

## Spectrum of three anyons in a harmonic-oscillator potential

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The Hamiltonian of three anyons in a harmonic-oscillator potential is diagonalized using a set of harmonic-oscillator states, by treating these as bosons or fermions with an appropriate statistical interaction. The interpolation between the bosonic and fermionic limits is established, and the spectrum of the low-lying states is displayed.

### I. INTRODUCTION

It is now well established that in two space dimensions, particles may obey fractional statistics and have multivalued wave functions.<sup>1-4</sup> The statistics of these particles, called anyons, is expected to continuously interpolate between normal Bose and Fermi limits. Recently it has also been suggested that anyons may be responsible for the occurrence of the fractional quantum Hall effect<sup>5</sup> and high-temperature superconductivity.<sup>6</sup> The quantum mechanics of anyons is complicated by the fact that their wave functions are multivalued. One manages to avoid this problem by treating anyons as bosons or fermions having single-valued wave functions but interacting via a long-range Aharonov-Bohm-type vector potential. While the many-anyon problem has been studied both in mean-field<sup>7</sup> as well as hydrodynamical<sup>8</sup> approaches, not much attention has been paid until recently to the exact solution of the few-anyon problem. The quantum spectrum of two anyons in a harmonic-oscillator potential<sup>1,9</sup> and on a circle<sup>10</sup> was solved early and used to calculate the second virial coefficient of a gas of anyons.<sup>10,11</sup> The two-anyon problem in an external magnetic field has also been studied in detail.<sup>11-13</sup> In a different approach, numerical lattice studies of few-anyon systems have been made on a square<sup>14</sup> and on a spherical surface.<sup>15</sup> Note, however, that the latter studies do not allow the bosons to overlap with one another ("hard-core bosons"), and are distinct in spirit from the ideal bosons confined in a harmonic oscillator that we consider here. Such a harmonic-oscillator three-anyon problem has been considered by Wu.<sup>9</sup>

It is particularly convenient to study few-anyon systems in a harmonic-oscillator potential because the analytical answers for the energies and the wave functions at the Bose and Fermi limits are exactly known. The three-anyon problem is especially interesting because of the following.

(a) Some of the states interpolate nonlinearly as the strength of the statistical interaction is varied, in contrast to the linear interpolation discovered analytically by Wu.<sup>9</sup>

(b) There is a crossing of levels not observed in the two-body harmonic-oscillator case. Let the strength of the statistical interaction be noted by  $\alpha$ , with the conven-

tion that for  $\alpha=0$  the particles are ideal bosons. With increasing strength of  $\alpha$ , the lowest energy state of the interaction bosonic system changes its structure completely at  $\alpha \approx 0.7$ .

The latter behavior of the ground state has also recently been obtained perturbatively by Khare and McCabe.<sup>16</sup> In an earlier letter,<sup>17</sup> we had numerically studied the low-lying eigenenergies ( $\leq 6\hbar\omega$ ) and their interpolation between the Bose and Fermi limits. Similar results were also presented by Sporre, Verbaarschot, and Zahed<sup>18</sup> employing a different method of calculation. In this paper, we present our formalism and calculations in more detail. In particular, we include in the basis a whole class of odd-parity states starting at  $8\hbar\omega$  (that were not included in Ref. 17), and establish the convergence of the anyonic ground-state energy.

This paper is organized as follows. In Sec. II, we discuss the Hamiltonian of three anyons in a harmonic-oscillator potential. We express the Hamiltonian in a reduced form entirely in terms of relative coordinates so that there are no center-of-mass excitations to contend with. In Sec. III, we introduce a very general method of constructing the symmetrized or antisymmetrized basis of states in terms of these relative coordinates. Our method is not specific to anyons and may be used elsewhere. In Sec. IV, we present the results of our numerical analysis with special emphasis on the structure of states and the interpolation patterns. Finally, in the Appendix, we briefly discuss an alternative method of constructing the polynomials of definite symmetry for three particles.

### II. HAMILTONIAN

The Lagrangian for the three-anyon problem is given by

$$L = \frac{m^2}{2} \sum_{i=1}^3 \dot{\mathbf{r}}_i^2 + \hbar\alpha \sum_{j>i=1}^3 \dot{\theta}_{ij} - \frac{m\omega^2}{2} \sum_{i=1}^3 r_i^2, \quad (1)$$

where  $\mathbf{r}_i$  denotes the position of the  $i$ th particle. The second term is the statistical interaction whose strength is given by  $\alpha$  ( $0 \leq \alpha \leq 1$ ) and

$$\theta_{ij} = \tan^{-1} \left( \frac{y_i - y_j}{x_i - x_j} \right). \quad (2)$$

