Symmetric bilinear forms over algebraic varieties

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Preface

Arround 1964 Cassels and Pfister initiated a theory of quadratic forms over function fields in arbitrary many variables, cf. the last chapter of Lam's Kingston lectures. The discovery that interesting theorems can be proved for quadratic forms over such complicated fields had a strong appeal to many algebraists, and in the last 12 years the theory of quadratic forms over fields has been pushed forward considerably. But despite the remarkable progress and the sometimes very ingenious arguments this algebraic theory of quadratic forms over fields is to my opinion not a fully adequate response to the general hope or belief that a rich and interesting theory of quadratic forms over function fields is possible. There should exist an "algebraic geometry of quadratic forms" to understand quadratic forms over function fields in much the same way as algebraic geomtry is known to be needed to understand function fields themselves.

For this algebraic geometry of quadratic forms the classical language of algebraic geometry is unsuited. Indeed, the classical language assumes the presence of an algebraically closed base field. This would spoil many interesting phenomena of quadratic forms, since -1 would be a square in every function ring. Thus we are urged to use Grothendieck's language of schemes.

We then should also admit schemes X with 2 not a unit in the ring $\mathcal{O}(X)$ of global functions. This means that we have to develop two theories, one for symmetric bilinear forms and one for truly quadratic forms, and that we have to study also the relations between these theories (e.g. tensor products of symmetric bilinear forms with quadratic forms, cf. [MH,p.111]*) for X affine).

Around 1967 I started - not always in the right way **) - such a theory for symmetric bilinear forms over schemes [K]. Since then good progress has been made in the local part of this theory (symmetric bilinear forms over local rings). But on the global side only few concrete results have been obtained.

The reason for this just seems to be that not too many mathematicians have spent much effort in this area. For example up to now nobody seems to have computed the Witt ring of the spheres Spec $\mathbb{R}[x_0,\dots,x_n]/(\sum_0^n x_i^2 - 1)$ over the field $\mathbb{R}$ of real numbers. In recent years many papers have been written on hermitian and quadratic forms over rings with involution, a subject which virtually includes the affine part of our theory. But most often the authors have been led by a totally different motivation, for example surgery theory.

The aim of these lectures is to stimulate interest in the global theory of symmetric bilinear forms over schemes. Quadratic forms are important as well but have not been included for lack of time. It is reasonable to study instead of symmetric bilinear

*) references at the end of the lectures.

**) In [K] the definition of the Witt ring W(X) is not the right one if X is not affine.

forms more generally hermitian forms over schemes with involution (cf. e.g. the introduction in [K_3]), but I refrained from this enlargement of the theory here to keep the complexity of our notions as low as possible.

It probably is a long way to push our theory so far that good applications can be made on a large scale to quadratic forms over function fields. Up to now such applications are visible only for curves, and here the role of complete schemes is impressive. But of course our theory should have - as the usual algebraic geometry - enough meaning by herself and allow other applications. The last chapter of the lectures seems to indicate that the Witt rings W(X) studied here are significant to understand the set of real points on an algebraic variety defined over the field $\mathbb{R}$ of real numbers. For another application see my talk "Real closures of algebraic varieties" at this conference. After all our lectures may well be regarded a chapter of algebraic K-theory as first visualized by Grothendieck in his work on the Riemann-Roch theorem.

The written version of the lectures is quite a bit longer than the oral version, since I had to replace a lot of sketches and handwaving by solid arguments. I further added two sections and three appendices, marked by asterisks, to the material of the oral lectures to illustrate and to round off the results displayed there. These sections and appendices are not necessary for an understanding of the other sections. But they are easily accessible to anybody who is acquainted with the main parts of

the lectures.

I want to thank the staff at Queen's and in particular Grace Orzech and Paulo Ribenboim for three very pleasant weeks at Kingston and my audience in the first two weeks for great endurance.

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Chapter I <u>Definition of the Grothendieck-Witt ring L(X) and the Witt ring W(X).</u>

§ 1 <u>Bilinear spaces</u>

Let X be an arbitrary scheme, not necessarily separated, i.e. a "prescheme" in the terminology of Grothendieck's blue books [EGA]. We denote by $\mathfrak{O}_X$ or simply by $\mathfrak{O}$ the structure sheaf of X. Thus for every open set $Z \subset X$ the set of sections $\mathfrak{O}(Z)$ of $\mathfrak{O}$ over Z is a commutative ring with 1 and for an open $Z' \subset Z$ the restriction map $f \to f|Z'$ from $\mathfrak{O}(Z)$ to $\mathfrak{O}(Z')$ is a ring homomorphism (mapping 1 to 1). For x a point of X we denote by $\mathfrak{O}_x$ the stalk of $\mathfrak{O}$ at x,

$$\mathfrak{O}_x = \varinjlim_{Z \ni x} \mathfrak{O}(Z)$$

with Z running through the open neighbourhoods of x. This is a local ring whose maximal ideal will be denoted by m_x.

