REAL CLOSURES OF SEMILOCAL RINGS, 
AND EXTENSION OF REAL PLACES

BY MANFRED KNEBUSCH
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(A) All rings in this announcement are commutative and with 1. For any ring $K$ we denote by $W(K)$ the Witt ring of nondegenerate symmetric bilinear forms over $K$.

DEFINITION 1. A signature $\sigma$ of $K$ is a ring homomorphism from $W(K)$ to $\mathbb{Z}$.

REMARK 1. If $K$ is a field, the signatures correspond uniquely with the orderings of $K$ [3], [9]. Thus Theorem 1 below generalizes the main results of Artin-Schreier's theory of ordered fields [1].

We consider pairs $(K, \sigma)$ with $K$ a connected ring and $\sigma$ a signature of $K$. There is an obvious notion of a homomorphism $(K, \sigma) \to (L, \tau)$ between pairs. We say that a homomorphism $\alpha: K \to L$ of rings is a (connected) covering, if $\alpha$ is the inductive limit of finite etale connected extensions of $K$, as studied in Galois theory. We say that a homomorphism $\alpha:(K, \sigma) \to (L, \tau)$ is a covering, if $K \to L$ is a covering.

DEFINITION 2. A real closure of a pair $(K, \sigma)$ is a covering $\alpha:(K, \sigma) \to (R, \rho)$ such that $(R, \rho)$ does not admit any coverings except isomorphisms.

By Zorn's lemma any pair $(K, \sigma)$ has at least one real closure.

THEOREM 1. Assume $\alpha:(K, \sigma) \to (R, \rho)$ is a real closure of a pair $(K, \sigma)$ with $K$ semilocal. Let $K_s$ denote the universal covering (= separable closure) of $K$.

(1) For any other real closure $\alpha':(K, \sigma) \to (R', \rho')$ there exists an isomorphism $\beta:(R, \rho) \cong (R', \rho')$ with $\alpha' = \beta \circ \alpha$.

(2) There does not exist any automorphism of $(R, \rho)$ leaving all elements of $K$ fixed except the identity.

(3) The Galois group of $K_s/R$ is a 2-group.

Assume in addition that 2 is a unit in $A$. Then even the following statements are true:

(3a) $K_s = R(\sqrt{-1})$.

(4) If $R'/K$ is any covering such that $[K_s:R'] = 2$, then $K_s = R'(\sqrt{-1})$ and $W(R') \cong \mathbb{Z}$. In particular $R'$ has a unique signature.

Thus if $K$ is semilocal with 2 a unit the signatures of $K$ correspond uniquely with the conjugacy classes of involutions in the Galois group of $K$.

REMARK 2. If $K$ is a Dedekind domain at least statement (1) of Theorem 1 remains true and $[K_s:R] \leq 2$.

The proofs of part (2) and (3) of Theorem 1 are essentially contained in
[7] (cf. [7a]). Also the proofs of the other parts and of Remark 2 depend
on this paper and on [5], [6]. The proof of (3a) and (4) proceeds by imitation
of a classical proof of the fundamental theorem of algebra. The main
point in the proof of (1) is to prove simultaneously the following trace
formula:

**THEOREM 2.** Let \( L \) be a finite covering of the semilocal ring \( K \) and let
\( \text{Tr}^*: W(L) \to W(K) \) denote the transfer map induced by the regular trace
\( \text{Tr} = \text{Tr}_{L/K} [11] \). Then, for any signature \( \sigma \) of \( K \) and any \( z \) in \( W(L) \),
\[
\sigma(\text{Tr}^*(z)) = \sum_{\tau|\sigma} \tau(z),
\]
where \( \tau \) runs through all signatures of \( L \) lying over \( \sigma \), with the convention
that the sum is zero if there are no such \( \tau \).

In [4], a proof of Theorem 1(1) and Theorem 2 over fields has been given
which, with the knowledge of the other parts of Theorem 1, immediately
generalizes to semilocal rings with 2 a unit. This proof also gives a good
idea of the techniques needed for the general case. Of course, no state­
ments about zeros of real polynomials are used (e.g., Sturm’s theorem).
The connection between Burnside and Witt rings, studied in [2], gives,
in the case that 2 is a unit, another approach to Theorem 1(1) [Dress, oral
communication].

For any ring \( A \), let \( 2^{-\infty}A \) denote the localization with respect to the
multiplicative system of powers of 2. Theorems 1 and 2 imply

**THEOREM 3.** Assume that \( K \to L \) is a finite covering of semilocal rings
and \( K \to K' \) is a homomorphism into a semilocal ring \( K' \).
(i) \( 2^{-\infty}W(L) \) is finite etale over \( 2^{-\infty}W(K) \) and is generated as module
over \( 2^{-\infty}W(K) \) by at most \([L:K]\) elements.
(ii) The kernel and cokernel of the natural map
\[
W(L) \otimes_{W(K)} W(K') \to W(L \otimes_K K')
\]
are 2-torsion groups.

