Journal für die reine und angewandte Mathematik

gegründet 1826 von

August Leopold Crelle

fortgeführt von
C. W. Borchardt, K. Weierstrass, L. Kronecker, L. Fuchs,
K. Hensel, L. Schlesinger, H. Hasse, H. Rohrbach

gegenwärtig herausgegeben von

Otto Forster · Willi Jäger · Martin Kneser
Horst Leptin · Samuel J. Patterson · Peter Roquette

unter Mitwirkung von
M. Deuring, P. R. Halmos, O. Haupt,
F. Hirzebruch, G. Köthe, K. Krickeberg, K. Prachar,
H. Reichardt, L. Schmetterer, B. Volkmann

JRMAA8

Band 335

Sonderdruck

Walter de Gruyter · Berlin · New York 1982

On the homology of algebraic varieties over real closed fields

By *Hans Delfs* and *Manfred Knebusch**) at Regensburg

Introduction

This paper is written in the terminology of semialgebraic spaces and semialgebraic maps over an arbitrary real closed field R, developed in [DK].[1]) We are only able to handle *affine* semialgebraic spaces, but this should suffice for most applications. Roughly an affine semialgebraic space M ([DK], § 7) is a semialgebraic subset of the set $V(R)$ of rational points of some affine algebraic variety V over R, "regarded without reference to the embedding $M \subset V(R)$", and a semialgebraic map $f: M \to N$ between such spaces is a continuous map which has a semialgebraic graph. Semialgebraic maps are our substitute for continuous maps in classical topology over $\mathbb{R}$. Notice that any semialgebraic subset M of the set of rational points of a *quasiprojective* variety over R is an affine semialgebraic space [DK], § 7.

§ 1 contains the basis of all our proofs in this paper. This is a careful examination of the roots of a finite system of polynomials in one variable whose coefficients are polynomials in a system of parameters varying in some semialgebraic subset of R^n. Something like our Lemma 1. 1 seems to be the heart of every semialgebraic method and occurs in the literature — mainly for $R = \mathbb{R}$ — at so many places that it is useless to write down references.

In § 2 we prove that it is possible to "triangulate" any affine semialgebraic space M and at the same time any finite family $M_1, \dots, M_r$ of semialgebraic subspaces of M with finitely many open simplices, cf. [L], [Hi] for $R = \mathbb{R}$. If the spaces $M, M_1, \dots, M_r$ are complete ([DK], § 9) this is also a triangulation in the traditional sense by closed simplices: We obtain a semialgebraic isomorphism $\psi: |K|_R \xrightarrow{\sim} M$ with $|K|_R$ the realization over R of an abstract finite simplicial complex K (§ 2, Def. 4 and 5) such that every $\psi^{-1}(M_j)$ is the realization $|L_j|_R$ of some subcomplex L_j of K.

*) Supported in part by DFG.

[1]) A survey on the main results in [DK] and § 2—§ 5 of the present paper has been given in [DK$_1$].

0075-4102/82/0335-0006$02.00

Let now (M, A) be a pair of complete affine semialgebraic spaces and let

$$\psi : (|K|_R, |L|_R) \xrightarrow{\sim} (M, A)$$

be a triangulation of this pair. Then it is tempting to define for any abelian group G "semialgebraic" homology groups $H_p(M, A; G)$ as the abstract simplicial groups $H_p(K, L; G)$, and to do the same in cohomology. The question whether this definition makes sense, i.e. whether these groups $H_q(M, A; G)$ and $H^q(M, A; G)$ are independent of the chosen triangulation and have similar properties as in the classical singular theory, has been answered affirmatively in the first author's thesis [D]. The classical approach by simplicial and singular chains is blocked up if the base field R is non archimedian, since the standard technique to make such simplices "small" by iterated barycentric subdivision is impossible. They just do not become small. But the — now also classical — approach by Alexander-Spanier cohomology and sheaf cohomology can be modified to work over any real closed field R. In [D] homology and cohomology groups $H_q(M, A; G)$, $H^q(M, A; G)$ with satisfactory properties are more generally defined for arbitrary pairs (M, A) of affine semialgebraic spaces. This does not cause serious additional difficulties, since every affine semialgebraic space M is the union of a filtered system $(M_\alpha | \alpha \in I)$ of complete subspaces M_α which are strong deformation retracts of M.

Some technical points in [D] are rather complicated. In § 3 we try to explain as far as possible why semialgebraic homology and cohomology groups exist and have good properties, without going into technical details. Full proofs should be published, but we feel that it now is more important to explicate how semialgebraic homology works and what can be done with it, and this is the theme of the present paper. Anyway, the thesis [D] is available on request for the interested reader.

If $\tilde{R}$ is a real closed field containing R then as a consequence of Tarski's principle we have an evident natural functor $M \to M(\tilde{R})$, called "base extension", from the category of semialgebraic spaces and maps over R to the corresponding category for $\tilde{R}$. This is explicated for affine semialgebraic spaces in § 4 and in general in [D], § 9. Any triangulation of a pair (M, A) over R yields by base extension a triangulation of the pair $(M(\tilde{R}), A(\tilde{R}))$. Thus we obtain natural isomorphisms

$$H_q(M, A; G) \xrightarrow{\sim} H_q(M(\tilde{R}), A(\tilde{R}); G)$$

for the semialgebraic homology groups and also for the cohomology groups (§ 4).

Starting from this fact we pursue the goal to transfer results from classical homology theory to algebraic varieties and their semialgebraic subsets over an arbitrary real closed field R. The idea is always to obtain a given situation over R by extension from a similar situation over the field R_0 of real algebraic numbers, then to make base extension from R_0 to R and to compare the (co)homology groups in these three cases.

In § 5 we establish in this way Alexander-Poincaré duality for, say, the space $V(R)$ of rational points of a smooth variety over R and its semialgebraic subsets, provided $V(R)$ is complete. Application to spheres yields a semialgebraic version of the generalized Jordan curve theorem and Brower's theorem of invariance of domain in the usual way. Similarly many other classical results can be transferred to arbitrary real closed fields, as for example the Borsuk-Ulam theorem and Lefschetz duality theory.

But all these results could equally well be established without reference to the field $R=\mathbb{R}$ directly by pursuing those classical proofs which work with simplicial methods. Thus our use of Tarski's principle here — concealed in the theory of base extension — is "innocent".

For reference to classical homology theory we need a book which stresses simplicial methods without using the modern approach by semisimplicial complexes. Semisimplicial complexes would cause unnecessary complications, since they automatically have infinitely many simplices, while all our geometric complexes have to be finite by the very nature of semialgebraic methods. We have chosen the excellent book of Maunder [M].

In the last two sections 7 and 8 we generalize some well known theorems on the homology of $V(\mathbb{R})$ for a possibly singular algebraic variety over $\mathbb{R}$ to varieties over arbitrary real closed fields. These are: Milnor's theorem on the sum of the Betti numbers of $V(\mathbb{R})$ ([Mi], cf. also [T]); Sullivan's theorem that all local Euler characteristics of $V(\mathbb{R})$ are odd ([Su], [BV], [H_1]); and the theorem on the existence of a fundamental class of $V(\mathbb{R})$ by Borel and Haefliger [BH], § 3. For the last topic we have to assume that $V(R)$ is complete, since we do not have an analogue of Borel-Moore homology at our disposal.

All known proofs of these results in the case $R=\mathbb{R}$ use transcendental tools in an essential way. Thus our use of Tarski's principle here is more remarkable and more problematic. There remains the task to find semialgebraic proofs which work equally well for all real closed fields.

The basis of our transfer method in § 7 and § 8 is a theorem on the generic local triviality of semialgebraic families (§ 6, Theorem 6. 4), which deserves independent interest. Such a theorem has been established before in the case $R=\mathbb{R}$ by R. Hardt [H]. Our theorem is more general even in the case $R=\mathbb{R}$, and the proof seems to be simpler than Hardt's proof even in this case. The greater generality makes the proof easier.

The results of § 1—§ 5 are already contained in the first author's thesis [D] up to very minor modifications. There is a further topic in [D] which we mention but do not explicate here for lack of space:[2]) There exists a purely algebraic description of the groups $H^q(V(R), G)$ for a quasiprojective variety V over R as "real-etale" cohomology groups, similar to M. Artin's description of the classical groups $H^q(V(C), G)$ for V an algebraic variety over C and G finite as etale cohomology groups [SGA 4, 3]; Exp. XI and XVI, 4. The real case is much easier than the complex case.

The second author gratefully acknowledges support by the Deutsche Forschungsgemeinschaft while working on parts of this paper during a two months stay at the University of California at Berkeley 1980. He also thanks this university and in particular Prof. T. Y. Lam for generous hospitality.

[2]) cf. also [CRC], where it has first been shown that the "real etale topos" of V and the semialgebraic topos of $V(R)$ are equivalent.

§ 1. The roots of a system of polynomials

Our goal in this section is to prove the following lengthy but fundamental lemma.

Lemma 1. 1. *Let $M_1, \ldots, M_r$ be semialgebraic subsets of R^n and let $P_1, \ldots, P_\rho$ be non zero polynomials in $n+1$ variables $X_1, \ldots, X_n, T$ with coefficients in the real closed field R. Assume that the leading coefficient $a_i(X_1, \ldots, X_n)$ of P_i as a polynomial in T does not vanish at any point of $M := M_1 \cup \cdots \cup M_r$. Then there exists a decomposition $M = A_1 \cup \cdots \cup A_d$ of M into disjoint semialgebraic subsets A_k and there exist semialgebraic functions λ_j^k, $1 \leqq j \leqq r(k)$, on every A_k, $1 \leqq k \leqq d$, with the following properties* i)—vii). *We denote by $\pi : R^{n+1} \to R^n$ the projection $(x_1, \ldots, x_n, t) \mapsto (x_1, \ldots, x_n)$ and by Z_{kj} the set $\{(x, \lambda_j^k(x)) \mid x \in A_k\}$.*

i) *For every k and i either $A_k \subset M_i$ or $A_k \cap M_i = \emptyset$ $(1 \leqq k \leqq d,\ 1 \leqq i \leqq r)$.*

ii) *For every $x \in A_k$ $\lambda_1^k(x) < \lambda_2^k(x) < \cdots < \lambda_{r(k)}^k(x)$ $(1 \leqq k \leqq d)$.*

iii) *For every derivative $\partial^l P_i(X, T)/\partial T^l$ of any of the polynomials P_i with respect to T and any k, $1 \leqq k \leqq d$, either $\partial^l P_i(x, T)/\partial T^l \equiv 0$ for every $x \in A_k$, or there exists a subset $J(i, l, k)$ of $\{1, \ldots, r(k)\}$ such that for every $x \in A_k$ the values $\lambda_j^k(x)$, $j \in J(i, l, k)$, are precisely all zeros of $\partial^l P_i(x, T)/\partial T^l$.*

iv) *For every k and l either $A_k \subset \bar{A}_l$ or $A_k \cap \bar{A}_l = \emptyset$ $(1 \leqq k,\ l \leqq d)$.*

v) *If $A_k \subset \bar{A}_l$ then for every set Z_{li}, $1 \leqq i \leqq r(l)$, there exists a unique j, $1 \leqq j \leqq r(k)$, such that*

$$\bar{Z}_{li} \cap \pi^{-1}(A_k) = Z_{kj}.$$

vi) *If $A_k \subset \bar{A}_l$ then for every set Z_{kj}, $1 \leqq j \leqq r(k)$, there exists some i, $1 \leqq i \leqq r(l)$, such that $Z_{kj} \subset \bar{Z}_{li}$ {so, by property* v), *$Z_{kj} = \bar{Z}_{li} \cap \pi^{-1}(A_k)$}.*

vii) *If $A_k \subset \bar{A}_l$ and $Z_{kj} \subset \bar{Z}_{li}$ then the function $f : A_k \cup A_l \to R$ defined by $f \mid A_k = \lambda_j^k$ and $f \mid A_l = \lambda_i^l$ is semialgebraic.*

To prove this we first recall that the real roots of a polynomial in one variable over R can be regarded as functions of the coefficients with semialgebraic graph ([C], § 1), and that the image of a semialgebraic set in R^{n+1} under π is a semialgebraic set in R^n (Tarski). Thus we can find a decomposition $M = A_1 \cup \cdots \cup A_d$ of M into disjoint semialgebraic sets with the following properties:

1) For every k and i either $A_k \subset M_i$ or $A_k \cap M_i = \emptyset$ $(1 \leqq k \leqq d,\ 1 \leqq i \leqq r)$.

2) For every k and i $(1 \leqq k \leqq d,\ 1 \leqq i \leqq \rho)$ and every $l \geqq 0$ the polynomial $\partial^l P_i(x, T)/\partial T^l$ either vanishes identically for every $x \in A_k$ or has $r(k, i, l)$ (independent of x) real roots

$$\mu_{1k}^{il}(x) < \mu_{2k}^{il}(x) < \cdots < \mu_{r(k,i,l),k}^{il}(x),$$

and the functions μ_{jk}^{il} on A_k have semialgebraic graphs.

3) Any two of these root functions μ_{jk}^{il}, μ_{vk}^{uw} on A_k either coincide or they do not meet, i.e. we have either $\mu_{jk}^{il}(x) = \mu_{vk}^{uw}(x)$ for all $x \in A_k$ or $\mu_{jk}^{il}(x) \neq \mu_{vk}^{uw}(x)$ for all $x \in A_k$.

We then find functions $\lambda_1^k, \ldots, \lambda_{r(k)}^k$ on A_k with semialgebraic graphs such that $\lambda_i^k(x) \neq \lambda_j^k(x)$ for all $x \in A_k$ if $i \neq j$ and such that for every root μ_{jk}^{il} there is some m, $1 \leqq m \leqq r(k)$, with $\lambda_m^k(x) = \mu_{jk}^{il}(x)$ for all $x \in A_k$. Since A_k is the disjoint union of the sets

$$\{x \in A_k \mid \lambda_{\sigma(1)}^k(x) < \lambda_{\sigma(2)}^k(x) < \cdots < \lambda_{\sigma(r(k))}^k(x)\}$$

with σ running through the permutations of $\{1, 2, \ldots, r(k)\}$, we may assume after a further subdivision of the sets A_k that for every $x \in A_k$

$$\lambda_1^k(x) < \cdots < \lambda_{r(k)}^k(x).$$

Each function λ_j^k is a simple root of one of the polynomials $\partial^l P_i / \partial T^l$ everywhere on A_k. Thus we derive from the implicit function theorem (cf. [DK], 6. 9), that λ_j^k is continuous in the strong topology, hence a semialgebraic function. The decomposition $(A_k \mid 1 \leqq k \leqq d)$ of M and the functions λ_j^k already have the properties i)—iii) stated in the lemma. These properties do not get lost if we subdivide the sets A_k further, which is necessary to obtain iv)—vii).

Remembering that the interior $\mathring{A}_k$ of any set A_k with respect to M is again semialgebraic (cf. [DK], 7. 7) we can assume after a subdivision of the A_k that every A_k either is open in M or has no interior points with respect to M.

We now define by induction on l semialgebraic sets B_{lj}, $1 \leqq j \leqq s(l)$. Each set B_{lj} will be contained in a (uniquely determined) set $A_{\alpha(l,j)}$. As sets B_{0j} we choose those sets A_k which are open in M. Then

$$B_0 := M \setminus \bigcup_{j=1}^{s(0)} B_{0j}$$

is closed in M and has no interior points with respect to M. Thus $\dim B_0 < \dim M$ (cf. [DK], § 8).

Assume that for some fixed $m \geqq 1$ the sets B_{lj}, $1 \leqq j \leqq s(l)$, are already defined for all $l < m$. We introduce for every $i < m$ the set

$$B_i := M \setminus \left(\bigcup_{l=0}^{i} \bigcup_{j=1}^{s(l)} B_{lj} \right).$$

Suppose B_{m-1} is not empty. We find a decomposition $(C_s \mid s \in I)$ of B_{m-1} into finitely many disjoint semialgebraic subsets C_s with the following properties:

a) Each set C_s is contained in some A_k.

b) If $l < m$ and $C_s \cap \bar{B}_{lj} \neq \emptyset$ then $C_s \subset \bar{B}_{lj}$.

c) If $C_s \subset A_k \cap \bar{B}_{lj}$, $l < m$, and if $Z_{ki} \cap \pi^{-1}(C_s)$ has non empty intersection with the closure of $Z_{\alpha(l,j),t} \cap \pi^{-1}(B_{lj})$ then $Z_{ki} \cap \pi^{-1}(C_s)$ is contained in that closure.

d) Either C_s is open in B_{m-1} or C_s has no interior points with respect to B_{m-1}.

We choose as sets B_{mj}, $1 \leqq j \leqq s(m)$, those sets C_s which are open in B_{m-1}. Then the set B_m, defined by the same formula as the B_i above, is the complement of $\bigcup_{j=1}^{s(m)} B_{mj}$ in B_{m-1} and thus has smaller dimension than B_{m-1}. This process of defining the sets B_{lj} stops if B_m becomes empty.

The family of these sets B_{lj} is a disjoint decomposition of M into disjoint semi-algebraic sets which refines the original decomposition $(A_k | 1 \leqq k \leqq d)$ of M. Changing notations we now call this decomposition $(A_k | 1 \leqq k \leqq d)$.

For this decomposition also the property iv) stated in the lemma holds true as a consequence of property b) above. Moreover as a consequence of c) above the following condition is fulfilled:

(*) If $A_k \subset \bar{A}_l$ and $Z_{kj} \cap \bar{Z}_{li} \neq \emptyset$, then $Z_{kj} \subset \bar{Z}_{li}$.

Consider now two different sets A_k, A_l with $A_k \subset \bar{A}_l$. As mentioned above each function λ_j^k is a simple root of one of the polynomials $\partial^m P_i / \partial T^m$. Hence we conclude from the implicit function theorem (cf. [DK], 6. 9) that for every set Z_{kj}, $1 \leqq j \leqq r(k)$, the intersection of Z_{kj} with the closure $\bar{Z}_{li}$ of at least one set Z_{li}, $1 \leqq i \leqq r(l)$, cannot be empty. Then it follows from the just stated property (*) that $Z_{kj} \subset \bar{Z}_{li}$. Thus our decomposition also fulfills vi).