More generally we regard - as nowadays usual - any sheaf F of abelian groups over X as a functor $Z \mapsto F(Z)$ from the category of open sets of X to the category of abelian groups. The morphisms of the first category are the inclusion maps $Z' \to Z$ existing whenever Z' is contained in Z. Thus for every pair of open sets Z',Z with $Z' \subset Z$ we have a "<u>restriction homomorphism</u>" from F(Z) to F(Z') which we denote usually by $f \to f|Z'$. That this functor F is a sheaf means by definition that the following

condition holds true:

<u>Sheaf condition:</u> If Z is an open set in X, and $\{Z_\alpha\}_{\alpha\in I}$ is an open covering of Z, and if $\{f_\alpha\}_{\alpha\in I}$ is a family of sections $f_\alpha \in F(Z_\alpha)$ with $f_\alpha | Z_\alpha \cap Z_\beta = f_\beta | Z_\alpha \cap Z_\beta$ whenever $Z_\alpha \cap Z_\beta$ is non empty, then there exists a unique section $f \in F(Z)$ over Z with $f | Z_\alpha = f_\alpha$ for all α.

Notice that this condition implies $F(\emptyset) = 0$ for the empty set $\emptyset$, and that for a functor F with $F(\emptyset) = 0$ the condition "$Z_\alpha \cap Z_\beta$ non empty" above can be dropped.

Our sheaf F of abelian groups is called an <u>$\mathcal{O}$-module</u> if every $F(Z)$ is an (unitary) $\mathcal{O}(Z)$-module and the restriction homomorphisms $F(Z) \to F(Z')$ are compatible with the ring homomorphisms $\mathcal{O}(Z) \to \mathcal{O}(Z')$ in the obvious sense. Then every stalk

$$F_x = \varinjlim_{x\in Z} F(Z)$$

is a module over the local ring $\mathcal{O}_x$. A <u>homomorphism</u> $\alpha: F \to F_1$ from F to another $\mathcal{O}$-module F_1 consists of a system of $\mathcal{O}(Z)$-module homomorphisms

$$\alpha_Z : F(Z) \to F_1(Z)$$

such that for every pair of open sets $Z' \subset Z$ the diagram

$$\begin{tikzcd} F(Z) \arrow[r, "\alpha_Z"'] \arrow[d, "\mathrm{res}"'] & F_1(Z) \arrow[d, "\mathrm{res}"] \\ F(Z') \arrow[r, "\alpha_{Z'}"'] & F_1(Z') \end{tikzcd}$$

with the restriction maps as the vertical arrows commutes. In this way we obtain the "category of $\mathcal{O}$-modules". Clearly a homomorphism $\alpha: F \to F_1$ is an <u>isomorphism</u> in our category if and only if all α_Z are bijective. It follows easily from the sheaf condition that α is already an isomorphism if the induced $\mathcal{O}_x$-homomorphisms

$$\alpha_x : F_x \to F_{1x}$$

on the stalks at all points x of X are bijective.

If Z is a fixed open subset of X then we obtain from every $\mathcal{O}_X$-module F an $\mathcal{O}_Z$-module $F|Z$ by restricting the functor F to the category of open subsets of Z, called the <u>restriction</u> of the $\mathcal{O}_X$-module F to Z.

After these preliminaries we come to our first definition.

<u>Definition 1.</u> A <u>vector bundle</u> E on X is an $\mathcal{O}$-module E which is locally free of finite rank, i.e. for every point x of X there exists an open neighbourhood Z of x such that the $\mathcal{O}_Z$-module $E|Z$ is isomorphic to the free $\mathcal{O}_Z$-module $\mathcal{O}_Z^{n_x}$ for some natural number n_x. We call n_x the <u>local rank</u> of E at x and call the locally constant function $x \to n_x$ on X the <u>rank</u> rkE of E. If X is connected then rkE of course is a constant.

<u>Remark.</u> Actually the word "vector bundle" here is an abuse of language. Vector bundles in the proper sense are defined in [EGA, II § 1.7]. The locally free $\mathcal{O}$-modules of finite rank are the sheafs of sections of these honest vector bundles and correspond with them in a unique way.

