On the local theory of signatures and reduced quadratic forms.

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Kleinstein and Rosenberg have proved [KL R_1, Prop. 6.4] that the space of signatures of a connected semilocal ring A is a "space of orderings" in the sense of Marshall, cf. the definition in § 2 below [1]. Thus Marshall's abstract theory of reduced Wittrings in $[M_i]$, $i = 0,1,2,3,4$, can be applied to semilocal rings and yields a very satisfactory generalization of the theory of reduced quadratic forms developed over fields by Becker, Köpping and L.Bröcker in the papers [Be K], [Be B] and [B] (cf. also the recent paper [Br M] of Brown and Marshall).

The goal of the present paper is to make explicit to a large extent what Marshall's theory means for semilocal rings, and to study the relations between the signatures, the reduced quadratic and bilinear forms over A and over the various residue class fields $A(\mathfrak{p})$ of the prime ideals $\mathfrak{p}$ of A. $\{A(\mathfrak{p})$ is the quotient field of $A/\mathfrak{p}.\}$ We thus hope to lay firm ground for further studies in the local theory of reduced quadratic forms.

The first three sections of the paper are of expository nature. After reviewing the notions "signature" and "reduced Witt ring " in § 1 we translate in § 2 Marshall's notions "form", "isotropy",

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1) Kleinstein and Rosenberg always assume that A has no residue class fields with only two elements. We shall remove this restriction.

"isometry" etc. over a space of orderings (X,G) into more concrete terms in the special case that (X,G) is the space of signatures of our semilocal ring A. In § 3 we do the same for the subspaces of (X,G) in the sense of Marshall. I wish to emphasize that nearly all results of § 2 and § 3 are explicitly or implicitly contained in the papers [Kl R], [Kl R_1] of Kleinstein and Rosenberg. But these authors use their own abstract framework including also nonreduced forms. It seems to be more laborious to identify in their papers what we need than to give a selfcontained exposition, as we do here. Moreover it would simply be inadequate to use two abstract theories to explain the rather elementary contents of § 2 and § 3.

With every signature σ of A there is associated a prime ideal $\mathfrak{p}(\sigma)$ of A in a canonical way, as has first been observed by Kanzaki and Kitamura in a special case [Ka Ki] and then in general in [K_2, § 4]. The sections § 4 - § 6 of the present paper are devoted to a detailed study of this relationship between prime ideals and signatures. In particular we obtain a "stratification" of the space X of signatures of A which reflects the relations between X and the signature spaces of $A(\mathfrak{p})$, $A/\mathfrak{p}$ and $A_\mathfrak{p}$ for the prime ideals $\mathfrak{p}$ of A. Our work here is by no means complete.

In § 7 we then prove the central theorem that all signatures σ in a nontrivial fan of X have the same associated prime ideal. This theorem enables us in § 8 to refine our stratification of X to a "P-structure" in the sense of Marshall [M_4, § 3] in a natural way. Moreover the theorem together with Marshall's theory yields various results about the relations between reduced forms over A and the fields $A(\mathfrak{p})$, as will be explained in § 9. In particular we shall see that a continuous function $f: X \to \mathbb{Z}$ can be represented by a bilinear or by a quadratic form over A, if this holds true over $A(\mathfrak{p})$ for the "restriction" of f to the signature space of $A(\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ of A.

In the last section § 10 we observe that with every semisignature (cf. [K_1, § 5] or [Kl R]) there is again associated a prime ideal $\mathfrak{p}(\sigma)$ in a canonical way. We then prove that a bilinear or quadratic form φ over A is weakly isotropic if and only if $\varphi \otimes A(\mathfrak{p})$ is weakly isotropic over the field $A(\mathfrak{p})$ for every prime ideal $\mathfrak{p}$. We also give a relative version of this theorem with respect to some saturated subgroup T (cf. § 3) of the group of units A^*. As far as I know this theorem has first been proved by L.Bröcker [B_2]. He uses a different method also involving semisignatures. The approach to weak isotropy via semisignatures goes in the field case back to Prestel [P_1]. Recently Brown and Marshall introduced over fields a new method avoiding semisignatures [Br M]. It would be interesting to generalize also their method to semilocal rings.

Contents

§ 1 Witt rings and signatures

Let A be a semilocal ring, i.e. a commutative ring with 1 which has only finitely many maximal ideals $m_1,\dots,m_g$. Without essential loss of generality we assume that A is connected, i.e. has only the idempotents 0 and 1.

The local algebraic theory of quadratic forms is concerned with bilinear spaces and quadratic spaces over A. A bilinear space is a - always finitely generated - projective A-module E equipped with a non degenerate symmetric bilinear form $B:E \times E \to A$, and a quadratic space is a projective A-module E equipped with a non degenerate quadratic form $q:E \to A$. That q is non degenerate means that the associated bilinear form $B_q(u,v) = q(u+v) - q(u) - q(v)$ is non degenerate. These notions make sense over an arbitrary commutative ring. In the present case we know that E is a free module over A. Moreover any bilinear space (E,B) which represents a unit is of the shape $\langle a_1,\dots,a_n\rangle$, i.e. has a base $e_1,\dots,e_n$ with $B(e_i,e_j) = a_i\delta_{ij}$ [Ba, Chap.I, Prop. 3.5]. Of course all a_i lie in the group of units A^*. Any quadratic space of even rank is an orthogonal sum of binary spaces [a,b], cf. [Ba, Chap.I, Prop. 3.4]. Here [a,b] denotes the quadratic space of rank 2 with basis e,f and $q(e) = a$, $q(f) = b$, $B_q(e,f) = 1$. Of course $1 - 4ab$ has to be a unit, since q is non degenerate.

If 2 is a unit in A then bilinear spaces and quadratic spaces are the same thing and any space is of the shape $\langle a_1,\dots,a_n\rangle$. If 2 is not a unit then all quadratic spaces are of even rank, hence orthogonal sums of spaces [a,b]. Even if you are only interested in bilinear spaces it is sometimes useful to work also with quadratic spaces and vice verse. Sections 2 and 3 of the present paper will give examples for this. We refer the reader to the first pages of Baeza's book [Ba] for elementary facts on quadratic and symmetric bilinear spaces, in particular for the definition of the tensor product of two bilinear spaces, a bilinear space with a quadratic space, and of two quadratic spaces, cf. also [MH, Appendix 1].

More modestly the local algebraic theory of quadratic forms is concerned with the Witt ring W(A) of bilinear spaces and the Witt ring Wq(A) of quadratic spaces over A. The elements of these rings are the "Witt classes" of bilinear spaces and of quadratic spaces respectively, i.e. the equivalence classes with respect to the Witt equivalence relation, denoted here by ~, cf. [Ba] for the details. The addition in these rings comes from the orthogonal sum and the multiplication from the tensor product of spaces.[2)] Moreover Wq(A) is a module over W(A) via the tensor product of bilinear spaces with quadratic spaces. The reader is advised to look into my lecture notes [K] for motivation to study the ring W(A) and to understand what is the global theory about corresponding to our local theory here.

Let me point out the most evident facts about the relations between W(A) and Wq(A), valid for A an arbitrary commutative ring. We have a natural ring homomorphism $z \to \tilde{z}$ from Wq(A) to W(A) mapping

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2) If 2 is not a unit in A then the ring Wq(A) has no unit element.

a Witt class $z = [E,q]$ to the Witt class $\tilde{z} = [E,B_q]$ of the associated bilinear space. Let $\mathfrak{a}$ denote the kernel of this map. For any two elements z,v in $Wq(A)$ we have

$$zv = \tilde{z}v = \tilde{v}z.$$

Thus $\mathfrak{a}Wq(A) = 0$. In particular $\mathfrak{a}$ is contained in the nil radical of $Wq(A)$. Let ξ_o denote the image of the Witt class of the positive definite quadratic space E_8 of rank 8 over Z under the natural map from $Wq(Z)$ to $Wq(A)$. We have $\tilde{\xi}_o = 8\cdot 1_{W(A)}$. Thus for any z in $\mathfrak{a}$

$$8z = \xi_o z = \tilde{z}\xi_o = 0,$$

i.e. $8\mathfrak{a} = 0$. Similarly also the cokernel of the map $Wq(A) \to W(A)$ is annihilated by 8, since for any z in $W(A)$

$$8z = \tilde{\xi}_o z = (z\xi_o)^{\sim}.$$

In the present paper we will study the reduced Witt rings

$$\overline{W}(A) := W(A)/\mathrm{Nil}\ W(A),\ \overline{W}q(A) = Wq(A)/\mathrm{Nil}\ Wq(A),$$

where "Nil" means the nil radical. Since the natural map from $Wq(A)$ to $W(A)$ has a nilpotent kernel $\mathfrak{a}$, it yields an <u>injection</u> from $\overline{W}q(A)$ to $\overline{W}(A)$. Henceforth we consider $\overline{W}q(A)$ as a subset of $\overline{W}(A)$. This subset is an ideal of $\overline{W}(A)$ containing $8\overline{W}(A)$.

Nil $W(A)$ is the intersection of the minimal prime ideals of $W(A)$. These ideals are fairly well understood. A <u>signature</u> σ of A is by definition a ring homomorphism σ from $W(A)$ to Z. Since the ring Z has no automorphisms except the identity the signatures σ are uniquely determined by their kernels $P_\sigma \subset W(A)$ and these kernels are precisely all prime ideals P of W(A) with $W(A)/P \cong \mathbb{Z}$. The following theorem holds true if A is an arbitrary connected commutative ring, cf. [KRW] for the semilocal case [3], and [D], [K] for the general case.

Theorem 1.1. If A has no signatures then the ideal I(A) consisting of the spaces of even rank is the only prime ideal of W(A) and hence the nil radical. Also $2^n W(A) = 0$ for some $n \geq 1$ in this case. If A has signatures σ then the corresponding kernels P_σ are precisely all minimal prime ideals of W(A).

If A is semilocal we also have

Theorem 1.2 [KRW]. The set of torsion elements z in W(A), i.e. elements z with $nz = 0$ for some $n \in \mathbb{N}$, coincides either with the nil radical Nil W(A) or the whole of W(A). Moreover any torsion element has 2-power order.

We denote the set of signatures of A by Sign(A). If Sign(A) is not empty we call A "formally real". This terminology is in good harmony with the usual definition for fields. Indeed, it can be shown for any commutative ring A that A is formally real if and only if -1 is not a sum of squares in A [K, Chap.III § 2].

Since now we always tacitly assume that A is formally real. Sign(A) can be identified with the spectrum of the ring $\mathbb{Q} \otimes_{\mathbb{Z}} W(A)$

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3) The "abstract theory" in [KRW] has recently been presented in a restricted, hence easier accessible way in [KK]. The treatment in [KK] suffices for all our purposes.

by associating with a signature σ the kernel $\mathbb{Q} \otimes P_\sigma$ of the induced homomorphism from $\mathbb{Q} \otimes W(A)$ to $W(A)$. Since $\mathbb{Q} \otimes W(A)$ has Krull dimension zero the topological space $\mathrm{Spec}(\mathbb{Q} \otimes W(A))$ is a Boolean space, i.e. a compact totally disconnected Hausdorff space. Thus by the above identification Sign(A) is a Boolean space X. Without referring to prime ideals the topology on X can be described as the coarsest topology such that all functions

$$\hat{z}: X \to \mathbb{Z}, \quad \sigma \to \sigma(z)$$

with z running through W(A) are continuous, $\mathbb{Z}$ being equipped with the discrete topology.

The ring homomorphism $z \to \hat{z}$ from W(A) to the ring $C(X,\mathbb{Z})$ of continuous $\mathbb{Z}$-valued functions on X clearly has the kernel Nil W(A). Thus $\overline{W}(A)$ embeds into $C(X,\mathbb{Z})$ and will often be regarded as a subring of $C(X,\mathbb{Z})$. The natural map from $\mathbb{Q} \otimes \overline{W}(A)$ to $C(X,\mathbb{Q})$ is bijective, cf. [AK, Th. 2.3]. This implies that $C(X,\mathbb{Z})/\overline{W}(A)$ is a torsion group. In order to describe $\overline{W}(A)$ and $\overline{W}q(A)$ as subsets of $C(X,\mathbb{Z})$ we are lead to the following problem.

<u>Representation problem.</u> For which continuous functions $f: X \to \mathbb{Z}$ does there exist a bilinear space or even a quadratic space E such that $f(\sigma) = \sigma(E)$ for all σ in X?

Here we of course mean by $\sigma(E)$ the value of σ on the Witt class [E] in the bilinear case and on the image of [E] in W(A) in the quadratic case. If $f(\sigma) = \sigma(E)$ for all σ in X then we say that f is <u>represented</u> by the space E.

A solution of the representation problem for A a field has been given by Becker and Bröcker [Be B] and subsequently by Brown and Marshall [Br M]. In § 9 we shall give a solution for A semilocal following the abstract approach of Marshall in [M_3] and [M_4].

§ 2 Elementary theory of reduced forms

Assume now again that A is semilocal and has the maximal ideals $m_1, \dots, m_g$. Assume for simplicity that A is connected but notice that everything done in this section remains true in the non connected case if we work only with free bilinear and quadratic spaces and read for W(A) the Witt ring of free bilinear spaces over A, cf. the end of § 2 in [KRW_1].

For A semilocal more insight into the space Sign(A) = X can be gained from the fact that W(A) is generated by the Witt classes [<a>] of the bilinear spaces <a> of rank one, $a \in A^*$. The value of a signature σ on an element [<a>] is ± 1, since $[\langle a\rangle]^2 = 1$ in W(A). This value will be simply denoted by $\sigma(a)$. The map $a \to \sigma(a)$ from A^* to the group $\{\pm 1\}$ is a character on the group A^*/A^{*2} of square classes, which we again denote by σ. Indeed, the homomorphism σ from W(A) to $\mathbb{Z}$ is uniquely determined by this character. We have $\sigma(-1) = -1$ for every σ in X since the space $\langle 1,-1\rangle$ has Witt class zero. Also for any σ in X the following property S_n holds true for all natural numbers $n \geq 1$ [KRW_1, Lemma 2.3].

S_n: If $a_1, \dots, a_n$ are finitely many units of A with $\sigma(a_1) = \dots = \sigma(a_n) = +1$ and $\lambda_1, \dots, \lambda_n$ are elements of A such that $b := \lambda_1^2 a_1 + \dots + \lambda_n^2 a_n$ is again a unit then $\sigma(b) = 1$.

Conversely, if all fields A/m_i have more than two elements, it can be shown that any character $\chi: A^* \to \{\pm 1\}$ with $\chi(-1) = -1$ which fulfills S_2 is a signature, cf. [KRW_1, Prop. 2.4] or [K, Chap.II, § 7]. If A is a field this means that the signatures $\sigma: A^* \to \{\pm 1\}$ are precisely the sign functions corresponding to the orderings α of A (total orderings compatible with addition and multiplication, $\sigma(a) = +1$ iff $a > 0$ with respect to α). Thus in the field case, as is well known, Sign(A) can be identified with the set of orderings of A. For a bilinear space E over A the value $\sigma(E)$ is the Sylvester signature of E with respect to α.

Coming back to our formally real connected semilocal ring A we choose once and for all a natural number h such that $2h - 1$ and $4h - 1$ are units in A. We denote by $\Pi(A)$ the set of all finite sums of elements $x^2 + xy + y^2h$ with x,y in A, and denote by $\Pi^*(A)$ the set $\Pi(A) \cap A^*$. As has been shown in [K_1, Th. 2.5] $\Pi^*(A)$ is precisely the set of all units a in A with $\sigma(a) = +1$ for all signatures σ of A. In particular $\Pi^*(A)$ is a subgroup of A^* independent of our choice of h. If $2 \in A^*$ then $\Pi(A)$ is the set of sums of squares in A, since

$$x^2 + xy + y^2h = (x + \tfrac{y}{2})^2 + (4h - 1)(\tfrac{y}{2})^2.$$

We introduce the group $G(A) := A^*/\Pi^*(A)$ and consider Sign(A) = X as a subset of the character group $\hat{G}(A)$ of G(A). Since X is not empty and $\sigma(-1) = -1$ for all σ in X the group G(A) is non trivial. $\hat{G}(A)$ has as the Pontrjagin dual of the discrete group G(A) a natural topology which makes $\hat{G}(A)$ a profinite group (= Boolean space which is a topological group). It is an easy exercise to verfy that X is a closed subset of $\hat{G}(A)$ and that G(A) induces on X the same topology as defined in § 1. Thus a basis of open sets of X is given by the sets

$$X(a_1,\dots,a_r) := \{\sigma \in X \mid \sigma(a_1) = \dots = \sigma(a_r) = 1\}$$

with $a_1,\dots,a_r$ finitely many elements in A^* [4]. These sets $X(a_1,\dots,a_r)$ are also closed in X.

M.Marshall starts in his papers $[M_i]$, $i = 0,1,2,3,4$ with the following abstract situation: G is an arbitrary group of exponent 2 equipped with a base point called "-1". There is given a subset X of the character group G having the following three properties:

O_1 : X is closed in $\hat{G}$.

O_2 : $\sigma(-1) = -1$ for all $\sigma \in X$.

O_3 : The intersection of the kernels of all $\sigma \in X$ in $\{1\}$. In other words, the set X generates the group $\hat{G}$.

Notice that all this holds true in our case $G = G(A)$, $X = \mathrm{Sign}(A)$. A <u>form</u> over G is just a finite tuple $\varphi = \langle g_1,\dots,g_n\rangle$ of elements g_i of G, and n is called the <u>dimension</u> dim φ of φ. For any $\sigma \in \hat{G}$ Marshall defines $\sigma(\varphi) := \sigma(g_1) +\dots+ \sigma(g_n)$. Forms over G are added and multiplied in the obvious way:

$$\langle g_1,\dots,g_n\rangle \perp \langle h_1,\dots,h_m\rangle = \langle g_1,\dots,g_n h_1,\dots,h_m\rangle,$$

$$\langle g_1,\dots,g_n\rangle \otimes \langle h_1,\dots,h_m\rangle = \langle g_1h_1,\dots,g_1h_m,g_2h_1,\dots,g_2h_m,\dots,g_nh_m\rangle.$$

Two forms φ,ψ over G are called <u>congruent</u> modulo X, written $\varphi \equiv \psi \bmod X$ or briefly $\varphi \equiv \psi$, if $\dim \varphi = \dim \psi$ and $\sigma(\varphi) = \sigma(\psi)$ for every σ in X. A congruence class of forms over G is called a <u>form over</u> (X,G). For any form φ over G we denote by $D_X(\varphi)$ the set of all g in G such that $\varphi \equiv \langle g\rangle \perp \psi \bmod X$ with some other form ψ. These

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[4] Remember that $\sigma(-a) = -\sigma(a)$ for σ in X.

elements g are the elements represented by φ over (X,G). Marshall calles the pair (X,G) a space of orderings, if in addition to the "trivial" axioms O_1, O_2, O_3 the following holds true:

O_4 : If φ_1, φ_2 are forms over G and g is an element of $D_X(\varphi_1 \perp \varphi_2)$ then there exist elements g_1 in $D_X(\varphi_1)$, g_2 in $D_X(\varphi_2)$ such that $g \in D_X(<g_1, g_2>)$.

It is for forms over spaces of orderings that Marshall is able to prove his theorems. We now want to explain that for A semilocal the pair (Sign A, G(A)) is indeed a space of orderings, a fact first proved by Kleinstein and Rosenberg under a mild restriction [KR_1, Prop. 6.4]. We then shall translate Marshall's main notions into the usual language of forms over A and shall apply his theory.

As above let A be a connected formally real semilocal ring, X = Sign(A), G = G(A). We call two bilinear spaces φ and ψ over A congruent, and write $\varphi \equiv \psi$, if dim φ = dim ψ [5] and $\sigma(\varphi) = \sigma(\psi)$ for all $\sigma \in X$. The latter condition means according to § 1 that φ and ψ have the same image in $\overline{W}(A)$. The congruence class $\overline{\varphi}$ of a bilinear space φ will be called a reduced form over A. This is a priori not the same thing as a form over (X,G) since φ has not necessarily an orthogonal basis.

The bilinear space $\varphi = (E,B)$ is called proper if φ has an orthogonal basis, i.e. $\varphi \cong <a_1, \dots, a_n>$ with some units $a_i \in A^*$. As pointed out in § 1 φ is proper if and only if there exists some x

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5) dim φ is the rank of the free module underlying φ.

in E with $B(x,x)$ a unit, which is always the case if $2 \in A^*$. The congruence class $\bar{\varphi}$ of a proper space φ is called a <u>proper reduced form</u> over A. Up to notation the proper reduced forms coincide with Marshall's forms over (X,G). Later in this section we shall see that in fact every reduced form is proper.

