28

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GENERIC SPLITTING OF QUADRATIC FORMS, II

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GENERIC SPLITTING OF QUADRATIC FORMS, II

By MANFRED KNEBUSCH

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We present applications of the theory developed in part I [5], studying forms with special properties in their splitting behaviour. The viewpoint of generic splitting is to a certain extent already implicitly contained in the work of Arason, Pfister, Elman, and Lam (cf. the references in [5]), and has led to important theorems. Nevertheless this viewpoint seems to generate many more problems than can be solved at this moment. In my opinion an essential task of the present paper is to raise interest in these problems, and I have written down some of them explicitly (4.13, 6.7 in part I, 8.3, 8.4, 10.6 in part II).

7. Excellent forms

Let φ be a non-split form over k, and let $(K_i,\ 0 \leqslant i \leqslant h)$ be a generic splitting tower of φ. Further let φ_i denote the kernel form of $\varphi \otimes K_i$.

DEFINITION 7.1. We fix a number r in $[0, h]$. We say that φ_r is *defined over* k, if there exists a form η over k with $\varphi_r \cong \eta \otimes K_r$. We then also say that φ_r is *defined by* η.

Of course φ_0 and φ_h are always defined over k. (Notice that $\varphi_h \cong d(\varphi) \otimes K_h$ if $\dim \varphi$ is odd.) Definition 7.1 does not depend on the choice of the generic splitting tower. Indeed, if $(K'_i,\ 0 \leqslant i \leqslant h)$ is another generic splitting tower of φ, then $\varphi_r \cong \eta \otimes K_r$ implies that $\ker(\varphi \otimes K'_r) \cong \eta \otimes K'_r$ by Proposition 3.1. Moreover the form η above is, up to isomorphism, uniquely determined by φ_r. In fact, more generally, the following proposition holds true.

PROPOSITION 7.2. *Let r be a number in $[1, h]$. Assume that η_1 and η_2 are anisotropic forms over k with $\dim \eta_i < \dim \varphi_{r-1}$ and $\eta_i \otimes K_r \sim \varphi \otimes K_r$ for $i = 1, 2$. Then $\eta_1 \cong \eta_2$.*

Proof. We assume, without loss of generality, that φ is anisotropic, and further that among all anisotropic forms δ over k with $\varphi \otimes K_r \sim \delta \otimes K_r$ the form η_1 has minimal dimension. We first settle the case where $r = 1$. Suppose η_1 and η_2 are not isomorphic. Then $\eta_1 \perp (-\eta_2)$ has a kernel form $\zeta \neq 0$, which satisfies $\zeta \otimes k(\varphi) \sim 0$. By Lemma 4.5 there exist a form ψ over k and some a in k^* such that

$$\varphi \perp \psi \cong a\zeta.$$

We obtain $\varphi \otimes k(\varphi) \sim (-\psi) \otimes k(\varphi)$. Furthermore

$$\dim \eta_1 + \dim \eta_2 \geqslant \dim \varphi + \dim \psi,$$

whence

$$\dim \eta_1 - \dim \psi \geqslant \dim \varphi - \dim \eta_2 > 0.$$

This contradicts the minimality of $\dim \eta_1$.

We now assume that $r \geqslant 2$ and proceed by induction on r. Suppose η_1 and η_2 are not isomorphic and hence not equivalent. Let s denote the maximal number in $[0, r-1]$ such that $\eta_1 \otimes K_s$ and $\eta_2 \otimes K_s$ are not equivalent. If $s > 0$ we apply the induction hypothesis for $r-s$ to the kernel forms of $\eta_1 \otimes K_s$ and $\eta_2 \otimes K_s$ and obtain a contradiction. Thus $s = 0$, and hence the kernel form ζ of $\eta_1 \perp (-\eta_2)$ again satisfies $\zeta \otimes k(\varphi) \sim 0$. From this we obtain a contradiction to the minimality of $\dim \eta_1$, as in the case where $r = 1$.

Examples 7.3. (i) If $\dim \varphi$ is even, $d(\varphi) \neq 1$, and the Brauer class $c(\varphi)$ is split by $k(\sqrt{d(\varphi)})$, then the kernel form φ_{h-1} has dimension 2 and is defined over k.

(ii) If $\dim \varphi$ is odd, $c(\varphi) \neq 1$, and $c(\varphi)$ is representable by a quaternion algebra, then φ_{h-1} has dimension 3 and is defined over k.

Proof. We know from Proposition 5.10 that φ_{h-1} has dimension 2 or 3 respectively. There exists a form η of dimension 2 respectively 3 over k with invariants $d(\eta) = d(\varphi)$, $c(\eta) = c(\varphi)$, as is easily verified. Since forms of dimension no greater than 3 are classified by these invariants [**9**], we have $\varphi_{h-1} \cong \eta \otimes K_{h-1}$.

Remark. The statements of Examples 7.3 can both be reversed, cf. Corollary 9.8 and Theorem 9.9.

Our main concern in the present section is to describe the anisotropic forms over an arbitrary field k for which all higher kernel forms are defined over k. We first recall and discuss a notion already introduced in Example 4.1.

Definition 7.4. A form φ over k is called a *Pfister neighbour*,† if there exist a Pfister form ρ, some a in k^*, and a form η with $\dim \eta < \dim \varphi$, such that

(*) $$\varphi \perp \eta \cong a\rho.$$

We call ρ the *associated* Pfister form of φ, and η the *complementary* form of φ, and say more specifically that φ is a *neighbour* of ρ. We call $\dim \eta$ the *codimension* of φ.

† The definition of Pfister neighbours here is slightly more general than the definition in Example 4.1.

To justify this terminology we have to check that the forms ρ and η are uniquely determined by φ and the relation (*). This is obvious if φ is isotropic, since then ρ must be isomorphic to $2^{n-1} \times H$ with 2^n the least 2-power above $\dim\varphi$. Assume now that φ is anisotropic. As already observed in Example 4.1, ρ must be anisotropic, and the fields $k(\rho)$ and $k(\varphi)$ have to be equivalent over k. Thus by Theorem 4.2 our form φ can be a neighbour of at most one Pfister form up to isomorphism. Moreover for every c in k^*, which is represented by φ, the form $a\rho$ in (*) must be isomorphic to $c\rho$. Thus by Witt's cancellation theorem η is also uniquely determined by the relation (*). This is also clear from Proposition 7.2, since $\varphi \otimes k(\varphi) \sim (-\eta) \otimes k(\varphi)$.

EXAMPLES 7.5. In § 5 we saw that the anisotropic Pfister neighbours of codimension at most 1 are precisely the anisotropic forms of height 1 (Theorem 5.8). The anisotropic Pfister neighbours of codimension 2 and 3 clearly have height no greater than 2, and thus have height 2. These types of forms of height 2 will be characterized in § 8 in a different way (Corollary 8.2).

LEMMA 7.6. *Assume that φ and η are anisotropic forms over k with $\dim\eta < \dim\varphi$ and $\varphi \otimes k(\varphi) \sim (-\eta) \otimes k(\varphi)$. Assume further that η is a Pfister neighbour. Then $\eta \otimes k(\varphi)$ is anisotropic.*

Proof. Suppose $\eta \otimes k(\varphi)$ is isotropic. Then there exists a place from $k(\eta)$ to $k(\varphi)$ over k. Now $\eta \otimes k(\eta)$ is equivalent to $(-\zeta) \otimes k(\eta)$ with ζ the complementary form of η. Thus $\eta \otimes k(\varphi) \sim (-\zeta) \otimes k(\varphi)$ also. Applying Proposition 7.2 with $r = 1$ we obtain the desired contradiction.

We now introduce the notion *excellent form* by the following inductive definition.

DEFINITION 7.7. All forms of dimension not greater than 1 are excellent. A form φ of dimension $n \geqslant 2$ is excellent if φ is a Pfister neighbour and the complementary form of φ is excellent.

In other words, a form φ is excellent if and only if there exists a sequence

$$\varphi = \eta_0, \eta_1, \ldots, \eta_t \tag{7.8}$$

of forms over k such that $\dim\eta_t \leqslant 1$ and each η_r with $0 \leqslant r < t$ is a Pfister neighbour with complementary form η_{r+1}. We call η_r the *rth complementary form* of φ $(0 \leqslant r \leqslant t)$.

PROPOSITION 7.9. *Let φ be an anisotropic excellent form over k. Then the sequence* (7.8) *has length $t = h(\varphi)$, and for every r in $[0, t]$ the rth kernel form of φ is defined over k by $(-1)^r\eta_r$.*

Proof. We proceed by induction on t. The assertion is trivial for $t \leqslant 1$. Assume that $t > 1$, and let $(K_r, 0 \leqslant r \leqslant h)$ denote a generic splitting tower of φ. Certainly $h \geqslant 1$. By Lemma 7.6 the kernel form of $\varphi \otimes K_1$ is $(-\eta) \otimes K_1$. Now $\eta \otimes K_1$ is again an anisotropic excellent form, whose sequence of complementary forms is

$$\eta_1 \otimes K_1, \eta_2 \otimes K_1, \ldots, \eta_t \otimes K_1.$$

Applying the induction hypothesis to $\eta_1 \otimes K_1$ we obtain $h = t$ and

$$\ker(\varphi \otimes K_r) \cong (-1)^r \eta_r \otimes K_r$$

for $2 \leqslant r \leqslant t$.

Example 7.10. Let n be a natural number not less than 1, and let 2^r denote the smallest 2-power above n, that is, with $n \leqslant 2^r$. Over any field k the form $n \times \langle 1 \rangle$ is a neighbour of the Pfister form $2^r \times \langle 1 \rangle$ with complementary form $(2^r - n) \times \langle 1 \rangle$. We see by induction that all forms $n \times \langle 1 \rangle$ over k $(n \geqslant 0)$ are excellent.

For any natural number $n \geqslant 2$ we denote by $r(n)$ the number not less than 1 such that $2^{r(n)}$ is the least 2-power not less than n, and by $c(n)$ the 'complementary number',

$$c(n) := 2^{r(n)} - n.$$

Clearly for any Pfister neighbour of dimension $n \geqslant 2$ the associated Pfister form has degree $r(n)$ and the complementary form has dimension $c(n)$. We define a function $h\colon \mathbf{N} \to \mathbf{N}$ inductively by

$$h(0) = h(1) = 0, \quad h(n) = h(c(n)) + 1$$

for $n \geqslant 2$. Finally we denote by $c_i(n)$, for $n \geqslant 0$ and any i in $[0, h(n)]$, the number inductively defined by

$$c_0(n) = n, \quad c_{i+1}(n) = c(c_i(n)).$$

Proposition 7.9 yields immediately by induction on n the following corollary.

Corollary 7.11. *Any anisotropic excellent form φ of dimension n has height $h(n)$. For every r in $[0, h(n)]$ the rth kernel form of φ has dimension $c_r(n)$.*

Remark 7.12. For any $r \geqslant 1$ we define a number $N(r)$ inductively by

$$N(1) = 0, \quad N(2) = 2, \quad N(r) = 2^r - N(r-1) \quad (r \geqslant 3).$$

We further introduce the numbers

$$N'(1) = 1, \quad N'(r) := N(r) + (-1)^r$$

for $r \geqslant 2$. One easily checks by induction on n for $r \geqslant 1$ that

(i) $h(n) \leqslant r-1$ for $0 \leqslant n < 2^r$,

(ii) $N(r)$ is the unique even natural number $n < 2^r$ with $h(n) = r-1$, and $N'(r)$ is the unique odd natural number $n < 2^r$ with $h(n) = r-1$.

In particular we can find over a real field k a form $n \times \langle 1 \rangle$ of any prescribed height (cf. Example 7.10).

By Proposition 7.9 all higher kernel forms of an excellent form over k are defined over k. We are now going to prove a converse of this statement. The main step is done by the following generalization of Theorem 5.8.