The third term in Eq. (1) denotes the one-body confinement potential which is chosen to be a harmonic oscillator. One may as well have chosen this to be a relative two-body confinement potential; the harmonic-oscillator parameter is then scaled by a factor  $\sqrt{3}$ . The Hamiltonian corresponding to Eq. (1) is given by

$$H = \frac{1}{2m} \sum_{i=1}^3 p_i^2 + \frac{m\omega^2}{2} \sum_{i=1}^3 r_i^2 - \frac{\alpha}{m} \sum_{j>i=1}^3 \frac{l_{ij}}{r_{ij}^2} + \frac{\alpha^2}{2m} \left[ \frac{(\mathbf{r}_{12} + \mathbf{r}_{13})^2}{r_{12}^2 r_{13}^2} + \cdots (\text{symm.}) \right], \quad (3)$$

where

$$l_{ij} = (\mathbf{r}_i - \mathbf{r}_j) \times (\mathbf{p}_i - \mathbf{p}_j).$$

The last term in Eq. (3) is identical to the more compact form given in Ref. 17. It is a simple exercise to eliminate the center-of-mass-dependent part of the Hamiltonian. Notice that the statistical interaction depends entirely on the relative coordinates. We choose the internal coordinates to be the Jacobi coordinates defined by

$$\rho = \frac{\mathbf{r}_1 - \mathbf{r}_2}{\sqrt{2}}, \quad (4)$$

$$\lambda = \frac{\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3}{\sqrt{6}}. \quad (5)$$

Before expressing  $H$  entirely in terms of relative coordinates, we note that  $H$  is manifestly symmetric under the permutation of particle indices. As long as one is interested in diagonalizing the Hamiltonian in a symmetric or antisymmetric basis, it is sufficient to consider only one of the three terms involving the statistical interaction, multiplied by 3. Thus the reduced Hamiltonian, after eliminating the center of mass, is given by

$$H_{\text{rel}} = \frac{1}{2m} (p_\rho^2 + p_\lambda^2) + \frac{m\omega^2}{2} (\rho^2 + \lambda^2) + \frac{3}{m\rho^2} \left[ -\alpha l_\rho + \frac{3\alpha^2}{4} \frac{(\sqrt{3}\rho + \lambda)^2}{(\rho + \sqrt{3}\lambda)^2} \right]. \quad (6)$$

The last term in  $H_{\text{rel}}$  introduces the three-body interactions. In principle we could diagonalize the Hamiltonian with any complete set of states. Here for simplicity we diagonalize  $H_{\text{rel}}$  in a harmonic-oscillator basis (which are eigenstates of  $H_{\text{rel}}$  at  $\alpha=0$ ). But first we outline the method of constructing the basis.

### III. THE BASIS

As mentioned already, it is natural to construct a basis (symmetric for bosons, antisymmetric for fermions) from the harmonic-oscillator eigenstates in the  $\rho, \lambda$  coordinates. Here we modify the method somewhat—instead of choosing the orthogonal eigenstates of the harmonic oscillator, an equivalent nonorthogonal basis is chosen. The reason for this is that it is much simpler to construct homogeneous polynomials of a given degree and symmetry than to construct the orthogonal three-anyon harmonic-oscillator eigenstates of a given symmetry

(which are inhomogeneous polynomials). In our problem, a harmonic-oscillator eigenstate with (relative) energy  $E_N = (N+2)\hbar\omega$  has the highest power  $N$  [i.e.,  $(\rho)^{N_1}(\lambda)^{N_2}, N_1 + N_2 = N$ ] in one of its terms. The set of all homogeneous polynomials of a given symmetry up to power  $N$  has a one-to-one correspondence to the set of the orthogonal eigenstates. In what follows, we outline this method.

Consider the four quantities

$$\rho_x, \rho_y, \lambda_x, \lambda_y,$$

where  $\rho$  and  $\lambda$  are defined in Eqs. (4) and (5). First we construct all homogeneous polynomials of  $N$ th degree in terms of these quantities that have definite permutation symmetry and definite angular momentum. That is, we classify these polynomials into irreducible subspaces under the rotation group in two dimensions,  $R_2$  and permutation group of three particles  $S_3$ . Under  $S_3$ , there are the following four irreducible representations: (i) symmetric (denoted by  $S$ ); (ii) mixed-symmetry  $\rho$  type (denoted by  $M_\rho$ ), which is antisymmetric under exchange of (say) particles 1 and 2, and has no particular symmetry with respect to particle 3; (iii) mixed-symmetry  $\lambda$  type (denoted by  $M_\lambda$ ), which is symmetric under the exchange of particles 1 and 2, and has again no particular symmetry with respect to particle 3; (iv) antisymmetric (denoted by  $A$ ).

The labels for mixed symmetry representations are a matter of choice and our convention follows from the definition of the relative coordinates in Eqs. (4) and (5). The method we follow is familiar from the quark models,<sup>19</sup> adapted to two space dimensions. The set of all polynomials of degree  $N$  have, in general, the form  $[\rho_x^i \rho_y^j \lambda_x^k \lambda_y^l]_{\text{perm}}$ ,  $N = i + j + k + l$ , where "perm" denotes permutations of the monomials. In Table I, we show a decomposition of these polynomials up to  $N=4$  under  $R_2$  and  $S_3$ . Note that the number of polynomials increases rather rapidly with  $N$ .

