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GENERIC SPLITTING OF QUADRATIC FORMS, I

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GENERIC SPLITTING OF QUADRATIC FORMS, I

By MANFRED KNEBUSCH

[Received 30 September 1974]

To Hel Braun on her sixtieth birthday

1. Introduction

The aim of the present paper is to pave the road towards a new chapter in the theory of quadratic forms over fields, which to my opinion well deserves the interests and efforts of the mathematicians working in this area. In a subsequent paper we shall give some applications of the theory developed here. In particular we shall describe all quadratic forms φ over a field k of characteristic not equal to 2, such that for every field L extending k the kernel form of $\varphi \otimes L$ can be defined over k, that is, is isomorphic to $\eta \otimes L$ for some form η over k. First examples of such forms are the Pfister forms and their subforms of codimension at most 3, as the reader can easily check (cf. Example 4.1 below).

Let $\varphi(x_1, \ldots, x_n)$ be a quadratic form over a field k of characteristic not equal to 2 in n $(n \geqslant 2)$ variables, which is not isomorphic to $\langle 1, -1 \rangle$. After fixing notation and recalling some results about specialization of forms we prove in §3 that the function field $k(\varphi)$ of the quadric $\varphi(x_1, \ldots, x_n) = 0$ is a 'generic zero field' of φ, that is, is a field L with the following universal property. For any field L' over k the form $\varphi \otimes L'$ is isotropic if and only if there exists a place $\lambda \colon L \to L' \cup \infty$ over k. Then we prove some elementary statements about $k(\varphi)$ and other generic zero fields.

In §4 we first study the question of how much information about φ is given by $k(\varphi)$. Then we ask for a lower bound of the degrees of transcendency of the generic zero fields of φ. Here our results are extremely incomplete and have only been included to stimulate interest in this difficult question.

Now let $j_0 < j_1 < \ldots < j_h$ be the ordered set of Witt indices (hyperbolic ranks) which occur for the forms $\varphi \otimes L$ with L running through all field extensions of K (in a universal domain). In §5 we construct a 'generic splitting tower',

$$K_0 = k \subset K_1 \subset \ldots \subset K_h,$$

of field extensions which have the following properties: (i) $\varphi \otimes K_s$ has Witt index j_s for $0 \leqslant s \leqslant h$; (ii) if L is a field over k and $\varphi \otimes L$ has Witt index j_r, then there exists a place λ from K_r to L over k, but if $r < h$ there exists no place from K_{r+1} to L over k. Furthermore the kernel form φ_r of $\varphi \otimes K_r$ has good reduction [8] with respect to every such place λ and the specialization $\lambda_*(\varphi_r)$ is the kernel form of $\varphi \otimes L$.

In particular the number h, the fields K_r, and all the kernel forms φ_r are 'essentially' uniquely determined by φ (cf. Corollary 5.3 for the precise statement). We call h the *height* of φ. The anisotropic forms of height 1 are, up to scalar factors, the anisotropic Pfister forms

$$\langle 1, a_1 \rangle \otimes \ldots \otimes \langle 1, a_r \rangle \quad (r \geqslant 1)$$

and their subforms of codimension 1 ($r \geqslant 2$), as has been independently proved by Adrian A. Wadsworth in his thesis [**15**]. If $h \geqslant 1$, that is, φ does not split, then φ_{h-1} has height 1, and thus φ_{h-1} is associated with a unique Pfister form τ over K_{h-1}, which we call the *leading form* of φ. We close § 5 with some elementary remarks on leading forms and the behaviour of generic splitting towers under extensions of the base field k.

In the last section a study is made of the dimension of leading forms, which leads to a natural filtration of the Witt ring $W(k)$ by ideals $J_n(k)$ ($n = 0, 1, 2, \ldots$). For $n \geqslant 1$ the ideal $J_n(k)$ is the set of all elements of $W(k)$ represented by forms of even dimension whose leading forms have dimension at least 2^n. A comparison of this filtration with the filtration by the powers of the fundamental ideal $I(k)$, consisting of the forms of even dimension, seems to be of central importance for the theory of quadratic forms over fields.

The reader may notice the analogy of some of our study, in particular §§ 3 and 4, with the work of Amitsur and Roquette [**1**, **12**, **13**] on generic splitting fields of central simple algebras over fields. These authors only consider fields which split a given algebra $\mathfrak{A}$ totally. Of course it also makes sense to study partial generic splitting of $\mathfrak{A}$, as is done for quadratic forms in the present paper. Recently my student and collaborator Ansgar Heuser obtained first results in this direction. He has shown that a central division algebra $\mathfrak{D}$ over k has a generic zero divisor field if and only if the degree of $\mathfrak{D}$ (which equals the square root of the dimension) is a power of a prime number p, and then $\mathfrak{D}$ even has a generic splitting tower. The p-primary division algebras of height 1 all have the degree p. Thus for algebras there is no interesting counterpart to the last section of our paper.

In June 1972 I gave a first lecture on generic splitting at King's College in London. This lecture contained most of the results of §§ 2–5 of the present paper. I take this opportunity of thanking this institution and in particular Professor A. Fröhlich for the stimulating discussions and the warm hospitality I received there.

2. Notation; specialization of forms

All fields in this paper have characteristic not equal to 2. We consider quadratic forms

$$\varphi(X_1, \ldots, X_n) = \sum_{i,j=1}^{n} a_{ij} X_i X_j \quad (a_{ij} = a_{ji})$$

over a field K in an arbitrary number $n \geqslant 0$ of variables, which we always tacitly assume to be non-singular, that is, $\det(a_{ij}) \neq 0$. ($\varphi = 0$ if $n = 0$.) We call n the *dimension* $\dim \varphi$ of φ and as usual abbreviate the polynomial φ by the symmetric matrix (a_{ij}). A diagonal form $a_1 X_1{}^2 + \ldots + a_n X_n{}^2$ will also be denoted by $\langle a_1, \ldots, a_n \rangle$.

We say that two forms† φ and ψ are *isomorphic*, $\varphi \cong \psi$, if they have the same dimension and ψ can be obtained from φ by a linear transformation of the variables. We call φ and ψ *similar* if $\varphi \cong a\psi$ for some a in K^*. We say that ψ is a *subform* of φ or that φ *represents* ψ, and write $\psi < \varphi$, if there exists a form χ such that φ is isomorphic to the orthogonal sum $\psi \perp \chi$. We say that ψ *divides* φ, and write $\psi | \varphi$, if there exists a form χ such that φ is isomorphic to the tensor product $\psi \otimes \chi$.

As has been shown by Witt [**16**] any form φ has a decomposition

$$\varphi \cong \varphi_0 \perp r \times H$$

with φ_0 *anisotropic*, that is, $\varphi_0(c) \neq 0$ for all $c \neq 0$ in K^s ($s = \dim \varphi_0$), and $r \times H$ denoting the orthogonal sum of r ($r \geqslant 0$) copies of the form $H = \langle 1, -1 \rangle$. By Witt's cancellation theorem [**16**, Satz 4] the number r and, up to isomorphism, the form φ_0 do not depend on the chosen decomposition of φ. As usual we call r the *index* $i(\varphi)$ of φ, and we call φ_0 a *kernel form* of φ, and write $\varphi_0 = \ker(\varphi)$. In contrast to old fashioned terminology we call two forms φ and ψ *equivalent*, and write $\varphi \sim \psi$, if $\ker(\varphi) \cong \ker(\psi)$. We say that a form φ is *isotropic* if $i(\varphi) > 0$, and that φ *splits* if $\dim \ker(\varphi) \leqslant 1$.

For any form $\varphi = (a_{ij})$ of dimension n over K we denote by $d(\varphi)$ the *discriminant* of φ, defined as the square class of $(-1)^{\binom{n}{2}} \det(a_{ij})$. We often regard $d(\varphi)$ as a form of dimension 1. We further denote by $c(\varphi)$ the '*Clifford invariant*' of φ, which, differing slightly from Witt's invariant in [**16**], is defined as follows. If n is even, $c(\varphi)$ is the class $[C(\varphi)]$ of the Clifford algebra $C(\varphi)$ of φ in the Brauer group $\mathrm{Br}(K)$ of K. If n is odd, $c(\varphi)$ is the class $[C^+(\varphi)]$ of the subalgebra $C^+(\varphi)$ of elements of even degree

† Instead of 'quadratic form' we say briefly 'form'.

in $C(\varphi)$. The invariants $d(\varphi)$ and $c(\varphi)$ do not alter if φ is replaced by an equivalent form. For the basic properties of $c(\varphi)$ see [**11**, §4].

We now recall some notation and a result from the paper [**8**], which will be needed in the sequel. Let $\lambda: K \to L \cup \infty$ be a place into another field L. (λ is allowed to be trivial, that is, to avoid the value ∞.) We say that a form φ over K has *good reduction* with respect to λ if there exists a symmetric matrix (a_{ij}) such that $\varphi \cong (a_{ij})$ for all $\lambda(a_{ij}) \neq \infty$, and $\det(\lambda(a_{ij})) \neq 0$. Then up to isomorphism the form $(\lambda(a_{ij}))$ over L does not depend on the choice of (a_{ij}) and is called the *reduction* or *specialization* $\lambda_*(\varphi)$ of φ with respect to λ. If φ has good reduction it is always possible to choose the matrix (a_{ij}) above as a diagonal matrix. If φ does not have good reduction we say that φ has *bad reduction* with respect to λ.

THEOREM 2.1 [8, Proposition 2.2]. *Assume that φ, ψ, χ are forms over K with $\varphi \cong \psi \perp \chi$, and that φ has good reduction with respect to $\lambda: K \to L \cup \infty$. If ψ has good reduction, then χ also has good reduction, and thus*

$$\lambda_*(\varphi) \cong \lambda_*(\psi) \perp \lambda_*(\chi).$$

3. Function fields of quadrics

In this section φ denotes a (non-singular quadratic) form of dimension $n > 1$ over a field k. For any field extension L of k φ may also be considered as a polynomial over L, and then will be denoted by $\varphi \otimes L$ or φ_L. The starting point of our paper is roughly the following question. Which indices $i(\varphi_L)$ and kernel forms $\ker(\varphi_L)$ can occur for a given form φ if L runs through all extensions of k (in some universal domain)? We call two field extensions K and L *equivalent* (over k), and write $K \sim L$ or more precisely $K \sim_k L$, if there exists a place from K to L over k and also a place from L to K over k. Over equivalent fields φ has the same behaviour.

PROPOSITION 3.1. *Let K and L be field extensions of k. Let ψ be the kernel form of φ_K, and let χ be the kernel form of φ_L. If there exists a k-place from K to L, then for every such place λ the form ψ has good reduction, and $\lambda_*(\psi) \sim \chi$. In particular, up to isomorphism, $\lambda_*(\psi)$ does not depend on the choice of λ. Furthermore $i(\varphi_L) \geqslant i(\varphi_K)$. If K and L are equivalent, then $i(\varphi_L) = i(\varphi_K)$ and $\lambda_*(\psi) \cong \chi$.*

This follows immediately from Theorem 2.1 and the facts that $\varphi \otimes K$ has good reduction with respect to every k-place λ from K to L and $\lambda_*(\varphi \otimes K) = \varphi \otimes L$. (Apply Theorem 2.1 with $\psi = r \times H$ and $r = i(\varphi \otimes K)$.)