It is possible to describe the congruence relation without using signatures. For this we work with the quadratic space $[1,h]$ with h the natural number from above. For the associated bilinear space $\begin{pmatrix} 2 & 1 \\ 1 & 2h \end{pmatrix}$ we have, cf. [K_1, § 3]:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2h \end{pmatrix} \perp \langle -1 \rangle \cong \langle 1, 2h-1, (2h-1)(1-4h) \rangle .$$

Applying any signature to this isometry we obtain $\sigma \begin{pmatrix} 2 & 1 \\ 1 & 2h \end{pmatrix} = +2$. Thus

$$\begin{pmatrix} 2 & 1 \\ 1 & 2h \end{pmatrix} \equiv \langle 1, 1 \rangle .$$

<u>Proposition 2.1.</u> For two bilinear spaces φ and ψ over A the following are equivalent: [6]

(i) $\varphi \equiv \psi$;

(ii) $\dim \varphi = \dim \psi$, $2^r \times \varphi \sim 2^r \times \psi$ for some $r \in \mathbb{N}$;

(iii) $2^r \times \varphi \otimes [1,h] \cong 2^r \times \psi \otimes [1,h]$ for some $r \in \mathbb{N}$.

<u>Proof.</u> (i) $\Leftrightarrow$ (ii) is clear from Theorem 1.2. (ii) $\Rightarrow$ (iii):

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6) $\sim$ means "Witt equivalent", $\cong$ means "isomorphic" (= isometric), $m \times \varphi$ means the orthogonal sum of m copies of φ.

Since $2^r \times \varphi \sim 2^r \times \psi$ the quadratic spaces $2^r \times \varphi \otimes [1,h]$ and $2^r \times \psi \otimes [1,h]$ are again Witt equivalent. Moreover they have the same rank. But over any semilocal ring Witt's cancellation theorem holds true for quadratic spaces, cf. [Ba]. Thus the two spaces are isomorphic.

(iii) $\Rightarrow$ (i): Applying a signature σ to the relation (iii) we obtain that $2^{r+1}\,\sigma(\varphi)$ and $2^{r+1}\,\sigma(\psi)$ are equal. Thus $\sigma(\varphi) = \sigma(\psi)$ for all signatures σ. Moreover (iii) implies that φ and ψ have the same rank. Thus $\varphi \equiv \psi$.

QED

Remark 2.2. If 2 is a unit in A then $4 \times \begin{pmatrix} 2 & 1 \\ 1 & 2h \end{pmatrix} \cong 8 \times \langle 1 \rangle$. Thus property (iii) then simply means $n \times \varphi \cong n \times \psi$ for some 2-power n.

For any bilinear space $\varphi = (E,B)$ over A we denote by $D(\varphi)$ the set of units a represented by φ, i.e. $a = B(x,x)$ with some $x \in E$. This set is non empty if and only if φ is proper. For a quadratic space $\varphi = (E,q)$ we denote by $D(\varphi)$ the set of units $a = q(x)$, $x \in E$. Finally we introduce for any bilinear space φ the set $\bar{D}(\varphi)$ consisting of all $b \in A^*$ such that $\varphi \cong \langle b \rangle \perp \psi$ with some bilinear space ψ. For φ proper we want to relate this set $\bar{D}(\varphi)$ to Marshall's set $D_X(\varphi)$ of the form φ over (X,G).

We make for some time the following hypothesis on A.

H(2): None of the fields A/m_i, $i = 1,\dots,g$, contains only two elements.

Later we shall remove this hypothesis in most statements.

The proposition stated now says in particular that $\bar{D}(\varphi)$ is the preimage of the subset $D_X(\varphi)$ of $G(A)$ in A^*.

<u>Proposition 2.3.</u> Let $\varphi = \langle a_1,\dots,a_n\rangle$ be a proper bilinear space over A, and let b be a unit of A. Under hypothesis H(2) the following are equivalent:

(i) $b \in \bar{D}(\varphi)$

(ii) $b \in D(m \times \varphi \otimes [1,h])$ for some $m \geq 1$; in other words, $b = t_1a_1 +\dots+ t_na_n$ with some $t_i \in \Pi(A)$.

(iii) $b = t_1a_1 +\dots+ t_na_n$ with some $t_i \in \Pi^*(A)$.

(iv) $\varphi \cong \langle b\rangle \perp \psi$ with some <u>proper</u> space ψ over A or $\psi = 0$.

<u>Proof.</u> (i) $\Rightarrow$ (ii): We have a congruence $\varphi \cong \langle b\rangle \perp \psi$ with some bilinear space ψ. According to Prop. 2.1 this implies

$$m \times \varphi \otimes [1,h] \cong m \times \langle b\rangle \otimes [1,h] \perp m \times \psi \otimes [1,h]$$

for some $m \geq 1$. Clearly b is represented by the space on the right hand side, hence also by $m \times \varphi \otimes [1,h]$.

(ii) $\Leftrightarrow$ (iii): This is evident from the transversality theorem 2.7 in [Ba K], since A has property H(2).

(ii) $\Rightarrow$ (iv): Consider a quadratic space $(E,q) = m \times \varphi \otimes [1,h]$ which represents b. We have an orthogonal decomposition $E = E_1 \perp\dots\perp E_n$ with

$$(E_i, q|E_i) \cong m \times \langle a_i\rangle \otimes [1,h].$$

Since the case $n = 1$ is trivial we assume $n \geq 2$. According to the transversality theorem cited above we have

$$b = q(z) + q(x_3) +\dots+ q(x_n)$$

with vectors $z \in E_1 \perp E_2$, $x_3 \in E_3, \ldots, x_n \in E_n$, such that the values $q(z), q(x_3), \ldots, q(x_n)$ are units $\{b = q(z)$ if $n = 2\}$. Also by the same transversality theorem

$$q(z) = q(x_1) + q(x_2)$$

with units $q(x_i)$, $x_i \in E_i$. We have $q(x_i) = a_i t_i$ with some $t_i \in \Pi^*(A)$, $i = 1, \ldots, n$, and

$$b = t_1 a_1 + \ldots + t_n a_n.$$

Moreover $c := t_1 a_1 + t_2 a_2$ is a unit. Now

$$\varphi \cong \langle t_1 a_1, t_2 q_2, \ldots, t_n a_n \rangle \cong$$

$$\cong \langle c, t_3 a_3, \ldots, t_n a_n, t_1 t_2 c a_1 a_2 \rangle$$

$$\cong \langle b \rangle \perp \chi \perp \langle t_1 t_2 c a_1 a_2 \rangle.$$

Thus $\varphi \cong \langle b \rangle \perp \psi$ with ψ proper. The implication (iv) $\Rightarrow$ (i) is trivial, and Proposition 2.3 is proved.

<u>Proposition 2.4.</u> Under hypothesis H(2) any bilinear space φ over A is congruent to a proper space ψ.

<u>Proof.</u> By the preceding proposition $\langle 1 \rangle \perp \varphi \cong \langle 1 \rangle \perp \psi$ with some proper space ψ. This implies $\varphi \cong \psi$.

Exploiting the implications (i) $\Leftrightarrow$ (ii) in Proposition 2.3 we obtain

<u>Theorem 2.5</u> [Kl R_1, Prop. 6.4]. Under the hypothesis H(2) the pair (Sign(A),G(A)) is a space of orderings.

Proof. Let φ_1 and φ_2 be proper spaces over A and let a be an element of $\bar{D}(\varphi_1 \perp \varphi_2)$. We have to find elements $b_i \in \bar{D}(\varphi_i)$, $i = 1,2$, such that $a \in \bar{D}(<b_1,b_2>)$. Now $a \in D(\rho_1 \perp \rho_2)$ with quadratic spaces

$$\rho_i := m \times \varphi_i \otimes [1,h].$$

By the transversality theorem 2.7 in [Ba K] $a = b_1 + b_2$ with units $b_i \in D(\rho_i)$. Then $b_1 \in \bar{D}(\varphi_1)$, $b_2 \in \bar{D}(\varphi_2)$ and even $a \in D(<b_1,b_2>)$.

QED

We now want to remove the hypothesis H(2) in the last three propositions. Thus assume that at least one residue class field A/m_i has only two elements. We consider the cubic extension

$$C := A[T]/(T^3 + 6T^2 + 29T + 1)$$

of A which is semilocal and connected and has only residue class fields with at least 7 elements, since the polynomial $T^3 + 6T^2 + 29T + 1$ is irreducible over all prime fields with less than 7 elements. The inclusion map $A \to C$ yields a ring homomorphism from W(A) to W(C) which sends the Witt class $[\varphi]$ of a bilinear space $\varphi = (E,B)$ to the Witt class $[\varphi \otimes C]$ of the base extension $\varphi \otimes C = (E \otimes_A C, B \otimes_A C)$. This ring homomorphism induces a continuous map $\alpha:\mathrm{Sign}(C) \to \mathrm{Sign}(A)$. For any τ in Sign(C) the signature $\alpha(\tau)$ is given by $\alpha(\tau)(a) = \tau(a)$, $a \in A^*$. Thus $\alpha(\tau)$ may be viewed as the "restriction" of τ to A.

As has been shown in [K_3, Lemma 4] the map α is <u>bijective</u>, hence a homeomorphism. Thus every signature σ of A has a unique extension σ' to C. Beside the equation $\sigma'(a) = \sigma(a)$ for $a \in A^*$

we have [K_3, Lemma 4]

$$\sigma'(c) = \sigma(N(c))$$

for every $c \in C^*$ with $N(c)$ the norm of c with respect to A, i.e. $N(c)$ = determinant of the A-linear map $x \mapsto cx$ from C to C. Thus the natural group homomorphism $a\,\Pi^*(A) \to a\,\Pi^*(C)$ from $G(A)$ to $G(C)$ is bijective with inverse map $c\,\Pi^*(C) \to N(c)\,\Pi^*(A)$. {Notice that $N(a) = a^3$ for a in A.} This altogehter shows that we have a natural isomorphism of pairs

$$(G(A),\ Sign(A)) \xrightarrow{\sim} (G(C),\ Sign(C))$$

in the sense of Marshall's papers. In particular we learn from Theorem 2.5, since C fulfills the hypothesis H(2):

<u>Theorem 2.5 a.</u> For every connected formally real semilocal ring A the pair (Sign(A),G(A)) is a space of orderings.

In this statement actually the word "connected" can be removed, cf. the first paragraph of § 2.

We now extend Proposition 2.4.

<u>Proposition 2.4 a.</u> For every bilinear space φ over A there exists a proper bilinear space ψ which is congruent to φ.

<u>Proof.</u> Assume again that A violates H(2), and consider the cubic extension C from above. By Prop. 2.4 we have

$$\varphi \otimes C \equiv \langle c_1, \ldots, c_n \rangle$$

with units $c_i \in C^*$. But

$$\langle c_1,\ldots,c_n\rangle \cong \langle Nc_1,\ldots,Nc_n\rangle$$

over C. This implies

$$\varphi \cong \langle Nc_1,\ldots,Nc_n\rangle$$

over A.

We finally extend Proposition 2.3.

Proposition 2.3 a. Let φ be a bilinear space over A and b a unit of A. Then the following are equivalent:

(i) $b \in \bar{D}(\varphi)$.

(ii) $b \in D(m \times \varphi \otimes [1,h])$ for some $m \geq 1$.

Proof. (i) $\Rightarrow$ (ii): The proof runs as the proof of (i) $\Rightarrow$ (ii) in Prop. 2.3.

(ii) $\Rightarrow$ (i): We assume that A violates H(2) and use again our cubic extension C. From (ii) we deduce that $b \in D(m \times \varphi \otimes [1,h] \otimes_A C)$. According to Proposition 2.3 this implies

$$\varphi \otimes C \cong \langle b,c_1,\ldots,c_m\rangle$$

with some $c_i \in C^*$. Thus

$$\varphi \otimes C \cong \langle b,N(c_1),\ldots,N(c_m)\rangle$$

and then

$$\varphi \cong \langle b,N(c_1),\ldots,N(c_m)\rangle$$

over A, hence $b \in \bar{D}(\varphi)$.

QED

We now give a description of the congruence relation for proper bilinear spaces analogous to the "chain equivalence" in the field case in [Be K]. We need the following easy lemma.

Lemma 2.6. Assume φ and ψ are congruent bilinear spaces over A. Then the signed determinants $d(\varphi)$ and $d(\psi)$ (cf. [K, Chap. II § 2]) differ only by a unit $t \in \Pi^*(A)$, $d(\psi) = td(\varphi)$.

Proof. Adding the space $\langle 1,-1\rangle$ to both φ and ψ we assume that φ and ψ are proper. Let $\varphi = \langle a_1,\ldots,a_n\rangle$. Then

$$d(\varphi) = (-1)^{\frac{n(n-1)}{2}} a_1 \ldots a_n A^{*2}.$$

Let σ be a signature of A and $\sigma(\varphi) = n - 2r$. Then

$$\sigma(d(\varphi)) = (-1)^{\frac{n(n-1)}{2} + r}.$$

Thus $\sigma(d(\varphi))$ is determined by $\sigma(\varphi)$, and the same holds true for $\sigma(d(\psi))$. We conclude that $\sigma(d(\varphi)) = \sigma(d(\psi))$ for all signature σ, which means $d(\psi) = td(\varphi)$ with $t \in \Pi^*(A)$.

QED

Using this lemma we clearly have the following description of congruence for binary proper spaces.

Proposition 2.7 a. Two binary spaces $\langle a_1,a_2\rangle$ and $\langle b_1,b_2\rangle$ are congruent if and only if $b_1 = t_1a_1 + t_2a_2$ and $b_2 = t_3a_1a_2b_1$ with elements t_1 and t_2 in $\Pi(A)$ and t_3 in $\Pi^*(A)$. Under the hypothesis H(2) also t_1 and t_2 can be chosen in $\Pi^*(A)$.

Applying the property O_4 it is now an easy exercise, done in $[M_o, \S\ 2]$, to prove

Proposition 2.7 b. Let $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_n)$ be n-tuples in A^* such that the spaces $\langle a_1,\ldots,a_n\rangle$ and $\langle b_1,\ldots,b_n\rangle$ are congruent. Then there exists a finite sequence of n-tuples in A^*, starting with $(a_1,\ldots,a_n)$ and ending with $(b_1,\ldots,b_n)$, such that any two adjacent n-tuples $(c_1,\ldots,c_n)$ and $(d_1,\ldots,d_n)$ differ either in one entry j or in two entries i,j and $d_j = tc_j$ with $t \in \Pi^*(A)$ in the first case, $\langle c_i,c_j\rangle \cong \langle d_i,d_j\rangle$ in the second case.

Next we examine how Marshall's notions "isotropy" and "kernel form" fit into the present context.

Definition (as in $[K_1, \S\ 5]$). A bilinear space φ over A is *weakly isotropic* if $m \times \varphi$ is isotropic for some $m \geq 1$. This means that the bilinear space $m \times \varphi = (E,B)$ contains a *primitive* vector x with $B(x,x) = 0$ [7]. If φ is not weakly isotropic we call φ *strongly anisotropic*.

The following proposition states in particular for a proper space φ that φ is weakly isotropic if and only if the reduced form φ over (X,G) is isotropic in the sense of Marshall $[M_2, \S\ 1]$.

Proposition 2.8. For any bilinear space φ over A the following are equivalent:

(i) φ is weakly isotropic.

(ii) The quadratic space $m \times \varphi \otimes [1,h]$ is isotropic for some $m \geq 1$.

---

7) A vector x of E is primitive if x is part of a free basis of the module E. For quadratic spaces over A isotropy is defined in the same way.

(iii) $\bar{D}(\varphi) = A^*$.

(iv) $\varphi \cong \langle 1,-1\rangle \perp \psi$ with some proper bilinear space ψ.

<u>Proof.</u> (i) $\Rightarrow$ (ii) is trivial. (ii) $\Rightarrow$ (i) is evident from the fact, proved in [K_1, p.65], that if for a bilinear space ψ the quadratic space $\psi \otimes [1,h]$ is isotropic then $6 \times \psi$ is isotropic.
(ii) $\Rightarrow$ (iii): This is evident from Prop. 2.3a since an isotropic quadratic space represents all units.
(iii) $\Rightarrow$ (ii): Also evident from Prop. 2.3a. Indeed, if $m \times \varphi \otimes [1,h]$ represents -1 and 1 then $2m \times \varphi \otimes [1,h]$ is isotropic.
(iii) $\Leftrightarrow$ (iv): φ is congruent to a proper space (Prop. 2.4 a), thus we may assume from the beginning that φ is proper. The equivalence (iii) $\Leftrightarrow$ (iv) is now contained in Marshall's observation [M_0, Lemma 2.9], that for any space of orderings (X,G) and any form ψ over G the following are equivalent:

a) $D_X(\psi) = G$,

b) $\psi \cong \langle 1,-1\rangle \perp \rho \bmod X$ with some form ρ over G.

We repeat the argument, since [M_0] is not published. (b) $\Rightarrow$ (a): Clearly $\langle 1,-1\rangle \cong \langle g,-g\rangle \bmod X$ for any $g \in G$. (a) $\Rightarrow$ (b): We have $\psi \cong \langle 1\rangle \perp \psi' \bmod X$ with some form ψ' over G. By O_4 there exists some y in $D_X(\psi')$ such that $-1 \in D_X(\langle 1,y\rangle)$. This implies $\langle 1,y\rangle \cong \langle -1,-y\rangle \bmod X$. Comparing signatures we obtain $\sigma(y) = -1$ for all $\sigma \in X$, hence $y = -1$ by O_3.

QED

<u>Remark.</u> In [K_1, Th. 5.13] still another characterization of weakly isotropic spaces has been given using "semisignatures". We shall come to this point in the last section of the paper.

For any bilinear space φ over A we have a congruence

$$\varphi \equiv r \times \langle 1,-1\rangle \perp \varphi_o$$

with φ_o strongly anisotropic and some $r \geq 1$, as follows from Proposition 2.8. Clearly the number r and the congruence class $\bar{\varphi}_o$ are uniquely determined by $\bar{\varphi}$. We call $\bar{\varphi}_o$ the <u>kernel form</u> of the reduced form $\bar{\varphi}$ and write $\bar{\varphi}_o = \ker(\bar{\varphi})$.

<u>Proposition 2.9.</u> Two bilinear spaces φ and ψ over A have the same image in the reduced Witt ring $\bar{W}(A)$, defined in § 1, if and only if $\ker(\bar{\varphi}) = \ker(\bar{\psi})$.

<u>Proof</u> (cf. [M_o, Lemma 2.12]). Since $\bar{\varphi}$ and $\ker(\bar{\varphi})$ have the same image in $\bar{W}(A)$ we may assume that φ and also ψ is strongly anisotropic. If $\varphi \equiv \psi$ then of course φ and ψ have the same image in $\bar{W}(A)$.

Assume now that φ and ψ have the same image in $\bar{W}(A)$. This means $\sigma(\varphi) = \sigma(\psi)$ for all $\sigma \in X$. In particular $\dim \varphi \equiv \dim \psi \bmod 2$. We have to show that $\dim \varphi = \dim \psi$. Then by the definition of congruence we know that $\varphi \equiv \psi$. Suppose that $\dim \varphi > \dim \psi$. Then $\dim \varphi = \dim \psi + 2r$ with $r > 0$, and

$$\varphi \equiv \psi \perp r \times \langle 1,-1\rangle,$$

since both sides have the same rank and signatures. This contradicts the fact that φ is strongly anisotropic (cf. Prop. 2.8). Similarly the inequality $\dim \varphi < \dim \psi$ would contradict the fact that ψ is strongly anisotropic. Thus indeed $\dim \varphi = \dim \psi$, q.e.d.

It should now be clear that $\overline{W}(A)$ is the same object as Marshall's Witt ring $W(X,G)$ of the space of orderings $(X,G) = (\mathrm{Sign}(A), G(A))$.

Under hypothesis H(2) we can read off from Proposition 2.7a/b a description of $\overline{W}(A)$ as a homomorphic image of the group ring $\mathbb{Z}[G(A)]$ analogous to the well known description of $W(A)$ as a homomorphic image of $\mathbb{Z}[A^*/A^{*2}]$ ([KRW, Th. 1.16]). Comparing these descriptions we see that $\overline{W}(A)$ is the quotient of $W(A)$ by the ideal generated by the spaces $<1,-t>$ with t in $\Pi^*(A)$. Thus we obtain

<u>Proposition 2.10.</u> Under hypothesis H(2) the nil radical of $W(A)$ is generated by the spaces $<1,-t>$ with t in $\Pi^*(A)$.

This had been proved in a different way in [K_1, § 2] and in the case $2 \in A^*$ already in [KRW_1, § 4].

§ 3 Congruence modulo a saturated group

In Marshall's theory of forms over (X,G) certain "subspaces" of (X,G) play a crucial role. We will recall the notion of subspace and then interpret this notion in the semilocal case.

Let (X,G) be a space of orderings and Y be a subset of X. We consider the subgroup $\Delta := Y^{\perp}$ of G, defined as the intersection of the kernels of all characters $\sigma \in Y$ ($\Delta = G$ if Y is empty), and the subgroup $\Delta^{\perp}$ of G consisting of all characters which vanish on Δ. Clearly $\Delta^{\perp}$ is the closed subgroup [Y] of G generated by Y in the topological sense. Marshall calls the set Y or the pair $(Y, G/\Delta)$ determined by Y a subspace of X if $Y = [Y] \cap X$. He proves that in this case $(Y, G/\Delta)$ is again a space of orderings [M_2, § 2]. Here of course the base point -1 of G/Δ is defined as the image of the base point -1 of G.