If X is <u>affine</u>, i.e. $X \cong \mathrm{Spec}(A)$ with A the commutative ring $\mathcal{O}(X)$, then every vector bundle E over X is uniquely determined by the A-module $P := E(X)$, since E is certainly "quasi-coherent" [EGA I § 1.4]. P is a finitely generated projective A-module. We obtain in this way an equivalence of the category of vector bundles over X and the category of finitely generated projective modules over A, cf. [Bb, II § 5].

For E a vector bundle over an arbitary scheme X we denote by E^* the <u>dual vector bundle</u> of E, defined as follows. $E^*(Z)$ is the set of homomorphisms from the $\mathcal{O}_Z$-module $E|Z$ to the $\mathcal{O}_Z$-module $\mathcal{O}_Z$. Any such homomorphism $\alpha : E|Z \to \mathcal{O}_Z$ can be multiplied with sections of $\mathcal{O}(Z)$ in an obvious way, and $E^*(Z)$ is an $\mathcal{O}(Z)$ module. Furthermore for $Z' \subset Z$ the homomorphism α yields a homomorphism α' from $E|Z'$ to $\mathcal{O}_{Z'}$ by looking only at the open subsets of Z'. This map $\alpha \mapsto \alpha'$ is the restriction map from $E^*(Z)$ to $E^*(Z')$. If $E|Z \cong \mathcal{O}_Z^n$, then also $E^*|Z \cong \mathcal{O}_Z^n$. Thus E^* is indeed again a vector bundle. For affine open sets Z of X we clearly can identify $E^*(Z)$ with the set of $\mathcal{O}(Z)$-linear maps from $E(Z)$ to $\mathcal{O}(Z)$,

$$E^*(Z) = \mathrm{Hom}_{\mathcal{O}(Z)}(E(Z), \mathcal{O}(Z)).$$

We also have

$$E_x^* = \mathrm{Hom}_{\mathcal{O}_x}(E_x, \mathcal{O}_x)$$

for any x in X.

For u in $E(Z)$ and α in $E^*(Z)$ we denote by $\langle u, \alpha \rangle$ the element $\alpha_Z(u)$ of $\mathcal{O}(Z)$. {Recall that α yields in particular a map α_Z from

$E(Z)$ to $\mathcal{O}(Z)\}$. Consider now the dual vector bundle E^{**} of E^*. Then we can easily establish a homomorphism $\kappa: E \to E^{**}$ in a unique way such that

$$\langle \alpha, \kappa(u) \rangle = \langle u, \alpha \rangle$$

for u in $E(Z)$, α in $E^*(Z)$. For any x in X the induced map

$$\kappa_x : E_x \to (E^{**})_x$$

coincides with the well known isomorphism from the free $\mathcal{O}_x$-module E_x onto its bidual. Thus κ is an isomorphism. We usually identify E and E^{**}.

We also introduce for a vector bundle E over X the sheaf $E \times_X E$ defined by

$$(E \times_X E)(Z) := E(Z) \times E(Z)$$

with obvious restriction maps. The stalk of $E \times_X E$ at a point x is $E_x \times E_x$ which justifies our fibre product notation to some extent.

<u>Definition 2.</u> A <u>symmetric bilinear form</u> B on the vector bundle E is a morphism

$$B: E \times_X E \to \mathcal{O}$$

in the category of sheaves over X, such that for every open Z the map

$$B_Z : E(Z) \times E(Z) \to \mathcal{O}(Z)$$

is a symmetric bilinear form on the $\mathcal{O}(Z)$ module $E(Z)$.

Clearly then B also yields a symmetric bilinear form B_x on the free $\mathcal{O}_x$-module E_x for every x in X.

B induces for each open Z map

$$\varphi_Z : E(Z) \to E^*(Z)$$

as follows. For u in E(Z) and v in E(Z'), $Z' \subset Z$, the homomorphism $\varphi_Z(u):E|Z \to \mathcal{O}_Z$ maps v onto $B_{Z'}(u|Z',v)$. These maps together constitute a homomorphism φ from the bundle E to the dual bundle E^*. For Z affine

$$\varphi_Z:E(Z) \to \mathrm{Hom}_{\mathcal{O}(Z)}(E(Z),\mathcal{O}(Z))$$

is the usual linear map from the projective module E(Z) over $\mathcal{O}(Z)$ to its dual module associated with the bilinear form B_Z. In general any homomorphism φ from E to E^* has an adjoint homomorphism $\varphi^t:E \xrightarrow[\sim]{\kappa} E^{**} \xrightarrow{\varphi^*} E^*$. The symmetric bilinear forms on E correspond in the way indicated above uniquely with the homomorphisms $\varphi:E \to E^*$ which are selfadjoint, $\varphi = \varphi^t$. We call B <u>non degenerate</u> if φ is an isomorphism.