We now ask: for which extensions L of k is the form $\varphi \otimes L$ isotropic?

DEFINITION 3.2. We call a field K over k a *generic zero field* of φ if

(a) $\varphi \otimes K$ is isotropic,

(b) for every field $L \supset k$ with $\varphi \otimes L$ isotropic there exists a place $\lambda\colon K \to L \cup \infty$ over k.

Once we know that there exists a generic zero field K, we also know that K is, up to equivalence, uniquely determined by φ. We did not exclude the case where φ itself is isotropic. In this case of course k is a generic zero field of φ.

There is an obvious candidate for a generic zero field of φ, namely the function field $k(\varphi)$ of the cone $\varphi = 0$, that is, the quotient field of $k[X_1, \ldots, X_n]/(\varphi)$. Here we must exclude the case where $\varphi \cong H$. All other forms φ are irreducible polynomials, as is easily seen. Let x_i denote the image of X_i in $k(\varphi)$. The following theorem says, in the special case where the place γ is trivial, that in fact not only is $k(\varphi)$ a generic zero field of φ but also $(x_1, \ldots, x_n)$ is a 'generic zero' of φ.

THEOREM 3.3. *Let $\varphi = (a_{ij})$ be a form of dimension at least 2 over k which is not isomorphic to H, and let $\gamma\colon k \to L \cup \infty$ be a place.*

(i) *Assume that φ represents an element c with $\gamma(c) \neq 0, \infty$. Then γ can be extended to a place $\lambda\colon k(\varphi) \to L \cup \infty$ if and only if either φ has good reduction with respect to γ and $\gamma_*(\varphi)$ is isotropic or φ has bad reduction.*

(ii) *Assume that no $\gamma(a_{ij})$ is infinite and that* $\det(\gamma(a_{ij})) \neq 0$. *Assume further that $(y_1, \ldots, y_n)$ is a zero not equal to $(0, \ldots, 0)$ of the quadratic form $(\gamma(a_{ij}))$ over L. Then there exists a place $\lambda\colon k(\varphi) \to L \cup \infty$ extending γ such that $\lambda(x_i) = y_i$ for the generators x_i of $k(\varphi)$.*

REMARKS 3.4. (a) Once this theorem is proved we know that statement (i) remains true with $k(\varphi)$ replaced by any other generic zero field K of φ.

(b) For any non-zero element a of k we have $k(\varphi) = k(a\varphi)$. Thus clearly the assumption about φ in part (i) of the theorem cannot be avoided.

(c) Not every homogeneous polynomial possesses a generic zero field. We give an example due to A. Heuser (cf. end of § 1). Let $\mathfrak{D}$ be a central division algebra over an arbitrary field k, whose dimension is not a power of a prime number. Then the norm form of $\mathfrak{D}$ has no generic zero field.

To prove Theorem 3.3 we need a lemma, which follows easily from general valuation theory [**5**, § 2, no. 4, Proposition 3, and § 8, no. 3, Theorem 1].

LEMMA 3.5. *Assume that K is a quadratic extension of a field E, where $K = E(\alpha)$ with $\alpha^2 = a$ in E and α not in E. Let $\mu\colon E \to L \cup \infty$ be a place.*

(i) *If* $\mu(a) \neq 0, \infty$ *and* L *contains an element* β *with* $\beta^2 = \mu(a)$, *then* μ *can be extended to a unique place* $\lambda\colon K \to L \cup \infty$ *with* $\lambda(\alpha) = \beta$.

(ii) *If* $\mu(ac^2) = 0$ *or* ∞ *for all* c *in* E, *then* μ *can be extended to a unique place* $\lambda\colon K \to L \cup \infty$.

Proof of Theorem 3.3. We first prove the second assertion. Let $\mathfrak{o}$ denote the valuation ring of γ. There exists a matrix A in $\mathrm{GL}(n, \mathfrak{o})$ such that the form $\varphi(Ax)$ is diagonal. This follows from the fact, mentioned in [8, § 1], that the 'space' over $\mathfrak{o}$ with matrix (a_{ij}) has an orthogonal basis. Thus we may assume from the beginning that $\varphi = \langle a_1, \ldots, a_n \rangle$ with

$$\gamma(a_i) = b_i \neq 0, \infty$$

for $1 \leqslant i \leqslant n$. We further assume without loss of generality that $y_n \neq 0$. Let E denote the subfield $k(x_1, \ldots, x_{n-1})$ of $k(\varphi)$ generated by the algebraically independent elements $x_1, \ldots, x_{n-1}$ over k. We extend γ to a place $\mu\colon E \to L \cup \infty$ with $\mu(x_i) = y_i$ for $1 \leqslant i \leqslant n-1$. This is easily done as follows. We have a unique place from E to a field $L(u_1, \ldots, u_{n-1})$ with indeterminates u_i, which extends γ and maps x_i to u_i for $1 \leqslant i \leqslant n-1$. Further we have for each i in $[1, n-1]$ a unique place from $L(u_1, \ldots, u_i)$ to $L(u_1, \ldots, u_{i-1})$, which is the identity on $L(u_1, \ldots, u_{i-1})$ and maps u_i to y_i. (Read $L(u_1, \ldots, u_{i-1}) = L$ if $i = 1$.) Composing all these places we obtain the desired extension $\mu\colon E \to L \cup \infty$ of γ. The element

$$x_n^2 = -a_n^{-1}(a_1 x_1^2 + \ldots + a_{n-1} x_{n-1}^2)$$

of E is mapped by μ to

$$-b_n^{-1}(b_1 y_1^2 + \ldots + b_{n-1} y_{n-1}^2) = y_n^2.$$

Thus by Lemma 3.5 the place μ extends to a unique place $\lambda\colon k(\varphi) \to L \cup \infty$ with $\lambda(x_n) = y_n$.

We now prove assertion (i). Consider first the case in which φ has good reduction. If γ can be extended to an L-valued place λ on $k(\varphi)$, then clearly

$$\gamma_*(\varphi) = \lambda_*(\varphi \otimes k(\varphi))$$

is isotropic. On the other hand we have just proved that if $\gamma_*(\varphi)$ is isotropic then γ can be extended to an L-valued place on $k(\varphi)$.

Consider now the case in which φ has bad reduction. According to the assumption about φ in the theorem we may assume that $\varphi = \langle a_1, \ldots, a_n \rangle$ with all $\gamma(a_i)$ finite, $\gamma(a_1) \neq 0$, and $\gamma(a_n d^2) = 0$ or ∞ for all d in k. Let $u_1, \ldots, u_{n-1}$ be indeterminates over L and as before let E denote the field $k(x_1, \ldots, x_{n-1})$. There is a (unique) place $\mu\colon E \to L(u_1, \ldots, u_{n-1}) \cup \infty$ which extends γ and maps x_i to u_i for $1 \leqslant i \leqslant n-1$. We want to show that $\mu(x_n^2 z^2)$ is 0 or ∞ for every z in E. Then we shall know from Lemma 3.5 that μ extends to a place α from $k(\varphi)$ to $L(u_1, \ldots, u_{n-1})$ and composing α

with an arbitrary place from $L(u_1, \ldots, u_{n-1})$ to L over L we obtain a place λ from $k(\varphi)$ to L extending γ, as desired.

Let z be an arbitrary non-zero element of E^*. We write $z = dfg^{-1}$, with d in k and polynomials f, g in $k[x_1, \ldots, x_{n-1}]$ such that for each of them all coefficients lie in the valuation ring $\mathfrak{o}$ of γ but not all lie in the maximal ideal of $\mathfrak{o}$. Then $\mu(f)$ and $\mu(g)$ are finite and non-zero, We obtain

$$\mu(x_n^2 z^2) = -[\gamma(a_1)u_1^2 + \ldots + \gamma(a_{n-1})u_{n-1}^2]\mu(f)^2\mu(g)^{-2}\mu(a_n^{-1}d^2).$$

Notice that all factors on the right-hand side except the last one are non-zero and finite. Now $\mu(a_n^{-1}d^2) = 0$ or ∞. Thus the same holds true for $\mu(x_n^2 z^2)$.

We close this section with some elementary observations about the function field $k(\varphi)$ and other generic zero fields of a form φ over k. We always assume that φ has dimension at least 3.

PROPOSITION 3.6. *$k(\varphi)$ is a regular field extension of k. For any generic zero field K of φ the subfield k is algebraically closed in K.*

Proof. For any field extension L of k the ring $k(\varphi) \otimes_k L$ is an integral domain, since φ is an absolutely irreducible polynomial. Thus $k(\varphi)$ is regular over k. In particular k is algebraically closed in $k(\varphi)$. If K is another generic zero field of φ, then there exists a place $\lambda: K \to k(\varphi) \cup \infty$ over k. This place must be trivial on the algebraic closure of k in K, which hence must coincide with k.

REMARK. For $\varphi = \langle 1, -a \rangle$ with $\varphi \not\cong H$ we have $k(\varphi) \cong k(\sqrt{a})(t)$ with a transcendental element t.

LEMMA 3.7. *If K is a generic zero field of φ and $y \in K^n$ is an arbitrary zero ($\neq 0$) of $\varphi \otimes K$, then every field K' with $k(y) \subset K' \subset K$ is again a generic zero field of φ.*

Proof. There exists a place from K to $k(y)$ over k. The inclusion map from K' to K is a place in the opposite direction.

Consider, for example, $K = k(\varphi)$ with generators $x_1, \ldots, x_n$ as above, and fix some index i with $1 \leqslant i \leqslant n$. Since $\varphi \otimes K$ has the zero

$$(x_1 x_i^{-1}, \ldots, x_n x_i^{-1}),$$

the function field

$$k(\varphi)_0 = k(x_1 x_i^{-1}, \ldots, x_n x_i^{-1})$$

of the projective variety $\varphi = 0$ is a generic zero field of φ. The field $k(\varphi)$ is purely transcendental over $k(\varphi)_0$ with generator x_i.

PROPOSITION 3.8. *φ is isotropic over k if and only if $k(\varphi)_0$ is a purely transcendental extension of k (cf. [**2**, **15**]).*

Proof. If $k(\varphi)_0$ is purely transcendental over k, then there exists a place from $k(\varphi)_0$ to k over k, and φ must be isotropic. Assume now that φ is isotropic. After a linear change of coordinates we have

$$\varphi(X_1, \ldots, X_n) = X_1X_2 + \psi(X_3, \ldots, X_n)$$

with ψ a quadratic form in the variables $X_3, \ldots, X_n$. The elements $x_3x_1^{-1}, \ldots, x_nx_1^{-1}$ of $k(\varphi)_0$ form a transcendency basis over k. They also generate $k(\varphi)_0$ since

$$x_2x_1^{-1} = -\psi(x_3x_1^{-1}, \ldots, x_nx_1^{-1}).$$

Since $k(\varphi)_0$ is regular over k, we have for any field extension L of k up to equivalence a unique free field composite $k(\varphi)_0 \cdot L$ over k. Obviously this composite coincides with $k(\varphi \otimes L)_0$. Thus we obtain from Proposition 3.8 the following corollary.