If Z is any subset of X then the set $Y := [Z] \cap X$ is a subspace of X and $Y^{\perp} = Z^{\perp}$ as is immediately verified. We call Y the subspace of (X,G) generated by Z. Another way to obtain all subspaces of (X,G) is to start with some non empty subset S of G. Then

$$X(S) := X \cap S^{\perp} = \{\sigma \in X \mid \sigma(S) = \{1\}\}$$

is a subspace of X and in fact the largest subspace Y with $Y^{\perp} \supset S$. This is again easily verified.

Let now A be as before a connected formally real semilocal ring and $G = G(A)$, $X = \mathrm{Sign}(A)$. We fix the following notations. For $Z \subset X$ we define

$$P(Z) := \{a \in A^* \mid \sigma(a) = 1 \text{ for all } \sigma \in Z\}.$$

In particular $P(\emptyset) = A^*$. For any one-point set $\{\sigma\} \subset X$ the group $P(\{\sigma\})$ is the kernel of σ in A^*. We shortly write $P(\sigma)$ for this and call $P(\sigma)$ the <u>signature domain</u> of σ. For $S \subset A^*$, $S \neq \emptyset$, we define

$$X(S) := \{\sigma \in X \mid \sigma(a) = 1 \text{ for all } a \in S\},$$

$$\hat{S} := P(X(S)).$$

We call the groups $P(Z)$, with Z running through the subsets of X, the <u>saturated subgroups</u> of A^*, and call $\hat{S}$ the <u>saturation</u> of the set S.

<u>Remark.</u> In $[K_1]$ the words "strictly saturated subgroup" and "strict saturation" are used instead. In $[K_1]$, $[KRW_1]$ and the papers of Kleinstein-Rosenberg the letters $\Gamma(Z)$, $V(S)$ are used instead of $P(Z)$, $X(S)$.

From the discussion in the abstract case above the following is clear.

<u>Proposition 3.1.</u> The saturated subgroups T of A^* correspond uniquely with the subspaces Y of (X,G) via $T = P(Y)$, $Y = X(T)$. The pairs $(Y, A^*/T)$ are again spaces of orderings. For any subset Z of X the set $Y := X(P(Z))$ is the subspace of (X,G) generated by Z. The corresponding saturated subgroup T is $P(Z)$. For any non empty subset S of A^* the saturation $\hat{S}$ is the smallest saturated subgroup of A^* containing S. The corresponding subspace of (X,G) is $X(S)$.

The following description of the saturation $\hat{S}$ has been given in [K_1, Th. 2.5], cf. also [KRW_1, § 2] for $2 \in A^*$.

<u>Theorem 3.2</u> ("Generalized Artin-Pfister theorem").
The saturation S of some non empty subset S of A* consists of all units

$$b = t_1 a_1 + \dots + t_n a_n$$

with t_i in $\Pi(A)$ and a_i a product of elements of S.

Under hypothesis H(2) the t_i can even be chosen in $\Pi^*(A)$ according to the transversality theorem 2.7 in [Ba K].

We now fix a saturated subgroup $T \neq A^*$. Then Marshall's notions of congruence of forms, representation of elements by forms, isotropy, kernel form, and Witt equivalence over the space of orderings $(X(T), A^*/T)$ read as follows.

<u>Definition 3.3.</u> Let φ be a bilinear space over A. Remember that we know from § 2 that φ is congruent to a proper space.

i) We call a bilinear space ψ over A <u>congruent to</u> φ <u>modulo</u> T, and write $\varphi \equiv \psi \bmod T$, if $\dim \varphi = \dim \psi$ and $\sigma(\varphi) = \sigma(\psi)$ for all σ in X(T). The congruence class modulo T of φ is denoted by φ_T.

ii) A unit b of A is <u>represented modulo</u> T by φ if $\varphi \equiv \langle b \rangle \perp \psi \bmod T$ with some bilinear space ψ. The set of these units is denoted by $D_T(\varphi)$.

iii) φ is called <u>isotropic modulo</u> T if $\varphi \equiv \langle 1,-1 \rangle \perp \psi \bmod T$ with some space ψ, and φ is called <u>anisotropic modulo</u> T otherwise.

iv) We have always a congruence

$$\varphi \equiv \varphi_0 \perp r \times \langle 1,-1\rangle \bmod T$$

with φ_0 anisotropic modulo T and some uniquely determined $r \geq 0$. The congruence class $(\varphi_0)_T$ is uniquely determined by φ_T. This class will be called the <u>kernel form of</u> φ <u>modulo</u> T and will be denoted by $\ker(\varphi_T)$.

v) A bilinear space ψ over A is called <u>Witt equivalent modulo</u> T to φ, written $\psi \sim \varphi$ mod T, if $\ker(\varphi_T) = \ker(\psi_T)$. The Witt equivalence class modulo T of φ is denoted by $[\varphi]_T$. The set of these classes is in an obvious way a commutative ring with 1, called the <u>Witt ring</u> $W_T(A)$ <u>of</u> A <u>modulo</u> T. This ring is the same as Marshall's ring $W(X(T),A^*/T)$.

It is easy to verify, cf. the proof of Proposition 2.9:

<u>Proposition 3.4.</u> Two bilinear spaces φ and ψ over A are Witt equivalent modulo T if and only if $\sigma(\varphi) = \sigma(\psi)$ for all σ in X(T).

This means according to § 1 that $W_T(A) = W(A)/c$ with c the intersection of all prime ideals P_σ, $\sigma \in X(T)$. We denote for any non empty subset S of A^* by $\mathfrak{a}(S)$ the ideal of W(A) generated by the spaces $\langle 1,-s\rangle$, $s \in S$. The following has been shown in $[K_1]$:

<u>Proposition 3.5 a</u> $[K_1$, Th. 2.4]. We assume H(2). We then have $c = \mathfrak{a}(T)$. Moreover if S is any subset of T with $\hat{S} = T$ then $c = \sqrt{\mathfrak{a}(S)}$, i.e. c is the set of all x in W(A) with $x^n \in \mathfrak{a}(S)$ for some $n \geq 1$.

Since $W(A)/\mathfrak{a}(S)$ is clearly a "real abstract Witt ring" in the sense of [KRW], the nil radical of this ring coincides with the set of torsion elements, and this set is a 2-group, cf. [KRW], [KK]. Thus we also have

<u>Proposition 3.5 b.</u> If $\hat{S} = T$ and H(2) is fulfilled, then $\mathfrak{c}$ is the set of all x in W(A) with $m x \in \mathfrak{a}(S)$ for some $m \geq 1$. The natural number m can be chosen as a 2-power.

We now obtain a statement similar to Proposition 2.1:

<u>Proposition 3.6.</u> Let S be a subset of T with $\hat{S} = T$. Then for two bilinear spaces φ and ψ over A the following are equivalent:

(i) $\varphi \equiv \psi \bmod T$

(ii) $\dim \varphi = \dim \psi$ and

$2^r \times \langle\langle s_1,\ldots,s_t\rangle\rangle \otimes \varphi \sim 2^r \times \langle\langle s_1,\ldots,s_t\rangle\rangle \otimes \psi$

with some $r \geq 0$ and suitable elements $s_1,\ldots,s_t$ of S. Here $\langle\langle s_1,\ldots,s_t\rangle\rangle$ denotes as usual the Pfister space $\langle 1,s_1\rangle \otimes\ldots\otimes \langle 1,s_t\rangle$.

(iii) $\dim \varphi = \dim \psi$ and

$$\langle t_1,\ldots,t_n\rangle \otimes \varphi \sim \langle t_1,\ldots,t_n\rangle \otimes \psi$$

with elements $t_1,\ldots,t_n$ of T.

(iv) $2^r \times \langle\langle s_1,\ldots,s_t\rangle\rangle \otimes \varphi \otimes [1,h] \cong$
$\cong 2^r \times \langle\langle s_1,\ldots,s_t\rangle\rangle \otimes \psi \otimes [1,h]$
for some $r \geq 0$ and elements $s_1,\ldots,s_t$ of S.

<u>Proof.</u> (i) $\Rightarrow$ (ii): We first assume that H(2) holds true. By the preceding proposition 3.5 b we have for some $r \geq 0$

$$2^r \times \varphi \sim 2^r \times \psi \perp \langle b_1,-b_1s_1\rangle \perp \ldots \perp \langle b_t,-b_ts_t\rangle$$

with elements b_i in A^* and s_i in S. Multiplying with the Pfister space $<<s_1,\dots,s_t>>$ we obtain

$$2^r \times <<s_1,\dots,s_t>> \otimes \varphi \sim 2^r \times <<s_1,\dots,s_t>> \otimes \psi,$$

as desired. Assume now that A violates H(2). We work with the cubic extension $C = A[T]/T^3 + 6T^2 + 29T + 1$ as in § 2. Let U denote the saturation of S in C^*. Then $\varphi \equiv \psi \bmod T$ implies $\varphi \otimes C \equiv \psi \otimes C \bmod U$. Since C fulfills H(2) this implies a relation

$$(*) \quad 2^m \times <<s_1,\dots,s_t>> \otimes \varphi \otimes C \sim 2^m \times <<s_1,\dots,s_t>> \otimes \psi \otimes C$$

over C with s_i in S. Thus the spaces $<<s_1,\dots,s_t>> \otimes \varphi \otimes C$ and $<<s_1,\dots,s_t>> \otimes \psi \otimes C$ have the same image in W(C). Since the natural map from W(A) to W(C) is bijective, the spaces $<<s_1,\dots,s_t>> \otimes \varphi$ and $<<s_1,\dots,s_t>> \otimes \psi$ over A have the same image in W(A). This means

$$(**) \quad 2^n \times <<s_1,\dots,s_t>> \otimes \varphi \sim 2^n \times <<s_1,\dots,s_t>> \otimes \psi.$$

{Actually applying the transfer map $s_*: W(C) \to W(A)$ described in [K_3, p.184] to the relation (*) it is possible to deduce from (*) the relation (**) with $n = m$}.

The implications (ii) ⇒ (iii) and (iii) ⇒ (i) are trivial. The implication (ii) ⇒ (iv) is clear in view of the cancellation law for quadratic spaces, cf. the proof of Prop. 2.1. The implication (iv) ⇒ (i) is again trivial, since $\sigma([1,h]) = +2$ for every signature σ of A.

QED

We now obtain by the same arguments as in the proof of Proposition 2.3 and 2.3.a:

<u>Proposition 3.7.</u> Assume that $T = \hat{S}$. Let $\varphi = \langle a_1,\dots,a_n \rangle$ be a proper bilinear space over A and b be a unit of A. The following are equivalent:

(i) $b \in D_T(\varphi)$

(ii) $b \in D(m \times \langle\langle s_1,\dots,s_t \rangle\rangle \otimes \varphi \otimes [1,h])$ with some $m \geq 1$ and elements s_i of S. If A fulfills H(2) then (i) and (ii) are also equivalent to

(iii) $b = t_1a_1 +\dots+ t_na_n$ with t_i in T.

In fact the proof is simpler than the proof of Proposition 2.3, since we know already that any bilinear space is congruent to a proper space.

It is now easily verified that the "chain equivalence description" of the congruence relation in Proposition 2.7a/7b remains true for congruence modulo T if we replace everywhere the group $\Pi^*(A)$ by T and the set $\Pi(A)$ by $T\cdot\Pi(A)$ (sums of elements $t(x^2+xy+y^2h)$ with $t \in T$, x,y in A).

Moreover we obtain an extension of Proposition 2.8 to isotropy modulo T by the same arguments as in the proof of that proposition.

<u>Proposition 3.8.</u> Let S be a subset of T with $S = T$. For a bilinear space φ over A the following are equivalent:

(i) φ is isotropic modulo T.

(ii) There exist elements $t_1,\dots,t_n$ in T such that $\langle t_1,\dots,t_n \rangle \otimes \varphi$ is isotropic.

(iii) $D_T(\varphi) = A^*$.

(iv) There exist elements $s_1,\dots,s_t$ in S and some $m \geq 1$ such that $m \times \langle\langle s_1,\dots,s_t\rangle\rangle \otimes \varphi$ is isotropic.

(v) There exist elements $s_1,\dots,s_t$ in S and some $m \geq 1$ such that $m \times \langle\langle s_1,\dots,s_t\rangle\rangle \otimes \varphi \otimes [1,h]$ is isotropic.

In the case that S is finite we read off from Propositions 3.6 - 3.8 the following statement, which by the way can be established completely within Marshall's abstract theory, cf. [M_1].

<u>Proposition 3.9.</u> Let φ be a bilinear space over A. Let $a_1,\dots,a_n$ be units of A and let ρ denote the Pfister space $\langle\langle a_1,\dots,a_n\rangle\rangle$.

1) The saturation T of the set $\{a_1,\dots,a_n\}$ coincides with $\overline{D}(\rho)$.

2) A bilinear space ψ is congruent to φ modulo T if and only if $\rho \otimes \psi$ is congruent to $\rho \otimes \varphi$.

3) $D_T(\varphi) = D(\rho \otimes \varphi)$.

4) φ is isotropic modulo T if and only if $\rho \otimes \varphi$ is weakly isotropic.

Notice that we now have gained a theory of forms modulo T over semilocal rings as good as the theory of Becker and Köpping in [Be K,§ 1 and § 2] over fields. The theory of round forms modulo T over fields in [Be K § 3] has already been established by Marshall in the abstract case [M_o, § 3], and hence goes through over any semilocal ring A, cf. also [Kl R_1, § 4]. A theory of "fans" analogous to [Be K, § 4] will be given in § 7.

§ 4 A stratification of the space of signatures

We postpone a further discussion of reduced forms until § 9, since we first need more insight into the space of signatures of our connected formally real semilocal ring A. This Boolean space will always be denoted by X. It turns out that X has a natural "stratification" by sets $X(\mathfrak{p})^{o}$ corresponding to the prime ideals $\mathfrak{p}$ of A.

For any signature σ of A we denote by $Q(\sigma)$ the set of all elements x in A which have a presentation

$$x = \lambda_1^2 a_1 + \ldots + \lambda_r^2 a_r$$

with $a_1, \ldots, a_r$ in $P(\sigma)$, (cf. definition in § 3), $\lambda_1, \ldots, \lambda_r$ in A, and $A\lambda_1 + \ldots + A\lambda_r = A$. The following remarkable fact has been proved by Kanzaki and Kitamura in a special case [Ka Ki] and then in [K_2, Appendix B] in general.

<u>Theorem 4.1.</u> A is the disjoint union of the sets $Q(\sigma)$, $-Q(\sigma) = (-1)Q(\sigma)$, and a prime ideal $\mathfrak{p}(\sigma)$. Moreover $Q(\sigma) + \mathfrak{p}(\sigma) = Q(\sigma)$.

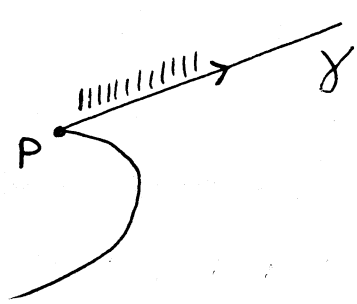
We call $\mathfrak{p} = \mathfrak{p}(\sigma)$ the <u>prime ideal associated</u> with σ. It is evident (cf. [K_2, § 4]) that there exists a <u>unique</u> signature $\bar{\sigma}$ of the field $A(\mathfrak{p})$ which "extends" σ, i.e. such that $\sigma: W(A) \to \mathbb{Z}$ is the composite of the natural map from $W(A)$ to $W(A(\mathfrak{p}))$ and of $\sigma: W(A(\mathfrak{p})) \to \mathbb{Z}$. We call $\bar{\sigma}$ the signature of $A(\mathfrak{p})$ "induced" by σ or "associated" with σ. The set $P(\sigma)$ in $A(\mathfrak{p})^*$ consists of all fractions as^{-2} with a,s images in $A/\mathfrak{p}$ of elements $a \in Q(\sigma)$, $s \in A \setminus \mathfrak{p}$. This is the set of positive elements with respect to $\bar{\sigma}$ if we regard $\bar{\sigma}$ as an ordering of the field $A(\mathfrak{p})$. It is further evident that $\mathfrak{p}$

contains every ideal $\mathfrak{a}$ of A which has the property that σ can be extended to the semilocal ring $A/\mathfrak{a}$. Indeed, such an ideal $\mathfrak{a}$ cannot meet $Q(\sigma)$ or $-Q(\sigma)$.

The geometry behind the fact that every signature comes along with a prime ideal still seems to be only partially understood. We give an example.

<u>Example 4.2.</u> We consider a complex smooth projective surface M defined over the field $\mathbb{R}$ of real numbers and a finite subset S of M stable under complex conjugation. The set $M(\mathbb{R})$ of real points on M is a compact real analytic surface. Let F be the function field $\mathbb{R}(M)$ consisting of all rational functions on M which are defined over $\mathbb{R}$, and let A be the semilocal ring consisting of all $f \in \mathbb{R}(M)$ which are defined in a neighbourhood of S, i.e. with pole divisors disjoint from S. Since A is a regular ring, every signature of A is the restriction of some ordering of F, cf. [CRW].

We now define an ordering τ on F as follows. Choose a real point P on M, choose an irreducible closed curve C in M which is defined over $\mathbb{R}$ and passes through P. Choose a real local branch $\gamma \subset M(\mathbb{R})$ of C at P. Choose an orientation of γ and a local orientation of the surface $\mathbb{R}(M)$ at P. We define a function $f \in F^*$ to be positive if f takes positive values near P on the left hand side of γ in positive direction.



This definition yields indeed an ordering of F which we regard as a signature τ of F. Notice that if C is <u>not</u> contained in the divisor of f then $\tau(f) = +1$ if and only if f has positive values on γ near P in positive direction. Moreover if P is <u>not</u> contained in the divisor of f then $\tau(f) = +1$ iff $f(P) > 0$. Let σ be the restriction of τ to A. What is the prime ideal $\mathfrak{p}(\sigma)$? We have three cases:

1) $P \in S$. Then for every $f \in A$ the divisor div(f) is disjoint from P. Thus $\sigma(f) = \operatorname{sign} f(P)$. Clearly the prime ideal $\mathfrak{p}(\sigma)$ is the maximal ideal of A corresponding to the point P.
2) $P \notin S$, but $C \cap S \neq \emptyset$. Then for every $f \in A$ the divisor div(f) does <u>not</u> contain C. We claim that $\mathfrak{p}(\sigma)$ is the prime ideal $\mathfrak{p}$ of A corresponding to the variety C through S, and that the ordering $\bar{\sigma}$ induced by σ on the function field $R(C) = A(\mathfrak{p})$ is defined as follows: A rational function f on C is positive if and only if f has positive values near P on γ in positive direction. Indeed, σ clearly extends to this signature $\bar{\sigma}$ on $A(\mathfrak{p})$, and σ cannot extend to a signature of any field $A/\mathfrak{m}_i$ with respect to a maximal ideal $\mathfrak{m}_i$, since for any real point Q in S we find some function f in A^* with $f(Q) < 0$ and $f(P)$ defined and > 0.
3) $C \cap S = \emptyset$. This is the hardest case. We claim that now $\mathfrak{p}(\sigma) = 0$, hence $\bar{\sigma}$ is the signature τ on F. Suppose there exists some prime ideal $\mathfrak{q} \neq 0$ of A and some signature ρ of $A(\mathfrak{q})$ which extends σ. The ideal $\mathfrak{q}$ cannot be a maximal ideal of A for the same reason as explained in the second case above. Let D denote the closed irreducible curve in M, defined over R, which passes through some point of S and corresponds to $\mathfrak{q}$. The signature ρ is an ordering of the field $R(D)$. As is well known there exists a real point P on D and a real oriented branch $\alpha \subset D(R)$ at P such that the func-

tions g in $R(D)$ which are positive with respect to ρ are precisely the functions which have positive values on α near P in positive direction. (For the proof consider the convex hull $\mathfrak{o}$ of R in $R(D)$ with respect to ρ, which is a discrete valuation ring of $R(D)$ and hence a local ring of the normalization $\tilde{D}$ of D, cf. § 6. This local ring corresponds to P and the branch α.) Now if $P \neq Q$ then we find a function f in A^*, defined at both P and Q, with $f(Q) > 0$ and $f(P) < 0$, hence $\rho(f) = +1$, $\tau(f) = -1$, a contradiction. Thus $P = Q$. Since α and γ are germs of different real analytic curves through P there exists some function f in A^* which is positive on α near P in positive direction and negative on γ near P in positive direction. Again we have $\rho(f) = +1$, $\tau(f) = -1$, a contradiction. Thus $\mathfrak{p}(\sigma) = 0$.

We return to an arbitrary connected formally real semilocal ring A. Let $\mathfrak{a}$ be an ideal of A and σ be a signature of A. We give a criterion that σ can be extended to a signature τ of $A/\mathfrak{a}$, i.e. that $\sigma: W(A) \to \mathbb{Z}$ factors through $\tau: W(A/\mathfrak{p}) \to \mathbb{Z}$. We introduce the subgroup $S(\mathfrak{a}) := A^* \cap (1+\mathfrak{a})$ of A^*.

<u>Theorem 4.3.</u> σ can be extended to $A/\mathfrak{a}$ if and only if $S(\mathfrak{a})$ is contained in the signature domain $P(\sigma)$.