A pair (E,B) consisting of a vector bundle E and a symmetric bilinear form B on E will be called a <u>bilinear bundle.</u> If B is known to be non degenerate it will be called a <u>bilinear space</u>. We obtain a category of bilinear bundles by adding the following definition.

<u>Definition 3.</u> A <u>morphism</u> from a bilinear bundle (E_1,B_1) to a bilinear bundle (E_2,B_2) is a homomorphism α from the vector bundle E_1 to E_2 such that

$$B_2(\alpha(u),\alpha(v)) = B_1(u,v)$$

for any open set Z and any u,v in $E_1(Z)$.

Here we simply wrote $\alpha(u),\alpha(v)$ instead of $\alpha_Z(u)$, $\alpha_Z(v)$ and B_i instead of B_{iZ}. Such notational simplifications will be made quite often, as long as no confusion is to be feared.

We now shall give other interpretations of the notion "bilinear space" in two special cases.

Example 1. Assume X is affine, $X = \mathrm{Spec}(A)$. Then a bilinear bundle (E,B) over X is uniquely determined by the pair (P,Φ) consisting of the projective A-mode $P := E(X)$ and the symmetric bilinear form

$$\Phi = B_X : P \times P \to A.$$

We call such a pair (P,Φ) a bilinear A-module, and we call (P,Φ) a bilinear space over A if Φ is non degenerate, i.e. if Φ induces an isomorphism from the A-module P to the dual A-module $P^* = \mathrm{Hom}_A(P,A)$. The category of bilinear bundles over X is equivalent to the category of bilinear modules over A, and under this equivalence, described above, the bilinear spaces over X correspond with the bilinear spaces over A.

In an analogous way we shall always transfer without further comment notions developed for bilinear bundles and spaces over X to bilinear modules and spaces over A if $X = \mathrm{Spec}(A)$.

Example 2. Let Y be an algebraic subset of the affine space $\mathbb{C}^N$ or the projective space $\mathbb{P}^N(\mathbb{C})$ which is defined over $\mathbb{R}$, i.e.

can be described as the set of zeros of a system of polynomials resp. forms with coefficients in $\mathbb{R}$. On Y we have an involution $y \mapsto \overline{y}$ mapping a point y to the point $\overline{y}$ which has affine resp. projective coordinates complex conjugate to those of y. For any subset A of Y we denote by $\overline{A}$ the image of A under the complex conjugation $y \mapsto \overline{y}$. With Y there is attached a scheme X in the following way. The points of X are the subsets $T \cup \overline{T}$ of Y with T running through the Zariski-closed irreducible subsets of Y, i.e. the closed subsets of Y which are defined over $\mathbb{R}$ and "$\mathbb{R}$-irreducible". For every Zariski-open subset W over Y we define a subset $\widetilde{W}$ of X as follows. $\widetilde{W}$ consists of all points $x = T \cup \overline{T}$ of X with $W \cap (T \cup \overline{T})$ non empty $\{$which implies $W \cap T$ non empty$\}$. These sets $\widetilde{W}$ are by definition the open subsets of X. Then the structure sheaf $\mathcal{O}_X = \mathcal{O}$ is defined as follows. $\mathcal{O}(\widetilde{W})$ is the ring of all regular functions $f: W \to \mathbb{C}$ which are compatible with complex conjugation: $f(\overline{y}) = \overline{f(y)}$ for all y in W. In this way we have obtained all "reduced algebraic schemes over $\mathbb{R}$" which are affine resp. projective.

We now assume for simplicity that Y is connected and consider a complex algebraic vector bundle $p: F \to Y$ which is "defined over $\mathbb{R}$". By this we mean roughly the following. Y has a covering $\{W_\alpha\}$ by Zariski-open subsets W_α, all stable under complex conjugation, and there exists a family $\{\lambda_\alpha\}$ of bijective mappings

$$\lambda_\alpha : p^{-1}(W_\alpha) \xrightarrow{\sim} W_\alpha \times \mathbb{C}^n$$

such that the diagrams

$$\begin{tikzcd} p^{-1}(W_\alpha) \arrow[rr, "\sim", "\lambda_\alpha"'] \arrow[dr, "\text{"}p\text{"}"'] & & W_\alpha \times \mathbb{C}^n \arrow[dl, "pr_1"] \\ & W_\alpha & \end{tikzcd}$$

commute and the transition maps

$$\lambda_\alpha \circ \lambda_\beta^{-1} : (W_\alpha \cap W_\beta) \times \mathbb{C}^n \to (W_\alpha \cap W_\beta) \times \mathbb{C}^n$$

are of the form

$$(y,z) \mapsto (y, \mu_{\alpha\beta}(y)z)$$

with functions $\mu_{\alpha\beta} : W_\alpha \cap W_\beta \to GL(n,\mathbb{C})$ all whose coefficients lie in $\mathcal{O}(\widetilde{W_\alpha \cap W_\beta})$.