The  $N=0$  polynomial is trivial and is simply 1. We shall denote the  $N=1$  polynomials according to their transformation properties under  $R_2$  and  $S_3$ ,

$$\phi_\rho^{N=1, L=\pm 1} \equiv \rho_x \pm i\rho_y = \rho_\pm, \quad (7)$$

$$\phi_\lambda^{N=1, L=\pm 1} \equiv \lambda_x \pm i\lambda_y = \lambda_\pm, \quad (8)$$

TABLE I. Decomposition of polynomials of  $N$ th degree under  $R_2$  and  $S_3$ .

$N$	$L$	$S$	$M_\rho$	$M_\lambda$	$A$
0	0	1	0	0	0
1	$\pm 1$	0	1	1	0
2	$\pm 2$	1	1	1	0
2	0	1	1	1	1
3	$\pm 3$	1	1	1	1
3	$\pm 1$	1	2	2	1
4	$\pm 4$	1	2	2	0
4	$\pm 2$	1	3	3	1
4	0	2	3	3	1

which are obviously of the mixed-symmetry type. If  $P(i, j)$  denotes the permutation of any two particles  $i$  and  $j$ , then it is easy to see that

$$P(1, 2)\phi_\rho^{1, L} = -\phi_\rho^{1, L}, \tag{9}$$

$$P(2, 3)\phi_\rho^{1, L} = \frac{1}{2}[\phi_\rho^{1, L} + \sqrt{3}\phi_\lambda^{1, L}], \quad L = \pm 1,$$

$$P(1, 2)\phi_\lambda^{1, L} = \phi_\lambda^{1, L}, \tag{10}$$

$$P(2, 3)\phi_\lambda^{1, L} = \frac{1}{2}[-\phi_\lambda^{1, L} + \sqrt{3}\phi_\rho^{1, L}], \quad L = \pm 1.$$

$P(1, 3)$  can always be obtained from the above transformations. Now suppose  $\phi_\rho^{N, L}$  and  $\phi_\lambda^{N, L}$ , for all  $N$  (polynomial degree) and  $L$  (angular momentum), transform like  $\phi_\rho^{1, L}$  and  $\phi_\lambda^{1, L}$ , then the following properties hold under  $S_3$ :

$$\phi_S^{N, L} = \phi_\rho^{N_1, L_1} \phi_\rho^{N_2, L_2} + \phi_\lambda^{N_1, L_1} \phi_\lambda^{N_2, L_2}, \tag{11}$$

$$\phi_\lambda^{N, L} = \phi_\rho^{N_1, L_1} \phi_\rho^{N_2, L_2} - \phi_\lambda^{N_1, L_1} \phi_\lambda^{N_2, L_2}, \tag{12}$$

$$\phi_\rho^{N, L} = \phi_\rho^{N_1, L_1} \phi_\lambda^{N_2, L_2} + \phi_\lambda^{N_1, L_1} \phi_\rho^{N_2, L_2}, \tag{13}$$

$$\phi_A^{N, L} = \phi_\rho^{N_1, L_1} \phi_\lambda^{N_2, L_2} - \phi_\lambda^{N_1, L_1} \phi_\rho^{N_2, L_2}, \tag{14}$$

where

$$N = N_1 + N_2, \quad L = L_1 + L_2.$$

Equations (11)–(14) provide a recursive algorithm for constructing all possible states of a given  $N$  and  $L$  starting from the elementary polynomials,  $\phi_\rho^{1, L}$  and  $\phi_\lambda^{1, L}$  given in Eqs. (7) and (8). In Table II we give the list of all poly-

nomials up to  $N=4$ . Note, however, that even though Eqs. (11)–(14) provide a global algorithm to construct polynomials of all possible symmetries under the permutation of three particles, not all such polynomials for a given  $N$  and  $L$  are distinct. The method therefore gets cumbersome for large  $N$  when the number of polynomials increases quickly. An alternative method is indicated in the Appendix.

For the  $L=0$  polynomials, however, there exists a simple method of constructing all symmetric polynomials.<sup>19</sup> Note that given two vectors  $\rho$  and  $\lambda$ , there are four two-dimensional scalars:  $\rho^2$ ,  $\lambda^2$ ,  $\rho \cdot \lambda$ , and  $\rho \times \lambda$ . The last one is a special case in two dimensions and is completely antisymmetric. All  $L=0$  symmetric polynomials are then combinations of the first three scalars. The square of  $\rho \times \lambda$ , which is symmetric, may be written entirely in terms of the first three. It is sufficient to construct three independent symmetric polynomials from which all other  $L=0$  polynomials may be obtained. These are

$$S_2 = \rho^2 + \lambda^2, \tag{15}$$

$$S_4 = (2\rho \cdot \lambda)^2 + (\rho^2 - \lambda^2)^2, \tag{16}$$

$$S_6 = (\rho^2 - \lambda^2)[(\rho^2 - \lambda^2)^2 - 3(2\rho \cdot \lambda)^2], \tag{17}$$

and

$$\phi_S^{N, 0} = S_2^i S_4^j S_6^k, \quad N = 2i + 4j + 6k. \tag{18}$$

$\phi_S^{N, 0}$  completely exhaust all  $L=0$  symmetric polynomials. The set of symmetric states involving  $S_2$  alone (i.e.,

TABLE II. The set of all polynomials up to  $N=4$ , classified according to their permutation symmetries arranged in different columns. The energy of the harmonic-oscillator state containing the polynomial with the highest power  $N$  is  $(N+2)\hbar\omega$ . The ‘-’ signifies state of that symmetry not present.