THEOREM 7.13. *Assume that φ and η are anisotropic forms over k such that $\varphi \otimes k(\varphi)$ has the kernel form $(-\eta) \otimes k(\varphi)$. Then φ is a Pfister neighbour and η is the complementary form of φ.*

Proof. We proceed by induction on $t := \dim \eta$. For $t \leqslant 1$ the assertion is true by Theorem 5.8. Thus we assume from now on that $t \geqslant 2$.

(i) First we show that the form $\tau = \varphi \perp \eta$ is anisotropic (cf. the proof of Theorem 5.8). Suppose τ is isotropic. Then we have decompositions

$$\varphi \cong \langle -c \rangle \perp \chi, \quad \eta \cong \langle c \rangle \perp \gamma$$

with some c in k^* and forms χ, γ. Since $\varphi \otimes k(\varphi) \sim (-\eta) \otimes k(\varphi)$ we have

$$\chi \otimes k(\varphi) \sim (-\gamma) \otimes k(\varphi).$$

Furthermore $\dim \chi > t-1 = \dim \gamma$, since $\dim \varphi > t$. Thus $\chi \otimes k(\varphi)$ is isotropic. Since χ is a subform of φ, this implies that the fields $k(\varphi)$ and $k(\chi)$ are equivalent over k, and we obtain

$$\chi \otimes k(\chi) \sim (-\gamma) \otimes k(\chi).$$

Furthermore $\gamma \otimes k(\chi)$ is anisotropic, since $\gamma \otimes k(\varphi)$ is anisotropic. By the induction hypothesis we conclude that $\chi \perp \gamma \cong a\rho$ with some Pfister form ρ and a in k^*. We have

$$k(\rho) \sim k(\chi) \sim k(\varphi),$$

and in particular $\rho \otimes k(\varphi) \sim 0$. By Lemma 4.5 $\varphi \perp \delta \cong b\rho$ with some form δ over k and b in k^*. Thus

$$\varphi \otimes k(\varphi) \sim (-\delta) \otimes k(\varphi).$$

Since $\varphi \otimes k(\varphi)$ is also equivalent to $(-\eta) \otimes k(\varphi)$ and

$$\dim \delta = \dim \gamma - 1 = \dim \eta - 2,$$

this contradicts Proposition 7.2 (or our hypothesis that $\eta \otimes k(\varphi)$ is anisotropic). The form $\tau = \varphi \perp \eta$ must be anisotropic.

(ii) Let L denote the field $k(\tau)$. We consider the forms $\tilde{\varphi} := \varphi \otimes L$ and $\tilde{\eta} := \eta \otimes L$ over L. As is easily seen

$$L(\tilde{\varphi}) \cong k(\varphi) \cdot k(\tau) \cong k(\varphi)(\tau \otimes k(\varphi))$$

over k, with $k(\varphi) \cdot k(\tau)$ the free composite of $k(\varphi)$ and $k(\tau)$ over k. Since $\tau \otimes k(\varphi)$ is isotropic, the field $L(\tilde{\varphi})$ is a purely transcendental extension of the field $k(\varphi)$. Thus

$$\tilde{\eta} \otimes L(\tilde{\varphi}) = (\eta \otimes k(\varphi)) \otimes L(\tilde{\varphi})$$

is anisotropic. Furthermore, since $k(\varphi) \subset L(\tilde{\varphi})$,

$$\tilde{\varphi} \otimes L(\tilde{\varphi}) = \varphi \otimes L(\tilde{\varphi}) \sim (-\eta) \otimes L(\tilde{\varphi}) = (-\tilde{\eta}) \otimes L(\tilde{\varphi}).$$

If $\tilde{\varphi}$ were anisotropic, then by step (i) of the proof $\tilde{\varphi} \perp \tilde{\eta} = \tau \otimes k(\tau)$ would be anisotropic, which is not true. Thus $\tilde{\varphi}$ is isotropic. This implies that

$$\varphi \otimes k(\tau) \sim (-\eta) \otimes k(\tau)$$

and $\tau \otimes k(\tau) \sim 0$, which means that τ is similar to a Pfister form (Theorem 5.8).

Now we prove the main result of this section.

THEOREM 7.14. *For an anisotropic form φ over k the following statements are equivalent:*

(i) *for any field $L \supset k$ there exists a form η over k such that $\eta \otimes L$ is the kernel form of $\varphi \otimes L$;*

(ii) *all higher kernel forms of φ are defined over k;*

(iii) *φ is excellent.*

Proof. That (i) implies (ii) is trivial, and that (ii) implies (i) is obvious from Theorem 5.1. We know from Proposition 7.9 that (iii) implies (ii). Thus we only have to show that (ii) implies (iii).

Let $(K_r, 0 \leqslant r \leqslant h)$ be a generic splitting tower of φ, and let η_r denote the form over k with

$$\ker(\varphi \otimes K_r) \cong (-1)^r \eta_r \otimes K_r \quad (0 \leqslant r \leqslant h),$$

in particular $\eta_0 = \varphi$. We choose for each η_r some a_r in k^* represented by η_r. We have to show that

$$\sigma_r := a_r \eta_r \perp a_r \eta_{r+1}$$

is a Pfister form for all r in $[0, h-1]$. We proceed by induction on h. The case where $h = 0$ is trivial and the case where $h = 1$ has been settled in Theorem 5.8. Assume from now on that $h > 1$. We know from Theorem 7.13 that σ_0 is a Pfister form. Further we know from the induction hypothesis, applied to the form $\eta_1 \otimes K_1$, that all $\sigma_i \otimes K_1$ with i in $[1, h-1]$ are Pfister forms.

We fix some r in $[1, h-1]$. If σ_r is isotropic, then $\sigma_r \otimes K_1 \sim 0$ and $\eta_r \otimes K_1 \sim (-\eta_{r+1}) \otimes K_1$, which is not true. Thus σ_r is anisotropic. Let $t = (t_1, \ldots, t_N)$ denote a sequence of $N = \dim \sigma_r$ indeterminates, let L denote the field $k(t)$, and let $\tilde{\varphi}$, $\tilde{\eta}_i$, $\tilde{\sigma}_i$ denote the forms $\varphi \otimes L$, $\eta_i \otimes L$, $\sigma_i \otimes L$ respectively. We have to show that the form

$$\gamma := (1, -\sigma_r(t)) \otimes \tilde{\sigma}_r$$

splits. Indeed, this means that σ_r is strongly multiplicative [7, § 2] and thus a Pfister form. Assume that γ does not split. Clearly $\gamma \otimes L(\tilde{\varphi})$ splits, since

$$\tilde{\sigma}_r \otimes L(\tilde{\varphi}) = \sigma_r \otimes (L \cdot k(\varphi))$$

is a Pfister form. Now $L(\tilde{\varphi})$ is equivalent over L to $L(\tilde{\sigma}_0)$, since $\tilde{\varphi}$ is a neighbour of the Pfister form $\tilde{\sigma}_0$. Thus $\gamma \otimes L(\tilde{\sigma}_0)$ splits. By Lemma 4.4 the form $\tilde{\sigma}_0$ divides γ. In particular $\dim \sigma_0 \leqslant 2 \dim \sigma_r$. But

$$\dim \sigma_0 > \dim \sigma_1 > \ldots > \dim \sigma_r$$

and all $\dim \sigma_i$ are powers of 2. Thus we have a contradiction if $r > 1$, and we obtain that all σ_r with $r > 1$ are indeed Pfister forms. In the case where $r = 1$ our assumption, that γ does not split, yields $\dim \sigma_0 = 2 \dim \sigma_1$ and then

$$\gamma = \langle 1, -\sigma_1(t) \rangle \otimes \tilde{\sigma}_1 \cong b\tilde{\sigma}_0$$

with some b in L^*. Now there exists some c in k^N with $\sigma_1(c) = 1$. Thus $b\tilde{\sigma}_0$ represents the element 1, and $b\tilde{\sigma}_0 \cong \tilde{\sigma}_0$. Substituting c for t in the equation

$$\langle 1, -\sigma_1(t) \rangle \otimes \tilde{\sigma}_1 \cong \tilde{\sigma}_0$$

we obtain by the principle of substitution [4, Corollary 2.5]

$$\langle 1, -1 \rangle \otimes \sigma_1 \cong \sigma_0.$$

Thus $\sigma_0 \sim 0$ and the neighbour φ of σ_0 is isotropic. This is the desired contradiction. σ_1 must also be a Pfister form.

REMARK 7.15. The proof shows, slightly more generally, the following. Assume that φ is an anisotropic form over k such that for some $s \geqslant 1$ in $[0, h(\varphi)]$ all rth kernel spaces of φ with r in $[0, s]$ are defined over k, and $\ker(\varphi \otimes K_r) \cong (-1)^r \eta_r \otimes K_r$. Then every η_r with r in $[0, s-1]$ is a Pfister neighbour with complementary form η_{r+1}.

We switch over to more elementary observations on excellent forms. If ψ is an excellent form over k and τ is a Pfister form over k, then $\tau \otimes \psi$ is again excellent, as follows from the definition 7.7 of excellent forms. Moreover if ζ_r is the rth complementary form of ψ then $\tau \otimes \zeta_r$ is the rth complementary form of $\tau \otimes \psi$.

The following question emerges: which Pfister forms divide a given excellent form? We use the following notation: φ is an excellent form of dimension at least 2; $\eta_0 = \varphi, \eta_1, \dots, \eta_t$ is the chain of complementary forms of φ; and finally ρ_r is the Pfister form associated with η_r for $0 \leqslant r \leqslant t-1$.

LEMMA 7.16. *If $0 \leqslant i < j \leqslant t-1$ then $\rho_i \cong \rho_j \otimes \mu_{ij}$ with some Pfister form μ_{ij} of dimension greater than* 1.

Proof. The assertion is trivial if $\rho_i \sim 0$. Assume that ρ_i is anisotropic. It suffices to consider the case where $j = i+1$. The field $k(\eta_{i+1})$ is equivalent to $k(\rho_{i+1})$ over k, and $k(\eta_{i+1})$ splits ρ_i. Thus $k(\rho_{i+1})$ splits ρ_i. It follows from Lemma 4.4 that ρ_{i+1} divides ρ_i, and it is also well known (cf. for example, [3]) that then ρ_i is the product of ρ_{i+1} and a Pfister form.

PROPOSITION 7.17. *Assume that our excellent form φ has even dimension.*

(i) *Every Pfister form γ, which divides φ, also divides all η_r. In particular $\gamma \mid \rho_{t-1}$.*

(ii) *There exists an excellent form ψ over k with $\varphi \cong \psi \otimes \rho_{t-1}$. The dimension of ψ is odd.*

Proof. We use the abbreviations $\rho := \rho_0$ and $\tau := \rho_{t-1}$. The first statement (i) follows by an easy induction on t, again with the use of Lemma 4.4. To prove the second statement (ii) we also proceed by induction on t. The case where $t = 1$ is trivial. Assume that $t > 1$. Let a denote an arbitrary element of k^* represented by η_1. We have $\varphi \perp \eta_1 \cong a\rho$. Further by the induction hypothesis $\eta_1 \cong \tau \otimes \zeta_1$ with an excellent form ζ_1 of odd dimension. Write $\zeta_1 \perp \zeta_2 \cong b\sigma$ with σ denoting the Pfister form associated with ζ_1. Clearly η_1 is a neighbour of $\sigma \otimes \tau$, hence $\rho_1 \cong \sigma \otimes \tau$. We obtain from Lemma 7.16 that

$$\rho \cong c\mu \otimes \sigma \otimes \tau$$

with some Pfister form μ of dimension at least 2 and c in k^*. Now ζ_1 is a subform of $b\sigma$ and does not coincide with $b\sigma$ since $t > 1$. Thus ζ_1 is a subform of $b\mu \otimes \sigma$ with

$$2 \dim \zeta_1 < \dim(\mu \otimes \sigma).$$

We obtain

$$b\mu \otimes \sigma \cong \psi \perp \zeta_1$$

with ψ a neighbour of $\mu \otimes \sigma$. Since ζ_1 is an excellent form of odd dimension, ψ is also an excellent form of odd dimension. Multiplying by τ we obtain

$$(\tau \otimes \psi) \perp \eta_1 \cong b\mu \otimes \sigma \otimes \tau \cong bc^{-1} \rho.$$

Here bc^{-1} can be replaced by the element a, and comparing this decomposition of $a\rho$ with the decomposition above we obtain $\varphi \cong \tau \otimes \psi$.