We can introduce a $\mathbb{C}$-antilinear involution $\tau : F \to F$ covering the involution $y \mapsto \overline{y}$ on Y such that over every W_α the involution $\tau | p^{-1}(W_\alpha)$ corresponds via λ_α to the standard involution $(y,z) \mapsto (\overline{y}, \overline{z})$ on $W_\alpha \times \mathbb{C}^n$. In this way an algebraic vector bundle over Y which is defined over $\mathbb{R}$ may be considered as a pair (F,τ) consisting of a complex algebraic vector bundle F over Y and a "locally trivial" $\mathbb{C}$-antilinear involution on F covering the complex conjugation on Y.

We now define an $\mathcal{O}_X$-module E as follows. For every open set W of Y stable under complex conjugation the $\mathcal{O}(\widetilde{W})$-module $E(\widetilde{W})$ consists of all sections $s : W \to p^{-1}(W)$ such that for any W_α meeting W the map

$$\lambda_\alpha \circ (s | W \cap W_\alpha) : W \cap W_\alpha \to p^{-1}(W \cap W_\alpha) \xrightarrow{\sim} (W \cap W_\alpha) \times \mathbb{C}^n$$

has coordinate functions lying in $\mathbb{C}(W \cap \widetilde{W_\alpha})$. Notice that this are precisely the regular sections s of the complex vector bundle F over W which have the property

$$s(\overline{y}) = \tau(s(y))$$

for every y in W. Using the maps λ_α we see that $E|\widetilde{W}_\alpha$ is isomorphic to $(\mathcal{O}_{W_\alpha})^n$. Thus E is a vector bundle over X. It is only an exercise to verify that we obtain in this way an equivalence from the category of algebraic vector bundles over Y which are defined over $\mathbb{R}$ to the category of vector bundles over X. It is further easily verified that the symmetric bilinear forms

$$B:E \times_X E \to \mathcal{O}_X$$

correspond uniquely with the regular maps

$$\beta:F \times_Y F \to \mathbb{C}$$

on the classical fibre product $F \times_Y F$ having the following properties:

a) For every y in Y the restriction of β to $p^{-1}(y) \times p^{-1}(y)$ is a symmetric bilinear form on the $\mathbb{C}$-vector space $p^{-1}(y)$.

b) For u,v in $p^{-1}(y)$

$$\beta(\tau u,\tau v) = \overline{\beta(u,v)}.$$

Moreover B is non degenerate if and only if our bilinear forms on the vector spaces $p^{-1}(y)$ all are non degenerate.

We return to our general theory. Two bilinear modules (E_1,B_1) and (E_2,B_2) over X can be added and multiplied. The

<u>orthogonal</u> <u>sum</u> $(E_1,B_1) \perp (E_2,B_2)$ is defined as the vector bundle $E_1 \oplus E_2$ having section modules

$$(E_1 \oplus E_2)(Z) := E_1(Z) \oplus E_2(Z)$$

equipped with the following bilinear form $B = B_1 \perp B_2$:

$$B_Z(u_1 \oplus u_2, v_1 \oplus v_2) = B_1(u_1,v_1) + B_2(u_2,v_2)$$

for sections u_1,v_1 in $E_1(Z)$ and u_2,v_2 in $E_2(Z)$. Similarly the <u>tensor</u> <u>product</u> $(E_1,B_1) \otimes (E_2,B_2)$ is defined as the vector bundle $E_1 \otimes_{\mathcal{O}} E_2$ equipped with the bilinear form $B_1 \otimes B_2$. Recall that the functor

$$Z \mapsto E_1(Z) \otimes_{\mathcal{O}(Z)} E_2(Z)$$

does not necessarily fulfill the sheaf condition, and that $E_1 \otimes_{\mathcal{O}} E_2$ is the sheaf associated to this "presheaf". But for Z an <u>affine</u> open subset of X we have

$$(E_1 \otimes_{\mathcal{O}} E_2)(Z) = E_1(Z) \otimes_{\mathcal{O}(Z)} E_2(Z).$$

$B_1 \otimes B_2$ can be characterized as the unique bilinear form B on X which fulfills

$$B_Z(u_1 \otimes u_2, v_1 \otimes v_2) = B_{1Z}(u_1,v_1)\, B_{2Z}(u_2,v_2)$$

for affine open sets Z and sections u_1,v_1 in $E_1(Z)$, and u_2,v_2 in $E_2(Z)$. The stalk of our bilinear bundle $(E_1,B_1) \otimes (E_2,B_2)$ at a point x is just the $\mathcal{O}_x$-module $E_{1x} \otimes_{\mathcal{O}_x} E_{2x}$ equipped with the bilinear form $B_{1x} \otimes B_{2x}$.

