

# **On the Extension of Real Places**

MANFRED KNEBUSCH

## On the Extension of Real Places<sup>1)</sup>

MANFRED KNEBUSCH

### Introduction

About twenty years ago S. Lang studied places  $\varphi: K \rightarrow R \cup \infty$  on a field  $K$  with values in a fixed real closed field  $R$  ([L]). One of his main results was the theorem, that any such place  $\varphi$  can be extended to an  $R$ -valued place on a suitable real closure of  $K$  ([L], Th. 6). Now the real closures of  $K$  correspond up to  $K$ -isomorphisms uniquely to the (total) orderings of  $K$ . Thus one may ask whether it is possible to obtain a more precise version of Lang's theorem by a more thorough analysis of the relations between orderings and real places. This question is the starting point of the present paper.

We say that an ordering  $\alpha$  of  $K$  lies over the place  $\varphi: K \rightarrow R \cup \infty$  or that  $\varphi$  and  $\alpha$  are compatible, if any element  $a$  of  $K$  which is positive with respect to  $\alpha$  has value  $\varphi(a) = \infty$  or  $\varphi(a) \geq 0$ . (Recall that  $R$  is ordered in a unique way). In §1 we first show that over any real place  $\varphi$  lies at least one ordering  $\alpha$ . Then we prove the following refinement of Lang's theorem:

**THEOREM 1.6.** *Assume that  $L$  is an algebraic field extension of  $K$ , that  $\beta$  is an ordering on  $L$  and that  $\varphi$  is an  $R$ -valued place on  $K$ , compatible with the restriction of  $\beta$  to  $K$ . Then there exists a unique  $R$ -valued place  $\psi$  on  $L$  extending  $\varphi$  and compatible with  $\beta$ .*

Harrison ([H]), and Leicht, Lorenz ([LL]) showed that the orderings  $\alpha$  of a field  $K$  correspond uniquely to the signatures  $\sigma$  of  $K$ , i.e. the ring homomorphisms  $\sigma: W(K) \rightarrow \mathbb{Z}$ , where  $W(K)$  denotes the Witt ring of non singular symmetric bilinear forms over  $K$  ([W]). As usual we denote for any  $a \neq 0$  in  $K$  by  $(a)$  the element of  $W(K)$  represented by the form  $B: K \times K \rightarrow K$ ,  $B(x, y) = axy$ . The signature  $\sigma$  corresponding to the ordering  $\alpha$  is characterized by  $\sigma(a) = +1$  if  $a > 0$  with respect to  $\alpha$ , and  $\sigma(a) = -1$  if  $a < 0$ . (Recall that  $W(K)$  is generated by the elements  $(a)$ .) We shall make strong use of this connection between orderings and Witt rings, and we shall always identify an ordering  $\alpha$  with the corresponding signature  $\sigma$ . The unique signature  $W(R) \rightarrow \mathbb{Z}$  will be denoted by  $\varrho$ .

As will be explained in §2, any  $R$ -valued place  $\varphi$  on  $K$  yields a well defined additive map  $\varphi_*: W(K) \rightarrow \mathbb{Z}$ , whose value on an element  $(a)$  is obtained in the following way: If the square class  $aK^{*2}$  contains an element  $a' = ab^2$  such that  $\varphi(a') \neq 0$  and  $\neq \infty$ , then  $\varphi_*(a) = \varrho(\varphi(a'))$  with an arbitrary choice of  $a'$ . If  $aK^{*2}$  contains no such elements,

<sup>1)</sup> The main results of this paper have been announced in [K<sub>2</sub>, part B].

then  $\varphi_*(a)=0$ . Obviously a signature  $\sigma$  lies over  $\varphi$  if and only if  $\sigma(a)=\varphi_*(a)$  for all  $a$  in  $K^*$  such that  $\varphi_*(a)\neq 0$ .

We prove in §2 the following counterpart of the theorem above:

**THEOREM 2.6.** *Assume that  $L$  is an arbitrary field extension of  $K$ , that  $\psi$  is an  $R$ -valued place on  $L$  and that  $\sigma$  is a signature of  $K$  lying over  $\psi \mid K$ . There exists a signature  $\tau$  of  $L$  lying over  $\psi$  and extending  $\sigma$  {i.e.  $\tau(a)=\sigma(a)$  for all  $a$  in  $K^*$ } if and only if  $\sigma(a)=\psi_*(a)$  for all  $a$  in  $K^*$  with  $\psi_*(a)\neq 0$ .*

We further prove in §2 a theorem about the real places in the field composites of an algebraic extension  $L_1/K$  and an arbitrary extension  $L_2/K$ .

Our work in §3 originates from the question, how many  $R$ -valued extensions has a given  $R$ -valued place  $\varphi$  of  $K$  in a finite field extension  $L/K$ . Recall that the regular trace  $\text{Tr}_{L/K}$  induces an additive map  $\text{Tr}_{L/K}^*: W(L) \rightarrow W(K)$  mapping the class of a symmetric bilinear space  $(E, B)$  over  $L$  to the class of the space  $(E, \text{Tr}_{L/K} \circ B)$  over  $K$  (cf [S]). We prove the following trace formula: For any  $x$  in  $W(L)$

$$\varphi_*(\text{Tr}_{L/K}^*(x)) = \sum_{\psi \mid \varphi} \psi_*(x)$$

where  $\psi$  runs through all  $R$ -valued places of  $L$  extending  $\varphi$ , with the convention that the right hand side is zero if there are no such places  $\psi$ . Applying this formula to the unit element (1) of  $W(L)$  one obtains that  $\varphi$  has exactly  $\varphi_*(\text{Tr}_{L/K}^*(1))$   $R$ -valued extensions to  $L$ .

The final section 4 gives an application of the theorem 1.6 cited above to the problem of extending an  $R$ -valued place  $\varphi$  on  $K$  to a field  $L$  which is finitely generated over  $K$  but not necessarily algebraic.

To prevent misunderstandings I remark that in this paper different places with the same valuation ring are never identified and that a place is allowed to be trivial, i.e. to avoid the value  $\infty$ .

## §1

We first recall some well known facts and notations (cf [L], [AS]). Assume that on a field  $K$  an ordering  $\sigma$  is given and that  $k$  is a subfield of  $K$ . An element  $a$  of  $K$  is called infinitely large over  $k$  (with respect to  $\sigma$ ) if there is no element  $c>0$  in  $k$  such that  $|a|<c$ . Here  $|a|$  denotes the element  $a$  if  $a\geq 0$  and  $-a$  if  $a\leq 0$  with respect to  $\sigma$ . The set of elements of  $K$ , which are not infinitely large over  $k$ , is a valuation ring of  $K$  ([AS], p. 95) which we call *the valuation ring*  $\mathfrak{o}=\mathfrak{o}(K/k, \sigma)$  *associated with*  $\sigma$  *over*  $k$ . Obviously the maximal ideal  $\mathfrak{m}$  of  $\mathfrak{o}$  is the set of all elements  $a$  in  $K$  which are infinitely small over  $k$ , i.e.  $|a|<c$  for all  $c>0$  in  $k$ .

Clearly  $k\subset \mathfrak{o}$ . A field  $k'$  with  $k\subset k'\subset K$  is called *archimedean over*  $k$  (with respect

to  $\sigma$ ) if  $k' \subset \mathfrak{o}$ . Then  $\mathfrak{o}$  is also the valuation ring associated with  $\sigma$  over  $k'$ . It is clear from general valuation theory that any algebraic extension  $k'$  of  $k$  in  $K$  is archimedean over  $k$ .

As follows from Zorn's lemma, there exists at least one intermediate field  $\tilde{k} \supset k$  which is *maximal* archimedean over  $k$ , i.e.  $\tilde{k}$  is archimedean over  $k$ , but no field  $k' \supset \tilde{k}$  different from  $\tilde{k}$  is archimedean over  $k$  ([L], p. 379). We say that  $k$  is *maximal archimedean* in  $K$ , if  $k = \tilde{k}$ . It is clear from general valuation theory that the field  $\mathfrak{o}/\mathfrak{m}$  is always algebraic over  $\tilde{k}$ .