$N$	$L$	$S$	$M_\rho$	$M_\lambda$	$A$
0	0	1	-	-	-
1	1	-	$\rho_+$	$\lambda_+$	-
2	2	$\rho_+^2 + \lambda_+^2$	$2\rho_+ \lambda_+$	$\rho_+^2 - \lambda_+^2$	-
0	2	$S_2 = \rho_+ \rho_- + \lambda_+ \lambda_-$	$\rho_+ \lambda_- + \rho_- \lambda_+$	$\rho_+ \rho_- - \lambda_+ \lambda_-$	$\rho_+ \lambda_- - \rho_- \lambda_+$
3	3	$\lambda_+(3\rho_+^2 - \lambda_+^2)$	$\rho_+(\rho_+^2 + \lambda_+^2)$	$\lambda_+(\rho_+^2 + \lambda_+^2)$	$\rho_+(\rho_+^2 - 3\lambda_+^2)$
1	3	$\rho_-(2\rho_+ \lambda_+) + \lambda_-(\rho_+^2 - \lambda_+^2)$	$\lambda_-(2\rho_+ \lambda_+) + \rho_-(\rho_+^2 - \lambda_+^2)$	$\rho_-(2\rho_+ \lambda_+) - \lambda_-(\rho_+^2 - \lambda_+^2)$	$-\lambda_-(2\rho_+ \lambda_+) + \rho_-(\rho_+^2 - \lambda_+^2)$
1	3	-	$\rho_-(\rho_+^2 + \lambda_+^2)$	$\lambda_-(\rho_+^2 + \lambda_+^2)$	-
4	4	$(\rho_+^2 + \lambda_+^2)^2$	$2\rho_+ \lambda_+(\rho_+^2 + \lambda_+^2)$	$(\rho_+^2 \lambda_+^2) - (\rho_+^2 - \lambda_+^2)$	-
4	4	-	$4\rho_+ \lambda_+(\rho_+^2 - \lambda_+^2)$	$(2\rho_+ \lambda_+)^2 - (\rho_+^2 - \lambda_+^2)$	-
2	4	$(\rho_+ \rho_- + \lambda_+ \lambda_-)(\rho_+^2 + \lambda_+^2)$	$(\rho_+ \lambda_- + \lambda_+ \rho_-)(\rho_+^2 + \lambda_+^2)$	$(\rho_+ \rho_- - \lambda_+ \lambda_-)(\rho_+^2 + \lambda_+^2)$	$(\rho_+ \lambda_- - \lambda_+ \rho_-)(\rho_+^2 + \lambda_+^2)$
2	4	-	$(2\rho_+ \lambda_+)(\rho_+ \rho_- - \lambda_+ \lambda_-)$ $+ (\rho_+ \lambda_- + \lambda_+ \rho_-)(\rho_+^2 - \lambda_+^2)$	$(2\rho_+ \lambda_+)(\rho_+ \lambda_- + \lambda_+ \rho_-)$ $- (\rho_+ \rho_- - \lambda_+ \lambda_-)(\rho_+^2 - \lambda_+^2)$	-
2	4	-	$(2\rho_+ \lambda_+)(\rho_+ \rho_- + \lambda_+ \lambda_-)$ $- (\rho_+ \lambda_- - \lambda_+ \rho_-)(\rho_+^2 - \lambda_+^2)$	$(2\rho_+ \lambda_+)(\rho_+ \lambda_- - \lambda_+ \rho_-)$ $+ (\rho_+ \rho_- + \lambda_+ \lambda_-)(\rho_+^2 - \lambda_+^2)$	-
0	4	$S_4 = (\rho_+ \lambda_- + \rho_- \lambda_+)^2$ $+ (\rho_+ \rho_- - \lambda_+ \lambda_-)^2$	$2(\rho_+ \lambda_- + \rho_- \lambda_+)(\rho_+ \rho_- - \lambda_+ \lambda_-)$	$(\rho_+ \lambda_- + \rho_- \lambda_+)^2$ $- (\rho_+ \rho_- - \lambda_+ \lambda_-)^2$	$(\rho_+ \lambda_- - \rho_- \lambda_+)(\rho_+ \rho_- + \lambda_+ \lambda_-)$
0	4	$S_2^2 = (\rho_+ \rho_- + \lambda_+ \lambda_-)^2$	$2\rho_+ \lambda_+(\rho_+^2 + \lambda_+^2)$	$(\rho_+^2 - \lambda_+^2)(\rho_+^2 + \lambda_+^2)$	-
0	4	-	$2\rho_- \lambda_-(\rho_+^2 + \lambda_+^2)$	$(\rho_+^2 - \lambda_+^2)(\rho_+^2 + \lambda_+^2)$	-