COROLLARY 3.9. *Assume that L is an arbitrary field extension of k. Then $\varphi \otimes L$ is isotropic if and only if the free composite $k(\varphi)_0 \cdot L$ is purely transcendental over L.*

Of course Proposition 3.8 and Corollary 3.9 remain true with $k(\varphi)_0$ replaced by $k(\varphi)$.

For any field L over k we call the degree of transcendency of L over k briefly the *dimension* of L over k and denote it by $\dim(L \mid k)$ or $\dim L$.

PROPOSITION 3.10. *Assume that L is a field over k with $\varphi \otimes L$ isotropic. If $\dim L \geqslant n-2$ with $n := \dim \varphi$, then $k(\varphi)_0$ can be embedded into L over k. If $\dim L \geqslant n-1$, then even $k(\varphi)$ can be embedded into L over k.*

This proposition follows immediately from Corollary 3.9 and a beautiful lemma of Roquette.

LEMMA 3.11 [**13**, p. 209]. *Let L_1 and L_2 be extensions of an infinite field k, and assume that L_1 has finite dimension over k and that $\dim L_1 \leqslant \dim L_2$. Assume further that L_1 can be embedded into a purely transcendental extension of L_2 over k. Then L_1 can be embedded into L_2 over k.*

Proof of Proposition 3.10 (cf. [**13**, p. 209]). If φ is isotropic then Proposition 3.10 follows from Proposition 3.8. Assume now that φ is anisotropic. Since we always assume that $n \geqslant 3$, the field k must be infinite. Now the free composite $E = k(\varphi)_0 \cdot L$ is purely transcendental over L by Corollary 3.9. Since $k(\varphi)_0$ embeds into E it can also be embedded into L, provided $\dim L \geqslant n-2$. The assertion about $k(\varphi)$ follows in the same way.

Notice that Proposition 3.10 remains true trivially if φ is anisotropic and has dimension 2.

4. Some remarks on generic zero fields

If φ and ψ are anisotropic forms over k then it may well happen that φ and ψ have equivalent generic zero fields, even if $\dim\varphi \neq \dim\psi$. It is also possible for $k(\varphi)_0$ and $k(\psi)_0$ to be isomorphic over k without φ and ψ being similar. We study an example.

EXAMPLE 4.1. A *Pfister form* φ is a form isomorphic to a product $\langle 1,a_1\rangle\otimes\ldots\otimes\langle 1,a_r\rangle$ of r $(r \geqslant 0)$ forms $\langle 1,a_i\rangle$ $(\varphi = \langle 1\rangle$ if $r = 0)$. The *degree* $\deg(\varphi)$ of φ is defined as this number r if φ is anisotropic, and defined as ∞ if φ is isotropic, in which case φ must split [**10**, § 2]. The reason for this convention will be apparent in § 6. We call a subform ψ of φ a *neighbour* of the Pfister form φ if ψ has dimension greater than $\frac{1}{2}\dim\varphi$. A very special example is (in the case where $r \geqslant 2$) the form ψ determined up to isomorphism by the equation $\varphi \cong \langle 1\rangle\perp\psi$, which we call the *pure part* of φ and denote by φ'.

(i) Let ψ be a neighbour of a Pfister form φ, where $\varphi \cong \psi\perp\eta$. Then the function fields $k(\varphi)_0$ and $k(\psi)_0$ are equivalent over k. Indeed, for every field L over k with φ_L isotropic we have $\psi_L\perp\eta_L \sim 0$, whence $\psi_L \sim -\eta_L$, and we see that ψ_L is also isotropic. Thus for any field L over k the form ψ_L is isotropic if and only if φ_L is isotropic. We shall study neighbours of Pfister forms more thoroughly in part II of this paper.

(ii) We choose a Pfister form χ of degree $r-1$ such that our Pfister form φ can be written $\varphi \cong \chi\perp a\chi$ with some a in k^*. For every non-zero subform η of χ the field $k(\chi\perp a\eta)_0$ can be embedded into $k(\varphi)_0$ over k, by Proposition 3.10. We shall now construct explicitly embeddings with the additional property that $k(\varphi)_0$ is purely transcendental over $k(\chi\perp a\eta)_0$.

The field $L := k(\varphi)_0$ has a presentation

$$L = k(x,y) = k(x_1,\ldots,x_n,y_1,\ldots,y_{n-1})$$

with $n = \dim\chi$, generators x_i, y_i, and the defining relation

$$(*) \qquad \chi(x)+a\chi(y,1) = 0.$$

Since χ is strongly multiplicative [**10**, § 2], there exists a matrix $T(y)$ in $\mathrm{GL}(n,k(y))$ such that, with a vector $X = (X_1,\ldots,X_n)$ of indeterminates,

$$\chi(T(y)X) = \chi(y,1)\chi(X).$$

(Think of X as a column vector.) Now consider the vector $z := T(y)^{-1}x$. We obtain from (*) that

$$\chi(z)+a = 0,$$

that is, $(z, 1)$ is a zero of the form $\psi := \chi \perp \langle a \rangle$. The field $K_0 := k(z)$ generated by this zero obviously has dimension at most $n-1$ over k. On the other hand $K_0(y) = k(x, y) = L$. Thus also $\dim(L \mid K_0) \leqslant n-1$. Since $\dim(L \mid k) = 2n-2$, we learn that $K_0 \mid k$ and $L \mid K_0$ both have dimension $n-1$. This implies that K_0 is k-isomorphic to $k(\psi)_0$ and L is purely transcendental over K_0 with $y_1, \ldots, y_{n-1}$ a basis of transcendency. We now consider the fields

$$K_i := k(z, y_1, \ldots, y_i)$$

for $0 \leqslant i \leqslant n-1$. ($K_0$ has the same meaning as before.) Let η be a fixed subform of χ of dimension $i+1$ with $0 \leqslant i \leqslant n-1$. Again by the strong multiplicativity of χ there exists a matrix A in $\mathrm{GL}(n, k(y_1, \ldots, y_i))$ such that

$$\chi(AX) = \eta(y_1, \ldots, y_i, 1)\chi(X).$$

Thus $(Az, y_1, \ldots, y_i, 1)$ is a zero of $\chi \perp a\eta$ in the field K_i which generates K_i over k. Our field K_i is isomorphic to $k(\chi \perp a\eta)_0$ over k, because it has the right dimension $n+i-1$.

(iii) In particular we see that two forms γ_1 and γ_2 of the same dimension greater than n with $\chi < \gamma_i < \varphi$ for $i = 1, 2$ have isomorphic function fields $k(\gamma_i)_0$.

In contrast to Example 4.1 we shall prove the following theorem.

THEOREM 4.2. *Let φ and ψ be anisotropic forms over k of dimension at least* 2.

(i) *If φ and ψ are both Pfister forms and $k(\varphi)$ is equivalent to $k(\psi)$ over k, then $\varphi \cong \psi$.*

(ii) *If φ is a Pfister form or the pure part of a Pfister form and $k(\varphi)$ is isomorphic to $k(\psi)$ over k, then φ is similar to ψ.*

The proof of this theorem is more or less an exercise since all the tools we use (see below) seem to be well known to the specialists (cf. [**2, 3, 6, 15**]). We give the full proof since most of the arguments will be used repeatedly in our paper. Part (ii) of the theorem has been independently proved by Wadsworth [**15**]. Wadsworth has also shown that two anisotropic forms of dimension 4 with isomorphic function fields are similar.

We first quote a general lemma, which is part of the 'norm theorem' in [**8**, § 4]. For any quadratic form φ over k we call an element a of k^* with $\varphi \cong a\varphi$ a *norm* of φ. The group of norms will be denoted by $N(\varphi)$.

LEMMA 4.3. *Assume that φ is a quadratic form over k and that $p(t)$ is an irreducible polynomial over k in variables $t_1, \ldots, t_r$, which is normed, that is, has leading coefficient* 1 *with respect to the lexicographical ordering of the* *monomials in the t_i. Let $k(p)$ denote the quotient field of $k[t]/(p)$. The following are equivalent:*

(i) *$p(t)$ is a norm of $\varphi \otimes k(t)$;*

(ii) *$\varphi \otimes k(p) \sim 0$.*

Part (i) of Theorem 4.2 is now an immediate consequence of the following lemma, which is essentially due to Elman and Lam [**6**, Theorem 1.4] and implicit in the paper [**3**] by Arason and Pfister.

LEMMA 4.4. *Let τ be a Pfister form of degree at least* 1, *and let φ be an anisotropic form over k. The following are equivalent:*

(i) *$\varphi \otimes k(\tau) \sim 0$;*

(ii) *τ divides φ;*

(iii) *there exists some form χ over k with $\varphi \sim \tau \otimes \chi$.*

Here the implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are trivial. We recall the proof that (i) $\Rightarrow$ (ii) from [**6**] for the convenience of the reader. Let $(t_1, \ldots, t_N)$ denote a set of $N = \dim \tau$ variables. Then (i) implies by Lemma 4.3 that $\tau(t_1, \ldots, t_N)$ is a norm of $\varphi \otimes k(t)$. Choose an element a in k^* which is represented by φ. Then $a\tau(t)$ is represented by $\varphi \otimes k(t)$. Thus by the subform theorem of Cassels and Pfister [**10**, p. 20] there exists a form ψ over k with $\varphi \cong a\tau \perp \psi$. Now by (i) also $\psi \otimes k(\tau) \sim 0$, since τ is a Pfister form. We obtain the implication (i) $\Rightarrow$ (ii) by induction on $\dim \varphi$.

To prove part (ii) of Theorem 4.2, and various other propositions, we need the following lemma.

LEMMA 4.5. *Let φ be a form over k of dimension at least* 2 *which is not isomorphic to H. Further let τ be a form over k which is not equivalent to* 0, *but for which $\tau \otimes k(\varphi) \sim 0$. Then φ is similar to a subform of τ.*

Proof. Replacing τ by its kernel form we assume that τ is anisotropic. Let $(t_1, \ldots, t_n) = t$ denote a vector of $n := \dim \varphi$ variables. We choose an element a of k^* represented by φ. Then the polynomial $p(t) = a\varphi(t)$ is normed after a suitable change of coordinates. Since $\tau \otimes k(p) \sim 0$ we obtain from Lemma 4.3 that $a\varphi(t)$ is a norm of $\tau \otimes k(t)$. Therefore $a\varphi(t)$ is represented by $\tau \otimes k(t)$. The subform theorem of Cassels and Pfister yields that $a\varphi < \tau$.

Proof of Theorem 4.2(ii). Assume first that φ is a Pfister form τ. Since $k(\tau) \cong k(\psi)$, certainly $\tau \otimes k(\psi)$ is isotropic, whence $\tau \otimes k(\psi) \sim 0$. By Lemma 4.5 the form ψ is similar to a subform of τ. But $\dim \tau = \dim \psi$ since the dimensions of $k(\tau)$ and $k(\psi)$ are equal. Thus ψ is similar to τ. Assume now that $\varphi = \tau'$ for a Pfister form τ. Then $\tau \otimes k(\psi) \sim 0$, since

$\tau' \otimes k(\psi)$ is isotropic. Lemma 4.5 implies that $\tau \cong a\psi \perp \chi$ with some a in k^* and a suitable form χ over k. Since $\dim \psi = \dim \tau'$, we obtain $\chi = \langle b \rangle$ with some b in k^*. Since τ is a Pfister form, $\tau \cong b\tau \cong ab\psi \perp \langle 1 \rangle$. Thus $\tau' \cong ab\psi$.