<u>Proof.</u> If $S(\mathfrak{a})$ is not contained in $P(\sigma)$ then certainly σ cannot be extended to $A/\mathfrak{a}$. Assume now $S(\mathfrak{a}) \subset P(\sigma)$ and suppose that nevertheless σ cannot be extended to $A/\mathfrak{a}$. This means that $\mathfrak{a}$ is not contained in $\mathfrak{p}(\sigma)$. So there exists some element x in $Q(\sigma) \cap \mathfrak{a}$.
By [K_2, Lemma B.2]

$$x = a + \lambda_1^2 a_1 + \ldots + \lambda_r^2 a_r$$

with $a, a_1, \ldots, a_r$ in $P(\sigma)$ and λ_i in A. The element $y := a^{-1}x$ lies in $\mathfrak{a}$ and has a presentation

$$y = 1 + \lambda_1^2 b_1 + \ldots + \lambda_r^2 b_r$$

with $b_1, \ldots, b_r$ again in $P(\sigma)$. We now choose a natural number $h > 1$ such that $1 - hy$ is a unit. This is easily done: Let l be the product of the "characteristic exponents" of all fields $A/\mathfrak{m}_i$. (The characteristic exponent of $A/\mathfrak{m}_i$ is p if this field has characteristic $p > 0$ and 1 otherwise.) Then choose some $m \in \mathbb{N}$ such that $mly \not\equiv 1 \mod \mathfrak{m}_i$ for all $\mathfrak{m}_i$ with $A/\mathfrak{m}_i$ of characteristic zero. Put $h = ml$. We have

$$1 - hy = -(h-1) - h(\lambda_1^2 b_1 + \ldots + \lambda_r^2 b_r).$$

Now $1 - hy$ is an element of $S(\mathfrak{a})$, hence of $P(\sigma)$ by assumption. Thus

$$-(h-1) = 1 - hy + h(\lambda_1^2 b_1 + \ldots + \lambda_r^2 b_r)$$

lies in $Q(\sigma)$. This is absurd. Therefore σ can be extended to $A/\mathfrak{a}$.

QED

<u>Remark 4.4.</u> The natural group homomorphism from A^* to $(A/\mathfrak{a})^*$ has the kernel $S(\mathfrak{a})$. This homomorphism is surjective by the Chinese remainder theorem. Thus the signature $\sigma: A^* \to \{\pm 1\}$ can have at most one extension $\tau: (A/\mathfrak{a})^* \to \{\pm 1\}$. We have proved the remarkable fact that if σ can be extended as a $\{\pm 1\}$-valued character to $(A/\mathfrak{a})^*$ then this extension is always again a signature. Our theorem also implies that the saturation $\widehat{S}(\mathfrak{a})$ of $S(\mathfrak{a})$ is precisely the preimage of $\Pi^*(A/\mathfrak{a})$ in A^*, i.e. the set of all units b in A^* which fulfill a congruence relation

$$b \equiv \sum_{i=1}^{r} (x_i^2 + x_i y_i + y_i^2 h) \bmod \mathfrak{a}$$

with some x_i, y_i in A.

Let $X^{\mathfrak{a}}$ denote the image of the restriction map from Sign$(A/\mathfrak{a})$ to X, which is injective and continuous. According to the just proved Theorem 4.3 the set $X^{\mathfrak{a}}$ is the subspace $X(S(\mathfrak{a}))$ of the space of orderings $(X,G(A))$, and this subspace is isomorphic to (Sign $A/\mathfrak{a}$, $G(A/\mathfrak{a})$). We shall usually identify $X^{\mathfrak{a}}$ with the signature space of $A/\mathfrak{a}$. Clearly $X^{\mathfrak{a}}$ is the union of the sets $X^{\mathfrak{p}}$ with $\mathfrak{p}$ running through the minimal prime ideals of A containing $\mathfrak{a}$, which are finitely many if A is noetherian.

We denote by $Z(\mathfrak{p})$ the space Sign $A(\mathfrak{p})$ of orderings of the field $A(\mathfrak{p})$ and by $X(\mathfrak{p})$ the image of $Z(\mathfrak{p})$ under the restriction map $\pi_{\mathfrak{p}}: Z(\mathfrak{p}) \to X$. The set $X(\mathfrak{p})$ is compact, hence closed in X. Finally we denote by $X(\mathfrak{p})^{o}$ the set of all σ in X with $\mathfrak{p}(\sigma) = \mathfrak{p}$ and by $Z(\mathfrak{p})^{o}$ the set of induced signatures $\bar{\sigma}$ on $A(\mathfrak{p})$ with σ running through $X(\mathfrak{p})^{o}$. The set $Z(\mathfrak{p})^{o}$ is the preimage of $X(\mathfrak{p})^{o}$ in $Z(\mathfrak{p})$ under $\pi_{\mathfrak{p}}$, and $\pi_{\mathfrak{p}}$ gives a bijection from $Z(\mathfrak{p})^{o}$ onto $X(\mathfrak{p})^{o}$. We have

(4.5) $$X^{\mathfrak{p}} = \bigcup_{\mathfrak{q} \supset \mathfrak{p}} X(\mathfrak{q}) = \bigcup_{\mathfrak{q} \supset \mathfrak{p}} X(\mathfrak{q})^{o}$$

with $\mathfrak{q}$ running through all prime ideals containing $\mathfrak{p}$. The latter union is disjoint. As a consequence

(4.6) $$X^{\mathfrak{p}} = X(\mathfrak{p})^{o} \cup \bigcup_{\mathfrak{q} \supsetneq \mathfrak{p}} X^{\mathfrak{q}} .$$

In the case that $A/\mathfrak{p}$ is regular the sets $X(\mathfrak{p})$ und $X^{\mathfrak{p}}$ coincide [CRW]. We shall see in § 6 that if A is a "geometric" semilocal ring the set $X(\mathfrak{p})^{o}$ is dense in the closed set $X(\mathfrak{p})$. Thus if in addition $A/\mathfrak{p}$ is regular the closure of $X(\mathfrak{p})^{o}$ is just the union of all $X(\mathfrak{q})^{o}$ with $\mathfrak{q} \supset \mathfrak{p}$. It seems thus reasonable to think of the family of sets $X(\mathfrak{p})^{o}$ as a "stratification" of the Boolean space X. This will be fully justified only if for <u>every</u> prime ideal $\mathfrak{p}$ of A with formally real residue class field $A(\mathfrak{p})$ the following question has a positive answer.

<u>Question.</u> Is $X(\mathfrak{p})$ the union of some sets $X(\mathfrak{q})^{o}$ with $\mathfrak{q} \supset \mathfrak{p}$?

Although this question remains open in our paper, we use the word "stratification" since now for simplicity. In § 8 we shall decompose the sets $X(\mathfrak{p})^{o}$ further into "fans", a very special sort of subspaces of $(X,G(A))$.

Absatz! Dann weiter auf S. 42.

Another natural closed subset of X related to a given prime ideal $\mathfrak{p}$ of A is the image $X_\mathfrak{p}$ of the restriction map from $\mathrm{Sign}(A_\mathfrak{p})$ to X. A signature σ of A can be extended to $A_\mathfrak{p}$ if and only if σ can be extended to some field $A(\mathfrak{q})$ with $\mathfrak{q} \subset \mathfrak{p}$. This means

$$(4.7) \qquad X_\mathfrak{p} = \bigcup_{\mathfrak{q}\subset\mathfrak{p}} X(\mathfrak{q}).$$

The "main part" of $X_\mathfrak{p}$ seems to be the subset

$$X_\mathfrak{p}^o := \bigcup_{\mathfrak{q}\subset\mathfrak{p}} X(\mathfrak{q})^o.$$

In the case that A is "geometric" this set is dense in $X_\mathfrak{p}$ according to the discussion above. The following easy lemma says in particular that every $\sigma \in X_\mathfrak{p}^o$ has a <u>unique</u> preimage in $\mathrm{Sign}(A_\mathfrak{p})$. This lemma is a special case of Lemma 3.4 in [K_4]. It will be needed later at several places.

<u>Lemma 4.8.</u> Let σ be a signature of A and $\mathfrak{p}$ be a prime ideal of A containing $\mathfrak{p}(\sigma)$. Then σ has a unique extension $\tilde{\sigma}$ to $A_\mathfrak{p}$ and $\mathfrak{p}(\tilde{\sigma}) = \mathfrak{p}(\sigma)A_\mathfrak{p}$. The set $Q(\tilde{\sigma})$ consists of the fractions qs^{-1} with q

and s in $Q(\sigma)$ and s in $A \setminus \mathfrak{p}$, and $P(\tilde{\sigma})$ consists of the fractions qs^{-1} with both q and s in $Q(\sigma) \cap (A \setminus \mathfrak{p})$.

We discuss rather briefly the intersection of our stratification with the subspace $X(T)$ of X for some subgroup T of A. Of course we assume that $X(T)$ is not empty. We introduce the multiplicative set $Q(T)$ in A consisting of all elements

$$x = \lambda_1^2 t_1 + \ldots + \lambda_r^2 t_r$$

with $t_1, \ldots, t_r$ in T, $\lambda_1, \ldots, \lambda_r$ in A and $A\lambda_1 + \ldots + A\lambda_r = A$. Since $Q(T)$ is contained in $Q(\sigma)$ for every $\sigma \in X(T)$ the zero element does not belong to $Q(T)$. Assume since now that T is saturated.

We say that A has the property H(2,3) if none of the residue class fields $A/\mathfrak{m}_i$ has only two or three elements.

<u>Remark 4.9.</u> Every element x of $Q(T)$ has a presentation

$$x = t_1 + \lambda_2^2 t_2 + \ldots + \lambda_r^2 t_r$$

with t_i in T and λ_i in A. Moreover if A has the property H(2,3) then $x = t_1 + t_2$ with suitable elements t_1, t_2 of T.

This can be proved in precisely the same way as has been done in [K_2, Appendix B] in the special case that T is the signature domain $P(\sigma)$ of some signature σ.

We are interested in ideals $\mathfrak{a}$ of A which do not meet the multiplicative set $Q(T)$. Under assumption H(2,3) this means that $\mathfrak{a}$ does not meet the set $1 + T$ according to the remark above. Recall from commutative algebra that the ideals $\mathfrak{a}$ which are maximal ideals disjoint from $Q(T)$ are prime ideals.

<u>Proposition 4.10.</u> Let $\mathfrak{a}$ be an ideal of A. Then

$$X(T) \cap X^{\mathfrak{a}} = X^{\mathfrak{a}}(\overline{T})$$

with $\overline{T}$ the image of T in $(A/\mathfrak{a})^*$. This intersection is empty if and only if $\mathfrak{a} \cap Q(T) \neq \emptyset$.

<u>Proof.</u> The first assertion is evident from the definitions. If $\mathfrak{a} \cap Q(T)$ is not empty then certainly $X(T) \cap X^{\mathfrak{a}} = \emptyset$. Assume now that $\mathfrak{a} \cap Q(T)$ is empty. We have to show that $X(T) \cap X^{\mathfrak{a}} = X^{\mathfrak{a}}(\overline{T})$ is not empty. This means by § 3 that the saturation of $\overline{T}$ in $(A/\mathfrak{a})^*$ is not the whole of $(A/\mathfrak{a})^*$. Suppose that -1 is in this saturation. By Theorem 3.2 we have in $A/\mathfrak{a}$ an equation

$$-1 = \bar{s}_1\bar{\mu}_1 + \ldots + \bar{s}_r\bar{\mu}_r$$

with s_i in T and μ_i in $\Pi(A)$, since every element of $\Pi(A/\mathfrak{a})$ can be liftet to an element of $\Pi(A)$. The elements $2\mu_i$ are sums of squares in A. Multiplying by 2 we thus obtain in $A/\mathfrak{a}$

$$-2 = \bar{t}_1\bar{\lambda}_1^2 + \ldots + \bar{t}_n\bar{\lambda}_n^2$$

with t_j in T, λ_j in A. Thus the element $2 + t_1\lambda_1^2 + \ldots + t_n\lambda_n^2$ of $Q(T)$ is contained in $\mathfrak{a}$, contrary to our assumption. This shows that the saturation of $\overline{T}$ in $(A/\mathfrak{a})^*$ is a proper subgroup, hence that $X(T) \cap X^{\mathfrak{a}}$ is not empty.

QED

For any prime ideal $\mathfrak{p}$ of A let $Q_{\mathfrak{p}}(T)$ denote the multiplicative set consisting of all elements

$$x = \lambda_1^2 t_1 + \ldots + \lambda_r^2 t_r$$

with t_i in T, λ_i in A, but not all λ_i in $\mathfrak{p}$.

Proposition 4.11. For any prime ideal $\mathfrak{p}$ the set $X(T) \cap X(\mathfrak{p})$ is the image of the subspace $Z(\mathfrak{p})(\bar{T})$ of $Z(\mathfrak{p}) = \operatorname{Sign} A(\mathfrak{p})$ under the restriction map $\pi_{\mathfrak{p}}: Z(\mathfrak{p}) \twoheadrightarrow X(\mathfrak{p})$, with $\bar{T}$ denoting again the image of T in $A/\mathfrak{p}$. Thus $\pi_{\mathfrak{p}}$ yields a bijection

$$Z(\mathfrak{p})^{\circ}(\bar{T}) := Z(\mathfrak{p})(\bar{T}) \cap Z(\mathfrak{p})^{\circ} \xrightarrow{\sim} X(T) \cap X(\mathfrak{p})^{\circ}.$$

The intersection $X(T) \cap X(\mathfrak{p})$ is empty if and only if $\mathfrak{p}$ meets the multiplicative set $Q_{\mathfrak{p}}(T)$.

The proof is similar to the proof of the preceding proposition. In the same vein we finally obtain

Remark 4.12. The intersection $X(T) \cap X_{\mathfrak{p}}$ is the image of the subspace $\operatorname{Sign}(A_{\mathfrak{p}})(\bar{T})$ of $\operatorname{Sign}(A_{\mathfrak{p}})$ with $\bar{T}$ the image of T in $A_{\mathfrak{p}}^*$. This intersection is empty if and only if the multiplicative set $Q_{\mathfrak{p}}(T)$ contains the element 0.

It would be more useful to have a good criterion that X(T) meets $X(\mathfrak{p})^{\circ}$ or $X_{\mathfrak{p}}^{\circ}$ instead of $X(\mathfrak{p})$ and $X_{\mathfrak{p}}$.

§ 5 Restriction of signatures

We discuss the "functorial" properties of our stratification. Let $\varphi: A \to B$ be a ring homomorphism from A to another connected formally real semilocal ring B. (Of course $\varphi(1) = 1$.) Let Y denote the space Sign B. The homomorphism φ induces a continuous map $\varphi^*: Y \to X$. For $\tau \in Y$ and $a \in A^*$ we have the formula $\varphi^*(\tau)(a) = \tau(\varphi(a))$. The image $\varphi^*(\tau)$ will be often denoted by $\tau|A$ and regarded as the "restriction" of τ to A. For any prime ideal $\mathfrak{q}$ of B we have

$$(5.1) \qquad \varphi^*(Y(\mathfrak{q})) \subset X(\varphi^{-1}(\mathfrak{q})).$$

Indeed, with $\mathfrak{p} := \varphi^{-1}(\mathfrak{q})$ we have a natural commutative diagram of rings

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ A(\mathfrak{p}) & \longrightarrow & B(\mathfrak{q}) \end{array} .$$

Thus if a signature τ of B extends to $B(\mathfrak{q})$ then the restriction $\tau|A$ extends to $A(\mathfrak{p})$. In the same way we see that

$$\varphi^*(Y_{\mathfrak{q}}) \subset X_{\varphi^{-1}(\mathfrak{q})}$$

and for any ideal $\mathfrak{b}$ of B that

$$\varphi^*(Y^{\mathfrak{b}}) \subset X^{\varphi^{-1}(\mathfrak{b})}.$$

Unfortunately in general φ^* does not map $Y(\mathfrak{q})^{\circ}$ into $X(\varphi^{-1}(\mathfrak{q}))^{\circ}$. We know from (5.1) that for any τ in T

$$\mathfrak{p}(\tau|A) \supset \varphi^{-1}(\mathfrak{p}(\tau)),$$

but we do not always have equality here. To get insight into this phenomenon we need a fact of independent interest.

Lemma 5.2. Assume A has no zero divisors. Let τ be a signature of the field of fractions F of A and let $\mathfrak{p}$ be the prime ideal of A associated with $\tau|A$. Let S be a multiplicative subset of $A\setminus\{0\}$. Thus $A \subset S^{-1}A \subset F$. The following are equivalent:

(i) $S \cap \mathfrak{p} = \emptyset$;

(ii) A is cofinal in $S^{-1}A$ with respect to τ, i.e. for any $b > 0$ in $S^{-1}A$ there exists some a in A with $a > b$ with respect to τ.

Remark 5.3. The case $S = A\setminus\{0\}$ means that $\mathfrak{p}(\tau|A) = 0$ if and only if A is cofinal in F with respect to τ. Thus the lemma gives an answer to our problem for φ the inclusion map $A \to F$.

Proof of Lemma 5.2. All inequalities are with respect to τ. Let σ denote the restriction $\tau|A$. We consider the negations of the properties (i), (ii) above.

(a) $S \cap \mathfrak{p} \neq \emptyset$;

(b) A is not cofinal in $S^{-1}A$.

(a) $\Rightarrow$ (b): We choose some s in $S \cap \mathfrak{p}$, $s > 0$. For every a in A we have

$$s^{-1} - a = s^{-1}(1-as) > 0$$

since $1-as$ lies in $1+\mathfrak{p}$, hence in $Q(\sigma)$.

(b) $\Rightarrow$ (a): There exists some $a > 0$ in A and some $s > 0$ in S such that $s^{-1}a > x$ for every $x > 0$ in A. Let $b > 0$ be some unit of A. Choosing $x = b^{-1}a$ we learn that $s < b$. We want to deduce from this inequality for all $b > 0$ in A^* that s lies in $\mathfrak{p}$, which will finish the proof. Let $a > 0$ be given in A. Using the Chinese remainder theorem we find some ξ in A with $a + \xi^2 \in A^*$. We have $s < (a+\xi^2)^{-1}$, hence

$$as < as + \xi^2 s < 1.$$

Thus $1 - as > 0$ for all a in A. According to Theorem 4.3 the signature σ can be extended from A to A/As. This means that $s \in \mathfrak{p}$.

QED

We also need the following fact, which has a trivial proof.

<u>Lemma 5.5.</u> Let σ be a signature of A and let $\mathfrak{a}$ be an ideal of A contained in $\mathfrak{p}(\sigma)$. Let $\tilde{\sigma}$ denote the unique extension of σ to $A/\mathfrak{a}$, cf. Remark 4.4. Then $\mathfrak{p}(\tilde{\sigma}) = \mathfrak{p}(\sigma)/\mathfrak{a}$.

Now we come back to our ring homomorphism $\varphi: A \to B$. Let τ be a signature of B and let σ denote the restriction $\tau|A$. Let $\mathfrak{q} := \mathfrak{p}(\tau)$ and $\mathfrak{p} := \varphi^{-1}(\mathfrak{q})$. We regard $\bar{A} := A/\mathfrak{p}$ as a subring of $\bar{B} := B/\mathfrak{q}$ and hence also as a subring of $B(\mathfrak{q})$. On $B(\mathfrak{q})$ we have the ordering $\bar{\tau}$.

<u>Theorem 5.6.</u> The following are equivalent:

(i) $\mathfrak{p} = \mathfrak{p}(\sigma)$.

(ii) $\bar{A}$ is cofinal in $B(\mathfrak{q})$ with respect to $\bar{\tau}$.

(iii) $\bar{A}$ is cofinal in $\bar{B}$ with respect to $\bar{\tau}$.

(iv) For every b in $Q(\tau)$ there exists some c in $Q(\tau)$ and some a in $P(\sigma)$ such that $b + c = \varphi(a)$.

(v) For every b in $P(\tau)$ we have elements $\lambda_1,\dots,\lambda_r$ in B, elements $c_1,\dots,c_r$ in $P(\tau)$, some μ in $\mathfrak{p}(\tau)$ and some a in $Q(\sigma)$ such that

$$b + \sum_{i=1}^{r} \lambda_i^2 c_i + \mu = \varphi(a).$$

<u>Proof.</u> Let $\tilde{\sigma}$ denote the restriction of $\bar{\tau}$ to $\bar{A}$, which is the unique signature of $\bar{A}$ extending σ. By Lemma 5.5 we have $\mathfrak{p}(\bar{\sigma}) = \mathfrak{p}(\sigma)/\mathfrak{p}$. Remark 5.3 now shows that (i) $\Leftrightarrow$ (ii). This remark also shows that $\bar{B}$ is cofinal in $B(\mathfrak{q})$, hence that (ii) $\Leftrightarrow$ (iii).
(iii) $\Rightarrow$ (iv): Every $\bar{b} > 0$ in $\bar{B}$ has a preimage b in $Q(\tau)$, and every $\bar{a} > 0$ in $\bar{A}$ has a preimage a in $Q(\sigma)$. Thus we obtain for every b in $Q(\tau)$ an equality

$$b + c + \mu = \varphi(a)$$

with $c \in Q(\tau)$, $\mu \in \mathfrak{p}(\tau)$, $a \in Q(\sigma)$. There exists some ξ in A such that $a + \xi^2$ is a unit in A. We have

$$b + c + \mu + \varphi(\xi)^2 = \varphi(a+\xi^2)$$

and $c + \mu + \varphi(\xi)^2$ lies in $Q(\tau)$. The implications (iv) $\Rightarrow$ (v) and (v) $\Rightarrow$ (iii) are trivial.