The ordering  $\sigma$  of  $K$  induces an ordering  $\bar{\sigma}$  of  $\mathfrak{o}/\mathfrak{m}$ , characterized in the following way ([AS], p. 95): An element  $\bar{a}$  of  $\mathfrak{o}/\mathfrak{m}$  is positive if and only if a preimage  $a$  in  $\mathfrak{o}$  is positive. (It does not matter which preimage is chosen.) If  $K$  is real closed then  $\tilde{k}$  is real closed, since  $\tilde{k}$  is algebraically closed in  $K$ . Thus in this case  $\tilde{k}$  maps bijectively onto  $\mathfrak{o}/\mathfrak{m}$  ([AS], p. 95).

All rings in this paper are commutative and have a unit element and all ring homomorphisms map 1 to 1. The unit group of a ring  $A$  is denoted by  $A^*$ . We further denote by  $W(A)$  the Witt ring of non degenerate symmetric bilinear forms over  $A$ , and for any homomorphism  $\alpha: A \rightarrow C$  into a ring  $C$  we denote by  $W(\alpha)$  the corresponding ring homomorphism from  $W(A)$  to  $W(C)$ . We refer the reader to [K], [KRW, §1], or [M] for these notions. For any element  $a$  in  $A$  we denote by  $(a)$  the element of  $W(A)$  which is represented by the form  $B: A \times A \rightarrow A$ ,  $B(x, y) = axy$ . These elements  $(a)$  form a subgroup  $Q(A)$  of  $W(A)$ , which will be identified with the group  $A^*/A^{*2}$  of square classes. If  $A$  is local, i.e.  $A$  has only one maximal ideal, the ring  $W(A)$  is generated by  $Q(A)$  (e.g. [KRW, §1]). In this paper only the Witt rings of fields and valuation rings will play a rôle.

As explained in the introduction, the signatures  $\sigma$  of a field  $K$ , i.e. the homomorphisms  $\sigma$  from the ring  $W(K)$  to  $\mathbb{Z}$ , correspond uniquely to the orderings of  $K$ . Let  $L$  be a field extension of  $K$  and  $i$  denote the inclusion map from  $K$  into  $L$ . For any signature  $\tau$  of  $L$  we denote by  $\tau|_K$  the signature  $\sigma = \tau \circ W(i)$  of  $K$ , and we say that  $\sigma$  is the *restriction of  $\tau$  to  $K$* , or that  $\tau$  is an *extension of  $\sigma$  to  $L$* . This terminology is compatible with the usual meaning of extension and restriction of orderings.

Throughout this paper  $R$  denotes a real closed field and  $\varrho$  denotes the signature of  $R$ . For a moment we forget about orderings of the field  $K$  and consider a place  $\varphi: K \rightarrow R \cup \infty$ . Let  $\mathfrak{o}$  denote the valuation ring of  $\varphi$ , i.e. the ring of all elements  $x$  in  $K$  with  $\varphi(x) \neq \infty$ . By composing the map  $W(\varphi|_{\mathfrak{o}})$  from  $W(\mathfrak{o})$  to  $W(R)$  with  $\varrho: W(R) \rightarrow \mathbb{Z}$  we obtain a ring homomorphism from  $W(\mathfrak{o})$  to  $\mathbb{Z}$  which we denote by  $\hat{\varphi}$ . On a generator  $(a)$  of  $W(\mathfrak{o})$ ,  $a$  in  $\mathfrak{o}^*$ , the map takes the value 1 if  $\varphi(a) > 0$  and  $-1$  if  $\varphi(a) < 0$ .

Since  $\mathfrak{o}$  is a Prüfer ring, the map  $W(i): W(\mathfrak{o}) \rightarrow W(K)$ , obtained from the inclusion map  $i: \mathfrak{o} \rightarrow K$  is injective ([K, Satz 11.1.1]; the reader may also consult [KRW<sub>1</sub>, Lemma 1.1] or [M, p. 93], where this fact is stated for Dedekind rings but proved for Prüfer rings). We shall always consider  $W(\mathfrak{o})$  as a subring of  $W(K)$ .

We say that a signature  $\sigma$  of  $K$  lies over  $\varphi$ , or that  $\varphi$  is compatible with  $\sigma$ , if  $\sigma$  extends  $\hat{\varphi}$ . Obviously this definition coincides with the definition given in the introduction.

**PROPOSITION 1.1.** [KRW<sub>2</sub>, 1.13.] *Over any place  $\varphi: K \rightarrow R \cup \infty$  lies at least one signature  $\sigma$  of  $K$ .*

Since this fact is central for the present work, we recall the proof given in [KRW<sub>2</sub>]: The kernel  $P$  of  $\hat{\varphi}$  is a minimal prime ideal of  $W(\mathfrak{o})$  [KRW]. Thus there exists at least one prime ideal  $Q$  of  $W(K)$  lying over  $P$  [B, Chap. II, §2, no. 6, Prop. 16]. Since  $W(\mathfrak{o})/P \cong \mathbb{Z}$  embeds into  $W(K)/Q$  we must have  $W(K)/Q \cong \mathbb{Z}$  ([LL], [H]). The only homomorphism  $\mathfrak{o}: W(K) \rightarrow \mathbb{Z}$  with kernel  $Q$  is the desired signature.

**LEMMA 1.2.** (cf [L], p. 382) *Let  $\varphi$  be an  $R$ -valued place on  $K$  and  $\sigma$  be a signature of  $K$  lying over  $\varphi$ . Assume further that  $a$  and  $b$  are elements of  $K$  and  $\varphi(a) \neq \infty$ . Then with respect to the orderings corresponding to  $\sigma$  and  $\varphi$  the following are true:*

- (i)  $b > a$  implies  $\varphi(b) = \infty$  or  $\varphi(b) \geq \varphi(a)$ .
- (ii)  $0 < b < a$  implies  $\varphi(b) \neq \infty$  and  $0 \leq \varphi(b) \leq \varphi(a)$ .

*Proof.* (i)  $b - a > 0$  implies  $\varphi(b - a) = \infty$  or  $\varphi(b - a) \geq 0$  and thus  $\varphi(b) = \infty$  or  $\varphi(b) \geq \varphi(a)$ .

(ii) This is clear if  $\varphi(b) = 0$ . Assume now  $\varphi(b) \neq 0$ . Then  $\varphi(ab^{-1}) = \varphi(a) \varphi(b^{-1}) \neq \infty$  and we obtain from  $ab^{-1} > 1$  and (i) that  $\varphi(a) \varphi(b^{-1}) \geq 1$ . Thus certainly  $\varphi(b) \neq \infty$ , and we obtain from  $b > 0$  that  $\varphi(b) > 0$ . Thus  $\varphi(a) \geq \varphi(b)$ . q.e.d.