We now switch over to a discussion of the minimum of the dimensions of the generic zero fields of a form φ over k. We always assume that φ has dimension $n \geqslant 3$.

DEFINITION 4.6. We call the minimum of the dimensions

$$\dim K = \dim(K \mid k)$$

of the generic zero fields K of φ the *degree of anisotropy* $A(\varphi)$ of φ. We further call a generic zero field K of dimension $A(\varphi)$ a *minimal generic zero field* of φ.

By Proposition 3.6 the degree of anisotropy $A(\varphi)$ is zero if and only if φ is isotropic. We assume from now up to the end of the section that φ is anisotropic.

Since $k(\varphi)_0$ has dimension $n-2$,

$$1 \leqslant A(\varphi) \leqslant n-2. \tag{4.7}$$

If φ is a Pfister form, we obtain from Example 4.1 that

$$A(\varphi) \leqslant \tfrac{1}{2}n-1. \tag{4.8}$$

Before we try to obtain further information about $A(\varphi)$ we state two rather obvious propositions.

PROPOSITION 4.9. *Assume that K is a minimal generic zero field of φ. Then K is finitely generated and regular over k. If L is any field over k such that $\varphi \otimes L$ is isotropic and $\dim L \geqslant A(\varphi)$, then K can be embedded over k into L.*

Proof. We consider first the case where $L = k(\varphi)$. There exists a place $\lambda\colon K \to k(\varphi) \cup \infty$ over k. Let $\tilde{K}$ denote the image field of K. Then $\varphi \otimes \tilde{K}$ is isotropic, and by Lemma 3.7 $\tilde{K}$ is a generic zero field of φ. Thus $\dim \tilde{K} \geqslant A(\varphi)$. The place λ must be an embedding of K into $k(\varphi)$. In particular K is finitely generated and regular over k, since this holds true for $k(\varphi)$. If now L is an extension of k with $\varphi \otimes L$ isotropic, then the free composite $k(\varphi)\cdot L$ is purely transcendental over L by Corollary 3.9. The field K can be embedded into $k(\varphi)\cdot L$ over k. By Roquette's Lemma 3.11 the field K can already be embedded into L provided $\dim L \geqslant A(\varphi)$.

PROPOSITION 4.10. *Assume that φ has good reduction with respect to a place $\lambda\colon k \to L \cup \infty$. Then $A(\lambda_*\varphi) \leqslant A(\varphi)$.*

Proof. Let K denote a minimal generic zero field of φ, and let F denote a minimal generic zero field of the form $\lambda_*(\varphi)$ over L. By Theorem 3.3 the place λ can be extended to a place $\mu: K \to F \cup \infty$. Let $\tilde{K}$ denote the image field of K. The form $\lambda_*(\varphi)$ has a non-trivial zero in the subfield $\tilde{K}L$ of F. Thus by Lemma 3.7 $\tilde{K}L$ is a minimal generic zero field of $\lambda_*(\varphi)$. The assertion is now obvious since $\dim(\tilde{K}L/L) \leqslant \dim(K/k)$.

By (4.7) and (4.8) we have $A(\varphi) = 1$ if $\dim \varphi = 3$ or if φ is a Pfister form of dimension 4.

PROPOSITION 4.11. *If* $\dim \varphi \geqslant 5$ *or if* $\dim \varphi = 4$ *and* φ *is not similar to a Pfister form, then* $A(\varphi) \geqslant 2$.

Proof. We assume without loss of generality that φ represents $\langle 1 \rangle$, and we choose some a in k^* such that φ represents $\langle 1, -a \rangle$. Let K be a minimal generic zero field of φ. Suppose $\dim K = 1$. According to Proposition 4.9 we can embed K over k into the field $L := k(\sqrt{a}, t)$ with an indeterminate t, and thus we assume that K is already a subfield of L. We consider the subfield $K \cdot k(\sqrt{a})$ generated by K and $k(\sqrt{a})$ in L. By Lüroth's theorem

$$K \cdot k(\sqrt{a}) = k(\sqrt{a}, u)$$

with some element of u which is purely transcendental over $k(\sqrt{a})$. Since $k(\sqrt{a})$ is separable over k the function field K/k must have genus zero [**4**, p. 291]. By a well-known theorem of Witt [**17**; **1**, p. 42; **4**, p. 302] there exists a Pfister form τ of degree 2 such that K is isomorphic over k to $k(\tau')_0$. Thus $k(\varphi)$ is equivalent to $k(\tau)$ over k, and in particular $\tau \otimes k(\varphi) \sim 0$. By Lemma 4.5 the form φ must be similar to a subform of τ. This contradicts our assumptions about φ.

We now show that over a real (that is, formally real) field k there exist forms with an arbitrary high degree of anisotropy. We call a form φ over k a *torsion form* if some multiple $m \times \varphi$ is hyperbolic. If k is non-real then all forms over k are torsion forms [**10**, p. 36], but if k is real then, for example, the forms $n \times \langle 1 \rangle$ are not torsion forms.

PROPOSITION 4.12. *Assume that* φ *is a Pfister form of degree* $r \geqslant 2$ *and not a torsion form. Then* $A(\varphi) \geqslant r-1$.

Proof. By a theorem of Pfister [**10**, § 5] there exists a real closure R of k such that $\varphi \otimes R$ does not split and therefore $\varphi \otimes R \cong 2^r \times \langle 1 \rangle$. By Proposition 4.10 it suffices to show that $A(\varphi \otimes R) \geqslant r-1$. Suppose $\varphi \otimes R$ has a generic zero field K of dimension at most $r-2$. Clearly K is non-real. By another theorem of Pfister [**10**, p. 70] every element of K^* is represented by the form $2^{r-2} \times \langle 1 \rangle$ over K. Thus the Pfister form

$\tau = 2^{r-1} \times \langle 1 \rangle$ over R has a non-trivial zero in K. By Lemma 4.5 $\varphi \otimes R$ must be similar to a subform of τ, which is impossible.

Considering Proposition 4.10 and the 'trivial' estimates (4.7) and (4.8) about $A(\varphi)$ it seems natural for us to pose the following questions.

QUESTIONS 4.13. Let k denote the field $k_0(u_1, \ldots, u_m)$ in $m \geqslant 2$ indeterminates u_i over an arbitrary field k_0.

(i) Has the form $\varphi = \langle 1, u_1, \ldots, u_m \rangle$ degree of anisotropy $m-1$?

(ii) Has the form $\tau = \langle 1, u_1 \rangle \otimes \ldots \otimes \langle 1, u_m \rangle$ degree of anisotropy $2^{m-1}-1$?

5. Generic splitting towers and leading forms

Let φ be a form over k. We construct a tower of fields

$$K_0 = k \subset K_1 \subset \ldots \subset K_h$$

in the following way. Decompose φ into a kernel form φ_0 and hyperbolic forms

$$\varphi \cong \varphi_0 \perp i_0 \times H.$$

If φ splits, that is, if $\dim \varphi_0 \leqslant 1$, we stop with $K_0 = k$. Otherwise we choose a generic zero field K_1 of φ_0 and decompose

$$\varphi_0 \otimes K_1 \cong \varphi_1 \perp i_1 \times H$$

with φ_1 anisotropic. If $\dim \varphi_1 \leqslant 1$ we stop. Otherwise we choose a generic zero field K_2 of φ_1 and decompose

$$\varphi_1 \otimes K_2 \cong \varphi_2 \perp i_2 \times H$$

with φ_2 anisotropic, and so on. We thus obtain a tower

$$K_0 = k \subset K_1 \subset \ldots \subset K_h,$$

a system of anisotropic forms φ_r over K_r, and a system of indices i_r, such that

$$\varphi \cong \varphi_0 \perp i_0 \times H,$$

$$\varphi_{r-1} \otimes K_r \cong \varphi_r \perp i_r \times H \quad (1 \leqslant r \leqslant h),$$

and $\dim \varphi_h \leqslant 1$. We call a tower constructed in this way a *generic splitting tower* of φ. This name is justified by the following theorem, with γ chosen there as a trivial place (see Example 5.2).

THEOREM 5.1. *Let $(K_r\colon 0 \leqslant r \leqslant h)$ be a generic splitting tower of φ with indices i_r and kernel forms φ_r (see above). Further let $\gamma\colon k \to L \cup \infty$ be a place and let $\mu\colon K_m \to L \cup \infty$ be an extension of γ for some m in $[0, h]$, which in the case where $m < h$ cannot be further extended to K_{m+1}.*

(i) *If φ has good reduction with respect to γ, then φ_m has good reduction with respect to μ and $\mu_*(\varphi_m)$ is a kernel form of $\gamma_*(\varphi)$. The index of $\gamma_*(\varphi)$ is $i_0 + \ldots + i_m$.*

(ii) *In any case, if c is any element of K_m^* represented by φ_m, then $c(\varphi \otimes K_m)$ and $c\varphi_m$ have good reduction with respect to μ and $\mu_*(c\varphi_m)$ is anisotropic.*

Proof. Assertion (ii) follows immediately from Theorem 3.3(i), applied to the form $c\varphi_m$ which represents the element 1. Assume now that φ has good reduction with respect to γ. Then $\varphi \otimes K_m$ has good reduction with respect to μ and $\mu_*(\varphi \otimes K_m) \cong \gamma_*(\varphi)$. Also, by Theorem 2.1, φ_m has good reduction with respect to μ, and

$$\gamma_*(\varphi) \cong \mu_*(\varphi_m) \perp (i_0 + \ldots + i_m) \times H.$$

Again by Theorem 3.3(i) the form $\mu_*(\varphi_m)$ is anisotropic.

EXAMPLE 5.2. Let L be an arbitrary field extension of k, and let m be maximal in $[0, h]$ such that there exists a place λ: $K_m \to L \cup \infty$ over k. Then $\varphi \otimes L$ has the precise index $i_0 + \ldots + i_m$. Thus K_m is 'generic' among all fields F over k with $i(\varphi \otimes F) \geqslant i_0 + \ldots + i_m$. Any place from some field K_r $(0 \leqslant r \leqslant m)$ to L over k can be extended to a place from K_m to L.

As an immediate consequence of Theorem 5.1 we see that the fields K_r, all indices i_r, and all kernel forms φ_r are essentially uniquely determined by φ.

COROLLARY 5.3. *Let $(K'_r$: $0 \leqslant r \leqslant h')$ be another generic splitting tower of φ with indices i'_r and kernel forms φ'_r. Then $h = h'$ and $i_r = i'_r$ for $0 \leqslant r \leqslant h$, and the fields K_r and K'_r are equivalent over k. For every place λ from K_r to K'_r over k the form φ_r has good reduction and $\lambda_*(\varphi_r) \cong \varphi'_r$ (cf. Proposition* 3.1).

