FRANZ VON KUTSCHERA

VALUATIONS FOR DIRECT PROPOSITIONAL LOGIC

In (1969) a subsystem of classical propositional logic – called *direct logic* since all indirect inferences are excluded from it – was formulated as a generalized sequential calculus. With its rules the truth-value (true (t) or false (f)) of the consequent can be determined from the truth-values of the premisses. In this way the semantical stipulations are formulated constructively as in a calculus of natural deduction, i.e. semantics itself is a formal system of rules. The rules for propositional composita define an extension of basic calculi containing only truth rules for atomic sentences. The sentences provable in the extensions of all basic calculi, i.e. in the propositional calculus itself, are the logically true sentences. The principle of bivalence is not presupposed for the basic calculi. That it holds for a specific calculus – and then also in its propositional extension – is rather a theorem to be proved by metatheoretic means.

In this approach a semantics in the usual sense, i.e. a semantics based on a concept of valuation, has no place since the calculus of direct logic itself is already conceived of as a system of semantical rules. Nevertheless it is of some interest to see that there is an adequate and intuitively plausible valuation-semantics for this calculus.

The language L used in what follows is the usual language of propositional logic with $\neg$, $\wedge$ and $\supset$ as basic operators.[1] A classical or *total* valuation of L is a function V mapping the set of all sentences of L into $\{t, f\}$ so that $V(\neg A) = t$ iff $V(A) = f$, $V(A \wedge B) = t$ iff $V(A) = V(B) = t$. In classical logic $\supset$ is definable by $\neg$ and $\wedge$; so these two truth conditions suffice. Partial valuations V that only map a subset of the set of sentences of L into $\{t, f\}$ can be defined in a number of ways. Following K. Fine in (1975) we cut down this number by the following principles. We write "$V(A) = u$" for "V is not defined for A". "$V(A) \neq u$" therefore means that $V(A) = t$ or $V(A) = f$.

(I) If $V(A_i) \neq u$ for all $1 \leq i \leq n$ and C is a propositional operator, then $V(C(A_1, \ldots, A_n)) \neq u$ and $V(C(A_1, \ldots, A_n)) = V'(C(A_1, \ldots, A_n))$ for each total valuation V' with $V(A_i) = V(A_i)$ for all i.

Erkenntnis **19** (1983) 253–260. 0165–0106/83/0191–0253 $00.80
Copyright © 1983 *by D. Reidel Publishing Co., Dordrecht, Holland, and Boston, U.S.A.*

Fine calls this the *Principle of Fidelity*. It says that the classical interpretation of propositional operators is to be retained. (I) implies the following truth conditions:

(a) If $V(A) = \text{f}$ then $V(\neg A) = \text{t}$.
If $V(A) = \text{t}$ then $V(\neg A) = \text{f}$.
(b) If $V(A) = V(B) = \text{t}$ then $V(A \wedge B) = \text{t}$.
If $V(A) = \text{f}$ and $V(B) \neq \text{u}$, or $V(B) = \text{f}$ and $V(A) \neq \text{u}$ then $V(A \wedge B) = \text{f}$.

If we define for partial valuations V and V':

DEFINITION 1. V' is an *extension* of V – for short $V' \geq V$ – iff for all atomic sentences A $V(A) \neq \text{u}$ implies $V'(A) = V(A)$,

we can state a second important principle, that of *Stability*, thus:

(II) If $V' \geq V$ and $V(A) \neq \text{u}$ then $V'(A) = V(A)$ for all sentences A of L.

From this we obtain:

(a′) If $V(\neg A) = \text{t}$ then $V(A) = \text{f}$.
If $V(\neg A) = \text{f}$ then $V(A) = \text{t}$.
(b′) If $V(A \wedge B) = \text{t}$ then $V(A) = V(B) = \text{t}$.
If $V(A \wedge B) = \text{f}$ then $V(A) = \text{f}$ or $V(B) = \text{f}$.

This means: If there are extensions of V that assign A different truth-values, $V(A)$ must be indeterminate; a compound sentence cannot be determinate if its truth value classically depends on that of one of its parts and that part is indeterminate.

As a third postulate we take Fine's *Principle of Maximal Definiteness*:

(III) If C is an n-place propositional operator $V(C(A_1, \ldots, A_n))$ is to be defined whenever that is possible according to (II).

The second condition in (b) may then be strengthened to: If $V(A) = \text{f}$ or $V(B) = \text{f}$ then $V(A \wedge B) = \text{f}$. In this way we arrive at the usual concept of a partial valuation:

DEFINITON 2. A *partial valuation* of L is a function V mapping a subset of the sentences of L into $\{t,f\}$ so that

(1) $V(\neg A) = t$ iff $V(A) = f$.
$V(\neg A) = f$ iff $V(A) = t$.