$j = k = 0$ ) are special since they have additional  $SO(4)$  invariance. As we show later, the eigenenergies of these states interpolate linearly between Bose and Fermi limits. The antisymmetric polynomials are then written as

$$\phi_{AS}^{N+2,0} = [\rho \times \lambda] S_2^i S_4^j S_6^k. \quad (19)$$

Unlike  $\phi_S^{N,0}$ , the above form does not exhaust all the antisymmetric polynomials. A new class of antisymmetric polynomials starts at  $N=6$  (with energy  $8\hbar\omega$ ), and has the structure

$$\tilde{\phi}_A^{6,0} = 2(\rho \cdot \lambda) [3(\rho^2 - \lambda^2)^2 - 4(\rho \cdot \lambda)^2]. \quad (20)$$

Higher-degree polynomials may then be generated by multiplying  $\tilde{\phi}_A^{6,0}$  with the symmetric form  $\phi_S^{N,0}$  given by Eq. (18), i.e.,

$$\tilde{\phi}_A^{N+6,0} = \tilde{\phi}_A^{6,0} \phi_S^{N,0}. \quad (21)$$

The harmonic-oscillator eigenstates may now be constructed from these homogeneous polynomials. The general form of such an eigenstate is given by

$$\psi_{S,A}^{N,L} = \sum_{n=N_{\min}}^N C_{n,L} \phi_{S,A}^{n,L} \exp \left[ -\frac{m\omega}{2\hbar} (\rho^2 + \lambda^2) \right], \quad (22)$$

where  $C_{n,L}$  are expansion coefficients and  $N_{\min}$  is the lowest polynomial degree for each  $L$ . Obviously when  $N = N_{\min}$ , the polynomial  $\phi_{S,A}$  completely defines the structure of the state. For example, the harmonic-oscillator fermionic eigenstate at  $6\hbar\omega$  is given by

$$\psi_A^{4,0} = \left[ 0.6667 - \frac{0.7454}{2^2} S_2 \right] (\rho \times \lambda) \exp \left[ -\frac{m\omega}{2\hbar} (\rho^2 + \lambda^2) \right], \quad (23)$$

with similar expressions for the other eigenstates. However, for computational convenience, we choose to diagonalize  $H_{\text{rel}}$  in a nonorthogonal basis provided by the set of all symmetric (or antisymmetric) polynomials of a given  $L$ . However, a problem arises with the bosonic wave functions since these do not necessarily vanish at the origin; as a result, the centrifugal terms in the Hamiltonian cause the energy to diverge when  $\alpha \neq 0$ . Following Wu,<sup>9</sup> we therefore introduce Jastrow-type correlations and define a nonorthogonal basis for the purpose of diagonalization,

$$\chi_{S,A}^{N,L} = N(\alpha) \prod_{i < j} |r_i - r_j|^\alpha \phi_{S,A}^{N,L} \exp \left[ -\frac{m\omega}{2\hbar} (\rho^2 + \lambda^2) \right], \quad (24)$$

where  $\alpha$  is the strength of the anyonic interaction and  $N(\alpha)$  is the normalization factor. The correlation factor as well as the Gaussian are symmetric and do not alter the symmetry of the state. The correlation factor, for  $\alpha \neq 0$ , introduces the zeros in the wave function necessary for avoiding the divergences.

#### IV. RESULTS AND DISCUSSION

We now turn to the results of diagonalizing the anyonic Hamiltonian given in Eq. (6). Since the Hamiltonian is rotationally invariant, we can diagonalize the Hamiltonian in a basis with well-defined angular momentum. As in any diagonalization, the finiteness of the basis poses a problem for the convergence of the eigenvalues, especially at  $\alpha=1$  and when the interpolation is nonlinear. We avoid this problem by diagonalizing  $H_{\text{rel}}$  both from the bosonic and fermionic limits and matching the results for interpolating states at  $\alpha = \frac{1}{2}$ . When the variation is linear, however, this is unnecessary, though it provides a check. The eigenvalues merge at  $\alpha = \frac{1}{2}$  only between interpolating states that differ by three units of angular momentum. To see this consider the wave function of the anyon

$$\psi_a = \prod_{i < j} e^{i\alpha\theta_{ij}} \psi_S^{N,L}, \quad (25)$$

when one starts with  $\alpha=0$  from the bosonic limit. Obviously the anyonic wave function is completely antisymmetric at  $\alpha=1$  (since  $\theta_{ij} = \theta_{ji} + \pi$ ) and carries an angular momentum  $(L + 3\alpha)$ . Alternatively, we could also start from the fermionic end in which case  $\alpha \rightarrow (1-\alpha)$  or, equivalently,  $\alpha \rightarrow -\alpha$  and  $\psi_S^{N,L} \rightarrow \psi_A^{N,L}$ ; the change in angular momentum is then given by  $L - 3\alpha$ , where  $0 \leq \alpha \leq 1$ .