DEFINITIONS 5.4. We call h the *height* $h(\varphi)$ of φ, and we call i_r the *r-th index* $i_r(\varphi)$ and φ_r an *r-th kernel form* of φ. Notice that $i_0(\varphi) = i(\varphi)$ and the 0th kernel form is the usual kernel form $\ker(\varphi)$. Any field extension of k which is equivalent over k to K_h will be called a *generic splitting field of* φ, and any field extension equivalent to K_{h-1} will be called a *leading field* of φ, provided $h \geqslant 1$.

We defined generic splitting towers by prescribing how they have to be constructed. A more intrinsic characterization of generic splitting towers of a form φ over k is given by the following remark.

REMARK 5.5. Let $j_0 < j_1 < \ldots < j_h$ be the sequence of all natural numbers which occur as Witt indices of the forms $\varphi \otimes L$ with L running through all extensions of k in some universal domain. Then a tower of fields

$$L_0 = k \subset L_1 \subset \ldots \subset L_h$$

is a generic splitting tower of φ if and only if it has the following three properties:

(a) $i(\varphi \otimes L_r) = j_r$ for $0 \leqslant r \leqslant h$;

(b) if L' is a field over k with $i(\varphi \otimes L') \geqslant j_r$, then there exists a place from L_r to L' over k;

(c) if again L' is a field over k and there exists a place from L_r to L' over k for some r with $1 \leqslant r \leqslant h$, then every place from some field L_s with $0 \leqslant s < r$ to L' over k can be extended to a place from L_r to L'.

The proof is easy and is left to the reader.

Theorem 5.1(i) contains the following information about the behaviour of these invariants under specializations.

COROLLARY 5.6. *Assume that φ is a form over k with good reduction with respect to a given place $\gamma: k \to k' \cup \infty$. Let*

$$(K_r: 0 \leqslant r \leqslant h) \quad \text{and} \quad (K'_s: 0 \leqslant s \leqslant h')$$

be generic splitting towers of φ and $\varphi' := \gamma_*(\varphi)$ respectively. For any s in $[0, h']$ let $r(s)$ denote the maximal number r in $[0, h]$ such that γ can be extended to a place from K_r to K_s. Then $h' \leqslant h$, and

$$0 \leqslant r(0) < r(1) < \ldots < r(h') = h.$$

We have

$$i_0(\varphi') = i_0(\varphi) + \ldots + i_{r(0)}(\varphi),$$

and for $1 \leqslant s \leqslant h'$

$$i_s(\varphi') = i_{r(s-1)+1}(\varphi) + \ldots + i_{r(s)}(\varphi).$$

Finally

$$\lambda_*(\ker(\varphi \otimes K_{r(s)})) \simeq \ker(\varphi' \otimes K'_s)$$

for every place λ from $K_{r(s)}$ to K'_s extending γ.

Proof. Apply Theorem 5.1 to the places

$$k \xrightarrow{\gamma} k' \cup \infty \hookrightarrow K'_s \cup \infty!$$

We determine the height and the indices of a 'generic form'.

EXAMPLE 5.7. Let $u_1, \ldots, u_n$ denote $n \geqslant 1$ indeterminates over an arbitrary field F, and let φ denote the form $\langle u_1, \ldots, u_n \rangle$ over the field $k := F(u_1, \ldots, u_n)$. Finally let m denote the largest natural number $[\frac{1}{2}n]$ below $\frac{1}{2}n$. Then φ is anisotropic, $h(\varphi) = m$, and $i_r(\varphi) = 1$ for $1 \leqslant r \leqslant m$.

Proof. Clearly there is no equation

$$\sum_{i=1}^{n} u_i h_i(u)^2 = 0$$

with polynomials $h_i(u)$ in $F[u_1, \ldots, u_n]$ which are not all zero. Thus φ is anisotropic. We shall construct a tower

$$L_0 = k \subset L_1 \subset \ldots \subset L_m$$

of algebraic field extensions of k such that $\varphi \otimes L_r$ has index r for $1 \leqslant r \leqslant m$. Since a priori $i(\varphi \otimes L') \leqslant m$ for any field L' over k, we then know from Theorem 5.1 that indeed $i_r(\varphi) = 1$ for $1 \leqslant r \leqslant m$ and $h(\varphi) = m$.

Consider for $1 \leqslant r \leqslant m$ the subfield

$$F_r := F(u_n, \sqrt{(-u_n u_{n-1})}, \ldots, u_{n-2r+2}, \sqrt{(-u_{n-2r+2}u_{n-2r+1})})$$

of the algebraic closure $\bar{k}$, and choose L_r as the purely transcendental extension $F_r(u_1, \ldots, u_{n-2r})$ in $\bar{k}$. Clearly

$$\varphi \otimes L_r \sim \langle u_1, \ldots, u_{n-2r} \rangle,$$

and the right-hand side is anisotropic over L_r, as shown above. Thus $i(\varphi \otimes L_r) = r$.

We now determine all forms of height 1. The following theorem has been proved independently by A. R. Wadsworth [**15**].

THEOREM 5.8. *An anisotropic form φ over k has height* 1 *if and only if φ is similar to a Pfister form of degree at least* 1 *or to the pure part* (*cf. Example* 4.1) *of a Pfister form of degree at least* 2.

The following proof coincides with Wadsworth's proof in the case where φ has even dimension, but is different from his proof for odd dimension. The method used here for odd dimension is susceptible to important generalizations which will be discussed in part II of this paper.

Proof. We have already stated in Example 4.1 that Pfister forms and pure parts of Pfister forms split in every field extension L of k over which they become isotropic. Assume now that φ has height 1 and consider first the case where φ has even dimension n. Then $\varphi \otimes k(\varphi) \sim 0$. We assume without loss of generality that φ represents 1. Let $t_1, \ldots, t_n$ be indeterminates over k. By Lemma 4.3 the element $\varphi(t_1, \ldots, t_n)$ of $k(t_1, \ldots, t_n)$ is a norm of $\varphi \otimes k(t_1, \ldots, t_n)$. In other words, φ is 'strongly multiplicative' and thus is a Pfister form [**10**, p. 26].

Now we consider the case where $\dim \varphi$ is odd and greater than 1. Replacing φ by a similar form we may assume that the discriminant (cf. §2) $d(\varphi) = 1$. We first show that φ does not represent the element 1. Otherwise we would have a decomposition $\varphi \cong \langle 1 \rangle \perp \chi$. We have $\varphi \otimes k(\varphi) \sim \langle 1 \rangle$ by our assumption on φ, hence $\chi \otimes k(\varphi) \sim 0$, and by Lemma 4.5 the form φ would be similar to a subform of χ, which is impossible.

Thus the form $\tau := \langle 1 \rangle \perp (-\varphi)$ is anisotropic. Let L be an arbitrary field extension of k. If $\varphi \otimes L$ is anisotropic, then also $h(\varphi \otimes L) = 1$, since $h(\varphi \otimes L) \leqslant h(\varphi)$ and $h(\varphi \otimes L)$ is not zero. Applying what we have just proved to $\varphi \otimes L$ instead of φ we see that $\tau \otimes L$ is anisotropic. If $\varphi \otimes L$ is isotropic, then $\varphi \otimes L \sim \langle 1 \rangle$ by our assumption on φ, whence $\tau \otimes L \sim 0$. Thus we have shown that $\tau \otimes L$ is either anisotropic or hyperbolic, that is, τ has height 1. Since τ represents 1, it must be a Pfister form, as shown above.

DEFINITION 5.9. Let φ be a form over k which does not split, and let F be a leading field of φ (cf. Definition 5.4). Then the kernel form ψ of $\varphi \otimes F$ has height 1. Thus by the theorem just proved ψ is similar to a Pfister form τ or to the pure part of a Pfister form τ over F. Of course τ is uniquely determined by ψ: if φ has even dimension, then

$$\tau \cong a\psi$$

for every a in F^* represented by ψ, and if φ has odd dimension,

$$\tau \cong \langle 1 \rangle \perp (-d)\psi$$

with d chosen in the square class $d(\varphi)$. We call τ the *leading form* of φ over F. If F_1 is another leading field of φ and τ_1 is the leading form of φ over F_1, then τ has good reduction with respect to every place λ from F to F_1 over k and $\lambda_*(\tau) \cong \tau_1$, as is easily deduced from Proposition 3.1 and the equations above. Thus the leading form does not depend essentially on the choice of F. Notice that τ has degree at least 2 if φ has odd dimension.

The leading form has the following connection with the discriminant $d(\varphi)$ and the Clifford invariant $c(\varphi)$ (cf. § 2).

PROPOSITION 5.10. *Let φ be a non-split form of dimension n over k. Let F be a leading field of φ, and let τ denote the leading form of φ over F.*

(i) *If n is even, $d(\varphi) = 1$, and $c(\varphi) = 1$, or if n is odd and $c(\varphi) = 1$, then* $\deg(\tau) \geqslant 3$.

(ii) *If n is even and $d(\varphi) \neq 1$, then*

$$\tau \cong \langle 1, -d(\varphi) \rangle \otimes F.$$

(iii) *If n is even, $d(\varphi) = 1$, and $c(\varphi) \neq 1$, or if n is odd and $c(\varphi) \neq 1$, then* $\deg(\tau) = 2$ *and thus the quaternion algebra over F with norm form τ represents* $c(\varphi) \otimes F$.

Proof. Let $(K_r \colon 0 \leqslant r \leqslant h)$ be the generic splitting tower defined inductively by

$$K_0 = k, \quad K_{r+1} = K_r(\varphi_r), \quad \varphi_r := \ker(\varphi \otimes K_r).$$

We may assume without loss of generality that $F = K_{h-1}$. Assertion (i) is obvious, since $d(\varphi_{h-1}) = 1$ and $c(\varphi_{h-1}) = 1$.

Assume that n is even and that $d(\varphi) \neq \langle 1 \rangle$. Let r be maximal in $[0, h]$ such that $d(\varphi) \otimes K_r \neq \langle 1 \rangle$. Clearly $r \leqslant h-1$. By Proposition 3.6 the field K_r would be algebraically closed in K_{r+1} if $\dim \varphi_r$ were greater than 2. Since $d(\varphi) \otimes K_{r+1} = \langle 1 \rangle$ we must have $r = h-1$ and $\dim \varphi_{h-1} = 2$. Clearly φ_{h-1} is similar to $\langle 1, -d(\varphi) \rangle \otimes K_{n-1}$.

Assume finally that $c(\varphi) \neq 1$ and for even n that in addition $d(\varphi) = 1$. Let r be maximal in $[0, h]$ such that $c(\varphi) \otimes K_r \neq 1$, and let D denote the division algebra over K_r representing the element $c(\varphi) \otimes K_r$ of the Brauer group $\mathrm{Br}(K_r)$. The division algebra D is split by $K_r(\varphi_r)$. Now $K_r(\varphi_r)$ has the form $E(\sqrt{-g})$ with E a purely transcendental field extension of K_r and some g in E^*. The division algebra $D \otimes E$ is split by $E(\sqrt{-g})$. Thus $D \otimes E$ has dimension 4 over E, and D has dimension 4 over K_r. Let σ denote the norm form of D, which is an anisotropic Pfister form of degree 2. We have $\sigma \otimes K_r(\varphi_r) \sim 0$. Thus by Lemma 4.5 φ_r is similar to a subform of σ. Now $\dim \varphi_{h-1} \geqslant 3$ if n is odd, and $\dim \varphi_{h-1} \geqslant 4$ if n is even, since then $d(\varphi_{h-1}) = 1$ and φ_{h-1} is similar to a Pfister form. We must have $r = h-1$ and $\sigma = \tau$.