(2) $V(A \wedge B) = t$ iff $V(A) = V(B) = t$.
$V(A \wedge B) = f$ iff $V(A) = f$ or $V(B) = f$.

The inferences valid in all partial valuations are that of minimal propositional logic. No sentence is true in all such valuations. For instance, $A \supset A$ cannot be true in a valuation V for which $V(A) = u$ according to (II). Intuitively, however, we would regard this sentence as true since $A \supset A$ will be true in any precisification of V that makes A true or false. $A \supset A$ is true whatever A means, and therefore we are inclined to regard $A \supset A$ as a logical truth. This idea suggests that we employ supervaluations as introduced by B. van Fraassen in (1970). If S_V is the set of all total extensions of V, then $S_V(A) = t/f$ iff for all V' in S_V $V'(A) = t/f$. This approach, however, is not satisfactory for the following reasons:

1. Supervaluations are not recursively defined. But it is a fundamental principle of semantics that the meaning of a compound expression is determined by that of its parts.

2. We cannot assume that every partial valuation has a total extension. If r is Russell's set, for instance, r $\in$ r cannot be regarded as either true or false. And the sentence "A is vague" would be false in every total valuation, and therefore in every supervaluation.

Fine has generalized van Fraassen's approach. His supervaluations are sets S of 3-valued valuations (with u as third truth value). On S a partial ordering is defined by $\leq$ and the V in S are recursively defined in the sense of modal logic by reference to the V' in S with $V' \geq V$. The following definition is based on this idea:

DEFINITION 3. A *D-valuation* of L is a triple $\mathfrak{M} = \langle I, S, V \rangle$ such that:

(1) I is a non-empty set of indices.

(2) For all $i \in I$ S_i is a subset of I such that
(a) $i \in S_i$
(b) $j \in S_i \wedge k \in S_j \supset k \in S_i$

(3) For all $i \in I$ V_i is a function mapping a subset of sentences of L into {t, f} so that
(a) If $j \in S_i$ then $V_j \geq V_i$,
(b) V_i fulfills the conditions (1) and (2) of D2.
(c) $V_i(A \supset B) =$ t iff for all $j \in S_i$ $V_j(B) =$ t if $V_j(A) =$ t.
$V_i(A \supset B) =$ f iff $V_i(A) =$ t and $V_i(B) =$ f.

It is easily seen that for D-valuations we then have: If $j \in S_i$ and $V_i(A) \neq$ u then $V_j(A) = V_i(A)$ for all sentences A of L. All V_i are partial valuations with the sole exception that $A \supset B$ now is not defined by $\neg (A \wedge \neg B)$. $V_i(A \supset B) =$ t is not equivalent with $V_i(A) =$ f or $V_i(B) =$ t; $V_i(A \supset B) =$ t can hold even if $V_i(A) =$ u and $V_i(B) \neq$ t. Especially we now have $V_i(A \supset A) =$ t for all i and all D-valuations.

D-valuations do not just assign truth values to sentences but also define inferential relations between them. For all meaning relations between atomic sentences of the type: "If $A_1, \ldots, A_m$ are true and $B_1, \ldots, B_n$ are false then C is true (false)," the set of the extensions V_j of V_i in a D-valuation can be so determined that these sentences $A_1 \wedge .. \wedge A_m \wedge \neg B_1 \wedge .. \wedge \neg B_n \supset (\neg)C$ come out true in i. By a suitable choice of I and the V_j in $\mathfrak{M}$ we can therefore capture all meaning relations – or *penumbral connections*, as Fine calls them – between the atomic sentences of L, just as we can distinguish analytic truths in intensional semantics by a suitable choice of the set of possible worlds. The concept of a D-valuation, then, results if we start out from partial valuations based on the principles of Fidelity, Stability and Maximal Definiteness and interpret implications in such a way that we can state all relations between truth values of sentences with them. This definition of the operator $\supset$ is in accordance with Fidelity, for if $V_i(A) \neq$ u $\neq V_i(B)$ $V_i(A \supset B) =$ t iff $V_i(A) =$ f or $V_i(B) = w$.