We continue to consider an excellent form φ of dimension at least 2 and to use the notation introduced before Lemma 7.16.

PROPOSITION 7.18. *Let s be a natural number with $1 \leqslant s \leqslant t$.*

(i) *If s is odd, then $\varphi \perp \eta_s$ is excellent and is a neighbour of ρ_0.*

(ii) *If s is even, then $\varphi \cong \psi \perp \eta_s$ with an excellent form ψ. If $s = 2$ and $\dim \rho_0 = 2 \dim \rho_1$, then ψ is similar to ρ_1. Otherwise ψ is a neighbour of ρ.*

Proof. We proceed by induction on s. We again use the abbreviation $\rho := \rho_0$ and denote by a a fixed element of k^* represented by η_1. The case where $s = 1$ is trivial. The case where $s = 2$, and $\dim \rho = 2 \dim \rho_1$ is also very easy. By Lemma 7.16 we have $\rho \cong \rho_1 \perp c\rho_1$ with some c in k^*. Further we know that

$$\varphi \perp \eta_1 \cong a\rho, \quad \eta_1 \perp \eta_2 \cong a\rho_1.$$

This yields $\varphi \cong ac\rho_1 \perp \eta_2$. From now on we assume that $s \geqslant 2$ and exclude the case where $s = 2$ and $\dim \rho = 2 \dim \rho_1$. Assume first that s is even. By the induction hypothesis

$$\gamma := \eta_1 \perp \eta_s$$

is excellent and a neighbour of ρ_1. Since $k(\gamma)$ and $k(\rho_1)$ are equivalent over k we obtain from Lemma 7.16 that $k(\gamma)$ splits ρ and then $\gamma \perp \psi \cong a\rho$ with some form ψ. Since $\gamma \cong \eta_1 \perp \eta_s$ this implies that $\varphi \cong \eta_s \perp \psi$. Clearly $\dim \gamma < \dim \rho_1$ if $s > 2$, and thus ψ is a neighbour of ρ in this case. If $s = 2$ then $\gamma \cong a\rho_1$, and ψ is again a neighbour of ρ, since we excluded the case where $\dim \rho = 2 \dim \rho_1$. The form ψ is excellent because γ is excellent.

Finally we consider the case where s is odd and $s \geqslant 3$. By the induction hypothesis $\eta_1 \cong \eta_s \perp \delta$ with excellent δ. Thus

$$(\varphi \perp \eta_s) \perp \delta \cong a\rho.$$

Clearly $\varphi \perp \eta_s$ is a neighbour of ρ, which is excellent since δ is excellent.

We mention a special case of this proposition.

COROLLARY 7.19. *Let φ be an excellent form of odd dimension $n \geqslant 3$, let ρ denote the Pfister form associated with φ and ρ_1 the Pfister form associated with the complementary form of φ. If $h(n)$ is odd then $\varphi \perp (-d(\varphi))$ is again excellent and a neighbour of ρ. If $h(n)$ is even there exists a decomposition*

$$\varphi \cong \psi \perp d(\varphi),$$

with ψ excellent. If $n = 2^r + 1$ for some $r \geqslant 1$ then ψ is similar to ρ_1. Otherwise ψ is again a neighbour of ρ.

We close this section with a discussion of the excellent forms of dimension less than 13, involving the invariants $d(\varphi)$ and $c(\varphi)$. We freely use the rules for computation of $c(\varphi)$ stated in [8, pp. 121 ff.]. If

τ is a quaternion form, that is, a Pfister form of dimension 4,† then $[\tau]$ denotes the Brauer class of the corresponding quaternion algebra, thus $c(\tau) = [\tau]$.

$\dim \varphi \leqslant 4$: all forms of dimension at least 3 are excellent. A four-dimensional form φ is excellent if and only if $d(\varphi) = 1$.

$\dim \varphi = 8$: φ is excellent, that is, similar to a Pfister form, if and only if $h(\varphi) \leqslant 1$, which is equivalent to $\deg(\varphi) \geqslant 3$. Thus by Proposition 5.10 φ is excellent if and only if $d(\varphi) = 1$ and $c(\varphi) = 1$ (cf. [8, p. 123]).

$\dim \varphi = 5$: $c(\varphi)$ is always representable as a product of at most two quaternion algebras, and if φ is anisotropic, then

$$h(\varphi) = 2, \quad i_1(\varphi) = i_2(\varphi) = 1.$$

The following statements are equivalent:

(i) φ is excellent;

(ii) $d(\varphi) < \varphi$;

(iii) $c(\varphi)$ is representable by a quaternion algebra.

Proof. That (i) implies (ii) follows from Corollary 7.19.

(ii) $\Rightarrow$ (iii): we have $\varphi \cong d(\varphi) \perp a\tau$ with τ a quaternion form and a in k^*. Thus $c(\varphi) = [\tau]$.

(iii) $\Rightarrow$ (i): assume that $c(\varphi) = [\tau]$. Then

$$\psi := \varphi \perp (-d(\varphi)) \otimes \tau'$$

has invariants $d(\psi) = 1$, $c(\psi) = 1$, and dimension 8. Thus ψ is similar to a Cayley form,‡ and φ is excellent.

$\dim \varphi = 6$: the following statements are equivalent:

(i) φ is excellent;

(ii) φ is divisible by $\langle 1, -d(\varphi) \rangle$;

(iia) φ is divisible by a binary form $\langle 1, -c \rangle$;

(iii) $c(\varphi)$ is split by $k(\sqrt{d(\varphi)})$.

If φ is anisotropic, then these statements are also equivalent to

(iv) $i_1(\varphi) = 2$.

Proof. That (i) implies (ii) is clear from Proposition 7.17. The implications (ii) $\Leftrightarrow$ (iia) and (ii) $\Rightarrow$ (iii) are trivial.

(iii) $\Rightarrow$ (ii): we may assume that φ is not hyperbolic. Let φ_0 denote the kernel form of φ. The invariants $d(\varphi_0)$ and $c(\varphi_0)$ become trivial over $K := k(\sqrt{d(\varphi)})$. Thus $\varphi_0 \otimes K$ has degree at least 3, that is, $\varphi_0 \otimes K$ splits.

† Some authors more generally call all forms which are similar to Pfister forms of dimension 4 quaternion forms.

‡ We use the term 'Cayley form' for Pfister forms of dimension 8.

This implies that $d(\varphi) \neq 1$ and

$$\varphi_0 \cong \langle 1, -d(\varphi)\rangle \otimes \chi$$

with some form χ over k, cf. Lemma 4.4 or [8, p. 123]. Since

$$d(\varphi_0) = d(\varphi) \neq 1$$

the form χ has dimension 1 or 3. Thus φ is also divisible by $\langle 1, -d(\varphi)\rangle$.

(iia) $\Rightarrow$ (i): if $\varphi \cong \langle 1, -c\rangle \otimes \psi$, then $\dim \psi = 3$ and ψ is excellent. Thus φ is also excellent.

For anisotropic φ the implications (iii) $\Leftrightarrow$ (iv) have already been established in Example 5.17.

$\dim \varphi = 7$: φ is excellent if and only if $c(\varphi) = 1$. Indeed, a neighbour of codimension 1 of a Pfister form of degree at least 3 has trivial Clifford invariant. On the other hand, if $c(\varphi) = 1$, then

$$\psi := \varphi \perp (-d(\varphi))$$

has dimension 8 and trivial invariants. Hence ψ is similar to a Pfister form.

$\dim \varphi = 9$: φ is excellent if and only if $d(\varphi) < \varphi$ and $c(\varphi) = 1$. Indeed, if φ is excellent, then $d(\varphi) < \varphi$ by Corollary 7.19, and that $c(\varphi) = 1$ is also easily verified. Assume now that $\varphi \cong d(\varphi) \perp \psi$ and $c(\varphi) = 1$. Then $c(\psi) = 1$ and $d(\psi) = 1$. Thus ψ is similar to a Pfister form.

$\dim \varphi = 10$: φ is excellent if and only if the following two conditions are fulfilled:

(a) φ is divisible by $\langle 1, -d(\varphi)\rangle$, and thus $c(\varphi)$ is split by $k(\sqrt{d(\varphi)})$;

(b) $\varphi > \eta$ with η the unique binary form having the invariants $d(\eta) = d(\varphi)$ and $c(\eta) = c(\varphi)$.

Proof. φ is excellent if and only if $\varphi \cong \langle 1, -d\rangle \otimes \psi$ with $\langle d\rangle = d(\varphi)$ and ψ excellent (Proposition 7.17). The assertion now follows from the previous description of excellent forms of dimension 5.

$\dim \varphi = 12$: φ is excellent if and only if φ is divisible by some quaternion form τ, and then $c(\varphi) = [\tau]$. This follows as in the analogous statement for the case where $\dim \varphi = 6$.

$\dim \varphi = 11$: φ is excellent if and only if $\psi := \varphi \perp (-d(\varphi))$ is excellent, whence if and only if $c(\varphi) = [\tau]$ for some quaternion form τ and τ divides ψ. Indeed, if φ is excellent then ψ is also excellent by Corollary 7.19. On the other hand, if ψ is excellent, then $\psi \cong a\tau \otimes \sigma'$ with some quaternion form σ. Thus φ is a neighbour of $\tau \otimes \sigma$ with complementary form

$$\eta := (-d(\varphi)) \perp a\tau.$$

η is excellent, hence so is φ.

8. Pfister neighbours and conjugate forms

We want to characterize anisotropic Pfister neighbours by intrinsic properties. Later we shall also deal with subforms of Pfister forms of degree n which have dimension 2^{n-1}.

PROPOSITION 8.1. *Let t be a natural number less than 5, and let φ be an anisotropic form over k. The following statements are equivalent:*

(i) *φ is a Pfister neighbour of codimension t;*

(ii) $\dim\varphi > t$, *and there exists an anisotropic form η of dimension t over k with $\varphi \otimes k(\varphi) \sim \eta \otimes k(\varphi)$;*

(iii) *the first kernel form of φ is defined over k and has dimension t.*

N.B. Of course then $-\eta$ is the complementary form of φ.

Proof. That (i) implies (ii) is evident, and that (iii) implies (i) is clear from Theorem 7.13. Assume now that (ii) holds true. We want to prove (iii). The form η is by Proposition 7.2 uniquely determined by φ, and we have to show that $\eta \otimes k(\varphi)$ is anisotropic. This is clear if $t \leqslant 3$ or if $t = 4$ and $d(\eta) = 1$, since then η is excellent and Lemma 7.6 applies. From now on we assume that $d(\eta) = \langle d \rangle \neq 1$ and $t = 4$. We consider the field $F := k(\sqrt{d})$ and the forms $\tilde{\varphi} := \varphi \otimes F$, $\tilde{\eta} := \eta \otimes F$. If $\tilde{\eta}$ were isotropic, then $\tilde{\eta}$ would split since $d(\tilde{\eta}) = 1$. Thus the form $\langle 1, -d \rangle$ would divide η (Lemma 4.4), which is a contradiction to $d(\eta) \neq 1$. Thus $\tilde{\eta}$ is anisotropic (cf. [9, Satz 14]). Suppose $\eta \otimes k(\varphi)$ is isotropic. Then

$$\tilde{\eta} \otimes F(\tilde{\varphi}) = \eta \otimes (F \cdot k(\varphi))$$

is also isotropic, and hence splits since it has discriminant 1. By Lemma 4.5 the form $\tilde{\varphi}$ is similar to a subform of $\tilde{\eta}$ which contradicts

$$\dim \varphi > \dim \eta.$$

Thus $\eta \otimes k(\varphi)$ is anisotropic.