The results for the  $L=0$  bosonic ground state and its radial excitations up to  $8\hbar\omega$  are shown in Fig. 1 along with the  $L=0$  fermionic ground state and its radial exci-

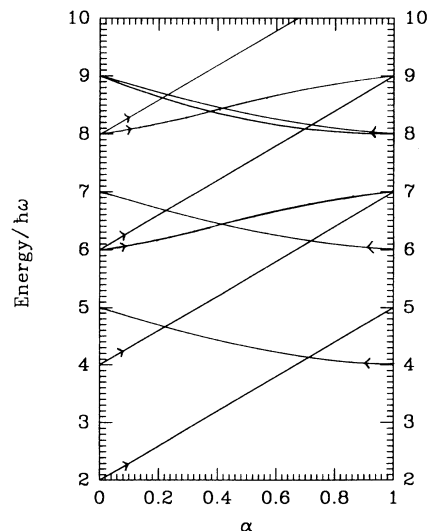


FIG. 1. The  $L=0$  bosonic and fermionic states are displayed with outgoing arrows as a function of  $\alpha$ . The value  $\alpha=0$  corresponds to noninteracting bosons in an oscillator while  $\alpha=1$  corresponds to noninteracting fermions. The beginning of each line as indicated by the arrow corresponds to the energy of the  $L=0$  Bose or Fermi states. While the low-lying bosonic states are linear, the corresponding fermionic states are nonlinear, and are quadratic for small  $(1-\alpha)$ . Only two of the three  $L=0$  states at  $8\hbar\omega$  at either end are shown.

tations. The structure of the bosonic  $L=0$  states is defined by the polynomials  $S_2, S_4, S_6$  [see Eq. (18)] and their degeneracies are given by the number of ways in which they may be combined to obtain an  $N$ th-degree polynomial. The symmetric ground state at  $2\hbar\omega$  ends up at  $5\hbar\omega$  missing the antisymmetric ground state at  $4\hbar\omega$ ; the level crossing occurs at  $\alpha \approx 0.7$ .

We find that the subset of  $L=0$  symmetric states constructed with  $S_2$  alone interpolate linearly in energy  $E=(N+2+3\alpha)\hbar\omega$ . To see this, consider the overlap function defined by

$$O_{SA}^{N,N'} = \left\langle \chi_A^{N',L'}(\alpha=0) \left| \prod_{i,j} e^{i\theta_{ij}} \right| \chi_S^{N,L}(\alpha=1) \right\rangle, \quad (26)$$

where  $\chi_S^{N,L}$  and  $\chi_A^{N',L'}$  have been defined earlier in Eq. (24). The quantity  $O_{SA}^{N,N'}$  essentially defines the overlap of an anyonic wave function at  $\alpha=1$  with pure oscillator states at  $\alpha=0$ . The phase factor then connects the symmetric states with the antisymmetric states. For symmetric states constructed with  $S_2$  alone [i.e.,  $N=2i, j=k=0$  in Eq. (18)], we get for  $L=0, L'=3$

$$O_{SA}^{N,N'} = - \frac{(i+i'+4)!}{\sqrt{(2i+4)!(2i'+4)!}} \quad (L=0, L'=3; j=k=j'=k'=0), \quad (27)$$

where  $i'$  corresponds to the power of  $S_2$  multiplying the lowest  $(5\hbar\omega)L=3$  fermionic state for the excited nodal states, analogously to Eq. (19). Note that  $O_{SA}^{N,N'} = -1$  when  $i=i'$ . That the overlaps are nonzero when  $i \neq i'$  is not worrisome, since the  $\chi$ 's are a nonorthogonal set. In fact, after an orthogonalization of the states on both sides in Eq. (26), the overlap matrix becomes diagonal with eigenvalues  $-1$ . [This is so because the overlap matrices between the nonorthogonal states on either side happen to be identical, up to a positive sign, with the matrix Eq. (27).] Obviously, there is a one-to-one correspondence between the bosonic  $L=0$  state with the  $S_2$  alone and a subset of the  $L'=3$  states at the fermionic limit. The interpolation appears to be linear in such cases and these are precisely some of the states singled out by the method used by Wu.<sup>9</sup> This simplicity is destroyed once other states involving  $S_4$  and  $S_6$  emerge—an example of this is the second state at  $6\hbar\omega$  shown in Fig. 1.