Remark 5.11. Let φ be a form of dimension at least 2 over k with $\varphi \not\cong H$. By the argument in the proof of Proposition 5.10(iii) a central division algebra D over k of dimension greater than 1 is split by $k(\varphi)$ if and only if D is a quaternion algebra and φ is similar to a subform of the norm form τ of D (cf. [2, §5, p. 48]). In particular the natural map from $\mathrm{Br}(k)$ to $\mathrm{Br}(k(\varphi))$ is injective if $\dim(\varphi) > 4$ or if $\dim(\varphi) = 4$ and $d(\varphi) \neq 1$. Now let φ be a form with $c(\varphi) \neq 1$. In general the class $c(\varphi)$ can be represented by a product of quaternion algebras, but not by a single quaternion algebra. All products of quaternion algebras occur in this way. Let F denote the field K_{h-1} occurring in the proof of Proposition 5.10. Then the natural map from $\mathrm{Br}(k)$ to $\mathrm{Br}(F)$ is injective, and by Proposition 5.10(iii) the class $c(\varphi) \otimes F$ can be represented by a quaternion algebra over F. More generally, by iterating this procedure we are able to construct for finitely many given products $\mathfrak{A}_1, \ldots, \mathfrak{A}_r$ of quaternion algebras over k a finitely generated regular field extension E of k, such that all $\mathfrak{A}_i$ are equivalent over E to quaternion algebras and nevertheless the natural map from $\mathrm{Br}(k)$ to $\mathrm{Br}(E)$ is injective.

There is a close relation between the leading forms of an odd-dimensional non-split form φ over k and the associated even-dimensional form $\psi := \varphi \perp (-d(\varphi))$.

Proposition 5.12. *Let $(K_i\colon 0 \leqslant i \leqslant h)$ be a generic splitting tower of φ, and let $(L_j\colon 0 \leqslant j \leqslant e)$ be a generic splitting tower of ψ. Further let τ denote* *the leading form of φ over K_{h-1}, and let σ denote the leading form of ψ over L_{e-1}. The following hold true:*

(i) *K_h and L_e are equivalent over k;*

(ii) *there exists a place $\lambda\colon L_{e-1} \to K_{h-1} \cup \infty$ over k;*

(iii) *σ has good reduction with respect to any such place λ and $\lambda_*(\sigma) \cong \tau$, and in particular, σ and τ have the same degree.*

Proof. For an arbitrary field extension E of k the form $\varphi \otimes E$ splits if and only if $\varphi \otimes E \sim -d(\varphi) \otimes E$, and this means that $\psi \otimes E \sim 0$. Thus $K_h \sim L_e$.

Clearly $\varphi \otimes K_{h-1}$ has the kernel form $-d(\varphi) \otimes \tau'$ and $\psi \otimes L_{e-1}$ has a kernel form $a\sigma$ with some $a \neq 0$ in L_{e-1}. We see that

$$(*) \qquad \psi \otimes K_{h-1} \sim -d(\varphi) \otimes \tau.$$

Thus $-d(\varphi) \otimes \tau$ is the kernel form of $\psi \otimes K_{h-1}$. In particular, by Theorem 5.1 $\dim \tau \geqslant \dim \sigma$. On the other hand,

$$\varphi \otimes L_{e-1} \sim d(\varphi) \otimes L_{e-1} \perp a\sigma,$$

and $\varphi \otimes L_{e-1}$ does not split, since $\psi \otimes L_{e-1}$ does not split. By the same theorem $\dim \tau' \leqslant \dim \sigma + 1$. Thus we obtain

$$\dim \sigma \leqslant \dim \tau \leqslant \dim \sigma + 2.$$

Now $\dim \tau \geqslant 4$ and by Proposition 5.10 $\dim \sigma \geqslant 4$ also. Since both dimensions are 2-powers, we must have $\dim \sigma = \dim \tau$. Again by Theorem 5.1 we obtain from (*) that there exists a place $\lambda\colon L_{e-1} \to K_{h-1} \cup \infty$. We choose such a place λ. The form $a\sigma$ has good reduction with respect to λ. In particular $a\sigma \cong b\sigma$ with some b such that $\lambda(b) \neq 0, \infty$. From Proposition 3.1 we obtain that

$$\lambda(b)\lambda_*(\sigma) \cong -d(\varphi) \otimes \tau.$$

Thus $\lambda_*(\sigma) \cong \tau$.

Let φ be a non-split form over k and let $(K_i\colon 0 \leqslant i \leqslant h)$ be a generic splitting tower of φ such that all fields K_i with $0 \leqslant i \leqslant h-1$ are regular over k, and in the case where $i_h(\varphi) > 1$ K_h is also regular over k. (If $i_h(\varphi) = 1$, that is, $d(\varphi) \neq 1$, then we have $K_h = K_{h-1}(\sqrt{d(\varphi)})$.) Such towers certainly exist (cf. Proposition 3.6). We want to construct from the K_i a generic splitting tower of $\varphi \otimes L$ for an arbitrary given field extension L of k.

We denote for $0 \leqslant r \leqslant h-1$, and in the case where $i_h(\varphi) > 1$ for $r = h$ also, by $L{\cdot}K_r$ the unique free composite of L and K_r over k. In the case where $i_h(\varphi) = 1$ we denote by $L{\cdot}K_h$ the field $(L{\cdot}K_{h-1})(\sqrt{d(\varphi)})$, which may coincide with $L{\cdot}K_{h-1}$. Let φ_r denote the kernel form of $\varphi \otimes K_r$ and let J

denote the set of all numbers i in $[0, h]$ such that the form

$$\tilde{\varphi}_i := \varphi_i \otimes (L \cdot K_i)$$

is anisotropic. Clearly $h \in J$. We write $J = \{r(0), \ldots, r(t)\}$ with $t \geqslant 0$ and

$$0 \leqslant r(0) < r(1) < \ldots < r(t) = h.$$

PROPOSITION 5.13. (i) *For every i in $[0, h] \setminus J$ the field $L \cdot K_{i+1}$ is equivalent to $L \cdot K_i$ over $L \cdot K_i$.*

(ii) *$\varphi \otimes L$ splits if and only if the field $L \cdot K_h$ is equivalent to L over L. Then $t = 0$.*

(iii) *Assume that $\varphi \otimes L$ does not split. Then $t \geqslant 1$ and*

$$L \subset L \cdot K_{r(1)} \subset \ldots \subset L \cdot K_{r(t)}$$

is a generic splitting tower of $\varphi \otimes L$.

Proof. (a) We first prove that for every i in $[0, h-1]$ the field $L \cdot K_{i+1}$ is a generic zero field of $\tilde{\varphi}_i$ over $L \cdot K_i$. This means that $L \cdot K_{i+1}$ is equivalent to $L \cdot K_i(\varphi_i) = (L \cdot K_i)(\tilde{\varphi}_i)$ over L. (Read $L \cdot K_i(\tilde{\varphi}_i) = L \cdot K_i$ if $i = h-1$ and $\tilde{\varphi}_i \cong H$.) Indeed, there exists a place λ from K_{i+1} to $K_i(\varphi_i)$ over K_i. Since λ is a place over k, this place extends in a unique way to a place $\tilde{\lambda}$ from $L \cdot K_{i+1}$ to $L \cdot K_i(\varphi_i)$ over L. Clearly $\tilde{\lambda}$ is a place over $L \cdot K_i$. In the same way we obtain a place from $L \cdot K_i(\varphi_i)$ to $L \cdot K_{i+1}$ over $L \cdot K_i$.

Now assertion (i) is evident. If $\varphi \otimes L$ splits, then $J = \{h\}$ and $t = 0$, and we obtain from (i) that $L \cdot K_h$ is equivalent to L over L. On the other hand, if $L \cdot K_h$ is equivalent to L over L, then clearly $\varphi \otimes L$ splits, since $\varphi \otimes (L \cdot K_h)$ splits. Thus assertion (ii) is proved.

(b) Assume now that $\varphi \otimes L$ does not split. Then by (i) and (ii) certainly $t \geqslant 1$. For every j in $[0, t]$ we denote by F_j the field $L \cdot K_{r(j)}$ and by ψ_j the kernel form of $\varphi \otimes F_j$. Further we denote by χ the kernel form of $\varphi \otimes L$. We know from part (a) of the proof that $L \cdot K_{r(j)+1}$ is a generic zero field of ψ_j over F_j, and that F_{j+1} is equivalent to $L \cdot K_{r(j)+1}$ over $L \cdot K_{r(j)+1}$, whence a fortiori over F_j. Thus F_{j+1} is a generic zero field of ψ_j over F_j for $0 \leqslant j \leqslant t-1$.

To complete the proof of (iii) it remains to show that F_1 is a generic zero field of χ over L. We know already that F_1 is a generic zero field of ψ_0 over F_0. Now by (i) the field F_0 is equivalent to L over L, hence $\psi_0 \cong \chi \otimes F_0$. Clearly $\chi \otimes F_1 \cong \psi_0 \otimes F_1$ is isotropic. Let L' be any field over L with $\chi \otimes L'$ isotropic. We have a place λ from F_0 to L' over L, and $\lambda_*(\psi_0) \cong \chi \otimes L'$. We learn from Theorem 3.3 that λ extends to a place from F_1 to L'. Thus indeed F_1 is a generic zero field of χ.

EXAMPLE 5.14. The form $\varphi \otimes L$ has the same height h as φ if and only if $J = [0, h]$, that is, all forms $\varphi_i \otimes (L \cdot K_i)$ are anisotropic. By a theorem of

Springer [14] this is certainly true if L/k is a finite extension of odd degree, since then all $L{\cdot}K_i/K_i$ have the same odd degree. By Corollary 5.6 the heights of φ and $\varphi \otimes L$ are also equal, if there exists a place from L to k over k. If $h(\varphi \otimes L) = h$, then $(L{\cdot}K_i\colon 0 \leqslant i \leqslant h)$ is a generic splitting tower of $\varphi \otimes L$.

COROLLARY 5.15. *Assume that $\varphi \otimes L$ does not split. Let m denote the degree of the leading form of $\varphi \otimes L$. Further let r denote the maximal number in $[0, h]$, such that $\varphi \otimes (L{\cdot}K_r)$ does not split. Then* $\dim \varphi_r = 2^m$ *if* $\dim \varphi$ *is even, and* $\dim \varphi_r = 2^m - 1$ *if* $\dim \varphi$ *is odd.*

Indeed, by Proposition 5.13 we may regard $\varphi_r \otimes (L{\cdot}K_r)$ as the highest non-split kernel form of $\varphi \otimes L$.

We give an application of Corollary 5.15.