Recalling Example 7.3 we obtain from this proposition the following results for $t = 2, 3$.

COROLLARY 8.2. *Assume that φ is an anisotropic form over k. Then φ is a Pfister neighbour of codimension 2 if and only if $h(\varphi) = 2$, $\dim\varphi$ is even, $d(\varphi) \neq 1$, and $c(\varphi)$ is split by $k(\sqrt{d(\varphi)})$. The form φ is a Pfister neighbour of codimension 3 if and only if $h(\varphi) = 2$, $\dim\varphi$ is odd, and $c(\varphi)$ is the Brauer class of a non-split quaternion algebra.*

One may ask whether Proposition 8.1 remains true for $t > 4$. This is clearly equivalent to the following question.

QUESTION 8.3. Assume that φ and η are anisotropic forms over k with $\dim\varphi > \dim\eta$ and $\varphi\otimes k(\varphi) \sim \eta\otimes k(\varphi)$. Is $\eta\otimes k(\varphi)$ anisotropic?

I cannot answer this question even under the stronger assumption that φ is a Pfister neighbour. This lack of knowledge compels us to pose the following weaker question.

QUESTION 8.4. Assume that φ and η are anisotropic forms over k with $\dim\eta < \dim\varphi$ and $\varphi\otimes k(\varphi) \sim (-\eta)\otimes k(\varphi)$. Is φ a Pfister neighbour?

N.B. If the answer is 'Yes', then η must be the complementary form of φ.

We give a partial answer to this question.

THEOREM 8.5. *Let t denote the dimension of η, and let 2^n denote the smallest 2-power not less than $\dim\varphi$. Assume that in addition one of the following conditions is fulfilled:*

(A) $t \leqslant 4$;

(B) $t = 5, \dim\varphi \neq 2^n-3$;

(C) $t = 6, \dim\varphi \neq 2^n-4$, *and* $\dim\varphi \neq 2^n-2$;

(D) $t \geqslant 7, 2^{n-1}+(t-6) \leqslant \dim\varphi \leqslant 2^n-t$.

Then φ is a Pfister neighbour.

We prove this theorem by induction on t, proceeding as in the proof of Theorem 7.13. The case where $t \leqslant 4$ is already settled. Now let $t > 4$. We assume that $\eta\otimes k(\varphi)$ is isotropic, since otherwise we have finished by Theorem 7.13.

As in the proof of this theorem we first show that the form $\tau := \varphi\perp\eta$ is anisotropic. Suppose τ is isotropic. We then have decompositions

$$\varphi \cong \langle -c\rangle\perp\chi, \quad \eta \cong \langle c\rangle\perp\gamma,$$

and we see, as in the proof of Theorem 7.13, that $k(\chi)$ is equivalent to $k(\varphi)$ and thus

$$\chi\otimes k(\chi) \sim (-\gamma)\otimes k(\chi).$$

From this we want to deduce, by use of the induction hypothesis, that

$$(*) \qquad \chi\perp\gamma \cong a\rho$$

with a in k^* and ρ a Pfister form. If $t = 5$, then $\dim\gamma = 4$, and we have no problem. If $t = 6$, then $\dim\varphi$ is even, whence $\dim\varphi \geqslant 2^{n-1}+2$. Thus 2^n is still the smallest 2-power above $\dim\chi$. Furthermore $\dim\chi \neq 2^n-3$ since we excluded the case where $\dim\varphi = 2^n-2$. Let $t = 7$. If $\dim\varphi > 2^{n-1}+1$, then 2^n is the smallest 2-power not less than $\dim\chi$ and $\dim\chi \leqslant 2^n-8$. If $\dim\varphi = 2^{n-1}+1$, then $\dim\chi = 2^{n-1}$. Thus for χ and γ

the condition (C) is fulfilled. Assume now that $t \geqslant 8$. Then

$$\dim \varphi \geqslant 2^{n-1} + (t-6) \geqslant 2^{n-1} + 2.$$

2^n is the smallest 2-power above $\dim \chi$ and clearly condition (D) for φ, η implies the same condition for χ, γ. Thus the induction hypothesis can be applied to χ and γ in all cases, and (*) is proved. As in the proof of Theorem 7.13 we obtain from (*) that $\varphi \otimes k(\varphi)$ is equivalent to $(-\delta) \otimes k(\varphi)$ for some form δ over k of strictly smaller dimension than η. This contradicts Proposition 7.2, and we learn that $\tau = \varphi \perp \eta$ is anisotropic.

We now want to show that $\varphi \otimes k(\tau)$ is isotropic. This will imply that $\varphi \otimes k(\tau) \sim (-\eta) \otimes k(\tau)$, whence $\tau \otimes k(\tau) \sim 0$, and our theorem will be proved.

Suppose $\varphi \otimes k(\tau)$ is anisotropic. As in the proof of Theorem 7.13 we use the notation $L := k(\tau)$, $\tilde{\varphi} := \varphi \otimes L$, $\tilde{\eta} := \eta \otimes L$. Since $L(\tilde{\varphi}) = k(\varphi) \cdot k(\tau)$ is a (purely transcendental) extension of $k(\varphi)$, we have

$$\tilde{\varphi} \otimes L(\tilde{\varphi}) \sim (-\tilde{\eta}) \otimes L(\tilde{\varphi}).$$

If $\tilde{\eta}$ were anisotropic, then applying what we have just proved to $\tilde{\varphi}, \tilde{\eta}$ instead of φ, η, we would obtain that $\tau \otimes k(\tau) = \tilde{\varphi} \perp \tilde{\eta}$ is anisotropic, which is not true. Thus $\tilde{\eta}$ is isotropic, and the fields L and $k(\eta)$ are equivalent over k. Let ζ denote the kernel form of $\tilde{\eta}$. We want to deduce a contradiction from the fact that

$$\tilde{\varphi} \otimes L(\tilde{\varphi}) \sim (-\zeta) \otimes L(\tilde{\varphi}),$$

by use of the induction hypothesis. If $t = 5$, then $\dim \zeta = 3$, since otherwise η would have height 1, which is impossible. If $t = 6$, then $i_1(\eta) \neq 2$, since otherwise η would be excellent, as was shown at the end of §7. Since $i_1(\eta) = 3$ is also impossible, we have $\dim \zeta = 4$. If $t \geqslant 7$, then $\dim \zeta = s \leqslant t-2$, and we obtain from our assumption (D) that

$$2^{n-1} + (s-6) < \dim \varphi < 2^n - s.$$

Thus we can apply the induction hypothesis in all cases to $\tilde{\varphi}$ and ζ. We obtain that $\tilde{\varphi} \perp \zeta$ is similar to a Pfister form, whence

$$\dim \varphi + \dim \zeta = 2^n.$$

But this contradicts our assumptions (B), (C), (D) on $\dim \varphi$. Thus $\tilde{\varphi}$ is isotropic and Theorem 8.5 is proved.

COROLLARY 8.6. *Let t be a natural number less than* 12. *Let φ be an anisotropic form over k and let 2^n denote the smallest 2-power such that $2^n \geqslant \dim \varphi$. Then the following are equivalent:*

(i) *φ is a Pfister neighbour of codimension t;*

(ii) $\dim \varphi = 2^n - t$, *and there exists an anisotropic form η of dimension t over k with $\varphi \otimes k(\varphi) \sim (-\eta) \otimes k(\varphi)$.*

Proof. That (i) implies (ii) is evident.

(ii) $\Rightarrow$ (i): one easily checks for $7 \leqslant t \leqslant 11$ that

$$2^n - t \geqslant 2^{n-1} + (t-6).$$

Thus we obtain from Theorem 8.5 that φ is a Pfister neighbour.

The first case we cannot handle by Theorem 8.5 is that where $t = 12$ and $\dim \varphi = 20$.

By the same methods we now study the phenomenon of 'conjugate forms'.

DEFINITION 8.7. Two forms φ and η over k are called *conjugate* if they have the same dimension and

$$(*) \qquad \varphi \perp (-\eta) \cong a\rho$$

with some Pfister form ρ and a in k^*. We then also say that φ and η are *half-neighbours* of the Pfister form ρ.

Clearly every form of 2-power dimension is conjugate to itself. Leaving this trivial case aside, we see that φ, η, and ρ must be anisotropic if (*) holds true.

THEOREM 8.8. *Let φ and η be anisotropic forms over k of dimension at least* 2. *The following are equivalent:*

(i) *φ and η are conjugate or $\varphi \cong \eta$;*

(ii) *$\varphi \otimes k(\varphi) \cong \eta \otimes k(\varphi)$ and $\varphi \otimes k(\eta) \cong \eta \otimes k(\eta)$;*

(iii) *for every field extension L of k the form $\varphi \otimes L$ is isotropic if and only if $\eta \otimes L$ is isotropic, and then $\varphi \otimes L \cong \eta \otimes L$.*

The implications (i) $\Rightarrow$ (ii) and (ii) $\Leftrightarrow$ (iii) are obvious. Thus we only have to show that (ii) implies (i). The following more general theorem holds true.

THEOREM 8.9. *Let φ and η be anisotropic forms over k of dimension at least* 2. *Assume that*

$$\varphi \otimes k(\varphi) \sim \eta \otimes k(\varphi), \quad \varphi \otimes k(\eta) \sim \eta \otimes k(\eta).$$

Then either $\varphi \cong \eta$ or $\varphi \perp (-\eta)$ is similar to a Pfister form.

Proof. We assume that $\dim \varphi \geqslant \dim \eta$ and we proceed by induction on $\dim \varphi$. If $\dim \varphi = 2$, that is, $\varphi \cong a\sigma$ with a in k^* and some Pfister form σ of degree 1, then $\eta \otimes k(\sigma) \sim 0$. By Lemma 4.4 the form σ divides η and thus $\eta \cong b\sigma$ with some b in k^*. Thus the assertion is evident for $\dim \varphi = 2$.

Now let $\dim \varphi > 2$. We again proceed as in the proof of Theorem 7.13. We show first that either the form $\tau := \varphi \perp (-\eta)$ is anisotropic or $\varphi \cong \eta$.

Suppose τ is isotropic. We then have decompositions

$$\varphi \cong \langle c\rangle \perp \chi, \quad \eta \cong \langle c\rangle \perp \gamma,$$

and we obtain from our assumptions

$$(*) \qquad \chi \otimes k(\varphi) \sim \gamma \otimes k(\varphi).$$

Since there exists a place from $k(\varphi)$ to $k(\chi)$ over k, this implies that

$$\chi \otimes k(\chi) \sim \gamma \otimes k(\chi).$$

Since $\dim \varphi \geqslant \dim \eta$ and $\varphi \otimes k(\eta)$ is equivalent to $\eta \otimes k(\eta)$ certainly $\varphi \otimes k(\eta)$ is isotropic. Thus we have a place from $k(\varphi)$ to $k(\eta)$ over k. If $\dim \gamma > 1$ then we also have a place from $k(\eta)$ to $k(\gamma)$ over k. Therefore we obtain from (*)

$$\chi \otimes k(\gamma) \sim \gamma \otimes k(\gamma),$$

provided $\dim \gamma > 1$. By the induction hypothesis we now conclude that in the case where $\dim \gamma > 1$ either $\chi \cong \gamma$, that is, $\varphi \cong \eta$, or

$$(**) \qquad \chi \perp (-\gamma) \cong a\rho$$

with some Pfister form ρ and a in k^*. Clearly (**) also holds true if $\dim \gamma = 1$.