Evidently it is possible to restrict the concept of a D-valuation in such a way that there is an index i_0 in I with $S_{i_0} = I$ (all V_j are then extensions of V_{i_0}), or that $j \in S_i$ is a partial ordering with $j \in S_i \wedge i \in S_j \supset i = j$, or that $V_j \geq V_i$ implies $j \in S_i$, without altering the resulting logic. As usual we say that a sentence A is *satisfied* by a D-valuation $\mathfrak{M} = \langle I, S, V \rangle$ iff $V_i(A) =$ t for all $i \in I$; that A is *D-true* iff A is satisfied by all D-valuations; that an interference $A_1, \ldots, A_n \rightarrow B$ is $\mathfrak{M}$-*valid* iff $V_i(A_1) = .. = V_i(A_n) =$ t implies $V_i(B) =$ t for all $i \in I$; and that the inference is *D-valid* iff it is valid in all D valuations.

If we call a D-valuation $\mathfrak{M} = \langle I, S, V \rangle$ *complete* iff there is a $j \in I$ such that V_j is a total valuation, then for every complete D-valuation there is a total valuation V' for which $V'(A) = V_{i_0}(A)$ for all sentences A with $V_{i_0}(A) \neq$ u, and vice versa. This also holds for *completable* D-valuations $\mathfrak{M}$, for which there is a $j \in I$ with $V_j(A) \neq$ u for all sentences A. A therefore is classically true iff no completable D-valuation satisfies $\neg A$.

We now want to show that the D-true sentences are exactly those that are provable in direct logic. This logic may be stated in the form of a calculus D^* with the following axioms and rules:

AXIOM 1. $A \supset (B \supset A)$
AXIOM 2. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
AXIOM 3. $\neg A \supset (A \supset B)$
AXIOM 4. $A \supset (\neg B \supset \neg (A \supset B))$
AXIOM 5a. $\neg (A \supset B) \supset A$
b. $\neg (A \supset B) \supset \neg B$
AXIOM 6. $A \supset \neg\neg A$
AXIOM 7. $\neg\neg A \supset A$
AXIOM 8. $A \supset (B \supset A \wedge B)$
AXIOM 9a. $A \wedge B \supset A$
b. $A \wedge B \supset B$
AXIOM 10a. $\neg A \supset \neg (A \wedge B)$
b. $\neg B \supset \neg (A \wedge B)$
AXIOM 11. $(\neg A \supset C) \supset ((\neg B \supset C) \supset ((\neg (A \wedge B) \supset C))$
RULE 1. $A, A \supset B \vdash B$.

In D^* $A \supset A$ is a theorem, and with it the deduction theorem may be proved in the usual way.

The soundness of D^* with respect to D-valuations is easily shown. All axioms of D^* are D-true, and with Rule 1 we obtain only D-true sentences from D-true premisses.

For the completeness proof we need the following stipulations and definitions: If $\mathfrak{B}$ is a set of sentences, $\mathfrak{B} \vdash A$ holds iff there is a finite subset $\mathfrak{B}'$ of $\mathfrak{B}$ from which A is derivable. $\mathfrak{B}$ is *consistent* iff not all sentences are derivable from $\mathfrak{B}$.

DEFINITION 4. A set of sentences $\mathfrak{B}$ is called *regular* iff

(a) If $\mathfrak{B} \vdash A$ then $A \in \mathfrak{B}$.
(b) If $\neg (A \wedge B) \in \mathfrak{B}$ then $\neg A \in \mathfrak{B}$ or $\neg B \in \mathfrak{B}$.
(c) $\mathfrak{B}$ is consistent.

DEFINITION 5. A *D-system* is a pair $\mathfrak{S} = \langle \mathrm{I}, \mathfrak{R} \rangle$ such that:

(a) I is a non-empty set of indices.
(b) For all $i \in I$ $\mathfrak{R}_i$ is a regular set of sentences.
(c) For all sentences A, B and all $i \in I$: if $A \supset B$ is not in $\mathfrak{R}_i$ there is a $j \in I$ with $\mathfrak{R}_i \subset \mathfrak{R}_j$, $A \in \mathfrak{R}_j$ and not $B \in \mathfrak{R}_j$.

We first prove two lemmata:

LEMMA 1. Every consistent set of sentences $\mathfrak{B}$ from which A is not derivable, can be extended to a regular set $\mathfrak{B}'$ not containing A.

