to a complex variable. $T^\mu(p,q)$ is then a holomorphic[1] function in the complex s-plane everywhere except for the positive real axis, where the poles (physical states) are located. Cauchy's Theorem then tells us that

$$\frac{1}{2\pi i}\oint_C \mathrm{d}s\, T^\mu(s) = 0 \tag{5.11}$$

for any closed path C that does not enclose any pole. Therefore, the integral along the path C shown in the left panel of Fig. 5.2 must vanish identically. Hence the integration along the two paths in the right panel of Fig. 5.2, C_2 and C_3, gives the same result

$$\int\limits_{C_2} \mathrm{d}s\, T^\mu(s) = \int\limits_{C_3} \mathrm{d}s\, T^\mu(s)\ . \tag{5.12}$$

Pushing the contour C_2 closer to the cut (indicated in red in the figure) has two effects: firstly, the integral over the small semi-circle goes to zero and we can neglect it; secondly, the path above and below the branch cut can be combined using Schwarz's Reflection Principle [106]

$$T(s+i\epsilon) - T(s-i\epsilon) \overset{\epsilon\to 0}{=} 2i\,\mathrm{Im}\, T(s)\ .$$

[1] To be precise, as T^μ is a combination of several Dirac structures, we should consider the coefficient functions of independent structures separately. These functions are then holomorphic in $s=(p-q)^2$. For simplicity we adopt this slightly sloppy language and refer to $T^\mu(p,q)$ itself as analytic.

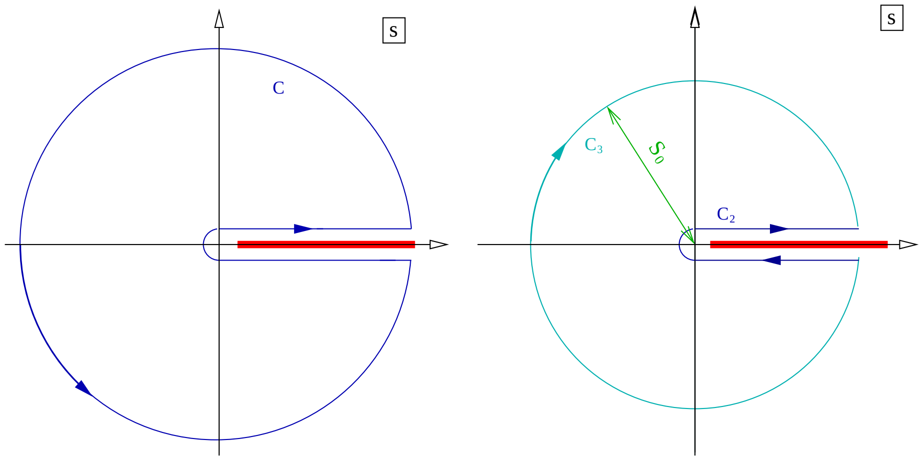


Figure 5.2: The complex $s = p'^2$ plane. The left panel shows the integration contour C. The integral along C vanishes due to Cauchy's Theorem as no pole is enclosed (the poles are located along the positive real axis and are indicated in red). In the right panel one can see the two different integration paths, C_2 in dark blue and C_3 in cyan. C_2 runs in a distance ϵ from the cut, while C_3 is a circle around the origin with radius S_0.

It follows that:

$$\frac{1}{2\pi i} \oint\limits_{|s|=S_0} \mathrm{d}s\, T(s) = \int_0^{S_0} \mathrm{d}s\, \frac{1}{\pi} \mathrm{Im}\, T(s)\ . \qquad (5.13)$$

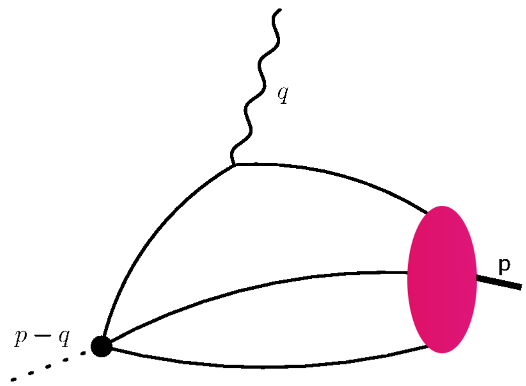
This equation relates the imaginary part of the correlation function on the branch cut, where the physical states are located, to an integral over a circle in the distance S_0 from the origin. If S_0 is much larger than Λ^2_{QCD} our QCD expansion will be valid everywhere on the circle except for the small region close to the cut. By choosing

$$m_N^2 < S_0 \lessapprox m_{\mathrm{Res}}^2\ ,$$

where m_{Res} is the mass of the lowest lying resonance, we can hope to pick up only the contribution of the ground state.

The statement that the integral over the imaginary part of $T(s)$ calculated in QCD reproduces the contribution of the hadronic states goes under the name of *quark-hadron duality*: the sum over the hadronic states reproduces free quark motion. Note that quark-hadron duality is *not* a local property. The QCD calculation does not give the exact shape of the complicated spectrum of nucleon resonances, which oscillates around the QCD result. These oscillations only drop out if averaged over a large interval. Fig. 5.3 shows an impressive example for this: the total cross section '$e^+ + e^- \to$ hadrons' normalized to the total cross section '$e^+ + e^- \to \mu^+ + \mu^-$'. The QCD result for this quantity is given by the straight, red lines. The agreement with the measured data is excellent for invariant masses larger than 1.5 GeV, even though the oscillations around the perturbative prediction are clearly visible. Below $\sqrt{s} = 1.5$ GeV the extrapolation of the theory result stays flat, whereas the actual data features a prominent resonance peak. An approximate pointwise correspondence of experiment and theory, local duality, is obviously not realized. However, the integrals over the measured spectrum and over the extrapolated theory prediction (from 0 to 1.5 GeV) are in very good agreement with each other.

Note that introducing the threshold S_0 is in fact a very intuitive procedure. Consider the diagram below, which represents the correlation function (5.1) on quark level.



The current η, represented by the dotted line, injects three quarks into the vacuum. One of the quarks interacts with a photon transferring the momentum q and recombines with the other two quarks to form a nucleon with momentum p. The red "blob" represents the distribution amplitude, which gives the propability for this recombination to happen. By restricting $s = (p-q)^2$ to values below S_0 the possible overlap of the injected quarks with states heavier than $\sqrt{S_0}$ is strongly suppressed as the states are relatively far off-shell (with exception of the wanted ground state). The same diagram can also help us understand how the nonperturbative soft contributions are naturally included in the sum rule approach. As the momentum $p'^2 = (p-q)^2$ is bounded from above by S_0, the large momentum transfer Q^2 must flow from the photon vertex along the quark line to the distribution amplitude. This quark then carries a very large momentum and therefore its momentum fraction (compared to the whole nucleon momentum p) is close to one, which is characteristic for soft contributions.

Restricting the integral over $s = p'^2$ to values below S_0 on the QCD side is then approximately equivalent to dropping the terms corresponding to the

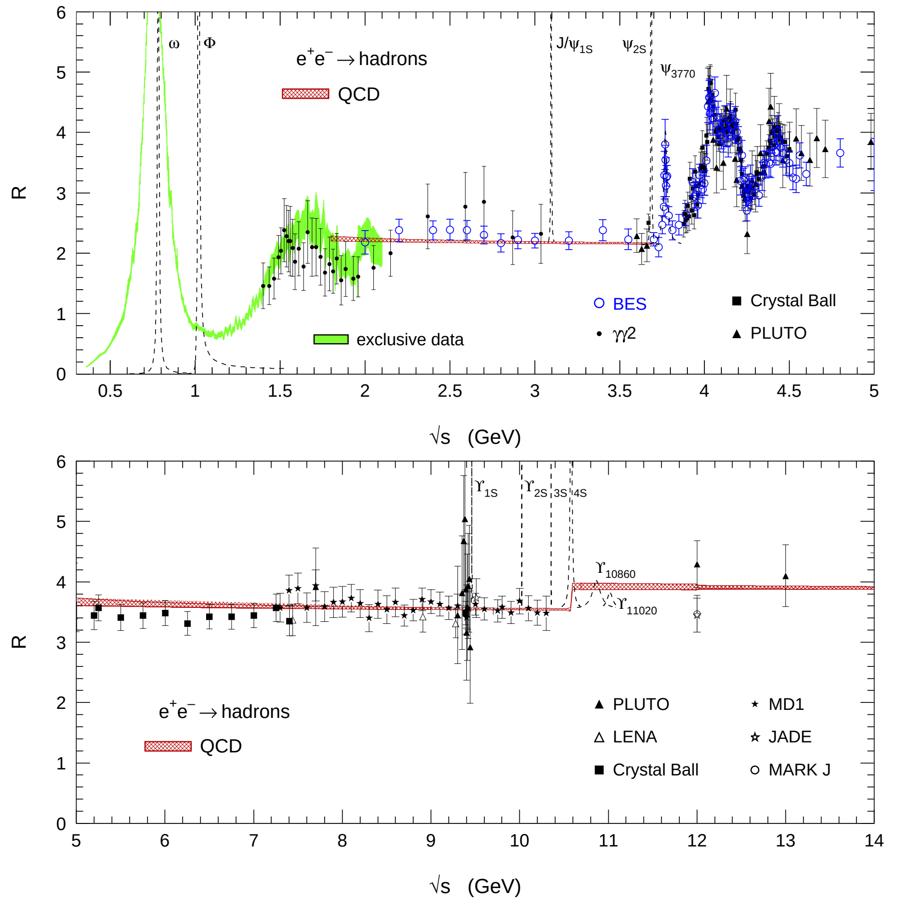


Figure 5.3: The cross section of $e^+ + e^- \to$ hadrons normalized to $e^+ + e^- \to \mu^+ + \mu^-$. The experimental data oscillates around the QCD prediction for low values of s and agrees very well for large values. The integral from 0 to 1.5 GeV of the extrapolated QCD result agrees with the same integral over the data – an example of (global) quark-hadron duality. For a single value of s theory and experiment can differ by more than a factor 2, which shows that pointwise or local quark-hadron duality is not realized. The figure is taken from [107].

resonances and the continuum in Eq. 5.5. Schematically:

$$\frac{1}{\pi}\int_0^{S_0} \mathrm{d}s\, \mathrm{Im}\, T^{\mu}_{\mathrm{QCD}} \approx \frac{\lambda_1}{(2\pi)^2} \frac{(\not{p}-\not{q}+m_N)\left[\gamma^\mu F_1(Q^2) - i\frac{\sigma^{\mu\nu}q_\nu}{m_N}F_2(Q^2)\right]}{m_N^2-(p-q)^2} \cdot N(P)\,. \quad (5.14)$$

This procedure is called *continuum subtraction* and is one of the central ideas in SVZ and light-cone sum rules.

It is possible to use a mathematical trick to further improve the approximate relation 5.14. In Eq. 5.8 the ground state contribution features a factor $1/(m_N^2 - p'^2)$, the lowest lying resonance has a factor $1/(m_{\mathrm{Res}}^2 - p'^2)$ and so on. The excited states are mildly suppressed by their higher mass, but the suppression is much too weak to make their contributions irrelevant. Making use of a Borel transformation defined via the Borel operator

$$\mathcal{B}_M := \lim_{\substack{p'^2,n\to\infty \\ p'^2/n=M^2}} \frac{1}{n!}(-p'^2)^{n+1}\frac{\partial^n}{(\partial p'^2)^n}\,, \quad (5.15)$$

where M is the so-called Borel parameter, it is possible to convert this suppression by a factor $\propto 1/(m_{\mathrm{Res}}^2 - p'^2)$ into an exponential suppression as

$$\mathcal{B}_M\left[1/(m_{\mathrm{Res}}^2 - p'^2)\right] = e^{-\frac{m_{\mathrm{Res}}^2}{M^2}}\,. \quad (5.16)$$

Applying $\mathcal{B}_M$ to both sides of Eq. 5.14 gives the final LCSR. Note that the equation holds for each independent Lorentz structure separately and one can choose structures that allow direct access to the form factors $F_1(Q^2)$ and $F_2(Q^2)$.

Let us briefly recapitulate the essential idea of LCSRs for baryon form factors:

- The central object is a correlation function. At least one of the participating hadrons is represented by a suitable interpolating current.
- The correlation function is expressed in two different ways: as a sum over hadron states (this representation includes the form factors) and via a QCD calculation on quark level; the nonperturbative long distance effects are absorbed into distribution amplitudes of increasing twist.
- Both representations can be related to each other using analyticity of the correlation function.
- The unwanted contributions of higher states can be removed by a continuum subtraction combined with a Borel transformation.
- By choosing specific Lorentz structures one can obtain sum rules for each form factor.

5.2 The $N\gamma \to N^\star$ Helicity Amplitudes

The nucleons are not the only light baryons whose quantum numbers can be realized by a combination of three u or d quarks. Today more than 20 excited states [108] have been classified and the study of these resonances is one of the main reseach areas of nuclear and low energy particle physics. While this provides a possible source of information on the "long and short-range interaction in the domain of quark confinement" [109] and a precision test for QCD in the low energy regime, electroexcitation of nucleon resonances also serves as a window to the inner structure of the nucleon itself.

For small momentum transfers $Q^2 < 3$ GeV^2 constituent quark models [110, 111, 112, 113, 114] tend to describe the data for the resonance and transition form factors measured in experiments at e.g. Jefferson Laboratory or at the Mainzer Mikrotron (MAMI) rather well. However, already at $Q^2 = 4$ GeV^2 large deviations from experiment occur. In view of the future 12 GeV upgrade of the Continuous Electron Beam Accelerator Facility (CEBAF), it is necessary to have theory predictions for momentum transfers Q^2 up to 12 GeV^2 [115], which is, coincidently, the rule of thumb for the range of applicability of the light-cone sum rule approach.

The three prime candidates among the various resonances are the $\Delta(1232)$, the Roper resonance ($P_{11}(1440)$) and the $N^\star$ or $S_{11}(1535)$. While the $N\gamma \to \Delta$ transition has been studied extensively in various models and approaches [116] – including LCSR [96, 117] – the $N^\star$, the parity partner of the nucleon[2], is much more elusive. "(Semi-) phenomenological approaches" [118] are generally based on Generalized Parton Distributions (GPDs) which are extracted by matching a phenomenological model to existing experimental data. Not only does this introduce some dependence on the model at hand, but it is also difficult to avoid a dangerous double counting of soft and hard contributions.

As light-cone sum rules avoid the ambiguities due to double counting and cover the region of momentum transfer up to 12 GeV^2, they seem to be the perfect candidate to examine the $N\gamma \to N^\star$ transition. However, as we will see shortly, sum rules suffer from other problems when applied to resonances.

5.2.1 Form Factors and Conventions

From the theoretical point of view, the electroproduction of the $N^\star$ is determined by the electromagnetic transition matrix element $\langle N^\star(p')|j^\nu_{\mathrm{em}}|N(p)\rangle$, where the electromagnetic vector current j^ν_{em} is sandwiched between a nucleon state $|N\rangle$ with momentum p and a $\langle N^\star|$ state with momentum $p' = p + q$, where q is the momentum transferred by the current. It can be parameterized in terms of two

[2] The $N^\star$ has the same quantum numbers as the nucleon, i.e. spin, isospin, charge, ..., but opposite parity and $m_N = 939$ GeV whereas $m_{N^\star} = 1535$ GeV [108].

form factors:

$$\langle N^\star(p')|j^\nu_{\rm em}|N(p)\rangle = \bar{N}^\star(p')\gamma_5 \left(\frac{G_1(Q^2)}{m_N^2}(\not{q}q^\nu - q^2\gamma^\nu) - i\frac{G_2(Q^2)}{m_N}\sigma^{\nu\mu}q_\mu\right) N(p) \,, \quad (5.17)$$

where $N^\star(p')$ and $N(p)$ are the $N^\star$ and the nucleon spinor, $Q^2 = -q^2$. $G_1(Q^2)$, $G_2(Q^2)$ are, respectively, the non-spin-flip and spin-flip form factors. This definition is very convenient, as the functional form of the parameterization closely resembles the standard definition of the matrix element for elastic nucleon-photon scattering, cp. (5.7),

$$\langle N(p')|j^\nu_{\rm em}|N(p)\rangle = \bar{N}(p+q)\left[\gamma^\nu F_1(Q^2) + i\frac{\sigma^{\nu\rho}q_\rho}{2m_P}F_2(Q^2)\right] N(p) \,, \quad (5.18)$$

where F_1 is the Dirac and F_2 the Pauli form factor.

However, in experimental particle physics a different definition is usually employed [109][3]:

$$\begin{aligned}&\langle N(p')|j^\nu_{\rm em}|N^*(p)\rangle = \\ &= \bar{N}^*(p')\left[(q^2\gamma^\nu - \not{q}q^\nu)\tilde{G}_1(Q^2) + (P^\star\cdot q\,\gamma^\nu - \not{q}P^{\star\nu})\tilde{G}_2(Q^2)\right]\gamma_5 N(p), \quad (5.19)\end{aligned}$$

where $P^\star = \frac{1}{2}(p+p')$ and $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\nu, \gamma_\mu]$. The form factors $\tilde{G}_1$ and $\tilde{G}_2$ are directly related to the so-called helicity amplitudes:

$$\mathcal{A}_{1/2}(Q^2) = +\left[2Q^2\tilde{G}_1(Q^2) - (m^2_{N^\star} - m^2_N)\tilde{G}_2(Q^2)\right] b \quad (5.20)$$

$$\mathcal{S}_{1/2}(Q^2) = -\left[2(m_{N^\star} - m_N)\tilde{G}_1(Q^2) + (m_{N^\star} + m_N)\tilde{G}_2(Q^2)\right] b\cdot\frac{|\vec{q}\,|}{\sqrt{2}} \quad (5.21)$$

with $b = e\sqrt{\frac{2m_{N^\star}(E+m_N)^2}{8m_N(m^2_{N^\star}-m^2_N)}}$, E being the nucleon energy, $\vec{q}$ the photon 3-momentum in the $N^\star$ rest frame and e the elementary charge. These helicity amplitudes can be obtained by measuring the total $N^\star(1535)$ cross section $\sigma_R(Q^2)$ at the resonance mass $m_{N^\star}$, e.g.

$$A_{1/2}(Q^2) = \sqrt{\frac{m_{N^\star}\Gamma_{N^\star}}{2m_N b_\eta}\sigma_R(Q^2)} \,, \quad (5.22)$$

where $\Gamma_{N^\star}$ is the width of the resonance, m_N the nucleon mass and $b_\eta \approx 0.55$ is the $N^\star \to \eta N$ branching fraction.