We now show that a relation (**) is impossible, and thus φ must be isomorphic to η. Indeed, we obtain from (*) and (**) that $\rho \otimes k(\varphi)$ splits; hence

$$\varphi \perp (-\delta) \cong b\rho$$

with some anisotropic form δ, and we have

$$\varphi \otimes k(\varphi) \sim \delta \otimes k(\varphi) \sim \eta \otimes k(\varphi).$$

This contradicts Proposition 7.2, since $\dim \eta = \dim \delta + 2$. Thus we finally draw the conclusion that either $\tau = \varphi \perp (-\eta)$ is anisotropic or $\varphi \cong \eta$.

We assume from now on that φ is not isomorphic to η and hence that τ is anisotropic. If $\varphi \otimes k(\tau)$ is isotropic then $\varphi \otimes k(\tau) \sim \eta \otimes k(\tau)$, whence $\tau \otimes k(\tau)$ splits, and we obtain the desired result that τ is similar to a Pfister form. We now consider the case where $\varphi \otimes k(\tau)$ is anisotropic. Then $\eta \otimes k(\tau)$ is also anisotropic. Indeed, otherwise $\varphi \otimes k(\tau)$ would be equivalent to $\eta \otimes k(\tau)$ and $\eta \otimes k(\tau)$ would be isotropic. This contradicts the fact that $\dim \varphi \geqslant \dim \eta$ and $\varphi \otimes k(\tau)$ is anisotropic.

We use the abbreviations $L := k(\tau)$, $\tilde{\varphi} := \varphi \otimes L$, $\tilde{\eta} := \eta \otimes L$. Since $L(\tilde{\varphi})$ is a field extension of $k(\varphi)$ we have

$$\tilde{\varphi} \otimes L(\tilde{\varphi}) = \varphi \otimes L(\tilde{\varphi}) \sim \eta \otimes L(\tilde{\varphi}) = \tilde{\eta} \otimes L(\tilde{\varphi}),$$

and for the same reasons we have

$$\tilde{\varphi} \otimes L(\tilde{\eta}) \sim \tilde{\eta} \otimes L(\tilde{\eta}).$$

But $\tilde{\varphi} \perp (-\tilde{\eta}) = \tau \otimes L$ is certainly isotropic. Thus as proved above $\tilde{\varphi} \cong \tilde{\eta}$. This means that $\tau \otimes k(\tau) \sim 0$, and τ is similar to a Pfister form.

In the case where $\dim \varphi > \dim \eta$ this theorem is a statement about Pfister neighbours.

COROLLARY 8.10. *Let φ and η be anisotropic forms over k with*

$$\dim \varphi > \dim \eta.$$

The following are equivalent:

(i) *φ is a Pfister neighbour with complementary form $-\eta$;*

(ii) *$\varphi \otimes k(\varphi) \sim \eta \otimes k(\varphi)$ and $\varphi \otimes k(\eta) \sim \eta \otimes k(\eta)$;*

(iii) *$\varphi \otimes k(\varphi) \sim \eta \otimes k(\varphi)$ and $\varphi \otimes k(\eta)$ is isotropic.*

Indeed, the equivalence of (ii) and (iii) is obvious, and the equivalence of (i) and (ii) is stated in the previous theorem.

If φ is a half-neighbour of a Pfister form ρ_1 then it may well happen that φ is a half-neighbour of still another Pfister form ρ_2, as is easily shown by examples. We study the relation between ρ_1 and ρ_2 in this case. We need the following result of Elman and Lam (see [**3**, Proposition 4.4] and its proof).

LEMMA 8.11. *Assume that ρ and γ are anisotropic Pfister forms of degree at least 1 and let τ be a maximal common Pfister divisor, that is, a Pfister form dividing ρ and γ such that every Pfister form η with $\tau | \eta$, $\eta | \rho$, $\eta | \gamma$ coincides with τ. Then*

$$i(\rho \perp (-\gamma)) = \dim \tau.$$

PROPOSITION 8.12. *Let φ be an anisotropic form over k. Assume that ρ_1 and ρ_2 are Pfister forms which are both half-neighbours of φ. Then ρ_1 and ρ_2 are linked, that is,*

$$\rho_1 \cong \sigma \otimes \langle 1, a_1 \rangle, \quad \rho_2 \cong \sigma \otimes \langle 1, a_2 \rangle,$$

with σ a Pfister form and a_1, a_2 in k^*.

Proof. Let b be an element of k^* represented by φ. Then we have equations

$$\varphi \perp (-\eta_1) \cong b\rho_1, \quad \varphi \perp (-\eta_2) \cong b\rho_2.$$

From this we obtain

$$b(\rho_1 \perp -\rho_2) \cong \varphi \otimes H \perp \eta_2 \perp (-\eta_1).$$

If φ has the dimension 2^n, then the ρ_i have degree $(n+1)$ and $\rho_1 \perp -\rho_2$ has index at least 2^n. The assertion follows from the preceding lemma.

We finally point out that neighbours and half-neighbours of Pfister forms behave very well with respect to specializations. Let $\lambda \colon k \to k' \cup \infty$

be an arbitrary place, and let φ be a form over k, which has good reduction with respect to λ. Further let σ be a Pfister form of degree $n \geqslant 1$ over k.

PROPOSITION 8.13. (i) *If σ has good reduction with respect to λ, then $\lambda_*(\sigma)$ is again a Pfister form.*

(ii) *If φ is a neighbour of σ, then σ has good reduction, and $\lambda_*(\varphi)$ is a neighbour of $\lambda_*(\sigma)$.*

(iii) *Assume that φ is a half-neighbour of σ. If σ has good reduction then $\lambda_*(\varphi)$ is a half-neighbour of $\lambda_*(\sigma)$. Otherwise $\lambda_*(\varphi)$ is a Pfister form.*

Proof. The first assertion follows from the fact that

$$h(\lambda_*(\sigma)) \leqslant h(\sigma) \leqslant 1,$$

cf. Corollary 5.6. To prove the other assertions we first retreat by standard arguments to the case where k is a finitely generated field (over its prime field), and hence a valuation associated with λ has finite rank [**2**, § 10, no. 3].† Then λ is a composite of places with valuations of rank at most 1. Thus we may assume that k' is the residue class ring $\mathfrak{o}/\mathfrak{m}$ of a valuation ring $\mathfrak{o}$ of k of rank 1 by its maximal ideal $\mathfrak{m}$, and that λ is the canonical place from k to k'. Let $v\colon k^* \twoheadrightarrow \Gamma$ denote a valuation corresponding to $\mathfrak{o}$. Assume that σ has bad reduction with respect to λ. We write

$$\sigma = \langle 1, a_1\rangle \otimes \ldots \otimes \langle 1, a_n\rangle,$$

and after a permutation of the factors we assume that for some r in $[1, n]$ the values $v(a_1), \ldots, v(a_r)$ have linearly independent images $\bar{v}(a_1), \ldots, \bar{v}(a_r)$ in the vector space $\Gamma/2\Gamma$ over the field of two elements, while the $\bar{v}(a_i)$ with $i > r$ are linearly dependent on $\bar{v}(a_1), \ldots, \bar{v}(a_r)$. Then we may replace the a_i with $i > r$ by units of $\mathfrak{o}$ and obtain

$$\sigma = \langle 1, a_1\rangle \otimes \ldots \otimes \langle 1, a_r\rangle \otimes \tau$$

with τ a Pfister form of degree $s := n - r$ having good reduction. Since $\mathfrak{o}$ has rank 1 and σ is assumed to have bad reduction the Pfister form $\lambda_*(\tau)$ does not split and hence is anisotropic [**6**, § 12].

We assume that φ is a neighbour or a half-neighbour of σ, and, without loss of generality, that φ represents 1. Then we have an equation

$$(*) \qquad \varphi \perp \eta \cong \sigma.$$

Again by [**6**, § 12] we can write

$$\eta \cong \perp_{\varepsilon} \langle a_1^{\varepsilon_1} \ldots a_r^{\varepsilon_r}\rangle \eta_\varepsilon$$

with $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ running through the multi-indices with coordinates $\varepsilon_i = 0$ or 1, all η_ε having good reduction, and the forms $\lambda_*(\eta_\varepsilon)$ with $\varepsilon \neq 0$

† Bourbaki uses the word 'height' instead of 'rank'.

being anisotropic $(0 = (0, \ldots, 0))$. Multiplying the equation (*) by $\langle a_1^{\varepsilon_1} \ldots a_r^{\varepsilon_r} \rangle$ and applying the additive map $\lambda_*: W(k) \to W(k')$ [4, § 3] we learn that $\lambda_*(\eta_\varepsilon)$ is isomorphic to $\lambda_*(\tau)$ for all $\varepsilon \neq 0$. Counting dimensions we see that $\varphi \perp \eta_0$ has dimension 2^s. If φ is a neighbour of σ this is clearly a contradiction, since $s \leqslant n-1$. If φ is a half-neighbour, then we must have $s = n-1$ and $\eta_0 = 0$, and applying our map $\lambda_*: W(k) \to W(k')$ to the equation (*) we obtain $\lambda_*(\varphi) \cong \lambda_*(\tau)$. All assertions of our proposition are now evident.

9. Forms with leading form defined over the base field

The primary source of this section has been the desire to describe all forms of height 2 with discriminant not equal to 1. This is possible (cf. Theorem 10.3) and leads to the study of 'forms with leading form defined over the base field'.

Assume that φ is a non-split form over k and F is a leading field of φ (cf. Definition 5.4). Let σ denote the leading form of φ over F.

Definition 9.1. We say that the leading form of φ is *defined over* k if there exists some form τ over k with $\sigma \cong \tau \otimes F$.

Clearly this property does not depend on the choice of F. Of course the leading form is defined over k if φ is excellent. By Proposition 5.10 the leading form is also defined over k if φ has even dimension and discriminant not equal to 1, or if $\dim \varphi$ is even, $d(\varphi) = 1$, and $c(\varphi)$ is representable by a quaternion algebra, or if $\dim \varphi$ is odd and $c(\varphi)$ is representable by a quaternion algebra. Moreover we have seen in § 6 that the leading form is defined over k if $\varphi \sim \gamma \perp \psi$ with γ a Pfister form of some degree $n < \infty$ and ψ is of degree at least $n+1$. Indeed, then $\sigma \cong \gamma \otimes F$ (Theorem 6.3).