Proof. Let $\neg (B \wedge C)_1, \neg (B \wedge C)_2, \ldots$ be a denumeration of all sentences of L of the form $\neg (B \wedge C)$. We set $\mathfrak{B}_0 = \mathfrak{B}$,

$$\mathfrak{B}_{n+1} = \begin{cases} \mathfrak{B}_n \cup \{\neg (B \wedge C)_{n+1} \supset \neg B\} \text{ if } \mathfrak{B}_n, \neg B \vdash A \text{ does not hold,} \\ \mathfrak{B} \cup \{\neg (B \wedge C)_{n+1} \supset \neg C\} \text{ if } \mathfrak{B}_n, \neg B \vdash A, \text{ but not } \mathfrak{B}_n, \neg C \vdash A, \\ \mathfrak{B}_n \text{ otherwise.} \end{cases}$$

$\mathfrak{B}''$ is to be the union of the sets $\mathfrak{B}_n$, and $\mathfrak{B}'$ the consequence set of $\mathfrak{B}''$. $\mathfrak{B}'$ is then closed with respect to derivability in D^*. We have then:

(1) For no n $\mathfrak{B}_n \vdash A$. This holds for $n = 0$ according to the condition of lemma 1, and if not $\mathfrak{B}_n \vdash A$ then not $\mathfrak{B}_{n+1} \vdash A$ in view of the definition of $\mathfrak{B}_{n+1}$, for if $\mathfrak{B}_n, \neg B \vdash A$ does not hold neither does $\mathfrak{B}_n, \{\neg (B \wedge C)_{n+1} \supset \neg B\} \vdash A$.

(2) Not $\mathfrak{B}'' \vdash A$. Otherwise there would be a finite subset $\mathfrak{B}^+$ of $\mathfrak{B}''$ with $\mathfrak{B}^+ \vdash A$. But if n is the greatest number such that $\neg (B \wedge C)_{n+1} \supset \neg B$ or $\neg (B \wedge C)_{n+1} \supset \neg C$ is in $\mathfrak{B}^+$, we would have $\mathfrak{B}_{n+1} \vdash A$ in contradiction to (1). A, then, is not in $\mathfrak{B}'$ and therefore $\mathfrak{B}'$ is also consistent.

(3) If $\neg (B \wedge C)_{n+1} \in \mathfrak{B}'$ then $\neg B \in \mathfrak{B}'$ or $\neg C \in \mathfrak{B}'$. For either $\neg (B \wedge C)_{n+1} \supset \neg B$ or $\neg (B \wedge C)_{n+1} \supset \neg C$ is in $\mathfrak{B}_{n+1}$ and therefore in $\mathfrak{B}'$ and therefore $\neg B$ or $\neg C$ in $\mathfrak{B}'$ since $\mathfrak{B}'$ is closed. Or $\mathfrak{B}_n, \neg B \vdash A$ and $\mathfrak{B}_n, \neg C \vdash A$. But then $\mathfrak{B}_n, \neg (B \wedge C) \vdash A$ according to Axiom

11 and the deduction theorem, and therefore $\mathfrak{B}', \neg (B \wedge C)_{n+1} \vdash A$. In view of (2) this is incompatible with $\neg (B \wedge C)_{n+1} \in \mathfrak{B}$, however.

$\mathfrak{B}'$ therefore is regular and in view of its construction $\mathfrak{B} \subset \mathfrak{B}'$.

LEMMA 2. If A is not provable in D^* there is a D-system $\langle I, \mathfrak{R} \rangle$ and an $i \in I$ such that A is not in $\mathfrak{R}_{i_0}$.

Proof. Let $\mathfrak{R}_i^+$ be the empty set, which is consistent and from which A is not derivable according to the assumption in Lemma 2.[2] As in Lemma 1 we extend $\mathfrak{R}_i^+$ to a regular set $\mathfrak{R}_{i_0}$ from which A is not derivable. For every sentence $B \supset C$ not in $\mathfrak{R}_{i_0}$ let $\mathfrak{R}_j^+$ be the set $\mathfrak{R}_{i_0} \cup \{B\}$. $\mathfrak{R}_{i_0}, B \vdash C$ cannot hold, for otherwise $\mathfrak{R}_{i_0} \vdash B \supset C$ and therefore $B \supset C \in \mathfrak{R}_{i_0}$. $\mathfrak{R}_j^+$, then, is a consistent set from which C is not derivable. It is extended to a regular set $\mathfrak{R}_j$ according to Lemma 1, and so forth. If I is a set of indices for all the sets $\mathfrak{R}_i$ we obtain in this way, $\langle I, \mathfrak{R} \rangle$ is a D-system and A is not in $\mathfrak{R}_{i_0}$.