PROPOSITION 5.16. *Let φ be a form of even dimension over k, and let h denote the height of φ. The following are equivalent:*

(i) *$d(\varphi) \neq 1$ and $c(\varphi)$ is not split by $k(\sqrt{d(\varphi)})$;*

(ii) *$h \geqslant 2$ and $i_{h-1}(\varphi) = i_h(\varphi) = 1$.*

Proof. Let $(K_r\colon 0 \leqslant r \leqslant h)$ be a generic splitting tower of φ, and let φ_r denote a kernel form of $\varphi \otimes K_r$. The condition $d(\varphi) \neq 1$ is equivalent to the condition $h \geqslant 1$ and $i_h(\varphi) = 1$, by Proposition 5.10. This will be assumed from now on.

(i) $\Rightarrow$ (ii): Let L denote the field $k(\sqrt{d(\varphi)})$. The form $\varphi \otimes L$ has discriminant 1 but Clifford invariant not equal to 1. Thus $\varphi \otimes L$ has height at least 1 and a leading form of degree 2, while φ has a leading form of degree 1. By Corollary 5.15 there is some r in $[0, h]$ with $\dim(\varphi_r) = 4$. We must have $h \geqslant 2$, $r = h-2$, and $i_{h-1}(\varphi) = 1$.

(ii) $\Rightarrow$ (i): φ_{h-2} has dimension 4. Hence $c(\varphi_{h-2})$ cannot be split by $K_{h-2}(\sqrt{d(\varphi)})$, since this would mean that φ_{h-2} is isotropic [16, p. 39]. A fortiori $c(\varphi)$ cannot be split by $k(\sqrt{d(\varphi)})$.

EXAMPLE 5.17. Let φ be an anisotropic 6-dimensional form over k. Then at least one of the invariants $d(\varphi)$, $c(\varphi)$ is not equal to 1, since the leading form of φ must have degree at most 2 (see [11, p. 123] for another proof). Further we know that $h(\varphi) = 2$ or 3. Here is a full list of the possibilities for the higher indices of φ.

(i) $h(\varphi) = 3 \Leftrightarrow i_1(\varphi) = i_2(\varphi) = i_3(\varphi) = 1 \Leftrightarrow d(\varphi) \neq 1$ and $c(\varphi)$ is not split by $k(\sqrt{d(\varphi)})$.

This follows immediately from Proposition 5.16.

(ii) $h(\varphi) = 2$, $i_1(\varphi) = 1$, and $i_2(\varphi) = 2 \Leftrightarrow d(\varphi) = 1$.

This is clear from Proposition 5.10. As a consequence of (i) and (ii) we have finally

(iii) $h(\varphi) = 2$, $i_1(\varphi) = 2$, and $i_2(\varphi) = 1 \Leftrightarrow c(\varphi)$ is split by $k(\sqrt{d(\varphi)})$.

We close this section with a rough study of the splitting behaviour of the forms which are divided by a Pfister form τ of degree at least 1.

PROPOSITION 5.18. *Let φ be a non-split form over k which is divided by a Pfister form τ of degree at least 1. We choose some form ψ over k with $\varphi \cong \psi\otimes\tau$. Let $(K_r\colon 0 \leqslant r \leqslant h)$ be a generic splitting tower of φ, and let φ_r denote the kernel form of $\varphi\otimes K_r$.*

(i) *The form τ divides every φ_r, that is, $\varphi_r \cong \tau\otimes\psi_r$ $(:= (\tau\otimes K_r)\otimes\psi_r)$ with some form ψ_r over K_r. Thus $\dim\tau$ divides $i_r(\varphi)$ for $0 \leqslant r \leqslant h-1$ and $\frac{1}{2}\dim\tau$ divides $i_h(\varphi)$.*

(ii) *For every r in $[0, h-1]$ and every factorization $\varphi_r \cong \tau\otimes\psi_r$ we have $\dim\psi_r \equiv \dim\psi \bmod 2$, and the square class $d(\psi_r)$ lies in $d(\psi)D^*(\tau\otimes K_r)$, with $D^*(\tau\otimes K_r)$ denoting the group of elements of K_r^* represented by $\tau\otimes K_r$.*

Proof. Let $k(\tau)\cdot K_r$ denote the unique free composite of $k(\tau)$ and K_r if $\deg(\tau) > 1$ and an arbitrary composite of $k(\tau)$ and K_r if $\deg(\tau) = 1$. We have

$$\varphi_r\otimes(k(\tau)\cdot K_r) \sim \varphi\otimes(k(\tau)\cdot K_r) \sim 0.$$

Since $k(\tau)\cdot K_r = K_r(\tau\otimes K_r)$ we obtain from Lemma 4.4 that $\tau\otimes K_r$ divides φ_r for $0 \leqslant r \leqslant h$. Now assertion (ii) easily follows from the fact that the annullator ideal of $\tau\otimes K_r$ in the Witt ring $W(K_r)$ of equivalence classes of forms over K_r is generated by the classes of forms $\langle 1, -\lambda\rangle$ with λ running through $D^*(\tau\otimes K_r)$ [**9**, § 4].

A statement about the leading form of φ will be proved at the end of § 6.

EXAMPLE 5.19. Let τ be an anisotropic Pfister form of degree at least 1 over an arbitrary field F. Let $u_1, \ldots, u_n$ be indeterminates over F and let m denote the number $[\frac{1}{2}n]$. The form $\varphi := \langle u_1, \ldots, u_n\rangle\otimes\tau$ over the field $k := F(u_1, \ldots, u_n)$ is anisotropic and has height m. We have $i_r(\varphi) = \dim\tau$ for $1 \leqslant r \leqslant m-1$, which implies that $i_m(\varphi) = \dim\tau$ in the case where n is even and $i_m(\varphi) = \frac{1}{2}\dim\tau$ in the case where n is odd.

Proof. This can be verified in the same way as Example 5.7. We use the tower $L_0 = k \subset L_1 \subset \ldots \subset L_m$ of algebraic field extensions of k constructed there, and see easily that

$$\ker(\varphi\otimes L_r) \cong \langle u_1, \ldots, u_{n-2r}\rangle\otimes\tau\otimes L_r$$

for $0 \leqslant r \leqslant m$. By Proposition 5.18 the dimensions of these forms must coincide with the dimensions of the higher kernel forms of φ.

6. The degree function

Let φ be a form over k which is not hyperbolic. As an immediate consequence of Theorems 5.1 and 5.8 we obtain the following proposition.

PROPOSITION 6.1. *Let L run through all field extensions of k in a universal domain with $\varphi\otimes L$ not hyperbolic. The minimum of the dimensions of the kernel forms of these $\varphi\otimes L$ is a 2-power 2^d ($d = 0$ if and only if* $\dim\varphi$ *is odd).*

We call d the *degree* $\deg(\varphi)$ of φ. If $\varphi\sim 0$ we put $\deg(\varphi) = \infty$. This notion generalizes the degree of a Pfister form introduced in Example 4.1. If φ is non-split and has even dimension, then $\deg(\varphi)$ coincides with the degree of a leading form of φ.

As usual we denote by $W(k)$ the Witt ring of k, whose elements are the equivalence classes $\{\varphi\}$ of forms φ over k. Since the degree of a form apparently depends only on its equivalence class, we have a well-defined function

$$\deg\colon W(k)\to \mathbf{N}\cup\infty.$$

For every $n\geqslant 0$ we denote by $J_n(k)$ the set of all $\{\varphi\}$ in $W(k)$ with $\deg(\varphi)\geqslant n$.

EXAMPLES 6.2. Clearly $J_1(k)$ coincides with the *fundamental ideal* $I(k)$ of $W(k)$, whose elements are the classes $\{\varphi\}$ with $\dim\varphi$ even. We see from Proposition 5.10 that $J_2(k)$ is the set of elements $\{\varphi\}$ in $I(k)$ with $d(\varphi) = 1$ and $J_3(k)$ is the set of elements $\{\varphi\}$ in $J_2(k)$ with $c(\varphi) = 1$. It is well known that $J_2(k)$ coincides with the second power $I^2(k)$ of $I(k)$ and $J_3(k)$ contains the ideal $I^3(k)$ [**11**, p. 122].

We want to prove that $J_n(k)$ is an ideal of $W(k)$ for every $n\geqslant 0$. For this we need a part of the following theorem, which deserves independent interest.

THEOREM 6.3. *Let φ be an orthogonal sum $a\tau\perp\psi$ with τ an anisotropic Pfister form of degree $n\geqslant 1$, a in k^*, and ψ a form of degree at least $n+1$ over k. Let E be a leading field of φ.*

(i) *The leading form of φ over E is $\tau\otimes E$.*

(ii) *If* $\deg(\psi)\geqslant n+2$, *then the kernel form of $\varphi\otimes E$ is $(a\tau)\otimes E$.*

Proof. (a) We may assume that ψ does not split. We first show that φ has degree n. Let $(L_i\colon 0\leqslant i\leqslant e)$ be a generic splitting tower of ψ. Suppose $\tau\otimes L_e\sim 0$. Let s be maximal in $[0,e-1]$ with $\tau\otimes L_s$ anisotropic. Then $\tau\otimes L_s(\psi_s)\sim 0$ with ψ_s the kernel form of $\psi\otimes L_s$. From this we obtain $b\psi_s<\tau\otimes L_s$ with some b in L_s^*, by Lemma 4.5, and hence $\deg\psi = \deg\psi_s\leqslant n$. This contradicts our assumptions. Therefore $\tau\otimes L_e$ is anisotropic. $\varphi\otimes L_e$

has the kernel form $(a\tau)\otimes L_e$ and thus $\deg\varphi \leqslant n$. Suppose φ has degree $m < n$. Let $(K_j\colon 0 \leqslant j \leqslant h)$ be a generic splitting tower of φ. The form $\varphi\otimes K_{h-1}$ has kernel form $b\rho$ with some b in K_{h-1}^* and ρ a Pfister form of degree m. Now

$$\psi\otimes K_{h-1} \sim b\rho\perp(-a)(\tau\otimes K_{h-1}).$$

The right-hand side has dimension $2^m+2^n < 2^{n+1}$. Thus $\psi\otimes K_{h-1}\sim 0$. We obtain $b\rho \sim a(\tau\otimes K_{h-1})$. Since $\dim\rho < \dim\tau$, the form $\tau\otimes K_{h-1}$ must be hyperbolic and thus also $\rho\sim 0$, which is a contradiction. This proves that $\deg\varphi = n$.