QED

If the equivalent properties (iv), (v) in Theorem 5.6 hold true then we say that A is <u>cofinal in</u> B with respect to τ (and φ).

<u>Example 5.7.</u> If $\varphi: A \to B$ is an integral homomorphism then property (iii) in Theorem 5.6 is easily verified for every signature τ of B. Thus A is cofinal in B for every τ and $\varphi^{-1}(\mathfrak{p}(\tau)) = \mathfrak{p}(\tau|A)$. The latter fact has already been proved in [K_4, § 3]. It implies that $\varphi^{-1}(Q(\tau)) = Q(\tau|A)$.

Example 5.8. If σ is any signature of A and $\mathfrak{p}$ is the associated prime ideal, then by Lemma 4.8 we have a unique extension $\tilde{\sigma}$ of σ to $A_\mathfrak{p}$. Now A is cofinal in $A_\mathfrak{p}$ with respect to $\tilde{\sigma}$, since the prime ideal $\mathfrak{p}(\tilde{\sigma}) = \mathfrak{p}A_\mathfrak{p}$ has the preimage $\mathfrak{p}$ in A. Thus for any multiplicative subset S of A which is disjoint from $\mathfrak{p}$ and produces a semilocal ring $S^{-1}A$ (cf. Remark 5.12 below) the set A is cofinal in $S^{-1}A$ with respect to the unique extension of σ to $S^{-1}A$. On the other hand, if τ is an extension of σ to a semilocal ring $S^{-1}A$ with $S \cap \mathfrak{p} \neq \emptyset$ then A is not cofinal in $S^{-1}A$ with respect to τ since the preimage of $\mathfrak{p}(\tau)$ in A must be different from $\mathfrak{p}$.

We now switch over to another problem. Let S_o be a multiplicative subset of our semilocal ring A such that $A_o := S_o^{-1}A$ is again semilocal. Let τ be a signature of A_o. For any multiplicative subset $S \subset S_o$ with $S^{-1}A$ semilocal we consider the restriction $\tau | S^{-1}A$ with respect to the natural map from $S^{-1}A$ to A_o. The associated prime ideal $\mathfrak{p}(\tau | S^{-1}A)$ has the shape $S^{-1}\mathfrak{p}(\tau, S^{-1}A)$ with a uniquely determined prime ideal $\mathfrak{p}(\tau, S^{-1}A)$ of A disjoint from S. What can be said about the totality of these prime ideals $\mathfrak{p}(\tau, S^{-1}A)$?

Let $S_o^{-1}\mathfrak{p}_o$ be the prime ideal of A_o associated with τ itself, $\mathfrak{p}_o = \mathfrak{p}(\tau, A_o)$. According to Lemma 4.8 the signature τ has a unique extension τ' to the localization $A_{\mathfrak{p}_o}$ of A_o with respect to $S_o^{-1}\mathfrak{p}_o$ and the prime ideal associated with τ' is $\mathfrak{p}_o A_{\mathfrak{p}_o}$. We may ask more generally for the prime ideal associated with $\tau' | S^{-1}A$ for any semilocal ring $S^{-1}A$ with S contained in $A \setminus \mathfrak{p}_o$ instead of S_o. Thus replacing τ by τ' we assume since now <u>that</u> $S_o = A \setminus \mathfrak{p}_o$ <u>and</u> $\mathfrak{p}_o A_{\mathfrak{p}_o}$ <u>is the prime ideal of</u> $A_{\mathfrak{p}_o}$ <u>associated with</u> τ.

Definition 5.9. We call a prime ideal $\mathfrak{p}$ of A associated with the signature τ of $A_o = A_{\mathfrak{p}_o}$ if $\mathfrak{p} \supset \mathfrak{p}_o$ and $\mathfrak{p}A_{\mathfrak{p}}$ is the prime ideal associated with $\tau|A_{\mathfrak{p}}$, i.e. $\tau|A_{\mathfrak{p}}$ can be extended to $A(\mathfrak{p})$. The set of these prime ideals will be denoted by $\mathrm{Ass}(\tau,A)$. By our assumption $\mathfrak{p}_o$ is an element of $\mathrm{Ass}(\tau,A)$.

Proposition 5.10. The set $\mathrm{Ass}(\tau,A)$ is totally ordered by inclusion. Thus this set is finite if $A/\mathfrak{p}_o$ has finite Krull dimension.

Proof. Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be prime ideals of A associated with τ. Suppose that neither $\mathfrak{p}_1 \subset \mathfrak{p}_2$ nor $\mathfrak{p}_2 \subset \mathfrak{p}_1$. Consider the ring $B := S^{-1}A$ with $S := A \setminus (\mathfrak{p}_1 \cup \mathfrak{p}_2)$ which has precisely two maximal ideals $\mathfrak{m}_1 := S^{-1}\mathfrak{p}_1$ and $\mathfrak{m}_2 := S^{-1}\mathfrak{p}_2$. Let $\mathfrak{r}$ be the prime ideal of B associated with $\tau|B$. The signature $\tau|B$ extends to the signature $\tau|B_{\mathfrak{m}_1}$ of $B_{\mathfrak{m}_1} = A_{\mathfrak{p}_1}$. By our assumption on $\mathfrak{p}_1$ this signature extends further to a signature of $B(\mathfrak{m}_1)$. Thus $\mathfrak{m}_1 \subset \mathfrak{r}$ which implies $\mathfrak{m}_1 = \mathfrak{r}$. For the same reason $\mathfrak{m}_2 = \mathfrak{r}$. This is the desired contradiction.

QED

Proposition 5.11. Let S be a multiplicative subset of A disjoint from $\mathfrak{p}_o$ with $S^{-1}A$ semilocal. Let $\mathfrak{p}$ be the prime ideal of A such that $S^{-1}\mathfrak{p}$ is associated with $\tau|S^{-1}A$. Then $\mathfrak{p}$ is the largest prime ideal in $\mathrm{Ass}(\tau,A)$ disjoint from S.

Remark 5.12. Let $\mathfrak{q}_1,\dots,\mathfrak{q}_t$ be the finitely many ideals in A which are maximal disjoint from S. Then S is contained in the multiplicative set $\Sigma := A \setminus (\mathfrak{q}_1 \cup \dots \cup \mathfrak{q}_t)$. On the other hand it is easily proved that every u in Σ is a divisor of some s in S with respect to A. Thus $S^{-1}A = \Sigma^{-1}A$.

Proof of Proposition 5.11. $\tau|S^{-1}A$ has a unique extension ρ to $A_\mathfrak{p}$ and ρ has the associated prime ideal $\mathfrak{p}A_\mathfrak{p}$, cf. Lemma 4.8. Now $\tau|S^{-1}A$ also extends to the residue class field $A(\mathfrak{p}_o) = (S^{-1}A)(S^{-1}\mathfrak{p}_o)$ of $S^{-1}A$. Thus $S^{-1}\mathfrak{p}_o \subset S^{-1}\mathfrak{p}$, which means $\mathfrak{p}_o \subset \mathfrak{p}$. We see that A_o is a localization of $A_\mathfrak{p}$ and conclude that $\rho = \tau|A_\mathfrak{p}$. Thus $\mathfrak{p}$ lies in $Ass(\tau,A)$. Let $\mathfrak{q}$ be any element of $Ass(\tau,A)$ disjoint from S. Then $\tau|S^{-1}A$ extends to $\tau|A_\mathfrak{q}$ and further to a signature on $A(\mathfrak{q}) = (S^{-1}A)(S^{-1}\mathfrak{q})$. Thus $S^{-1}\mathfrak{q}$ is contained in $S^{-1}\mathfrak{p}$, i.e. $\mathfrak{q}$ is contained in $\mathfrak{p}$.

QED

Remark 5.13. In the situation of Proposition 5.11 the following is a priori clear: If $S^{-1}\mathfrak{p}$ is the prime ideal associated with $\tau|S^{-1}A$ and $\bar{\tau}$ is the signature induced by τ on $A(\mathfrak{p}_o)$ then $\bar{\tau}|S^{-1}(A/\mathfrak{p}_o)$ has as associated prime ideal $S^{-1}(\mathfrak{p}/\mathfrak{p}_o)$, cf. Lemma 5.5. Thus in the whole study we could have replaced from the beginning A by the ring $A/\mathfrak{p}_o$ and A_o by the field $A(\mathfrak{p}_o)$.

Our results, Propositions 5.10 and 5.11, can be globalized. Let V be a scheme. Assume that V is irreducible and reduced, which is no essential restriction in view of Remark 5.13. Let τ be a signature of the field F of rational functions on V.

Definition 5.14. We call a point x of V associated with τ if the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_x = \mathcal{O}_{V,x}$ is associated with the signature $\tau|\mathcal{O}_x$ of $\mathcal{O}_x$. This just means that $\tau|\mathcal{O}_x$ can be extended to the residue class field $\kappa(x) = \mathcal{O}_x/\mathfrak{m}_x$. We denote the set of these points by $Ass(\tau,V)$. For any $x \in Ass(\tau,V)$ we also call the reduced irreducible subscheme $\overline{\{x\}}$ of V with generic point x a <u>subscheme associated</u> with τ. By definition the scheme V itself is associated with τ.

<u>Theorem 5.15.</u> Assume that V is quasiprojective, i.e. isomorphic to an open subscheme of a projective space P^N_Λ over some commutative ring Λ. The set of subschemes of V associated with a given signature τ of F is totally ordered by inclusion.

<u>Proof.</u> Let x_1 and x_2 be points in Ass(τ,V), and suppose that neither $\overline{\{x_1\}} \subset \overline{\{x_2\}}$ nor $\overline{\{x_2\}} \subset \overline{\{x_1\}}$. We consider the semilocal subring $A := \mathfrak{o}_{x_1} \cap \mathfrak{o}_{x_2}$ of F which has two maximal ideals $\mathfrak{p}_1, \mathfrak{p}_2$ corresponding to the points x_1, x_2. We have $\mathfrak{o}_{x_1} = A_{\mathfrak{p}_1}$, $\mathfrak{o}_{x_2} = A_{\mathfrak{p}_2}$ and we see that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ both lie in Ass(τ,A). This contradicts Proposition 5.10.

The assumption that V is quasiprojective had just been made to guarantee that A is semilocal. In § 6 we shall remove this assumption.

<u>Theorem 5.16.</u> Let C be a finite set in V which is contained in an open affine subset of V. We denote by A the subring $\mathfrak{o}_{V,C}$ of F consisting of all rational functions on V which are defined in some neighbourhood of C. Let Z be the smallest subscheme of V associated with τ which meets the set C, and let $\mathfrak{p}$ be the prime ideal of A defined by Z. Then $\mathfrak{p}$ is the prime ideal associated with the signature $\tau | A$.

For the proof one simply applies Proposition 5.11 to the semilocal A and the signature τ of the quotient field F of A(take $\mathfrak{p}_o = 0$).

§ 6 Connection with valuations

Let τ be an ordering of a field F. For any subfield k of F we denote by $\mathfrak{o}(\tau,F/k)$ the convex hull of the subset k in F with respect to τ. It is well known that $\mathfrak{o}(\tau,F/k)$ is a valuation ring of F, cf. [Kr], [P]. The maximal ideal $\mathfrak{m}(\tau,F/k)$ is the set of all elements in F which are infinitely small relative to k with respect to τ [loc.cit.].

We call a valuation ring $\mathfrak{o}$ of F, i.e. with quotient field F, compatible with τ, if the following three properties hold true, which turn out to be equivalent, cf. [P] or [KW].

(i) $1 + \mathfrak{m} \subset P(\tau)$ for the maximal ideal $\mathfrak{m}$ of $\mathfrak{o}$.

(ii) $\mathfrak{o}$ is convex in F with respect to τ.

(iii) There exists a subfield k of F such that $\mathfrak{o} = \mathfrak{o}(\tau,F/k)$.

As field k we can choose in fact any maximal subfield k of $\mathfrak{o}$.

All valuation rings compatible with τ contain $\mathfrak{o}_0 := \mathfrak{o}(\tau,F/\mathbb{Q})$, and thus the set of these rings is totally ordered by inclusion. Moreover any subring $\mathfrak{o}$ of F which contains $\mathfrak{o}_0$ is a valuation ring and is compatible with τ. It is also well known [Kr,P], that τ induces an ordering $\bar{\tau}$ on the residue class field $\mathfrak{o}/\mathfrak{m}$ of any valuation ring $\mathfrak{o}$ compatible with τ. Regarding τ and $\bar{\tau}$ as signatures we have $\tau(a) = \bar{\tau}(\bar{a})$ for any unit a of $\mathfrak{o}$. This means in our terminology that $\mathfrak{m} = \mathfrak{p}(\tau|\mathfrak{o})$ and that $\bar{\tau}$ is the signature induced by $\tau|\mathfrak{o}$ on $\mathfrak{o}/\mathfrak{m}$ in the sense of § 4.

Let now A be a semilocal subring of F with quotient field F. The ring $\mathfrak{o} := A\cdot\mathfrak{o}_0$ generated by A and $\mathfrak{o}_0$ in F is the smallest valuation ring of F compatible with τ and containing A. Looking at property (ii) above we see that $\mathfrak{o}$ coincides with the convex hull $\tilde{A}$ of A with respect to τ. (Notice that A contains the convex hull $\mathfrak{o}_0$ of $\mathbb{Z}$ in F and thus is a valuation ring.) Let $\mathfrak{m}$ denote the maximal ideal of $\mathfrak{o} = \tilde{A}$.

<u>Theorem 6.1.</u> The prime ideal $\mathfrak{p}(\sigma)$ associated with the signature $\sigma := \tau|A$ is the center $\mathfrak{m} \cap A$ of $\mathfrak{o}$ in A. The signature $\bar{\tau}$ of $\mathfrak{o}/\mathfrak{m}$ extends the signature $\bar{\sigma}$ of $A(\mathfrak{p}(\sigma))$.

<u>Proof.</u> The first assertion follows from Theorem 5.6 since $A/\mathfrak{m} \cap A$ is cofinal in $\mathfrak{o}/\mathfrak{m}$ with respect to $\bar{\tau}$. The second assertion then is evident.

We return to the situation considered at the end of § 5. Thus F is the field of rational functions on a reduced irreducible scheme V and τ is a signature on F. Theorem 6.1 has the following consequence:

<u>Corollary 6.2.</u> A point x of V is associated with τ if and only if there exists a valuation ring $\mathfrak{o}$ of F compatible with τ which has the center x on V, i.e. $\mathfrak{o}$ contains the local ring $\mathcal{O}_x$ of x and the maximal ideal $\mathfrak{m}$ of $\mathfrak{o}$ contains the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_x$.

From this corollary we obtain another proof of Theorem 5.15, and we can even remove the assumption that V is quasiprojective in this theorem. Indeed, any valuation ring $\mathfrak{o}$ of F has at most one center x on V, cf. [EGA, I 8.2.2] and the set of valuation rings of F compatible with τ is totally ordered by inclusion.

For x in Ass(τ,V) we can in some sense determine the smallest valuation ring of F compatible with τ and dominating $\mathfrak{O}_x$. Notice that $\mathfrak{O}_x$ contains a copy of the field $\mathbb{Q}$.

Proposition 6.3. Let $x \in V$ be associated with τ and let k be a maximal subfield of the ring $\mathfrak{O}_x$. Then $\mathfrak{o}(\tau,F/k)$ is the smallest valuation ring of F which is compatible with τ and dominates $\mathfrak{O}_x$.

Proof. Put $\mathfrak{o} = \mathfrak{o}(\tau,F/k)$, $\mathfrak{m} = \mathfrak{m}(\tau,F/k)$. Clearly $\mathfrak{o}$ is contained in any valuation ring $\mathfrak{o}'$ of F which is compatible with τ and contains $\mathfrak{O}_x$, since k can be extended to a maximal subfield k' of $\mathfrak{o}'$ and $\mathfrak{o}'$ coincides with $\mathfrak{o}(\tau,F/k')$. It remains to show that $\mathfrak{o}$ dominates $\mathfrak{O}_x$, i.e. $\mathfrak{o} \supset \mathfrak{O}_x$ and $\mathfrak{m} \supset \mathfrak{m}_x$. For any λ in k^* with $\tau(\lambda) = +1$ we have $\lambda + \mathfrak{m}_x \subset \mathfrak{O}_x^*$ and then $\tau(\lambda+\mathfrak{m}_x) = \{+1\}$ since $\mathfrak{m}_x$ is the prime ideal associated with $\tau|\mathfrak{O}_x$. Thus $\mathfrak{m}_x$ is infinitely small relative to k (with respect to τ, as always), i.e. $\mathfrak{m}_x \subset \mathfrak{m}$. Suppose $\mathfrak{O}_x$ is not contained in $\mathfrak{o}$. Then there exists a unit u of $\mathfrak{O}_x$ with $u \notin \mathfrak{o}$. Since the field k(u) cannot be contained in $\mathfrak{O}_x$, due to the maximality of k, we have $k[u] \cap \mathfrak{m}_x \neq \{0\}$. Thus we have an equation

$$u^d + \lambda_1 u^{d-1} + \dots + \lambda_d = c$$

with $\lambda_1,\dots,\lambda_d$ in k and c in $\mathfrak{m}_x$, $d > 0$. The elements $\lambda_1,\dots,\lambda_{d-1},\lambda_d - c$ all lie in $\mathfrak{o}$. Since $\mathfrak{o}$ is integrally closed this implies $u \in \mathfrak{o}$, a contradiction. Thus $\mathfrak{O}_x \subset \mathfrak{o}$, q.e.d.

This valuation ring $\mathfrak{o}$ can be determined in a more constructive way as follows:

<u>Remark 6.4.</u> Let L be some subfield of $\mathfrak{o}_x$, and assume as before $x \in \mathrm{Ass}(\tau,V)$. Choose some transcendency basis $(\bar{t}_i | i \in I)$ of the field $\kappa(x) = \mathfrak{o}_x/\mathfrak{m}_x$ over L. Choose then a set of representatives $(t_i | i \in I)$ of this basis in $\mathfrak{o}_x$. We introduce the subfield $k_o := L(t_i | i \in I)$ of F. The ring $L[t_i | i \in I]$ has zero intersection with $\mathfrak{m}_x$. Thus k_o is contained in $\mathfrak{o}_x$. We claim that $\mathfrak{o}(\tau,F/k_o)$ coincides with the valuation ring considered in Proposition 6.3. Indeed, extend k_o to a maximal subfield k of $\mathfrak{o}_x$. Since k injects into $\kappa(x)$ the extension k/k_o is algebraic. Thus k lies in the convex hull $\mathfrak{o}(\tau,F/k_o)$ of k_o and we have $\mathfrak{o}(\tau,F/k_o) = \mathfrak{o}(\tau,F/k)$.

We now are able to fill a gap in § 4.

<u>Definition 6.5.</u> We call a connected semilocal ring A <u>geometric</u> if A is isomorphic to the semilocal ring $\mathfrak{o}_{V,C}$ of some quasiprojective scheme V of finite type over some field k with respect to some finite subset C of V. In other terms, A is geometric if there exists some finitely generated algebra A_o over some field k and finitely many prime ideals $\mathfrak{q}_1,\ldots,\mathfrak{q}_r$ of A_o such that $A \cong S^{-1}A_o$ with $S = A_o \setminus (\mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_r)$.

<u>Theorem 6.6.</u> For any prime ideal $\mathfrak{p}$ of a geometric semilocal ring A the stratum $X(\mathfrak{p})^o$ is dense in the closed set $X(\mathfrak{p})$ (cf. notations of § 4).

Recall that we have a continuous map $\pi_\mathfrak{p}: Z(\mathfrak{p}) \to X(\mathfrak{p})$ from the space of signatures $Z(\mathfrak{p})$ of the field $A(\mathfrak{p})$ onto $X(\mathfrak{p})$. The preimage $Z(\mathfrak{p})^o$ of $X(\mathfrak{p})^o$ in $Z(\mathfrak{p})$ consists precisely of all orderings τ of $A(\mathfrak{p})$ such that the subring $A/\mathfrak{p}$ is cofinal in $A(\mathfrak{p})$ with respect to τ, as is clear from Remark 5.3 and Lemma 5.5. If we know that $Z(\mathfrak{p})^{o}$ is dense in $Z(\mathfrak{p})$ then Theorem 6.6 is evident. Thus it suffices to prove the following lemma.

<u>Lemma 6.7.</u> Let A be a geometric semilocal ring without zero divisors, and let F denote the field of fractions of A. The set of orderings τ of F such that A is cofinal in F with respect to τ is dense in the space Z of all orderings of F.