We now check the validity of v). Let again two different sets A_k and A_l be given with $A_k \subset \bar{A}_l$. From property (*) and the assumption that the leading coefficient $a_m(X_1, \ldots, X_n)$ of P_m vanishes nowhere on M $(1 \leqq m \leqq \varrho)$ we derive that $\bar{Z}_{li} \cap \pi^{-1}(A_k)$ is the union of those Z_{kj} which are contained in $\bar{Z}_{li}$. Thus to prove v) it suffices to show that for a given point x in A_k there is a unique $t \in R$ with $(x, t) \in \bar{Z}_{li}$.

First we shall explain that there is at least one such value t. The leading coefficients of all non zero derivatives $\partial^l P_m / \partial T^l$, considered as polynomials in T, do not vanish at x. Thus it follows from a well known elementary estimate that all roots of these polynomials are bounded in an open bounded neighbourhood U of x in R^n. Therefore the closure $\overline{Z_{li} \cap \pi^{-1}(U)}$ is a closed bounded subset of R^{n+1}. The image of this set under π is closed in R^n (cf. [DK], 9. 4). It contains $A_l \cap U$, and we conclude:

$$x \in \overline{A_l \cap U} \subset \pi(\bar{Z}_{li}).$$

Now we are left to show the following claim:

(**) For every i, $1 \leqq i \leqq r(l)$, and every $x \in A_k$ there exists at most one point $z \in \bar{Z}_{li}$ with $\pi(z) = x$.

Choose a non zero polynomial $Q = \partial^m P_u / \partial T^m$ of minimal degree d in T with $Q(y, \lambda_i^l(y)) = 0$ for all $y \in A_l$. We prove the claim by induction on d. Since the leading coefficient of Q vanishes nowhere on M the degree of $Q(y, T)$ is d for every $y \in M$. If $d = 1$ the polynomial $Q(x, T)$ is linear and the claim is evident. Now assume $d > 1$, and that the claim is true for degree $< d$. Suppose that (x, t_1) and (x, t_2) are different points of $\bar{Z}_{li}$, say $t_1 < t_2$. There exists a root t of $\partial Q(x, T)/\partial T$ with $t_1 < t < t_2$ and $Q(x, t) \neq 0$. This value t is a simple root of some derivative $\partial^\mu Q(x, T)/\partial T^\mu$, $\mu \geqq 1$. The degree of the polynomial $S := \partial^\mu Q / \partial T^\mu$ in T is smaller than d. By the implicit function theorem there exists an open semialgebraic neighbourhood U of x in R^n and some $\varepsilon > 0$ such that the set of zeros of $S(X_1, \ldots, X_n, T)$ in $U \times]t - \varepsilon, t + \varepsilon[$ is the graph $\Gamma(\varphi)$ of a semialgebraic function $\varphi : U \to]t - \varepsilon, t + \varepsilon[$.

We now choose a path $\alpha: [0, 1] \to R^{n+1}$ with $\alpha(0) = (x, t_1)$ and $\alpha(]0, 1]) \subset Z_{li}$. This is possible by the curve selection lemma [DK], 12. 1. There exists a unique connected component D of $A_l \cap U$ which contains $\pi \circ \alpha(]0, a[)$ for some $a \in]0, 1[$. Since D is connected we have $\varphi | D = \lambda_j^l | D$ for some j, $1 \leqq j \leqq s(l)$. Now λ_j^l is a simple root of S everywhere on A_l. We further have

$$\lim_{\substack{y \to x \\ y \in D}} \lambda_j^l(y) = t.$$

Certainly $j \neq i$, since $Q(x, t) \neq 0$. Thus either $\lambda_i^l > \lambda_j^l$ everywhere on A_l or $\lambda_i^l < \lambda_j^l$ everywhere on A_l. But $\pi \circ \alpha(]0, a[)$ is contained in D, and

$$\lim_{\substack{s \to 0 \\ s > 0}} \lambda_i^l(\pi \circ \alpha(s)) = t_1 < t.$$

Thus certainly $\lambda_i^l < \lambda_j^l$ on A_l {i.e. $i < j$}.

We now choose a path $\gamma: [0, 1] \to R^n$ with $\gamma(0) = (x, t_2)$ and $\gamma(]0, 1[) \subset Z_{li}$, again by use of the curve selection lemma. We introduce the semialgebraic set

$$N := \{(\delta(s), \lambda_j^l \circ \delta(s)) | s \in]0, 1]\},$$

where $\delta = \pi \circ \gamma$. This set N is bounded in R^{n+1} because the roots of all non zero derivatives of $P_n(X, T)$ with respect to T are bounded over $\delta([0, 1])$. Thus $\pi(\bar{N})$ is closed and contains $\delta(]0, 1])$. We conclude that there is a point $(x, \tilde{t}) \in \bar{N}$. Since $\lambda_j^l > \lambda_i^l$, moreover

$$\lim_{s \to 0} \lambda_i^l(\delta(s)) = t_2$$

and $(x, \tilde{t}) \in \bar{N}$, we conclude that $\tilde{t} \geqq t_2$. Thus the points (x, t) and $(x, \tilde{t})$ of $\bar{Z}_{lj}$ are certainly different. But λ_j^l is a root of S on A_l, and S has degree $< d$ with respect to T. This contradicts our induction hypothesis, and the claim (**) is proved. Property v) is now verified.

It remains to check property vii). So assume $A_k \subset \bar{A}_l$, $k \neq l$, and $Z_{kj} \subset \bar{Z}_{li}$, and let $f: A_k \cup A_l \to R$ be the function whose restriction to A_k is λ_j^k and to A_l is λ_i^l. Obviously f has a semialgebraic graph, and this graph is closed in $(A_k \cup A_l) \times R$. It now suffices to check that the graph is locally bounded. Then we may conclude from [DK], 9. 10 or [B], Prop. 8. 13. 8 that f is continuous.

It is clear from our construction that A_l is disjoint from $\bar{A}_k$. (This is also a formal consequence of property iv).) Thus A_l is open in $A_l \cup A_k$, and it suffices to study f in a neighbourhood of a given point x of A_k. Now λ_i^l and then also λ_j^k is a root of a non zero derivative $Q := \partial^m P_\mu / \partial T^m$ of some P_μ. Since the leading coefficient of Q does not vanish at x the roots of $Q(y, T)$ are bounded for all y in a neighbourhood of x in R^n.

Thus f is indeed a semialgebraic function on $A_k \cup A_l$, and Lemma 1. 1 is completely proved.

§ 2. Triangulation of affine semialgebraic spaces

We shall prove in this section that any affine semialgebraic space can be triangulated. We first explain what is meant here by a triangulation.

Definition 1. An *open n-simplex S over R* is the interior of the convex closure of $n+1$ affinely independent points $e_0, \ldots, e_n$ in some affine space R^m:

$$S = \left\{ \sum_{i=0}^{n} t_i e_i \,\middle|\, t_i \in R,\ t_i > 0,\ \sum_{i=0}^{n} t_i = 1 \right\}.$$

The points $e_0, \ldots, e_n$, which are uniquely determined by S, are called the *vertices* of S. (In the case $n=0$ the open simplex with vertex e_0 is the one point set $\{e_0\}$.)

A *closed n-simplex* over R is the closure $\bar{S}$ of an open n-simplex S in its embedding space R^m. We also call S the *interior* of $\bar{S}$. The *faces* of $\bar{S}$ are the convex hulls of the non empty subsets of $\{e_0, e_1, \ldots, e_n\}$. They are again closed simplices in R^m. The *open faces* of $\bar{S}$ are the interior of the faces of $\bar{S}$, and these sets are also called the open faces of S.

Definition 2. A *geometric simplicial complex over R* is a pair $(X, (S_i, i \in I))$ consisting of a semialgebraic subset X of an affine space R^m and a finite family of pairwise disjoint open simplices S_i in R^m such that the following hold true.

i) X is the union of all S_i, $i \in I$.

ii) The intersection $\bar{S}_i \cap \bar{S}_j$ of the closure of any two simplices S_i, S_j is either empty or a face of $\bar{S}_i$ as well as of $\bar{S}_j$.

The geometric simplicial complex $(X, (S_i, i \in I))$ is called *complete*, if the semialgebraic space X is complete [DK], § 9. This means:

iii) For every $j \in I$, all open faces of S_j are again members of the family $(S_i, i \in I)$.

For every geometric simplicial complex $(X, (S_i, i \in I))$ in R^n the closure $\bar{X}$ of X in R^n can be made a complete simplicial complex $(\bar{X}, (S_i, i \in \bar{I}))$ in a unique way such that $I \subset \bar{I}$ and the new open simplices S_i, $i \in \bar{I} \setminus I$, are precisely all open faces of the S_i, $i \in I$, which are not contained in X. We call $(\bar{X}, (S_i, i \in \bar{I}))$ the completion of $(X, (S_i, i \in I))$.

Later we shall often briefly write "X" for a geometric simplicial complex $(X, (S_i, i \in I))$.

Definition 3. Let M be a semialgebraic space over R and let $M_1, \ldots, M_r$ be semialgebraic subsets of M. A *triangulation* of M is a triple $(X, (S_i, i \in I), \psi)$ consisting of a geometric simplicial complex $(X, (S_i, i \in I))$ and a semialgebraic isomorphism $\psi: X \xrightarrow{\sim} M$ from the semialgebraic space X onto M. We then call the sets $\psi(S_i)$ the *open simplices of M* with respect to this triangulation. We call the triangulation (X, ψ) a *simultaneous triangulation of M, $M_1, \ldots, M_r$*, if in addition every M_j is a union of simplices $\psi(S_i)$.

Clearly any semialgebraic space M which can be triangulated must be affine. The following strong converse holds true:

Theorem 2.1. *Let M be an affine semialgebraic space and let $M_1, \ldots, M_m$ be finitely many semialgebraic subsets of M. Then there exists a simultaneous triangulation of $M, M_1, \ldots, M_m$.*

M is isomorphic to a semialgebraic subspace of some affine space R^n, hence by stereographic projection also isomorphic to a semialgebraic subspace of the unit sphere S^n in R^{n+1}. Thus Theorem 2.1 is a consequence of the following more precise result.

Theorem 2.2. *Let $M_1, \ldots, M_m$ be bounded semialgebraic subsets of R^n. Then there exists a complete geometric simplicial complex $(X, (S_i, i \in I))$ in R^n with X convex and a semialgebraic automorphism κ of R^n such that:*

i) *$\kappa(x) = x$ for every $x \in R^n \setminus X$.*

ii) *Each set M_j, $1 \leqq j \leqq m$, is the union of some sets $\kappa(S_i)$, $i \in I$.*

This triangulation theorem 2.2 is well known in the case $R = \mathbb{R}$, cf. [Hi], [L]. All proofs in the literature use analytic tools. This destroys the possibility simply to copy one of these proofs in the general case, and leaves us with the need to give a complete proof of Theorem 2.2. The investigation of the roots of a system of polynomials in § 1 will enable us to avoid all "analytic conclusions". But the main idea in our proof will be the same as in Hironaka's proof [Hi].

We prove Theorem 2.2 by induction on n. The case $n=1$ is trivial since each M_j is a finite union of bounded intervals and points.

Assume now that the assertion is true for $n \geqq 1$. We want to show that it is true also for $n+1$. Let $M_1, \ldots, M_m$ be bounded subsets of R^{n+1}. We can make the following two assumptions (cf. [Hi]):

1) *M_j is closed in R^{n+1}*, $1 \leqq j \leqq m$. Indeed, if M_j is not closed, then we replace M_j by the finite family $M_{j0}, M_{j1}, \ldots, M_{j,s(j)}$ defined as follows. $M_{j0} := \bar{M}_j$, $M_{j,k+1} := \overline{M'_{j,k+1}}$, where $M'_{j,0} = M_j$, $M'_{j,k+1} = M_{jk} \setminus M'_{jk}$, $s(j)$ = the first index s such that M_{js} is empty. Since $\dim M_{j,k+1} < \dim M_{j,k}$ ([DK], 8.11) such an index $s(j)$ exists. It is easily checked that every M_j is the disjoint union of the sets $M_{j0} \setminus M_{j1}$, $M_{j2} \setminus M_{j3}$, etc.

2) *M_j has no interior points*, i.e. $\dim M_j \leqq n$ $(1 \leqq j \leqq m)$. Indeed, we may replace M_j by its boundary ∂M_j which is again a semialgebraic set ([DK], 7.7): Suppose we have found a complete simplicial complex $(X, (S_i, i \in I))$, with X convex, and a semialgebraic automorphism κ of R^{n+1} fulfilling the properties of Theorem 2.2 with respect to $\partial M_1, \ldots, \partial M_m$. Then ∂M_j and therefore, since X is convex, M_j is contained in $X = \bigcup S_i = \bigcup \kappa(S_i)$. M_j is the union of ∂M_j and $M_j \setminus \partial M_j$ because M_j is closed in R^{n+1}. For all $i \in I$ $\kappa(S_i)$ is either contained in ∂M_j or $\kappa(S_i) \cap \partial M_j = \emptyset$. Since $\kappa(S_i)$ is connected we conclude that then also either $\kappa(S_i) \subset M_j$ or $\kappa(S_i) \cap M_j = \emptyset$.

Let $P_1,\ldots,P_s \in R[X_1,\ldots,X_n,T]$ be non zero polynomials such that each set M_i is a finite union of sets of the form

$$\{x \in R^{n+1} \mid P_j(x)=0,\ P_{j_k}(x)>0,\ k=1,\ldots,\rho\}.$$

After performing an appropriate coordinate transformation of the form $X_1=X_1'+a_1\cdot T'$, $X_2=X_2'+a_2\cdot T',\ldots,X_n=X_n'+a_n\cdot T'$, $T=T'$ with elements $a_i\in R$ we may assume that the polynomials $P_1,\ldots,P_s$ are normed in the last variable T.

Let $\pi: R^{n+1}\to R^n$ denote the natural projection $(x_1,\ldots,x_n,t)\mapsto(x_1,\ldots,x_n)$. We apply Lemma 1.1 to the semialgebraic sets $\pi(M_1),\ldots,\pi(M_m)$ and the polynomials $P_1,\ldots,P_s$. We choose a decomposition $R^n=\bigcup_{k=1}^{e} A_k$ of R^n into disjoint semialgebraic sets A_k and semialgebraic functions λ_i^k, $1\leqq i\leqq r(k)$, on every A_k with the properties listed in Lemma 1.1. For $r\in R$, $r>0$, we define

$$\bar{B}_r(0):=\left\{(x_1,\ldots,x_n)\in R^n \mid \sum_{i=1}^{n} x_i^2\leqq r^2\right\}.$$

We choose some $r>0$ such that

$$\bigcup_{i=1}^{m} \pi(M_i)\subset \bar{B}_r(0).$$

Subdividing the sets A_k, if necessary, we assume that

$$\bar{B}_r(0)=\bigcup_{k\in J_1} A_k,\quad \bar{B}_{2r}(0)=\bigcup_{k\in J_2} A_k$$

with index sets $J_1\subset J_2\subset\{1,\ldots,e\}$.

We apply the induction hypothesis to the family $(A_k \mid k\in J_2)$. We have a complete geometric simplicial complex $(X,(B_\alpha,\alpha\in I))$ in R^n, with X convex, and a semialgebraic automorphism κ of R^n such that:

i) κ is the identity on $R^n\setminus X$.

ii) Each set A_k, $k\in J_2$, is a union of some simplices $\kappa(B_\alpha)$.

Define $C_\alpha:=\kappa(B_\alpha)$ and the index sets

$$I_1:=\{\alpha\in I \mid C_\alpha\subset \bar{B}_r(0)\},\quad I_2:=\{\alpha\in I \mid C_\alpha\in \bar{B}_{2r}(0)\}.$$

Each simplex C_α, $\alpha\in I_2$, is contained in a set A_k, $k\in J_2$. Let

$$\xi_1^\alpha<\xi_2^\alpha<\cdots<\xi_{s(\alpha)}^\alpha$$

be the restrictions of the semialgebraic functions $\lambda_1^k<\cdots<\lambda_{r(k)}^k$ to C_α, and let

$$W_{\alpha i}:=\{(x,\xi_i^\alpha(x)) \mid x\in C_\alpha\},\quad (\alpha\in I_2,\ 1\leqq i\leqq s(\alpha)).$$

Since all the polynomials $\dfrac{\partial^l P_k}{\partial T^l}$ are normed in T (up to a factor which is a natural number), it follows from a well known estimate of the absolute values of roots of polynomials that there is some $c>0$ in R with $|\xi_i^\alpha(x)|<c$ for all $\alpha\in I_2$, $1\leqq i\leqq s(\alpha)$, and all $x\in C_\alpha$. Every set M_j, $1\leqq j\leqq m$, is the union of some sets $W_{\alpha l}$, $\alpha\in I_1$. Indeed, since M_j has no interior points, at least one polynomial P_k vanishes everywhere on M_j, and all polynomials P_k have constant signs $\{-1,+1 \text{ or } 0\}$ on every set $W_{\alpha l}$, since $W_{\alpha l}$ is connected.

We introduce the automorphism $\tilde{\kappa} := \kappa \times \mathrm{id}_R$ of R^{n+1}. After a barycentric subdivision of the simplices B_α we may assume that for all pairs $(\alpha, i) \neq (\alpha, j)$ either $\pi(\bar{W}_{\alpha i} \cap \bar{W}_{\alpha j})$ is empty or is the image $\bar{C}_\beta = \kappa(\bar{B}_\beta)$ of a proper face $\bar{B}_\beta$ of $\bar{B}_\alpha$. We have

$$\pi(\tilde{\kappa}^{-1}(W_{\alpha i})) = B_\alpha \quad (1 \leqq i \leqq s(\alpha)).$$

According to Lemma 1. 1. v) above every vertex of B_α there lies a uniquely determined point of $\tilde{\kappa}^{-1}(\bar{W}_{\alpha i})$. Let $S_{\alpha i}$ denote the open (straight!) simplex of R^{n+1} spanned by these "vertices" of $\tilde{\kappa}^{-1}(\bar{W}_{\alpha i})$. We claim that for $(\alpha, i) \neq (\beta, j)$ the intersection $S_{\alpha i} \cap S_{\beta j}$ is empty. This is evident if $\alpha \neq \beta$. If $\alpha = \beta$ and, say, $i < j$, then for every vertex p of B_α the vertices of $S_{\alpha i}$ and $S_{\beta j}$ above p have last coordinates $t_{pi} \leqq t_{pj}$. For at least one vertex p of B_α we have strict inequality, since otherwise $\pi(\bar{W}_{\alpha i} \cap \bar{W}_{\alpha j})$ could not be a proper "face" of $\kappa(\bar{B}_\alpha)$. Thus indeed $S_{\alpha i} \cap S_{\alpha j}$ is empty.