Using Gordon's Identity for axial currents

$$(m_{N^\star} - m_N)\overline{N}^\star(p')\gamma_\nu\gamma_5 N(p) = \overline{N}^\star(p')\left[(p'+p)_\nu + i\sigma_{\nu\rho}q^\rho\right] N(p) \quad (5.23)$$

[3]Note that in [109] the definition of the γ_5 matrix differs by a sign from our definition.

one can relate the "tilded" and "untilded" form factors. The helicity amplitudes can be rewritten as [109]

$$\mathcal{A}_{1/2} = e\sqrt{\frac{Q^2 + (m_{N^\star} - m_N)^2}{2m_N^5(m_{N^\star}^2 - m_N^2)}} \left[Q^2 G_1(Q^2) + m_n(m_{N^\star} - m_N)G_2(Q^2)\right] \quad (5.24)$$

$$\mathcal{S}_{1/2} = e\sqrt{\frac{Q^2 + (m_{N^\star} - m_N)^2}{4m_N^5(m_{N^\star}^2 - m_N^2)}} C\left[(m_N - m_{N^\star})G_1(Q^2) + m_N G_2(Q^2)\right] \; , \quad (5.25)$$

where $C = \sqrt{1 + \frac{(Q^2 - m_{N^\star}^2 - m_N^2)^2}{2Q^2 m_{N^\star}^2}}$ is a kinematic factor.

5.2.2 LCSR for $\gamma N \to N^\star$ with Nucleon DAs?

Following [117], where the transition $N\gamma \to \Delta$ was investigated, we start with the correlation function

$$\Pi^\nu = i \int \mathrm{d}^4x \; e^{-iqx} \langle 0|\eta_{N^\star}(0) j^\nu_{\mathrm{em}}(x)|N(p)\rangle \; , \quad (5.26)$$

where $\eta_{N^\star}$ represents a suitable current for the $N^\star$ resonance. The natural choice for $\eta_{N^\star}$ is a Ioffe-like current [101], which is adjusted for the different parity by adding (or removing) one γ_5 matrix; the parity operator $\mathcal{P} = i\gamma_0$ anticommutes with γ_5 [37]. The current for the $N^{\star,+}$ then has the form

$$\eta_{N^\star}(x) := \epsilon^{ijk} \left(u^a(x)\mathcal{C}\gamma_\mu u^j(x)\right) \gamma^\mu d^k(x) \quad (5.27)$$

where i, j, k are color indices, $\mathcal{C}$ is the charge conjugation matrix and ϵ^{ijk} is the totally antisymmetric tensor, which is necessary to obtain an $SU(3)_{\mathrm{color}}$ singlet. The processes $p\gamma \to N^{\star,+}$ and $n\gamma \to N^{\star,0}$ are connected by isospin [117]; therefore, we only have to consider $p\gamma \to N^{\star,+}$.

However, it turns out that our argument that adding an additional γ_5 changes a current creating nucleon into a current creating a $N^\star$ is inherently flawed. This can be seen in two ways:

A relativistic spinor is not an eigenstate of the parity operator $\mathcal{P}$ – this is only true in the non-relativistic limit [37] – and therefore parity is not a good quantum number for a current that creates baryons with arbitrary momentum p. As a consequence $\eta_{N^\star}$ must have non-vanishing overlap with the nucleon state. After the standard Borel transformation this contribution is enhanced by a factor $e^{m_{N^\star}^2/m_N^2} \approx 15$ and is likely to dominate the correlation function, thus spoiling any predictions for the $N\gamma \to N^\star$ transition.

For more descriptive argument one can consider the current-current correlation function

$$i \int \mathrm{d}^4x \; e^{-ipx} \langle 0|\eta_{N^\star}(0)\overline{\eta}_{N^\star}(x)|0\rangle \quad (5.28)$$

which has to be evaluated to determine the coupling λ_{N^*} of the current to the $N^\star$ state. The perturbative QCD contribution to the correlator is straightforward to obtain by applying Wick's theorem to Eq. (5.28). Up to maybe a sign it must have the exactly same form as the corresponding contribution to the nucleon-nucleon correlator, because the two additional γ_5 matrices will eventually meet and vanish due to $\gamma_5^2 = \mathbb{1}$. Hence, the leading term in the OPE has "forgotten" about the parity of the original currents and the OPE will automatically contain contributions due to the (much lighter) nucleon and Roper states. This was already stated in [119] where the obtained quark level results for the SVZ sum rules for $N^\star$ distribution amplitudes were identical to the result for the nucleon. However, in [119] the consequence that the nucleon contribution is therefore automatically included was not reached. Instead a higher continuum threshold S_0 is suggested to accommodate for the higher $N^\star$ mass. Obviously, this approach is wrong, as it corresponds to summing over several resonances and the result cannot represent the $N^\star$ alone.

We see that it is not feasible to use the ansatz of Eq. (5.26) to gain information on $\gamma p \to N^{\star,+}$. In fact, there seems to be no current that can be used to generate an $N^\star$ in the light-cone sum rules approach, as it is difficult to separate parity partners. The only option left is to use the standard Ioffe current $\eta_I(x)$ for the nucleon, see Eq. (5.3), and keep the $N^\star$ as an on-shell state:

$$\Pi^\nu = i \int \mathrm{d}^4x \, e^{-iqx} \langle 0|\eta(0) j^\nu_{\mathrm{em}}(x)|N^\star(P)\rangle \, . \tag{5.29}$$

However, this approach requires the distribution amplitudes for the $N^\star$, i.e. light-ray matrix elements ($n^2 = 0$) of the form

$$\langle 0|u_\alpha(a_1 n) u_\beta(a_2 n) d_\gamma(a_3 n)|N^\star(P)\rangle \, ,$$

which then again cannot be determined using the SVZ sum rules. The $N^\star$ DAs must be provided by a different method.

Recently a detailed lattice QCD analysis of nucleon wave functions was performed by the QCDSF collaboration[4] [74]. As a byproduct of this calculation information on the $N^\star$ [120, 118] can be obtained. While parity still has to be used to separate nucleon and $N^\star$, lattice QCD can work in the low momentum regime, where parity is an "almost-good" quantum number and the contributions of wrong parity are expected to be of the order of 5% [118] – well below the statistical uncertainties. However, before one can use lattice QCD to calculate the DAs, they first have to be defined in a proper way.

[4] The details of the approach can also be found in Nikolaus Wakentin's PhD thesis [73].

5.2.3 The $N^\star$ Distribution Amplitudes

In [65] an involved analysis of the distribution amplitudes of the nucleon was performed. It was shown that the most general decomposition of the leading Fock state matrix element is given by

$$4\langle 0|\epsilon_{ijk} u_\alpha^{i'}(a_1 x)[a_1 x, a_0 x]_{i'i} u_\beta^{j'}(a_2 x)[a_2 x, a_0 x]_{j'j} d_\gamma^{k'}(a_3 x)[a_3 x, a_0 x]_{k'k}|N(p)\rangle = \\ = \sum_{i=1}^{24} (\Gamma_3)^i_{\alpha\beta}(\Gamma_4^i)_{\gamma\rho} N_\rho(p)\; F_i(a_1, a_2, a_3)\,, \tag{5.30}$$

where $\Gamma_{3/4}$ are 24 independent Dirac structures and the F_i are functions of the light-ray coordinates a_1, a_2, a_3, which can be expanded in terms of local operators, see App. E for details. The path-ordered exponentials assure gauge invariance of the matrix elements; in the following, we do not show them explicitly even though their presence is always implied. The 24 functions F_i can be related using isospin symmetry, identity of two quark flavors and Fierz identities. Only 8 independent distribution amplitudes remain, one leading twist-3 DA (4.5), three twist-4 DAs (4.8), three twist-5 DAs and one twist-6 DA.

As a first attempt we define the general $N^\star$ three-quark matrix element as

$$4\langle 0|\epsilon_{ijk} u_\alpha^i(a_1 x) u_\beta^j(a_2 x) d_\gamma^k(a_3 x)|N^\star(p)\rangle = \\ = \sum_i (\Gamma_3)^i_{\alpha\beta} \left(\Gamma_4^i \gamma_5\right)_{\gamma\rho} N_\rho(p)\; \widetilde{F}_i(a_1, a_2, a_3)\,, \tag{5.31}$$

i.e. we add one additional γ_5 matrix (compared to the definition of the nucleon matrix element). This will generate different relations between the $\widetilde{F}_i$ compared to the F_i as the Fierz transformations are different for the modified Lorentz structures. Thus, the choice (5.31) requires a rederivation of all relations given in [65]. Therefore, we take a different path.

Our second option is the modification of the left-hand side of Eq. (5.30) by including an additional γ_5:

$$4\langle 0|\epsilon_{ijk} u_\alpha^i(a_1 x) u_\beta^j(a_2 x) \left(\gamma_5 d^k(a_3 x)\right)_\gamma |N^\star(p)\rangle = \\ = \sum_i (\Gamma_3)^i_{\alpha\beta} \left(\Gamma_4^i\right)_{\gamma\rho} N_\rho(p)\; \approx{F}_i(a_1, a_2, a_3)\,. \tag{5.32}$$

Obviously, this preserves all relations of the $\stackrel{\approx}{F}_i$ due to Fierz identities as we leave the right-hand side untouched. However, by adding the γ_5 matrix to only one quark, one gives it a somewhat special status. This upsets the isospin relations and forces us to reexamine those given in [65] – again additional work we want to avoid.

This leads us to a third possible definition, which not only allows us to keep all relations derived in [65], but also has additional merits to be discussed shortly. The idea is the following: we do need to change the parity of (5.30), which can be achieved by adding an uneven number of γ_5 matrices. At the same time, we want to keep all symmetries (isospin,...). The easiest way to achieve this, while leaving the r.h.s. of (5.30) as it is, is to introduce three additional γ_5 in the left-hand side. In this case each quark is adorned with one γ_5 and the isospin relations do not change. We define this three-quark matrix element as

$$
\begin{aligned}
&4\langle 0|\epsilon_{ijk}\left(\gamma_5 u^i\right)_\alpha (a_1 x)\left(\gamma_5 u^j\right)_\beta (a_2 x)\left(\gamma_5 d^k\right)_\gamma (a_3 x)|N^\star(P)\rangle = \\
&= \sum_i (\Gamma_3)^i_{\alpha\beta}\left(\Gamma^i_4\right)_\gamma^{\ \rho} N^\star{}_\rho(P)\, F_i(a_1,a_2,a_3) = \\
&= \quad \mathcal{S}_1 m_{N^\star} C_{\alpha\beta}(\gamma_5 N^\star)_\gamma + \mathcal{S}_2 m^2_{N^\star} C_{\alpha\beta}\left(\not{x}\gamma_5 N^\star\right)_\gamma \\
&\quad + \mathcal{P}_1 m_{N^\star}(\gamma_5 C)_{\alpha\beta} N^\star_\gamma + \mathcal{P}_2 m^2_{N^\star}(\gamma_5 C)_{\alpha\beta}\left(\not{x}N^\star\right)_\gamma \\
&\quad + \left(\mathcal{V}_1 + \frac{x^2 m^2_{N^\star}}{4}\mathcal{V}_1^M\right)(\not{P}C)_{\alpha\beta}(\gamma_5 N^\star)_\gamma + \mathcal{V}_2 m_{N^\star}\left(\not{P}C\right)_{\alpha\beta}\left(\not{x}\gamma_5 N^\star\right)_\gamma \\
&\quad + \mathcal{V}_3 m_{N^\star}(\gamma_\mu C)_{\alpha\beta}(\gamma^\mu\gamma_5 N^\star)_\gamma + \mathcal{V}_4 m^2_{N^\star}(\not{x}C)_{\alpha\beta}(\gamma_5 N^\star)_\gamma \\
&\quad + \mathcal{V}_5 m^2_{N^\star}(\gamma_\mu C)_{\alpha\beta}(i\sigma^{\mu\nu}x_\nu\gamma_5 N^\star)_\gamma + \mathcal{V}_6 m^3_{N^\star}(\not{x}C)_{\alpha\beta}(\not{x}\gamma_5 N^\star)_\gamma \\
&\quad + \left(\mathcal{A}_1 + \frac{x^2 m^2_{N^\star}}{4}\mathcal{A}_1^M\right)(\not{P}\gamma_5 C)_{\alpha\beta} N^\star_\gamma + \mathcal{A}_2 m_{N^\star}(\not{P}\gamma_5 C)_{\alpha\beta}(\not{x}N^\star)_\gamma \\
&\quad + \mathcal{A}_3 m_{N^\star}(\gamma_\mu\gamma_5 C)_{\alpha\beta}(\gamma^\mu N^\star)_\gamma + \mathcal{A}_4 m^2_{N^\star}(\not{x}\gamma_5 C)_{\alpha\beta} N^\star_\gamma \\
&\quad + \mathcal{A}_5 m^2_{N^\star}(\gamma_\mu\gamma_5 C)_{\alpha\beta}(i\sigma^{\mu\nu}x_\nu N^\star)_\gamma + \mathcal{A}_6 m^3_{N^\star}(\not{x}\gamma_5 C)_{\alpha\beta}(\not{x}N^\star)_\gamma \\
&\quad + \left(\mathcal{T}_1 + \frac{x^2 m^2_{N^\star}}{4}\mathcal{T}_1^M\right)(P^\nu i\sigma_{\mu\nu}C)_{\alpha\beta}(\gamma^\mu\gamma_5 N^\star)_\gamma \\
&\quad + \mathcal{T}_2 m_{N^\star}(x^\mu P^\nu i\sigma_{\mu\nu}C)_{\alpha\beta}(\gamma_5 N^\star)_\gamma + \mathcal{T}_3 m_{N^\star}(\sigma_{\mu\nu}C)_{\alpha\beta}(\sigma^{\mu\nu}\gamma_5 N^\star)_\gamma \\
&\quad + \mathcal{T}_4 m_{N^\star}(P^\nu\sigma_{\mu\nu}C)_{\alpha\beta}(\sigma^{\mu\rho}x_\rho\gamma_5 N^\star)_\gamma + \mathcal{T}_5 m^2_{N^\star}(x^\nu i\sigma_{\mu\nu}C)_{\alpha\beta}(\gamma^\mu\gamma_5 N^\star)_\gamma \\
&\quad + \mathcal{T}_6 m^2_{N^\star}(x^\mu P^\nu i\sigma_{\mu\nu}C)_{\alpha\beta}(\not{x}\gamma_5 N^\star)_\gamma + \mathcal{T}_7 m^2_{N^\star}(\sigma_{\mu\nu}C)_{\alpha\beta}(\sigma^{\mu\nu}\not{x}\gamma_5 N^\star)_\gamma \\
&\quad + \mathcal{T}_8 m^3_{N^\star}(x^\nu\sigma_{\mu\nu}C)_{\alpha\beta}(\sigma^{\mu\rho}x_\rho\gamma_5 N^\star)_\gamma\,, \qquad (5.33)
\end{aligned}
$$

where we inserted the explicit expressions for the Lorentz structures Γ_3^i and Γ_4^i. The 24 functions $\mathcal{S}_i$, $\mathcal{P}_i$, $\mathcal{V}_i$, $\mathcal{A}_i$ and $\mathcal{T}_i$ depend only on the scalar product $P \cdot z$ and the light-ray coordinates a_i, $i = 1, 2, 3$.

The functions fulfill the same relations, see App. E, as their namesakes for the nucleon distribution amplitudes in [65]. Up to next-to-leading order in conformal spin their functional form depends on only 8 independent parameters: $f_{N^\star}$, $\lambda_1^\star$, $\lambda_2^\star$, $A_1^{u\star}$, $V_1^{d\star}$, $f_2^{d\star}$, $f_1^{d\star}$ and $f_1^{u\star}$ in the notation[5] of [65].

Let us define the leading twist distribution amplitude $\Phi_3^\star$ via

$$
\begin{aligned}
\langle 0|\epsilon^{ijk}\left(u_i^\uparrow(a_1 n)\,\mathcal{C}\not{n}u_j^\downarrow(a_2 n)\right)\not{n}d_k^\uparrow(a_3 n)|N^\star(P)\rangle &= \\
&= \frac{1}{2} f_{N^*}\, pn\not{n}\, N^{\star\uparrow}(P)\int \mathcal{D}x\, e^{-ipn\sum x_i a_i}\,\Phi_3^\star(x_i)\,, \qquad (5.34)
\end{aligned}
$$

[5] We added an asterisk to avoid confusion with the corresponding parameters of the NDAs.