<u>Proof.</u> Let k be the subfield of A occuring in Definition 6.5. Then F is a finitely generated field extension of k. Let $f_1,\dots,f_r$ be elements of F^* such that the open set

$$Z(f_1,\dots,f_r) = \{\tau \in Z \mid \tau(f_1) = \dots = \tau(f_r) = 1\}$$

is not empty. We have to show that $Z(f_1,\dots,f_r)$ contains a signature τ such that A is cofinal in F with respect to τ. Since $Z(f_1,\dots,f_r)$ is not empty the field $K := F(\sqrt{f_1},\dots,\sqrt{f_r})$ is formally real. More precisely the elements of $Z(f_1,\dots,f_r)$ are the restrictions of the orderings of K to F. The integral closure B of A in K is finite over A [Bb, § 3 n° 2, Th.2], hence again semilocal, and A is cofinal in B with respect to every ordering of K. Thus replacing A by B we may assume from the beginning that A is integrally closed, and we have only to find one ordering τ of F such that A is cofinal in F with respect to τ. Moreover replacing k by the algebraic closure of k in K we may assume from the beginning that the subfield k of A is algebraically closed in F and hence the field extension F/k is regular.

We choose an ordering τ_o of F and introduce the real closure R of k with respect to τ_o. We regard A as the semilocal ring $\mathfrak{o}_{V,C}$ of some integral affine algebraic scheme V over k at some finite subset C of V. Consider the scheme $V \otimes_k R$ which is again integral. $A \otimes_k R$ is the semilocal ring of the finite set C' of points of $V \otimes_k R$ lying over points of C. The function field $F \cdot R$ of $V \otimes_k R$ is formally real, and for every ordering τ of $F \cdot R$ the subring A of $F \cdot R$ is cofinal in the subring $A \otimes_k R$. Thus we may replace A by $A \otimes_k R$ and assume without loss of generality that in addition to the assumptions above the base field k is a real closed field R.

We regard as before A as a semilocal ring $\mathfrak{o}_{V,C}$ of some affine integral algebraic scheme V over R. Let Z denote the Zariski closure of the finite set C in V. According to E.Artin's specialization theory, cf. [L, Theorem 8], there exists a regular point x in $V \setminus Z$ which is rational over R. We choose a regular system of parameters $t_1, \dots, t_n$ of the regular local ring $B := \mathfrak{o}_{V,x}$. Let W be the closed integral subscheme of V passing through x which defines the prime ideal Bt_1 of B, and let $\mathfrak{o}$ denote the local ring of the generic point of W in V. This is a discrete valuation ring of F, whose residue class field $\mathfrak{o}/\mathfrak{m}$ is the function field R(W) of W. Since Z is not contained in W, the ring $A = \mathfrak{o}_{V,C}$ is not contained in $\mathfrak{o}$. Now the field $\mathfrak{o}/\mathfrak{m}$ is formally real since W possesses the regular point x which is rational over R. Thus there exists some ordering τ of F compatible with $\mathfrak{o}$, cf. [Kr] or [P]. But $\mathfrak{o}$ is the maximal proper valuation ring of F compatible with τ since $\mathfrak{o}$ has rank one. Thus the convex hull of A in F with respect to τ must be the whole of F.

QED

Notice that some finiteness condition about A is necessary in Theorem 6.6. If we take for example a non archmedian real closed field R and choose for A the local ring $\mathfrak{o}(\rho, R/\mathbb{Q})$ with ρ the unique ordering of R, then X is a one point set, and $X = X(\mathfrak{p})$ for $\mathfrak{p} = (0)$. But $X(\mathfrak{p})^{o}$ is empty.

§ 7 Fans

If G is any nontrivial group of exponent 2 with base point "-1", and X is the set of all characters $\sigma \in G$ with $\sigma(-1) = -1$ then according to Marshall the pair (X,G) is a space of orderings ([M_2, 3.6 and 3.7]; this seems not to be proved explicitly in Marshall's papers, but the proof is easy). Such a space (X,G) is called a fan. Marshall gives in [M_3, § 2] two other conditions for a space of orderings to be a fan:

Lemma 7.1. For any non empty space of orderings (X,G) the following are equivalent:

(i) (X,G) is a fan.

(ii) For every element $g \neq -1$ of G the set $D_X(<1,g>)$ contains only the elements 1 and g

(iii) For any three characters $\sigma_1, \sigma_2, \sigma_3$ in X the product $\sigma_1 \sigma_2 \sigma_3$ lies again in X.

If (X,G) is a space of orderings then a subset Y of X is called a fan in X, if Y is a subspace of X and $(Y, G/Y^\perp)$ is a fan.

Lemma 7.2. Let (X,G) be a space of orderings. A non empty subset Y of X is a fan if and only if Y is closed in X and for any three elements $\sigma_1, \sigma_2, \sigma_3$ of Y the product $\sigma_1 \sigma_2 \sigma_3$ lies again in Y.

Indeed, these conditions are clearly necessary for Y to be a fan in X. On the other hand, if they are fulfilled then the closed subgroup [Y] generated by Y in G is the union of Y and $\sigma_0 Y$ for some arbitrary chosen σ_0 in Y. Thus certainly $[Y] \cap X = Y$, which means that Y is a subspace of X. By Lemma 7.1 the space of orderings $(Y,G/Y^{\perp})$ is a fan.

Fans are of utmost importance in the theory of reduced forms as is clear from Marshall's abstract representation theorem [M_3, Th. 3.5] which we shall recall in § 9.

We return to our connected formally real semilocal ring A and the space of orderings $(X,G) = (\mathrm{Sign}\,A, G(A))$. The fans in (X,G) will be called <u>fans of</u> A. We further call for any fan V of A the saturated subgroup P(V) the <u>fan domain</u> of V. Recall that P(V) is the intersection of all signature domains $P(\sigma)$ with σ in Y and that any signature domain $P \supset P(Y)$ is one of the $P(\sigma)$ with σ in Y, since Y is a subspace of (X,G). Moreover by the definition of fans a saturated subgroup T of A^* is a fan domain if and only if any subgroup P of A^* with $T \subset P$, $-1 \notin P$, $(A^*:P) = 2$ is a signature domain. From Lemma 7.1 and Proposition 3.7 we read off the following characterization of fan domains.

<u>Proposition 7.3.</u> Suppose A fulfills H(2). A saturated subgroup T of A^* is a fan domain if and only if for every a in A^* with $a \notin -T$ we have

$$(T+aT) \cap A^* = T \cup aT.$$

<u>Remark.</u> If A violates H(2) then we have to replace this equation by

$$(U+aU) \cap A^* = T \cup aT$$

with $U = T\cdot\Pi(A)$, cf. Prop. 3.7.

This is completely analogous to the field case [Be K § 4]. We now come to the central theorem of our paper that the fans of A are all "induced" in a canonical way from the residue class fields $A(\mathfrak{p})$.

<u>Theorem 7.4.</u> Let V be a fan of A containing at least four elements.

i) All $\sigma \in V$ have the same associated prime ideal $\mathfrak{p}$.

ii) The set of induced signatures $\bar{V} := \{\bar{\sigma} | \sigma \in V\}$ on the field $A(\mathfrak{p})$ is again a fan.

To prove this theorem we first observe that it suffices to consider the case that A fulfills the hypothesis H(2,3), i.e. has only residue class fields with at least 4 elements. Indeed, assume that the theorem is proved in this case but that A violates H(2,3). Then we know that the theorem holds true for the ring $C = A[T]/T^3 + 6T^2 + 29T + 1$. As observed in § 2 the natural morphism of pairs (Sign A,G(A)) $\rightarrow$ (Sign C,G(C)) is an isomorphism of spaces of orderings. Bearing in mind Example 5.7 we see that the statement of our theorem also holds true for A.

Thus we assume since now that A fulfills H(2,3). The following lemma will be of great use for us.

<u>Lemma.</u> Let $\sigma_1,\sigma_2,\sigma_3,\sigma_4$ be signatures of A with $\sigma_1 \sigma_2 \sigma_3 \sigma_4 = 1$ in $\hat{G}$. Let $\mathfrak{p}_i = \mathfrak{p}(\sigma_i)$, $P_i = P(\sigma_i)$, $Q_i = Q(\sigma_i)$. Then

$$(Q_2 \cup \mathfrak{p}_2) \cap (Q_3 \cup \mathfrak{p}_3) \cap (Q_4 \cup \mathfrak{p}_4) \subset Q_1 \cup \mathfrak{p}_1.$$

<u>Proof of the lemma.</u> Suppose that this inclusion relation is wrong. Then choose some

$$x \in (-Q_1) \cap (Q_2 \cup \mathfrak{p}_2) \cap (Q_3 \cup \mathfrak{p}_3) \cap (Q_4 \cup \mathfrak{p}_4).$$

Suppose we have in the field $A(\mathfrak{p}_1)$ the inequality $\bar{x} \leq -1$ with respect to the ordering $\bar{\sigma}_1$. Take some ξ in A such that

$$a := 1 + (1+\xi^2)x$$

is a unit in A. This is possible by the Chinese remainder theorem. In the field $A(\mathfrak{p}_1)$ we have $\bar{a} < 0$ with respect to $\bar{\sigma}_1$. Thus $\sigma_1(a) = -1$. But clearly $a \in Q_i$ for $i = 2,3,4$, hence $\sigma_2(a) = \sigma_3(a) = \sigma_4(a) = +1$. This contradicts our relation $\sigma_1 \sigma_2 \sigma_3 \sigma_4 = 1$, and we learn that $-\bar{x} < 1$ in $A(\mathfrak{p}_1)$ with respect to σ_1. Write $-x = u + v$ with u and v in P_1 (Remark 4.9). The element $y := xv^{-2}$ lies again in $-Q_1$ and in all $Q_i \cup \mathfrak{p}_i$ with $i = 2,3,4$. Thus also $-\bar{y} < 1$ in $A(\mathfrak{p}_1)$. But we have with respect to $\bar{\sigma}_1$

$$1 > -\bar{x} > \bar{v} > 0, \text{ hence } -\bar{y} > 1.$$

This contradiction shows that our lemma holds true.

We now prove part (i) of the theorem. It is easily deduced from Lemma 7.2 that any two elements (in fact three elements) of V are contained in a four element subfan of V. Thus we assume without loss of generality that V consists of four different signatures $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ with the relation $\sigma_1 \sigma_2 \sigma_3 \sigma_4 = 1$ in $\hat{G}$. As in the lemma we use the abbreviations $(i = 1,2,3,4)$ $\mathfrak{p}_i = \mathfrak{p}(\sigma_i)$, $P_i = P(\sigma_i)$, $Q_i = Q(\sigma_i)$.

Suppose that the intersection $\mathfrak{p}_2 \cap \mathfrak{p}_3 \cap \mathfrak{p}_4$ is <u>not</u> contained in $\mathfrak{p}_1$. Then clearly $(-Q_1) \cap \mathfrak{p}_2 \cap \mathfrak{p}_3 \cap \mathfrak{p}_4$ would be non empty, contradicting our lemma. Thus

$$\mathfrak{p}_2 \cap \mathfrak{p}_3 \cap \mathfrak{p}_4 \subset \mathfrak{p}_1.$$

This implies that $\mathfrak{p}_1$ contains at least one of the prime ideals $\mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4$. Of course this remains true for any permutation of the four prime ideals $\mathfrak{p}_i$.

Since now we assume without loss of generality that $\mathfrak{p}_1$ is a minimal element of the set $\{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4\}$ with respect to inclusion. After renumbering the sequence $\mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4$ we assume that $\mathfrak{p}_2 \subset \mathfrak{p}_1$, hence $\mathfrak{p}_2 = \mathfrak{p}_1$. We want to show that also $\mathfrak{p}_3 = \mathfrak{p}_4$. We distinguish three cases.

Case 1: $\mathfrak{p}_1 \not\subset \mathfrak{p}_3$. Then $\mathfrak{p}_4 \subset \mathfrak{p}_3$. If $\mathfrak{p}_4$ is different from $\mathfrak{p}_3$ then we must have $\mathfrak{p}_1 \subset \mathfrak{p}_4$ in contradiction to $\mathfrak{p}_1 \not\subset \mathfrak{p}_3$. Thus $\mathfrak{p}_3 = \mathfrak{p}_4$.

Case 2: $\mathfrak{p}_1 \not\subset \mathfrak{p}_4$. Interchanging the roles of σ_3 and σ_4 we see again $\mathfrak{p}_3 = \mathfrak{p}_4$.

Case 3: $\mathfrak{p}_1 \subset \mathfrak{p}_3$ and $\mathfrak{p}_1 \subset \mathfrak{p}_4$. Suppose $\mathfrak{p}_4 \not\subset \mathfrak{p}_3$. We choose some element x in $\mathfrak{p}_4$ which does not lie in $\mathfrak{p}_3$ and hence also not in $\mathfrak{p}_1$. We have $x \in \varepsilon_i Q_i$ for $i = 1,2,3$ with suitable $\varepsilon_i = \pm 1$. Since any three different signatures are linearly independent in the $\mathbb{F}_2$-vector space $\hat{G}$, there exists some unit c with $\sigma_1(c) = -\varepsilon_1$, $\sigma_2(c) = \varepsilon_2$, $\sigma_3(c) = \varepsilon_3$. The element cx lies in $(-Q_1) \cap Q_2 \cap Q_3 \cap \mathfrak{p}_4$ in contradiction to our lemma. Thus $\mathfrak{p}_4 \subset \mathfrak{p}_3$ and for the same reason $\mathfrak{p}_3 \subset \mathfrak{p}_4$.

We now have proved in all cases that $\mathfrak{p}_1 = \mathfrak{p}_2$ and $\mathfrak{p}_3 = \mathfrak{p}_4$. We want to show that $\mathfrak{p}_3$ is contained in $\mathfrak{p}_1$, which means $\mathfrak{p}_3 = \mathfrak{p}_1$ by the minimality of $\mathfrak{p}_1$. Suppose $\mathfrak{p}_3 \not\subset \mathfrak{p}_1$. Then there exists some x in $\mathfrak{p}_3$ with $x \in -Q_1$, hence $x \in \varepsilon Q_2$ with $\varepsilon = +1$ or $\varepsilon = -1$. Choose some unit c in A with $\sigma_1(c) = +1$, $\sigma_2(c) = \varepsilon$. Then the element cx lies in

$(-Q_1) \cap Q_2 \cap \mathfrak{p}_3 \cap \mathfrak{p}_4$ contradicting our lemma. Thus $\mathfrak{p}_3 = \mathfrak{p}_1$ and part (i) of Theorem 7.4 is proved.

We now prove part (ii) of this theorem. Let $\mathfrak{p}$ denote the prime ideal associated with all signatures $\sigma \in V$. The set $\bar{V}$ is the pre-image of V under the restriction map $\pi_{\mathfrak{p}}: Z(\mathfrak{p}) \to X$. Since $\pi_{\mathfrak{p}}$ is continuous $\bar{V}$ is a closed subset of the space $Z(\mathfrak{p})$ of signatures of $A(\mathfrak{p})$. We have to show that $\bar{V}$ is a fan of $A(\mathfrak{p})$. According to Lemma 7.2 it suffices to verify that for any three signatures $\sigma_1, \sigma_2, \sigma_3$ in V the product $\bar{\sigma}_1 \bar{\sigma}_2 \bar{\sigma}_3$ lies in $\bar{V}$. Now the product $\sigma_4 := \sigma_1 \sigma_2 \sigma_3$ lies in V. Thus it suffices to verify that $\bar{\sigma}_1 \bar{\sigma}_2 \bar{\sigma}_3 \bar{\sigma}_4 = 1$.

From our lemma we learn that $Q_2 \cap Q_3 \cap Q_4 \subset Q_1$. Suppose the relation $\bar{\sigma}_1 \bar{\sigma}_2 \bar{\sigma}_3 \bar{\sigma}_4$ is not true. We choose some x in $A \setminus \mathfrak{p}$ such that for the image $\bar{x}$ in $A(\mathfrak{p})$

$$(*) \qquad \bar{\sigma}_1(\bar{x})\, \bar{\sigma}_2(\bar{x})\, \bar{\sigma}_3(\bar{x})\, \bar{\sigma}_4(\bar{x}) = -1.$$

Since any three different signatures are independent, there exists a unit a in A with $\sigma_i(a) = \bar{\sigma}_i(\bar{x})$ for $i = 2,3,4$. The element $y = ax$ lies in $Q_2 \cap Q_3 \cap Q_4$, since $\bar{\sigma}_2(\bar{y}) = \bar{\sigma}_3(\bar{y}) = \bar{\sigma}_4(\bar{y}) = +1$. Thus y also lies in Q_1 and $\bar{\sigma}_1(\bar{y}) = +1$. But we have

$$\sigma_1(a)\, \sigma_2(a)\, \sigma_3(a)\, \sigma_4(a) = +1.$$

Comparing this with the equation (*) we obtain $\bar{\sigma}_1(\bar{y}) = -1$, a contradiction. Theorem 7.4 is completely proved.

For a non trivial fan V of A we call the prime ideal $\mathfrak{p}$ associated with any σ in V the <u>prime ideal associated with the fan</u> V, and we write $\mathfrak{p} = \mathfrak{p}(V)$.

Multiplying by $u \in P_1$ we obtain

$$u - \xi^2 \in Q_1.$$

On the other hand

$$a = \xi^2 - u - v,$$

thus

$$\xi^2 - u = a + v \in Q_1.$$

This is a contradiction: Q_1 cannot contain both $\pm (u-\xi^2)$. Thus

$$Q_2 \cap Q_3 \cap Q_4 \subset Q_1.$$

(This implies $\bar{Q}_2 \cap \bar{Q}_3 \cap \bar{Q}_4 \subset \bar{Q}_1$.)

Now suppose the relation $\bar{\sigma}_1 \bar{\sigma}_2 \bar{\sigma}_3 \bar{\sigma}_4 = 1$ is not true. We choose some x in $A \setminus \mathfrak{p}$ such that

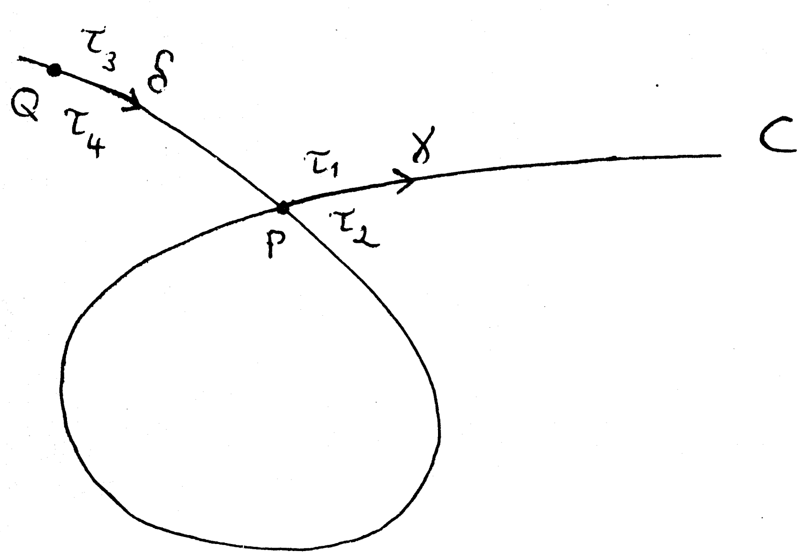
(*) $$\bar{\sigma}_1(\bar{x}) \bar{\sigma}_2(\bar{x}) \bar{\sigma}_3(\bar{x}) \bar{\sigma}_4(\bar{x}) = -1.$$

We find a unit a in A with $\sigma_i(a) = \bar{\sigma}_i(\bar{x})$ for $i = 2,3,4$, since any three different signatures are "independent". The element $y = ax$ lies in $Q_2 \cap Q_3 \cap Q_4$. We have $\sigma_1(a)\ \sigma_2(a)\ \sigma_3(a)\ \sigma_4(a) = +1$. Thus we obtain from (*) that $\bar{\sigma}_1(\bar{y}) = -1$, i.e. $y \in -Q_1$. This contradicts our preceding result that $Q_2 \cap Q_3 \cap Q_4$ is contained in Q_1, and we have finally verified that $\sigma_1\ \sigma_2\ \sigma_3\ \sigma_4 = 1$. Theorem 7.4 is completely proved.

For a non trivial fan V of A we call the prime ideal $\mathfrak{p}$ associated with any σ in V the <u>prime ideal associated with the fan</u> V, and we write $\mathfrak{p} = \mathfrak{p}(V)$.

We give an example in which the geometry behind the fact that every non trivial fan is associated with a prime ideal is visible.

Example 7.4. We go back the geometric situation (M,S,C,P,γ) described in Example 4.2. Let τ_1 be the ordering of F considered in that example and let τ_2 be the "opposite" ordering. The positive elements with respect to τ_2 are those functions which take positive values near P at the right hand side of γ in positive direction of γ. Choose another real point Q on the curve C, a local orientation of $M(\mathbb{R})$ at Q and an oriented real branch δ of C at Q. Let τ_3,τ_4 be the orderings of F defined as before starting with Q and δ.



Then $\{\tau_1,\tau_2,\tau_3,\tau_4\}$ is a fan of the field F. Indeed, if a function $f \in F^*$ has equal signs $\tau_1(f) = \tau_2(f)$ then the divisor div(f) contains C with even multiplicity (perhaps zero). Thus $\tau_3(f) = \tau_4(f)$. If $\tau_1(f) = -\tau_2(f)$ then div(f) contains C with odd multiplicity, hence $\tau_3(f) = -\tau_4(f)$. Thus $\tau_1\,\tau_2\,\tau_3\,\tau_4 = 1$ [8].