**PROPOSITION 1.3.** (cf. [L], Th. 5). *Let  $\varphi$  be an  $R$ -valued place on  $K$  and  $k$  be a subfield of  $K$  on which  $\varphi$  is trivial. Assume that  $R$  is archimedean over  $\varphi(k)$ . Then for a signature  $\tau$  on  $K$  the following are equivalent:*

- (i)  $\tau$  lies over  $\varphi$ .
- (ii) *The valuation ring  $\mathfrak{o}$  of  $\varphi$  coincides with the valuation ring  $\mathfrak{o}(K/k, \tau)$  of  $\tau$  over  $k$ . The homomorphism  $\bar{\varphi}: \mathfrak{o}/\mathfrak{m} \rightarrow R$  induced by  $\varphi$  on the residue class field of  $\sigma$  is order preserving with respect to the ordering  $\bar{\tau}$ , induced by  $\tau$  on  $\mathfrak{o}/\mathfrak{m}$ , and  $\varrho$ .*

**PROOF.** a) Let  $\mathfrak{o}'$  denote the ring  $\mathfrak{o}(K/k, \tau)$  and  $\mathfrak{m}'$  denote the maximal ideal of  $\mathfrak{o}'$ . We first assume  $\mathfrak{o}' = \mathfrak{o}$  and analyze the situation in this case. That  $\bar{\varphi}$  is order preserving with respect to  $\bar{\tau}$  and  $\varrho$  means in the language of quadratic forms that the map  $\varrho \circ W(\bar{\varphi})$  from  $W(\mathfrak{o}/\mathfrak{m})$  to  $\mathbb{Z}$  coincides with  $\bar{\tau}$ . We denote the canonical map from  $\mathfrak{o}$  onto  $\mathfrak{o}/\mathfrak{m}$  by  $\alpha$ . Clearly  $\tau \mid W(\mathfrak{o}) = \bar{\tau} \circ W(\alpha)$ . Now  $W(\alpha)$  is surjective, since all generators  $(\bar{a})$ ,  $\bar{a}$  in  $(\mathfrak{o}/\mathfrak{m})^*$ , can be lifted to elements  $(a)$  in  $W(\mathfrak{o})$ . Therefore the equation  $\bar{\tau} = \varrho \circ W(\bar{\varphi})$  is equivalent to  $\tau \mid W(\mathfrak{o}) = \varrho \circ W(\varphi \mid \mathfrak{o}) = \hat{\varphi}$ . Thus (ii)  $\Rightarrow$  (i) is clear, and to prove (i)  $\Rightarrow$  (ii) it only remains to be shown that if  $\tau$  lies over  $\varphi$  the rings  $\mathfrak{o}'$  and  $\mathfrak{o}$  coincide.

b) Assume that  $\tau$  lies over  $\varphi$ . We first show that  $\mathfrak{m} \subset \mathfrak{m}'$ . Let  $a$  be an element of  $K$

which is not in  $\mathfrak{m}'$  and let  $b = |a|$  with respect to  $\tau$ . There exists some  $c > 0$  in  $k$  with  $b > c$ . By Lemma 1.2(i) we obtain  $\varphi(b) = \infty$  or  $\varphi(b) \geq \varphi(c) \geq 0$ . Since  $\varphi(c) \neq 0$ , certainly  $\varphi(b) \neq 0$  and thus  $\varphi(a) \neq 0$ , i.e.  $a$  lies not in  $\mathfrak{m}$ . This proves  $\mathfrak{m} \subset \mathfrak{m}'$ .

Now we show  $\mathfrak{m}' \subset \mathfrak{m}$ . Then  $\mathfrak{o} = \mathfrak{o}'$  will be clear. Assume  $a$  is an element of  $\mathfrak{m}'$  and without loss of generality  $a > 0$ . For any  $c > 0$  in  $k$  we have  $0 < a < c$  and thus by Lemma 1.2(ii)  $0 \leq \varphi(a) \leq \varphi(c)$ . Since  $R$  is archimedean over  $\varphi(k)$  the value  $\varphi(a)$  must be zero i.e.  $a$  lies in  $\mathfrak{m}$ . q.e.d.

**EXAMPLE 1.4.** If  $k$  is a maximal subfield of  $K$  such that a given place  $\varphi: K \rightarrow R \cup \infty$  is trivial on  $k$  then for any signature  $\tau$  of  $K$  the conditions (i) and (ii) of Proposition 1.3 are equivalent. In fact, all values of  $\varphi$  lie in the algebraic closure  $R'$  of  $\varphi(k)$  in  $R$ , which is archimedean over  $\varphi(k)$ . Replace  $R$  by  $R'$ !

**COROLLARY 1.5.** Let  $\sigma$  be a signature on a field  $K$  and  $k$  be a subfield of  $K$ , whose algebraic closure  $k'$  in  $K$  is maximal archimedean with respect to  $\sigma$ . Further assume that  $\chi: k \rightarrow R$  is an order preserving homomorphism with respect to  $\sigma|_k$  and  $\varrho$ . Then there exists a unique place  $\varphi: K \rightarrow R \cup \infty$  with the following properties:

- (i)  $\varphi$  is compatible with  $\sigma$ ,
- (ii)  $\varphi|_k = \chi$ ,
- (iii)  $\varphi$  is zero dimensional over  $k$ , i.e. all values  $\neq \infty$  of  $\varphi$  are algebraic over  $\chi(k)$ .

*Proof.* We replace  $R$  by the algebraic closure of  $\varphi(k)$  in  $R$  and then forget condition (iii). According to Proposition 1.3 the valuation ring of the place  $\varphi$  to be constructed must coincide with  $\mathfrak{o} = \mathfrak{o}(K/k, \tau)$ . Now, by the assumption about  $k'$ , the residue class ring  $\mathfrak{o}/\mathfrak{m}$  of  $\mathfrak{o}$  is algebraic over  $k$ . Thus it follows from a well known theorem of Artin and Schreier [AS, Satz 8] that there exists a unique order preserving homomorphism  $\beta: \mathfrak{o}/\mathfrak{m} \rightarrow R$  with respect to  $\bar{\tau}$  and  $\varrho$  which extends  $\chi$  (cf [K<sub>1</sub>, Cor. 5.1], where this is proved by similar methods as are used in the present paper). By Proposition 1.3 the place which has the valuation ring  $\mathfrak{o}$  and induces on  $\mathfrak{o}/\mathfrak{m}$  the map  $\beta$  fulfills the conditions (i) and (ii) and is the only  $R$ -valued place with these properties. q.e.d.

From Proposition 1.3 and Corollary 1.5 we obtain

**THEOREM 1.6.** (cf [L], Th. 6). Assume that  $L$  is an algebraic field extension of  $K$ , that  $\tau$  is a signature of  $L$  and that  $\varphi$  is an  $R$ -valued place of  $K$ , compatible with  $\tau|_K$ . Then there exists a unique  $R$ -valued place  $\psi$  of  $L$  extending  $\varphi$  and compatible with  $\tau$ .

*Proof.* We chose a maximal subfield  $k$  of  $K$  on which  $\varphi$  is trivial. Then  $\varphi$  is zero dimensional over  $k$ . By Example 1.4 the valuation ring  $\mathfrak{o}$  of  $\varphi$  coincides with the valuation ring  $\mathfrak{o}(K/k, \sigma)$  associated with the restriction  $\sigma$  of  $\tau$  to  $K$ . Since  $L/K$  is algebraic, the residue class ring of  $\mathfrak{o}(L/k, \tau)$  is algebraic over the residue class ring of  $\mathfrak{o}(K/k, \sigma)$ . Thus the algebraic closure  $k'$  of  $k$  in  $L$  is maximal archimedean with respect to  $\tau$ . Now we can apply Cor. 1.5 to  $\sigma$  and the homomorphism  $\chi = \varphi|_k$  from  $k$  to  $R$  and also to

$\tau$  and  $\chi$ . Clearly  $\varphi$  is the place of  $K$  corresponding to  $\sigma$  and  $\chi$  in the sense of Cor. 1.5. We denote by  $\psi$  the place of  $L$  corresponding to  $\tau$  and  $\chi$ . Since  $\psi \mid K$  must correspond to  $\sigma$  and  $\chi$  we have  $\psi \mid K = \varphi$ . On the other hand any  $R$ -valued extension  $\psi'$  of  $\varphi$  to  $L$  is zero dimensional over  $k$ . Thus if  $\psi'$  is compatible with  $\tau$ , Cor. 1.5 yields  $\psi' = \psi$ .

## §2

We want to study the orderings of a field  $K$  which lie over a given  $R$ -valued place of  $K$ . For this purpose we first consider more generally an arbitrary valuation ring  $\mathfrak{o}$  with maximal ideal  $\mathfrak{m}$ , residue class field  $k = \mathfrak{o}/\mathfrak{m}$ , and quotient field  $K$ . For any  $a$  in  $\mathfrak{o}$  we denote by  $\bar{a}$  the image in  $k$ . The following proposition has been proved in this generality in [K<sub>3</sub>, §3] (cf [Sp], [M, Chap. V] if  $\mathfrak{o}$  is discrete, and [K, §12] if  $\mathfrak{o}$  has rank one).