The completeness of D^* can now be proved in the following way: If A is not a theorem of D^*, there is a D-system $\langle I, \mathfrak{R} \rangle$ and an index $i_0 \in I$ such that A is not in $\mathfrak{R}_{i_0}$. For all $i \in I$ we define sets S_i and functions V_i by

(a) $j \in S_i$ iff $\mathfrak{R}_i \subset \mathfrak{R}_j$

(b) $V_i(B) = \text{t}$ iff $B \in \mathfrak{R}_i$

$V_i(B) = \text{f}$ iff $\neg B \in \mathfrak{R}_i$, for all sentences B.

Then $\langle I, S, V \rangle$ is a D-valuation. By (a) conditions (2a,b) from Definition 3 are satisfied, and also condition (3a). By (b) this also holds for (3b)-(3c):

$V_i(\neg B) = \text{t}$ iff $\neg B \in \mathfrak{R}_i$ iff $V_i(B) = \text{f}$.
$V_i(\neg B) = \text{f}$ iff $\neg\neg B \in \mathfrak{R}_i$ iff $B \in \mathfrak{R}_i$ (cf. A6,A7) iff $V_i(B) = \text{t}$.
$V_i(B \wedge C) = \text{t}$ iff $B \wedge C \in \mathfrak{R}_i$ iff $B, C \in \mathfrak{R}_i$ (cf. A8,A9) iff $V_i(B) = V_i(c) = \text{t}$.
$V_i(B \wedge C) = \text{f}$ iff $\neg (B \wedge C) \in \mathfrak{R}_i$ iff $\neg B \in \mathfrak{R}_i$ or $c \in \mathfrak{R}_i$ (in view of the regularity of $\mathfrak{R}_i$ and Axiom 10) iff $V_i(B) = \text{f}$ or $V_i(C) = \text{f}$).
If $V_i(B \supset C) = \text{t}$ then $B \supset C \in \mathfrak{R}_i$. If $j \in S_i$ then according to (a) $B \supset C \in \mathfrak{R}_j$. If $V_j(B) = \text{t}$ and therefore $B \in \mathfrak{R}_j$ then $C \in \mathfrak{R}_j$ in view of Rule 1 and the closure of $\mathfrak{R}_j$, hence $V_j(C) = \text{t}$.

If, on the other hand, for all $j \in S_i$ and $V_j(B) = \text{t}$ we have $V_j(C) = \text{t}$, we also have $C \in \mathfrak{R}_j$ in case of $B \in \mathfrak{R}_j$. Then $B \supset C$ must be in $\mathfrak{R}_i$ and therefore $V_i(B \supset C) = \mathrm{t}$, for otherwise there would be a j such that $\mathfrak{R}_i \subset \mathfrak{R}_j$ and not $C \in \mathfrak{R}_j$.

$V_i(B \supset C) = \mathrm{f}$ iff $\neg (B \supset C) \in \mathfrak{R}_i$ iff B, $\neg C \in \mathfrak{R}_i$ (cf. A4,A5) iff $V_i(B) = \mathrm{t}$ and $V_i(C) = \mathrm{f}$.

$\mathfrak{M} = \langle I, S, V \rangle$, then, is a D-valuation, and since A is not in $\mathfrak{R}_{i_0}$ $V_{i_0}(A) \neq \mathrm{t}$, i.e. $\mathfrak{M}$ does not satisfy A.

For the direct version of predicate logic the completeness proof is much more complicated – mainly because $\bigvee y(A[y] \supset \bigwedge x A[x])$ and $\bigvee y(\neg \bigwedge x A[x] \supset \neg A[y])$ are not theorems of this logic. The structure of the valuation-concept that fits direct logic, however, already becomes sufficiently clear from the propositional case.

NOTES

[1] The completeness of the system of operators $\{\neg, \wedge, \supset\}$ in the framework of direct logic has been proved in (1969).

[2] We cannot set $\mathfrak{R}_{i0}^{+} = \{\neg A\}$ or $\mathfrak{R}_{i0}^{+} = \{A \supset \neg A\}$ for these sets may be inconsistent as the cases of $A = B \vee \neg B$ and $A = (B \supset C) \supset (B \supset \neg B) \vee C$ shown.

REFERENCES

Fine, K.: 1975,, 'Vagueness, truth, and logic', *Synthese* **30**, 265–300.

Fraassen, B. van: 1970, 'Presuppositions, supervaluations, and free logic, in K. Lambert (ed.), *Philosophical Problems in Logic*, D. Reidel, Dordrecht, 67–91.

Kutschera, F. v.: 1969, 'Ein verallgemeinerter Widerlegungsbegriff für Gentzenkalküle', *Archiv f. math. Logik und Grundlagenforschung* **12**, 104–118.

Manuscript received 5 January 1982