Another way to put the definitions of $B_1 \perp B_2$ and $B_1 \otimes B_2$ is the following. The natural map from the tensor product

$E_1^* \otimes_{\mathcal{O}} E_2^*$ to the bundle $(E_1 \otimes E_2)^*$ is an isomorphism since the E_i are locally free. We identify $E_1^* \otimes_{\mathcal{O}} E_2^*$ with $(E_1 \otimes_{\mathcal{O}} E_2)^*$ in this way. Similarly we identify $(E_1 \oplus E_2)^*$ with $E_1^* \oplus E_2^*$, regarding the linear forms on $E_i|Z$ as linear forms on $(E_1 \oplus E_2)|Z$ which are zero on $E_j|Z$ $\{(i,j) = (1,2)$ or $= (2,1)\}$. Let $\varphi_i : E_i \to E_i^*$ be the homomorphism associated with B_i. Then $B_1 \perp B_2$ is the bilinear form corresponding to

$$\begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} : E_1 \oplus E_2 \to E_1^* \oplus E_2^*$$

and $B_1 \otimes B_2$ is the bilinear form corresponding to

$$\varphi_1 \otimes \varphi_2 : E_1 \otimes E_2 \to E_1^* \otimes E_2^*.$$

From this description of $B_1 \perp B_2$ and $B_1 \otimes B_2$ it is evident that these forms are non degenerate if both B_1 and B_2 are non degenerate.

We now discuss a special type of bilinear spaces over X. Let (a_{ij}) be a symmetric $n \times n$-matrix with coefficients a_{ij} in the ring $\mathcal{O}(X)$. We take the free bundle $\mathcal{O}^n$, and denote by $e_1, \dots, e_n$ the standard basis of $\mathcal{O}^n$, i.e. the global sections

$$e_i = (0, \dots, \underset{i}{1}, \dots 0)$$

in $\mathcal{O}^n(X) = \mathcal{O}(X)^n$. We introduce on $\mathcal{O}^n$ a symmetric bilinear form B as follows. Let Z be open in X, and let

$$u = (u_1, \dots, u_n) \in \mathcal{O}(Z)^n$$
$$v = (v_1, \dots, v_n) \in \mathcal{O}(Z)^n$$

be sections in $\mathcal{O}^n(Z)$. We put

$$B_Z(u,v) := \sum_{i,j=1}^{n} (a_{ij}|Z)\, u_i v_j.$$

Clearly B is the unique bilinear form on $\mathcal{O}^n$ with $B(e_i,e_j) = a_{ij}$. We abbreviate this bilinear bundle (E,B) by the matrix (a_{ij}). Clearly (a_{ij}) is a space if and only if the determinant of this matrix is a unit in $\mathcal{O}(X)$. In this way we obtain up to isomorphism all free spaces, i.e. all spaces (E,B) with a free vector bundle $E \cong \mathcal{O}^n$.

Two free spaces (or free bilinear bundles) (a_{ij}) and (a'_{ij}) are isomorphic if and only if there exists an $n \times n$-matrix S with coefficients in $\mathcal{O}(X)$ and determinant a unit of $\mathcal{O}(X)$ such that

$$(a'_{ij}) = S^t(a_{ij})S.$$

The proof is the same as the usual proof over fields.

A "diagonal" free bilinear bundle (a_{ij}), $a_{ij} = a_i\delta_{ij}$, will also be denoted by $\langle a_1,\ldots,a_n\rangle$. We have

$$\langle a_1,\ldots,a_n\rangle = \langle a_1\rangle \perp \ldots \perp \langle a_n\rangle.$$

The tensor product of two diagonal free bilinear bundles $\langle a_1,\ldots,a_n\rangle$ and $\langle b_1,\ldots,b_m\rangle$ is isomorphic to

$$\langle a_1b_1,\ldots,a_1b_m,a_2b_1,\ldots,a_2b_m,\ldots\ a_nb_m\rangle.$$

§ 2 Subbundles.

In this section we develop some elementary linear algebra dealing with "subbundles" of a bilinear space.