**PROPOSITION 2.1.** *There exists a unique additive map  $\partial: W(K) \rightarrow W(k)$  such that  $\partial(a) = (\bar{a})$  for every  $a$  in  $\mathfrak{o}^*$  and  $\partial(a) = 0$  for every  $(a)$  in  $Q(K)$  which lies not in  $Q(\mathfrak{o})$ , i.e. with  $aK^{*2} \cap \mathfrak{o}^*$  empty.<sup>2</sup>*

Let  $v: K^* \rightarrow \Gamma$  denote a valuation corresponding to  $\mathfrak{o}$  with value group  $\Gamma$ . This valuation induces a map  $\tilde{v}$  from  $Q(K) = K^*/K^{*2}$  to  $\Gamma/2\Gamma$ . We chose a subgroup  $M$  of  $Q(K)$  such that  $\tilde{v}$  gives a bijection from  $M$  to  $\Gamma/2\Gamma$ . Such a subgroup  $M$  clearly exists, since  $Q(K)$  and  $\Gamma/2\Gamma$  are vector spaces over the field of two elements. We call  $M$  a *group of representatives* for  $\Gamma/2\Gamma$ . Any element  $z$  of  $W(K)$  can be written – possibly in different ways – in the form  $z = \sum_{m \in M} x_m m$  with  $x_m$  in  $W(\mathfrak{o})$  and only finitely many  $x_m \neq 0$ , since this is true for the generators  $(a)$  of  $W(K)$ . We denote for any  $x$  in  $W(\mathfrak{o})$  by  $\bar{x}$  the image under the natural map from  $W(\mathfrak{o})$  to  $W(k)$ . Then one immediately computes for  $m$  in  $M$ :

$$\partial(mz) = \bar{x}_m. \quad (2.2)$$

In particular  $\partial(mz)$  is zero for nearly all  $m$  in  $M$ . Thus we have a map

$$\Delta: W(K) \rightarrow W(k) [M]$$

into the group ring of  $M$  over  $W(k)$ , defined by

$$\Delta(z) = \sum_{m \in M} \partial(mz) m. \quad (2.3)$$

It is clear from (2.2) that  $\Delta$  is a ring homomorphism which is surjective, since the

<sup>2)</sup> In [K<sub>3</sub>] this map  $\partial$  is denoted by  $\varphi_*$  with  $\varphi$  the place  $K \rightarrow k \cup \infty$  corresponding to  $\mathfrak{o}$ . In the present paper  $\varphi_*$  will have a slightly different meaning.

natural map  $W(\mathfrak{o}) \rightarrow W(k)$  is surjective. It is further clear from (2.2) that  $\Delta(z)=0$  if and only if all  $x_m$  lie in the kernel of  $W(\mathfrak{o}) \rightarrow W(k)$ , which we denote by  $W(\mathfrak{o}, \mathfrak{m})$ . Summarizing we obtain

**PROPOSITION 2.4.**  $\Delta: W(K) \rightarrow W(k) [M]$  is a ring-epimorphism whose kernel is the ideal of  $W(K)$  generated by  $W(\mathfrak{o}, \mathfrak{m})$ .

*Remarks.* i) It has been shown in [K, §12] that the set  $W(\mathfrak{o}, \mathfrak{m})$  itself is an ideal of  $W(K)$  if  $\mathfrak{o}$  has rank one.

ii) It is not difficult to prove for any local ring  $\mathfrak{o}$  with maximal ideal  $\mathfrak{m}$  that  $W(\mathfrak{o}, \mathfrak{m})$  is generated as an ideal by the elements  $1 - (1 + d)$  with  $d$  in  $\mathfrak{m}$ . We shall not need this fact.

We now consider a real place  $\varphi: K \rightarrow R \cup \infty$  and denote by  $\mathfrak{o}$  the valuation ring of  $\varphi$ . We continue to use the notations  $\mathfrak{m}, k, v, \Gamma, M$  with respect to  $\mathfrak{o}$  as above.  $\varphi$  induces a homomorphism  $\bar{\varphi}$  from  $k$  into  $R$ . The composite map

$$\varphi_* = \varrho \circ W(\bar{\varphi}) \circ \partial: W(K) \rightarrow W(k) \rightarrow W(R) \rightarrow \mathbb{Z}$$

has the description given in the introduction, and the restriction of  $\varphi_*$  to  $W(\mathfrak{o})$  is the ring homomorphism  $\hat{\varphi}$  considered in §1. As an easy consequence of Proposition 2.4 we obtain

**THEOREM 2.5.** The signatures  $\sigma: W(K) \rightarrow \mathbb{Z}$  lying over a given place  $\varphi: K \rightarrow R \cup \infty$  correspond uniquely to the characters  $\chi: M \rightarrow \{\pm 1\}$  by the following formulas:

$$\chi(m) = \sigma(m)$$

for  $m$  in  $M$ , and

$$\sigma(z) = \sum_{m \in M} \chi(m) \varphi_*(mz)$$

for  $z$  in  $W(K)$ .

*Remark.* Except the last formula this has already been proved by Krull in a different way [Kr, p. 189].

*Proof.* Clearly  $\hat{\varphi}$  vanishes on  $W(\mathfrak{o}, \mathfrak{m})$ . Thus any signature  $\sigma$  lying over  $\varphi$  must vanish on the ideal generated by  $W(\mathfrak{o}, \mathfrak{m})$  in  $W(K)$ . By Prop. 2.4 such a signature must have the form  $\sigma = \alpha \circ \Delta$  with a uniquely determined ring homomorphism  $\alpha$  from  $W(k) [M]$  to  $\mathbb{Z}$ . These homomorphisms  $\alpha$  correspond uniquely to the pairs  $(\alpha_0, \chi)$  consisting of a homomorphism  $\alpha_0: W(k) \rightarrow \mathbb{Z}$  and a character  $\chi: M \rightarrow \{\pm 1\}$ , the correspondence being given by  $\alpha_0 = \alpha \mid W(k)$  and  $\chi = \alpha \mid M$ . From the definition (2.3) of  $\Delta$  we obtain for the signature  $\sigma = \alpha \circ \Delta$  and  $z$  in  $W(K)$ :

$$\sigma(z) = \sum_{m \in M} \chi(m) (\alpha_0 \circ \partial)(mz).$$



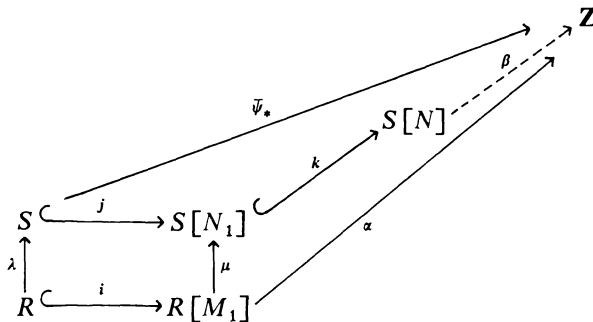
Such a signature  $\sigma$  coincides with  $\hat{\phi}$  on  $W(\mathfrak{o})$  if and only if  $\alpha_0(\bar{z}) = \hat{\phi}(z)$  for all  $z$  in  $W(\mathfrak{o})$ . This means  $\alpha_0 \circ \partial = \phi_*$ . Theorem 2.5 is now obvious. q.e.d.