Let $T_{\alpha i}$ denote the image of $S_{\alpha i}$ under $\tilde{\kappa}$. Notice that for points $(x, t) \in S_{\alpha i}$ and $(x, t') \in S_{\alpha, i+1}$ resp. for points $(x, t) \in T_{\alpha i}$ and $(x, t') \in T_{\alpha, i+1}$ we have $t < t'$. The sets $T_{\alpha i}$, $S_{\alpha i}$, $W_{\alpha i}$ are all contained in $X \times]-c, c[$.

We now introduce semialgebraic automorphisms g_α of $C_\alpha \times [-c, c]$ for $\alpha \in I_2$. These g_α are "vertical shiftings" (cf. [Hi], 1. 9). We use the following notations:

$$W_{\alpha 0} := T_{\alpha 0} := C_\alpha \times \{-c\},$$

$$W_{\alpha, s(\alpha)+1} := T_{\alpha, s(\alpha)+1} := C_\alpha \times \{c\},$$

$$\xi_0^\alpha(x) := -c, \ \xi_{s(\alpha)+1}^\alpha(x) := c$$

for $x \in C_\alpha$. Let (x, t) be a point in $C_\alpha \times [-c, c]$ and assume that (x, t) lies between $T_{\alpha i}$ and $T_{\alpha, i+1}$ $(0 \leqq i \leqq s(\alpha))$. If $(x, t_i(x))$ is the point of $T_{\alpha i}$ above x and $(x, t_{i+1}(x))$ is the point of $T_{\alpha, i+1}$ above x then there is a unique $u \in [0, 1]$ with

$$t = (1-u)\, t_i(x) + u\, t_{i+1}(x).$$

We define

$$g_\alpha(x, t) := \left(x, (1-u)\, \xi_i^\alpha(x) + u\, \xi_{i+1}^\alpha(x)\right).$$

Obviously g_α is a semialgebraic automorphism of $C_\alpha \times [-c, c]$ which maps $T_{\alpha, i}$ onto $W_{\alpha, i}$. These shiftings g_α, $\alpha \in I_2$, fit together to a bijective map

$$g : \bar{B}_{2r}(0) \times [-c, c] \to \bar{B}_{2r}(0) \times [-c, c].$$

g has a semialgebraic graph. By use of Lemma 1. 1, v)—vii) it is easily checked that g is also continuous. Thus g is semialgebraic. Since $\bar{B}_{2r}(0) \times [-c, c]$ is a complete space, g is in fact a semialgebraic automorphism of $\bar{B}_{2r}(0) \times [-c, c]$ ([DK], 9. 8).

We modify g outside $\bar{B}_r(0) \times [-c, c]$ to an automorphism g' of $\bar{B}_{2r}(0) \times [-c, c]$, which keeps every point of the boundary of $\bar{B}_{2r}(0) \times [-c, c]$ fixed, as follows:

$$g'(x, t) := \begin{cases} g(x, t) & \text{for } x \in \bar{B}_r(0), \\ (1-s)\, g(x, t) + s(x, t) & \text{for } x \in \bar{B}_{2r}(0),\ \|x\| = (1+s)\, r,\ s \in [0, 1]. \end{cases}$$

We extend g' by the identity to a semialgebraic automorphism $\tilde{g}$ of R^{n+1}.

We choose some $d>c$ in R, introduce the points $P_+:=(0,d)$, $P_-:=(0,-d)$ and form the cones

$$Q_+:=\{(1-u)P_+ + u(x,c) \mid x\in X,\ u\in[0,1]\},$$
$$Q_-:=\{(1-u)P_- + u(x,-c) \mid x\in X,\ u\in[0,1]\},$$

over $X\times\{c\}$ resp. $X\times\{-c\}$ with vertices P_+ resp. P_-. The automorphism $\tilde{g}\circ\tilde{\kappa}$ of R^{n+1} maps $X\times[-c,c]$ onto itself and is the identity on (boundary of X) $\times[-c,c]$.

We now obtain the desired automorphism κ' of R^{n+1} as follows. We introduce the convex set

$$L:=(X\times[-c,c])\cup Q_+\cup Q_-$$

and define

$$\kappa'(y):=\begin{cases} y & \text{for } y\notin L,\\ \tilde{g}\circ\tilde{\kappa}(y) & \text{for } y\in X\times[-c,c],\\ (1-u)P_+ + u\tilde{\kappa}(x,c) & \text{for } y=(1-u)P_+ + u(x,c)\in Q_+,\\ (1-u)P_- + u\tilde{\kappa}(x,-c) & \text{for } y=(1-u)P_- + u(x,-c)\in Q_-. \end{cases}$$

κ' is indeed a semialgebraic automorphism of R^{n+1} which outside L is the identity. κ' maps every simplex $S_{\alpha i}$ with $\alpha\in I_1$, $1\leqq i\leqq s(\alpha)$, onto the set $W_{\alpha i}$. Adding to the simplices $S_{\alpha i}$, $\alpha\in I_2$, further simplices in an evident way we obtain a triangulation of L. By construction every set M_j, $1\leqq j\leqq m$, is a union of some sets $\kappa'(S_{\alpha i})$ with $\alpha\in I_1$. Thus κ' does what we want, and Theorem 2. 2 is proved.

In order to work with triangulations efficiently we now recall some classical terminology adapted to an arbitrary real closed field R instead of the field $\mathbb{R}$ of real numbers.

Definition 4. a) An *abstract simplicial complex* K (= simplicial complex in [S], = simplicial scheme in [G]) is a pair $(E(K), S(K))$ consisting of a set $E(K)$, whose elements are called the *vertices* of K, and a set $S(K)$ of finite non empty subsets of $E(K)$, which are called the *simplices* of K, such that:

i) For every $e\in E(K)$ the set $\{e\}$ is an element of $S(K)$.

ii) Every non empty subset τ of some $\sigma\in S(K)$ is again an element of $S(K)$.

If a simplex $\sigma\in S(K)$ consists of $n+1$ elements we say that σ has dimension n or that σ is an n-simplex.

b) A *simplicial map* $\alpha: K\to L$ between abstract simplicial complexes K and L is a map $\alpha: E(K)\to E(L)$ such that the image $\alpha(\sigma)$ of any set $\sigma\in S(K)$ is an element of $S(L)$. If in addition α is bijective and also α^{-1} is simplicial, then α is called an isomorphism between K and L.

In the sequel we assume always tacitly that all occurring abstract simplicial complexes have only *finitely many vertices*. Other abstract simplicial complexes will not play any role in this paper. The *dimension* $\dim K$ of an abstract simplicial complex K is defined as the maximum of the dimensions of the simplices of K.

With any geometric simplicial complex $(X, (S_i, i \in I))$ in R^n we associate an abstract simplicial complex $K(X) = (E(X), S(X))$ in the following way: $E(X)$ is the union of the sets $E(S_i)$ of vertices of all open simplices S_i. A non empty subset σ of $E(X)$ is an element of $S(X)$ if and only if σ is a subset of $E(S_i)$ for some $i \in I$.

Definition 5. A realization of an abstract simplicial complex over R is a *complete* geometric complex $(X, (S_i, i \in I)) = X$ over R together with an isomorphism $\alpha : K \xrightarrow{\sim} K(X)$.

As in the case $R = \mathbb{R}$ it can be shown, cf. e.g. [M], Th. 2.3.16:

Proposition 2.3. *Every n-dimensional abstract simplicial complex has a realization in* R^{2n+1}.

If $(X, \alpha : K \xrightarrow{\sim} K(X))$ is a realization of an abstract simplicial complex K over R which is kept fixed during the considerations then we usually identify the vertices of K with the vertices of X via α, i.e. we regard α as the identity. We then use the notations $|K|_R$ or $|K|$ for the geometric complex X and talk of "the" realization of K over R. For any geometric complex $(Y, (S_i, i \in I))$ in R^n the realization $|K(Y)|$ of $K(Y)$ can and will be chosen as the completion $(\bar{Y}, (S_i, i \in \bar{I}))$ of Y, cf. Definition 2. In particular, if Y is complete, $|K(Y)| = Y$.

Definition 6. a) The *realization* $|\alpha| = |\alpha|_R$ *over* R of a *simplicial map* $\alpha : K \to L$ between abstract simplicial complexes K and L is the unique map $\varphi : |K|_R \to |L|_R$ which coincides with α on the vertices of $|K|_R$ and whose restriction to every closed simplex of $|K|_R$ is affine. This map $|\alpha| = |\alpha|_R$ is clearly semialgebraic.

b) Let X and Y be geometric simplicial complexes. A map $\varphi : X \to Y$ is called simplicial, if φ extends (uniquely) to a continuous map $\bar{\varphi} : \bar{X} \to \bar{Y}$ between the completions $\bar{X}$, $\bar{Y}$ of X, Y and if $\bar{\varphi}$ is the realization $|\alpha|$ of a simplicial map α between the abstract simplicial complexes $K(\bar{X}) = K(X)$ and $K(\bar{Y}) = K(Y)$.

Definition 7. a) Let $K = (E(K), S(K))$ be an abstract simplicial complex. For any two simplices σ, τ of K we write $\sigma < \tau$ if σ is a proper face of τ, i.e. $\sigma \subset \bar{\tau}$ and $\sigma \neq \tau$. The *barycentric subdivision* K' of K is the following abstract simplicial complex: $E(K')$ is the set $S(K)$. {Think of any simplex $|\sigma| \subset |K|$ as replaced by its barycenter.} $S(K')$ is the set of (finite) subsets of $E(K')$ which are totally ordered by inclusion. We write any r-simplex of K' as a chain $\sigma_0 < \sigma_1 < \cdots < \sigma_r$ $(\sigma_i \in S(K))$. A realization $|K|_R$ of K over R immediately yields a realization $|K'|_R$ of K' over R by barycentric subdivision of all simplices of $|K|_R$ in the evident geometric way. A vertex $\sigma \in E(K') = S(K)$ is then realized as the barycenter of the simplex $|\sigma|$.

b) For every complete geometric simplicial complex X over R we denote by X' the barycentric subdivision in the geometric sense. Thus $X' = |K(X)'|_R$. For a non complete geometric complex X we denote by X' the set X equipped with the partition by all open simplices of $(\bar{X})'$ which are contained in X. Then X' is again a geometric simplicial complex, called the barycentric subdivision of X.

c) The barycentric subdivision of a triangulation $(X, (S_i, i \in I), \psi)$ of a semialgebraic space M is the map $\psi : X \xrightarrow{\sim} M$ together with the barycentric subdivision $X' = (X, (S_i', i \in I'))$ of the geometric complex $(X, (S_i, i \in I))$.

To investigate the homology of a semialgebraic subset A of an affine semialgebraic space M we shall later use a simultaneous triangulation

$$\psi : X = \bigcup (S_i \mid i \in I) \xrightarrow{\sim} M$$

of M and A. Difficulties may arise from the fact that the complete subcomplex of $\bar{X}$ generated by the vertices with image in A is in bad relation to the family of all open simplices S_i with $\psi(S_i) \subset A$. Thus we shall usually need a triangulation of M which is "good" on A. We now explain the precise meaning of the word "good" here.

Definition 8. A simultaneous triangulation

$$\psi : X = \bigcup (S_i \mid i \in I) \xrightarrow{\sim} M$$

of M and A is called *good on A*, if the following properties are fulfilled:

a) For every $i \in I$ with $\psi(S_i) \subset A$ at least one vertex of S_i is contained in X and is mapped by ψ into A.

b) If all vertices of an open simplex S of $\bar{X}$ are contained in X and are mapped by ψ into A then $S \subset X$ and $\psi(S) \subset A$.

The following fact is easily shown.

Proposition 2.4. *Let $\psi : X \xrightarrow{\sim} M$ be a simultaneous triangulation of the semialgebraic space M and some semialgebraic subsets $M_1, \dots, M_r$ of M. Then the first barycentric subdivision $\psi : X' \xrightarrow{\sim} M$ of this triangulation is good on M and good on every M_i, $1 \leqq i \leqq r$.*

An indication that good triangulations are useful is given by Proposition 2.5 below.

Definition 9. a) Let M, N be semialgebraic spaces over R and $f, g : M \rightrightarrows N$ be semialgebraic maps from M to N. A *semialgebraic homotopy* from f to g is a semialgebraic map $F : M \times [0, 1] \to N$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for every $x \in M$. Here of course $[0, 1]$ means the unit interval in R, as before.

b) A semialgebraic subset A of a semialgebraic space M is called a (semialgebraic) *strong deformation retract* of M if there exists a semialgebraic homotopy $F : M \times [0, 1] \to M$ with $F(x, 0) = x$, $F(x, 1) \in A$ for every $x \in M$ and $F(a, t) = a$ for every $a \in A$, $t \in [0, 1]$.

In the same way a lot of elementary terminology and elementary observations in classical homotopy theory takes over to our semialgebraic setting.

Proposition 2.5. *Let $\psi : X \xrightarrow{\sim} M$ be a simultaneous triangulation of a semialgebraic space M and a semialgebraic subspace A of M, which is good on A. Let L denote the simplicial subcomplex of $K(X)$ generated by the set of vertices*

$$F(A) := \{p \in E(X) \mid p \in X, \psi(p) \in A\},$$

i.e.

$$E(L) := F(A),\ S(L) := \{\sigma \in S(X) \mid \sigma \subset F(A)\}.$$

Then $\psi(|L|)$ is a strong deformation retract of A.

Proof. We denote our geometric simplicial complex X more precisely by $(X, (S_i, i \in I))$. Replacing M by the set X we assume that ψ is the identity. We introduce the index set J' consisting of all $i \in I$ with $S_i \subset A$ and the subset J consisting of all $i \in I$ with $\bar{S}_i \subset A$. Since the triangulation is good on A the space $|L|$ is the union of all $\bar{S}_i$ with $i \in J$. Also A is the union of all S_i with $i \in J'$.

Let some $i \in J'$ be fixed. Since the triangulation is good on A the intersection $\bar{S}_i \cap |L|$ is not empty and $\bar{S}_i \cap |L| = \bar{S}_j$ with some $j = j(i) \in J$. Let $e_0, \ldots, e_r$ denote the vertices of S_j and $e_0, \ldots, e_r, \ldots, e_s$ denote the vertices of S_i. {N.B.: $r = s$ iff $i \in J$ iff $i = j$.} We introduce the set

$$\tilde{S}_i := \left\{ \sum_{k=0}^{s} t_k e_k \in \bar{S}_i \;\middle|\; \sum_{k=0}^{r} t_k > 0 \right\}$$

and the map $\chi_i : \tilde{S}_i \to \bar{S}_j$,

$$\chi_i\left(\sum_{k=0}^{s} t_k e_k\right) := \left(\sum_{k=0}^{r} t_k\right)^{-1} \left(\sum_{k=0}^{r} t_k e_k\right).$$

Clearly χ_i is semialgebraic and $\chi_i | \bar{S}_j$ is the identity of $\bar{S}_j$. In particular if $i \in J$ then $\tilde{S}_i = \bar{S}_i$ and χ_i is the identity on $\bar{S}_i$.

Let A' denote the union of all sets $\tilde{S}_i$, $i \in J'$. This is a semialgebraic subset of $\bar{X}$ which contains A. All the maps χ_i, $i \in J'$, fit together and yield a semialgebraic map $\chi : A' \to |L|$ which is the identity on $|L|$. We then have a homotopy $F : A' \times [0, 1] \to A'$ from $\mathrm{id}_{A'}$ to χ defined by

$$F(x, t) := (1 - t)\, x + t \chi(x).$$

F maps $A \times I$ into A and the restriction $F | A \times I$ is a homotopy from id_A to $\chi | A$. Moreover $F(x, t) = x$ for every $x \in |L|$ and $t \in [0, 1]$, as desired. Q.E.D.

Definition 10. We call L the *abstract core* and $\psi(|L|)$ the *geometric core* of A with respect to the good triangulation ψ.

§ 3. Semialgebraic homology and cohomology

In this section we want to explain — omitting most technical details — that a reasonable homology and cohomology theory exists for affine semialgebraic spaces over any real closed field R, thus in particular for the space $X(R)$ of real points of a quasiprojective variety X over R.

It seems to be easier to start with cohomology instead of homology (cf. Introduction).

Let (M, A) be a pair consisting of an affine semialgebraic space M over R and a semialgebraic subset A of M. We choose a *good triangulation of the pair* (M, A). By this we mean a simultaneous triangulation $(X, (S_i, i \in I), \psi)$ of M and A which is good on M and good on A, cf. § 2. Let K and L denote the abstract cores of M and A with respect to ψ (Def. 10 in § 2). In the case $R = \mathbb{R}$ the classical singular cohomology groups $H^p(M; G)$ and $H^p(A; G)$ with coefficients in some abelian group G coincide with the cohomology groups $H^p(K; G)$ and $H^p(L; G)$ of the abstract simplicial complexes K and L, since $\psi(|K|)$ and $\psi(|L|)$ are strong deformation retracts of M and A respectively (Prop. 2. 5). Comparing the long cohomology sequences of the pairs (M, A) and $(|K|, |L|)$ we see that also $H^p(M, A; G)$ coincides with the simplicial cohomology group $H^p(K, L; G)$.

If R is an arbitrary real closed base field then the cohomology groups $H^p(M, A; G)$ which we want to define should again coincide with the simplicial groups $H^p(K, L; G)$. Just defining $H^p(M, A; G) := H^p(K, L; G)$ would leave us with the difficult task to prove that these groups do not depend on the choice of the good triangulation of (M, A). Thus it seems better to define the groups $H^p(M, A; G)$ in a more theoretical way, which leaves no doubts that these groups are true invariants of the pair (M, A), and to verify a posteriori that the groups $H^p(M, A; G)$ are canonically isomorphic to the simplicial groups $H^p(K, L; G)$. This has been done in the paper [D], which contains complete proofs of all results we shall state in this section and further details.