	N	$N^\star(1535)$
$f_N \cdot 10^3[\text{GeV}^2]$	3.234(63)(86)	4.544(117)(223)
$-\lambda_1 \cdot 10^3[\text{GeV}^2]$	35.57(65)(136)	37.55(101)(768)
$\lambda_2 \cdot 10^3[\text{GeV}^2]$	70.02(128)(268)	191.9(44)(391)
φ^{100}	0.3999(37)(139)	0.4765(33)(155)
φ^{010}	0.2986(11)(52)	0.2523(20)(32)
φ^{001}	0.3015(32)(106)	0.2712(41)(136)
A_1^u	0.1013(61)(223)	0.2242 (62)(258)
V_1^d	0.3015 (32)(106)	0.2712 (41)(136)

Table 5.1: Lattice results for the leading conformal spin parameters $\lambda_1^{(\star)}$, $\lambda_2^{(\star)}$ and $f_N^{(\star)}$ for nucleon and $N^\star$ as well as for the first moments φ^{lmk} of the twist-3 distribution amplitude (see text for their definition). The results were obtained by the QCDSF collaboration [120, 118]. The numbers for A_1^u and V_1^d follow from Eq. (5.38). All values correspond to a scale of $\mu = 1$ GeV.

where $P^2 = m_{N^\star}^2$. Note that we included an additional sign compared to the definition of the nucleon DA Φ_3 in [65] (Eq. (2.26) therein). This is necessary to keep the analogy $f_N \leftrightarrow f_{N^\star}$ as the sign cancels the one arising due to commuting the additional γ_5 matrices on the left-hand side of Eq. (5.33). We can use the standard definitions for the normalization constants $f_{N^\star}$ and f_N

$$\langle 0|\epsilon^{ijk}(u_i C\not{n} u_j)(0)\gamma_5 \not{n} d_k(0)|N(P)\rangle = f_N pn\not{n}\, N(P) \tag{5.35}$$

$$\langle 0|\epsilon^{ijk}(u_i C\not{n} u_j)(0)\gamma_5 \not{n} d_k(0)|N^\star(P)\rangle = f_{N^\star} pn\gamma_5\not{n}\, N^\star(P). \tag{5.36}$$

These two constants can be calculated directly on the lattice. Furthermore, it is possible to evaluate the moments

$$\varphi^{lmn} = \int \mathcal{D}x\;\, x_1^l x_2^m x_3^n\, \Phi^{(\star)}(x_i) \tag{5.37}$$

of the distribution amplitudes for $l+m+n \le 2$, see [74]. φ^{000} is equal to 1 by definition and the other results are shown in Table 5.1. These moments are related to the parameters $A_1^{u\star}$ and $V_1^{d\star}$ by [121]

$$A_1^{u\star} = 2\varphi^{100} + \varphi^{001} - 1\,, \tag{5.38}$$

$$V_1^{d\star} = \varphi^{001}\,. \tag{5.39}$$

We could define the twist-4 distribution amplitudes in analogy to the definitions in [65], but it is more convenient to make use of the result for the Wandzura-Wilczek contributions (4.168) and separate genuine twist-4 and WW parts already in the definition.

$$\langle 0|\epsilon^{ijk}\Big(u_i^\uparrow(a_1 n)\mathcal{C}\not{n} u_j^\downarrow(a_2 n)\Big)\not{p} d_k^\uparrow(a_3 n)|N^\star(P)\rangle =$$

$$= \frac{1}{4} pn\, \not{p}\, N^{\star\uparrow}(P) \int \mathcal{D}x\, e^{-ipn\sum x_i a_i} \left[f_{N^\star} \Phi_4^{N^\star,WW}(x_i) + \lambda_1^\star \Phi_4^{N^\star}(x_i) \right], \quad (5.40)$$

$$\langle 0|\epsilon^{ijk} \left(u_i^\uparrow(a_1 n) \mathcal{C} \not{n} \gamma_\perp \not{p} u_j^\downarrow(a_2 n) \right) \gamma^\perp \not{n} d_k^\uparrow(a_3 n) |N^\star(P)\rangle =$$

$$= -\frac{1}{2} pn\, \not{n}\, m_{N^\star} N^{\star\uparrow}(P) \int \mathcal{D}x\, e^{-ipn\sum x_i a_i} \left[f_{N^\star} \Psi_4^{N^\star,WW}(x_i) - \lambda_1^\star \Psi_4^{N^\star}(x_i) \right], \quad (5.41)$$

$$\langle 0|\epsilon^{ijk} \left(u_i^\uparrow(a_1 n) \mathcal{C} \not{p} \not{n} u_j^\uparrow(a_2 n) \right) \not{n} d_k^\uparrow(a_3 n) |N^\star(P)\rangle =$$

$$= \frac{\lambda_2^\star}{12} pn \not{n}\, m_{N^\star} N^{\star\uparrow}(P) \int \mathcal{D}x\, e^{-ipn\sum x_i a_i}\, \Xi_4^{N^\star}(x_i), \quad (5.42)$$

where the subscript $\perp$ indicates the two-dimensional plane perpendicular to the light-cone. The normalization constants $\lambda_1^\star$, $\lambda_2^\star$ and $f_{N^\star}$ were assigned using Eqs. (4.170), (4.168) and (4.177). $\lambda_1^\star$ and $\lambda_2^\star$ can be calculated on the lattice [120, 118] via the matrix elements

$$\langle 0|\epsilon^{ijk} (u_i \mathcal{C} \gamma_\mu u_j)(0) \gamma_5 \gamma^\mu d_k(0) |N^\star(P)\rangle = \lambda_1^\star m_{N^\star} \gamma_5 N^\star(P) \quad (5.43)$$

$$\langle 0|\epsilon^{ijk} (u_i \mathcal{C} \sigma_{\mu\nu} u_j)(0) \gamma_5 \sigma^{\mu\nu} d_k(0) |N^\star\rangle(P)\rangle = \lambda_2^\star m_{N^\star} \gamma_5 N^\star(P). \quad (5.44)$$

There is currently no reliable lattice data for the higher moments of the twist-4 distribution amplitudes and we have to work with the twist-4 normalization factors alone.

However, it is still useful to restore the WW contributions to the parameters $f_1^{d\star}$ and $f_1^{u\star}$ using (4.177). With the values for $f_{N^\star}$ and $\lambda_1^\star$ in Table 5.1 we obtain

$$f_1^{d\star} = \frac{3}{10} - \frac{1}{6} \frac{f_{N^\star}}{\lambda_1^\star} \approx 0.320\,,$$
$$f_1^{u\star} = \frac{1}{10} - \frac{1}{6} \frac{f_{N^\star}}{\lambda_1^\star} \approx 0.120\,. \quad (5.45)$$

To conclude this section, one can say that the synthesis of lattice-determined matrix elements, RGE-analysis of the Wandzura-Wilczek contributions and the existing distribution amplitudes for the nucleon enabled us to achieve the first-time determination of the $N^\star$ distribution amplitudes.

5.2.4 LCSRs for the $N\gamma \to N^\star$ Transition

With the $N^\star$ distribution amplitudes established, everything is in place to calculate the LCSRs for the $N\gamma \to N^\star$ transition and determine the Q^2 dependence of the form factors G_1 and G_2. However, it turns out that it is not necessary to repeat the whole analysis which was performed for $N\gamma \to N$, for the $N^\star$ case. In fact, it is possible to restore the result for the form factors G_1 and G_2 from the already known sum rules for F_1 and F_2.

First observe that we can cast the left-hand side of the three quark matrix element (5.33) into the same form as the left-hand side of (5.30) by multiplying with $(\gamma_5)_{\alpha'\alpha}(\gamma_5)_{\beta'\beta}(\gamma_5)_{\gamma'\gamma}$.

$$
\begin{aligned}
&4\langle 0|\epsilon_{ijk}\left(u^i\right)_\alpha(a_1x)\left(u^j\right)_\beta(a_2x)\left(d^k\right)_\gamma(a_3x)|N^\star(P)\rangle = \\
&= -\left(\mathcal{V}_1+\frac{x^2m_{N^\star}^2}{4}\mathcal{V}_1^M\right)(\not{P}C)_{\alpha\beta}\left(\gamma_5\gamma_5N^\star\right)_\gamma - \mathcal{V}_2m_{N^\star}(\not{P}C)_{\alpha\beta}\left(\gamma_5\not{x}\gamma_5N^\star\right)_\gamma \\
&\quad -\mathcal{V}_3m_{N^\star}(\gamma_\mu C)_{\alpha\beta}\left(\gamma_5\gamma^\mu\gamma_5N^\star\right)_\gamma - \mathcal{V}_4m_{N^\star}^2(\not{x}C)_{\alpha\beta}\left(\gamma_5\gamma_5N^\star\right)_\gamma \\
&\quad -\mathcal{V}_5m_{N^\star}^2(\gamma_\mu C)_{\alpha\beta}\left(i\gamma_5\sigma^{\mu\nu}x_\nu\gamma_5N^\star\right)_\gamma - \mathcal{V}_6m_{N^\star}^3(\not{x}C)_{\alpha\beta}\left(\gamma_5\not{x}\gamma_5N^\star\right)_\gamma \\
&\quad -\left(\mathcal{A}_1+\frac{x^2m_{N^\star}^2}{4}\mathcal{A}_1^M\right)(\not{P}\gamma_5C)_{\alpha\beta}\left(\gamma_5N^\star\right)_\gamma - \mathcal{A}_2m_{N^\star}(\not{P}\gamma_5C)_{\alpha\beta}\left(\gamma_5\not{x}N^\star\right)_\gamma \\
&\quad -\mathcal{A}_3m_{N^\star}(\gamma_\mu\gamma_5C)_{\alpha\beta}\left(\gamma_5\gamma^\mu N^\star\right)_\gamma - \mathcal{A}_4m_{N^\star}^2(\not{x}\gamma_5C)_{\alpha\beta}\left(\gamma_5N^\star\right)_\gamma \\
&\quad -\mathcal{A}_5m_{N^\star}^2(\gamma_\mu\gamma_5C)_{\alpha\beta}\left(i\gamma_5\sigma^{\mu\nu}x_\nu N^\star\right)_\gamma - \mathcal{A}_6m_{N^\star}^3(\not{x}\gamma_5C)_{\alpha\beta}\left(\gamma_5\not{x}N^\star\right)_\gamma \\
&\quad +\ldots\ ,
\end{aligned}
\tag{5.46}
$$

where we do not explicitly display the scalar, pseudo-scalar and tensor structures. The reason for this will become clear shortly. All changes on the right-hand side are indicated by red color and the additional γ_5 attached to the Lorentz structure with the spinor is always kept to the very left side of the Dirac string even if it can be removed by using $\gamma_5^2 = \mathbb{1}$. Then, up to an overall factor $(-\gamma_5)$ and the replacement $m_N \leftrightarrow m_{N^\star}$, Eq. (5.46) is identical to the parameterization of the nucleon matrix element $\langle 0|u_\alpha(a_1)u_\beta(a_2)d_\gamma(a_3)|N(P)\rangle$. This is the first hint that one may be able to reuse the existing calculations [67, 104] for the $N\gamma \to N$ form factors to obtain the $N\gamma \to N^\star$ form factors.

However, the two transition matrix elements, see Eqs. (5.17) & (5.18),

$$
\langle N^\star(P')|j_{\rm em}^\nu|N(P)\rangle = \bar{N}^\star(P')\gamma_5\left(\frac{G_1(Q^2)}{m_N^2}(\not{q}q_\nu - q^2\gamma_\nu) - \frac{iG_2(Q^2)}{m_N}\sigma_{\nu\mu}q^\mu\right)N(P)
$$

and

$$
\langle N(P')|j_{\rm em}^\nu|N(P)\rangle = \bar{N}(P')\left[\gamma^\nu F_1(Q^2) + i\frac{\sigma^{\nu\rho}q_\rho}{2m_P}F_2(Q^2)\right]N(P)
$$

do not have the same Lorentz structures. There is no term $\not{q}q^\nu$ multiplying the Dirac form factor F_1, and one has to make use of a standard trick to remove the unwanted Lorentz structures: we introduce an auxiliary vector z^μ with $z^2 = 0$, $q\cdot z = 0$ and saturate all free Lorentz indices in the correlation function $\Pi^\nu(P,q)$, see (5.29), with z. That is, we consider

$$
z_\nu\Pi^\nu = i\int \mathrm{d}^4x\, e^{-iqx}\langle 0|\eta(0)\ \ z\cdot j_{\rm em}(x)\ \ |N^\star(P)\rangle\ . \tag{5.47}
$$

This procedure[6] removes the spurious $\not{q}q_\nu$ from the transition matrix element

[6] In [67, 104] the very same strategy was employed to reduce the number of independent Lorentz structures. Therefore, if one wants to keep the "symmetry" with the LCSR calculation for the electromagnetic form factors of the nucleon, the projection with z^ν has to be introduced anyway and the removal of all terms proportional to $\not{q}q^\nu$ is a welcome side effect.

as

$$z_\nu \langle N^\star(P')|j^\nu_{\rm em}|N(P)\rangle = \bar{N}^\star(P')\gamma_5 \left(\frac{Q^2 G_1(Q^2)}{m_N^2} \not{z} - i \frac{G_2(Q^2)}{m_N} \sigma_{\nu\mu} z^\nu q^\mu \right) N(P) \, . \tag{5.48}$$

This expression is very similar to the matrix element $z_\nu \langle N(P')|j^\nu_{\rm em}|N(P)\rangle$, if one performs the identifications

$$\frac{Q^2 G_1(Q^2)}{m_N^2} \longleftrightarrow F_1(Q^2) \qquad -2G_2(Q^2) \longleftrightarrow F_2(Q^2) \, . \tag{5.49}$$

The only remaining obvious difference is the appearance of the $N^\star$ spinor instead of a nucleon spinor and the additional γ_5 matrix in Eq. (5.48).

We can now formulate a simple strategy to acquire the light-cone sum rules for the form factors $G_1(Q^2)$ and $G_2(Q^2)$. The basis is the careful calculation of the nucleon form factors in [67] and [104], which is, to the best of our knowledge, the de facto handbook for LCSR calculations involving baryon distribution amplitudes.

We advocate the following course of action:

- We follow through the calculation of [67], but include the γ_5 matrix in the distribution amplitudes, see Eq. (5.46), and in the transition matrix element, cf. (5.48).
- Furthermore, one must replace all nucleon masses **in the quark level calculation** of the correlation function by $m_{N^\star}$.
- In both representations of the correlator, via hadronic states and form factors or via free quarks and distribution amplitudes, the additional γ_5 matrix can always be commuted to the very left using the relation

  $$\gamma_\mu \gamma_5 = -\gamma_5 \gamma_\mu \, ,$$

  i.e. each commutation amounts to one additional minus sign.
- Once the γ_5 has reached the left-most position we can drop it without consequences[7]; the only remnants of the $N^\star$ DAs are signs and the replacement $m_N \to m_{N^\star}$ in the light-cone expansion.
- The overall sign received by each individual term is given by

  $$(-1)^{(\#\,\text{of commutations})+1} \, ,$$

  where the "+1" stems from the overall sign in Eq. (5.46). We see that our well-considered definition of the $N^\star$ DAs turns the determination of the sum rules into a mere counting exercise.

[7]Both representations feature a γ_5 matrix, once it reaches its "final destination" it can be treated as a common factor.

- As the final sum rules for the nucleon form factors do not receive contributions due to scalar, pseudo-scalar and tensor DAs, the $N^\star$ form factors do neither. This is ultimately the reason why we did not include these terms explicitly in Eq. (5.46).
- Finally, one replaces the form factors F_1 and F_2 according to (5.49) to obtain the LCSR expressions for $G_1(Q^2)$ and $G_2(Q^2)$.

One sees that the LCSRs for G_1 and G_2 follow, virtually without any serious calculation, from already existing sum rules. This is a feature of our specific choice of the DAs and a different definition, like Eq. (5.31) or Eq. (5.32), does not permit such a shortcut and would have forced us to essentially repeat the derivations of [65] and [67] from the scratch.

The LCSRs for the form factors – pre Borel transformation and continuum subtraction – have the form

$$\frac{2\lambda_1 Q^2 G_1(Q^2)}{m_N^2(m_n^2 - P^2)} = 4\frac{m_{N^*}}{m_N}\left\{ e_d \int_0^1 \mathrm{d}x_3 \left[\frac{\rho_{\mathcal{A}}^{d,1}}{q_3^2} + \frac{\rho_{\mathcal{A}}^{d,2}}{q_3^4}\right] + e_u \int_0^1 \mathrm{d}x_2 \left[\frac{\rho_{\mathcal{A}}^{u,1}}{q_2^2} + \frac{\rho_{\mathcal{A}}^{u,2}}{q_2^4}\right]\right\} \tag{5.50}$$

$$\frac{2\lambda_1 G_2(Q^2)}{m_N^2 - P^2} = 4\left\{ e_d \int_0^1 \mathrm{d}x_3 \left[\frac{\rho_{\mathcal{B}}^{d,1}}{q_3^2} + \frac{\rho_{\mathcal{B}}^{d,2}}{q_3^4}\right] + e_u \int_0^1 \mathrm{d}x_2 \left[\frac{\rho_{\mathcal{B}}^{u,1}}{q_2^2} + \frac{\rho_{\mathcal{B}}^{u,2}}{q_2^4}\right]\right\} \tag{5.51}$$

with $q_i = q - x_i P$ and

$$\rho_{\mathcal{A}}^{d,1} = \widetilde{\widetilde{\mathcal{V}}}_2^{(3)} + x_3 \widetilde{\mathcal{V}}_3^{(3)} \tag{5.52}$$

$$\rho_{\mathcal{A}}^{d,2} = 2m_{N^*}^2 x_3^2\, \widetilde{\widetilde{\mathcal{V}}}_5^{(3)} + Q^2\, \widetilde{\widetilde{\mathcal{V}}}_2^{(3)} \tag{5.53}$$

$$\rho_{\mathcal{A}}^{u,1} = \widetilde{\widetilde{\mathcal{A}}}_2^{(2)} - \widetilde{\widetilde{\mathcal{V}}}_2^{(2)} + x_2(\widetilde{\mathcal{A}}_3^{(2)} - \widetilde{\mathcal{V}}_1^{(2)} + 3\widetilde{\mathcal{V}}_3^{(2)}) \tag{5.54}$$

$$\begin{aligned}\rho_{\mathcal{A}}^{u,2} =& Q^2(\widetilde{\widetilde{\mathcal{A}}}_2^{(2)} + \widetilde{\widetilde{\mathcal{V}}}_2^{(2)}) - m_{N^\star}^2 x_2(\mathcal{V}_1^{M(u)} + 4\,\widetilde{\widetilde{\mathcal{V}}}_6^{(2)}) \\ & - 2m_{N^\star}^2 x_2^2(\widetilde{\widetilde{\mathcal{A}}}_5^{(2)} + \widetilde{\widetilde{\mathcal{V}}}_4^{(2)} - 2\,\widetilde{\widetilde{\mathcal{V}}}_5^{(2)})\end{aligned} \tag{5.55}$$

$$\rho_{\mathcal{B}}^{d,1} = -\frac{1}{2}\widetilde{\mathcal{V}}_1^{(3)} \tag{5.56}$$

$$\rho_{\mathcal{B}}^{d,2} = m_{N^\star}^2 x_3(\widetilde{\widetilde{\mathcal{V}}}_2^{(3)} - 2\,\widetilde{\widetilde{\mathcal{V}}}_5^{(3)}) - \frac{1}{2}m_{N^\star}^2 \mathcal{V}_1^{M(d)} \tag{5.57}$$

$$\rho_{\mathcal{B}}^{u,1} = \frac{1}{2}\left(\widetilde{\mathcal{A}}_1^{(2)} + \widetilde{\mathcal{V}}_1^{(2)}\right) \tag{5.58}$$

$$\begin{aligned}\rho_{\mathcal{B}}^{u,2} =& m_{N^\star}^2 x_2\left(\widetilde{\widetilde{\mathcal{A}}}_2^{(2)} + 2\,\widetilde{\widetilde{\mathcal{A}}}_5^{(2)} + \widetilde{\widetilde{\mathcal{V}}}_2^{(2)} + 2\,\widetilde{\widetilde{\mathcal{V}}}_4^{(2)} - 4\,\widetilde{\widetilde{\mathcal{V}}}_5^{(2)}\right) \\ & + \frac{1}{2}m_{N^\star}^2\left(\mathcal{A}_1^{M(u)} + \mathcal{V}_1^{M(u)}\right)\,.\end{aligned} \tag{5.59}$$

The calligraphic functions are defined in App. E. The Borel transformation and subsequent continuum subtraction can be performed using the substitution

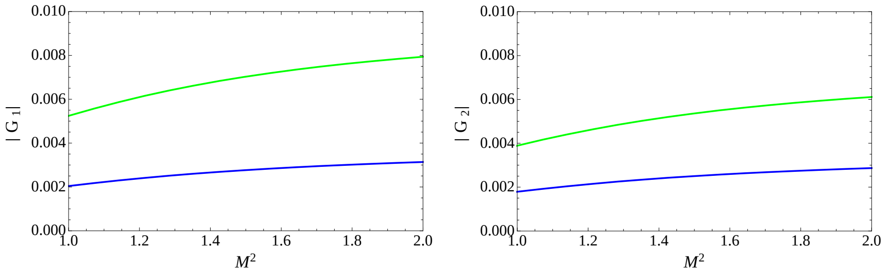


Figure 5.4: The dependence of the form factors G_1 and G_2 on the Borel parameter M in the Borel window. The left panel shows G_1, the right panel G_2 for two different values for Q^2 each ($Q^2 = 5$ GeV^2 in green, $Q^2 = 7$ GeV^2 in blue).

rules [104, 67]

$$\int_0^1 \mathrm{d}x \, \frac{\rho(x)}{(q-xP)^2} \quad \to \quad - \int_{x_0}^1 \mathrm{d}x \, \rho(x) \exp\left(-\frac{\bar{x}\left(\frac{Q^2}{x}+M^2\right)}{M^2}\right) \tag{5.60}$$

$$\int_0^1 \mathrm{d}x \, \frac{\rho(x)}{(q-xP)^4} \quad \to \quad - \frac{\rho(x_0)}{Q^2+x_0M^2} \exp\left(-\frac{s_0}{M^2}\right) + \frac{1}{M^2}\int_{x_0}^1 \mathrm{d}x \, \frac{\rho(x)}{x^2} \exp\left(-\frac{\bar{x}\left(\frac{Q^2}{x}+M^2\right)}{M^2}\right) \tag{5.61}$$

$$\frac{1}{m_N^2-P^2} \quad \to \quad \exp\left(-\frac{m_N^2}{M^2}\right) , \tag{5.62}$$

where

$$x_0 = \frac{1}{2M^2}\left(M^2-Q^2-S_0+\sqrt{(M^2-Q^2-S_0)^2+4M^2Q^2}\right) . \tag{5.63}$$

M is the Borel parameter and S_0 the continuum threshold.