(B) For arbitrary valuation rings (at least) it is also possible to study by
the same methods the behavior of certain signatures in “ramified cover­
ings”. This leads to results about real places of fields. From now on \( R \)
denotes a fixed real closed field and \( K \) denotes a field of characteristic
zero.

**DEFINITION 3.** A signature \( \sigma \) of \( K \) and a place \( \phi: K \to R \cup \infty \) are
compatible, if \( \phi(a) \geq 0 \) or \( \phi(a) = \infty \) for all \( a \) in \( K \) which are positive with
respect to \( \sigma \) (cf. Remark 1).

One easily proves that for any \( R \)-valued place \( \phi \) of \( K \) there exists at
least one signature of \( K \) which is compatible with \( \phi \).
THEOREM 4. Assume that $L$ is an algebraic field extension of $K$, that $\tau$ is a signature of $L$ and $\phi$ is an $R$-valued place of $K$, compatible with the restriction $\tau|K$ of $\tau$ to $K$. Then there exists a unique $R$-valued place $\psi$ of $L$ extending $\phi$ and compatible with $\tau$.

This theorem refines a result of Lang [8, Theorem 6] which says that any $R$-valued place of $K$ can be extended to an $R$-valued place of at least one real closure of $K$.

For any $a$ in $K^*$ we denote as usual by $(a)$ the element of $W(K)$ represented by the one-dimensional symmetric bilinear form with matrix $(a)$. It can be shown by well-known arguments (e.g., [10, Chapter V, proof of Lemma 1.2]) that any place $\phi: K \to R \cup \infty$ yields a well-defined additive map $\phi_*$ from $W(K)$ to $Z$, whose value on an element $(a)$ is obtained in the following way: Let $\rho$ denote the unique signature of $R$. If there is some $b$ in $K$ such that $\phi(ab^2) \neq 0$ and $\neq \infty$ then $\phi_*(a) = \rho(\phi(ab^2))$ with an arbitrary choice of such an element $b$. Otherwise $\phi_*(a) = 0$. Clearly a signature $\sigma$ of $K$ is compatible with $\phi$ if and only if $\sigma(a) = \sigma_*(a)$ for all $a$ in $K$ with $\phi_*(a) \neq 0$.

As a counterpart of Theorem 4 we have

THEOREM 5. Assume that $L$ is an arbitrary field extension of $K$, that $\psi$ is an $R$-valued place of $L$ and $\sigma$ is a signature of $K$ compatible with $\psi|K$. There exists a signature $\tau$ of $L$ compatible with $\psi$ and extending $\sigma$ (i.e., $(K, \sigma) \to (L, \tau)$ is a homomorphism) if and only if $\sigma(a) = \psi_*(a)$ for all $a$ in $K$ with $\psi_*(a) \neq 0$.

THEOREM 6. Assume $L_1$ is an algebraic field extension of $K$ and $L_2$ is an arbitrary field extension of $K$. Further assume that, on each $L_1$, an $R$-valued place $\phi_1$ is given and $\phi_1|K = \phi_2|K$. Then the following are equivalent:

(i) There exists a field composite $F$ of $L_1$ and $L_2$ over $K$ and an $R$-valued place $\psi$ on $F$ extending both $\phi_1$ and $\phi_2$.
(ii) $\phi_1(a) = \phi_2(a)$ for all $a$ in $K^*$ such that both $\phi_1(a)$ and $\phi_2(a)$ are not zero.

THEOREM 7. Let $L$ be a finite algebraic field extension of $K$ and $\phi$ an $R$-valued place of $K$. Then, for any $z$ in $W(L)$,

$$\phi_*(\text{Tr}_{L_1/K}(z)) = \sum_{\psi|\phi} \psi_*(z),$$

where $\psi$ runs through all $R$-valued places of $L$ extending $\phi$.

Thus $\phi$ has $\phi_*(\text{Tr}_{L_1/K}(1))$ $R$-valued extensions to $L$.

Detailed proofs will appear elsewhere. I wish to thank A. Dress and A. Rosenberg for very helpful discussions.
REFERENCES


MATHEMATISCHE INSTITUT, UNIVERSITÄT DES SAARLANDES, D-66 SAARBRÜCKEN IM STADTwald, FEDERAL REPUBLIC OF GERMANY