On the other hand, the overlaps of the  $L=0$  fermionic states with the  $L=3$  bosonic states (or equivalently  $L=-3$  when  $\theta_{ij} \rightarrow -\theta_{ij}$ ) are not so simple, even when only the subset of radial excitations involving  $S_2$  [ $j=k=0$  in Eq. (19)] are considered. The overlaps defined as in Eq. (26) with these subsets of states (interchanging  $S$  and  $A$ ) are given by

$$O_{AS}^{N,N'}(1) = \frac{3\sqrt{3}(i+i'+5)!}{\sqrt{35(2i+6)!(2i'+4)!}} \quad (L=0, L'=3; j=k=j'=k'=0), \quad (28)$$

$$O_{AS}^{N,N'}(2) = \frac{(i+i'+6)!}{\sqrt{7(2i+6)!(2i'+6)!}} \quad (L=0, L'=3; j'=1, j=k=k'=0), \quad (29)$$

where  $i, j, k, \dots$  are again the powers of  $S_2, S_4,$  and  $S_6$  according to Eqs. (18) and (19). The first overlap  $O_{AS}^{N,N'}(1)$ , Eq. (28), refers to the first series of  $L=3$  bosonic states ( $j'=0$ ) starting from  $5\hbar\omega$ , and the second overlap  $O_{AS}^{N,N'}(2)$ , Eq. (29), refers to the second series of states ( $j'=1$ ) starting from  $7\hbar\omega$ . These overlaps are not equal to 1 even when  $i=i'$ . A summary of the overlap distribution for the lowest states is shown in Fig. 2. The percentages shown in this figure may be taken to be probabilities, since these have been calculated with orthogonalized states. Note from this figure that the lowest  $L=-3$  bosonic state has only 73% overlap with the two  $L=0^-$  fermionic states; the rest are spread thinly over a large number of states. The nonlinearity in the interpolation primary arises from the spread in these overlaps. This renders diagonalization all the more difficult in a finite-size basis. Indeed, for the convergence of the lowest spin-zero state (in the fermionic basis) at  $\alpha=0.5$ , we had to include states of the  $\tilde{\phi}_A$  type up to  $N=26$ .

In Fig. 3, we show the results of diagonalization for all states with  $L \leq 3$  up to  $6\hbar\omega$ . It will be seen from the spectrum that the time-reversed states  $L$  and  $-L$  are split in energy by the anyonic interaction. As in the case of  $L=0$  states, the nonlinearly interpolating states suffer large mixing of the basis states for nonzero  $\alpha$ , while the linear ones remain pure. The large number of level crossings result in the bunching of quantum states at  $\alpha = \frac{1}{2}$  and  $\frac{2}{3}$  and, we suspect, even at  $\alpha = \frac{1}{3}$  if we go high enough in energy. Our calculation shows that the harmonic-oscillator basis is not a very good one for diagonalizing the long-range statistical interaction. For example, com-

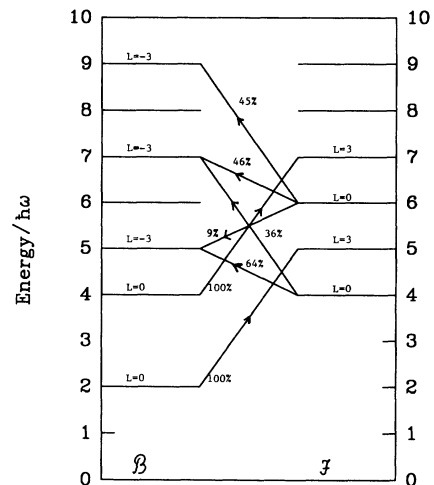


FIG. 2. The overlap function  $O^{N,N'}$  for low-lying  $L=0$  states. The beginning and the end of the arrow indicate the initial and final states in Eq. (26). The percentage figures indicated along each line denote the probability of the overlap calculated with orthogonalized states. A spread in the probability over many states results in nonlinear interpolation. Note that the lowest-lying  $L=-3$  bosonic state has only 73% overlap with the lowest two  $L=0$  states at the other end. Its overlap with the other three  $L=0$  fermionic states at  $8\hbar\omega$  totals only to about 5% and is not shown.

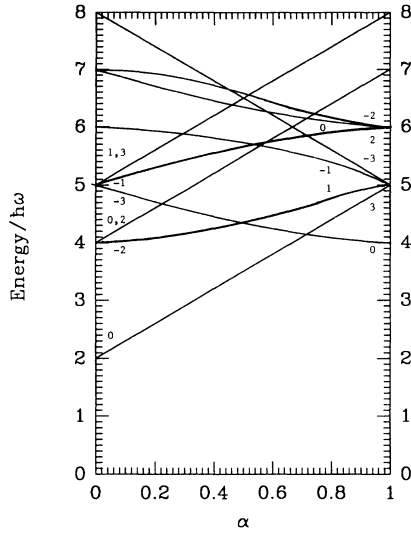


FIG. 3. The low-lying energy levels of the Hamiltonian Eq. (6) are displayed as a function of  $\alpha$ . The angular momentum of the states is labeled at the ends. At the left end,  $\alpha=0$  corresponds to the noninteracting bosons in an oscillator while  $\alpha=1$  corresponds to noninteracting fermions. In the case of nonlinear interpolation, the curves correspond to the best fit to the numerical eigenvalues for various values of  $\alpha$ . In the figure, all states up to  $5\hbar\omega$  are shown. At  $6\hbar\omega$ , there are only three states at the fermionic end (all shown), and six at the bosonic end (only one shown). The number of states increases rapidly from  $7\hbar\omega$ .