Proposition 9.2. *Assume that the leading form of our form φ is defined over k.*

(i) *If φ has even dimension there exists up to isomorphism a unique form τ over k with $\tau \otimes F \cong \sigma$.*

(ii) *If φ has odd dimension there exists up to isomorphism a unique form τ over k with $\tau \otimes F \cong \sigma$ and $\langle 1 \rangle < \tau$.*

(iii) *In both cases τ is a Pfister form.*

Proof. We choose a generic splitting tower $(K_i, 0 \leqslant i \leqslant h)$ of φ and assume that $F = K_{h-1}$. Let τ_1 and τ_2 be forms over k with

$$\tau_1 \otimes K_{h-1} \cong \tau_2 \otimes K_{h-1} \cong \sigma.$$

Suppose τ_1 and τ_2 are not isomorphic. We must have $h \geqslant 2$. Let s denote the maximal number in $[0, h-2]$ with $\tau_1 \otimes K_s \not\cong \tau_2 \otimes K_s$, and let ζ denote the kernel form of $[\tau_1 \perp (-\tau_2)] \otimes K_s$. Clearly $\zeta \neq 0$. For $0 \leqslant r \leqslant h$ we denote the kernel form of $\varphi \otimes K_r$ by φ_r. Since ζ is split by $K_s(\varphi_s)$, we obtain by Lemma 4.5 an equation

$$\varphi_s \perp \psi \cong c\zeta$$

with some form ψ over K_s and some c in K_s^*. Clearly

$$\varphi_s \otimes K_{s+1} \sim (-\psi) \otimes K_{s+1},$$

and thus $\dim \psi \geqslant \dim \varphi_{s+1}$. We obtain

$$(*) \qquad 2 \dim \tau_1 \geqslant \dim \zeta \geqslant \dim \varphi_s + \dim \varphi_{s+1} \geqslant 2 \dim \varphi_{s+1} + 2.$$

If $\dim \varphi$ is even, this is a contradiction, and thus $\tau_1 \cong \tau_2$. Assume now that $\dim \varphi$ is odd and that τ_1 and τ_2 both represent $\langle 1 \rangle$. Then $\tau_1 \perp (-\tau_2)$ is isotropic. Using (*) we see that

$$2 \dim \tau_1 \geqslant \dim \zeta + 2 \geqslant 2 \dim \varphi_{s+1} + 4,$$

which is a contradiction. Thus $\tau_1 \cong \tau_2$ in this case too. Of course it is also possible to find some τ over k with $\langle 1 \rangle < \tau$ and $\tau \otimes K_{h-1} \cong \sigma$. Indeed, if η is a form over k with $\eta \otimes K_{h-1} \cong \sigma$ and c is some element of k^* represented by η, then $c\eta$ is such a form τ. We have proved (i) and (ii).

Now let τ denote the unique form over k with $\langle 1 \rangle < \tau$ and $\tau \otimes K_{h-1} \cong \sigma$, and let $t = (t_1, \ldots, t_n)$ denote a sequence of $n = \dim \tau$ indeterminates over k. We consider the forms $\tilde{\varphi} := \varphi \otimes k(t)$ and $\tilde{\tau} := \tau \otimes k(t)$. By Proposition 5.13 the tower $(K_i(t), 0 \leqslant i \leqslant h)$ is a generic splitting tower of $\tilde{\varphi}$, and $\tilde{\tau} \otimes K_{h-1}(t)$ is the leading form of $\tilde{\varphi}$ over $K_{h-1}(t)$. Since this leading form is a Pfister form, we have

$$\tilde{\tau} \otimes K_{h-1}(t) \cong [\tau(t)\tilde{\tau}] \otimes K_{h-1}(t).$$

Now $\tau(t)\tilde{\tau}$ also represents 1. Thus, as proved above, $\tau(t)\tilde{\tau} \cong \tilde{\tau}$. This means that τ is strongly multiplicative and hence a Pfister form.

Remark. If $\dim \varphi$ is odd then the condition $\langle 1 \rangle < \tau$ in Proposition 9.2 cannot be omitted. Consider, for example, a form $\varphi = \rho \perp \langle a \rangle$ with some Pfister form ρ such that $\langle 1, a \rangle \otimes \rho$ is anisotropic. Then φ is a neighbour of $\langle 1, a \rangle \otimes \rho$ with complementary form $a\rho'$. The leading form of φ is $\rho \otimes K_1$. Since $\langle 1, a \rangle \otimes \rho \otimes K_1 \sim 0$ we have $(-a\rho) \otimes K_1 \cong \rho \otimes K_1$. But $-a\rho$ is not isomorphic to ρ.

Definition 9.3. Let φ be a non-split form over k, whose leading form σ is defined over k. Let τ be the unique Pfister form over k with $\tau \otimes K_{h-1} \cong \sigma$. Then we say that σ is *defined by* τ over k.

We want to obtain information about those forms whose leading forms are defined over k. It suffices to regard even-dimensional forms, since the following proposition holds true.

PROPOSITION 9.4. *Let φ be a non-split odd-dimensional form over k, and let τ be a non-split Pfister form over k. The leading form of φ is defined over k by τ if and only if the leading form of $\psi := \varphi \perp (-d(\varphi))$ is defined over k by τ.*

Proof. If the leading form of ψ is defined over k by τ then according to Proposition 5.12 the same holds true for the leading form of φ. We assume now that the leading form of φ is defined over k by τ and, without loss of generality, that $d(\varphi) = -1$. Let F be a leading field of φ, let E be a leading field of ψ, and let σ denote the leading form of ψ over E. By Proposition 5.12 $\dim \sigma = \dim \tau$. Suppose σ is not isomorphic to $\tilde{\tau} := \tau \otimes E$. Then $\sigma \otimes E(\tilde{\tau})$ is anisotropic, $\psi \otimes E$ has the kernel form $a\sigma$ with some a in E^*, and $\varphi \otimes E(\tilde{\tau})$ is equivalent to

$$\alpha := a(\sigma \otimes E(\tilde{\tau})) \perp \langle -1 \rangle.$$

Let T be any field extension of $E(\tilde{\tau})$ such that $\alpha \otimes T$ is isotropic. Then the kernel form of $\varphi \otimes T$, which coincides with the kernel form of $\alpha \otimes T$, has dimension no greater than $\dim \tau'$. Thus there exists a place from F to T over k. Since $\varphi \otimes F$ is equivalent to $\tau' \otimes F$ and $\tau \otimes T$ splits, we obtain

$$\alpha \otimes T \sim \varphi \otimes T \sim \tau' \otimes T \sim \langle -1 \rangle,$$

whence $\sigma \otimes T \sim 0$. Choosing $T = E(\tilde{\tau})$ we see that α itself is certainly anisotropic. Now our study of α shows that α has height 1, and hence is isomorphic to the pure part of a Pfister form. But

$$\dim \alpha + 1 = \dim \tau + 2$$

is not a 2-power, since $\dim \tau$ is a 2-power greater than 2. This is the desired contradiction, which proves that $\sigma \cong \tau \otimes E$.

There always exists a place from E to F over k (Proposition 5.12). One may ask under which circumstances the fields E and F are actually equivalent.

REMARK 9.5. Let $(K_r, 0 \leqslant r \leqslant h)$ be a generic splitting tower of our odd-dimensional form φ and let $(L_s, 0 \leqslant s \leqslant e)$ be a generic splitting tower of $\psi := \varphi \perp \langle -d \rangle$ with $\langle d \rangle = d(\varphi)$. Further let φ_r denote the kernel form of $\varphi \otimes K_r$ and ψ_s denote the kernel form of $\psi \otimes L_s$. Assume that the leading form of φ—hence also the leading form of ψ—is defined over k by τ. Then the following are equivalent:

(i) $\dim \varphi_{h-2} > \dim \tau + 1$;

(ii) $L_{e-1} \sim K_{h-1}$ over k;

(iii) ψ_{e-1} represents $-d$, hence $\psi_{e-1} \cong (-d)\tau \otimes L_{e-1}$.

If (i)–(iii) do *not* hold true, then L_{e-1} is equivalent to K_{h-2} over k.

Proof. We always have a place from L_{e-1} to K_{h-1} over k, and since

$$\varphi \otimes L_{e-1} \sim \psi_{e-1} \perp \langle d \rangle,$$

we also have a place from K_{h-2} to L_{e-1} over k. Furthermore this place can be extended to a place from K_{h-1} to L_{e-1} if and only if the right-hand side is isotropic. This proves the implications (ii) $\Leftrightarrow$ (iii) and (i) $\Rightarrow$ (ii). (iii) $\Rightarrow$ (ii) gezeigt! ? Finally the condition (iii) implies $\varphi \otimes L_{e-1} \sim (-d)\tau' \otimes L_{e-1}$, and then there exists a place from K_{h-1} to L_{e-1} over k. Thus the equivalence of (i), (ii), and (iii) is evident. Assume now that

$$\dim \varphi_{h-2} = \dim \tau + 1.$$

We have

$$(*) \qquad \varphi_{h-2} \perp (d\tau' \otimes K_{h-2}) \cong d\tau \otimes \gamma$$

with γ a Pfister form over K_{h-2} (Theorem 7.13), and counting dimensions we see that γ is a binary form $\langle 1, b \rangle$. The relation (*) implies that

$$\psi \otimes K_{h-2} \sim bd\tau \otimes K_{h-2}.$$

Thus there exists a place from L_{e-1} to K_{h-2} over k, and these fields are equivalent.

For even-dimensional forms we have the following result.

THEOREM 9.6. *Let τ be an anisotropic Pfister form of degree $n \geqslant 1$, and let φ be a non-split even-dimensional form over k. The following are equivalent:*

(i) *the leading form of φ is defined over k by τ;*

(ii) $\varphi \equiv \tau \bmod J_{n+1}(k)$.

(Recall that $J_r(k)$ denotes the ideal of all forms of degree at least r.)

Here the implication (ii) $\Rightarrow$ (i) has already been proved in § 6, cf. Theorem 6.3. Assume now that the leading form of φ is defined over k by τ. We have to show that the form $\psi := \varphi \perp (-\tau)$ has degree greater than n. Suppose $\deg(\psi) = n$. Let E denote a leading field of ψ which is regular over k. The kernel form of $\psi \otimes E$ is a product $a\rho$ with ρ a Pfister form of degree n over E and some a in E^*. Now we learn from Proposition 6.11 that $\varphi \otimes k(\tau)$ has degree greater than n. Thus $\psi \otimes k(\tau)$ also has degree greater than n. Again by Proposition 6.11 we see that ρ is split by $E \cdot k(\tau) = E(\tau \otimes E)$. Thus $\tau \otimes E$ divides ρ, and since both forms have the same dimension, we obtain that $\rho \cong \tau \otimes E$. This implies that

$$\varphi \otimes E \sim \tau \otimes E \perp a\rho \cong \langle 1, a \rangle \otimes \rho.$$

Thus $\varphi \otimes E$ has degree greater than n. We want to deduce from this that $\tau \otimes E$ splits. This would be a contradiction to the fact that $\tau \otimes E \cong \rho$. Thus Theorem 9.6 will be proved if we verify the following lemma.

LEMMA 9.7. *Let φ be a non-split even-dimensional form over k, whose leading form is defined over k by τ. Let L be a field extension of k such that $\varphi \otimes L$ has strictly larger degree than φ. Then $\tau \otimes L$ splits.*

Proof. Let $(K_r, 0 \leqslant r \leqslant h)$ be the generic splitting tower of φ defined by

$$K_0 = k, \quad K_{r+1} = K_r(\varphi_r),$$

with φ_r the kernel form of $\varphi \otimes K_r$ for $0 \leqslant r \leqslant h-1$. We know from Proposition 6.11 that τ splits over $K_{h-1} \cdot L$. Suppose $\tau \otimes L$ is anisotropic. Certainly $h \geqslant 2$. Since $K_{r+1} \cdot L$ is the function field of $\varphi_r \otimes K_r \cdot L$ over $K_r \cdot L$ and $\dim \varphi_r > \dim \tau$ for $0 \leqslant r \leqslant h-2$, we see, again by Proposition 6.11, that all forms $\tau \otimes K_r \cdot L$ with $0 \leqslant r \leqslant h-1$ are anisotropic. But $\tau \otimes K_{h-1} \cdot L$ splits. Thus $\tau \otimes L$ splits.

In the special case where $\deg \varphi = 2$ Theorem 9.6 yields the following corollary.

COROLLARY 9.8. *Let τ be an anisotropic quaternion form over k, and let φ be a non-split form over k. The following are equivalent:*

(i) *the leading form of φ is defined over k by τ;*

(ii) $c(\varphi) = [\tau]$.

Proof. That (ii) implies (i) is already clear from Proposition 5.10. The proof of the implication (i) $\Rightarrow$ (ii) can be reduced to the case where φ has even dimension by the use of Proposition 9.4. By Theorem 9.5 the form $\psi := \varphi \perp (-\tau)$ has degree greater than 2. We have $c(\psi) = 1$, whence $c(\varphi) = [\tau]$.