**THEOREM 2.6.** *Let  $L$  be an arbitrary field extension of  $K$  and  $\psi: L \rightarrow R \cup \infty$  a real place. Further let  $\sigma$  be a signature of  $K$ . Then the following are equivalent:*

- (i) *There exists a signature  $\tau$  of  $L$  which lies over  $\psi$  and extends  $\sigma$ .*
- (ii)  *$\sigma(a) = \psi_*(a)$  for all  $a$  in  $K^*$  such that  $\psi_*(a) \neq 0$ .*

*Proof.* (i)  $\Rightarrow$  (ii) is evident. We now assume (ii), which in particular implies that  $\sigma$  lies over the restriction  $\varphi = \psi|_K$ . We denote by  $\mathfrak{o}$  and  $\mathfrak{D}$  the valuation rings of  $\varphi$  resp.  $\psi$ , and by  $\bar{\mathfrak{o}}$  and  $\bar{\mathfrak{D}}$  their residue class fields, further by  $w: L^* \rightarrow \Gamma'$  a valuation corresponding to  $\bar{\mathfrak{D}}$  with value group  $\Gamma'$  and by  $\Gamma$  the value group  $w(K^*)$  of  $w|_{K^*}$ . We chose a group  $M$  of representatives for  $\Gamma/2\Gamma$  in  $Q(K)$ . Let  $M_0$  denote the subgroup of all  $m$  in  $M$  with  $w(m)$  in  $2\Gamma'$ , and let  $M_1$  denote an arbitrary chosen subgroup of  $M$  such that  $M = M_0 \times M_1$ . The map from  $M_1$  into  $\Gamma'/2\Gamma'$  induced by  $w$  is injective. Thus also the natural map from  $M_1$  into  $Q(L)$  is injective, and we can chose a group  $N$  of representatives of  $\Gamma'/2\Gamma'$  in  $Q(L)$  which contains the image  $N_1$  of  $M_1$ . We further chose a subgroup  $N_0$  of  $N$  such that  $N = N_0 \times N_1$ . Finally let  $\Delta$  denote the map (2.3) from  $W(K)$  onto  $W(\bar{\mathfrak{o}})[M]$  and  $\Delta'$  the analogous map from  $W(L)$  onto  $W(\bar{\mathfrak{D}})[N]$ .

By the proof of Theorem 2.5 we have  $\sigma = \alpha \circ \Delta$  with a homomorphism  $\alpha$  from  $W(\bar{\mathfrak{o}})[M]$  to  $\mathbf{Z}$  which extends the homomorphism  $\bar{\phi}_* = \varphi \circ W(\bar{\phi})$  from  $W(\bar{\mathfrak{o}})$  to  $\mathbf{Z}$ . Similarly  $\tau$  must have the form  $\tau = \beta \circ \Delta'$  with a homomorphism  $\beta$  from  $W(\bar{\mathfrak{D}})[N]$  to  $\mathbf{Z}$  which we have to construct. Since the natural map from  $Q(K)$  to  $Q(L)$  maps  $M_0$  into  $Q(\mathfrak{D}) \subset W(\mathfrak{D})$ , there is an obvious map  $\lambda$  from the ring  $R := W(\bar{\mathfrak{o}})[M_0]$  to the ring  $S := W(\bar{\mathfrak{D}})$ . Combining  $\lambda$  with the bijection  $M_1 \simeq N_1$  induced by  $Q(K) \rightarrow Q(L)$  we obtain a map  $\mu$  from the ring  $R[M_1] = W(\bar{\mathfrak{o}})[M]$  to  $S[N_1]$ . Consider now the diagram



with inclusion maps  $i, j, k$ . That  $\tau$  lies over  $\psi$  means that the upper triangle is commutative, and that  $\tau$  extends  $\sigma$  means that the lower triangle is commutative. Our hypothesis (ii) means  $\bar{\psi}_* \circ \lambda = \alpha \circ i$ . Now the square with the arrows  $i, \mu, \lambda, j$  is a pushout

{i.e. gives a description of  $S[N_1]$  as the tensor product of  $S$  and  $R[M_1]$  over  $R$ }. Thus there is a unique map  $\gamma: S[N_1] \rightarrow \mathbf{Z}$  with  $\gamma \circ j = \bar{\psi}_*$  and  $\gamma \circ \mu = \alpha$ . The only condition which  $\beta$  has to fulfill is the commutativity of the following triangle:

$$\begin{array}{ccc} S[N_1] & \xrightarrow{\gamma} & \mathbf{Z} \\ \downarrow \iota & \searrow \beta & \\ S[N] & & \end{array}$$

Since  $S[N] = S[N_1][N_0]$  clearly such a homomorphism  $\beta$  exists. q.e.d.

*Remark 2.7.* More precisely the extensions of  $\gamma: S[N_1] \rightarrow \mathbf{Z}$  to  $\mathbf{Z}$ -valued homomorphisms  $\beta$  of  $S[N]$  correspond bijectively to the characters of  $N_0$ . The map from  $N_0$  to  $\Gamma'/(2\Gamma' + \Gamma)$  induced by  $w$  is bijective. Thus the signatures  $\tau$  of  $L$  extending  $\sigma$  and lying over  $\psi$  correspond bijectively to the characters of  $\Gamma'/(2\Gamma' + \Gamma)$  (in a non-canonical way), if there are any such signatures  $\tau$ .

We close this section with an application of the last two theorems.

**THEOREM 2.8.** *Assume that  $L_1$  and  $L_2$  are field extensions of a field  $K$  and that  $L_1$  is algebraic over  $K$ . Further assume that on each  $L_i$  an  $R$ -valued place  $\varphi_i$  is given such that  $\varphi_1$  and  $\varphi_2$  coincide on  $K$ . Then the following are equivalent:*

(i) *There exists a field composite  $F$  of  $L_1$  and  $L_2$  over  $K$  and an  $R$ -valued place  $\psi$  on  $F$  extending both  $\varphi_1$  and  $\varphi_2$ .*

(ii)  *$\varphi_{1*}(a) = \varphi_{2*}(a)$  for all  $a$  in  $K^*$  such that both  $\varphi_{1*}(a)$  and  $\varphi_{2*}(a)$  are not zero.*

*Proof.* (i)  $\Rightarrow$  (ii) is trivial. We now assume that (ii) holds. We first construct on each  $L_i$  a signature  $\sigma_i$  lying over  $\varphi_i$  such that  $\sigma_1 \mid K = \sigma_2 \mid K$ : Let  $\varphi$  denote the restriction  $\varphi_1 \mid K = \varphi_2 \mid K$ , let  $\Gamma$  denote the value group of a valuation of  $K$  corresponding to  $\varphi$ , and  $M$  denote a group of representatives of  $\Gamma/2\Gamma$  in the group  $Q(K)$ . Further let  $A_i$  ( $i=1, 2$ ) denote the subgroup of all  $m$  in  $M$  such that  $\varphi_{i*}(m) \neq 0$  and let  $\chi_i$  denote the character  $\varphi_{i*} \mid A_i$  of  $A_i$ . By hypothesis  $\chi_1$  and  $\chi_2$  coincide on  $A_1 \cap A_2$ . Thus it is possible to choose a character  $\chi$  of  $M$  with  $\chi \mid A_i = \chi_i$  for  $i=1, 2$ . Let  $\sigma$  denote the signature of  $K$  lying over  $\varphi$  and corresponding to the character  $\chi$  as explained in Theorem 2.5. Clearly  $\sigma(m) = \varphi_{1*}(m)$  for any  $m$  in  $M$  with  $\varphi_{1*}(m) \neq 0$  and thus  $\sigma(a) = \varphi_{1*}(a)$  for any  $a$  in  $K^*$  with  $\varphi_{1*}(a) \neq 0$ . The same holds with  $\varphi_2$  instead of  $\varphi_1$ . By Theorem 2.6 there exists a signature  $\sigma_i$  on each  $L_i$  which lies over  $\varphi_i$  and extends  $\sigma$ .

We now obtain a field composite  $F$  of  $L_1$  and  $L_2$  over  $K$  and a signature  $\tau$  on  $F$  extending both  $\sigma_1$  and  $\sigma_2$  in the following way: Let  $S$  be a real closure of  $L_2$  with respect to  $\sigma_2$  and  $\gamma$  denote the signature of  $S$ . There exists a (unique) homomorphism  $f: L_1 \rightarrow S$  over  $K$  which is compatible with  $\sigma_1$  and  $\gamma$ , i.e.  $\sigma_1 = \gamma \circ W(f)$ . {Apply [AS], Satz 8. This is also a special case of our Theorem 1.6: Extend the trivial place  $K \hookrightarrow S$

to an  $S$ -valued place on  $L_1$  compatible with  $\sigma_1$ .} The field composite  $F := f(L_1) L_2 \subset S$  and the signature  $\tau := \gamma \upharpoonright F$  have the desired properties.