8) I learned this example of a four element fan and the ones given below from L.Bröcker.

Assume now that $C \cap S = \emptyset$. Let N be some algebraic variety over $\mathbb{R}$ which contains M as a closed subvariety. Let B denote the semilocal ring of N at the finite set S. We have a natural ring homomorphism

$$B = \mathcal{O}_{N,S} \longrightarrow\!\!\!\!\rightarrow A = \mathcal{O}_{M,S} \longrightarrow F$$

from B to F. Let ρ_i denote the restriction of the signature τ_i of F to B ($i = 1,2,3,4$). Then $V := \{\rho_1,\rho_2,\rho_3,\rho_4\}$ is a fan of B. It follows from Example 4.2 and our general theory that $\mathfrak{p}(V)$ is the prime ideal of B defined by the closed irreducible subvariety M of N and $\overline{V} = \{\tau_1,\tau_2,\tau_3,\tau_4\}$. We obtain similar examples replacing Q by P and taking for δ another real branch of C at P or the branch γ with reversed orientation.

We want to describe the prime ideal $\mathfrak{p}(V)$ associated with a non trivial fan V (i.e. V has at least 4 elements) of a connected formally real semilocal ring A starting with the fan domain $T = P(V)$.

<u>Proposition 7.5.</u> $\mathfrak{p}(V)$ is the set of all x in A such that $(1+Ax) \cap A^*$ is contained in T.

<u>Proof.</u> $(1+Ax) \cap A^*$ is contained in T if and only if this set is contained in $P(\sigma)$ for every $\sigma \in V$. By Theorem 4.3 this means that every $\sigma \in V$ can be extended to A/Ax, hence $Ax \subset \mathfrak{p}(\sigma)$ for every σ in V.

QED

<u>Remark.</u> The proof shows that Proposition 7.5 remains true for an arbitrary subspace V of $(\mathrm{Sign}(A),G(A))$ if we then denote by $\mathfrak{p}(V)$ the intersection of all prime ideals $\mathfrak{p}(\sigma)$, $\sigma \in V$.

We give a second description of the prime ideal $\mathfrak{p}(V)$. Any ideal $\mathfrak{a}$ of A which is maximally disjoint from the multiplicative set Q(V) is a prime ideal [9]. Here again V could be any subspace of Sign(A). But for V a non trivial fan much more holds true.

Theorem 7.6. Let V be a non trivial fan of A. Then $\mathfrak{p}(V)$ is disjoint from Q(V) and contains any other ideal $\mathfrak{a}$ of A which is disjoint from Q(V).

N.B. If A fulfills H(2,3) then in this statement Q(V) can be replaced by 1 + T, cf. Remark 4.9.

Proof. We have to show for every element x of A:

$$x \in \mathfrak{p}(V) \Leftrightarrow (Ax) \cap Q(V) = \emptyset.$$

The implication "⇒" is trivial, since the image $\bar{Q}(V)$ of Q(V) in the field $A(\mathfrak{p}(V))$ cannot contain zero. Let now x be an element of A with (Ax) ∩ Q(V) empty. Suppose x is not contained in $\mathfrak{p}(V)$. Then by Theorem 4.3

$$(1+Ax) \cap A^* \not\subset P(\sigma)$$

for every σ in V. Thus the saturation of the group $T\cdot[(1+Ax) \cap A^*]$ is the whole of A^* and contains in particular -1. We have an equation

$$-1 = (1+a_1x)t_1\lambda_1 +\ldots+ (1+a_rx)t_r\lambda_r$$

with t_i in T, a_i in A, and λ_i in $\Pi(A)$, cf. Theorem 3.2. This equation implies

$$-2x(a_1t_1\lambda_1 +\ldots+ a_rt_r\lambda_r) = 2 + 2\lambda_1t_1 +\ldots+ 2\lambda_rt_r.$$

9) cf. § 4 for the definition of Q(V).

Since the elements $2\lambda_i$ are sums of squares the right hand sides lies in Q(V). This contradicts our hypothesis that Ax does not meet Q(V). Thus x lies in $\mathfrak{p}(V)$.

QED

§ 8 A P-structure on the space of signatures

The main result Theorem 7.4 of the preceding section enables us to introduce on the space of signatures of our semilocal ring A a natural "P-structure" in the sense of Marshall $[M_4]$ which refines the stratification given in § 4.

Definition 8.1 $[M_4$, § 3]. A P-structure on a space of orderings (X,G) is a partition

$$X = \bigsqcup_{\alpha \in I} X(\alpha)$$

of the set X into fans X(α) such that every fan V of X meets at most two sets X(α).

If (X,G) is the space of signatures of some field K then a well known natural P-structure

$$X = \bigsqcup_{\lambda \in M(K)} X(\lambda)$$

on X = Sign K is defined as follows (cf.[BeB],[BrM]: M(k) is the set of $\mathbb{R}$-valued places $\lambda: K \to \mathbb{R} \cup \infty$ on K. For every $\lambda \in M(K)$ the set $X(\lambda)$ consists of all $\sigma \in X$ which are "compatible" with λ. Here we call an ordering σ compatible with λ if $\lambda(a) \geq 0$ or $\lambda(a) = \infty$ for all elements a of K which are positive with respect to σ. In other words, σ is compatible with λ if and only if σ is compatible with the valuation ring $\mathfrak{o}_\lambda$ of λ (cf. § 6) and the injection from the residue class field $\mathfrak{o}_\lambda/\mathfrak{m}_\lambda$ to $\mathbb{R}$ induced by λ is order preserving with respect to $\bar{\sigma}$ and the unique ordering of $\mathbb{R}$. This implies that $\bar{\sigma}$ is an archimedian ordering, hence that $\mathfrak{o}_\lambda$ is the smallest valuation ring $\mathfrak{o}(\sigma,K/Q)$ of K compatible with σ. We then see immediately that for every $\sigma \in X$ there exists a unique $\mathbb{R}$-valued place λ compatible with σ. Thus X is indeed the disjoint union of all $X(\lambda)$. Moreover it is classical that every $X(\lambda)$ is a fan of X, cf. [Kr], [P] (in particular non empty). Finally every fan V of X meets at most two sets $X(\lambda)$. This follows from Bröcker's important theorem [B_1], that for every fan V of X there exists a valuation ring $\mathfrak{o}$ of K and a fan $\bar{V}$ of the residue class field $\mathfrak{o}/\mathfrak{m}$ of $\mathfrak{o}$ consisting of at most two elements, such that every $\sigma \in V$ is compatible with $\mathfrak{o}$ and induces a signature $\bar{\sigma} \in \bar{V}$ on $\mathfrak{o}/\mathfrak{m}$.

We come back to our connected formally real semilocal ring A. We use the notations $X = \text{Sign } A$, $X(\mathfrak{p}), X(\mathfrak{p})^{o}, Z(\mathfrak{p}) = \text{Sign } A(\mathfrak{p}), Z(\mathfrak{p})^{o}$, etc. introduced in § 4. For every prime ideal $\mathfrak{p}$ of A we denote the set $M(A(\mathfrak{p}))$ of $\mathbb{R}$-valued places on the field $A(\mathfrak{p})$ briefly by $M(\mathfrak{p})$ and the fan consisting of all those orderings $\sigma \in Z(\mathfrak{p})$, which are compatible with a given $\lambda \in M(\mathfrak{p})$, briefly by $Z(\mathfrak{p},\lambda)$. Thus

$$Z(\mathfrak{p}) = \bigsqcup_{\lambda \in M(\mathfrak{p})} Z(\mathfrak{p},\lambda)$$

is our natural P-structure on $Z(\mathfrak{p})$. We introduce the subset $M(\mathfrak{p})^{o}$ of $M(\mathfrak{p})$ consisting of all λ in $M(\mathfrak{p})$ such that the subring $\mathfrak{o}_{\lambda}\cdot A/\mathfrak{p}$ of $A(\mathfrak{p})$ generated by $\mathfrak{o}_{\lambda}$ and $A/\mathfrak{p}$ is the whole of $A(\mathfrak{p})$. We conclude from § 5 that

$$(8.1) \qquad Z(\mathfrak{p})^{o} = \bigsqcup_{\lambda \in M(\mathfrak{p})^{o}} Z(\mathfrak{p},\lambda).$$

Indeed, let τ be an ordering of $A(\mathfrak{p})$ and σ the restriction of τ to A. Let λ be the $\mathbb{R}$-valued place of $A(\mathfrak{p})$ associated with τ. Then

$o_\lambda \cdot A/\mathfrak{p}$ is the convex hull of $A/\mathfrak{p}$ in $A(\mathfrak{p})$ with respect to τ, cf. § 6. By definition τ lies in $Z(\mathfrak{p})^\circ$ if and only if $\mathfrak{p} = \mathfrak{p}(\sigma)$, which according to Theorem 5.6 means that $o_\lambda \cdot A/\mathfrak{p}$ is the whole of $A(\mathfrak{p})$, i.e. that λ lies in $M(\mathfrak{p})^\circ$.

For any λ in $M(\mathfrak{p})$ we denote by $X(\mathfrak{p},\lambda)$ the image of $Z(\mathfrak{p},\lambda)$ under the restriction map $\pi_\mathfrak{p}: Z(\mathfrak{p}) \to X(\mathfrak{p})$. This set $X(\mathfrak{p},\lambda)$ is closed in X and is a fan according to Lemma 7.2. For λ in $Z(\mathfrak{p})^\circ$ moreover $\pi_\mathfrak{p}$ gives a homeomorphism from $Z(\mathfrak{p},\lambda)$ onto $X(\mathfrak{p},\lambda)$, and we have

$$X(\mathfrak{p})^\circ = \bigsqcup_{\lambda \in M(\mathfrak{p})^\circ} X(\mathfrak{p},\lambda). \tag{8.2}$$

We introduce the set $M(A)$ consisting of all pairs $(\mathfrak{p},\lambda)$ with $\mathfrak{p}$ a prime ideal of A and λ in $M(\mathfrak{p})^\circ$,

$$M(A) := \bigsqcup_{\mathfrak{p}} \{\mathfrak{p}\} \times M(\mathfrak{p})^\circ.$$

We then obtain from (8.2) the partition

$$X = \bigsqcup_{(\mathfrak{p},\lambda) \in M(A)} X(\mathfrak{p},\lambda) \tag{8.3}$$

of X into fans $X(\mathfrak{p},\lambda)$, which refines our previous stratification of X.

<u>Theorem 8.4.</u> The partition (8.3) is a P-structure on X.

<u>Proof.</u> Let V be a fan of A. We have to show that V intersects at most two of the sets $X(\mathfrak{p},\lambda)$, $(\mathfrak{p},\lambda) \in M(A)$. If V has one or two

elements this is trivial. Assume now that V has more than two elements. We know from Theorem 7.4 that $V \subset X(\mathfrak{p})^{\circ}$ for a suitable prime ideal $\mathfrak{p}$ and that the preimage $\overline{V}$ of $X(\mathfrak{p})^{\circ}$ in $Z(\mathfrak{p})^{\circ}$ is a fan of $A(\mathfrak{p})$. Thus we know from field theory that $\overline{V}$ meets at most two of the sets $Z(\mathfrak{p},\lambda)$ with λ in $M(\mathfrak{p})$. This implies that V meets at most two of the sets $X(\mathfrak{p},\lambda)$ with λ in $M(\mathfrak{p})^{\circ}$.

QED

Let $T(\mathfrak{p},\lambda)$ denote the fan domain of $X(\mathfrak{p},\lambda)$ in A^*. Theorem 8.4 implies that the reduced Witt ring $\overline{W}(A)$ is a certain "amalgamated product" of the Witt rings $W_{T(\mathfrak{p},\lambda)}(A)$ with $(\mathfrak{p},\lambda)$ running through $M(A)$, see [M_4, Theorem 3.12] for the details.

For any space of orderings (X,G) which is a fan the Witt ring $W(X,G)$ is isomorphic to the group ring $\mathbb{Z}[G/\{\pm 1\}]$ (not quite canonically, cf. [M_2, 3.8]). In particular

$$W_{T(\mathfrak{p},\lambda)}(A) \cong \mathbb{Z}[A^*/T(\mathfrak{p},\lambda)\cdot\{\pm 1\}].$$

This Witt ring is naturally isomorphic to the Witt ring of the fan $Z(\mathfrak{p},\lambda)$ of $A(\mathfrak{p})$, hence isomorphic to the group ring $\mathbb{Z}[\Gamma_\lambda/2\Gamma_\lambda]$ with Γ_λ denoting the value group $A(\mathfrak{p})^*/\mathfrak{o}_\lambda^*$ of λ. We have in fact a natural group homomorphism

$$A^*/T(\mathfrak{p},\lambda)\cdot\{\pm 1\} \to \Gamma_\lambda/2\Gamma_\lambda$$

induced by the natural homomorphism from A^* to $A(\mathfrak{p})^*$ and the valuation belonging to λ. From the discussion just made it can be concluded that this group homomorphism is in fact an isomorphism.

It would be interesting to have a closer look at the quotient topology on M(A) with respect to the natural surjection $X \to M(A)$ which collapses every fan $X(\mathfrak{p},\lambda)$ to the point λ. For example it would be desirable to know whether Marshall's axioms P_3, P_4 (cf. [M_4, § 3]) hold true for our P-structure on X. We do not enter into this.

§ 9 Representation of functions on the signature space by forms

We recall Marshall's abstract representation theorem ([M_3, Th. 3.5],[M_4]).

Theorem 9.1. Let (X,G) be an arbitrary space of orderings and let $f:X \to \mathbb{Z}$ be a continuous function. We regard W(X,G) as a subring of $C(X,\mathbb{Z})$ in the canonical manner, cf. § 1. The following are equivalent:

(i) $f \in W(X,G)$.

(ii) $f|V \in W(V)$ for every finite fan V in X.

(iii) $\sum_{\sigma \in V} f(\sigma) \equiv 0 \bmod |V|$ for all finite fans V in X.

Here $|V|$ denotes the cardinality of V, which is a 2-power. This theorem solves the representation problem for bilinear spaces over an arbitrary connected formally real semilocal ring A posed in § 1, since we "know" what the fans of A are by Theorem 7.3 and by Bröcker's valuation theoretic description of fans over fields [B_1].

There remains the question which continuous functions $f:\text{Sign } A \to \mathbb{Z}$ can be represented by quadratic spaces. It turns out that nothing new happens.

Theorem 9.2. Every bilinear space of even rank over A is congruent to the bilinear space associated with some quadratic space over A. Thus every continuous function $f:X \to 2\mathbb{Z}$ which is representable by a bilinear space is representable by a quadratic space.

If 2 is a not a unit in A then by this theorem $\bar{W}q(A)$ coincides with the ideal $\bar{I}(A)$ of $\bar{W}(A)$ consisting of the reduced bilinear spaces of even rank. If $2 \in A^*$ then of course $\bar{W}q(A) = \bar{W}(A)$. Notice that over other commutative connected rings things may be quite different. For example $W(\mathbb{Z}) = \bar{W}(\mathbb{Z})$ can be identified with $\mathbb{Z}$ and $Wq(\mathbb{Z}) = \bar{W}q(\mathbb{Z}) = 8\mathbb{Z}$.

Theorem 9.2 is a consequence of the following lemma.

<u>Lemma 9.3.</u> For every unit a of A there exists some element b in A such that $1 - 4b \in A^*$ and

$$\langle 1, a\rangle \cong \begin{pmatrix} 2 & 1 \\ 1 & 2b \end{pmatrix}.$$

Indeed, assume this lemma holds true. Let φ be a bilinear space of rank 2m over A. As shown in § 2, φ is congruent to a proper space $\langle c_1, \ldots, c_{2m}\rangle$. Now

$$\langle c_1, \ldots, c_{2m}\rangle \cong \perp_{i=1}^{m} \langle c_{2i-1}\rangle \otimes \langle 1, a_i\rangle$$

with $a_i = c_{2i-1}c_{2i}$. Using Lemma 9.3 we obtain

$$\varphi \cong \perp_{i=1}^{m} \langle c_{2i-1}\rangle \otimes \begin{pmatrix} 2 & 1 \\ 1 & 2b_i \end{pmatrix}$$

with suitable $b_i \in A$ such that $1 - 4b_i \in A^*$. The bilinear space on the right hand side is associated with the quadratic space

$$\perp_{i=1}^{m} \langle c_{2i-1}\rangle \otimes [1, b_i].$$

It remains to verify Lemma 9.3. The ring A/4A is not formally real. Thus the group $\Pi^*(A/4A)$ is the whole unit group of A/4A. Every element of this group can be lifted to an element of A lying in $\Pi(A)$.

Thus for our given unit a we have a congruence $-a \equiv u \bmod 4A$ with some $u \in \Pi(A)$. Now choose some ξ in A with $\xi \equiv 1 \bmod \mathfrak{m}_i$ for all maximal ideals $\mathfrak{m}_i$ which contain u and $\xi \equiv 0 \bmod \mathfrak{m}_i$ for the other $\mathfrak{m}_i$. Notice that if $2 \in \mathfrak{m}_i$ then $u \equiv -a \bmod \mathfrak{m}_i$, hence $u \notin \mathfrak{m}_i$. Thus $t := u + 4\xi^2$ is a unit in A. We have $t \in \Pi^*(A)$ and $t \equiv -a \bmod 4A$, hence $a = t(4b-1)$ with some b in A. Thus

$$\langle 1,a\rangle \equiv \langle 1,4b-1\rangle.$$

We now show that

$$\langle 1,4b-1\rangle \equiv \begin{pmatrix} 2 & 1 \\ 1 & 2b \end{pmatrix}$$

which will finish the proof of the lemma. Take any signature σ of A. Let $\mathfrak{p}$ be the prime ideal associated with σ and $\bar{\sigma}$ the signature induced by σ on $A(\mathfrak{p})$. Since $2 \neq 0$ in $A(\mathfrak{p})$ we have over this field

$$\begin{pmatrix} 2 & 1 \\ 1 & 2\bar{b} \end{pmatrix} \cong \langle 2, 2(4\bar{b}-1)\rangle.$$

Applying $\bar{\sigma}$ to both sides we obtain

$$\sigma\begin{pmatrix} 2 & 1 \\ 1 & 2b \end{pmatrix} = 1 + \sigma(4b-1) = \sigma(\langle 1,4b-1\rangle).$$

Thus indeed $\begin{pmatrix} 2 & 1 \\ 1 & 2b \end{pmatrix}$ is congruent to $\langle 1,4b-1\rangle$, and the lemma is proved.

Returning to bilinear forms we draw some consequences from Marshall's representation theorem (= Theorem 9.1) and our knowledge about fans from § 7. We use the notations $X = \text{Sign } A$, $X(\mathfrak{p})$, $\pi_{\mathfrak{p}}$, $Z(\mathfrak{p}) = \text{Sign } A(\mathfrak{p})$, etc. from § 4.

<u>Theorem 9.4.</u> Let $f:X \to \mathbb{Z}$ be a continuous function which either has only even or only odd values. Assume that for every $\mathfrak{p} \in \mathrm{Spec}(A)$ the function $f \circ \pi_{\mathfrak{p}}: Z(\mathfrak{p}) \to \mathbb{Z}$ can be represented by a bilinear space over the field $A(\mathfrak{p})$. Then f can be represented by a bilinear space over A.

<u>Proof.</u> This follows immediately from Theorem 9.1 and the fact proved in § 7 that for every fan V over A with more than two elements there exists some prime ideal $\mathfrak{p}$ and some fan $\bar{V}$ over $A(\mathfrak{p})$ which is mapped bijectively onto V under $\pi_{\mathfrak{p}}$.

More generally we obtain by the same arguments (recall notations from § 3 and § 4)

<u>Theorem 9.4a.</u> Let T be a saturated proper subgroup of A^*. Let $f:X(T) \to \mathbb{Z}$ be a continuous function which either has only even or only odd values. Assume that for every prime ideal $\mathfrak{p}$ of A which does not meet the multiplicative set $Q_{\mathfrak{p}}(T)$ (cf. 4.11) the function

$$f \circ \pi_{\mathfrak{p}}: Z(\mathfrak{p})(\bar{T}) \to \mathbb{Z}$$

can be represented by a congruence class modulo $\bar{T}$ of bilinear forms over $A(\mathfrak{p})$. Then f can be represented by a congruence class modulo T of bilinear forms over A.

The <u>reduced stability index</u> st(A) of A is defined as the smallest natural number n such that every function $f \in C(X, 2^n\mathbb{Z})$ lies in $\bar{W}(A)$. ($\mathrm{st}(A) = \infty$ if no such n exists.) It can be shown that st(A) is also the smallest n such that $\bar{I}^{n+1}(A) = 2\bar{I}^n(A)$, cf. [$M_3$, Th. 4.2] and for the field case [B, Satz 3.17]. It follows from Theorem 9.1 that st(A) is the smallest number n such that every fan V of A has at most 2^n elements ([M_3, Th. 4.4], cf. [B_1, § 2] for the field case). Theorem 7.4 now immediately implies

<u>Theorem 9.5.</u> $st(A) \leq \sup(st(A(\mathfrak{p})) | \mathfrak{p} \in \operatorname{Spec} A)$.