We now study along the same lines the forms with the stronger property that the highest non-split kernel form is defined over the base field.

THEOREM 9.9. *Let τ be an anisotropic Pfister form of degree $n \geqslant 1$, and let φ be an even-dimensional form of height $h \geqslant 1$. Let a be an element of k^*. The following are equivalent:*

(i) *the $(h-1)$th kernel form of φ is defined over k by $a\tau$;*

(ii) $\varphi \equiv a\tau \bmod J_{n+2}(k)$.

Proof. The implication (ii) $\Rightarrow$ (i) has already been stated in Theorem 6.3. Now let $(K_i, 0 \leqslant i \leqslant h)$ denote a generic splitting tower of φ, and let φ_i denote the kernel form of $\varphi \otimes K_i$. We assume that $\varphi_{h-1} \cong a\tau \otimes K_{h-1}$, and we have to show that $\psi := \varphi \perp (-a\tau)$ has degree greater than $n+1$. By Theorem 7.13 we know that

$$(*) \qquad \varphi_{h-2} \perp (-a\tau) \otimes K_{h-2} \cong -a\sigma$$

with σ an anisotropic Pfister form of degree $r \geqslant n+2$ over K_{h-2}. Suppose ψ has degree $m < r$. Let E denote a leading field of ψ. The kernel form of

$\psi \otimes E$ is a product $b\rho$ with b in E^* and ρ a Pfister form of degree m. We have

$$\varphi \otimes E \sim b\rho \perp (a\tau) \otimes E.$$

The right-hand side has dimension $2^m + 2^n$ which is not greater than $2^{r-1} + 2^{r-2}$ and thus is certainly no greater than $\dim \varphi_{h-2}$. Thus there exists a place λ from K_{h-2} to E over k, and we obtain from the relation (*) above that

$$\psi \otimes E \sim -a\lambda_*(\sigma).$$

Thus $\rho \cong \lambda_*(\sigma)$, and $m = r$, which is absurd since we supposed $m < r$. This proves that $\deg \psi = r \geqslant n+2$.

REMARK 9.10. Assume that the highest non-split kernel form of our non-split even-dimensional form φ is defined over k by $a\tau$. Let $(K_i, 0 \leqslant i \leqslant h)$ be a generic splitting tower of φ, and let $(L_j, 0 \leqslant j \leqslant e)$ be a generic splitting tower of $\psi := \varphi \perp (-a\tau)$. Further let ρ be the leading form of ψ over L_{e-1}. Analogous to Proposition 5.12 the following hold true:

(i) L_e and K_{h-1} are equivalent over k;

(ii) there exists a place λ: $L_{e-1} \to K_{h-2} \cup \infty$ over k, and for any such place the Pfister form $\lambda_*(\rho)$ is anisotropic and φ_{h-2} is a neighbour of $\lambda_*(\rho)$.

Proof. (i) Since $\psi \otimes K_{h-1} \sim 0$ we have a place from L_e to K_{h-1} over k, and since $\varphi \otimes L_e \sim (a\tau) \otimes L_e$ we also have a place in the opposite direction.

(ii) We use the notation from the proof of Theorem 9.9 with $E = L_{e-1}$. We proved there that $\deg \rho = \deg \sigma$ and $\psi \otimes K_{h-2} \sim -a\sigma$. Thus we have a place λ from L_{e-1} to K_{h-2}, and for any such place $\lambda_*(\rho) = \sigma$.

EXAMPLE 9.11. Let ρ and τ be anisotropic Pfister forms over k of degrees $n \geqslant 2$ and $m \geqslant 0$ respectively. Further let d be an element of k^* not represented by τ. We regard the form

$$\varphi := (\rho' \perp \langle d \rangle) \otimes \tau$$

and assume that φ is anisotropic. Let $(K_i, 0 \leqslant i \leqslant h)$ be a generic splitting tower of φ, and let φ_i be the kernel form of $\varphi \otimes K_i$. Clearly

$$\varphi \equiv -\langle 1, -d \rangle \otimes \tau \mod I^{n+m}(k),$$

and thus we know, a priori, that the leading form of φ is always defined over k by $\langle 1, -d \rangle \otimes \tau$, and that in the case where $n \geqslant 3$ even the form φ_{h-1} is defined over k. A closer look at φ shows that

$$(*) \qquad \varphi_1 \cong \chi' \otimes \langle 1, -d \rangle \otimes (\tau \otimes K_1)$$

with χ a Pfister form over K_1 of degree $n-1$, cf. the proof below. Thus φ_1 is excellent, and $h = 3$ if $n \geqslant 3$, while $h = 2$ if $n = 2$. The form φ_1 is certainly not defined over k. Indeed, otherwise φ would be excellent (Theorem 7.14). But $\dim \varphi$ is a 2-power.

Proof of (*). $\rho \otimes \tau \otimes K_1$ is anisotropic. Indeed, otherwise φ would be similar to a subform of $\rho \otimes \tau$. This would imply that $\varphi \cong \rho \otimes \tau$ and then $d\tau \cong \tau$, in contrast to our assumption about d. A fortiori, $\rho' \otimes \tau$ and $d\tau$ remain anisotropic over K_1, while their sum φ becomes isotropic. Thus there exists some b in K_1^* represented by $\tilde{\tau} := \tau \otimes K_1$, such that $-db$ is represented by $\rho' \otimes \tau \otimes K_1$. Now $\rho \otimes \tau \otimes K_1$ splits over the function field of $\langle 1, -db \rangle \otimes \tilde{\tau} = \langle 1, -d \rangle \otimes \tilde{\tau}$, hence

$$\rho \otimes \tau \otimes K_1 \cong \langle 1, -d \rangle \otimes \tilde{\tau} \otimes \chi$$

with some Pfister form χ over K_1 of degree $n-1$. One easily deduces from this that

$$\varphi \otimes K_1 \sim \langle 1, -d \rangle \otimes \chi' \otimes \tilde{\tau}.$$

The right-hand side is a subform of $\rho \otimes \tau \otimes K_1$ and is thus anisotropic. This yields that

$$\varphi_1 \cong \langle 1, -d \rangle \otimes \chi' \otimes \tilde{\tau}$$

as asserted above.

In this example φ is *not* divisible by $\sigma := \langle 1, -d \rangle \otimes \tau$, in contrast to the behaviour of excellent forms (Proposition 7.17). Indeed, otherwise σ would also divide $\rho \otimes \tau$, since $\rho \otimes \tau \sim \varphi \perp \sigma$. We should obtain

$$\rho \otimes \tau \cong \langle 1, -d \rangle \otimes \tau \otimes \chi$$

with χ a Pfister form of degree $n-1$ over k, and thus φ would be isotropic, cf. the preceding proof.

As further examples we discuss the forms of dimension less than 10 with leading form defined over the base field. φ always denotes an *anisotropic* form, and if φ has even dimension we assume in addition that $d(\varphi) = 1$. If the leading form of φ is defined over k then we want to represent φ as a combination or a subform of simpler forms in such a way that this property becomes evident.

If $\dim \varphi \leqslant 5$ and the leading form is defined over k then φ is excellent. The case where $\dim \varphi = 6$ does not occur, since any 6-dimensional form φ with $d(\varphi) = 1$ and $c(\varphi)$ representable by a quaternion algebra must be isotropic ([**8**, p. 123; **1**], cf. Example 10.2).

EXAMPLE 9.12. Assume that $\dim \varphi = 8$ (and $d(\varphi) = 1$). The following are equivalent:

(i) the leading form of φ is defined over k;
(ii) $c(\varphi) = [\tau]$ with τ a quaternion form;
(iii) φ is divisible by a binary form $\langle 1, -a\rangle$.
In this case $h(\varphi) \leqslant 2$, and $i_1(\varphi) = i_2(\varphi) = 2$ if $h(\varphi) = 2$.

Proof. We have $c(\varphi) = 1$ if and only if φ is similar to a Pfister form (cf. end of § 7). In this case the assertions (i)–(iii) hold true. From now on we assume that $c(\varphi) \neq 1$, that is, $\deg \varphi = 2$. The implications (i) $\Leftrightarrow$ (ii) are evident from Corollary 9.8.

(iii) $\Rightarrow$ (i): assume that $\varphi \cong \langle 1, -a\rangle \otimes \chi$. We can write

$$\chi \cong b(\rho' \perp \langle d\rangle)$$

with some quaternion form ρ and b, d in k^*. According to Example 9.11 the form φ has height 2 and leading form defined by $\langle 1, -d\rangle \otimes \langle 1, -a\rangle$.

(i) $\Rightarrow$ (iii): clearly $h(\varphi) = 2$ and $i_1(\varphi) = 2$, since any 6-dimensional form of determinant 1 with leading form defined over the base field is isotropic. We choose some g in k^* such that $\varphi \otimes k(\sqrt{g})$ becomes isotropic. Then this form has index greater than 1, and thus φ contains a subform $\langle 1, -g\rangle \otimes \chi$ with $\dim \chi = 2$ [8, p. 123]. We write this subform as $a_1\rho_1$ with ρ_1 a quaternion form. Since $d(\varphi) = 1$ we have

$$\varphi \cong a_1\rho_1 \perp a_2\rho_2$$

with ρ_2 another quaternion form. We want to show that ρ_1 and ρ_2 have a common divisor $\langle 1, -a\rangle$, that is, are linked. Then assertion (iii) will be proved.

Suppose ρ_1 and ρ_2 are not linked. Then $\alpha := \rho_1' \perp -\rho_2'$ is anisotropic (cf. Lemma 8.11). The form $\varphi \otimes k(\tau)$ has degree at least 3 (Proposition 6.11). Regarding its Clifford invariant we see that $\rho_1 \otimes k(\tau)$ and $\rho_2 \otimes k(\tau)$ are isomorphic. Thus τ divides α. But this is absurd since $\dim \tau = 4$ and $\dim \alpha = 6$. We learn that ρ_1 and ρ_2 are linked.

EXAMPLE 9.13. Assume that $\dim \varphi = 7$. The following are equivalent:
(i) the leading form of φ is defined over k;
(ii) $c(\varphi) = [\tau]$ with τ a quaternion form;
(iii) φ is a subform of an anisotropic product $\langle 1, -a\rangle \otimes \eta$ with $\dim \eta = 4$.
In this case either $h(\varphi) = 1$ or $h(\varphi) = 3$. If $h(\varphi) = 3$ the first kernel form of φ is not defined over k.

Proof. The leading form of φ is defined over k if and only if the same holds true for the form $\psi := \varphi \perp (-d(\varphi))$ (Proposition 9.4). Thus the implications (i) $\Leftrightarrow$ (ii) and (iii) $\Rightarrow$ (i) are clear from the preceding example, and the implication (i) $\Rightarrow$ (iii) will be also clear if we have shown that ψ is anisotropic under the assumption (i). Suppose ψ is isotropic. Then the kernel form of ψ must have dimension 6. But there does not exist any anisotropic form of dimension 6 with leading form defined over the base field and discriminant 1. Thus indeed ψ is anisotropic.

Assume finally that (i)–(iii) hold true. If the first kernel form is defined over k then φ is excellent, hence $h(\varphi) = 1$. Otherwise certainly $h(\varphi) \geqslant 3$, whence $h(\varphi) = 3$.

EXAMPLE 9.14. Assume that $\dim \varphi = 9$. Then the leading form of φ is defined over k if and only if one of the following statements holds true:

(A) φ is excellent, whence φ is similar to $\rho \perp \langle a \rangle$ with a in k^* and ρ a Cayley form (cf. § 7);

(B) φ is similar to $\rho' \perp \chi$ with ρ a Cayley form and χ a binary form not representing 1;

(C) φ is a subform of an anisotropic product $\langle 1, -a \rangle \otimes \eta$ with $d(\eta) = 1$, $\dim \eta = 6$.