pared to the results of Ref. 18 there is a consistent overestimation of the eigenvalue of the ground state of three fermions as a function of  $\alpha$  due to the basis being not large enough (typically the size is about 11 states for each  $L$ , except as noted above for  $L=0$ ). On the other hand, we have the advantage of knowing the unperturbed energies at both limits in this basis and of eliminating the center-of-mass excitations everywhere. The interpolation pattern of the low-lying states is clearly established from Figs. 1 and 3. Recently Khare and McCabe<sup>16</sup> have calculated the ground-state energy of three anyons in a harmonic-oscillator potential near fermionic statistics in perturbation theory. They show at the fermionic limit that the second-order perturbation contribution from the  $l_\rho$  term [see Eq. (6)] mixes in the odd-parity states  $\tilde{\phi}_A$  with the ground state and lowers the energy. We find a similar result when the basis is enlarged to  $N=26$  for  $L=0$ . For example, for  $\alpha=0.9$ , the  $L=0$  ground-state energy converges to  $4.015\hbar\omega$  compared to their perturbative value of  $4.013\hbar\omega$ .

To summarize, we have outlined a very general method of constructing three-particle harmonic-oscillator states in terms of the internal coordinates with definite permutation symmetry and angular momentum in two dimensions. With this basis we have diagonalized the relative anyonic Hamiltonian and obtained the interpolation pattern<sup>22</sup> of the low-lying anyonic states, including the fermionic ground state that was missing in the calculation of Ref. 9. For many states we have checked the interpolation from bosonic as well as from the fermionic limits.

For some of the states that interpolate nonlinearly, we miss the correct energy at  $\alpha=1$ ; however, diagonalizing from both limits and matching the energies around  $\alpha=\frac{1}{2}$  establishes the interpolation.

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#### APPENDIX

We illustrate an alternative method of constructing symmetric and antisymmetric polynomials. This elegant method was used to subsequently check the states used in the diagonalization of the anyonic Hamiltonian. The notation and method are similar to that introduced by Kramer and Moshinsky<sup>20</sup> and Laughlin.<sup>21</sup> To this end we introduce the complex internal coordinates

$$Z_1 = \lambda_+ + i\rho_+, \quad (\text{A1})$$

$$Z_2 = \lambda_+ - i\rho_+. \quad (\text{A2})$$

Under permutations they have the property

$$P(1,2) \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \quad (\text{A3})$$

$$P(2,3) \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 0 & e^{2\pi i/3} \\ e^{-2\pi i/3} & 0 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}. \quad (\text{A4})$$

Note that  $(Z_1 Z_2)$  has a transformation property identical to that of  $(Z_1^* Z_2^*)$ . As a result we have the elementary symmetric polynomials, when  $N=L$  (apart from the trivial  $\phi_S^{0,0}$ ),

$$\phi_S^{2,2} = Z_1 Z_2, \quad (\text{A5})$$

$$\phi_S^{3,3} = Z_1^3 + Z_2^3, \quad (\text{A6})$$

and elementary antisymmetric polynomials

$$\phi_A^{2,0} = Z_1 Z_1^* - Z_2 Z_2^*, \quad (\text{A7})$$

$$\phi_A^{3,3} = Z_1^3 - Z_2^3. \quad (\text{A8})$$

It is now a simple exercise to write down the polynomials corresponding to the highest  $L$  for each  $N$  (see, for example, Ref. 21),

$$\phi_{S,A}^{N,N} = [Z_1 Z_2]^j [Z_1^{3k} \pm Z_2^{3k}], \quad N=2j+3k \quad (\text{A9})$$

where the  $\pm$  refers to  $S$  or  $A$  polynomials.

One can generate the  $-N \leq L \leq N$  polynomials by using the step-down operator,

$$\mathcal{L}_- = \left[ Z_1^* \frac{\partial}{\partial Z_2} + Z_2^* \frac{\partial}{\partial Z_1} \right], \quad (\text{A10})$$

which does not alter the symmetry of the polynomials. For example, using this operator, we have

$$\phi_S^{2,0} = \mathcal{L}_- \phi_S^{2,2} = Z_1 Z_1^* + Z_2 Z_2^*, \quad (\text{A11})$$

$$\phi_S^{3,1} = \mathcal{L}_- \phi_S^{3,3} = Z_1^2 Z_2^* + Z_2^2 Z_1^*, \quad (\text{A12})$$

etc. Note that unlike in three dimensions, the operator  $\mathcal{L}_-$  always reduces the angular momentum by two units. It is easy to compare these examples with the ones given in Table II.

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