5.2.5 Numerical Results

For the numerical evaluation of the sum rules for the form factors G_1 and G_2 we need a value for the continuum threshold S_0 in the nucleon channel and the Borel parameter. All other parameters can be taken from Table 5.1. The standard choice of the duality interval S_0 for the Ioffe current is $S_0 = (1.5\text{ GeV})^2$ [101, 105]. For the Borel parameter we choose the middle of the Borel Window

$$1\text{ GeV}^2 < M^2 < 2\text{ GeV}^2 ,$$

that is $M^2 = 1.5$ GeV^2. The dependence on this choice is only moderate as can be seen in Fig. 5.4. The plots for the form factors G_1 and G_2 are shown in Fig. 5.5. We varied the parameters in the distribution amplitudes within the

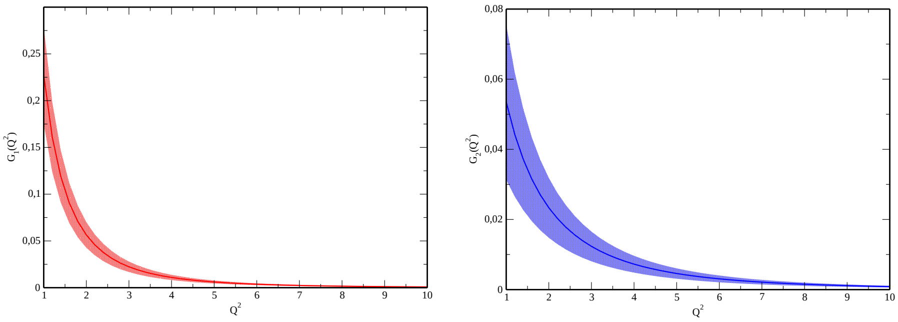


Figure 5.5: LCSR predictions for the Q^2 dependence of the from factors G_1 (left panel) and G_2 (right panel). The solid lines correspond to the result obtained using the central values for the nonperturbative parameters. The shaded bands show the uncertainties.

ranges given in Table 5.1 and added the effects in quadrature for each value of Q^2 to obtain the error bands. These uncertainties are larger than the variation of the sum rules in the Borel window, which gives an estimate of the intrinsic uncertainties of the sum rule. Hence, we only show the error band due to lattice uncertainties.

Both form factors exhibit a strong Q^2 dependence. The rapid growth for small values of Q^2 is an artifact; it corresponds to the breakdown of the light-cone expansion in powers of $m^2_{N^\star}/Q^2$, which is no longer valid for $Q^2 < 2$ GeV^2. The strong suppression for large values of Q^2, however, is typical for electromagnetic form factors and can also be observed in the form factors of the $N \to N\gamma$ and $\Delta \to N\gamma$ transitions [117, 67]. To isolate the more subtile structures in the Q^2 dependence, it is useful to remove the so-called dipole behavior by considering

$$\frac{Q^2 G_1(Q^2)}{G_{\text{dipole}}(Q^2)} \quad \text{and} \quad \frac{G_2(Q^2)}{G_{\text{dipole}}(Q^2)} \,, \tag{5.64}$$

where

$$G_{\text{dipole}}(Q^2) = \frac{1}{(1 + Q^2/\mu_0^2)} \quad \text{with} \quad \mu_0^2 = 0.71 \text{ GeV}^2 \,. \tag{5.65}$$

This dipole formula was first obtained when fitting the electromagnetic form factors of the nucleon, but it turns out that the same dipole shape is also an excellent approximation for $Q^2 G_1(Q^2)$ and $G_2(Q^2)$. This can be seen in Fig. 5.6, where an exact dipole would correspond to a constant. Fig. 5.6 also gives a better impression of the uncertainties than Fig. 5.5, as the overall kinematic factors tend to compress the error bands for large values of Q^2.

Perturbative QCD can make predictions on the behavior of $G_1(Q^2)$ for very

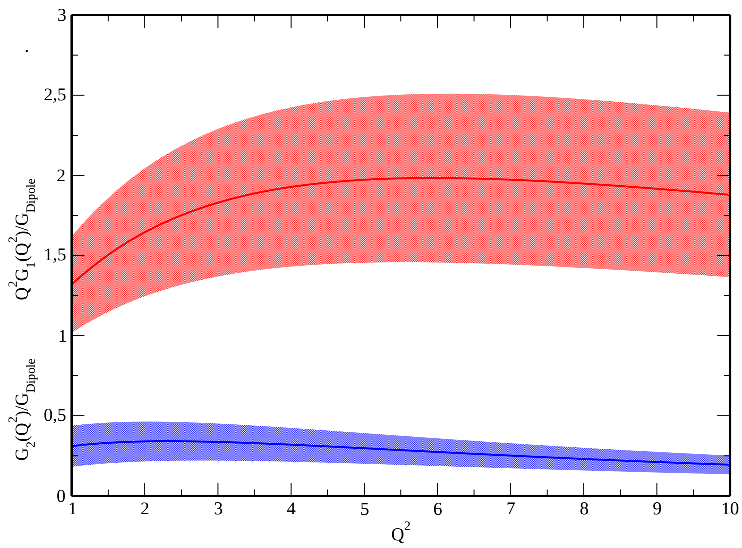


Figure 5.6: Q^2 dependence of the form factors G_2 and $Q^2 G_1$ beyond the leading dipole behavior. The solid lines correspond to the sum rule results, the shaded band to the uncertainties. $Q^2 G_1(Q^2)/G_{\rm dipole}$ is shown in red, $G_2(Q^2)/G_{\rm dipole}$ in blue.

large values of Q^2. The asymptotic form is given by [122]

$$G_1(Q^2) \propto \frac{1}{(Q^2)^3} \, .$$

As we do not take radiative corrections to the sum rules into account, they will not be able to reproduce this behavior for $Q^2 \to \infty$. However, in the region of intermediate momentum transfer, which is relevant for comparisons with experiment, the soft contributions are expected to dominate. Fitting the central values of the LCSR prediction for G_1 in the interval 5 GeV^2 $< Q^2 <$ 10 GeV^2 with the function $\frac{C_1}{(Q^2)^n}$ yields $n = 3.1$, which is rather close to what pQCD predicts for very large Q^2.

To compare our results with experiment, it is necessary to consider the helicity amplitudes $\mathcal{A}_{1/2}$ and $\mathcal{S}_{1/2}$. They can be obtained from the form factors via the relations (5.20). One sees immediately that $\mathcal{A}_{1/2}$ is dominated by G_1, which is enhanced by a factor

$$\frac{Q^2}{m_N(m_{N^\star} - m_N)} \approx 1.8 \text{ GeV}^{-2} \cdot Q^2$$

compared to the contribution proportional to G_2. The situation is reversed for $\mathcal{S}_{1/2}$. Here the form factor G_1 is suppressed relative to the G_2 contribution by a factor

$$\frac{(m_N - m_{N^\star})}{m_N} \approx -0.64 \, .$$

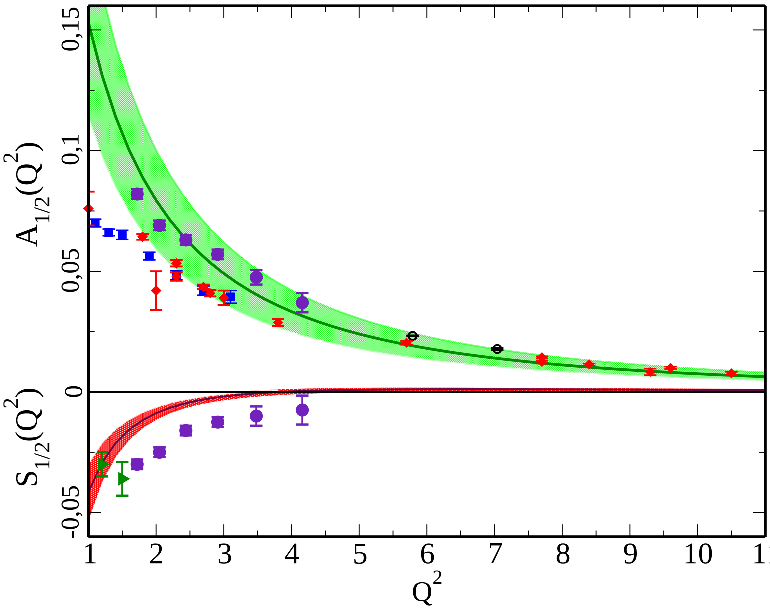


Figure 5.7: The LCSR predictions for the helicity amplitudes $\mathcal{A}_{1/2}$ (in green) and $\mathcal{S}_{1/2}$ (in red). The uncertainties are indicated by the shaded areas. The blue data points are taken from [131], the red diamonds from [132], the green triangles from [133] and the black circles are from [129]. The most recent data, indicated by purple discs, is from [130]. The plot is taken from [118].

Thus, $\mathcal{S}_{1/2}$ is given as the difference of two sum rules (for different Lorentz structures) that are (numerically) almost equal and large cancelations are expected to arise. In such situations LCSRs are known to be very unreliable. The reason for this is the following: The uncertainties due the nonperturbative parameters usually compensate each other. That is, a small increase of e.g. $\lambda_1^\star$ affects both sum rules, the one for G_1 as well as the one G_2, similarly and $\mathcal{S}_{1/2}$, the difference of the two sum rules, is rather stable with respect to the change in $\lambda_1^\star$. However, the intrinsic uncertainties of the LCSR approach for the two sum rules are virtually independent and tend to destabilize the final result. An example for such large cancelations can be found in the light-cone sum rule determination of the quantity R_{EM}, the ratio of the electric quadrupole amplitude to the magnetic dipole amplitude of the $\Delta \to N\gamma$ transition. The sum rule predictions range from $R_{EM} = 0.2$ to $R_{EM} = -0.15$ [123, 96, 117], whereas experiments [124, 125, 126, 127, 128] yield $-0.01 > R_{EM} > -0.05$ for values of Q^2 up to 4 GeV^2. Therefore, our result for $\mathcal{S}_{1/2}$ has to be taken with great caution.

Figure 5.7 shows both $\mathcal{A}_{1/2}$ and $\mathcal{S}_{1/2}$ as a function of Q^2. The error bands again only take the lattice uncertainties into account. The sum rule for $\mathcal{A}_{1/2}$ is expected to be only weakly affected by sum rule instabilities and exhibits an excellent agreement with the most recent JLab data (open black circles [129] and purple filled circles [130]) for $Q^2 > 2$ GeV^2. On the other hand, the LCSR result for $\mathcal{S}_{1/2}$ shows a deviation from experimental data and does not seem to favor a specific sign for $\mathcal{S}_{1/2}$: below $Q^2 = 4$ GeV^2 $\mathcal{S}_{1/2}$ is negative, above 4 GeV^2 positive. This peculiar behavior is probably a consequence of large cancelations spoiling the quality of the sum rule. However, in defense of our sum rule one should note that the experimental data only shows the pure statistical errors and does not take into account any systematic uncertainties due to the challenging task of separating events of different parity. Note that, while the general shape of the helicity amplitudes is predominately generated by the kinematic prefactors in (5.20), the overall normalization depends only on the LCSR result and the good agreement with experiment in the case of $\mathcal{A}_{1/2}$ is a first success of our synthetic strategy. This is especially true as there is currently no other method that can describe the form factors at large momentum transfers without relying on phenomenological input.

Naturally, there is still vast room for improvements of the technique from the sum rule side. The inclusion of α_s corrections to the sum rules can be expected to increase precision and stability, see [134] for a first step in this direction. Furthermore, the study of resonances requires a more stringent treatment of power corrections proportional to the resonance mass. These corrections play a more prominent role than in the nucleon case. This additional effort is necessary in order to keep up with the expected improvements on the lattice side (due to e.g. larger lattices or smaller pion masses); these can only bear fruit if LCSR uncertainties do not become the limiting factor.

5.3 Summary

The aim of this chapter was to present new possibilities for the study of form factors of excited light baryons in the light-cone sum rule formalism. After sketching the general concepts of LCSRs, we focused on the case of the electromagnetic form factors of the $N\gamma \to N^\star$ transition, which is one of the hot topics to be studied with the new CLAS12 detector. We found that even though the $N^\star$ is not a nucleon resonance per se – it has a different parity – one cannot define a simple interpolating current that does not have finite overlap with the nucleon state. As the nucleon is much lighter than the $N^\star$, the Borel transformation, a central tool of the sum rule method, enhances the nucleon contribution considerably. This makes it impossible to avoid a contamination of the sum rules.

A possible way out of this dilemma is using $N^\star$ instead of nucleon distribution amplitudes and generating the nucleon part of the transition via a standard interpolating current. As one cannot employ SVZ sum rules to gain insight into these DAs – this again requires a current for the $N^\star$ – we had to use the results of a lattice QCD calculation of $N^\star$ matrix elements. This necessitated the definition of the novel $N^\star$ distribution amplitudes. While there was a certain freedom in choosing this definition, we found that there exists one specific choice, which preserves all relations derived for the nucleon DAs.

This turned out to be a boon, as it allowed us to find a simple way to relate the sum rules for the electromagnetic form factors of the nucleon to the sum rules for the $N\gamma \to N^\star$ transition form factors. While our results for the Q^2 dependence of the form factors $G_1(Q^2)$ and $G_2(Q^2)$ still feature rather large uncertainties, which predominantly stem from the lattice, our result for the helicity amplitude $\mathcal{A}_{1/2}$ shows (within errors) a very good agreement with the experimental data.

あしたのことをいうと天井のねずみが笑う。

– *Japanese Proverb*

Prediction is very difficult, especially about the future.

– *Niels Bohr*

6 Conclusion

Summary

The central themes of this thesis were baryon distribution amplitudes and their applications to hard exclusive processes in QCD.

Chapter 2 served as a short recapitulation of some fundamental concepts of QCD and generic quantum field theories. After introducing the Lagrangian density of QCD, we presented two important ideas: dimensional regularization and renormalization group equations.

Next, some more sophisticated tools required for our renormalization group analysis of baryon distribution amplitudes were presented. First, the spinor formalism was explained. It treats spinor and Lorentz indices on the same footing and thus simplifies the classification of generic operators with respect to their transformation properties under the Lorentz group. This can be used to project a quantum operator onto a specific light-cone twist. As we intended to make use of the conformal symmetry of the 1-loop renormalization group equations, we turned to the construction of an one-particle operator basis with good conformal properties in Chap. 3.2. Making use of the advantages of the spinor formalism and of the equations of motion, we could remove all unwanted field components from the basis. At the end of the chapter, we showed how a multi-particle operator basis of light-ray fields can be obtained.

In Chapter 4, we devised an approach for the renormalization of higher-twist baryon operators. The method is based on the approach of [61] for twist-3 baryon operators and uses the manifest conformal invariance of the RGEs; this is realized in an operator basis following our construction principles, see also Chap. 3. Starting with twist 4, quasipartonic and non-quasipartonic operators can mix under renormalization and we focused on this special case. Using the conformal symmetry of the RGEs as a starting point, we determined the general functional form of the renormalization kernels. This provided a first check of our results, for which we gave one explicit example of how they were obtained. An additional, novel check for the twist-4 2-to-3 kernels was constructed in Sect. 4.4. As it only makes use of Poincaré invariance, it is expected to hold beyond 1-loop order. The spectrum of anomalous dimensions was determined using numerical methods and the Schrödinger equation-like structure of the RGEs. It turned out that the chiral three-quark sector is integrable and we were able to give an explicit expression for the conserved charge. Our results for the multiplicatively renormalizable operators allowed a stringent separation of genuine twist-4 and Wandzura-Wilczek contributions for an important special case: the nucleon distribution amplitudes.

In the last chapter, we gave a sample application of baryon distribution amplitudes. To this end, the light-cone sum rule method, an approach providing a direct connection of experimentally accessable form factors and distribution amplitudes, was introduced. Anticipating the 12 GeV upgrade of CEBAF, the $N\gamma \to N^\star$ transition provided a natural first example, as this process will be studied in great detail in near future. However, applying LCSRs to processes including an excited hadron state proves problematic, as no feasible interpolating current can be found. As a possible way out, we suggested using lattice data to extract $N^\star$ distribution amplitudes which we derived in full analogy to the existing definition of the nucleon distribution amplitudes. Thus, it was possible to map the sum rules for the $N\gamma \to N^\star$ transition form factors onto the already existing sum rules for the nucleon electromagnetic form factors. Using the lattice results for the distribution amplitudes provided by the QCDSF collaboration, we could, for the first time, estimate the two helicity amplitudes in the region of intermediate momentum transfer and found a good agreement with experiment.

Main Results

Our conformal one-particle operator basis, see Eq. (3.63), constitutes one of the central novelties of this thesis. It can be used to construct a multi-particle operator basis for generic processes, each of which has well-defined transformation properties under the conformal group: the prerequisite for manifestly conformally invariant renormalization group equations.