Let E be a vector bundle over X. Assume for every open subset Z of X there is given an $\mathcal{O}(Z)$-submodule V(Z) of E(Z), and that for open sets $Z' \subset Z$ the restriction homomorphism from E(Z) to E(Z') maps V(Z) into V(Z'). If then then functor $V: Z \to V(Z)$ on the category of open subsets of X fulfills the sheaf condition (cf. § 1) we call V an $\mathcal{O}_X$ -<u>submodule</u> of E.

<u>Definition.</u> We call an $\mathcal{O}$-submodule V of E a <u>subbundle</u> of E, if V is locally a direct summand of E. This means that every point x of X has an open neighbourhood Z such that

(*) $$E|Z \cong (V|Z) \oplus W$$

with W a suitable $\mathcal{O}_Z$ -submodule of $E|Z$.

Clearly then V and E/V are vector bundles, since direct summands of locally free $\mathcal{O}_Z$ -modules of finite rank are again locally free. On the other hand, if we only know that E/V is a vector bundle then we have a splitting (*) over <u>every affine</u> open subset Z of X. Indeed, the canonical projection from $E|Z$ onto $(E/V)|Z$ has a section, since $(E/V)|Z$ corresponds to a projective module over $\mathcal{O}(Z)$.

We now regard a bilinear bundle (E,B) over X with associated homomorphism $\varphi: E \to E^*$. For every $\mathcal{O}$-submodule V of E we define another $\mathcal{O}$-submodule $V^{\perp}$ as follows. If Z is an open subset of X then $V^{\perp}(Z)$ consists of all sections s in $E(Z)$ such that $s|Z'$ is orthogonal to $V(Z')$ with respect to $B_{Z'}$ for every open $Z' \subset Z$. It suffices for this to know that the germ s_x of s at every point x of Z is orthogonal to V_x. Thus every stalk $(V^{\perp})_x$ consists of all t in E_x which are orthogonal to V_x, i.e.

$$(V^{\perp})_x = (V_x)^{\perp}.$$

The $\mathcal{O}$-module $V^{\perp}$ is the kernel of the homomorphism

$$E \xrightarrow{\varphi} E^* \xrightarrow{i^*} V^*$$

with i^* the dual of the inclusion homomorphism i from V to E, as is immediately verified. *)

Assume now B is <u>non degenerate</u>, and V is a <u>subbundle</u> of E. Then φ is an isomorphism and i^* is an epimorphism. Thus φ induces an isomorphism

$$\alpha : E/V^{\perp} \xrightarrow{\sim} V^*,$$

and in particular $V^{\perp}$ is again a subbundle of E.

We identify $(E/V)^*$ with the $\mathcal{O}$-submodule of E^* whose sections over any open $Z \subset X$ are the linear forms $\lambda : E|Z \to \mathcal{O}_Z$ which vanish on $V|Z$. By definition $V^{\perp}$ is the inverse image of $(E/V)^*$ under the isomorphism $\varphi : E \xrightarrow{\sim} E^*$. Thus we obtain from φ an isomorphism

$$\beta : V^{\perp} \xrightarrow{\sim} (E/V)^*.$$

*) i^* means restriction of the linear forms on E to V.

The isomorphisms α and β correspond to bilinear maps

(1) $$V \times_X (E/V^{\perp}) \to \mathcal{O}$$

(2) $$V^{\perp} \times_X (E/V) \to \mathcal{O}$$

which are perfect dualitites. These bilinear maps are of course just the maps obtained from B by "restriction" in the obvious way. Since $V^{\perp}$ is again a subbundle of E we also have a perfect duality

(3) $$V^{\perp\perp} \times_X E/V^{\perp} \to \mathcal{O}.$$

induced by B. V is an $\mathcal{O}$-submodule of $V^{\perp\perp}$. Comparing (1) and (3) we see that actually $V = V^{\perp\perp}$. Let us summarize our observations.

<u>Proposition 1.</u> Let V be a subbundle of the bilinear space (E,B), and let $\varphi: E \xrightarrow{\sim} E^*$ denote the isomorphism from E to E* associated with B.

i) $V^{\perp}$ is a subbundle of E. There exists a unique isomorphism α from $E/V^{\perp}$ to E* such that the diagram

$$\begin{tikzcd} E \arrow[r, "\varphi", "\sim"'] \arrow[d, two heads] & E^* \arrow[d, two heads] \\ E/V^{\perp} \arrow[r, "\sim", "\alpha"'] & V^* \end{tikzcd}$$

with canonical vertical surjections commutes.

ii) There exists a unique isomorphism β from $V^{\perp}$ to (E/V)* such that the diagram

$$\begin{tikzcd} V^{\perp} \arrow[r, "\beta", "\sim"'] \arrow[d, hook] & (E/V)^* \arrow[d, hook] \\ E \arrow[r, "\sim", "\varphi"'] & E^* \end{tikzcd}$$

with canonical vertical injections commutes.

iii) $V^{\perp\perp} = V$.