We work on the affine semialgebraic space M with the "semialgebraic site" M_{sa}. This is a site in the sense of Grothendieck defined as follows (cf. [DK], § 7 for motivation). The objects of the category of M_{sa} are the open semialgebraic subsets of M. The morphisms are the inclusion maps between these sets. The coverings of an open semialgebraic set $U \subset M$ are the *finite* families $(U_i, i \in I)$ of open semialgebraic subsets of U with $U = \bigcup (U_i, i \in I)$[3]). Thus an (abelian) sheaf on M_{sa} is an assignment $U \rightarrow F(U)$ of an abelian group $F(U)$ to every open semialgebraic $U \subset M$ fulfilling the usual sheaf conditions with respect to finite coverings in the usual sense. We call these sheaves the *semialgebraic* (abelian) *sheaves* on M. For any such sheaf F we denote by $H^q(M, F)$ the q-th cohomology group $H^q(M_{sa}, F)$ in the sense of Grothendieck, defined by use of an injective resolution of F. It can be shown using standard arguments that the groups $H^p(M, F)$ can be computed using the more general flabby resolutions instead of injective resolutions [D], 6. 8.

Let now A be a *closed* semialgebraic subset of M and G be some abelian group. We define a semialgebraic sheaf $G_{M,A}$ on M as follows:

$$G_{M,A}(U) := G^{\pi_0(U, U \cap A)}.$$

Here $\pi_0(U, U \cap A)$ denotes the finite (!) set of all connected components of U which do not meet the set A, and $G^{\pi_0(U, U \cap A)}$ denotes as usual the abelian group of all maps from $\pi_0(U, U \cap A)$ to G. For any open semialgebraic set $V \subset U$ we define the restriction

[3]) The use of this site had already been suggested by Brumfiel [B, p. 248].

map $G_{M,A}(U) \to G_{M,A}(V)$ as the map induced by the natural map from $\pi_0(V)$ to $\pi_0(U)$. It is evident that $G_{M,A}$ is indeed a semialgebraic sheaf. If A is empty we simply write G_M instead of $G_{M,\emptyset}$. We define cohomology groups ($p \geqq 0$)

$$H^p(M, G) := H^p(M, G_M),$$

$$H^p(M, A; G) := H^p(M, G_{M,A}).$$

For any semialgebraic map $f: (M, A) \to (N, B)$ we obtain functorial group homomorphisms $f^*: H^p(N, B, G) \to H^p(M, A, G)$ in the following way. We have an evident morphism $\alpha: f^* G_{N,B} \to G_{M,A}$ of semialgebraic sheaves on M, which induces a homomorphism

$$\alpha_*: H^p(M, f^* G_{N,B}) \to H^p(M, G_{M,A}).$$

The adjunction homomorphism $\beta: G_{N,B} \to f_* f^* G_{N,B}$ induces a homomorphism

$$\beta_*: H^p(N, G_{N,B}) \to H^p(N, f_* f^* G_{N,B}).$$

We further have the edge homomorphism

$$\gamma: H^p(N, f_* f^* G_{N,B}) \to H^p(M, f^* G_{N,B})$$

from the Leray spectral sequence

$$H^p(N, R^q f_*(f^* G_{N,B})) \Rightarrow H^{p+q}(M, f^* G_{N,B}).$$

We then define

$$f^* = \alpha_* \circ \gamma \circ \beta_*: H^p(N, G_{N,B}) \to H^p(M, G_{M,A}).$$

If A is a closed semialgebraic subset of M and B is a closed semialgebraic subset of A we have an evident exact sequence of semialgebraic sheaves (i denotes the inclusion $A \to M$)

$$0 \to G_{M,A} \to G_{M,B} \to i_* G_{A,B} \to 0,$$

which induces a long exact sequence

$$\cdots \to H^p(M, A; G) \to H^p(M, B; G) \to H^p(A, B; G) \xrightarrow{\delta} H^{p+1}(M, A; G) \to \cdots. \tag{3. 1}$$

Choosing a suitable flabby resolution of $G_{M,A}$ it can be shown — as in the classical case $R = \mathbb{R}$ for the topological site — that the groups $H^p(M, G_{M,A}) = H^p(M, A; G)$ coincide with the homology groups of an "Alexander-Spanier cochain complex". The Alexander-Spanier cochains are defined as in the classical case ([S], Chap. 6, § 4) except that now only finite coverings by open semialgebraic subsets of M are admitted. Using this description the following theorem can be proved [D], 7. 1.

Theorem 3. 2. *Any two homotopic semialgebraic maps $f_0, f_1: (M, A) \rightrightarrows (N, B)$ induce the same homomorphisms*

$$f_0^* = f_1^*: H^p(N, B; G) \to H^p(M, A; G).$$

The proof of this theorem seems to be the hard part in building up semialgebraic cohomology. The now existing proof is in principle similar to the classical proof ([S], Chap. 6, § 5) but considerably more complicated. In the whole theory this proof seems to be the place where the nonarchimedian pathologies of the field R are most apparent.

An affine semialgebraic space M has good separation properties similar to a paracompact topological space. They imply that for any semialgebraic sheaf F over M the Grothendieck cohomology groups $H^p(M, F)$ coincide with the Čech cohomology groups $\check{H}^p(M, F)$ ([D], 5. 2, cf. [G], II, 5. 12). Using this fact and Theorem 3. 2 finally the following desired theorem comes out [D], 8. 4.

Theorem 3. 3. *Let A be a closed semialgebraic subset of M. Let K be the core of M and L the core of A with respect to some good triangulation of (M, A). Then we have natural isomorphisms*

$$H^p(K, L; G) \xrightarrow{\sim} H^p(|K|, |L|; G) \xrightarrow{\sim} H^p(M, A; G).$$

Here the second isomorphism is clear from (3. 1) and Theorem 3. 2 (cf. the beginning of this section), and the first isomorphism drops out from a computation of $\check{H}^p(|K|, G_{|K|, |L|})$ using the covering of $|K|$ by the open stars of the vertices of K. These stars are — again by Theorem 3. 2 — acyclic semialgebraic sets.

Up to now we have admitted only pairs (M, A) with A closed in M. It is now easy to define cohomology groups $H^p(M, A; G)$ for arbitrary semialgebraic subsets A of affine semialgebraic spaces M and to restate the essential properties for these groups. We denote by $I(A)$ the set of all complete semialgebraic subspaces C of A such that C is a strong deformation retract of A.

Lemma 3. 4. *Given any complete semialgebraic subset E of A there exists some $C \in I(A)$ with $C \supset E$.*

Proof. Choose a good triangulation of the pair (A, E). Then the geometric core of A contains E and is an element of $I(A)$ by Proposition 2. 5.

In particular $I(A)$ is not empty, and for any two sets $C_1 \in I(A)$, $C_2 \in I(A)$ there exists a third set $C_3 \in I(A)$ containing $C_1 \cup C_2$. We regard $I(A)$ as an inductively ordered set by the inclusion relation. We define

$$H^p(M, A; G) := \varprojlim_{C \in I(A)} H^p(M, C; G).$$

If $C_1 \in I(A)$, $C_2 \in I(A)$ and $C_1 \subset C_2$, then by Theorem 3. 2 and the long exact cohomology sequences of the triples $(A, C_1, \emptyset)$, $(A, C_2, \emptyset)$ we have $H^p(A, C_1; G) = 0$, $H^p(A, C_2; G) = 0$ for all $p \geqq 0$. Thus by the long cohomology sequence of the triple (A, C_2, C_1) we have $H^p(C_2, C_1; G) = 0$ for all $p \geqq 0$. Then the long sequence of the triple (M, C_2, C_1) shows that all natural homomorphisms $H^p(M, C_2; G) \to H^p(M, C_1, G)$ are isomorphisms. Thus we obtain

Proposition 3. 5. *For any $C \in I(A)$ the natural map $H^p(M, A; G) \to H^p(M, C; G)$ is an isomorphism for every $p \geqq 0$ and abelian group G.*

Given a semialgebraic map $f: (M, A) \to (N, B)$ between pairs of affine semialgebraic spaces it is now easy to define the homomorphisms

$$f^*: H^p(N, B; G) \to H^p(M, A; G).$$

Choose some $C \in I(A)$. Then $f(C)$ is a complete semialgebraic subset of B [DK], § 9. Choose some $D \in I(B)$ which contains $f(C)$, which is possible by Lemma 3.4. We have a natural homomorphism

$$f^*: H^p(N, D; G) \to H^p(M, C; G)$$

and we define the map f^* from $H^p(N, B; G)$ to $H^p(M, A; G)$ such that the diagram

$$\begin{array}{ccc} H^p(N, D; G) & \xrightarrow{f^*} & H^p(M, C; G) \\ \cong \uparrow & & \cong \uparrow \\ H^p(N, B; G) & \xrightarrow{f^*} & H^p(M, A; G) \end{array}$$

commutes.

It is now trivial to establish the long exact cohomology sequence (3.1) for three affine spaces $M \supset A \supset B$ and to extend the theorems 3.2 and 3.3 to any pair of affine spaces.

Remark. The procedure in [D] is slightly different. There the cohomology groups $H^p(M, A; G)$ are defined from the beginning by use of Alexander-Spanier cochains for *all* pairs (M, A) of affine semialgebraic spaces. The long sequence (3.1) and the homotopy axiom Theorem 3.2 are directly verified for all these pairs. But only for A closed in M the group $H^p(M, A; G)$ is identified with the p-th cohomology group of a sheaf, namely $G_{M,A}$. We do not know whether a useful interpretation of $H^p(M, A; G)$ as sheaf cohomology exists in general.

We now discuss homology. Let (M, A) be a pair of affine semialgebraic spaces. We call a good triangulation $(X', (S_i', i \in I'), \psi')$ of (M, A) a *refinement* of another good triangulation $(X, (S_i, i \in I), \psi)$ of (M, A), and write $\psi < \psi'$, if every simplex $\psi'(S_i')$ is contained in some simplex $\psi(S_j)$. It is clear from § 2 that any two good triangulations of (M, A) have a common refinement. Thus the set of all good triangulations of (M, A) carries an inductive ordering.

Let $\psi < \psi'$ be two good triangulations of (M, A). Let K, L be the abstract cores of M and A with respect to ψ and K', L' the abstract cores with respect to ψ'. Then $\psi(|K|) \subset \psi'(|K'|)$ and $\psi(|L|) \subset \psi'(|L'|)$. Let $\lambda: (|K|, |L|) \hookrightarrow (|K'|, |L'|)$ be the semialgebraic map corresponding to the inclusion map from $(\psi(|K|), \psi(|L|))$ to $(\psi'(|K'|), \psi'(|L'|))$ via ψ and ψ'. For every vertex $e' \in E(K')$ we choose a vertex $\mu(e') \in E(K)$ of the open simplex of X whose image under ψ contains $\psi'(e')$. If $e' \in E(L')$ we choose $\mu(e')$ in $E(L)$. All this is possible since the triangulation ψ is good on M and A. Now μ is a simplicial map from (K', L') to (K, L), cf. [D], p. 116. It is easily seen that $|\mu|: |K'| \to |K|$ is a homotopy inverse of $\lambda: |K| \to |K'|$, and that $|\mu|: |L'| \to |L|$ is a homotopy inverse of $\lambda: |L| \to |L'|$. Comparing the long exact cohomology sequences of the pairs (K, L) and (K', L') by the maps μ^* coming from μ we see that

$$\mu^*: H^p(K, L; G) \to H^p(K', L'; G)$$

is an isomorphism for every $p \geqq 0$ and every abelian group G. One now deduces in a rather formal way that also the induced maps in simplicial homology

$$\mu_*: H_p(K', L'; G) \to H_p(K, L; G)$$

are isomorphisms, cf. [D], § 8. Of course, both the homomorphisms μ^* and μ_* do not depend on the choice of μ, but only on ψ and ψ'.

We define the *semialgebraic homology* groups of (M, A) as follows.

$$(3.6) \qquad H_p(M, A; G) := \varprojlim_{\psi} H_p(K, L; G).$$

Here ψ runs through the inductive system of all good triangulations of (M, A), and K, L are the abstract cores of M and A with respect to ψ. The transition maps $H_p(K', L'; G) \to H_p(K, L; G)$ are the isomorphisms μ_* described above.

For these groups $H_p(M, A; G)$ functoriality is easily established. Let

$$f: (M, A) \to (N, B)$$

be a semialgebraic map. Let $(Y, (T_j, j \in J), \psi)$ be a good triangulation of (N, B) with abstract cores K_1, L_1 of N and B. According to § 2 we find a good triangulation $(X, (S_i, i \in I), \varphi)$ of (M, A) such that every set $f^{-1}(\psi(T_j))$ is a union of sets $\varphi(S_i)$.

Let K and L be the abstract cores of M and A. For every vertex $e \in E(K)$ we choose a vertex $g(e) \in E(K_1)$ of the open simplex T_j whose image under ψ contains $f \circ \varphi(e)$. If $e \in E(L)$ then we choose $g(e)$ in $E(L_1)$. It turns out that $g: (K, L) \to (K_1, L_1)$ is a simplicial map, and that the induced maps

$$g_*: H_p(K, L; G) \to H_p(K_1, L_1; G)$$

do not depend on the choice of g [D], p. 122f. We define

$$f_*: H_p(M, A; G) \to H_p(N, B; G)$$

such that the diagram

$$\begin{array}{ccc} H_p(K, L; G) & \xrightarrow{g_*} & H_p(K_1, L_1; G) \\ \cong \uparrow & & \cong \uparrow \\ H_p(M, A; G) & \xrightarrow{f_*} & H_p(N, B; G) \end{array}$$

commutes. This map f_* does not depend on the choice of the triangulations ψ and φ [D], loc. cit.

The analogues of the long exact sequence (3. 1) and of Theorem (3. 2) for the semialgebraic homology groups are easily verified. The analogue of Theorem 3. 3 is true ex definitiones:

$$H_p(K, L; G) \xrightarrow{\sim} H_p(|K|, |L|; G) \xrightarrow{\sim} H_p(M, A; G)$$

for any good triangulation of (M, A) with cores K and L.

It is clear from our simplicial description of semialgebraic cohomology and homology that

$$H^0(M, A; G) \cong H_0(M, A; G) \cong G^{\pi_0(M, A)}$$

and that

$$H^p(M, A; G) = H_p(M, A; G) = 0$$

for $p > \dim(M \setminus A)$.

We finally emphasize a fact which is evident from our simplicial description of semialgebraic homology and cohomology and Proposition 2. 5 (cf. beginning of this section).

Theorem 3. 7. *For every pair (M, A) of affine semialgebraic spaces over the field $\mathbb{R}$ of real numbers the semialgebraic homology groups $H_p(M, A; G)$ coincide with the classical singular homology groups $H_{p,cl}(M, A; G)$. The same holds true in cohomology.*

§ 4. Extension of the base field

Let $\tilde{R}$ be another real closed field which contains the real closed field R. There exists a natural functor "base extension" from the category of affine semialgebraic spaces over R to the category of affine semialgebraic spaces over $\tilde{R}$. This is a consequence of Tarski's principle to transfer elementary statements from R to $\tilde{R}$, as will now be explicated.

Let M be a semialgebraic subset of R^n. We describe M by finitely many equalities and strict inequalities of polynomials:

$$M = \bigcup_{i=1}^{r} \{x \in R^n \mid g_i(x) = 0, f_{ij}(x) > 0, j = 1, \ldots, s_i\}.$$

We define a semialgebraic subset $\tilde{M}$ of $\tilde{R}^n$ using the same equalities and inequalities:

$$\tilde{M} := \bigcup_{i=1}^{r} \{x \in \tilde{R}^n \mid g_i(x) = 0, f_{ij}(x) > 0, j = 1, \ldots, s_i\}.$$

According to Tarski's principle this set $\tilde{M}$ depends only on M but not on the chosen description of M. Also $\tilde{M} \cap R^n = M$, thus M is a subset of $\tilde{M}$ uniquely determined by $\tilde{M}$.

Permanently using Tarski's principle we obtain

Lemma 4. 1. *Let $M \subset R^n$ and $N \subset R^m$ be semialgebraic subsets of R^n and R^m, let $A \subset M$, $B \subset N$ be semialgebraic subsets of M and N respectively, and let $f: M \to N$ be a semialgebraic map with graph $\Gamma(f) \subset M \times N \subset R^{n+m}$. Let $g: N \to L$ be a semialgebraic map from N to a semialgebraic subset L of R^r. Let finally $i: A \to M$ denote the inclusion map from A to M.*

i) $(M \times N)^{\sim} = \tilde{M} \times \tilde{N}$.

ii) *$\Gamma(f)^{\sim}$ is the graph of a semialgebraic map $\tilde{f}: \tilde{M} \to \tilde{N}$.*

iii) *f is injective (surjective, resp. open, resp. closed) if and only if $\tilde{f}$ is injective (surjective, resp. open, resp. closed).*

iv) $\tilde{f}(\tilde{A}) = f(A)^{\sim}$, $\tilde{f}^{-1}(\tilde{B}) = f^{-1}(B)^{\sim}$, $(f|A)^{\sim} = \tilde{f}|\tilde{A}$.

v) $(g \circ f)^{\sim} = \tilde{g} \circ \tilde{f}$.

vi) *$\tilde{A}$ is contained in $\tilde{M}$ and $\tilde{i}: \tilde{A} \to \tilde{M}$ is the inclusion map.*

We now establish a functor E from the category $\mathscr{C}(R)$ of affine semialgebraic spaces over R to $\mathscr{C}(\tilde{R})$. We choose for every affine semialgebraic space M a fixed embedding $\chi_M : M \to R^n$ into some R^n. We then define

$$E(M) := \chi_M(M)^\sim .$$

For any semialgebraic map $f: M \to N$ we have a unique map $\chi(f): \chi_M(M) \to \chi_N(N)$ which corresponds to f via χ_M and χ_N. We define

$$E(f) := \widetilde{\chi(f)} .$$

Clearly E is a functor from $\mathscr{C}(R)$ to $\mathscr{C}(\tilde{R})$. It is easily checked that E preserves finite fibre products.

For every affine semialgebraic space M over R we have a unique injective map α_M from the set M to the set $E(M)$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\alpha_M} & E(M) \\ {\scriptstyle \chi_M}\big\downarrow & & \big\downarrow \\ R^n & \longrightarrow & \tilde{R}^n \end{array} ,$$

where the unnotated arrows are inclusion maps, is commutative. Denoting by v and $\tilde{v}$ the forget functors from the categories $\mathscr{C}(R)$, $\mathscr{C}(\tilde{R})$ to the category of sets it is easily checked that $M \mapsto \alpha_M$ is a natural transformation $\alpha: v \to \tilde{v} \circ E$.