The second main result is given in Eqs. (4.49), (4.51), (4.54), (4.57), (4.58) and (4.61), where the twist-4 evolution kernels for baryon operators of generic flavor structure are presented. These kernels allow, for the first time, the determination of the mixing of quasipartonic and non-quasipartonic operators to all orders in conformal spin.

Finally, the light-cone sum rules for the electromagnetic form factors of the $N\gamma \to N^\star$ transition, Eqs. (5.50) and (5.51), represent the first theory determination of these form factors in the region of intermediate momentum transfer as well as the first application of the new $N^\star$ distribution amplitudes.

Outlook

The analysis of renormalization kernels based on conformal symmetry and our one-particle light-ray operators basis are not limited to the case of twist-4 baryon distribution amplitudes. The next natural step is the calculation of twist-4 corrections to the structure functions in deep inelastic lepton-hadron scattering [62]. This analysis is expected to be much more involved than our presentation for twist-4 baryon operators, as the number of independent kernels is much larger. Therefore, it will be necessary to consider the relations between the kernels of different twists (see Chap. 4.4) in more detail, as they suggest some "hidden" connections; it may very well be possible to restore the twist-4 2-to-3 kernels from already known 2-to-2 kernels. Other possible applications are, of course, baryon operators of twist 5 and 6, as our method is independent of the twist considered. At the moment, there is, however, no reason to undertake such an involved calculation, as lattice QCD, the most promising option for twist-5 distribution amplitudes with next-to-next-to-leading order accuracy in conformal spin, is not expected to yield results with the necessary precision in the near future.

The synthetic lattice-LCSR approach allows for the calculation of transition form factors of hadron resonances, which are in the focus of the near future JLab experimental program. One of the next steps is the reanalysis of the $N\gamma \to \Delta$ transition using Δ distribution amplitudes. This may shed some light on the peculiar observation that LCSR feature a rather strong deviation from experiment if nucleon distribution amplitudes and a good agreement if photon distribution amplitudes are used [135]. In the next few years distribution amplitudes for the baryon octet and decouplet will become available, opening up various new possibilities for the study of form factors using LCSRs.

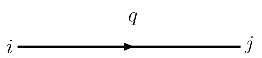
A

Feynman Rules

In this appendix we present the Feynman Rules used in the calculation of the various diagrams for the renormalization kernels. Note that we only give the rules in the light-cone gauge ($n_\mu A^\mu = 0$, $n^2 = 0$ and gauge parameter $\zeta = 0$).

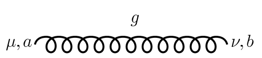
Propagators

- propagator of a quark q with mass m_q and momentum p



$$\frac{1}{i}\delta^{ab}\frac{\left[\not{p} + m_q\right]_{ij}}{m_q^2 - p^2}$$

- propagator of a gluon g with momentum k

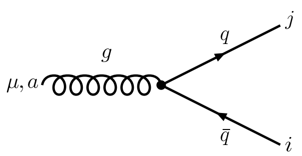


$$-i\left(g_{\mu\nu} - \frac{n_\mu k_\nu + n_\nu k_\mu}{nk}\right)\frac{\delta^{AB}}{k^2}$$

Here $a, b = 1, 2, 3$ denote the color indices of the quarks, $A, B = 1, \ldots, 8$ the color indices of the gluons and μ, ν are Lorentz indices.

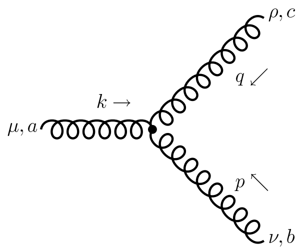
Vertices

- the quark-gluon vertex m_q and momentum p



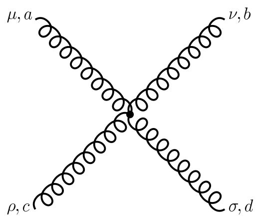
$$ig\gamma_\mu T^A_{ij}$$

- three-gluon vertex



$$gf^{ABC}\big[g_{\mu\nu}(k-p)_\rho + g_{\nu\rho}(p-q)_\mu + g_{\rho\nu}(q-k)_\mu\big]$$

- four-gluon vertex



$$-ig\Big[f^{ABE}f^{CDE}\left(g_{\nu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) + f^{ACE}f^{BDE}\left(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) + f^{ADE}f^{BCE}\left(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma}\right)\Big]$$

Here A, B, C, D, E are gluon color indices, i, j quark color indices and the Greek letters represent Lorentz indices. The arrows denote the direction of the mometa q, k, p. f^{ABC} are the structure constants of $SU(3)_{\text{color}}$.

Additional Rules for virtual particle loops

- for each internal loop we need a momentum integration

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4}$$

- each fermion loop comes with an additional factor -1 and introduces a Dirac trace

B

Calculations in Spinor Formalism

B.1 DAs – From Lorentz to Spinor Notation

We will show explicitly how to convert distribution amplitudes given in the "standard" notation to the spinor formalism. The simplest example is the twist-3 nucleon DA.

It is defined as

$$\begin{aligned}\langle 0|\epsilon^{ijk}(u_i^{\downarrow}(z_1 n)\, C \not{n} u_j^{\uparrow}(z_2 n))\, \not{n} d_k^{\downarrow}(z_3 n)|N(P)\rangle = \\ = -\frac{1}{2} f_N (pn)\, \not{n} N^{\downarrow}(p) \int \mathcal{D}x\, e^{-iPn\sum_i x_i z_i}\, \Phi_3(x_i,\mu)\,,\end{aligned} \tag{B.1}$$

compare Eq. (4.5). After replacing all Dirac matrices by the matrix representations given in (3.16), one obtains for the left-hand side:

$$\begin{aligned}&\langle 0|\epsilon^{ijk} \begin{pmatrix}\psi_\alpha^{u,i}\\ 0\end{pmatrix} \begin{pmatrix}-\epsilon_{\alpha\beta} & 0\\ 0 & -\epsilon^{\dot\alpha\dot\beta}\end{pmatrix} \begin{pmatrix}0 & n_{\beta\dot\delta}\\ \bar n^{\dot\beta\delta} & 0\end{pmatrix} \begin{pmatrix}0\\ \bar\chi_{u,i}^{\dot\delta}\end{pmatrix} \begin{pmatrix}0 & n_{\rho\dot\rho}\\ \bar n^{\dot\rho\rho} & 0\end{pmatrix} \begin{pmatrix}\psi_\rho^{d,k}\\ 0\end{pmatrix} |N(P)\rangle \\ &= \langle 0| - \epsilon^{ijk}\psi_\alpha^{u,i}\epsilon_{\alpha\beta} n_{\beta\dot\delta}\bar\chi_{u,i}^{\dot\delta} \cdot \bar n^{\dot\rho\rho}\psi_\rho^{d,k}|N(P)\rangle = \ldots\,.\end{aligned} \tag{B.2}$$

Using the properties of the ϵ tensor, cf. Eq. (3.11), we get

$$\begin{aligned}\ldots &= \langle 0|\epsilon^{ijk}\psi_{u,i}^{\beta} n_{\beta\dot\delta}\bar\chi_{u,j}^{\dot\delta}\cdot \bar n^{\dot\rho\rho}\psi_\rho^{d,k}|N(P)\rangle \overset{3.37}{=} \langle 0|\epsilon^{ijk}\psi_{u,i}^{\beta}\lambda_\beta\bar\lambda_{\dot\delta}\bar\chi_{u,j}^{\dot\delta}\cdot\bar\lambda^{\dot\rho}\lambda^\rho\psi_\rho^{d,k}|N(P)\rangle \\ &\overset{3.50}{=} \langle 0|\epsilon^{ijk}\psi_+^{u,i}\bar\chi_+^{u,j}\psi_+^{d,k}|N(P)\rangle\ \bar\lambda^{\dot\rho}\,.\end{aligned} \tag{B.3}$$

The right-hand side of Eq. (B.1) takes the form:

$$\begin{aligned}-\frac{1}{2} f_N p^\mu n_\mu \begin{pmatrix}0 & n_{\rho\dot\rho}\\ \bar n^{\dot\rho\rho} & 0\end{pmatrix} \begin{pmatrix}N_\rho(P)\\ 0\end{pmatrix} \int \mathcal{D}x\, e^{-iPn\sum_i x_i z_i}\, \Phi_3(x_i,\mu) = \\ = -\frac{1}{2} f_N (pn)\bar\lambda^{\dot\rho}\lambda^\rho N_\rho^{(\downarrow)}(P) \int \mathcal{D}x\, e^{-iPn\sum_i x_i z_i}\, \Phi_3(x_i,\mu)\,.\end{aligned} \tag{B.4}$$

Comparing left and right hand side, one finds that both exhibit the same transformation behavior. It is encoded in the spinor $\bar\lambda^{\dot\rho}$. Dropping the spinors from both sides we finally arrive at

$$\langle 0|\epsilon^{ijk}\psi_+^{u,i}(z_1 n)\bar{\chi}_+^{u,j}(z_2 n)\psi_+^{d,k}(z_3 n)|N(P)\rangle = \\ = -\frac{1}{2} f_N(pn) N_+^{(\downarrow)}(P) \int \mathcal{D}x \, e^{-iPn\sum_i x_i z_i} \, \Phi_3(x_i,\mu)\,. \tag{B.5}$$

B.2 The Amplitude D_4

In this section we will derive the relation (4.24). Starting point is the matrix element:

$$\mathbb{M} := \langle 0|\epsilon^{ijk}[0,z_1]\psi_+^{i,(u)}(z_1)[0,z_2]\psi_+^{j,(d)}(z_2)[0,z_3]D_{-+}\bar{\chi}_+^{j,(u)}(z_3)|N(p)\rangle \tag{B.6}$$

where i,j,k are color indices, the superscripts (u) and (d) indicate the quark flavor and $p \propto \tilde{n}$ is the nucleon momentum. $\mathbb{M}$ is – up to a factor (-2) and the trivial exchange $z_2 \leftrightarrow z_3$ – equal to the left-hand side of Eq. (4.24).

It is possible to replace the derivative ∂_{-+} by a generator of translations $\hat{P}_{-+}$ using

$$\partial_{-+}\bar{\chi}_+^{j,(u)}(z_3) = \frac{1}{i}\left[\hat{P}_{-+},\bar{\chi}_+^{j,(u)}\right](z_3)\,. \tag{B.7}$$

This allows us to rewrite (B.6) in the form

$$\begin{aligned}\mathbb{M} :=& \frac{1}{i}\langle 0|\left[\hat{P}_{-+},\epsilon^{ijk}[0,z_1]\psi_+^{i,(u)}(z_1)[0,z_2]\psi_+^{j,(d)}(z_2)[0,z_3]\bar{\chi}_+^{j,(u)}(z_3)\right]|N(p)\rangle \\ &+ i\langle 0|\left[\hat{P}_{-+},\epsilon^{ijk}[0,z_1]\psi_+^{i,(u)}(z_1)[0,z_2]\psi_+^{j,(d)}(z_2)[0,z_3]\right]\bar{\chi}_+^{j,(u)}(z_3)|N(p)\rangle \\ &- ig\langle 0|\epsilon^{ijk}[0,z_1]\psi_+^{i,(u)}(z_1)[0,z_2]\psi_+^{j,(d)}(z_2)[0,z_3]A_{-+}(z_3)\bar{\chi}_+^{j,(u)}(z_3)|N(p)\rangle\end{aligned} \tag{B.8}$$

The term in the first line on the right-hand side is equal to zero. The action of $\hat{P}_{-+}$ on the whole operator just gives the total momentum in a direction perpendicular to the light-cone. As the nucleon momentum p is light-like, this matrix element must vanish.

The path-ordered exponents $[0,z_i]$ contain the quantum field A and the operator $\hat{P}$ acts on this field, too. It is useful to calculate $\left[\hat{P}_\mu,[0,z]\right]$ separately as this expression appears quite frequently. Let us first recall the definition

$$[0,z] = \mathrm{Pexp}\left(-ig\int_0^1 \mathrm{d}\tau (zn)_\mu A^\mu(\tau z)\right)\,, \tag{B.9}$$

where we do not show the color structure explicitly. For our purpose one only has to keep in mind that the gluon fields A do not commute. The action of $\hat{P}_\mu$ on $[0,z]$ is then given by

$$\left[\hat{P}_\mu,[0,z]\right] = \left[\hat{P}_\mu,[0,\Delta\tau\cdot z][\Delta\tau\cdot z,2\Delta\tau\cdot z]\ldots[(N-1)\Delta\tau\cdot z,z]\right]_{\substack{\Delta\tau\cdot N=1\\ N\to\infty,\,\Delta\tau\to 0}} =$$

$$
\begin{aligned}
&= \int_0^1 \mathrm{d}\tau [0,\tau z] z n_\nu \left[\hat{P}_\mu, -igA^\nu(\tau z)\right][\tau z, z] = \\
&= g\int_0^1 \mathrm{d}\tau [0,\tau z] z n_\nu \left(\frac{\partial}{\partial(\tau z^\mu)} A^\nu(\tau z)\right)[\tau z, z] \\
&= g\int_0^1 \mathrm{d}\tau\ [0,\tau z] z n_\nu (\partial_\mu A^\nu - \partial^\nu A_\mu)(\tau z)[\tau z, z] + \\
&\quad + g\int_0^1 \mathrm{d}\tau\ [0,\tau z] \left(\frac{\partial}{\partial \tau} A^\mu(\tau z)\right)[\tau z, z] = \\
&= \quad g\int_0^1 \mathrm{d}\tau\ [0,\tau z] z_\nu (\partial_\mu A^\nu - \partial^\nu A_\mu)(\tau z)[\tau z, z] + \\
&\quad + g\int_0^1 \mathrm{d}\tau\ \left(\frac{\partial}{\partial \tau}[0,\tau z] A^\mu(\tau z)[\tau z, z]\right) \\
&\quad - g\int_0^1 \mathrm{d}\tau\ \left(\frac{\partial}{\partial \tau}[0,\tau z]\right) A^\mu(\tau z)[\tau z, z] \\
&\quad - g\int_0^1 \mathrm{d}\tau\ [0,\tau z] A^\mu(\tau z) \left(\frac{\partial}{\partial \tau}[\tau z, z]\right)
\end{aligned}
\tag{B.10}
$$

where we used integration by parts to obtain the last line. With the help of the standard formulas [33]

$$
\frac{\partial}{\partial(\tau z_\rho)}[0,\tau z] = -ig[0,\tau z]A^\rho(\tau z) \qquad \frac{\partial}{\partial(\tau z_\rho)}[\tau z, z] = igA^\rho(\tau z)[\tau z, z] \tag{B.11}
$$

for the gauge links Eq. (B.10) can be greatly simplified. After reshuffling some terms we get

$$
\begin{aligned}
\left[\hat{P}_\mu, [0,z]\right] =& g\int_0^1 \mathrm{d}\tau\ [0,\tau z] x^\nu \left[\partial_\mu A_\nu - \partial_\nu A_\nu + ig[A_\nu, A_\mu]\right](\tau z)[\tau z, z] \\
&+ g[0,z]A_\mu(z) - gA_\mu(0)[0,z] \\
=& g\int_0^1 [0,tz] z n^\nu F_{\mu\nu}(\tau z)\tau z, z] + g[0,z]A_\mu(z) - gA_\mu(0)[0,z]
\end{aligned}
\tag{B.12}
$$

With Eqs. (B.12) and (B.7) one can rewrite Eq. (B.8) such that all generators $\hat{P}$ can either be replaced with ordinary derivatives (if $\hat{P}$ acts directly on a quark field) or give rise to additional gluon fields. Further, we can now safely refrain from displaying the gauge links, since there are no quantum operators left. This leads to the somewhat lengthy expression

$$
\begin{aligned}
\mathbb{M} =& ig\int_0^1 \mathrm{d}\tau\ z_1^\nu \langle 0|\epsilon^{ijk} \left[F_{-+\nu}(\tau z_1)\psi_+^{(u)}(z_1)\right]^i \psi_+^{j,(d)}(z_2)\bar{\chi}_+^{k,(u)}(z_3)|N(p)\rangle \\
&+ ig\langle 0|\epsilon^{ijk}\left[A_{-+}\psi_+^{(u)}\right]^i(z_1)\psi_+^{j,(d)}(z_2)\bar{\chi}_+^{k,(u)}|N(p)\rangle \\
&- ig\langle 0|\epsilon^{ijk}\left[A_{-+}(0)\psi_+^{(u)}(z_1)\right]^i \psi_+^{j,(d)}(z_2)\bar{\chi}_+^{k,(u)}|N(p)\rangle \\
&+ ig\int_0^1 \mathrm{d}\tau\ z_2^\nu \langle 0|\epsilon^{ijk}\psi_+^{i,(u)}(z_1)\left[F_{-+\nu}(\tau z_2)\psi_+^{(d)}(z_2)\right]^j \bar{\chi}_+^{k,(u)}(z_3)|N(p)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+ ig\langle 0|\epsilon^{ijk}\psi_+^{i,(u)}(z_1)\left[A_{-+}\psi_+^{(d)}\right]^j(z_2)\bar{\chi}_+^{k,(u)}(z_3)|N(p)\rangle \\
&- ig\langle 0|\epsilon^{ijk}\psi_+^{i,(u)}(z_1)\left[A_{-+}(0)\psi_+^{(d)}(z_2)\right]^j\bar{\chi}_+^{k,(u)}(z_3)|N(p)\rangle \\
&+ ig\int_0^1 \mathrm{d}\tau\; z_3^\nu\langle 0|\epsilon^{ijk}\psi_+^{i,(u)}(z_1)\psi_+^{(d)}(z_2)^j\left[F_{-+\nu}(\tau z_2)\bar{\chi}_+^{(u)}(z_3)\right]^k|N(p)\rangle \\
&+ ig\langle 0|\epsilon^{ijk}\psi_+^{i,(u)}(z_1)\psi_+^{j,(d)}(z_2)\left[A_{-+}\bar{\chi}_+^{k,(u)}\right]^k(z_3)|N(p)\rangle \\
&- ig\langle 0|\epsilon^{ijk}\psi_+^{i,(u)}(z_1)\psi_+^{j,(d)}(z_2)\left[A_{-+}(0)\bar{\chi}_+^{k,(u)}(z_3)\right]^k|N(p)\rangle \\
&- ig\langle 0|\epsilon^{ijk}\psi_+^{i,(u)}(z_1)\psi_+^{j,(d)}(z_2)\left[A_{-+}\bar{\chi}_+^{(u)}\right]^k(z_3)|N(p)\rangle \\
&- \langle 0|\epsilon^{ijk}\left(\partial_{-+}\psi_+^{i,(u)}\right)(z_1)\psi_+^{j,(d)}(z_2)\bar{\chi}_+^{k,(u)}(z_3)|N(p)\rangle \\
&- \langle 0|\epsilon^{ijk}\psi_+^{i,(u)}(z_1)\left(\partial_{-+}\psi_+^{j,(d)}\right)(z_2)\bar{\chi}_+^{k,(u)}(z_3)|N(p)\rangle\;,
\end{aligned}
\tag{B.13}
$$

where we used the shorthand notation $z_i^\mu := z_i \cdot n^\mu$. The number of terms can be reduced in the following way:

- The terms in the third, sixth and ninth line sum to zero. This is a direct consequence of Eq. (4.17).
- The terms in the eighth and tenth line cancel each other.
- The terms in the second and eleventh line as well as the terms in the fifth and twelfth line can be combined using $D_{-+} = \partial_{-+} - igA_{-+}$
- The term in the seventh line can be rewritten using Eq. (4.17).