In case A the leading form is defined by ρ, while in cases B and C it is defined by a quaternion form τ. In case B the form $\psi := \varphi \perp (d(\varphi) \otimes \tau')$ is isotropic while in case C the form ψ is anisotropic.

Proof. It can be immediately checked that for the 9-dimensional forms of the types A, B, C the statements just made hold true. Assume now that the leading form of φ is defined over k by τ, that φ is not excellent, and finally, without loss of generality, that $d(\varphi) = 1$. Clearly τ has degree 2. If the form $\psi := \varphi \perp \tau'$ is anisotropic, then ψ is a product $\langle 1, -a \rangle \otimes \eta$ with $\dim \eta = 6$ and $d(\eta) = 1$ according to [8, pp. 123 ff.], since ψ has dimension 12 and trivial invariants. Assume now that ψ is isotropic. Then, again by [8, p. 123], the kernel form of ψ is similar to a non-split Cayley form ρ, hence

$$\varphi \sim g\rho \perp (-\tau')$$

with some g in k^*. The right-hand side is isotropic. Thus there exists some b in k^* represented by τ' such that bg is represented by ρ. We have $g\rho \cong b\rho$ and $\tau \cong \langle 1, b \rangle \otimes \langle 1, -c \rangle$ with some c in k^*, and we obtain

$$\varphi \sim b\rho' \perp c\langle 1, b \rangle.$$

Since both sides have the same dimension they are isomorphic.

10. On some forms of height 2

In this section φ always denotes an *anisotropic* form over k. We are interested in the forms φ with leading form defined over k and $h(\varphi) = 2$. We consider only even-dimensional forms. Indeed, if φ has odd dimension, height 2, and leading form defined over k by τ, then the first kernel form of φ is defined over k by $-d(\varphi) \otimes \tau'$, and thus φ is excellent, a case settled already in § 7.

LEMMA 10.1. *Assume that* $\dim\varphi$ *is even,* $h(\varphi) = 2$, *and that the leading form of* φ *is defined over* k *by a Pfister form* τ *of degree* $n \geqslant 1$.

(i) φ *is excellent if and only if* $\varphi\otimes k(\tau)$ *is isotropic. Then* $\varphi \cong a\tau\otimes\rho'$ *with* ρ *a Pfister form of degree greater than* 1 *and* a *in* k^*.

(ii) *If* φ *is not excellent, then* $\varphi\otimes k(\tau)$ *is similar to a Pfister form. In particular* $\dim\varphi = 2^N$ *with* $N > n$.

Proof. Assume first that φ is excellent, that is,

$$\varphi\perp a\tau \cong a\sigma$$

with σ a Pfister form of degree greater than $n+1$ and some a in k^*. Since $\sigma\otimes k(\tau)$ splits there exists a Pfister form ρ such that $\sigma \cong \tau\otimes\rho$. Cancellation yields $\varphi \cong a\tau\otimes\rho'$. In particular $\varphi\otimes k(\tau)$ splits. Assume now that $\varphi\otimes k(\tau)$ is isotropic. Then there exist places from $k(\varphi)$ to $k(\tau)$ over k. Applying such a place to $\varphi\otimes k(\varphi)$ we obtain with some c in $k(\tau)^*$

$$\varphi\otimes k(\tau) \sim c(\tau\otimes k(\tau)) \sim 0.$$

Thus there exists a form ψ over k with $\varphi \cong \tau\otimes\psi$. The dimension of ψ must be odd, since otherwise φ would have degree greater than n. Since the form $\psi\perp -d(\psi)$ lies in $I^2(k)$ we have

$$\varphi \equiv d(\psi)\otimes\tau \mod I^{n+2}(k).$$

Thus the first kernel form of φ is defined over k by $d(\psi)\otimes\tau$ (Theorem 6.3), and φ is excellent. Assertion (i) is proved.

Assume now that $\tilde{\varphi} := \varphi\otimes k(\tau)$ is anisotropic. By Proposition 5.13 the form $\tilde{\varphi}$ has height less than 2. Thus $h(\tilde{\varphi}) = 1$ and $\tilde{\varphi}$ is similar to a Pfister form.

EXAMPLE 10.2. A special case of this lemma is the following result due to Pfister [8, p. 123] and Albert [**1**]. Let φ be a form of dimension 6 with discriminant 1. Assume that $c(\varphi)$ can be represented by a quaternion algebra. Then φ is isotropic. Indeed, φ has height less than 3 and the leading form of φ is defined by a quaternion form over k. If φ were anisotropic, then by Lemma 10.1 either $\dim\varphi$ or $\dim\varphi+4$ would be a 2-power. This is not true.

THEOREM 10.3. *The even-dimensional anisotropic forms of height* 2 *with discriminant not equal to* 1 *are precisely the anisotropic forms* φ *with* $\dim\varphi = 4$ *and* $d(\varphi) \neq 1$ *and the anisotropic products* $a\langle 1, -b\rangle\otimes\rho'$ *with* a, b *in* k^* *and* ρ *a Pfister form of degree greater than* 1.

Proof. $d(\varphi) \neq 1$ means that φ has degree 1 and that the leading form of φ is defined over k (Proposition 5.10). As stated in Lemma 10.1 the anisotropic products $a\langle 1, -b\rangle\otimes\rho'$ are the even-dimensional anisotropic excellent forms of height 2 and degree 1. Furthermore it is trivial that the anisotropic forms φ with $\dim \varphi = 4$ and $d(\varphi) \neq 1$ have height 2 and are not excellent (Example 9.11 with $\tau = \langle 1 \rangle$). Assume now that φ is an anisotropic form over k with $d(\varphi) = \langle d \rangle \neq 1$ and $h(\varphi) = 2$, and that φ is not excellent. By Lemma 10.1 the form $\tilde{\varphi} := \varphi \otimes k(\sqrt{d})$ is an anisotropic Pfister form. Now

$$c(\tilde{\varphi}) = c(\varphi) \otimes k(\sqrt{d}) \neq 1,$$

since otherwise φ would be excellent according to Corollary 8.2. Thus $\dim \varphi = \dim \tilde{\varphi} = 4$.

It would be desirable to describe the even-dimensional forms φ of height 2 with leading form defined over k by a Pfister form τ of degree $n > 1$ in a way similar to Theorem 10.3. Up to now I have not been able to do this, even for $n = 2$. Nevertheless I want to push the study of these forms a small step further to stimulate interest in this very concrete problem.

Excluding the excellent forms we assume that φ is an anisotropic form of height 2 with leading form defined over k by a Pfister form τ of degree $n > 1$ and that the dimension of φ is a 2-power 2^N. Clearly $N > n$. We choose some g in k^* represented by φ. We further choose a *maximal Pfister divisor* μ of φ, that is, a Pfister form μ dividing φ such that there does not exist a Pfister form ω different from μ with $\mu | \omega$ and $\omega | \varphi$. According to Lemma 9.7 τ splits over $k(\mu)$ and hence μ also divides τ. Thus we have decompositions

$$g\varphi \cong \mu \otimes \psi, \quad \tau \cong \mu \otimes \gamma \tag{10.4}$$

with ψ a form representing 1 [**3**, Theorem 1.4] and γ a Pfister form of degree at least 1. We write $\psi \cong \langle 1 \rangle \perp \psi'$ and introduce the form

$$\alpha := \mu \otimes [\psi' \perp (-\gamma')].$$

Clearly

$$g\varphi \perp (-\tau) \cong (\mu \otimes H) \perp \alpha.$$

PROPOSITION 10.5. *The form α is anisotropic. Thus $g\varphi \perp (-\tau)$ has index $\dim \mu$, and in particular the degree of μ is a number r independent of the choice of μ with $0 \leqslant r \leqslant n-1$.*

Proof. According to Lemma 10.1 τ does not divide φ and thus $r \leqslant n-1$. Suppose α is isotropic. There exists some b in k^* such that $-b$ is represented by $\mu \otimes \gamma'$ and by $\mu \otimes \psi'$. We then can modify ψ' in such a way that ψ' represents $-b$ [**3**], and we obtain

$$\tau \cong \omega \otimes \delta, \quad g\varphi \cong \omega \perp (\mu \otimes \chi)$$

with $\omega := \mu \otimes \langle 1, -b \rangle$, some Pfister form δ, and some other form χ over k.

(Notice that τ becomes isotropic over $k(\omega)$.) The form $\varphi\otimes k(\omega)$ is isotropic. Thus with some c in $k(\omega)^*$

$$\varphi\otimes k(\omega) \sim c\tau\otimes k(\omega) \sim 0,$$

and ω divides φ. This contradicts the maximality of μ.

Clearly the anisotropic products $\mu\otimes\chi$ with μ a Pfister form and χ a form of dimension 4 with $d(\chi)$ not represented by μ are precisely all forms of the type considered here with $N = n+1$, $r = n-1$ (cf. Example 9.11). In view of Theorem 10.3 we pose the following question.

QUESTION 10.6. Let $n \geqslant 2$. Do there exist anisotropic forms φ of height 2 and degree n with leading form defined over the base field and $\dim\varphi = 2^N$, where $N \geqslant n+2$?

To study this question one may assume in addition that $r = n-1$ for these hypothetical forms φ. Indeed, the following lemma holds true.

LEMMA 10.7. $r < n-1$ *if and only if* $\varphi\otimes k(\alpha)$ *is anisotropic.*

In this case $\tilde{\varphi} = \varphi\otimes k(\alpha)$ is again a form of the type studied here, but with a maximal Pfister divisor of degree greater than r according to Proposition 10.5. By iterating this procedure we obtain from φ a form with the same invariants N and n but $r = n-1$.

The proof of Lemma 10.7 is easy. Assume first that $\varphi\otimes k(\alpha)$ is anisotropic. A maximal Pfister divisor of this form has degree $\tilde{r} > r$, as just stated. On the other hand, we still have $\tilde{r} \leqslant n-1$. Thus $r < n-1$.

Assume now that $r < n-1$. Clearly $\alpha\otimes k(\tau) \sim g\varphi\otimes k(\tau)$. But φ has strictly smaller dimension than α since $r < n-1$. Thus $\alpha\otimes k(\tau)$ is isotropic, and there exists a place from $k(\alpha)$ to $k(\tau)$ over k. Since $\varphi\otimes k(\tau)$ is anisotropic $\varphi\otimes k(\alpha)$ is also anisotropic. This completes the proof of Lemma 10.7.

In the case where $n = 2$ we automatically have $r = 1$.

PROPOSITION 10.8. *Let φ be an anisotropic form of height 2, and assume that the leading form of φ is defined by a quaternion form τ over k. Then φ is divisible by some binary form.*

Proof. We choose some a in k^* such that $\varphi\otimes k(\sqrt{a})$ becomes isotropic. If this form splits we are through. Otherwise this form has a kernel form of dimension 4. Thus we have a decomposition

$$\varphi \cong (\langle 1, -a\rangle\otimes\chi)\perp\eta$$

with $\dim\eta = 4$ and $\dim\chi$ even [8, p. 123]. Now η has determinant 1 and thus is similar to a quaternion form ρ. Considering Clifford invariants we obtain

$$[\rho][\tau] = [a, d(\chi)].$$

Thus ρ and τ are linked [8, p. 124]. Let ζ be a common divisor of ρ and τ of dimension 2. Then $\varphi \otimes k(\zeta)$ is isotropic, and hence with some c in $k(\zeta)^*$

$$\varphi \otimes k(\zeta) \sim c\tau \otimes k(\zeta) \sim 0.$$

Thus ζ divides φ.

This proposition may be regarded as a generalization of Example 9.12.

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