We usually write $M(\tilde{R})$ instead of $E(M)$ and $f_{\tilde{R}}$ instead of $E(f)$, and we call $M(\tilde{R})$ and $f_{\tilde{R}}$ the semialgebraic space resp. the semialgebraic map over $\tilde{R}$ obtained from M resp. f by *base extension from R to $\tilde{R}$*. We usually regard M as a subset of $M(\tilde{R})$ via the injection α_M. Also, if A is a semialgebraic subspace of M, we regard $A(\tilde{R})$ as a subspace of $M(\tilde{R})$ via the injective map $i_{\tilde{R}}$ obtained by base extension from the inclusion map $i: A \hookrightarrow M$.

Notice that if V is a quasiprojective variety over R, then the space obtained from $V(R)$ by base extension "is" the space $V(\tilde{R})$ of rational points of V over $\tilde{R}$.

Remark. It is fairly obvious that base extension can more generally be established for arbitrary semialgebraic spaces over R and the semialgebraic maps between them, cf. [D], § 9.

If K is an abstract simplicial complex, then clearly

$$|K|_{\tilde{R}} = |K|_R(\tilde{R}) .$$

Also, if $\alpha: K \to L$ is an abstract simplicial map, the realization $|\alpha|_{\tilde{R}}$ is the base extension of the realization $|\alpha|_R$ to $\tilde{R}$. If $(X, (S_i, i \in I))$ is a geometric simplicial complex over R, then $(X(\tilde{R}), S_i(\tilde{R}), i \in I))$ is a geometric simplicial complex over $\tilde{R}$. Finally, if $(X, (S_i, i \in I), \psi)$ is a good triangulation of a pair (M, A) of affine semialgebraic spaces, then by Tarski's principle $(X(\tilde{R}), (S_i(\tilde{R}), i \in I), \psi_{\tilde{R}})$ is a good triangulation of $(M(\tilde{R}), A(\tilde{R}))$.

M and $M(\tilde{R})$ clearly have the same abstract core K with respect to these triangulations. Also A and $A(\tilde{R})$ have the same abstract core L. According to § 3 we have for every $q \geqq 0$ and every abelian group G natural isomorphisms

$$\gamma : H_q(K, L; G) \xrightarrow{\sim} H_q(M, A; G),$$
$$\tilde{\gamma} : H_q(K, L; G) \xrightarrow{\sim} H_q(M(\tilde{R}), A(\tilde{R}); G).$$

Thus we have an isomorphism

$$\beta_{M,A} = \tilde{\gamma} \circ \gamma^{-1} : H_q(M, A; G) \xrightarrow{\sim} H_q(M(\tilde{R}), A(\tilde{R}); G).$$

It is easily checked that $\beta_{M,A}$ does not depend on the choice of the good triangulation $(X, (S_i, i \in I), \psi)$, and that for every semialgebraic map $f: (M, A) \to (N, B)$ the diagram

$$\begin{array}{ccc} H_q(M, A; G) & \xrightarrow[\sim]{\beta_{M,A}} & H_q(M(\tilde{R}), A(\tilde{R}); G) \\ \downarrow f_* & & \downarrow (f_{\tilde{R}})_* \\ H_q(N, B; G) & \xrightarrow[\sim]{\beta_{N,B}} & H_q(N(\tilde{R}), B(\tilde{R}); G) \end{array}$$

commutes. The maps $\beta_{M,A}$ are also compatible with the long exact homology sequences (cf. (3. 1)) for three spaces $B \subset A \subset M$ over R and the corresponding spaces

$$B(\tilde{R}) \subset A(\tilde{R}) \subset M(\tilde{R}).$$

The same observations hold true in semialgebraic cohomology. Thus we may state

Theorem 4. 2. *For every pair (M, A) of affine semialgebraic spaces we have canonical isomorphisms*

$$H_q(M, A; G) \xrightarrow{\sim} H_q(M(\tilde{R}), A(\tilde{R}); G),$$
$$H^q(M, A; G) \xrightarrow{\sim} H^q(M(\tilde{R}), A(\tilde{R}); G).$$

In the case $q = 0$, $A = \emptyset$ this theorem yields in particular

Corollary 4. 3. *An affine semialgebraic space M over R is connected if and only if $M(\tilde{R})$ is connected. More generally, if $M_1, \ldots, M_r$ are the connected components of M, then $M_1(\tilde{R}), \ldots, M_r(\tilde{R})$ are the connected components of $M(\tilde{R})$.*

It is not difficult to generalize this statement to arbitrary semialgebraic spaces, for example by use of Lemma 9. 12 in [DK].

Theorem 4. 2 opens the road to transfer many results from classical homology theory to semialgebraic homology theory without any serious labour. We give examples for this in the present and the next section. In all these examples Theorem 4. 2 seems to be not really necessary, since we could as well adapt classical proofs using triangulations and staying over a fixed real closed field R. But this would often be more laborious.

We always denote by R_0 the real closure of the field $\mathbb{Q}$ of rational numbers. This field R_0 is contained in any other real closed field. We obtain from Theorem 4. 2 and Theorem 3. 7 as an important consequence

Corollary 4. 4. *Let V be a quasiprojective variety defined over R_0. Then for every real closed field R the semialgebraic homology groups $H_p(V(R), G)$ are canonically isomorphic to the classical singular homology groups $H_{p,cl}(V(\mathbb{R}), G)$. The same holds true in cohomology.*

Example 4. 5. Let $S^n(R)$ denote the unit sphere in R^{n+1}. For any abelian group G

$$H_0(S^n(R), G) \cong H^0(S^n(R), G) \cong G,$$
$$H_n(S^n(R), G) \cong H^n(S^n(R), G) \cong G.$$
$$H_q(S^n(R), G) = H^q(S^n(R), G) = 0 \quad \text{for} \quad q \neq 0, n.$$

Theorem 4. 6 (Excision). *Let A and B be semialgebraic subsets of an affine semi-algebraic space M. Suppose there exist open semialgebraic subsets U and V of M with $U \subset A$, $V \subset B$ and $U \cup V = M$. Then for every abelian group G and every $q \geqq 0$ the natural maps*

$$H_p(B, A \cap B; G) \to H_p(M, A; G)$$

are isomorphisms.

Proof. This holds true in the case $R = \mathbb{R}$, cf. [M], 8. 2. 1. For a proof in general we choose a simultaneous triangulation of M, A, B, U, V. From this triangulation it is evident that there exists an affine semialgebraic space M_0 over R_0 together with semialgebraic subsets A_0, B_0, U_0, V_0 of M_0 such that U_0 and V_0 are open in M_0, $U_0 \cup V_0 = M_0$, $U_0 \subset A_0$, $V_0 \subset B_0$, and a semialgebraic isomorphism $\varphi : M_0(R) \xrightarrow{\sim} M$ which maps $A_0(R)$, $B_0(R)$, $U_0(R)$, $V_0(R)$ onto A, B, U, V respectively. We have a natural commutative diagram (omitting the coefficient group G)

$$\begin{array}{ccc} H_p(B_0, A_0 \cap B_0) & \xrightarrow{\alpha_1} & H_p(M_0, A_0) \\ \cong \downarrow & & \downarrow \cong \\ H_p(B, A \cap B) & \xrightarrow{\alpha_2} & H_p(M, A) \end{array}$$

and a second such diagram where the lower horizontal arrow α_2 is replaced by the natural map

$$\alpha_3 : H_p(B_0(\mathbb{R}), A_0(\mathbb{R}) \cap B_0(\mathbb{R})) \to H_p(M_0(\mathbb{R}), A_0(\mathbb{R})).$$

Now $U_0(\mathbb{R})$ and $V_0(\mathbb{R})$ are open in $M_0(\mathbb{R})$ and they cover together the space $M_0(\mathbb{R})$, as is clear by Tarski's principle or from the triangulation above. Thus α_3 is an isomorphism. From the second diagram we learn that α_1 is an isomorphism and then from the first diagram, that α_2 is an isomorphism. Q.E.D.

Remark 4.7. If A and B are closed semialgebraic subsets of an affine complete semialgebraic space M with $M = A \cup B$, then again

$$H_p(B, A \cap B; G) \to H_p(M, A; G)$$

is an isomorphism for every $p \geqq 0$ and every G. Indeed, choosing a simultaneous triangulation of M, A, B we may assume that $M = |K|_R$, $A = |L_1|_R$, $B = |L_2|_R$ with K an abstract simplicial complex and subcomplexes L_1, L_2 of K such that $K = L_1 \cup L_2$. It is well known and evident that the maps

$$H_p(L_2, L_1 \cap L_2; G) \to H_p(K, L_1; G)$$

are isomorphisms [M], p. 118.

§ 5. Alexander-Poincaré duality

We consider again affine semialgebraic spaces over a fixed real closed field R.

Definition 1. Let K be an abstract simplicial complex and x be a point of the realization $|K|$ over R. The *simplicial neighbourhood* $N_K(x)$ of x with respect to K is the union of all *closed* simplices of $|K|$ which contain x. The *link* $Lk_K(x)$ of x with respect to K is the union of all closed simplices of $|K|$ which are contained in $N_K(x)$ but do not contain x.

Notice that there exists a semialgebraic isomorphism φ from $N_K(x)$ onto the cone over $Lk_K(x)$ which maps x to the new vertex of the cone [M], p. 43. If x is a vertex of K then φ may even be chosen as a *simplicial* isomorphism.

As in the case $R = \mathbb{R}$ the following important fact can be verified in an elementary way, cf. [M], 2.4.5 and [DK], 9.8.

Proposition 5.1. *Let K and L be abstract simplicial complexes and let $f: |K| \to |L|$ be an injective semialgebraic map. Let x be some point of $|K|$ such that $f(x)$ lies in the interior of $f(|K|)$ with respect to $|L|$. Then there exists a semialgebraic homotopy equivalence from $Lk_K(x)$ to $Lk_L(f(x))$. In particular, if two triangulations $\varphi: |K| \xrightarrow{\sim} M$, $\psi: |L| \xrightarrow{\sim} M$ of a complete semialgebraic space M are given, then the links of any point $x \in M$ with respect to φ and ψ are homotopy equivalent, $Lk_K(\varphi^{-1}(x)) \simeq Lk_L(\psi^{-1}(x))$.*

Definition 2 (cf. [DK], § 13). A *semialgebraic n-manifold* M over R is a semialgebraic space M over R in which every point x of M has an open semialgebraic neighbourhood isomorphic to some open semialgebraic subset of R^n. Clearly then $\dim M = n$.

Example. For any irreducible n-dimensional algebraic variety V over R the space $V(R)_{reg}$ of regular real points of V is a semialgebraic n-manifold. This is a consequence of the implicit function theorem, cf. [DK], Proof of Prop. 8.6.

As an application of Proposition 5.1 we obtain

Corollary 5.2. *Let K be an abstract simplicial complex. Assume that $|K|$ is an n-dimensional semialgebraic manifold. Then the link $Lk_K(x)$ of every point x of $|K|$ is homotopy equivalent to the unit sphere $S^{n-1}(R)$.*

Proof (cf. [M], 3. 4. 3). Let Δ denote the abstract n-dimensional simplex, and $\dot{\Delta}$ the boundary subcomplex of Δ. Let p denote the center of $|\Delta|$. Then $Lk_{\Delta}(p) \cong S^{n-1}(R)$. Now choose a semialgebraic isomorphism of $|\Delta|$ onto a neighbourhood B of x in $|K|$, which maps p to x. This is possible, since $|K|$ is a manifold. Then by Proposition 5. 1

$$S^{n-1}(R) \cong Lk_{\Delta}(p) \simeq Lk_K(x).$$

Definition 3. Let M be a complete affine semialgebraic space over R, and let $\psi: |K| \xrightarrow{\sim} M$ be a triangulation of M. Then M is called a *homology n-manifold* over R, if for every $x \in |K|$ and every $q \geqq 0$ the group $H_q(Lk_K(x), \mathbb{Z})$ is isomorphic to $H_q(S^{n-1}(R), \mathbb{Z})$, i.e. $H_q(Lk_K(x), \mathbb{Z}) \cong \mathbb{Z}$ for $q=0$ and $q=n$ and $H_q(Lk_K(x), \mathbb{Z}) = 0$ else.

Notice that according to Proposition 5. 1 the choice of the triangulation is irrelevant for the homology groups of the links. Thus Definition 3 makes sense.

According to Corollary 5. 2 every complete affine semialgebraic n-manifold over R is a homology n-manifold. In particular for every quasiprojective variety V over R without singular real points the space $V(R)$ is a homology n-manifold provided $V(R)$ is complete.

Proposition 5. 3. *Let M be an affine semialgebraic space over R and let $\tilde{R}$ be a real closed field containing R. Then $M(R)$ is a homology n-manifold if and only if $M(\tilde{R})$ is a homology n-manifold.*

Proof. We may regard M as a semialgebraic subset of some R^N. Then M is complete if and only if M is closed and bounded in R^N, and $M(\tilde{R})$ is complete if and only if $M(\tilde{R})$ is closed and bounded in $\tilde{R}^N$ [DK], § 9. Thus it is evident by Tarski's principle that M is complete if and only if $M(\tilde{R})$ is complete.

Assume now that M is complete. We choose a triangulation $\psi: |K|_R \xrightarrow{\sim} M$ of M, and we obtain by base extension a triangulation $\psi_{\tilde{R}}: |K|_{\tilde{R}} \xrightarrow{\sim} M(\tilde{R})$ of $M(\tilde{R})$. For any point x of $|K|_R$ the link $Lk_{K,\tilde{R}}(x)$ of x in $|K|_{\tilde{R}}$ is obtained from the link $Lk_{K,R}(x)$ in $|K|_R$ by base extension. If $M(\tilde{R})$ is a homology n-manifold, then

$$H_q(Lk_{K,R}(x), \mathbb{Z}) \cong H_q(Lk_{K,\tilde{R}}(x), \mathbb{Z}) \cong H_q(S^{n-1}(\tilde{R}), \mathbb{Z}) \cong H_q(S^{n-1}(R), \mathbb{Z})$$

for every $q \geqq 0$ and $x \in |K|_R$. Thus M is a homology n-manifold.

Assume now that M is a homology n-manifold. Let x be a point in $|K|_{\tilde{R}}$. We choose a point $y \in |K|_R$ which lies in the same open simplex of $|K|_{\tilde{R}}$ as x. Then

$$Lk_{K,\tilde{R}}(x) = Lk_{K,\tilde{R}}(y) = Lk_{K,R}(y)(\tilde{R}),$$

and we see as above that $Lk_{K,\tilde{R}}(x)$ has the same homology groups as $S^{n-1}(\tilde{R})$. Thus $M(\tilde{R})$ is a homology n-manifold. Q.E.D.

Remark 5. 4. Let M be an affine semialgebraic space over R and let $\tilde{R}$ be a real closed field containing R. Then it can be shown that M is an n-manifold if and only if $M(\tilde{R})$ is an n-manifold. We do not need this fact.

Proposition 5. 5. *Let M be a connected homology n-manifold over R. Then $H_n(M, \mathbb{Z}/2) \cong \mathbb{Z}/2$ and $H_n(M, \mathbb{Z}) \cong \mathbb{Z}$ or $H_n(M, \mathbb{Z}) = 0$.*

This is well known in the case $R = \mathbb{R}$ [M], 5. 3. 6, and can be immediately generalized to arbitrary R by use of Proposition 5. 3.

Definition 4. A connected homology n-manifold M is called *orientable* if $H_n(M, \mathbb{Z}) \cong \mathbb{Z}$. An arbitrary homology n-manifold M is called orientable if every connected component of M is orientable.

Notation. Let L be a subcomplex of a fixed abstract simplicial complex K. Then we denote by $\bar{L}$ the abstract subcomplex of the barycentric subdivision K' of K which consists of all simplices of K' which do *not* have any vertex in L'.

Theorem 5. 6. (Combinatorial Alexander-Poincaré duality theorem). *Let $L_1 \subset K_1$ be subcomplexes of an abstract complex K. Assume that $|K|_R$ is a homology n-manifold. Then we have for every $q \geqq 0$ a natural isomorphism*

$$H^q(K_1, L_1; \mathbb{Z}/2) \xrightarrow{\sim} H_{n-q}(\bar{L}_1, \bar{K}_1; \mathbb{Z}/2).$$

(Of course $H_q(\ldots) = 0$ if $q < 0$.) *If $|K|_R$ is orientable we have natural isomorphisms*

$$H^q(K_1, L_1; G) \xrightarrow{\sim} H_{n-q}(\bar{L}_1, \bar{K}_1; G)$$

for every abelian group G.

This is well known in the case $R = \mathbb{R}$ [M], Theorem 5. 3. 13. In general $|K|_R$ is a homology n-manifold iff $|K|_{\mathbb{R}}$ is a homology n-manifold and clearly $|K|_R$ is orientable iff $|K|_{\mathbb{R}}$ is orientable. Thus the theorem remains true over any real closed field R. In fact the proof in [M] by its combinatorial nature directly generalizes to any real closed field. In particular the explicit description of the duality isomorphisms in [M] takes over to any real closed field R.

Let now M be a homology n-manifold over R and let $B \subset A$ be semialgebraic subsets of M. We choose a triangulation $\psi: |K| \xrightarrow{\sim} M$ which is good on A and B. Without loss of generality we assume that $M = |K|$ and ψ is the identity mapping. Let K_1 and L_1 be the abstract cores of A and B respectively. Thus $|K_1|$ and $|L_1|$ are the largest complete subcomplexes of $|K|$ contained in A and B respectively, and

$$(1) \qquad H^q(A, B; G) \cong H^q(K_1, L_1; G)$$

for every $q > 0$ and every abelian group G. Now the first barycentric subdivision of our triangulation is also good on $M \setminus A$ and $M \setminus B$ (cf. Prop. 2. 4). Clearly $\bar{K}_1$ and $\bar{L}_1$ are the abstract cores of $M \setminus |K_1|$ and $M \setminus |L_1|$ with respect to the barycentric subdivision of ψ. Thus

$$(2) \qquad H_q(\bar{L}_1, \bar{K}_1; G) \cong H_q(M \setminus |L_1|, M \setminus |K_1|; G).$$

The triangulation ψ is good on all the sets $M \setminus |L_1|$, $M \setminus B$, $M \setminus |K_1|$, $M \setminus A$. The sets $M \setminus |L_1|$ and $M \setminus B$ have the same abstract core L_2 and the sets $M \setminus |K_1|$ and $M \setminus A$ the same abstract core K_2 with respect to ψ. Thus

$$(3) \quad H_q(M \setminus |L_1|, M \setminus |K_1|; G) \cong H_q(L_2, K_2; G) \cong H_q(M \setminus B, M \setminus A; G).$$

Taking into account the isomorphisms (1), (2), (3) we obtain from Theorem 5. 6 a weak version of a semialgebraic Alexander-Poincaré duality theorem.