A further simplification is possible by using the Fierz identity (3.12), equations of motion and Eq. (3.20) to replace

$$
\begin{aligned}
D_{-+}\psi_+(z_i) &\to D_{++}\psi_-(z_i) = 2\frac{\partial}{\partial z_i}\psi_-(z_i) \\
z_i^\mu F_{-+\mu} &= \frac{1}{2}z_1 F_{-+,++} \to z_1\bar{f}_{++}\;.
\end{aligned}
\tag{B.14}
$$

Using the abbreviations for the matrix elements introduced in (4.23) one finally obtains

$$
\begin{aligned}
\mathbb{M} =& -2\frac{\partial}{\partial z_1}\varphi_1(z_1,z_2,z_3) - 2\frac{\partial}{\partial z_2}\varphi_2(z_1,z_2,z_3) \\
&+ \int_0^1 \mathrm{d}\tau\;\left(z_1\varphi_1^g(z_1,z_2,z_3,\tau z_1) - z_3\varphi_1^g(z_1,z_2,z_3,\tau z_3)\right) \\
&+ \int_0^1 \mathrm{d}\tau\;\left(z_2\varphi_1^g(z_1,z_2,z_3,\tau z_2) - z_3\varphi_2^g(z_1,z_2,z_3,\tau z_3)\right) = \\
=& -2\frac{\partial}{\partial z_1}\varphi_1(z_1,z_2,z_3) - 2\frac{\partial}{\partial z_2}\varphi_2(z_1,z_2,z_3) \\
&+ \int_0^1 \mathrm{d}\tau\int_{z_3}^{z_1}\mathrm{d}s\;\frac{\partial}{\partial s}\left(s\varphi_1^g(z_1,z_2,z_3,\tau s)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^1 \mathrm{d}\tau \int_{z_3}^{z_2} \mathrm{d}s\, \frac{\partial}{\partial s}\left(s\varphi_2^g(z_1,z_2,z_3,\tau s)\right) = \\
&\overset{t:=\tau s}{=} -2\frac{\partial}{\partial z_1}\varphi_1(z_1,z_2,z_3) - 2\frac{\partial}{\partial z_2}\varphi_2(z_1,z_2,z_3) \\
&+\int_{z_2}^{z_1} \mathrm{d}s\, \frac{\partial}{\partial s}\left(\int_0^s \mathrm{d}t\; \varphi_1^g(z_1,z_2,z_3,t)\right) \\
&+\int_{z_3}^{z_1} \mathrm{d}s\, \frac{\partial}{\partial s}\left(\int_0^s \mathrm{d}t\; \varphi_2^g(z_1,z_2,z_3,t)\right) \\
&= -2\frac{\partial}{\partial z_1}\varphi_1(z_1,z_2,z_3) - 2\frac{\partial}{\partial z_2}\varphi_2(z_1,z_2,z_3) \\
&+\int_{z_2}^{z_1} \mathrm{d}s\; \varphi_1^g(z_1,z_2,z_3,s) + \int_{z_3}^{z_1} \mathrm{d}s\; \varphi_2^g(z_1,z_2,z_3,s) = \\
&\overset{s:=\bar{\tau} z_{1/2}+\tau z_3}{=} -2\frac{\partial}{\partial z_1}\varphi_1(z_1,z_2,z_3) - 2\frac{\partial}{\partial z_2}\varphi_2(z_1,z_2,z_3) \\
&- z_{13}\int_1^0 \mathrm{d}\tau\; \varphi_1^g(z_1,z_2,z_3,z_{13}^\tau) - z_{23}\int_1^0 \mathrm{d}\tau\; \varphi_2^g(z_1,z_2,z_3,z_{23}^\tau) = \\
&= -2\left[\frac{\partial}{\partial z_1}\varphi_1(z_1,z_2,z_3) + \frac{\partial}{\partial z_2}\varphi_2(z_1,z_2,z_3)\right. \\
&\left. -\frac{1}{2}\int_0^1 \mathrm{d}\tau\; \left(z_{13}\varphi_1^g(z_1,z_2,z_3,z_{13}^\tau) + z_{23}\varphi_2^g(z_1,z_2,z_3,z_{23}^\tau)\right)\right] \qquad \text{(B.15)}
\end{aligned}
$$

Comparing this result with the definition of $\mathbb{M}$, Eq. (B.6), one arrives at equation (4.24).

C Conformally Invariant Kernels

We collect all renormalization kernels for twist-4 baryon operators that are allowed by conformal symmetry. All kernels are taken from [61] and [62].

C.1 Two-Particle Kernels

For the mapping $(j_1, j_2) \to (i_1, i_2)$ with $j_1 + j_2 - i_1 - i_2 = 0$ the most general form for the kernel K is given by

$$[K^{i_1 i_2}_{j_1 j_2}\varphi](z_1, z_2) = \int_0^1 \mathrm{d}\alpha \int_0^1 \mathrm{d}\beta\ \bar{\alpha}^{i_1+j_1-2}\alpha^{i_2-j_2}\,\bar{\beta}^{i_2+j_2-2}\beta^{i_1-j_1}\ \kappa\left(\frac{\alpha\beta}{\bar{\alpha}\bar{\beta}}\right)\varphi(z^{\alpha}_{12}, z^{\beta}_{21})\,. \tag{C.1}$$

The invariant function κ can be divided into two classes.

The first class contains those kernels, where a δ function cancels one of the integrations. This class encompasses the following kernels:

- $i_1 = j_1$ and $i_2 = j_2$ and $\kappa(x) \propto \delta(x)$

$$[\mathcal{H}^{v}_{12}\varphi](z_1, z_2) = \int_0^1 \frac{\mathrm{d}\alpha}{\alpha}\Big\{\bar{\alpha}^{2j_1-1}[\varphi(z_1, z_2) - \varphi(z^{\alpha}_{12}, z_2)] + \bar{\alpha}^{2j_2-1}[\varphi(z_1, z_2) - \varphi(z_1, z^{\alpha}_{21})]\Big\}\,. \tag{C.2}$$

- $i_1 = j_1$, $i_2 = j_2$ and $\kappa(x) \propto \delta(1 - x)$

$$[\mathcal{H}^{d}_{12}\varphi](z_1, z_2) = \varphi(z^{\alpha}_{12}, z^{\beta}_{21}) = \int_0^1 \mathrm{d}\alpha\,\bar{\alpha}^{2j_1-1}\,\alpha^{2j_2-1}\,\varphi(z^{\alpha}_{12}, z^{\alpha}_{12})\,, \tag{C.3}$$

- $i_1 = j_2$, $i_2 = j_1$, $j_1 > j_2$ and $\kappa(x) \propto \delta(x)$

$$[\mathcal{H}^{e}_{12}\varphi](z_1, z_2) = \int_0^1 d\alpha\,\bar{\alpha}^{2j_2-1}\,\alpha^{2(j_1-j_2)-1}\,\varphi(z^{\alpha}_{12}, z_2) \tag{C.4}$$

where the subscripts $_{12}$ indicate that the kernel acts on the coordinates of the 1st and 2nd particle.

In the second class the function $\kappa(x)$ is proportional to a Heaviside-Θ-function and both integrals remain intact.

- $i_1 = j_1$ & $i_2 = j_2$ and $\kappa(x) \propto \Theta(x)$

$$[\mathcal{H}_{12}^{+}\varphi](z_1, z_2) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, \bar{\alpha}^{2j_1-2}\, \bar{\beta}^{2j_2-2}\, \varphi(z_{12}^{\alpha}, z_{21}^{\beta})\,, \tag{C.5}$$

- $i_1 = j_1$ & $i_2 = j_2$ and $\kappa(x) \propto \Theta(1-x)$

$$[\mathcal{H}_{12}^{-}\varphi](z_1, z_2) = \int_0^1 d\alpha \int_{\bar{\alpha}}^{1} d\beta\, \bar{\alpha}^{2j_1-2}\, \bar{\beta}^{2j_2-2}\, \varphi(z_{12}^{\alpha}, z_{21}^{\beta})\,. \tag{C.6}$$

C.2 Three-Particle Kernels

The kernels mapping the representation $(1, 1, 3/2)$ onto $(1, 1/2)$ or $(1/2, 1)$ are much more complicated. First of all, the sum of the conformal spin is not conserved. This can be corrected by introducing additional factors z_{ij}^2. In addition to that, there are two independent conformal ratios; we have to introduce an invariant function $\tau(x, y)$. The most general 2-to-3 kernel takes the form

$$[\mathcal{H}_{2\to 3}f](z_1, z_2) = z_{12}^2 \int_0^1 \mathrm{d}\alpha \int_0^1 \mathrm{d}\beta \int_0^1 \mathrm{d}\gamma\; \bar{\beta}^{2j_2-1}\beta^{2j_1-1}\tau\left(\frac{\alpha\gamma}{\bar{\alpha}\bar{\gamma}}, \frac{\gamma\bar{\beta}}{\beta\bar{\gamma}}\right) f(z_{12}^{\alpha}, z_{21}^{\gamma}, z_{21}^{\beta})\,. \tag{C.7}$$

The list of all possible kernels is extensive and we restrict ourselves to those kernels that actually appear in the calculations. The notation follows [62].

$$[\mathcal{V}_{12(3)}^{(1)}\, f](z_1, z_2) = z_{12}^2 \int_0^1 d\alpha \int_{\bar{\alpha}}^1 d\beta\, \frac{\bar{\alpha}\bar{\beta}}{\alpha}\, f(z_{12}^{\alpha}, z_2, z_{21}^{\beta})\,, \tag{C.8}$$

$$[\mathcal{V}_{12(3)}^{(2)}\, f](z_1, z_2) = z_{12}^2 \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, \beta\, f(z_{12}^{\alpha}, z_2, z_{21}^{\beta})\,, \tag{C.9}$$

$$[\mathcal{V}_{12(3)}^{(3)}\, f](z_1, z_2) = z_{12}^2 \int_0^1 d\beta \int_{\beta}^1 d\gamma\, \frac{\beta\bar{\gamma}}{\gamma}\left(\frac{\bar{\gamma}}{\gamma} - 2\frac{\bar{\beta}}{\beta}\right) f(z_1, z_{21}^{\gamma}, z_{21}^{\beta})\,, \tag{C.10}$$

$$[\mathcal{V}_{12(3)}^{(4)}\, f](z_1, z_2) = z_{12}^2 \int_0^1 d\beta\, \bar{\beta} \left\{ f(z_1, z_2, z_{21}^{\beta}) + \frac{\bar{\beta}}{\beta} \int_0^{\beta} d\gamma f(z_1, z_{21}^{\gamma}, z_{21}^{\beta}) \right\}, \tag{C.11}$$

$$[\mathcal{V}_{12(3)}^{(a)} f](z_1, z_2) = z_{12}^2 \int_0^1 d\alpha \int_{\bar{\alpha}}^1 d\beta \int_0^{\bar{\alpha}} d\gamma\, \bar{\beta}\, f(z_{12}^{\alpha}, z_{21}^{\gamma}, z_{21}^{\beta})\,, \tag{C.12}$$

$$[\mathcal{V}_{12(3)}^{(b)} f](z_1, z_2) = z_{12}^2 \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \int_0^{\bar{\alpha}} d\gamma\, \bar{\beta}\, f(z_{12}^{\alpha}, z_{21}^{\gamma}, z_{21}^{\beta})\,, \tag{C.13}$$

$$[\mathcal{V}_{12(3)}^{(c)} f](z_1, z_2) = z_{12}^2 \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \int_0^{\beta} d\gamma\, \bar{\beta}\, f(z_{12}^{\alpha}, z_{21}^{\gamma}, z_{21}^{\beta})\,, \tag{C.14}$$

which correspond to the choices

$$\tau^{(1)}(x,y) = \delta(x)\,\theta(x/y-1), \qquad \tau^{(2)}(x,y) = \delta(y)\,\theta(1-x/y), \tag{C.15}$$

$$\tau^{(3)}(x,y) = \delta(x)\,\theta(y-1)\left(\frac{1}{y}-2\right), \quad \tau^{(4)}(x,y) = \theta(1-y)\delta(x/y)\,(1+\delta(y)), \tag{C.16}$$

$$\tau^{(a)}(x,y) = \theta(1-x)\theta(x/y-1), \qquad \tau^{(b)}(x,y) = \theta(1-x)\theta(1-x/y), \tag{C.17}$$

$$\tau^{(c)}(x,y) = \theta(1-y)\theta(1-x/y)\,. \tag{C.18}$$

D Multiplicatively Renormalizable Baryon Operators of Twist 4

Before we are able to write down the expressions for the multiplicatively renormalizable operators (MRO), it is necessary to define some simple local operator basis. The MROs can be represented as a linear combination of these operators.

For the chiral case we choose:

$$
\begin{aligned}
Q_1^{(k_1,k_2,k_3)} =& \epsilon^{ijk}[(n\cdot D)^{k_1}\psi_-^a]^i\,[(n\cdot D)^{k_2}\psi_+^b]^j\,[(n\cdot D)^{k_3}\psi_+^c]^k\,,\\
Q_2^{(k_1,k_2,k_3)} =& \epsilon^{ijk}[(n\cdot D)^{k_1}\psi_+^a]^i\,[(n\cdot D)^{k_2}\psi_-^b]^j\,[(n\cdot D)^{k_3}\psi_+^c]^k\,,\\
Q_3^{(k_1,k_2,k_3)} =& \epsilon^{ijk}[(n\cdot D)^{k_1}\psi_+^a]^i\,[(n\cdot D)^{k_2}\psi_+^b]^j\,[(n\cdot D)^{k_3}\psi_-^c]^k\,,
\end{aligned}
\tag{D.1}
$$

and

$$
\begin{aligned}
G_1^{(k_1,k_2,k_3,k_4)} =& ig\epsilon^{ijk}\,(\mu\lambda)\,[(nD)^{k_4}\bar{f}_{++}(nD)^{k_1}\psi_+^a]^i\,[(nD)^{k_2}\psi_+^b]^j\,[(nD)^{k_3}\psi_+^c]^k\,,\\
G_2^{(k_1,k_2,k_3,k_4)} =& ig\epsilon^{ijk}(\mu\lambda)\,[(nD)^{k_1}\psi_+^a]^i\,[(nD)^{k_4}\bar{f}_{++}(nD)^{k_2}\psi_+^b]^j\,[(nD)^{k_3}\psi_+^c]^k\,.
\end{aligned}
\tag{D.2}
$$

The chiral spectrum has the advantage that the anomalous dimensions in the $\varepsilon = e^{\pm\frac{2\pi}{3}}$ sector are double degenerate. Therefore, any linear combination of the corresponding eigenfunctions satisfies the evolution equation equally well. For the nucleon operators it is useful to construct an eigenbasis that reflects the identity of two quark flavors. The linear combination

$$
\Psi^{\pm}_{N,q} = (1\pm\mathcal{P}_{12})\Psi^{\varepsilon}_{N,q}\,,\quad \varepsilon = e^{\pm i2\pi/3}
\tag{D.3}
$$

is symmetric in the first and second quark and we will use $\Psi^{\pm}_{N,q}$ instead of $\Psi^{\varepsilon}_{N,q}$ with $\varepsilon = e^{\pm i2\pi/3}$.[1]

The chiral basis of MROs for $N \le 2$ takes the form [62]:

$$
\mathbb{O}^{chiral,+}_{0,0} = Q_1^{(000)} + Q_2^{(000)} - 2Q_3^{(000)}\,,
$$

[1] This choice does not change the fact that we are working with arbitrary flavor structures. However, the nucleon operators take a more convenient form.

$$
\begin{aligned}
\mathbb{O}^{chiral,+}_{1,1} =& Q_1^{(100)} - \frac{3}{2}Q_1^{(010)} + Q_1^{(001)} - \frac{3}{2}Q_2^{(100)} + Q_2^{(010)} + Q_2^{(001)} + \frac{1}{2}Q_3^{(100)} \\
&+ \frac{1}{2}Q_3^{(010)} - 2Q_3^{(001)} \\
\mathbb{O}^{chiral,+}_{2,0} =& Q_1^{(200)} + 2Q_1^{(020)} + Q_1^{(002)} - 4Q_1^{(110)} + 2Q_1^{(101)} - 4Q_1^{(011)} + 2Q_2^{(200)} \\
&+ Q_2^{(020)} + Q_2^{(002)} - 4Q_2^{(110)} - 4Q_2^{(101)} + 2Q_2^{(011)} - 3Q_3^{(200)} - 3Q_3^{(020)} \\
&- 2Q_3^{(002)} + 8Q_3^{(110)} + 2Q_3^{(101)} + 2Q_3^{(011)} - \frac{7}{12}G_1^{(0000)} - \frac{7}{12}G_2^{(0000)}, \\
\mathbb{O}^{g,chiral,+}_{2,0} =& \frac{3}{2}\left(G_1^{(0000)} + G_2^{(0000)}\right), \qquad E^{g,chiral}_{2,0} = 19/3
\end{aligned} \tag{D.4}
$$

and

$$
\begin{aligned}
\mathbb{O}^{chiral,-}_{0,0} =& Q_1^{(000)} - Q_2^{(000)}, \\
\mathbb{O}^{chiral,-}_{1,1} =& 3Q_1^{(100)} + \frac{1}{2}Q_1^{(010)} - 2\,Q_1^{(001)} - \frac{1}{2}Q_2^{(100)} - 3\,Q_2^{(010)} + 2\,Q_2^{(001)} \\
&- \frac{5}{2}Q_3^{(100)} + \frac{5}{2}Q_3^{(010)}, \\
\mathbb{O}^{chiral,-}_{2,0} =& Q_1^{(200)} + \frac{4}{3}Q_1^{(020)} + \frac{5}{3}Q_1^{(002)} - 2\,Q_1^{(101)} - 4\,Q_1^{(011)} - \frac{4}{3}Q_2^{(200)} \\
&- Q_2^{(020)} - \frac{5}{3}Q_2^{(002)} + 4Q_2^{(101)} + 2Q_2^{(011)} + \frac{1}{3}Q_3^{(200)} - \frac{1}{3}Q_3^{(020)} \\
&- 2Q_3^{(101)} + 2Q_3^{(011)} - \frac{7}{36}G_1^{(0000)} + \frac{7}{36}G_2^{(0000)}, \\
\mathbb{O}^{g,chiral,-}_{2,0} =& \frac{1}{2}\left(G_1^{(0000)} - G_2^{(0000)}\right), \qquad E^{g,chiral}_{2,0} = 19/3\,.
\end{aligned} \tag{D.5}
$$

For the multiplicatively renormalizable operators of the lowest dimension ($N \leq 2$) in the $\varepsilon = 1$ sector one finds [62]

$$
\begin{aligned}
\mathbb{O}^{chiral,1}_{1,0} =& Q_1^{(010)} - Q_1^{(001)} - Q_2^{(100)} + Q_2^{(001)} + Q_3^{(100)} - Q_3^{(010)}, \\
\mathbb{O}^{chiral,1a}_{2,1} =& Q_1^{(020)} - Q_1^{(002)} - 6Q_1^{(110)} + 6Q_1^{(101)} - Q_2^{(200)} + Q_2^{(002)} + 6Q_2^{(110)} \\
&- 6Q_2^{(011)} + Q_3^{(200)} - Q_3^{(020)} - 6Q_3^{(101)} + 6Q_3^{(011)}, \\
\mathbb{O}^{chiral,1b}_{2,1} =& Q_1^{(200)} - \frac{1}{2}Q_1^{(020)} - \frac{1}{2}Q_1^{(002)} - Q_1^{(110)} - Q_1^{(101)} + 2\,Q_1^{(011)} \\
&- \frac{1}{2}Q_2^{(200)} + Q_2^{(020)} - \frac{1}{2}Q_2^{(002)} - Q_2^{(110)} + 2\,Q_2^{(101)} - Q_2^{(011)} \\
&- \frac{1}{2}Q_3^{(200)} - \frac{1}{2}Q_3^{(020)} + Q_3^{(002)} + 2\,Q_3^{(110)} - Q_3^{(101)} - Q_3^{(011)}\,.
\end{aligned} \tag{D.6}
$$

The anomalous dimensions not presented here can be found in Table 4.1.