(b) Since $\psi\otimes K_h \sim (-a\tau)\otimes K_h$ and $\deg(\psi) > n$, we obtain $\psi\otimes K_h\sim 0$ and $\tau\otimes K_h \sim 0$. Let s be maximal in $[0, h-1]$ with $\tau\otimes K_s$ anisotropic. Then again by Lemma 4.5 $b\varphi_s < \tau\otimes K_s$ with some b in K_s^*. Since φ has degree n clearly $s = h-1$ and

$$\varphi_{h-1}\cong b(\tau\otimes K_{h-1}).$$

Thus $\tau\otimes K_{h-1}$ is the leading form of φ. Furthermore

$$\psi\otimes K_{h-1} \sim [\varphi\perp(-a\tau)]\otimes K_{h-1} \sim \langle b, -a\rangle\otimes\tau\otimes K_{h-1}.$$

If $\deg(\psi) \geqslant n+2$, then this implies that

$$\psi\otimes K_{h-1}\sim 0 \quad\text{and}\quad \varphi_{h-1}\cong a(\tau\otimes K_{h-1}).$$

THEOREM 6.4. *$J_n(k)$ is an ideal of $W(k)$ for every $n \geqslant 0$.*

Proof. (a) We first want to prove that all $J_n(k)$ are closed under addition. This is equivalent to the statement

$$(*)\qquad \deg(\varphi_1\perp\varphi_2) \geqslant \operatorname{Min}(\deg\varphi_1, \deg\varphi_2)$$

for any two forms φ_1, φ_2 over k. If φ_1 or φ_2 has odd dimension or if $\varphi_1\perp\varphi_2\sim 0$ the assertion (*) is trivial. We now exclude these cases. We can find a field extension L of k such that

$$\ker(\varphi_1\otimes L\perp\varphi_2\otimes L) = a\rho$$

with ρ a Pfister form of degree $n := \deg(\varphi_1\perp\varphi_2)$ and some a in L^*. Now

$$\deg(\varphi_i\otimes L) \geqslant \deg\varphi_i \quad (i = 1, 2)$$

by definition of the degree function. Thus it suffices to prove the assertion for the forms $\tilde\varphi_i := \varphi_i\otimes L$ instead of the φ_i. If $\deg\tilde\varphi_2 > n$, then we obtain from the fact that

$$\tilde\varphi_1 \sim a\rho\perp(-\tilde\varphi_2),$$

by Theorem 6.3, that $\tilde\varphi_1$ has degree n. Thus in any case

$$\operatorname{Min}(\deg\tilde\varphi_1, \deg\tilde\varphi_2) \leqslant n.$$

(b) $W(k)$ is additively generated by 1-dimensional forms $\langle a\rangle$. Clearly $\deg(\varphi) = \deg(a\varphi)$ for every form φ. Thus every $J_n(k)$ is stable under multiplication by elements of $W(k)$.

As an immediate consequence of Theorem 6.4 we obtain the following corollary.

COROLLARY 6.5. *Let φ and ψ be forms over k with* $\deg\varphi \neq \deg\psi$. *Then*

$$\deg(\varphi\perp\psi) = \mathrm{Min}(\deg\varphi, \deg\psi).$$

The nth power $I^n(k)$ of the fundamental ideal $I(k)$ is additively generated by the Pfister forms of degree n, since $I(k)$ is additively generated by the Pfister forms of degree 1. Thus Theorem 6.4 yields another corollary.

COROLLARY 6.6. $I^n(k) \subset J_n(k)$ *for every* $n \geqslant 0$.

This is precisely the same statement as the 'Hauptsatz' of Arason and Pfister in [**3**]. Thus Theorem 6.4 may be regarded as a generalization of Arason and Pfister's result.

QUESTION 6.7. Is $I^n(k) = J_n(k)$ for every $n \geqslant 0$?

We know only that this is so for $n \leqslant 2$. An answer to this question would be very important progress in our knowledge about quadratic forms. Attacks on this problem for $n = 3$ have been made by Pfister [**11**, Satz 14], Elman and Lam [**7**], and Arason [**2**, §§3 and 4].

REMARK 6.8. Assume for arbitrary $n \geqslant 0$ that φ is a form over k of degree at least $n+1$, and that

$$\varphi \equiv a_1\rho_1\perp\ldots\perp a_s\rho_s \mod I^{n+1}(k)$$

with s ($s \leqslant 3$) Pfister forms ρ_i of degree at most n. Then φ lies in $I^{n+1}(k)$.

Proof. We may assume that $n \geqslant 1$ and also that all ρ_i have degree at least 1. The case where $s = 1$ is impossible by Corollary 6.5 (or Theorem 6.3). Assume that $s = 2$. The form $\rho_1\otimes k(\rho_2)$ has degree at least $n+1$, hence it is hyperbolic, and also $\rho_2\otimes k(\rho_1)$ is hyperbolic. Thus $\rho_1 \cong \rho_2$ by Lemma 4.4, and $\varphi \equiv a_1\langle 1, a_1a_2\rangle\rho_1 \mod I^{n+1}(k)$. As seen before, φ lies in $I^{n+1}(k)$. Assume finally that $s = 3$ and without loss of generality that

$$m := \deg\rho_1 \geqslant \deg\rho_2 \geqslant \deg\rho_3.$$

Let F denote the field $k(\rho_1)$. We have

$$\varphi\otimes F \equiv \rho_2\otimes F\perp\rho_3\otimes F \mod I^{n+1}(F),$$

and we learn from the case where $s \leqslant 2$ that $\rho_2 \otimes F \cong \rho_3 \otimes F$. Let ζ denote the kernel form of $\rho_2 \perp (-\rho_3)$ which has dimension at most $2^{m+1}-2$. We may assume that $\zeta \neq 0$. By Lemma 4.4 the form ρ_1 divides ζ and thus $\zeta \cong c\rho_1$ with some c in k^*. We obtain

$$\begin{aligned}\varphi &\equiv a_1\rho_1 \perp a_2\zeta \perp \langle a_2, a_3\rangle \rho_3 \\ &\equiv \langle a_1, a_2c\rangle \rho_1 \perp \langle a_2, a_3\rangle \rho_3 \quad \operatorname{mod} I^{n+1}(k).\end{aligned}$$

As shown before, φ lies in $I^{n+1}(k)$.

PROPOSITION 6.9. *For arbitrary $m \geqslant 0$ and $n \geqslant 0$*

$$I^m(k)\cdot J_n(k) \subset J_{m+n}(k).$$

Proof. It suffices to show this for $m = 1$. Since $I(k)$ is generated by the forms $\langle 1, -a\rangle$, we only have to prove that for a fixed form φ over k, which is not hyperbolic, and some a in k^* the form $\alpha := \langle 1, -a\rangle \otimes \varphi$ has strictly larger degree than φ. This is trivial if φ has odd dimension. We assume from now on that $\dim \varphi$ is even and proceed by induction on $h(\varphi)$. The case where $h(\varphi) = 1$ is evident. Assume that $h(\varphi) > 1$ and without loss of generality that α does not split. Let F be a leading field of α, and let ρ be the leading form of α over F. Since $\deg(\varphi \otimes F) \geqslant \deg(\varphi)$ it suffices to prove that $\deg(\rho) > \deg(\varphi \otimes F)$. Furthermore $h(\varphi \otimes F) \leqslant h(\varphi)$. Thus replacing k by F and φ by a form similar to $\varphi \otimes F$ we have retreated to the case where

$$(*) \qquad \langle 1, -a\rangle \otimes \varphi \sim \rho$$

with an anisotropic Pfister form ρ. Assume first that $\rho \otimes k(\varphi)$ is anisotropic. Then we obtain from the fact that

$$\langle 1, -a\rangle \otimes (\varphi \otimes k(\varphi)) \sim \rho \otimes k(\varphi),$$

by the induction hypotheses, that $\deg(\rho)$ is strictly larger than $\deg(\varphi)$, since $\deg(\rho) = \deg(\rho \otimes k(\varphi))$, $\deg(\varphi) = \deg(\varphi \otimes k(\varphi))$, and

$$h(\varphi \otimes k(\varphi)) = h(\varphi) - 1.$$

We now consider the remaining case in which $\rho \otimes k(\varphi)$ splits. By Lemma 4.5 $b\varphi \perp \zeta \cong \rho$ with b in k^* and some form ζ over k. Suppose $\zeta = 0$. Then we obtain from $(*)$ that

$$\langle 1, -a, -b\rangle \otimes \varphi \sim 0.$$

This is a contradiction, since $\langle 1, -a, -b\rangle$ is not a zero divisor in $W(k)$ [**10**, p. 36]. Thus $\zeta \neq 0$. We obtain that $\dim \varphi < \dim \rho$ and a fortiori that $\deg(\varphi) < \deg(\rho)$.

COROLLARY 6.10. *Assume that φ is an odd-dimensional form and that ψ is an arbitrary form over k. Then*

$$\deg(\varphi \otimes \psi) = \deg(\psi).$$

Proof.

$$\varphi \otimes \psi \sim \psi \perp [\varphi \perp \langle -1 \rangle] \otimes \psi.$$

By Proposition 6.9 the second summand on the right-hand side has strictly larger degree than ψ, and the assertion follows from Corollary 6.5.

Corollary 6.10 is a refinement of the theorem that forms of odd dimension are non-zero divisors in $W(k)$.

We now discuss briefly the behaviour of the degree function under field extensions.

PROPOSITION 6.11. *Let φ be a non-split even-dimensional form over k. Let F be a leading field of φ which is regular over k, and let τ denote the leading form of φ over F.*

(i) *For any field L over k we have* $\deg(\varphi \otimes L) > \deg(\varphi)$ *if and only if τ splits in the free composite $F \cdot L$.*

(ii) *Let L be a generic zero field of some form χ of dimension greater than* 1 *over k. Then* $\deg(\varphi \otimes L) > \deg(\varphi)$ *if and only if $\chi \otimes F$ is similar to a subform of τ. In particular the natural map*

$$J_n/J_{n+1}(k) \to J_n/J_{n+1}(L)$$

is injective for all n with $2^n < \dim \chi$.

Proof. Assertion (i) follows immediately from Proposition 5.13. Then (ii) is a consequence of Lemma 4.5, since $F \cdot L$ is equivalent to

$$F \cdot k(\chi) = F(\chi \otimes F).$$

(Assume that $\chi \not\lesssim H$ without loss of generality. Then also $\chi \otimes F \not\lesssim H$.)

Finally we mention a mild application of the theory developed in this section, a supplement to the previous Proposition 5.18.

PROPOSITION 6.12. *Let φ be a non-split form over k, and let τ be a Pfister form of degree at least* 1 *over k dividing φ, where $\varphi \cong \tau \otimes \psi$. Then, with the notation introduced in Proposition* 5.18, *if* $\dim \psi$ *is odd,*

$$\varphi_{h-1} \cong d(\psi) \otimes \tau \otimes K_{h-1}.$$

Now assume that $\dim \psi$ *is even. If $d(\psi)$ is represented by τ we have* $\deg(\varphi) \geqslant \deg(\tau) + 2$. *Otherwise*

$$\varphi_{h-1} \cong a\langle 1, -d(\psi) \rangle \otimes \tau \otimes K$$

with some a in K_{h-1}^*.

Proof. In the first case

$$\psi \otimes \tau \sim d(\psi) \otimes \tau \perp \chi \otimes \tau$$

with $\chi := \psi \perp (-d(\psi))$ in $I^2(k)$, and in the second case

$$\psi \otimes \tau \sim \langle 1, -d(\psi)\rangle \otimes \tau \perp \eta \otimes \tau$$

with $\eta := \psi \perp \langle -1, d(\psi)\rangle$ in $I^2(k)$. By Corollary 6.6 the forms $\chi \otimes \tau$ and $\eta \otimes \tau$ both have degree at least $\deg(\tau) + 2$, and our assertions follow from Theorem 6.3.

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