Theorem 5. 7. *Let $B \subset A$ be semialgebraic subsets of a homology n-manifold M over R. Then for every $q \geqq 0$*

$$H^q(A, B; \mathbb{Z}/2) \cong H_{n-q}(M \setminus B, M \setminus A; \mathbb{Z}/2).$$

If M is orientable, then for every G

$$H^q(A, B; G) \cong H_{n-q}(M \setminus B, M \setminus A; G).$$

Remark 5. 8. Using the classical simplicial definitions it is possible to define cup- and cap-products in our semialgebraic (co)homology theory which do not depend on the chosen triangulations. Assume for simplicity that M is connected. If we choose a generator η_M of $H_n(M, \mathbb{Z})$ if M is orientable resp. of $H_n(M, \mathbb{Z}/2)$ if M is not orientable, then the duality isomorphisms in Theorem 5. 7 are induced by the cap-product with this "fundamental class" η_M. In particular they are independent of the chosen triangulation (cf. [M]). But Theorem 5. 7 as it stands suffices for the applications we have in mind.

For any affine semialgebraic space M over R we define the *reduced homology group* $\tilde{H}_q(M, G)$ in the usual way as the kernel of the natural homomorphism from $H_q(M, G)$ to $H_q(*, G)$ induced by the map from M to the one point space $*$. Similarly we define $\tilde{H}^q(M, G)$ as the cokernel of the natural homomorphism from $H^q(*, G)$ to $H^q(M, G)$. Of course $\tilde{H}_q(M, G) = H_q(M, G)$ and $\tilde{H}^q(M, G) = H^q(M, G)$ for $q > 0$. Writing down the long exact homology and cohomology sequences of the pair $(M, \{a\})$ for some $a \in M$, we see that

$$\tilde{H}_q(M, G) \xrightarrow{\sim} H_q(M, a; G), \quad H^q(M, a; G) \xrightarrow{\sim} \tilde{H}^q(M, G)$$

for every $q \geqq 0$ and G.

Corollary 5.9 (Alexander duality). *Let A be a non empty proper semialgebraic subset of the sphere $S^n(R)$. Then for every $q \geqq 0$ and every G there exists an isomorphism*

$$\tilde{H}^q(A, G) \xrightarrow{\sim} \tilde{H}_{n-q-1}(S^n(R) \setminus A, G).$$

Proof. Applying Theorem 5. 7 to $M = S^n(R)$, A, and $B := \{a\}$ with some point $a \in A$ we obtain

$$\tilde{H}^q(A, G) \cong H_{n-q}(S^n(R) \setminus \{a\}, S^n(R) \setminus A; G).$$

Now $S^n(R) \setminus \{a\}$ is semialgebraically isomorphic to R^n by stereographic projection. Thus $S^n(R) \setminus \{a\}$ is contractible. The long reduced exact homology sequence for $(S^n(R) \setminus \{a\}, S^n(R) \setminus A)$ yields

$$H_{n-q}(S^n(R) \setminus \{a\}, S^n(R) \setminus A; G) \cong \tilde{H}_{n-q-1}(S^n(R) \setminus A, G). \qquad \text{Q.E.D.}$$

We mention some applications of the duality theorems 5. 7 and 5. 9 to the geometry of spheres obtained in the same way as in the classical theory.

Theorem 5. 10. *Let M be a homology n-manifold over R which is not orientable. Then there does not exist an injective semialgebraic map from M to $S^{n+1}(R)$.*

Proof. We may assume that M is connected. Suppose $f: M \longrightarrow S^{n+1}(R)$ is an injective semialgebraic map. Then f is a semialgebraic isomorphism from M onto the semialgebraic subspace $N := f(M)$ of $S^{n+1}(R)$, cf. [DK], 9. 8. By Alexander duality

$$\tilde{H}^0(S^{n+1}(R) \setminus N, \mathbb{Z}) \cong \tilde{H}_n(N, \mathbb{Z}) = 0.$$

Thus also $\tilde{H}_0(S^{n+1}(R)\setminus N, \mathbb{Z})=0$, and we obtain, again by Alexander duality,

$$0=\tilde{H}_0(S^{n+1}(R)\setminus N, \mathbb{Z}/2)\cong \tilde{H}^n(N, \mathbb{Z}/2)\cong \mathbb{Z}/2,$$

a contradiction. (Notice that $n>0$.) Q.E.D.

Theorem 5. 11. *Let M be a closed semialgebraic subset of $S^{n+1}(R)$, $n\geqq 1$. Assume that M is a homology n-manifold with k connected components. Then $S^{n+1}(R)\setminus M$ has $k+1$ connected components.*

Proof. M is orientable by Theorem 5. 10. Alexander duality yields

$$\tilde{H}_0(S^{n+1}(R)\setminus M, \mathbb{Z})\cong \tilde{H}^n(M, \mathbb{Z})=H^n(M, \mathbb{Z}).$$

By a special case of Poincaré duality

$$H^n(M, \mathbb{Z})\cong H_0(M, \mathbb{Z}).$$

This gives the theorem. {Theorem 5. 10 is not really necessary for the proof, since we could equally work with $\mathbb{Z}/2$ as coefficients.}

Corollary 5.12 (Generalized Jordan curve theorem). *Let $f: S^n(R)\to S^{n+1}(R)$ be an injective semialgebraic map with image M. Then $S^{n+1}(R)\setminus M$ has precisely two connected components C_1 and C_2. Both open sets C_1 and C_2 have the boundary M.*

Proof. f is a semialgebraic isomorphism from $S^n(R)$ onto M. The first assertion is contained in Theorem 5. 11. By elementary dimension theory the set $C_1\cup C_2$ is dense in $S^{n+1}(R)$, cf. [DK], 13. 1. Let x be a given point in M. Suppose x is not contained in the boundary of C_1. Then x is contained in the boundary of C_2. Let $M'=M\setminus\{x\}$. This is a contractible space. Clearly $S^{n+1}(R)\setminus M'$ has two connected components C_1 and $C_2\cup\{x\}$. But by Alexander duality

$$\tilde{H}_0(S^{n+1}(R)\setminus M', \mathbb{Z})\cong \tilde{H}^n(M', \mathbb{Z})=0.$$

This is a contradiction. Thus every point of M lies in the boundary of C_1.

By a well known classical argument [Do], p. 79 we obtain from the generalized Jordan curve theorem

Theorem 5.13 (Invariance of domain). *Let U be an open semialgebraic subset of R^n and let $f: U\to R^n$ be an injective semialgebraic map. Then $f(U)$ is again open (and semialgebraic) in R^n.*

§ 6. Local triviality of semialgebraic families

The last two sections may foster the impression that it is a more or less trivial matter to transfer results on the homology of real algebraic varieties to algebraic varieties over any real closed field R. This impression is misleading, at least at our present state of knowledge. Up to now we have only applied the "obvious" transfer method. The success of this method depends on an extraordinary thorough combinatorial understanding of the given classical situation.

We now provide the tools for another transfer method, to be explicated in the next sections.

Lemma 6. 1. *Let $P_1(X, T), \ldots, P_s(X, T)$ be non zero polynomials in*

$$R[X_1, \ldots, X_n, T_1, \ldots, T_r] \quad (n \geqq 1, r \geqq 1).$$

Then there exists a linear transformation

$$T_1 = T_1' + a_1 T_r', \quad T_2 = T_2' + a_2 T_r', \ldots, \quad T_{r-1} = T_{r-1}' + a_{r-1} T_r', \quad T_r = a_r T_r'$$

over R $(a_r \neq 0)$ such that the transformed polynomials $\tilde{P}_k(X, T') := P_k(X, T)$ are of the form

$$\tilde{P}_k(X, T') = b_{0k}(X) T_r'^{m_k} + b_{1k}(X, T_1', \ldots, T_{r-1}') T_r'^{m_k - 1} + \cdots + b_{m_k, k}(X, T_1', \ldots, T_{r-1}')$$

with non zero polynomials $b_{0k}(X) \in R[X_1, \ldots, X_n]$.

More precisely it is true that the set of points $(a_1, \ldots, a_r) \in R^r$ which yield such a transformation contains a non empty Zariski open subset of R^r. The easy proof may be left to the reader. We shall use the lemma only in the following weaker form, which is needed to take benefit from Lemma 1. 1.

Corollary 6. 2. *Let $P_1(X, T), \ldots, P_s(X, T)$ be non zero polynomials in*

$$R[X_1, \ldots, X_n, T_1, \ldots, T_r].$$

Then after performing a suitable linear transformation of the coordinates $T_1, \ldots, T_r$ there exists a non empty Zariski open subset U of R^n such that the leading coefficient $b_{0k}(X, T_1, \ldots, T_{r-1})$ of every polynomial $P_k(X, T)$ with respect to T_r has no zeros on $U \times R^{r-1}$.

We now consider semialgebraic subsets of a product $R^n \times R^m$. We denote by $pr_1 : R^n \times R^m \to R^n$ the natural projection of $R^n \times R^m$ to the first factor.

Proposition 6. 3. *Let N be an open semialgebraic subset of R^n and $M_1, \ldots, M_r$ be semialgebraic subsets of $R^n \times R^m$. Then there exists an open semialgebraic subset U of R^n with $\dim (R^n \setminus U) < n$ such that for every connected component B of $U \cap N$ the following holds true:*

B is contractible into every point $y \in B$. Given a point $y \in B$, the retraction $r : B \to \{y\}$ and a (semialgebraic) homotopy $H : B \times [0, 1] \to B$ between the identity and r, there exists a retraction $s : B \times R^m \to \{y\} \times R^m$ and a homotopy $K : (B \times R^m) \times [0, 1] \to B \times R^m$ between the identity and s with the following properties:

i) *The diagram*

$$\begin{array}{ccc} (B \times R^m) \times [0, 1] & \xrightarrow{K} & B \times R^m \\ \Big\downarrow {\scriptstyle pr_1 \times id} & & \Big\downarrow {\scriptstyle pr_1} \\ B \times [0, 1] & \xrightarrow{H} & B \end{array}$$

commutes.

ii) *s and K yield for every M_j by restriction a retraction*

$$s_j : (B \times R^m) \cap M_j \to (\{y\} \times R^m) \cap M_j$$

and a homotopy

$$K_j : [(B \times R^m) \cap M_j] \times [0, 1] \to (B \times R^m) \cap M_j$$

between the identity and s_j.

iii) *For every* $x \in B$ *the retraction* s *gives by restriction a semialgebraic isomorphism*

$$s_x : \{x\} \times R^m \xrightarrow{\sim} \{y\} \times R^m$$

which maps $(\{x\} \times R^m) \cap M_j$ *onto* $(\{y\} \times R^m) \cap M_j$ *for* $1 \leqq j \leqq r$.

iv) *If* H *does not move the point* y *then* K *does not move any point in* $\{y\} \times R^m$.

Proof. We use induction on m. For $m=0$ the assertion follows from the triangulation theorem 2. 1. Indeed, choose a simultaneous triangulation of $R^n, N, M_1, \ldots, M_r$ and define U as the union of all open n-simplices in R^n.

Assume now that $m \geqq 1$ and that the proposition is already proved for $m-1$. Let $P_k(X_1, \ldots, X_n, T_1, \ldots, T_m)$ be non zero polynomials in $R[X_1, \ldots, X_n, T_1, \ldots, T_m]$, $1 \leqq k \leqq s$, such that each M_j, $1 \leqq j \leqq r$, is a finite union of sets of the form

$$\{(x, t) \in R^n \times R^m \mid P_{k_0}(x, t) = 0, P_{k_1}(x, t) > 0, \ldots, P_{k_h}(x, t) > 0\}$$

or

$$\{(x, t) \in R^n \times R^m \mid P_{k_1}(x, t) > 0, \ldots, P_{k_h}(x, t) > 0\}$$

with $k_0, \ldots, k_h \in \{1, 2, \ldots, s\}$. After a suitable linear transformation of the coordinates $T_1, \ldots, T_m$ we may assume that the leading coefficient $a_k(X_1, \ldots, X_n, T_1, \ldots, T_{m-1})$ of every polynomial P_k with respect to T_m has no zeros on $U_1 \times R^{m-1}$, with U_1 a non empty Zariski open subset of R^n (Corollary 6. 2). We now apply Lemma 1. 1. We let the one set $U_1 \times R^{m-1}$ play the role of the family $(M_i \mid i = 1, \ldots, r)$ in Lemma 1. 1, $(X_1, \ldots, X_n, T_1, \ldots, T_{m-1})$ the role of $(X_1, \ldots, X_n)$, T_m the role of T and $P_1, \ldots, P_s$ the role of $P_1, \ldots, P_\rho$. Choose a decomposition $U_1 \times R^{m-1} = \bigcup_{k=1}^{d} A_k$ and semialgebraic functions λ_i^k, $1 \leqq i \leqq r(k)$, as described in Lemma 1. 1. Without loss of generality we assume that $\lambda_1^k(x) < 0$ and $\lambda_{r(k)}^k(x) > 0$ for all $x \in A_k$. {Consider if necessary in addition to the polynomials $P_1, \ldots, P_s$ the polynomials $T_m + 1$, $T_m - 1$.}

We apply our induction hypothesis to the open semialgebraic subset $N \cap U_1$ of R^n and the semialgebraic subsets $A_1, \ldots, A_d$ of $R^n \times R^{m-1}$. Thus there exists a semialgebraic open subset $U \subset U_1$ of R^n with $\dim(R^n \setminus U) < n$ such that for every connected component B of $U \cap N$ the following holds true: B is contractible into every point $y \in B$. Given a point $y \in B$, the retraction $r : B \to \{y\}$ and a homotopy $H : B \times [0, 1] \to B$ between the identity and r, there exist a retraction $s' : B \times R^{m-1} \to \{y\} \times R^{m-1}$ and a homotopy $K' : (B \times R^{m-1}) \times [0, 1] \to B \times R^{m-1}$ between the identity and s', which have the properties i)—iv) stated in the proposition with R^m replaced by R^{m-1} and the M_j replaced by the A_k, $1 \leqq k \leqq d$.

We write $B\times R^m$ as the disjoint union $B\times R^m=\bigcup_{l=1}^{N} C_l$, where C_l runs through the following finitely many semialgebraic sets (recall ii) in Lemma 1. 1):

$$D_{ki}:=\{(x,\lambda_i^k(x))\mid x\in A_k\cap(B\times R^{m-1})\},\quad 1\leqq k\leqq d,\quad 1\leqq i\leqq r(k);$$

$$E_{ki}:=\{(x,t)\mid x\in A_k\cap(B\times R^{m-1}),\ \lambda_i^k(x)<t<\lambda_{i+1}^k(x)\},\quad 1\leqq k\leqq d,\quad 1\leqq i\leqq r(k)-1;$$

$$F_k:=\{(x,t)\mid x\in A_k\cap(B\times R^{m-1}),\ t<\lambda_1^k(x)\},\quad 1\leqq k\leqq d;$$

$$G_k:=\{(x,t)\mid x\in A_k\cap(B\times R^{m-1}),\ t>\lambda_{r(k)}^k(x)\},\quad 1\leqq k\leqq d.$$

We conclude from Lemma 1. 1 iii) that the polynomials $P_1,\dots,P_s$ have constant sign $\{+1,-1,\text{or }0\}$ on each C_l, hence every set $M_j\cap(B\times R^m)$ is the union of some sets C_l. We lift the homotopies $K'\mid[A_k\cap(B\times R^{m-1})]\times[0,1]$, $1\leqq k\leqq d$, to homotopies

$$K_l: C_l\times[0,1]\to C_l,\quad 1\leqq l\leqq N,$$

as follows:

1) $C_l=D_{ki}$:
$K_l((x,\lambda_i^k(x)),t):=(K'(x,t),\lambda_i^k(K'(x,t)))$, $(t\in[0,1])$.

2) $C_l=E_{ki}$:
$K_l((x,\mu\lambda_i^k(x)+(1-\mu)\lambda_{i+1}^k(x)),t):=(K'(x,t),\mu\lambda_i^k(K'(x,t))$
$+(1-\mu)\lambda_{i+1}^k(K'(x,t)))$, $(t\in[0,1],\ 0<\mu<1)$.

3) $C_l=F_k$:
$K_l((x,\mu\lambda_1^k(x)),t):=(K'(x,t),\mu\lambda_1^k(K'(x,t)))$, $(\mu>1)$.

4) $C_l=G_k$:
$K_l((x,\mu\lambda_{r(k)}^k(x)),t):=(K'(x,t),\mu\lambda_{r(k)}^k(K'(x,t)))$, $(\mu>1)$.

The maps K_l fit together to a semialgebraic homotopy

$$K:(B\times R^m)\times[0,1]\to B\times R^m,$$

as is easily verified using the properties iv)—vii) of Lemma 1. 1. {It is evident that K has a semialgebraic graph. Thus only continuity has to be checked.} Define $s(x):=K(x,1)$ for $x\in B\times R^m$. Then the retractions $s: B\times R^m\to\{y\}\times R^m$ and the homotopy K have the desired properties i)—iv), as follows from the corresponding properties of s' and K'.

Q.E.D.

Remark. We shall need from Proposition 6. 3 only the existence of the retraction s with the properties ii) and iii). The homotopy statements in i), ii), iv) have been included to make the idea of the proof "moving the fibre" more transparent.

We now prove a stronger and at the same time simpler theorem than Proposition 6. 3 on the "local triviality" of semialgebraic families.

Theorem 6. 4. *Let $f: M \to N$ be a semialgebraic map between affine semialgebraic spaces, and let $M_1, \ldots, M_r$ be semialgebraic subsets of M. Then there exists a decomposition $N = \bigcup_{\lambda=1}^{u} B_\lambda$ of N into finitely many disjoint semialgebraic subsets B_λ, for every B_λ some affine semialgebraic space F_λ with semialgebraic subsets $F_{\lambda,1}, \ldots, F_{\lambda,r}$, and semialgebraic isomorphisms $\varphi_\lambda : f^{-1}(B_\lambda) \xrightarrow{\sim} B_\lambda \times F_\lambda$ such that for every $\lambda \in \{1, \ldots, u\}$ the following holds true:*

a) *The diagram*

$$\begin{array}{ccc} f^{-1}(B_\lambda) & \xrightarrow{\varphi_\lambda} & B_\lambda \times F_\lambda \\ & {\scriptstyle f|f^{-1}(B_\lambda)} \searrow \quad \swarrow {\scriptstyle pr_1} & \\ & B_\lambda & \end{array}$$

commutes.

b) $\varphi_\lambda(M_j \cap f^{-1}(B_\lambda)) = B_\lambda \times F_{\lambda j}$ *for* $1 \leqq j \leqq r$.