Next, we consider the operators involving quark fields of mixed chirality. Note that the spectrum is not degenerate, as the permutation symmetry does not extend to the Hamiltonian of mixed chirality. We define:

$$
\begin{aligned}
\mathcal{Q}_1^{(k_1,k_2,k_3)} =& \epsilon^{ijk}\,[(n\cdot D)^{k_1}\psi_-^a]^i\,[(n\cdot D)^{k_2}\psi_+^b]^j\,[(n\cdot D)^{k_3}\bar\chi_+^c]^k\,, \\
\mathcal{Q}_2^{(k_1,k_2,k_3)} =& \epsilon^{ijk}\,[(n\cdot D)^{k_1}\psi_+^a]^i\,[(n\cdot D)^{k_2}\psi_-^b]^j\,[(n\cdot D)^{k_3}\bar\chi_+^c]^k\,,
\end{aligned}
$$

$$\mathcal{Q}_3^{(k_1,k_2,k_3)} = \frac{1}{2}\epsilon^{ijk}\,[(n\cdot D)^{k_1}\psi_+^a]^i\,[(n\cdot D)^{k_2}\psi_+^b]^j\,[(n\cdot D)^{k_3}\bar{\chi}_+^{3/2,c}]^k\,, \tag{D.7}$$

where $\bar{\chi}_+^{3/2} \equiv \bar{\chi}_+^{(3/2,0)} = -(\mu D\bar{\lambda})\bar{\chi}_+ \equiv -D_{\mu\dot{\lambda}}\bar{\chi}_+$, cf. Eq. (3.63), and

$$\begin{aligned}
\mathcal{G}_1^{(k_1,k_2,k_3,k_4)} =& ig\epsilon^{ijk}\,(\mu\lambda)\,[(nD)^{k_4}\bar{f}_{++}(nD)^{k_1}\psi_+^a]^i\,[(nD)^{k_2}\psi_+^b]^j\,[(nD)^{k_3}\bar{\chi}_+^c]^k\,,\\
\mathcal{G}_2^{(k_1,k_2,k_3,k_4)} =& ig\epsilon^{ijk}\,(\mu\lambda)\,[(nD)^{k_1}\psi_+^a]^i\,[(nD)^{k_4}\bar{f}_{++}(nD)^{k_2}\psi_+^b]^j\,[(nD)^{k_3}\bar{\chi}_+^c]^k\,.
\end{aligned} \tag{D.8}$$

The multiplicatively renormalizable operators of the lowest dimension read [62]

$$\begin{aligned}
\mathbb{O}_{0,0}^{mixed} =& \mathcal{Q}_1^{(000)} - \mathcal{Q}_2^{(000)}\,,\\
\mathbb{O}_{1,0}^{mixed} =& \mathcal{Q}_1^{(100)} + \mathcal{Q}_1^{(010)} - \frac{3}{2}\mathcal{Q}_1^{(001)} - \mathcal{Q}_2^{(100)} - \mathcal{Q}_2^{(010)} + \frac{3}{2}\mathcal{Q}_2^{(001)}\,,\\
\mathbb{O}_{1,1}^{mixed} =& \mathcal{Q}_1^{(100)} - \mathcal{Q}_1^{(010)} + \frac{1}{2}\mathcal{Q}_1^{(001)} - \mathcal{Q}_2^{(100)} + \mathcal{Q}_2^{(010)} + \frac{1}{2}\mathcal{Q}_2^{(001)} + \mathcal{Q}_3^{(000)}\,,\\
\mathbb{O}_{2,1}^{mixed} =& \mathcal{Q}_1^{(200)} - \mathcal{Q}_1^{(020)} - \frac{2}{3}\mathcal{Q}_1^{(002)} - 2\mathcal{Q}_1^{(101)} + 3\mathcal{Q}_1^{(011)} - \mathcal{Q}_2^{(200)}\\
&+ \mathcal{Q}_2^{(020)} - \frac{2}{3}\mathcal{Q}_2^{(002)} - 2\mathcal{Q}_2^{(011)} + 3\mathcal{Q}_2^{(101)}\\
&+ \mathcal{Q}_3^{(100)} + \mathcal{Q}_3^{(010)} - \frac{4}{3}\mathcal{Q}_3^{(001)} + \frac{91}{282}\mathcal{G}_1^{(0000)} + \frac{91}{282}\mathcal{G}_1^{(0000)}\,,\\
\begin{pmatrix} \mathbb{O}_{2,0}^{mixed} \\ \mathbb{O}_{2,2}^{mixed} \end{pmatrix} =& \mathcal{Q}_1^{(200)} + \mathcal{Q}_1^{(020)} + \frac{4}{27}\left(4 \pm \sqrt{43}\right)\mathcal{Q}_1^{(002)} + \frac{4}{9}\left(-5 \pm \sqrt{43}\right)\mathcal{Q}_1^{(110)}\\
&+ \frac{2}{9}\left(1 \mp 2\sqrt{43}\right)\mathcal{Q}_1^{(101)} - \frac{1}{9}\left(17 \pm 2\sqrt{43}\right)\mathcal{Q}_1^{(011)} - \mathcal{Q}_2^{(200)} - \mathcal{Q}_2^{(020)}\\
&- \frac{4}{27}\left(4 \pm \sqrt{43}\right)\mathcal{Q}_2^{(002)} - \frac{4}{9}\left(-5 \pm \sqrt{43}\right)\mathcal{Q}_2^{(110)}\\
&- \frac{2}{9}\left(1 \mp 2\sqrt{43}\right)\mathcal{Q}_2^{(011)} + \frac{1}{9}\left(17 \pm 2\sqrt{43}\right)\mathcal{Q}_2^{(101)}\\
&+ \frac{1}{9}\left(19 \mp 2\sqrt{43}\right)\left[\mathcal{Q}_3^{(100)} - \mathcal{Q}_3^{(010)}\right]\\
&+ \frac{1}{234}\left(33 \mp 16\sqrt{43}\right)\left(\mathcal{G}_1^{(0000)} - \mathcal{G}_2^{(0000)}\right).\\
\mathbb{O}_{2,0}^{g,mixed} =& \mathcal{G}_1^{(0000)} - \mathcal{G}_2^{(0000)}\,, \qquad E_{2,0}^{g,mixed} = 7\,,\\
\mathbb{O}_{2,1}^{g,mixed} =& 3\left(\mathcal{G}_1^{(0000)} + \mathcal{G}_2^{(0000)}\right)\,, \qquad E_{2,1}^{g,mixed} = 79/9\,.
\end{aligned} \tag{D.9}$$

The corresponding eigenvalues (anomalous dimensions) are listed in Table 4.2.

E

Nucleon Distribution Amplitudes

In this appendix we give the explicit expressions for all three-particle nucleon distribution amplitudes up to twist-6. Furthermore, we include the relations due to isospin and Fierz transformations for the the full set of the 24 amplitudes $\mathcal{S}_i$, $\mathcal{P}_i$, $\mathcal{A}_i$, $\mathcal{V}_i$ and $\mathcal{T}_i$. Details as to how these relations were obtained can be found in the original publications [65, 97, 67]. Our presentation follows [121].

Starting point is the general Lorentz decomposition of the light-ray matrix element $\langle 0|\epsilon^{ijk} u_\alpha^i(a_1 x) u_\beta^j(a_2 x) d_\gamma^k(a_3 x)|N(P)\rangle$:

$$
\begin{aligned}
&4\langle 0|\epsilon^{ijk} u_\alpha^i(a_1 x) u_\beta^j(a_2 x) d_\gamma^k(a_3 x)|N(P)\rangle \\
=\ & \mathcal{S}_1 m_N C_{\alpha\beta}\left(\gamma_5 N\right)_\gamma + \mathcal{S}_2 m_N^2 C_{\alpha\beta}\left(\not{x}\gamma_5 N\right)_\gamma \\
&+ \mathcal{P}_1 m_N \left(\gamma_5 C\right)_{\alpha\beta} N_\gamma + \mathcal{P}_2 m_N^2 \left(\gamma_5 C\right)_{\alpha\beta}\left(\not{x} N\right)_\gamma \\
&+ \left(\mathcal{V}_1 + \frac{x^2 m_N^2}{4}\mathcal{V}_1^M\right)\left(\not{P} C\right)_{\alpha\beta}\left(\gamma_5 N\right)_\gamma + \mathcal{V}_2 m_N \left(\not{P} C\right)_{\alpha\beta}\left(\not{x}\gamma_5 N\right)_\gamma \\
&+ \mathcal{V}_3 m_N \left(\gamma_\mu C\right)_{\alpha\beta}\left(\gamma^\mu\gamma_5 N\right)_\gamma + \mathcal{V}_4 m_N^2 \left(\not{x} C\right)_{\alpha\beta}\left(\gamma_5 N\right)_\gamma \\
&+ \mathcal{V}_5 m_N^2 \left(\gamma_\mu C\right)_{\alpha\beta}\left(i\sigma^{\mu\nu} x_\nu \gamma_5 N\right)_\gamma + \mathcal{V}_6 m_N^3 \left(\not{x} C\right)_{\alpha\beta}\left(\not{x}\gamma_5 N\right)_\gamma \\
&+ \left(\mathcal{A}_1 + \frac{x^2 m_N^2}{4}\mathcal{A}_1^M\right)\left(\not{P}\gamma_5 C\right)_{\alpha\beta} N_\gamma + \mathcal{A}_2 m_N \left(\not{P}\gamma_5 C\right)_{\alpha\beta}\left(\not{x} N\right)_\gamma \\
&+ \mathcal{A}_3 m_N \left(\gamma_\mu\gamma_5 C\right)_{\alpha\beta}\left(\gamma^\mu N\right)_\gamma + \mathcal{A}_4 m_N^2 \left(\not{x}\gamma_5 C\right)_{\alpha\beta} N_\gamma \\
&+ \mathcal{A}_5 m_N^2 \left(\gamma_\mu\gamma_5 C\right)_{\alpha\beta}\left(i\sigma^{\mu\nu} x_\nu N\right)_\gamma + \mathcal{A}_6 m_N^3 \left(\not{x}\gamma_5 C\right)_{\alpha\beta}\left(\not{x} N\right)_\gamma \\
&+ \left(\mathcal{T}_1 + \frac{x^2 m_N^2}{4}\mathcal{T}_1^M\right)\left(P^\nu i\sigma_{\mu\nu} C\right)_{\alpha\beta}\left(\gamma^\mu\gamma_5 N\right)_\gamma + \mathcal{T}_2 m_N \left(x^\mu P^\nu i\sigma_{\mu\nu} C\right)_{\alpha\beta}\left(\gamma_5 N\right)_\gamma \\
&+ \mathcal{T}_3 m_N \left(\sigma_{\mu\nu} C\right)_{\alpha\beta}\left(\sigma^{\mu\nu}\gamma_5 N\right)_\gamma + \mathcal{T}_4 m_N \left(P^\nu \sigma_{\mu\nu} C\right)_{\alpha\beta}\left(\sigma^{\mu\rho} x_\rho \gamma_5 N\right)_\gamma \\
&+ \mathcal{T}_5 m_N^2 \left(x^\nu i\sigma_{\mu\nu} C\right)_{\alpha\beta}\left(\gamma^\mu\gamma_5 N\right)_\gamma + \mathcal{T}_6 m_N^2 \left(x^\mu P^\nu i\sigma_{\mu\nu} C\right)_{\alpha\beta}\left(\not{x}\gamma_5 N\right)_\gamma \\
&+ \mathcal{T}_7 m_N^2 \left(\sigma_{\mu\nu} C\right)_{\alpha\beta}\left(\sigma^{\mu\nu}\not{x}\gamma_5 N\right)_\gamma + \mathcal{T}_8 m_N^3 \left(x^\nu \sigma_{\mu\nu} C\right)_{\alpha\beta}\left(\sigma^{\mu\rho} x_\rho \gamma_5 N\right)_\gamma\ ,
\end{aligned}
\tag{E.1}
$$

where C is the charge conjugation matrix and the 24 calligraphic functions depend on the coordinates a_1, a_2, a_2 and on $P \cdot x$. Functions adorned with the superscript M are the so-called x^2-corrections which were derived in [66]. The calligraphic functions do in general *not* correspond to Lorentz structures of definite light-cone twist. However, they can be written as linear combinations of a new set of 24 functions with fixed twist [65]:

$$\begin{aligned}
\mathcal{S}_1 &= S_1\,, & 2(P\cdot x)\,\mathcal{S}_2 &= S_1 - S_2\,,\\
\mathcal{P}_1 &= P_1\,, & 2(P\cdot x)\,\mathcal{P}_2 &= P_2 - P_1\,,\\
\mathcal{V}_1 &= V_1\,, & 2(P\cdot x)\,\mathcal{V}_2 &= V_1 - V_2 - V_3\,,\\
2\mathcal{V}_3 &= V_3\,, & 4(P\cdot x)\,\mathcal{V}_4 &= -2V_1 + V_3 + V_4 + 2V_5\,,\\
4(P\cdot x)\mathcal{V}_5 &= V_4 - V_3\,, & 4\,(P\cdot x)^2\,\mathcal{V}_6 &= V_2 + V_3 + V_4 + V_5 - V_6 - V_1\,,\\
\mathcal{A}_1 &= A_1\,, & 2(P\cdot x)\mathcal{A}_2 &= A_2 - A_3 - A_1\,,\\
2\mathcal{A}_3 &= A_3\,, & 4(P\cdot x)\mathcal{A}_4 &= -2A_1 - A_3 - A_4 + 2A_5\,,\\
4(P\cdot x)\mathcal{A}_5 &= A_3 - A_4\,, & 4\,(P\cdot x)^2\,\mathcal{A}_6 &= A_1 - A_2 + A_3 + A_4 - A_5 + A_6\,,\\
\mathcal{T}_1 &= T_1\,, & 2(P\cdot x)\mathcal{T}_2 &= T_1 + T_2 - 2T_3\,,\\
2\mathcal{T}_3 &= T_7\,, & 2(P\cdot x)\mathcal{T}_4 &= T_1 - T_2 - 2T_7\,,\\
2(P\cdot x)\mathcal{T}_5 &= T_5 + 2T_8 - T_1\,, & 4\,(P\cdot x)^2\,\mathcal{T}_6 &= 2T_2 - 2T_3 - 2T_4\\
& & &\quad + 2T_5 + 2T_7 + 2T_8\,,\\
4(P\cdot x)\mathcal{T}_7 &= T_7 - T_8\,, & 4\,(P\cdot x)^2\,\mathcal{T}_8 &= -T_1 + T_2 + T_5 - T_6 + 2T_7 + 2T_8\,.
\end{aligned} \tag{E.2}$$

The calligraphic functions are much more convenient for explicit calculations. However, the non-calligraphic functions having fixed collinear twist allow for a conformal expansion.

Due to the identity of two quark flavors, the functions in (E.2) exhibit the symmetry relations [65]:

$$\begin{aligned}
V_i(a_1,a_2,a_3) &= V_i(a_2,a_1,a_3)\,, & T_i(a_1,a_2,a_3) &= T_i(a_2,a_1,a_3)\,,\\
S_i(a_1,a_2,a_3) &= -S_i(a_2,a_1,a_3)\,, & P_i(a_1,a_2,a_3) &= -P_i(a_2,a_1,a_3)\,,\\
A_i(a_1,a_2,a_3) &= -A_i(a_2,a_1,a_3)\,.
\end{aligned} \tag{E.3}$$

The fact that nucleon and $N^\star$ have isospin $I = 1/2$ gives rise to [65]

$$\begin{aligned}
2T_1(a_1,a_2,a_3) =&[V_1 - A_1](a_1,a_3,a_2) + [V_1 - A_1](a_2,a_3,a_1)\,,\\
[T_3 + T_7 + S_1 - P_1]\,(a_1,a_2,a_3) =&[V_3 - A_3](a_3,a_1,a_2) + [V_2 - A_2](a_2,a_3,a_1)\,,\\
2T_2(a_1,a_2,a_3) =&[T_3 - T_7 + S_1 + P_1](a_3,a_1,a_2)\\
&+ [T_3 - T_7 + S_1 + P_1](a_3,a_2,a_1)\,,\\
[T_4 + T_8 + S_2 - P_2]\,(a_1,a_2,a_3) =&[V_4 - A_4](a_3,a_1,a_2) + [V_5 - A_5](a_2,a_3,a_1)\,,\\
2T_5(a_1,a_2,a_3) =&[T_4 - T_8 + S_2 + P_2](a_3,a_1,a_2)\\
&+ [T_4 - T_8 + S_2 + P_2](a_3,a_2,a_1)\,,
\end{aligned}$$

$$2T_6(a_1, a_2, a_3) = [V_6 - A_6](a_1, a_3, a_2) + [V_6 - A_6](a_2, a_3, a_1)\,. \tag{E.4}$$

The relations (E.4) would not be valid if (5.31) or (5.32) had been implemented as definitions for the $N^\star$ DAs.