By Theorem 1.6 there is a unique  $R$ -valued place  $\psi$  on  $F$  which extends  $\varphi_2$  and is compatible with  $\tau$ . Clearly  $\psi \upharpoonright L_1$  is compatible with  $\sigma_1$  and extends  $\varphi$ . Thus by the same theorem  $\psi \upharpoonright L_1 = \varphi_1$ . q.e.d.

*Remark 2.9.* In the proof just completed we used the fact that for given signatures  $\sigma_1, \sigma_2$  on  $L_1$  and  $L_2$  with  $\sigma_1 \upharpoonright K = \sigma_2 \upharpoonright K$  there exists a field composite  $F$  of  $L_1$  and  $L_2$  over  $K$  and a signature  $\tau$  on  $F$  with  $\tau \upharpoonright L_1 = \sigma_1$  and  $\tau \upharpoonright L_2 = \sigma_2$ . This remains true if both  $L_1$  and  $L_2$  are arbitrary field extensions of  $K$  with  $F$  a free field composite. Further it can be shown that for given  $\sigma_1$  and  $\sigma_2$  up to equivalence only one such free composite  $F$  and only one such  $\tau$  exists. These facts are closely related to the following theorem (see also [K<sub>2</sub>, Th. 3]): Denote by  $A$  the total quotient ring of  $L_1 \otimes_K L_2$ . The kernel and the cokernel of the obvious map from  $W(L_1) \otimes_{W(K)} W(L_2)$  to  $W(A)$  are 2-primary torsion groups. I omit the proofs since we do not need these results in the present paper.

In the situation of Theorem 3.9 it may happen that there exists more than one  $R$ -valued place  $\psi$  on a field composite  $F$  of  $L_1$  and  $L_2$  which extends both  $\varphi_1, \varphi_2$ , as shows the following

**EXAMPLE 2.10.** Let  $\varphi$  be an  $R$ -valued place on a field  $K$  and let  $a$  and  $c$  be elements of  $K$  such that  $\varphi_*(a) = 0$ ,  $\varphi_*(c) = 1$ ,  $c$  not a square. For example let  $K = R(t)$  with one indeterminate  $t$ , let  $\varphi$  be the place over  $R$  with  $\varphi(t) = 0$  and  $a = t$ ,  $c = 1 + t^2$ . Using the trace formula proved in the next section (see also Introduction) one easily checks that there is exactly one  $R$ -valued place  $\varphi_1$  on  $L_1 := K(\sqrt{a})$  and one  $R$ -valued place  $\varphi_2$  on  $L_2 := K(\sqrt{ac})$  which extend  $\varphi$  (Of course this also follows from general valuation theory). The field  $F = K(\sqrt{a}, \sqrt{ac})$  is the only composite of  $L_1$  and  $L_2$  over  $K$ . But  $\varphi_2$  has – by the same trace formula – exactly two extensions  $\psi, \psi'$  to  $F$  with values in  $R$ , which both must also extend  $\varphi_1$ .

### §3

Assume that  $L$  is a finite extension of degree  $n$  of a field  $K$  and that an  $R$ -valued place  $\varphi$  is given on  $K$ . Let  $\text{Tr}^*: W(L) \rightarrow W(K)$  denote the transfer map from  $W(L)$  to  $W(K)$  with respect to the regular trace  $\text{Tr} = \text{Tr}_{L/K}$  ([S], cf. Introduction). The goal of this section is to prove the following trace formula:

**THEOREM 3.1.** *For every  $x$  in  $W(L)$*

$$\varphi_*(\text{Tr}^*(x)) = \sum_{\psi \upharpoonright \varphi} \psi_*(x),$$

*with the sum taken over all  $R$ -valued places  $\psi$  on  $L$  which extend  $\varphi$ .*

*N.B.* The sum is finite, since  $\varphi$  has at most  $n$  extensions  $\psi$  to  $L$  [B, Chap VI, §8 no. 3, Th. 1].

We shall deduce this theorem from our results about the connection between signatures and real places in the previous sections and from the following trace formula for signatures, which is a consequence of Artin-Schreier's theory of real closures (see [K<sub>1</sub>, §5]):

**PROPOSITION 3.2.** *For every signature  $\sigma$  of  $K$  and every element  $x$  of  $W(L)$*

$$\sigma(\mathrm{Tr}^*(x)) = \sum_{\tau \mid \sigma} \tau(x),$$

where  $\tau$  runs through the finite set of all signatures of  $L$  which extend  $\sigma$

Let  $\psi_i$ ,  $1 \leq i \leq r$ , denote the  $R$ -valued places of  $L$  which extend  $K$  ( $r=0$ , if there are no such places). We chose a valuation  $v: K^* \rightarrow \Gamma$  with value group  $\Gamma$  corresponding to the place  $\varphi$ , and for each  $\psi_i$ ,  $1 \leq i \leq r$ , a corresponding valuation  $w_i: L^* \rightarrow \Gamma_i$  extending  $v$  with value group  $\Gamma_i \supset \Gamma$ . Since  $\sum_i (\Gamma_i: \Gamma) \leq n$  [B, loc. cit.], all  $(\Gamma_i: \Gamma)$  are finite.

To prove Theorem 3.1 we have to surmount some technical difficulties, which arise from the fact that  $\Gamma$  may not be finitely generated. To get an idea of the proof the reader is advised to follow first the proof under the additional assumption that  $\Gamma$  is finitely generated. Then it is clear (and follows from Lemma 3.3 below), that  $\Gamma/2\Gamma$  and all  $\Gamma_i/2\Gamma_i$  have the same finite cardinality. The proof goes through with the choice  $M_0 = M$ ,  $M_1 = \{1\}$ ,  $N_{i0} = N_i$  below, and Lemma 3.3 and 3.4 may be skipped.

**LEMMA 3.3.** *For each  $i$  with  $1 \leq i \leq r$  the kernel and the cokernel of the natural map  $\alpha_i: \Gamma/2\Gamma \rightarrow \Gamma_i/2\Gamma_i$  are finite and have the same cardinality.*

*Proof.* Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \Gamma & \rightarrow & \Gamma_i & \rightarrow & \Gamma_i/\Gamma \rightarrow 0 \\ & & \downarrow 2 & & \downarrow 2 & & \downarrow 2 \\ 0 & \rightarrow & \Gamma & \rightarrow & \Gamma_i & \rightarrow & \Gamma_i/\Gamma \rightarrow 0 \end{array}$$

of exact sequences, where the vertical arrows denote the homotheties  $x \mapsto 2x$ . Since  $\Gamma_i/\Gamma$  is finite, the kernel  $A_i$  and the cokernel  $B_i$  of  $\Gamma_i/\Gamma \xrightarrow{2} \Gamma_i/\Gamma$  have the same finite cardinality. Now the snake lemma gives an exact sequence

$$0 \rightarrow A_i \rightarrow \Gamma/2\Gamma \xrightarrow{\alpha_i} \Gamma_i/2\Gamma_i \rightarrow B_i \rightarrow 0$$

which makes the assertion obvious. q.e.d.

In  $Q(K) \cong K^*/K^{*2}$  we chose a group  $M$  of representatives of  $\Gamma/2\Gamma$ . Then after fixing an element  $x$  of  $W(L)$  we chose a decomposition  $M = M_0 \times M_1$  with the following properties:

- a)  $M_0$  is finite,
- b)  $M_0$  contains the representatives of the elements of all finite subsets  $\text{Ker } \alpha_i$ ,  $1 \leq i \leq r$ , of  $\Gamma/2\Gamma$ ,
- c)  $\varphi_*(m \cdot \text{Tr}^*(x)) = 0$  for  $m$  in  $M$  but not in  $M_0$ ,
- d)  $\psi_{i*}(m \cdot x) = 0$  for  $1 \leq i \leq r$  and all  $m$  in  $M_1$  with  $m \neq 1$ .