Remarks. 1) It is not necessary to state anew contractibility for the B_λ, since it is evident from the triangulation theorem 2. 1 that the B_λ can always be chosen as contractible into every point.

2) The theorem states *generic* local triviality. Indeed, if N is embedded into some R^n, then it follows from the theorem that there exists some Zariski closed subset Z of R^n such that $N \setminus Z$ is not empty and the family $M, M_1, \ldots, M_r$ lies trivially over every connected component of $N \setminus Z$ with respect to f. The theorem contains nothing more.

3) A theorem similar to Theorem 6. 4 has in the case $R = \mathbb{R}$ first been proved by R. Hardt [H], p. 295. Hardt states generic local triviality only for the map f itself without considering subsets of M. His theorem contains a further statement which we also could prove but do not need here. Varchenko [V] and Wallace [W] have proven our theorem for $R = \mathbb{R}$ with homeomorphisms φ_λ instead of semialgebraic isomorphisms. The proofs of all three authors are rather different from our proof.

Proof. We proceed by induction on $n := \dim N$. The case $n = 0$ is trivial. Assume $n > 0$. Triangulating N and regarding the partition of N into open simplices we retreat to the case that N is the open standard n-simplex in R^n. We choose a fixed embedding of M into some R^m. We have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow[\alpha]{\sim} & \Gamma'(f) \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle pr_1|\Gamma'(f)} & \\ & N & \end{array}$$

with $pr_1 : R^n \times R^m \to R^n$ the natural projection and α the evident isomorphism $x \mapsto (f(x), x)$ from M to the "switched graph" $\Gamma'(f)$ of f. Replacing $M, f, M_1, \ldots, M_r$ by $N \times R^m$, pr_1, $\alpha(M)$, $\alpha(M_1), \ldots, \alpha(M_r)$ we assume since now that $M = N \times R^m$ and $f = pr_1$. We apply Proposition 6. 3 and gain an open semialgebraic subset $N_1 = U \cap N$ with the properties stated there and $\dim(N\setminus N_1)<n$. Let $(B_\lambda|1\leqq\lambda\leqq v)$ denote the family of connected components of N_1. We choose in every B_λ a point y_λ and introduce the semialgebraic subsets $F_{\lambda 1},\ldots,F_{\lambda r}$ of $F_\lambda:=R^m$ defined by

$$\{y_\lambda\}\times F_{\lambda j}=(\{y_\lambda\}\times R^m)\cap M_j.$$

We then choose a homotopy $H_\lambda: B_\lambda\times[0,1]\to B_\lambda$ from the identity to the retraction $r_\lambda: B_\lambda\to\{y_\lambda\}$ and gain a homotopy on $B_\lambda\times R^m$ over H_λ and a retraction

$$s_\lambda: B_\lambda\times R^m\to\{y_\lambda\}\times R^m$$

with the properties i)—iv) stated in Proposition 6. 3. Write

$$s_\lambda(x,t)=(y_\lambda,\sigma_\lambda(x,t))$$

with a semialgebraic map $\sigma_\lambda: B_\lambda\times R^m\to R^m$. Now define $\varphi_\lambda: B_\lambda\times R^m\to B_\lambda\times R^m$ by

$$\varphi_\lambda(x,t)=(x,\sigma_\lambda(x,t)).$$

φ_λ is by property iii) a bijective semialgebraic map. By Theorem 5. 13 (Invariance of domain) φ_λ is also open, hence a semialgebraic isomorphism. Again by property iii) φ_λ maps $(B_\lambda\times R^m)\cap M_j$ onto $B_\lambda\times F_{\lambda,j}$. Thus properties a) and b) of the theorem hold true for these φ_λ. Application of the induction hypothesis to the restriction

$$f^{-1}(N\setminus N_1)\to N\setminus N_1$$

of f and the subsets $M_j\cap f^{-1}(N\setminus N_1)$, $1\leqq j\leqq r$, of $f^{-1}(N\setminus N_1)$ finishes the proof.

Remark. The theorem on the invariance of domain can be avoided in this proof by embedding $R^n\times R^m$ into $R^n\times S^m\subset R^n\times R^{m+1}$ and then applying Proposition 6. 3 to $R^n\times R^{m+1}$, the subset $R^n\times S^m$ and the images of the M_j. It can then be seen in an elementary way that the bijective semialgebraic map $\varphi_\lambda: B_\lambda\times S^m\to B_\lambda\times S^m$ obtained from the retraction s_λ is an isomorphism. Indeed, every point $x\in B_\lambda$ has a complete neighbourhood N in B_λ and φ_λ yields a bijection from $N\times S^m$ to $N\times S^m$ which must be an isomorphism by [DK], 9. 8.

We close this section with an application of Theorem 6. 4 to the local geometry of semialgebraic subsets of R^n of independent interest. We denote for any $\varepsilon>0$ by D_ε the closed ball of radius ε with center at the origin 0, and by S_ε the boundary of D_ε. For any semialgebraic subset M of R^n we denote by M_ε the intersection $M\cap S_\varepsilon$. For a semialgebraic subset N of S_ε we denote by $C(N)$ the cone

$$C(N):=\{tx|0\leqq t\leqq 1, x\in N\}$$

over N with vertex 0.

Theorem 6.5 (cf. [W] for $R=\mathbb{R}$). *Let $M_1,\ldots,M_r$ be semialgebraic subsets of R^n which all contain the point 0. Then there exists an element $\eta>0$ in R and for every $\varepsilon\in\,]0,\eta[$ a semialgebraic automorphism φ_ε of D_ε with the following properties:*

i) *φ_ε preserves the euclidean norm, $\|\varphi_\varepsilon(x)\|=\|x\|$ for every $x\in D_\varepsilon$.*

ii) *$\varphi_\varepsilon(x)=x$ for every $x\in S_\varepsilon$.*

iii) *φ_ε maps $M_j\cap D_\varepsilon$ onto $C(M_{j,\varepsilon})$ for $1\leqq j\leqq r$.*

Proof. We apply Theorem 6.4 to the semialgebraic map $f: R^n \to [0, \infty[$ defined by $f(x) = \|x\|$ and the semialgebraic sets $M_1, \ldots, M_r$ in R^n. We obtain a finite sequence $0 < \eta_1 < \eta_2 < \cdots < \eta_t$ in R such that the family $(R^n, M_1, \ldots, M_r)$ is semialgebraically trivial over every interval $]0, \eta_1[,]\eta_1, \eta_2[, \ldots,]\eta_t, \infty[$ with respect to f. In particular there exists for every $\varepsilon \in]0, \eta_1[$ a semialgebraic isomorphism

$$\psi_\varepsilon : D_\varepsilon \setminus \{0\} \xrightarrow{\sim}]0, \varepsilon] \times S_\varepsilon, \, x \mapsto (\|x\|, s_\varepsilon(x)),$$

with a retraction $s_\varepsilon : D_\varepsilon \setminus \{0\} \twoheadrightarrow S_\varepsilon$ which maps every set $M_j \cap (D_\varepsilon \setminus \{0\})$ onto $M_{j,\varepsilon}$. Composing ψ_ε with the semialgebraic isomorphism

$$]0, \varepsilon] \times S_\varepsilon \xrightarrow{\sim} D_\varepsilon \setminus \{0\}, (t, y) \mapsto \varepsilon^{-1} t y$$

we obtain a semialgebraic automorphism $\tilde{\varphi}_\varepsilon$ of $D_\varepsilon \setminus \{0\}$ which preserves the norm and is the identity on S_ε and maps $M_j \cap (D_\varepsilon \setminus \{0\})$ onto $C(M_{j,\varepsilon}) \setminus \{0\}$ for $1 \leqq j \leqq r$. This automorphism $\tilde{\varphi}_\varepsilon$ extends to a semialgebraic automorphism φ_ε of D_ε with $\varphi_\varepsilon(0) = 0$, and φ_ε has all the required properties.

Remark 6.6. Let M be a complete semialgebraic subset of R^n (complete for simplicity) and let $M_1, \ldots, M_r$ be closed semialgebraic subsets of M which all contain the origin 0. Then the local conic structure

$$(M \cap D_\varepsilon, M_1 \cap D_\varepsilon, \ldots, M_r \cap D_\varepsilon) \xrightarrow{\sim} (C(M_\varepsilon), C(M_{1,\varepsilon}), \ldots, C(M_{r,\varepsilon}))$$

stated in Theorem 6.5 can be identified with the local conic structure given by a suitable simultaneous triangulation $\psi : |K| \xrightarrow{\sim} M$ of $M, M_1, \ldots, M_r$ (cf. § 2) which has 0 as a vertex. Indeed, let $\mathring{D}_\varepsilon$ denote the interior $D_\varepsilon \setminus S_\varepsilon$ of D_ε. We can choose a triangulation

$$\varphi : |L| \xrightarrow{\sim} M \setminus \mathring{D}_\varepsilon$$

of $M \setminus \mathring{D}_\varepsilon$ which simultaneously triangulates all the sets $M_\varepsilon, M_j \setminus \mathring{D}_\varepsilon, M_{j,\varepsilon}$ $(1 \leqq j \leqq r)$. Adding the cones over all closed simplices of M_ε to this triangulation we obtain by use of Theorem 6.5 a simultaneous triangulation $\psi : |K| \xrightarrow{\sim} M$ of $M, M_1, \ldots, M_r$ such that the link of 0 in $M, M_1, \ldots, M_r$ is $M_\varepsilon, M_{1,\varepsilon}, \ldots, M_{r,\varepsilon}$ respectively and the simplicial neighbourhood of 0 is $M \cap D_\varepsilon, M_1 \cap D_\varepsilon, \ldots, M_r \cap D_\varepsilon$ respectively.

§ 7. Generalization of a theorem of Milnor

Theorem 7.1. *Let $f_1, \ldots, f_t$ be polynomials in n variables over a real closed field R, all of degree $\leqq d$. Let*

$$M := \mathscr{Z}(f_1, \ldots, f_t; R) = \{x \in R^n \mid f_i(x) = 0, 1 \leqq i \leqq t\}$$

denote the set of common zeros of these polynomials in R^n. Let further

$$h_q(M) := \dim H_q(M, K)$$

denote the q-th Betti number of M with respect to an arbitrarily chosen field K of coefficients. Then

$$\sum_{q \geqq 0} h_q(M) \leqq d(2d-1)^{n-1}$$

In particular M has at most $d(2d-1)^{n-1}$ connected components.

This theorem has been proved by Milnor [M] in the case $R=\mathbb{R}$. A similar theorem has been proved by Thom [T]. Both authors use Morse theory as their main tool. We now want to explain how it can be seen starting from the known case $R=\mathbb{R}$ that the theorem holds true over an arbitrary real closed field R.

As before let R_0 denote the real closure of the field $\mathbb{Q}$ of rational numbers, i.e. the algebraic closure of $\mathbb{Q}$ in $\mathbb{R}$. It is evident that the theorem holds true over R_0. Indeed, for any system $f_1, \ldots, f_t$ of polynomials in $R_0[X_1, \ldots, X_n]$ the semialgebraic space $\mathscr{X}(f_1, \ldots, f_t; R_0)$ yields the semialgebraic space $\mathscr{X}(f_1, \ldots, f_t; \mathbb{R})$ by base extension from R_0 to $\mathbb{R}$. Thus these two spaces have the same Betti numbers.

We fix natural numbers $n \geqq 1$, $d \geqq 1$, $t \geqq 1$. We introduce the vector space $P=P(n,d)$ over R_0 consisting of all polynomials in $R_0[X_1, \ldots, X_n]$ of degree $\leqq d$. We further introduce the algebraic subset

$$V=V(n,d,t):=\{(x, f_1, \ldots, f_t) \in R_0^n \times P^t \mid f_1(x)=\cdots=f_t(x)=0\}$$

of $R_0^n \times P^t$ and the natural projection

$$\pi: V \to P^t,\ (x, f_1, \ldots, f_t) \mapsto (f_1, \ldots, f_t).$$

According to Theorem 6.4 there exists a finite partition $(A_\alpha \mid \alpha \in I)$ of P^t into semialgebraic sets and commutative diagrams

$$\begin{array}{ccc} \pi^{-1}(A_\alpha) & \xrightarrow[\varphi_\alpha]{\sim} & A_\alpha \times F_\alpha \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & A_\alpha & \end{array}$$

with suitable affine semialgebraic spaces F_α over R_0 and semialgebraic isomorphisms φ_α. Subjecting all spaces and maps to base extension from R_0 to R we obtain the partition $(A_\alpha(R) \mid \alpha \in I)$ of $P(R)^t$ into semialgebraic sets and commutative diagrams

$$\begin{array}{ccc} \pi_R^{-1}(A_\alpha(R)) & \xrightarrow[\varphi_{\alpha,R}]{\sim} & A_\alpha(R) \times F_\alpha(R) \\ & {\scriptstyle\pi_R}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & A_\alpha(R) & \end{array}$$

with semialgebraic isomorphisms $\varphi_{\alpha,R}$. Notice that $P(R)$, $V(R)$ and π_R have the same meaning as P, V, π with R_0 replaced by R.

Let now $(f_1,\dots,f_t)$ be a given point in $P(R)^t$. Then

$$\pi_R^{-1}(f_1,\dots,f_t)=\mathscr{Z}(f_1,\dots,f_t;R)\times\{(f_1,\dots,f_t)\}\cong\mathscr{Z}(f_1,\dots,f_t;R).$$

Let α denote the index in I with $(f_1,\dots,f_t)\in A_\alpha(R)$. Choose any point $(g_1,\dots,g_t)\in A_\alpha$. The fibres of π_R at $(f_1,\dots,f_t)$ and $(g_1,\dots,g_t)$ are semialgebraically isomorphic. Thus

$$(7.2.)\qquad \mathscr{Z}(f_1,\dots,f_t;R)\cong\mathscr{Z}(g_1,\dots,g_t;R)=\mathscr{Z}(g_1,\dots,g_t;R_0)(R).$$

In particular, $\mathscr{Z}(f_1,\dots,f_t;R)$ has the same Betti numbers as $\mathscr{Z}(g_1,\dots,g_t;R_0)$, and thus the sum of these Betti numbers is $\leq d(2d-1)^{n-1}$.

§ 8. Local Euler characteristic and fundamental class

For any pair (M,A) of affine semialgebraic spaces over R we denote as usual by $\chi(M,A)$ the Euler characteristic of the pair with respect to the coefficient group $\mathbb{Z}$,

$$\chi(M,A):=\sum_{q\geq 0}(-1)^q\, rk\, H_q(M,A;\mathbb{Z}).$$

We are interested in the "local Euler characteristic" $\chi(M,M\setminus\{x\})$ of M at a point $x\in M$. If M is embedded into some R^n with $x=0$, and if $\varepsilon>0$ is chosen in R small enough such that $M\cap D_\varepsilon$ has the conic structure described in Theorem 6.5, then by well known arguments $(M_\varepsilon:=M\cap S_\varepsilon)$

$$(8.1)\qquad \chi(M,M\setminus\{x\})=1-\chi(M_\varepsilon).$$

Indeed, by the excision theorem 4.6

$$H_q(M,M\setminus\{x\})\cong H_q(M\cap D_\varepsilon,(M\cap D_\varepsilon)\setminus\{x\})\cong H_q(C(M_\varepsilon),C(M_\varepsilon)\setminus\{0\}),$$

and since $C(M_\varepsilon)$ is contractible

$$\chi(C(M_\varepsilon),C(M_\varepsilon)\setminus\{0\})=1-\chi(C(M_\varepsilon)\setminus\{0\})=1-\chi(M_\varepsilon).$$

In the same way we see that if x has a complete neighbourhood N in M and $\psi:|K|\xrightarrow{\sim}N$ is a triangulation of N then (cf. § 5, Def. 1)

$$(8.2)\qquad \chi(M,M\setminus\{x\})=1-\chi(Lk_K(\psi^{-1}(x))).$$

Theorem 8.3. *Let V be a quasiprojective variety over R. Then for every $x\in V(R)$ the local Euler characteristic $\chi(V(R),V(R)\setminus\{x\})$ is odd.*

In the case $R=\mathbb{R}$ this theorem is due to Sullivan [Su]. A complete proof has been given by Burghelea and Verona in [BV], § 4 and then another proof by Hardt [H_1]. All these authors consider more generally analytic sets and use transcendental tools.

Starting from the case $R=\mathbb{R}$ we now prove Theorem 8.3 in general. Again, it is already clear that the theorem holds true for $R=R_0$, since then base extension from R to $\mathbb{R}$ is possible. For arbitrary R we retreat by excision to the case (notations from § 7)

$$V(R)=\mathscr{Z}(f_1,\dots,f_t;R)\subset R^n$$

with some polynomials $f_1, \dots, f_t$ in $R[X_1, \dots, X_n]$. Then by (7. 2) there exist polynomials $g_1, \dots, g_t$ in $R_0[X_1, \dots, X_n]$ such that for the semialgebraic space $M := \mathscr{Z}(g_1, \dots, g_t; R_0)$ over R_0 we have a semialgebraic isomorphism $\alpha: V(R) \xrightarrow{\sim} M(R)$. For any point $x \in V(R)$

$$\chi(V(R), V(R) \setminus \{x\}) = \chi(M(R), M(R) \setminus \{\alpha(x)\}).$$

We now make use of a lemma which we shall prove below.

Lemma 8. 4. *Let N be an affine semialgebraic space over some real closed field R and let $\tilde{R}$ be some real closed field containing R. Then for every point $x \in N(\tilde{R})$ there exists a semialgebraic automorphism φ of $N(\tilde{R})$ such that $\varphi(x) \in N$.*

According to this lemma there exists a point $y \in M$ such that

$$\chi(M(R), M(R) \setminus \{\alpha(x)\}) = \chi(M(R), M(R) \setminus \{y\}) = \chi(M, M \setminus \{y\}),$$

which is an odd number.