It turns out to be convenient to define a momentum space representation of the calligraphic and non-calligraphic functions:

$$F(a_1, a_2, a_3, P\cdot z) = \int \mathcal{D}x\; e^{-iPx\sum_i x_i a_i} F(x_1, x_2, x_3)\,. \tag{E.5}$$

x_i corresponds to the longitudinal momentum fraction carried by the ith quark. These functions are related to the "tilded" ones in the final sum rules for $N\gamma \to N^\star$ (5.52) – (5.59) by [104]

$$\begin{aligned}
\widetilde{\mathcal{F}}_i^{(3)}(x_3) &:= \int_0^{1-x_3} dx_1\; \mathcal{F}(x_1, 1-x_1-x_3, x_3) \\
\widetilde{\widetilde{\mathcal{F}}}_i^{(3)}(x_3) &:= \int_1^{x_3} dx_3' \int_0^{1-x_3'} dx_1\, \mathcal{F}(x_1, 1-x_1-x_3', x_3') \\
\widetilde{\widetilde{\widetilde{\mathcal{F}}}}_i^{(3)}(x_3) &:= \int_1^{x_3} dx_3' \int_1^{x_3'} dx_3'' \int_0^{1-x_3''} dx_1\, \mathcal{F}(x_1, 1-x_1-x_3'', x_3'')
\end{aligned} \tag{E.6}$$

$$\begin{aligned}
\widetilde{\mathcal{F}}_i^{(2)}(x_2) &:= \int_0^{1-x_2} dx_1\; \mathcal{F}(x_1, x_2, 1-x_1-x_2) \\
\widetilde{\widetilde{\mathcal{F}}}_i^{(2)}(x_2) &:= \int_1^{x_2} dx_2' \int_0^{1-x_2'} dx_1 \mathcal{F}(x_1, x_2', 1-x_1-x_2') \\
\widetilde{\widetilde{\widetilde{\mathcal{F}}}}_i^{(2)}(x_2) &:= \int_1^{x_2} dx_2' \int_1^{x_2'} dx_2'' \int_0^{1-x_2''} dx_1 \mathcal{F}(x_1, x_2'', 1-x_1-x'')
\end{aligned} \tag{E.7}$$

Conformal Expansion of the Distribution Amplitudes

In [65] the distribution amplitudes were expanded to second order in conformal spin. Note that this expansion does not make use of the multiplicatively renormalizable operators defined in App. D.

All expressions are taken from [121], as [65] features serveral typos.

- Twist 3

$$\begin{aligned}
V_1(x_i, \mu) &= 120 x_1 x_2 x_3 \left[\phi_3^0(\mu) + \phi_3^+(\mu)(1-3x_3)\right]\,, \\
A_1(x_i, \mu) &= 120 x_1 x_2 x_3 (x_2 - x_1)\phi_3^-(\mu)\,, \\
T_1(x_i, \mu) &= 120 x_1 x_2 x_3 \Big[\phi_3^0(\mu) - \frac{1}{2}\left(\phi_3^+ - \phi_3^-\right)(\mu)(1-3x_3)\Big]
\end{aligned} \tag{E.8}$$

- Twist 4

$$V_2(x_i, \mu) = 24 x_1 x_2 \left[\phi_4^0(\mu) + \phi_4^+(\mu)(1-5x_3)\right]\,,$$

$$
\begin{aligned}
A_2(x_i,\mu) =& 24x_1x_2(x_2-x_1)\phi_4^-(\mu)\,,\\
T_2(x_i,\mu) =& 24x_1x_2\left[\xi_4^0(\mu)+\xi_4^+(\mu)(1-5x_3)\right]\,,\\
V_3(x_i,\mu) =& 12x_3\left[\psi_4^0(\mu)(1-x_3)+\psi_4^+(\mu)(1-x_3-10x_1x_2)\right.\\
&\left.+\psi_4^-(\mu)(x_1^2+x_2^2-x_3(1-x_3))\right]\,,\\
A_3(x_i,\mu) =& 12x_3(x_2-x_1)\left[\left(\psi_4^0+\psi_4^+\right)(\mu)+\psi_4^-(\mu)(1-2x_3)\right]\,,\\
T_3(x_i,\mu) =& 6x_3\left[(\phi_4^0+\psi_4^0+\xi_4^0)(\mu)(1-x_3)\right.\\
&+(\phi_4^++\psi_4^++\xi_4^+)(\mu)(1-x_3-10x_1x_2)\\
&\left.+(\phi_4^--\psi_4^-+\xi_4^-)(\mu)(x_1^2+x_2^2-x_3(1-x_3))\right],\\
T_7(x_i,\mu) =& 6x_3\left[(\phi_4^0+\psi_4^0-\xi_4^0)(\mu)(1-x_3)\right.\\
&+(\phi_4^++\psi_4^+-\xi_4^+)(\mu)(1-x_3-10x_1x_2)\\
&\left.+(\phi_4^--\psi_4^--\xi_4^-)(\mu)(x_1^2+x_2^2-x_3(1-x_3))\right],\\
S_1(x_i,\mu) =& 6x_3(x_2-x_1)\left[(\phi_4^0+\psi_4^0+\xi_4^0+\phi_4^++\psi_4^++\xi_4^+)(\mu)\right.\\
&\left.+(\phi_4^--\psi_4^-+\xi_4^-)(\mu)(1-2x_3)\right],\\
P_1(x_i,\mu) =& 6x_3(x_1-x_2)\left[(\phi_4^0+\psi_4^0-\xi_4^0+\phi_4^++\psi_4^+-\xi_4^+)(\mu)\right.\\
&\left.+(\phi_4^--\psi_4^--\xi_4^-)(\mu)(1-2x_3)\right]
\end{aligned}
\tag{E.9}
$$

- Twist 5

$$
\begin{aligned}
V_4(x_i,\mu) =& 3\left[\psi_5^0(\mu)(1-x_3)+\psi_5^+(\mu)(1-x_3-2(x_1^2+x_2^2))\right.\\
&\left.+\psi_5^-(\mu)\left(2x_1x_2-x_3(1-x_3)\right)\right],\\
A_4(x_i,\mu) =& 3(x_2-x_1)\left[-\psi_5^0(\mu)+\psi_5^+(\mu)(1-2x_3)+\psi_5^-(\mu)x_3\right],\\
T_4(x_i,\mu) =& \frac{3}{2}\left[(\phi_5^0+\psi_5^0+\xi_5^0)(\mu)(1-x_3)\right.\\
&+\left(\phi_5^++\psi_5^++\xi_5^+\right)(\mu)(1-x_3-2(x_1^2+x_2^2))\\
&\left.+\left(\phi_5^--\psi_5^-+\xi_5^-\right)(\mu)\left(2x_1x_2-x_3(1-x_3)\right)\right],\\
T_8(x_i,\mu) =& \frac{3}{2}\left[(\phi_5^0+\psi_5^0-\xi_5^0)(\mu)(1-x_3)\right.\\
&+\left(\phi_5^++\psi_5^+-\xi_5^+\right)(\mu)(1-x_3-2(x_1^2+x_2^2))\\
&\left.+\left(\phi_5^--\psi_5^--\xi_5^-\right)(\mu)\left(2x_1x_2-x_3(1-x_3)\right)\right],\\
V_5(x_i,\mu) =& 6x_3\left[\phi_5^0(\mu)+\phi_5^+(\mu)(1-2x_3)\right],\\
A_5(x_i,\mu) =& 6x_3(x_2-x_1)\phi_5^-(\mu)\,,\\
T_5(x_i,\mu) =& 6x_3\left[\xi_5^0(\mu)+\xi_5^+(\mu)(1-2x_3)\right],\\
S_2(x_i,\mu) =& \frac{3}{2}(x_2-x_1)\left[-\left(\phi_5^0+\psi_5^0+\xi_5^0\right)(\mu)\right.\\
&+\left(\phi_5^++\psi_5^++\xi_5^+\right)(\mu)(1-2x_3)\\
&\left.+\left(\phi_5^--\psi_5^-+\xi_5^-\right)(\mu)x_3\right],\\
P_2(x_i,\mu) =& \frac{3}{2}(x_2-x_1)\left[-\left(-\phi_5^0-\psi_5^0+\xi_5^0\right)(\mu)\right.
\end{aligned}
$$

$$+\left(-\phi_5^+ - \psi_5^+ + \xi_5^+\right)(\mu)(1-2x_3)$$
$$+\left(-\phi_5^- + \psi_5^- + \xi_5^-\right)(\mu)x_3\Big] \tag{E.10}$$

- Twist 6

$$V_6(x_i,\mu) = 2\left[\phi_6^0(\mu) + \phi_6^+(\mu)(1-3x_3)\right],$$
$$A_6(x_i,\mu) = 2(x_2-x_1)\phi_6^-,$$
$$T_6(x_i,\mu) = 2\Big[\phi_6^0(\mu) - \frac{1}{2}\left(\phi_6^+ - \phi_6^-\right)(1-3x_3)\Big]. \tag{E.11}$$

The various scale-dependent parameters ϕ, ξ and ψ are related to the eight independent parameters $f_N, \lambda_1, \lambda_2, f_1^u, f_1^d, f_2^d, A_1^u, V_1^d$ which were already introduced in Chapter 4. Note that the difference between the distribution amplitudes for $N^\star$ and nucleon is mainly hidden in these eight numbers. The relations are [121]:

$$\phi_3^0 = \phi_6^0 = f_N, \qquad \phi_4^0 = \phi_5^0 = \frac{1}{2}(f_N + \lambda_1),$$
$$\xi_4^0 = \xi_5^0 = \frac{1}{6}\lambda_2, \qquad \psi_4^0 = \psi_5^0 = \frac{1}{2}(f_N - \lambda_1),$$
$$\phi_3^- = \frac{21}{2} f_N A_1^u, \qquad \phi_3^+ = \frac{7}{2} f_N (1-3V_1^d), \tag{E.12}$$

$$\phi_4^+ = \frac{1}{4}\left[f_N(3-10V_1^d) + \lambda_1(3-10f_1^d)\right],$$
$$\phi_4^- = -\frac{5}{4}\left[f_N(1-2A_1^u) - \lambda_1(1-2f_1^d-4f_1^u)\right],$$
$$\psi_4^+ = -\frac{1}{4}\left[f_N(2+5A_1^u-5V_1^d) - \lambda_1(2-5f_1^d-5f_1^u)\right],$$
$$\psi_4^- = \frac{5}{4}\left[f_N(2-A_1^u-3V_1^d) - \lambda_1(2-7f_1^d+f_1^u)\right],$$
$$\xi_4^+ = \frac{1}{16}\lambda_2(4-15f_2^d), \qquad \xi_4^- = \frac{5}{16}\lambda_2(4-15f_2^d), \tag{E.13}$$

$$\phi_5^+ = -\frac{5}{6}\left[f_N(3+4V_1^d) - \lambda_1(1-4f_1^d)\right],$$
$$\phi_5^- = -\frac{5}{3}\left[f_N(1-2A_1^u) - \lambda_1(f_1^d - f_1^u)\right],$$
$$\psi_5^+ = -\frac{5}{6}\left[f_N(5+2A_1^u-2V_1^d) - \lambda_1(1-2f_1^d-2f_1^u)\right],$$
$$\psi_5^- = \frac{5}{3}\left[f_N(2-A_1^u-3V_1^d) + \lambda_1(f_1^d - f_1^u)\right],$$
$$\xi_5^+ = \frac{5}{36}\lambda_2(2-9f_2^d), \qquad \xi_5^- = -\frac{5}{4}\lambda_2 f_2^d, \tag{E.14}$$

$$\phi_6^+ = \frac{1}{2}\left[f_N(1-4V_1^d) - \lambda_1(1-2f_1^d)\right],$$
$$\phi_6^- = \frac{1}{2}\left[f_N(1+4A_1^u) + \lambda_1(1-4f_1^d-2f_1^u)\right]. \tag{E.15}$$

The Mass or x^2-Corrections

The x^2 corrections correspond to contributions due to deviations of the quark positions from the light-ray. If the nucleon or $N^\star$ was massless there would be no such deviation. Therefore, these corrections are also referred to as mass corrections. There are all in all three different x^2-terms in (E.1), but only the vector and axialvector ones contribute to the sum rules for the form factor. Hence, we omit the tensor term. All formulas are taken from [121]; this work provides the most recent compilation of all relevant expressions.

For the vector structure V_1 one defines

$$\mathcal{V}_1^{M(u)}(x_2) = \int_0^{1-x_2} dx_1 V_1^M(x_1, x_2, 1-x_1-x_2) = \frac{x_2^2}{24}\left(f_N C_f^u + \lambda_1 C_\lambda^u\right) ,$$
$$\mathcal{V}_1^{M(d)}(x_3) = \int_0^{1-x_3} dx_1 V_1^M(x_1, 1-x_1-x_3, x_3) = \frac{x_3^2}{24}\left(f_N C_f^d + \lambda_1 C_\lambda^d\right) \tag{E.16}$$

with

$$\begin{aligned}
C_f^u =&(1-x_2)^3\big[113 + 495x_2 - 552x_2^2 - 10A_1^u(1-3x_2) \\
&\quad + 2V_1^d(113 - 951x_2 + 828x_2^2)\big],\\
C_\lambda^u =&-(1-x_2)^3\big[13 - 20f_1^d + 3x_2 + 10f_1^u(1-3x_2)\big],\\
C_f^d =&-(1-x_3)\big[1441 + 505x_3 - 3371x_3^2 + 3405x_3^3 - 1104x_3^4 \\
&- 24V_1^d\big(207 - 3x_3 - 368x_3^2 + 412x_3^3 - 138x_3^4\big)\big] - 12(73 - 220V_1^d)\ln(x_3),\\
C_\lambda^d =&-(1-x_3)\Big[11 + 131x_3 - 169x_3^2 + 63x_3^3 - 30f_1^d(3 + 11x_3 - 17x_3^2 + 7x_3^3)\Big] \\
&- 12(3 - 10f_1^d)\ln(x_3)\,.
\end{aligned} \tag{E.17}$$

For the axial DA A_1 one defines

$$\mathcal{A}_1^{M(u)}(x_2) = \int_0^{1-x_2} dx_1 A_1^M(x_1, x_2, 1-x_1-x_2) = \frac{x_2^2}{24}(1-x_2)^3\left(f_N D_f^u + \lambda_1 D_\lambda^u\right) ,$$
$$\mathcal{A}_1^{M(d)}(x_3) = \int_0^{1-x_3} dx_1 A_1^M(x_1, 1-x_1-x_3, x_3) = 0\,, \tag{E.18}$$

with

$$\begin{aligned}
D_f^u &= 11 + 45x_2 - 2A_1^u(113 - 951x_2 + 828x_2^2) + 10V_1^d(1 - 30x_2)\,,\\
D_\lambda^u &= 29 - 45x_2 - 10f_1^u(7 - 9x_2) - 20f_1^d(5 - 6x_2)\,.
\end{aligned} \tag{E.19}$$

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Dear God.
We paid for all this stuff ourselves,
so thanks for nothing.

Bart Simpson

Acknowledgements

And now to the most pleasurable part of writing this thesis:
Thanking all the people who made it possible!

- First of all, I have to thank my advisor Prof. Dr. Vladimir Braun for, now that I think about it, quite a lot of things. Apart from keeping me off the streets for 3 more years by offering the opportunity to work on and write this dissertation, his comments, suggestions and ideas were indispensable for all projects I was involved in. Furthermore, I have to thank him for his patience in explaining and discussing various aspects of field theory, as well as for the friendly and relaxed atmosphere in his group. Last but not least, he provided opportunities for many nice trips to schools and conferences such as Herbstschule Maria Laach, DPG Tagung in Freiburg and Munich, Cracow School of Theoretical Physics, JLab Workshop on Hard Exclusive Processes and the GHP Workshop in Denver.

- Special thanks are due to Alexander Lenz, who shared his office with me for three years and therefore had to endure more of my annoying questions than anybody else. He was also the first to be asked to: proofread drafts, talks and proceedings, suggest synonyms or remind me of a definition. However, he cunningly escaped proofreading larger parts of this thesis by going on "Elternzeit". Even though our big Λ paper is still "under construction" – since 2007, the collaboration was always pleasurable and instructional. I also have to thank him for many interesting discussions and for providing numerous ideas for nice short weekend and semester projects (such as D_0 mixing and the 4th family standard model).

- I am very much indebted to Alexander Manashov, who served as my main advisor for everything related to integrability, conformal symmetry, renormalization and tricks to solve integrals. He patiently found and explained (if necessary several times) the mistakes in my calculations. The collaboration with him was always interesting and educational. I further have to thank him for spotting so many typos.

- Furthermore, I have to thank my "brothers" Markus Bobrowski and Johann Riedl (who was stuck in an office with me for one year) for everything related to CP violation in the D_0 system, families and generations – the more the merrier.

- Nikolaus Warkentin was instrumental in providing the lattice results for the $N^\star$ distribution amplitudes and was still answering questions regarding physics even though he was not longer involved in research.

- Further thanks go to Tobias Lautenschlager for asking many questions on the relations in the spinor formalism, which helped finding some sign errors.

- Christian Hagen deserves thanks for explaining the basic concepts of Lattice QCD to me in such a way that I could at least claim to understand them.

- Heidi Decock and Monika Maschek took care of all administrative issues.

- I further have to thank the QCDSF collaboration in person of M. Göckeler, R. Horsley, T. Kaltenbrunner, Y. Nakamura, D. Pleiter, P.E.L. Rakow, A. Schäfer, G. Schierholz, H. Stuben and J.M. Zanotti for providing the lattice data for the distribution amplitudes.

- The past and present SysAdmins Dieter Hierl, Christian Ehmann and Martin Hetzenegger keept all systems running and Stefan Solbrig provided detailed information on LaTeX.

- For proofreading this manuscript I especially indebted to: Andreas Dürmeier, who even checked the bibliography, and Christian Rohrwild.

- My work was supported by DFG (grant 9209070) for which I am grateful.

Now to the things unrelated to physics, which turned out to be important, too.

- I thank my mother and father for supporting me throughout the 21 years my education took until now. They were always there for me.

- Sophia provided emotional support and did not destroy any drafts of this thesis accidentally.