Clearly such a decomposition of  $M$  does exist. By property b) the map  $Q(K) \rightarrow Q(L)$  is injective on  $M_1$ . Thus we regard  $M_1$  also as a subgroup of  $Q(L)$ . By property b) further all maps  $w_i: Q(L) \rightarrow \Gamma_i/2\Gamma_i$  are injective on  $M_1$ . For each  $i$  with  $1 \leq i \leq r$  we chose a group  $N_i$  of representatives of  $\Gamma_i/2\Gamma_i$  in  $Q(L)$  which contains  $M_1$ . Then we chose a decomposition  $N_i = N_{i0} \times M_1$ .

LEMMA 3.4. All  $N_{i0}$ ,  $1 \leq i \leq r$ , have the same cardinality as  $M_0$ .

*Proof.* For each  $i$ ,  $1 \leq i \leq r$ , we have a commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \rightarrow & \tilde{v}(M_1) & \rightarrow & \Gamma/2\Gamma & \rightarrow & C \rightarrow 0 \\ & & \downarrow & & \downarrow z_i & & \downarrow \\ 0 & \rightarrow & \tilde{w}_i(M_1) & \rightarrow & \Gamma_i/2\Gamma_i & \rightarrow & D_i \rightarrow 0 \end{array}$$

with groups  $C$  and  $D_i$  which have the cardinalities  $|C| = |M_0|$  and  $|D_i| = |N_{i0}|$ . Let  $X_i$  denote the kernel and  $Y_i$  the cokernel of the map  $C \rightarrow D_i$ . Clearly the map  $\tilde{v}(M_1) \rightarrow \tilde{w}_i(M_1)$  is bijective. Thus  $X_i \cong \text{Ker } \alpha_i$  and  $Y_i \cong \text{Coker } \alpha_i$ . Lemma 3.4 now follows from Lemma 3.3 and the exact sequence

$$0 \rightarrow X_i \rightarrow C \rightarrow D \rightarrow Y_i \rightarrow 0. \quad \text{q.e.d.}$$

After these preparations we consider sets  $S$  and  $T_i$ ,  $1 \leq i \leq r$ , of signatures, defined as follows:  $S$  = set of all signatures  $\sigma$  of  $K$  lying over  $\varphi$  with  $\sigma(m) = 1$  for all  $m$  in  $M_1$ ; further  $T_i$  = set of all signatures  $\tau$  of  $L$  lying over  $\psi_i$  with  $\tau(m) = 1$  for all  $m$  in  $M_1$ . We regard the group  $\hat{M}_0$  of characters of  $M_0$  as the subgroup of characters of  $M$  which are trivial on  $M_1$ , and  $\hat{N}_{i0}$  as the group of characters of  $N_i$  which are trivial on  $M_1$ . Theorem 2.5 gives a bijection from  $S$  to  $\hat{M}_0$  mapping each  $\sigma$  in  $S$  to the corresponding character of  $M$ , and in the same way a bijection from  $T_i$  to  $\hat{N}_{i0}$ . Thus we know from Lemma 3.4 that the sets  $S$ ,  $T_i$ ,  $1 \leq i \leq r$ , are all finite and have the same cardinality  $|M_0|$ .

Now Theorem 1.6 says, that for each signature  $\tau$  of  $L$ , whose restriction  $\tau|_K$  lies over  $\varphi$ , there exists a unique  $R$ -valued place  $\psi$  of  $L$  which extends  $\varphi$  and is compatible with  $\tau$ . Thus the union  $T = \bigcup_i T_i$  of all  $T_i$   $\{T = \emptyset \text{ if } r = 0\}$  is the set of all signatures  $\tau$  of  $L$  whose restrictions  $\tau|_K$  belong to  $S$ , and furthermore this union is disjoint,  $T_i \cap T_j = \emptyset$  for  $i \neq j$ .

Theorem 3.1 will now come out by computing the finite sum  $\sum_{\tau \in T} \tau(x)$  with our

fixed element  $x$  in two different ways. On the one hand,

$$\sum_{\tau \in T} \tau(x) = \sum_{\sigma \in S} \sum_{\tau \mid \sigma} \tau(x)$$

Thus by Prop. 3.2 with  $z = \text{Tr}^*(x)$ :

$$\sum_{\tau \in T} \tau(x) = \sum_{\sigma \in S} \sigma(z).$$

By the last formula in Theorem 2.5 this sum equals

$$\sum_{\chi \in M_0} \left( \sum_{m \in M} \chi(m) \varphi_*(mz) \right).$$

By the property c) of the decomposition  $M = M_0 \times M_1$  we may replace  $M$  by  $M_0$  in the interior sum. Then interchanging the summations we obtain

$$\sum_{\tau \in T} \tau(x) = |M_0| \varphi_*(z). \quad (3.5)$$

On the other hand

$$\sum_{\tau \in T} \tau(x) = \sum_{i=1}^r \sum_{\tau \in T_i} \tau(x).$$

{Read zero for the right hand side if  $r=0$ }. Again by Theorem 2.5

$$\sum_{\tau \in T_i} \tau(x) = \sum_{\chi \in N_{i0}} \left( \sum_{m \in N_i} \chi(m_0) \psi_{i*}(mx) \right),$$

where  $m_0$  denotes the component of  $m$  in  $N_{i0}$ . Since  $N_{i0}$  is finite, we may interchange the summations and obtain for the right hand side

$$\sum_{m \in N_i} \psi_{i*}(mx) \left( \sum_{\chi \in N_{i0}} \chi(m_0) \right).$$

The interior sum is zero if  $m_0 \neq 1$ . But if  $m_0 = 1$ , then  $m \in M_1$  and by property d) of our decomposition of  $M$  the factor  $\psi_{i*}(mx)$  vanishes except for  $m=1$ . Thus this sum reduces to  $|N_{i0}| \psi_{i*}(x)$ , and we obtain

$$\sum_{\tau \in T} \tau(x) = \sum_{i=1}^r |N_{i0}| \psi_{i*}(x). \quad (3.6)$$

Theorem 3.1 now follows from (3.5), (3.6), and Lemma (3.4).

If  $\varphi$  is a discrete place, a more direct and more geometric proof of Theorem 3.1 without resorting to signatures follows easily from the lemma on p. 322 in [G]. It would be desirable to have a similar proof in the general case. The main difficulty

seems to be that one has to work with quadratic forms on modules, which in general are not finitely generated.

#### §4

This section does not use the content of §2 and §3. Assume that  $\varphi: K \rightarrow R \cup \infty$  is a place from a field  $K$  to a real closed field  $R$  and that  $L$  is a *finitely generated* field extension of  $K$ . We ask for a criterion that  $\varphi$  is extendable to an  $R$ -valued place of  $L$ . Let  $\mathfrak{o}$  denote the valuation ring of  $\varphi$  and as in §1 let  $\hat{\varphi}$  denote the homomorphism from  $W(\mathfrak{o})$  to  $\mathbf{Z}$  induced by  $\varphi$ .