<u>Question 9.6.</u> Does equality hold if A is a geometric semilocal ring (cf. Definition 6.5)?

If A is a formally real semilocal ring of some algebraic scheme V of dimension n over some field k, then according to Bröcker [B_1, Satz 4.8] we have

$$st(A(\mathfrak{p})) \leq st(k) + n+1$$

for every prime ideal $\mathfrak{p}$ of A. Thus

$$st(A) \leq st(k) + n+1. \tag{9.7}$$

If in addition st(L) = st(k) for all formally real algebraic extensions L of k, e.g. k is real closed, then we know from [B_1, 4.7] that even $st(A(\mathfrak{p})) \leq st(k) + n$ for all $\mathfrak{p}$, hence

$$st(A) \leq st(k) + n. \tag{9.8}$$

We now give an application of our main result Theorem 7.4 to the reduced Witt ring $\overline{W}(A)$ of a <u>regular</u> semilocal formally real ring A. Let F be the quotient field of A. Then $\overline{W}(A)$ injects into $\overline{W}(F)$, since all signatures of A can be extended to F [CRW]. We regard $\overline{W}(A)$ as a subring of $\overline{W}(F)$. Let $\mathfrak{m}_1,\ldots,\mathfrak{m}_g$ as always denote the maximal ideals of A and let B_i denote the localization $A_{\mathfrak{m}_i}$. We also regard all Witt rings $\overline{W}(B_i)$ as subrings of $\overline{W}(F)$. The ring $\overline{W}(A)$ is contained in all rings $\overline{W}(B_i)$.

<u>Theorem 9.9.</u> $\overline{W}(A) = \overline{W}(B_1) \cap \ldots \cap \overline{W}(B_g)$.

To prove this we need the following

<u>Lemma 9.10.</u> Let τ_1 and τ_2 be signatures of F with $\tau_1|A = \tau_2|A$. Then there exist some $i \in \{1,\ldots,g\}$ such that $\tau_1|B_i = \tau_2|B_i$.

<u>Proof.</u> Let $\mathfrak{p}$ be the prime ideal of A associated with the signature $\sigma := \tau_1|A = \tau_2|A$. We choose some maximal ideal $\mathfrak{m}_i \supset \mathfrak{p}$. Then σ has a <u>unique</u> extension $\tilde{\sigma}$ to B_i, cf. Lemma 4.8. Thus $\tau_1|B_i$ and $\tau_2|B_i$ both coincide with $\tilde{\sigma}$.

QED

We now come to the proof of Theorem 9.9. For sake of clearness we do <u>not</u> regard $\overline{W}(B_i)$ as a subring of $\overline{W}(F)$, but denote the image of an element ψ of $\overline{W}(B_i)$ in $\overline{W}(F)$ by $\psi \otimes F$. We use similar notations for images of elements of $\overline{W}(A)$ in the $\overline{W}(B_i)$ and $\overline{W}(F)$. Let $\varphi_1,\ldots,\varphi_g$ be elements of $\overline{W}(B_1),\ldots,\overline{W}(B_g)$ respectively such that

$$\varphi_1 \otimes F = \varphi_2 \otimes F = \ldots = \varphi_g \otimes F.$$

We have to find an element φ in $\overline{W}(A)$ such that $\varphi \otimes B_i = \varphi_i$. We define a function $f:\mathrm{Sign}(A) \to \mathbb{Z}$ as follows: $f(\sigma) = \tau(\varphi_1 \otimes F)$ with τ some extension of σ to F. This function f is well defined. Indeed, if also $\tau' \in \mathrm{Sign}\, F$ extends σ then by the preceding lemma $\tau|B_i = \tau'|B_i$ for some i, hence

$$\tau(\varphi_1 \otimes F) = \tau(\varphi_i \otimes F) = \tau'(\varphi_i \otimes F) = \tau'(\varphi_1 \otimes F).$$

The restriction map π: Sign $F \to$ Sign A is identifying. The $\mathbb{Z}$-valued continuous function $g := \varphi_1 \otimes F$ on Sign F (recall that we regard reduced forms as functions on the signature space) admits the factorization $g = f \circ \pi$. Thus f is continuous. We now want to apply Theorem 9.1 to verify that f lies in $\overline{W}(A)$. Clearly f admits either only even or only odd values. Let V be a fan of A consisting of 2^n elements with $n \geq 2$. Let $\mathfrak{p}$ be the prime ideal of A associated with V and $\overline{V}$ be the fan of $A(\mathfrak{p})$ induced by V, cf. Theorem 7.4. We choose some $\mathfrak{m}_i \supset \mathfrak{p}$. Let W be the image of V under the restriction map from Sign A to Sign B_i, which is again a fan. We have a commutative triangle of restriction maps

$$\begin{array}{ccc} \text{Sign } A & \longrightarrow & \text{Sign } B_i \\ & \searrow \quad \swarrow & \\ & \text{Sign } A(\mathfrak{p}) & \end{array}$$

The map from V to $\overline{V}$ is bijective, hence also the map from V to W is bijective. Thus W is a fan of B_i consisting of 2^n elements. Let ρ be an element of W, let σ be the restriction $\rho|A$ and let τ be some extension of ρ to F. Then

$$f(\sigma) = \tau(\varphi_1 \otimes F) = \tau(\varphi_i \otimes F) = \rho(\varphi_i).$$

Thus

$$\sum_{\sigma \in V} f(\sigma) = \sum_{\rho \in W} \rho(\varphi_i) \equiv 0 \bmod 2^n.$$

According to Theorem 9.1 the function f is represented by some $\varphi \in \overline{W}(A)$. Let σ be any signature of A, let ρ be an extension of σ

to some B_i and let τ be a further extension of ρ to F. Then

$$\rho(\varphi \otimes B_i) = \sigma(\varphi) = f(\sigma) = \tau(\varphi_i \otimes F) = \rho(\varphi_i).$$

Thus $\varphi \otimes B_i = \varphi_i$,   q.e.d.

§ 10 Weakly isotropic forms

A semisignature σ of A is an additive map from W(A) to $\mathbb{Z}$ which maps the Witt class of every space <a> of rank one to ± 1, cf. [K_1, § 5]. Such a map σ vanishes on the torsion part of W(A) and hence can also be regarded as a map from the reduced Witt ring $\overline{W}(A)$ to $\mathbb{Z}$, cf. Th. 1.2. Sometimes it is more convenient to regard σ as a map from the set A* of units of A to $\{\pm 1\}$, as we are accustomed to for signatures. We denote all these maps by the same letter σ. For any bilinear space E over A of rank n we have $|\sigma(E)| \leq n$ (use Prop. 2.4a). We call E positive definite with respect to σ if $\sigma(E) = n$, negative definite if $\sigma(E) = -n$, and indefinite else.

Our interest in semisignatures stems from the following fact.

Proposition 10.1. A bilinear space E over A is weakly isotropic if and only if E is indefinite with respect to every semisignature of A.

This has been proved under the hypothesis H(2) that A has no residue class fields with only two elements in [K_1, § 5] and again in [KlR, § 3]. The hypothesis H(2) can be removed as follows. Assume that A violates H(2) and let C denote the cubic extension $A[T]/T^3 + 6T^2 + 29T + 1$ which fulfills H(2). As observed in § 2 the natural map from the space of orderings (Sign A,G(A)) to (Sign C,G(C)) is an isomorphism. Since "weak isotropy" is a notion within the theory of spaces of orderings, cf. § 2, a bilinear space E over A is weakly isotropic if and only if $E \otimes_A C$ is weakly isotropic. The reduced Witt rings $\overline{W}(A)$ and $\overline{W}(C)$ are canonically isomorphic, since these rings depend only on the corresponding spaces of orderings. Thus there is a natural one-one correspondence between the semisignatures of A and the semisignatures of C. The assertion in Proposition 10.1 holds true for A since it holds true for C.

Let now σ be a fixed semisignature of A. We want to explain that there exists a prime ideal $\mathfrak{p} = \mathfrak{p}(\sigma)$ of A associated with σ in a completely analogous way as for signatures. We denote by $P(\sigma)$ the set of all a in A^* with $\sigma(a) = +1$ and by $Q(\sigma)$ the set of all elements

$$x = \lambda_1^2 a_1 + \dots + \lambda_r^2 a_r$$

with a_i in $P(\sigma)$ and λ_i in A such that $A\lambda_1 + \dots + A\lambda_r = A$. The set $Q(\sigma)$ does contain the zero element. Indeed, the spaces $\langle a_1, \dots, a_r \rangle$ with all a_i in $P(\sigma)$ are positive definite for σ hence anisotropic.

<u>Lemma 10.2.</u> Every element x of $Q(\sigma)$ has a presentation

$$x = a_1 + \lambda_2^2 a_2 + \dots + \lambda_r^2 a_r$$

with a_i in $P(\sigma)$ and λ_i in A.

The proof coincides with the proof of the same fact for signatures in [K_2, p. 86]. Notice that in this proof the element $a_1^{-1} a_2$ now is not necessarily contained in $P(\sigma)$. Nevertheless the elements $c^{-1} a_1$, $c^{-1} a_2$ in the cited proof lie in $P(\sigma)$, since

$$c^{-1} a_1 = c^{-2}(\xi^2 a_1 + \xi^2 a_2), c^{-1} a_2 = c^{-2}(\xi^2 a_2 + \eta^2 a_2^2 a_1^{-1}).$$

<u>Lemma 10.3.</u> Let x and y be elements of A with $x + y \in Q(\sigma)$. Then $x \in Q(\sigma)$ or $y \in Q(\sigma)$.

Again the proof in the special case of signatures [K_2, p.87] goes through for semisignatures with very small modifications. Do not divide by a_1 at the beginning of the proof!

Let $\mathfrak{p}$ denote the complement of $Q(\sigma) \cup (-Q(\sigma))$ in A.

<u>Theorem 10.4.</u> $\mathfrak{p}$ is a prime ideal of A, and $Q(\sigma) + \mathfrak{p} = Q(\sigma)$. There exists a unique semisignature $\bar{\sigma}$ of the field $A(\mathfrak{p})$ which extends σ. If σ can be extended to $A/\mathfrak{a}$ for some ideal $\mathfrak{a}$ then $\mathfrak{a} \subset \mathfrak{p}$.

<u>Proof.</u> (cf. [K_2, p.87] for the first three steps.) We use the abbreviations $P := P(\sigma)$ and $Q := Q(\sigma)$.

i) We prove that $\mathfrak{p}$ is an additive subgroup of A. It follows from the definition of $\mathfrak{p}$ that $-\mathfrak{p} = \mathfrak{p}$ and that $0 \in \mathfrak{p}$. Let x and y be elements of $\mathfrak{p}$. By Lemma 10.3 the sum $x + y$ neither lies in Q nor in $-Q$. Thus $x + y \in \mathfrak{p}$.

ii) We prove $Q + \mathfrak{p} = Q$. Let x be an element of Q and y be an element of $\mathfrak{p}$ and let $z := x + y$. Suppose z lies in $-Q$. Then $y = z - x$ also lies in $-Q$, a contradiction. Suppose z lies in $\mathfrak{p}$. Then $x = z - y$ also lies in $\mathfrak{p}$, a contradiction. Thus $z \in Q$.

iii) We show that $\lambda^2\mathfrak{p} \subset \mathfrak{p}$ for all $\lambda \in A$. Let $\lambda \in A$ and $x \in \mathfrak{p}$ be given. We choose some μ in A such that $\lambda^2 + \mu^2$ is a unit of A. Then $(\lambda^2+\mu^2)(\pm Q) = \pm Q$. Thus $(\lambda^2+\mu^2)\mathfrak{p} = \mathfrak{p}$. Suppose $\lambda^2 x$ lies in $-Q$. Then

$$\mu^2 x = (\lambda^2+\mu^2)x - \lambda^2 x \in \mathfrak{p} + Q = Q.$$

We have equations

$$-\lambda^2 x = \sum_{i=1}^{s} \gamma_i^2 a_i$$

$$\mu^2 x = \sum_{j=1}^{t} \delta_j^2 b_j$$

with a_i, b_j in P and

$$\sum_{i=1}^{s} \gamma_i A = A, \quad \sum_{j=1}^{s} \delta_j A = A.$$

We deduce from this the relation

$$\sum_{i=1}^{s} \gamma_i^2 \mu^2 a_i + \sum_{j=1}^{t} \delta_j^2 \lambda^2 b_j = 0.$$

Since

$$\sum_i \gamma_i \mu A + \sum_j \delta_j \lambda A = \mu A + \lambda A = A,$$

the relation shows that the bilinear space $\langle a_1,\ldots,a_s, b_1,\ldots,b_t\rangle$ is isotropic, hence indefinite with respect to σ. This contradicts the fact that all a_i and b_j lie in P. Thus $\lambda^2 x$ does not lie in $-Q$. Replacing x by $-x$ we see that $\lambda^2 x$ does not lie in Q either. The element $\lambda^2 x$ must lie in $\mathfrak{p}$.

iv) We now show that for every $\lambda \in A$ and $x \in \mathfrak{p}$ the element λx lies in $\mathfrak{p}$. Then we know that $\mathfrak{p}$ is an ideal of A. We have

$$2\lambda x = (1+\lambda)^2 x - x - \lambda^2 x \in \mathfrak{p}.$$

If λx would be contained in Q or $-Q$ then the same would be true for $2\lambda x$. Thus $\lambda x \in \mathfrak{p}$.

v) We show that $x^3 \in Q$ for every x in Q. Indeed, by Lemma 10.2 we have

$$x = a + \sum_{i=1}^{n} \lambda_i^2 a_i = a + x_1$$

with a and all a_i in P. This implies

$$x^3 = a^2 \cdot a + 3 \sum_{i=1}^{n} (\lambda_i a)^2 a_i + 3x_1^2 a + \sum_{i=1}^{n} (\lambda_i x_1)^2 a_i \in Q.$$

vi) Since now we assume without loss of generality that $1 \in P$. Indeed, we can replace σ by $-\sigma$ without changing any essentials. We show that $x^2 \in Q$ for every x in Q. If x^2 would lie in $-Q$ then also the element zero would lie in $-Q$. Thus x^2 lies in Q or in $\mathfrak{p}$. Suppose $x^2 \in \mathfrak{p}$. Then $x^3 \in \mathfrak{p}$, which contradicts step (v). Thus $x^2 \in Q$.

vii) Suppose there exist elements x and y in $A \setminus \mathfrak{p}$ with $xy \in \mathfrak{p}$. Eventually replacing x by $-x$ or y by $-y$ we may assume that x and y lie in Q. From the equation above (step v) for x^3 we obtain

$$x^3 y^2 = (ay)^2 y + 3 \sum_{i=1}^{n} (\lambda_i ay)^2 a_i + 3(x_1 y)^2 a + \sum_{i=1}^{n} (\lambda_i x_1 y)^2 a_i.$$

This sum lies in $\mathfrak{p}$. A priori all terms of the sum lie in $Q \cup \mathfrak{p}$. Since $Q + \mathfrak{p} = Q$, all terms of the sum must lie in $\mathfrak{p}$. In particular $a^3 y^2 \in \mathfrak{p}$. Since a is a unit $y^2 \in \mathfrak{p}$. But according to step (vi) the element y^2 lies in Q. This contradiction proves that $\mathfrak{p}$ is a prime ideal.

The proof of the remaining assertions in Theorem 10.4 is now easy and is the same as in the special case of signatures, cf. [K_2].

We call $\mathfrak{p}$ the <u>prime ideal associated with</u> the semisignature σ and write $\mathfrak{p} = \mathfrak{p}(\sigma)$. We call $\bar{\sigma}$ the semisignature (= quadratic semiordering in the sense of Prestel [P_1]) of the field $A(\mathfrak{p})$ induced by σ.

We now come to the main result of this section.

Theorem 10.5. Let E be a bilinear space over A and assume that $E \otimes_A A(\mathfrak{p})$ is weakly isotropic for every prime ideal $\mathfrak{p}$ of A. Then E is weakly isotropic.

Proof. Let σ be a semisignature of A, let $\mathfrak{p} = \mathfrak{p}(\sigma)$ and let $\bar{\sigma}$ denote the semisignature of $A(\mathfrak{p})$ induced by σ. Then, if E is of rank n,

$$|\sigma(E)| = |\bar{\sigma}(E \otimes A(\mathfrak{p}))| < n$$

since $E \otimes A(\mathfrak{p})$ is weakly isotropic. Thus E is indefinite with respect to all semisignatures, which by Proposition 10.1 means that E is weakly isotropic.

QED

Remark. This theorem has first been proved by L.Bröcker [B_2]. For our proof it suffices to know that every semisignature σ of A can be extended to some residue class field $A(\mathfrak{p})$. Bröcker shows this using a method invented by A.Prestel, cf. [P, Lemma 1.4].

Corollary 10.6. Let be given a bilinear space φ over A and some unit a of A. Assume that for every prime ideal $\mathfrak{p}$ of A the image a of a in $A(\mathfrak{p})$ lies in $\bar{D}(\varphi \otimes A(\mathfrak{p}))$. Then a lies in $\bar{D}(\varphi)$.

This follows from Theorem 10.5 since $a \in \bar{D}(\varphi)$ if and only if $\varphi \perp \langle -a \rangle$ is weakly isotropic.

<u>Remark 10.7.</u> We call a <u>quadratic</u> space E over A weakly isotropic if $m \times E$ is isotropic for some $m \geq 1$. The statement in Theorem 10.5 remains true for E a quadratic instead of bilinear space. Indeed, assume that $E \otimes A(\mathfrak{p})$ is weakly isotropic for all $\mathfrak{p}$. Then it follows from Theorem 10.5 that the bilinear space $\tilde{E}$ associated with E is weakly isotropic. Thus $m \times \tilde{E}$ is isotropic for some $m \geq 1$. This implies trivially that the quadratic space $2m \times E$ is isotropic, cf. [K_3, p.188].

We want to generalize Theorem 10.5 to forms modulo some proper saturated subgroup T of A^*.

<u>Definition 10.8.</u> A semisignature σ of A is <u>compatible</u> with T if the map $\sigma: W(A) \to \mathbb{Z}$ factors through the canonical surjection from $W(A)$ onto $W_T(A)$. If A fulfills $H(2)$ this means according to Proposition 3.5a that the map $\sigma: A^* \to \{\pm 1\}$ is constant on all cosets aT, $a \in A^*$ [7]. We call such a semisignature also a T-semisignature.

The following proposition has been proved by Kleinstein and Rosenberg ([KlR, Th. 3.10]; the hypothesis $H(2)$ there can be removed in the same way as above in Prop. 10.1).

<u>Proposition 10.9.</u> Let E be a bilinear space over A which is indefinite with respect to every T-semisignature of A. Then E is isotropic modulo T.

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7) Actually hypothesis $H(2)$ is not needed here. It is not difficult to prove that the kernel of $\overline{W}(A) \twoheadrightarrow W_T(A)$ is always additively generated by the reduced forms $\langle a,-at \rangle$, $a \in A^*$, $t \in T$.

Remark. We could define semisignatures over any space of orderings (X,G) and establish a proposition analogous to Prop. 10.1 for the forms over (X,G), which would then contain Prop. 10.9 as a special case. We do not enter into this.

Let now $\mathfrak{p}$ be a prime ideal of A which is disjoint from the multiplicative set $Q_{\mathfrak{p}}(T)$, cf. notations in § 4. Let $T(\mathfrak{p})$ denote the saturated subgroup of $A(\mathfrak{p})^*$ generated by the image $\bar{T}$ of T in $A/\mathfrak{p}$. It is easily seen that $T(\mathfrak{p})$ consists of all fractions $\bar{a}\bar{s}^{-2}$ with a in $Q_{\mathfrak{p}}(T)$ and s in $A \setminus \mathfrak{p}$. In particular $T(\mathfrak{p}) \neq A(\mathfrak{p})^*$.

Theorem 10.10. Let E be a bilinear space over A such that $E \otimes A(\mathfrak{p})$ is isotropic modulo $T(\mathfrak{p})$ for every prime ideal $\mathfrak{p}$ of A. Then E is isotropic modulo T.

Proof. By Proposition 10.9 it suffices to verify that E is indefinite with respect to every T-semisignature of A. Let such a T-semisignature σ be given. Let $\mathfrak{p} = \mathfrak{p}(\sigma)$ be the associated prime ideal and let $\bar{\sigma}$ denote the induced semisignature of $A(\mathfrak{p})$. The set $Q_{\mathfrak{p}}(T)$ is contained in $Q(\sigma)$, hence disjoint from $\mathfrak{p}$. Moreover it is immediately checked that $T(\mathfrak{p}) \cdot P(\bar{\sigma}) = P(\bar{\sigma})$, which means that $\bar{\sigma}$ is compatible with $T(\mathfrak{p})$. Since $E \otimes A(\mathfrak{p})$ is isotropic modulo $T(\mathfrak{p})$, this space is certainly indefinite with respect to $\bar{\sigma}$. Now $\sigma(E) = \bar{\sigma}(E \otimes A(\mathfrak{p}))$. Thus E is indefinite with respect to σ. Our theorem is proved.

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