The proof of Lemma 8. 4 is easy. Choosing a triangulation of N over R we may assume that N is a geometric simplicial complex in some space R^m. Let S be the open simplex of N such that $S(\tilde{R})$ contains x. Choose some point y in S. Then the closed simplex $\bar{S}(\tilde{R})$ over $\tilde{R}$ may be regarded as the cone over $(\partial \bar{S})(\tilde{R})$ with new vertex x and also as the cone over $(\partial \bar{S})(\tilde{R})$ with new vertex y. Thus we have an evident semialgebraic automorphism χ of $\bar{S}(\tilde{R})$ which maps x to y and keeps every point of $(\partial \bar{S})(\tilde{R})$ fixed. Extend χ by the identity to a semialgebraic automorphism ψ of $\bar{N}(\tilde{R})$. Then ψ maps $N(\tilde{R})$ to $N(\tilde{R})$ and $\varphi := \psi | N(\tilde{R})$ is an automorphism of $N(\tilde{R})$ as desired.

As already observed by Sullivan Theorem 8. 3 deserves particular interest in connection with the following purely combinatorial lemma.

Lemma 8. 5. *Let K be an arbitrary abstract simplicial complex and let K' denote the first barycentric subdivision of K. For any $k \leqq \dim K$ we introduce the chain*

$$S_k := \Sigma(\sigma_0 < \cdots < \sigma_k)$$

in the oriented abstract chain complex $C_k(K', Z)$ with $(\sigma_0 < \cdots < \sigma_k)$ running through all k-simplices of K' (cf. § 2, Def. 7 for notations). *Write*

$$\partial S_k = \Sigma a(\tau_0 < \cdots < \tau_{k-1})(\tau_0 < \cdots < \tau_{k-1})$$

with $(\tau_0 < \cdots < \tau_{k-1})$ running through the $(k-1)$-simplices of K'. Then for every such $(k-1)$-simplex

$$a(\tau_0 < \cdots < \tau_{k-1}) \equiv \chi(Lk_{K'}(\tau_{k-1})) \bmod 2.$$

The proof is a "pleasant combinatorial exercise" [Su] which may safely be left to the reader.

Let now V be any quasiprojective variety over R and let M be a connected component of $V(R)$. Assume that M is complete and has dimension n. Let $\psi: |K| \xrightarrow{\sim} M$ be a triangulation of M. According to (8. 2) and Theorem 8. 3 all the chains S_k given in Lemma 8. 5 yield cycles $\bar{S}_k$ modulo 2. Taking the image of the class $[\bar{S}_k] \in H_k(K', Z/2)$ under the natural isomorphism from $H_k(K', Z/2)$ to $H_k(M, Z/2)$ we obtain homology classes

$$S_k(M, \psi) \in H_k(M, Z/2).$$

Clearly

$$S_0(M, \psi) = \chi(M) \bmod 2.$$

We now have a closer look at the top cycle $\bar{S}_n \in C_n(K', \mathbb{Z}/2)$ for a given triangulation $\psi: |K| \xrightarrow{\sim} M$. Replacing V by the Zariski closure of M, if necessary, we assume that $\dim V = n$. Then the intersection $M \cap V_{\mathrm{reg}}(R)$ of M with the Zariski-open subset V_{reg} of regular points of V is not empty. We denote by $N_1, \ldots, N_r$ the connected components of $M \cap V_{\mathrm{reg}}(R)$ and by $\bar{N}_1, \ldots, \bar{N}_r$ the closures of $N_1, \ldots, N_r$ in M. Let $\partial \bar{N}_i$ be the boundary of $\bar{N}_i$. We assume furthermore that ψ is a simultaneous triangulation of M, $\bar{N}_1, \ldots, \bar{N}_r, \partial \bar{N}_1, \ldots, \partial \bar{N}_r$ which is good on all these spaces. Let K_i be the abstract core of $\bar{N}_i$ and L_i be the abstract core of $\partial \bar{N}_i$ $(1 \leqq i \leqq r)$. We identify M with $|K|$, $\bar{N}_i$ with $|K_i|$, $\partial \bar{N}_i$ with $|L_i|$ $(1 \leqq i \leqq r)$ via ψ and identify $H_n(K', \mathbb{Z}/2)$ with $H_n(M, \mathbb{Z}/2)$, $H_n(K_i', L_i'; \mathbb{Z}/2)$ with $H_n(\bar{N}_i, \partial \bar{N}_i; \mathbb{Z}/2)$.

Clearly $\bigcup_{i=1}^{r} \bar{N}_i$ is the union of all closed n-simplices of M. Let $\bar{T}$ be any chain in $C_n(K', \mathbb{Z}/2)$. We denote by $\bar{T}_i$ the sum of all n-simplices which appear in $\bar{T}$ and are contained in $\bar{N}_i$. Obviously we have

$$\bar{T} = \bar{T}_1 + \cdots + \bar{T}_r.$$

Assume now that $\bar{T}$ is a cycle. Then the boundary of $\bar{T}_i$ is contained in $\partial \bar{N}_i$. Thus we get natural homomorphisms

$$\alpha_i: H_n(M, \mathbb{Z}/2) \to H_n(\bar{N}_i, \partial \bar{N}_i; \mathbb{Z}/2)$$

mapping $[\bar{T}]$ to $[\bar{T}_i]$. We notice in passing that α_i is the composite of the natural map from $H_n(M, \mathbb{Z}/2)$ to $H_n(M, M \setminus N_i; \mathbb{Z}/2)$ with the excision isomorphism from $H_n(M, M \setminus N_i; \mathbb{Z}/2)$ to $H_n(\bar{N}_i, \partial \bar{N}_i; \mathbb{Z}/2)$ (cf. Remark 4. 7). We call α_i the "restriction map to $\bar{N}_i$".

We consider the homomorphism

$$\alpha = (\alpha_1, \ldots, \alpha_r): H_n(M, \mathbb{Z}/2) \to \bigoplus_{i=1}^{r} H_n(\bar{N}_i, \partial \bar{N}_i; \mathbb{Z}/2).$$

Since $\dim M = n$ the abstract simplicial complex K' does not contain any $(n+1)$-simplices. We conclude that every non zero cycle in $C_n(K', \mathbb{Z}/2)$ (resp. in $C_n(K_i', L_i'; \mathbb{Z}/2)$) yields a non zero element of $H_n(M, \mathbb{Z}/2)$ (resp. of $H_n(\bar{N}_i, \partial \bar{N}_i; \mathbb{Z}/2)$). From this remark it is evident that the map α is injective. To study the question whether α is surjective we need the following

Lemma 8. 6. *Let $\bar{T} \in C_n(K_i', L_i'; \mathbb{Z}/2)$ be a non zero cycle. Then $\bar{T}$ is the sum of all n-simplices of K' which are contained in $\bar{N}_i$, i.e. $\bar{T} = \bar{S}_{n,i}$.*

Proof. Let A_1 be the union of those closed simplices which appear in $\bar{T}$ and let A_2 be the union of those closed n-simplices which do not appear in $\bar{T}$. Assume that A_2 is not empty. Then $B_1 := A_1 \cap N_i$ and $B_2 := A_2 \cap N_i$ are both closed non empty semialgebraic subsets of N_i with $N_i = B_1 \cup B_2$. Since N_i is connected, the intersection $B_1 \cap B_2$ is not empty, hence contains an open simplex D of $|K'|$. The barycentric subdivision ψ' of ψ is good on N_i, therefore at least one vertex e of D lies in N_i and then also in $B_1 \cap B_2$. Let K'', K_i'', L_i'' be the second barycentric subdivision of K, K_i, L_i, i.e., the first barycentric subdivision of K', K_i', L_i'. The simplicial neighbourhood $N_{K''}(e)$ of e is completely contained in N_i. Since N_i is an n-dimensional semialgebraic manifold we conclude as in the proof of Corollary 5. 2 that the link $Lk_{K''}(e)$ of e with respect to K'' is homotopy equivalent to the unit sphere $S^{n-1}(R)$. From the long exact homology sequence of the pair $(N_{K''}(e), Lk_{K''}(e))$ we learn that

$$H_n(N_{K''}(e), Lk_{K''}(e); \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

Let $\bar{Z}_1$ be the sum of those n-simplices of $N_{K''}(e)$ which are contained in B_1, and let $\bar{Z}_2$ be the sum of those n-simplices of $N_{K''}(e)$ which are contained in B_2. Both chains are non zero. Since $\bar{Z}_1$ is the "local part" of the barycentric subdivision of $\bar{T}$ at e and since the boundary of $\bar{T}$ is contained in $\partial \bar{N}_i$, the boundary of $\bar{Z}_1$ must be contained in $Lk_{K''}(e)$, i.e., $\bar{Z}_1$ is a cycle in $C_n(N_{K''}(e), Lk_{K''}(e); \mathbb{Z}/2)$. $\bar{Z}_2$ is the "local complement" at e of $\bar{Z}_1$ in the barycentric subdivision of $\bar{S}_n$ which is a cycle in $C_n(K'', \mathbb{Z}/2)$, hence we conclude that also $\bar{Z}_2$ is a cycle. Since K'' does not contain $(n+1)$-simplices $\bar{Z}_1$ and $\bar{Z}_2$ yield two different non zero homology classes in $H_n(N_{K''}(e), Lk_{K''}(e); \mathbb{Z}/2)$ and we obtain a contradiction. Thus A_2 is empty, which means that $\bar{T}$ is the union of all closed n-simplices contained in $\bar{N}_i$.

Lemma 8. 6. implies that

$$H_n(\bar{N}_i, \partial \bar{N}_i; \mathbb{Z}/2) = \{0, [\bar{S}_{n,i}]\} \cong \mathbb{Z}/2.$$

Thus the homology class $S_n(M, \psi)$ in $H_n(M, \mathbb{Z}/2)$ has the property that its restriction to $\bar{N}_i$ generates $H_n(\bar{N}_i, \partial \bar{N}_i; \mathbb{Z}/2)$ for every $i \in \{1, \ldots, r\}$, and since α is injective, it is the only homology class in $H_n(M, \mathbb{Z}/2)$ with this property. In particular $S_n(M, \psi)$ does not depend on the chosen triangulation ψ. Henceforth we write $S_n(M)$ instead of $S_n(M, \psi)$ and call $S_n(M)$ the *fundamental class of M*.

Assume now that V is *normal*. Then the codimension of the complement of $M \cap V_{reg}(R)$ in M is at least 2. Thus $\partial \bar{N}_i$ has dimension at most $n-2$. This implies that every chain in $C_n(K', \mathbb{Z}/2)$ whose boundary is contained in $\bigcup_{i=1}^{r} \partial \bar{N}_i$ is in fact a cycle. In particular the natural map

$$H_n(\bar{N}_i, \mathbb{Z}/2) \to H_n(\bar{N}_i, \partial \bar{N}_i; \mathbb{Z}/2)$$

is an isomorphism. Now the natural maps from $H_n(\bar{N}_i, \mathbb{Z}/2)$ to $H_n(M, \mathbb{Z}/2)$ yield a section of α, and we see that α is bijective.

Altogether we have proved

Theorem 8.7. *Let V be a quasiprojective variety over a real closed field R. Let M be a connected component of $V(R)$ which is a complete semialgebraic space of dimension n. Assume that also* $\dim V = n$. *Let $N_1, \ldots, N_r$ denote the connected components of the set of regular points in M. We introduce the natural restriction map α_i from $H_n(M, \mathbb{Z}/2)$ to $H_n(\bar{N}_i, \partial \bar{N}_i; \mathbb{Z}/2)$, $1 \leqq i \leqq r$, and the homomorphism*

$$\alpha = (\alpha_1, \ldots, \alpha_r) : H_n(M, \mathbb{Z}/2) \to \bigoplus_{i=1}^{r} H_n(\bar{N}_i, \partial \bar{N}_i; \mathbb{Z}/2).$$

Then

$$H_n(\bar{N}_i, \partial \bar{N}_i; \mathbb{Z}/2) \cong \mathbb{Z}/2,$$

and the image $\alpha_i(S_n(M))$ of the fundamental class

$$S_n(M) \in H_n(M, \mathbb{Z}/2)$$

is the generator of $H_n(\bar{N}_i, \partial \bar{N}_i; \mathbb{Z}/2)$ $(1 \leqq i \leqq r)$. Moreover the map α is injective. If V is normal, then

$$H_n(\bar{N}_i, \partial \bar{N}_i; \mathbb{Z}/2) = H_n(\bar{N}_i, \mathbb{Z}/2)$$

and the map α is bijective.

If V is normal, then we call the generator of $H_n(\bar{N}_i, \mathbb{Z}/2)$ the *fundamental class* $S_n(\bar{N}_i)$ *of* $\bar{N}_i$. Identifying $H_n(M, \mathbb{Z}/2)$ with the direct sum of the $H_n(\bar{N}_i, \mathbb{Z}/2)$ via the canonical maps $H_n(\bar{N}_i, \mathbb{Z}/2) \to H_n(M, \mathbb{Z}/2)$ we have

$$S_n(M) = S_n(\bar{N}_1) + \cdots + S_n(\bar{N}_r).$$

We mention another property of the fundamental class.

Proposition 8.8. *Let V be a quasiprojective variety over a real closed field R. Let M be a connected component of $V(R)$ which is a complete semialgebraic space of dimension n. Then for all $x \in V_{reg}(R) \cap M$ the image of the fundamental class $S_n(M)$ of M under the natural homomorphism*

$$H_n(M, \mathbb{Z}/2) \to H_n(M, M \setminus \{x\}, \mathbb{Z}/2)$$

is a non zero generator of $H_n(M, M \setminus \{x\}, \mathbb{Z}/2)$.

Proof. We may assume that V is the Zariski-closure of M. We fix a point x in $V_{reg}(R) \cap M$. Let $\psi : |K| \xrightarrow{\sim} M$ be a triangulation as above. Assume that x is a vertex and that the simplicial neighbourhood $N_{K'}(x)$ of x is contained in $V_{reg}(R) \cap M$. $N_{K'}(x) \setminus \{x\}$ is homotopy equivalent to $Lk_{K'}(x)$. By excision we get

$$H_n(M, M \setminus \{x\}; \mathbb{Z}/2) \cong H_n(N_{K'}(x), N_{K'}(x) \setminus \{x\}; \mathbb{Z}/2) \cong H_n(N_{K'}(x), Lk_{K'}(x); \mathbb{Z}/2).$$

Using similar arguments as in the proof of Lemma 8.6 we see that

$$H_n(N_{K'}(x), Lk_{K'}(x); \mathbb{Z}/2) \cong \mathbb{Z}/2$$

and is generated by the sum of those n-simplices of K' which are contained in $N_{K'}(x)$. The assertion follows now immediately.

The role and the meaning of the homology classes $S_k(M, \psi)$ with $0 < k < n$, defined above, seems to be still mysterious even in the case $R = \mathbb{R}$. We do not know whether these classes are independent of the choice of the triangulation.

Added in proof. We now know that the semialgebraic space $V(R)$ of real points of any algebraic variety V over R (separated, of finite type) is affine, cf. § 2 of our forthcoming paper "Retractions and homotopy extension in semialgebraic spaces". Thus the standard assumption in the present paper, that all occurring varieties are quasi-projective, is not necessary.

References

[SGA 4, 3] *M. Artin, A. Grothendieck*, et al., Théorie des topos et cohomologie étale des schémas, Lecture Notes Math. **305**, Berlin-Heidelberg-New York 1973.

[BH] *A. Borel, A. Haefliger*, La classe fondamentale d'un espace analytique, Bull. Soc. Math. France **89** (1961), 461—513.

[B] *G. W. Brumfiel*, Partially ordered rings and semialgebraic geometry, Cambridge 1979.

[BV] *D. Burghelea, A. Verona*, Local homological properties of analytic sets, manusc. math. **7** (1972), 55—66.

[C] *P. J. Cohen*, Decision procedures for real and *p*-adic fields, Comm. Pure Appl. Math. **22** (1969), 131—151.

[CRC] *M. F. Coste-Roy, M. Coste*, Le spectre étale réel d'un anneau est spatial, C. R. Acad. Sci. Paris **290** (1980), 91—94.

[D] *H. Delfs*, Kohomologie affiner semialgebraischer Räume, Diss. Univ. Regensburg 1980.

[DK] *H. Delfs, M. Knebusch*, Semialgebraic topology over a real closed field. II, Basic theory of semi-algebraic spaces, Math. Z. **178** (1981), 175—213.

[DK_1] *H. Delfs, M. Knebusch*, Semialgebraic topology over a real closed field, Proc. Special Session on Real Algebraic Geometry, San Francisco 1981; to appear in Contemporary Mathematics, publ. Amer. Math. Soc.

[Do] *A. Dold*, Lectures on algebraic topology, Berlin-Heidelberg-New York 1972.

[G] *R. Godement*, Théorie des faisceaux, Paris 1964.

[H] *R. M. Hardt*, Semi-algebraic local triviality in semialgebraic mappings, Amer. J. Math. **102** (1980), 291—302.

[H_1] *R. M. Hardt*, Sullivan's local Euler characteristic theorem, manuscripta math. **12** (1974), 87—92.

[Hi] *H. Hironaka*, Triangulation of algebraic sets, Proc. Amer. Math. Soc. **29** (1975), 165—186.

[L] *S. Lojasiewicz*, Triangulation of semi-analytic sets, Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 449—474.

[M] *C. R. F. Maunder*, Algebraic Topology, London 1970.

[Mi] *J. Milnor*, On the Betti numbers of real varieties, Proc. Amer. Math. Soc. **15** (1964), 275—280.

[S] *E. H. Spanier*, Algebraic Topology, New York 1966.

[Su] *D. Sullivan*, Combinatorial invariants of analytic spaces, Proc. Liverpool Singularities Sympos. I, Lecture Notes Math. **192**, Berlin-Heidelberg-New York 1971, 165—168.

[T] *R. Thom*, Sur l'homologie des variétés algébriques réelles, Differential and Combinatorial topology (Sympos. Marston Morse, S. Chairns ed.), Princeton 1965.

[V] *A. N. Varčenko*, Theorem on the topological equisingularity of families of algebraic varieties and families of polynomial mappings, Izv. Akad. Nauk SSR Ser. Mat. **36** (1972), 957—1019, translated in: Math. USSR Izvestija **6** (1972), 949—1008.

[W] *A. H. Wallace*, Linear sections of algebraic varieties, Indiana Univ. Math. J. **20** (1971), 1153—1162.

Fakultät für Mathematik, Universität Regensburg, Universitätsstr. 31, D-8400 Regensburg

Eingegangen 8. Dezember 1981