**PROPOSITION 4.1.**  *$\varphi$  is extendable to an  $R$ -valued place of  $L$  if and only if  $\hat{\varphi}$  vanishes on the kernel of the natural map  $W(\mathfrak{o}) \rightarrow W(L)$ .*

*Proof.* We denote this kernel by  $W(\mathfrak{o}, L)$ .

i) Assume that  $\psi: L \rightarrow R \cup \infty$  is an extension of  $\varphi$ . Then for every  $Z$  in  $W(\mathfrak{o})$  clearly  $\hat{\varphi}(Z) = \hat{\psi}(Z_L)$  with  $Z_L$  the image of  $Z$  in  $W(L)$  under the natural map  $W(K) \rightarrow W(L)$ . Thus  $\hat{\varphi}(Z) = 0$  if  $Z$  is in  $W(\mathfrak{o}, L)$ .

ii) Now assume  $\hat{\varphi}(W(\mathfrak{o}, L)) = 0$ . We first construct a signature  $\tau$  of  $L$  such that  $\tau \mid K$  lies over  $\varphi$ . We proceed as in the proof of Prop. 1.1. The kernel  $P$  of the ring-homomorphism  $\hat{\varphi}$  from  $W(\mathfrak{o})$  onto  $\mathbf{Z}$  contains  $W(\mathfrak{o}, L)$  and thus yields a prime ideal  $\bar{P}$  of  $W(\mathfrak{o})/W(\mathfrak{o}, L)$ , which is a subring of  $W(L)$  in a natural way.  $\bar{P}$  must be minimal [KRW]. Thus we can find a minimal prime ideal  $Q$  of  $W(L)$  lying over  $\bar{P}$ . Since  $W(\mathfrak{o})/P \cong \mathbf{Z}$  embeds into  $W(L)/Q$  we have  $W(L)/Q \cong \mathbf{Z}$ . The only homomorphism  $\tau: W(L) \rightarrow \mathbf{Z}$  with kernel  $Q$  is a signature whose restriction  $\sigma = \tau \mid K$  lies over  $\varphi$ .

Let  $S$  be a real closure of  $L$  with respect to  $\tau$ . Then the algebraic closure  $K'$  of  $K$  in  $S$  is a real closure of  $K$  with respect to  $\sigma$ . By Theorem 1.6 there exists a unique  $R$ -valued place  $\gamma$  of  $K'$  which extends  $\varphi$ . On the other hand the composite field  $L' = LK'$  in  $S$  is a finitely generated formally real field extension of  $K'$ . Thus by a well known theorem of Artin and Lang ([L, Theorem 7], see also [K<sub>1</sub>, §6]) there exists a place  $\chi: L' \rightarrow K' \cup \infty$  which is the identity on  $K'$ . The  $R$ -valued place  $\gamma \circ \chi$  on  $L'$  extends  $\gamma$  and thus its restriction to  $L$  extends  $\varphi$ . q.e.d.

For any field extension  $L/K$  we denote by  $W(K, L)$  the kernel of the natural map from  $W(K)$  to  $W(L)$ .

**THEOREM 4.2.** *Let  $L \supset K \supset k$  be three fields such that  $L/k$  is finitely generated. Then the following are equivalent:*

i) *Any place  $\varphi: K \rightarrow R \cup \infty$  into a real closed field  $R$  which is trivial on  $k$  can be extended to an  $R$ -valued place of  $L$ .*

ii) *The statement i) with the additional condition inserted, that  $R$  is algebraic over  $\varphi(k)$ .*

iii)  $W(K, L)$  is a torsion group.

Notice that in statement (iii) the field  $k$  does not occur.

*Proof.* (i)  $\Rightarrow$  (ii) is trivial, and (iii)  $\Rightarrow$  (i) follows from the previous proposition 4.1. To prove (ii)  $\Rightarrow$  (iii) we assume that there exists an element  $Z$  in  $W(K, L)$  which is not torsion. We have to show that there exists a real closure  $R$  of  $k$  and a place  $\varphi: K \rightarrow R \cup \infty$  which is the identity on  $k$ , such that  $\varphi$  cannot be extended to an  $R$ -valued place of  $L$ . We write

$$Z = \sum_{i=1}^n (a_i)$$

with elements  $(a_i)$  in  $Q(K)$ . Since  $Z$  is not a torsion element there exists a signature  $\sigma$  of  $K$  such that

$$\sigma(Z) = \sum_{i=1}^n \sigma(a_i) \neq 0$$

([P, Satz 22], see also [LL], [KRW]). Let  $S$  denote a real closure of  $K$  with respect to  $\sigma$  and  $R$  denote the algebraic closure of  $k$  in  $S$ . Finally let  $K'$  denote the field composite  $KR$  in  $S$ , which is finitely generated over  $R$ . According to Artin and Lang [L, Th. 8, p. 387] there exists a place  $\gamma: K' \rightarrow R \cup \infty$ , which is the identity on  $R$ , such that all  $\gamma(a_i)$ ,  $1 \leq i \leq n$  are finite and not zero and  $\hat{\gamma}(a_i) = \sigma(a_i)$ . Let  $\varphi$  denote the restriction of  $\gamma$  to  $K$  and  $\mathfrak{o}$  the valuation ring of  $\varphi$ . Clearly  $Z \in W(\mathfrak{o})$  and  $\hat{\varphi}(Z) = \sigma(Z) \neq 0$ . By Proposition 4.1 this place  $\varphi$  can not be extended to  $L$ . q.e.d.

## REFERENCES

- [AS] ARTIN, E. and SCHREIER, O., *Algebraische Konstruktion reeller Körper*, Hamburger Abh. 5 (1926), 85–99.
- [B] BOURBAKI, N., *Éléments de mathématique*, Algèbre commutative, Hermann, Paris.
- [G] GEYER, W. D., HARDER, G., KNEBUSCH, M., SCHARLAU, W., *Ein Residuensatz für symmetrische Bilinearformen*, Invent. Math. 11 (1970), 319–328.
- [H] HARRISON, D. K., *Witt rings, Lecture notes*, Dept. of Math., Univ. of Kentucky, Lexington, Kentucky, 1970.
- [K] KNEBUSCH, M., *Grothendieck- und Wittringe von nichtausgearteten symmetrischen Bilinearformen*, Sitzber. Heidelber. Akad. Wiss. 1969/70, 3. Abhandlung, pp. 93–157 (also obtainable as separate volume from Springer Verlag).
- [K<sub>1</sub>] KNEBUSCH, M., *On the uniqueness of real closures and the existence of real places*, Comment. Math. Helv. 47 (1972), 260–269.
- [K<sub>2</sub>] KNEBUSCH, M., *Real closures of semi-local rings, and extension of real places*, Bull. Amer. Math. Soc. 79 (1973), 78–81.
- [K<sub>3</sub>] KNEBUSCH, M., *Specialization of quadratic and symmetric bilinear forms, and a norm theorem*, to appear in Acta Arithmetica 24 (1973).
- [KRW] KNEBUSCH, M., ROSENBERG, A., and WARE, R., *Structure of Witt-rings and quotients of abelian group rings*, Amer. J. Math. 94 (1972), 119–155.



- [KWR<sub>1</sub>] KNEBUSCH, M., ROSENBERG, A., and WARE, R., *Grothendieck- and Witt-rings of hermitian forms over Dedekind rings*, Pacific J. Math. 43 (1972), 657–673.
- [KWR<sub>2</sub>] KNEBUSCH, M., ROSENBERG, A., and WARE, R., *Signatures on semi-local rings*, to appear in J. of Algebra 26 (1973).
- [Kr] KRULL, W., *Allgemeine Bewertungstheorie*, J. reine angew. Math. 167 (1931), 160–196.
- [L] LANG, S., *The theory of real places*, Annals Math. 57 (1953), 378–391.
- [LL] LEICHT, J. and LORENZ, F., *Die Primideale des Wittschen Ringes*, Invent. math. 10 (1970), 82–88.
- [M] MILNOR, J., and HUSEMOLLER, D., *Symmetric bilinear forms*, Ergebnisse Math. 73, Springer, Berlin – Heidelberg – New York 1973.
- [P] PFISTER, A., *Quadratische Formen in beliebigen Körpern*, Invent. math. 1 (1966), 116–132.
- [S] SCHARLAU, W., *Zur Pfisterschen Theorie der quadratischen Formen*, Invent. math. 6 (1969), 327–328.
- [Sp] SPRINGER, T. A., *Quadratic forms over fields with a discrete valuation I*, Indag. Math. 17 (1955), 352–362.
- [W] WITT, E., *Theorie der quadratischen Formen in beliebigen Körpern*, J. reine angew. Math. 176 (1937), 31–44.

Mathematisches Institut  
der Universität des Saarlandes  
BRD, 66–Saarbrücken 15

Received January 8, 1973