We assume now that (E,B) is a bilinear bundle and that F is an $\mathcal{O}$-submolule of E which is again a vector bundle. We further assume that the bilinear form $B|F$ is non degenerate. The form B may be degenerate. Let $\varphi : E \to E^*$ and $\psi : F \xrightarrow{\sim} F^*$ denote the homomorphisms associated with B and $B|F$, and let $i : F \to E$ denote the inclusion homomorphism from F to E. As observed above $F^{\perp}$ is the kernel of the homomorphism $i^* \circ \varphi$ from E to F^*, hence also the kernel of the homomorphism $p := \psi^{-1} \circ i^* \circ \varphi$ from E to F. More concretely, if u is a section of E over some affine open set Z then by definition $p(u)$ is the unique section v of F over Z with $B(u,w) = B(v,w)$ for all w in $F(Z)$.

Since $\psi = i^* \circ \varphi \circ i$ we have $p \circ i = id_F$. Thus E is the direct sum of F and $F^{\perp}$. Moreover $i \circ p$ is a projection operator on E with image F, the "orthogonal projection" from E onto F. We summarize:

<u>Proposition 2.</u> Assume F is an $\mathcal{O}$-submodule of a bilinear bundle (E,B), further that F is a vector bundle and $B|F$ is non degenerate. Then E is the direct sum of F and $F^{\perp}$. In particular F is a subbundle of E. We have a canonical orthogonal projection from E

onto F.

<u>Definition.</u> We call a morphism α from a bilinear bundle (E_1,B_1) to a bilinear bundle (E_2,B_2) an <u>isometry</u> if α is injective *) and $\alpha(E_1)$ is a subbundle of E_2.

<u>Proposition 3.</u> If B_1 is non degenerate then every morphism α from (E_1,B_1) to (E_2,B_2) is an isometry.

<u>Proof.</u> By Proposition 2 it suffices to show that α is an injective homomorphism from E_1 to E_2. This will be true if we know that the maps $\alpha_x : E_{1x} \to E_{2x}$ on the stalks are injective. {Apply the sheaf condition!} Now if $\alpha_x(u) = 0$ for some u in E_{1x} then

$$B_{1x}(u,v) = B_{2x}(\alpha_x(u),\alpha_x(v)) = 0$$

for every v in E_{1x}. Thus u = 0.

Assume now (E,B) is a bilinear <u>space</u>.

<u>Definition.</u> A <u>totally isotropic subbundle</u> V of (E,B) is a subbundle V of E such that B is zero on $V \times_X V$. In other terms, $V^{\perp} \supset V$.

A totally isotropic subbundle V of (E,B) will also be called a <u>sublagrangian</u> of (E,B). {"Lagrangians" will be introduced in the next section.} For every such sublagrangian the $\mathcal{O}$-module $V^{\perp}/V$ is a subbundle of the vector bundle E/V, since $V^{\perp}$ is locally a direct summand of E. From B we obtain in an evident

*) This means that $\alpha_Z : E_1(Z) \to E_2(Z)$ is injective for every open $Z \subset X$.

way a symmetric bilinear form $\overline{B}$ on $V^{\perp}/V$ such that the natural projection $p:V^{\perp} \to V^{\perp}/V$ is a morphism from the bilinear bundle $(V^{\perp}, B|V^{\perp})$ to $(V^{\perp}/V,\overline{B})$.

<u>Proposition 4.</u> $(V^{\perp}/V,\overline{B})$ is a bilinear space.

<u>Proof.</u> Let $\varphi:E \xrightarrow{\sim} E^*$ denote again the homomorphism associated with B and $\overline{\varphi}:V^{\perp}/V \to (V^{\perp}/V)^*$ the homomorphism associated with $\overline{B}$. We obtain from the definitions a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] \arrow[d, "\gamma"', "\cong"] & V^{\perp} \arrow[r, "p"] \arrow[d, "\beta"', "\cong"] & V^{\perp}/V \arrow[r] \arrow[d, "\overline{\varphi}"] & 0 \\
0 \arrow[r] & (E/V^{\perp})^* \arrow[r] & (E/V)^* \arrow[r] & (V^{\perp}/V)^* \arrow[r] & 0
\end{tikzcd}$$

with exact rows and isomorphisms β and γ as described in Proposition 1 ii. Thus also $\overline{\varphi}$ is an isomorphism.

§ 3 Metabolic spaces

We now introduce a special type of bilinear spaces. Let (U,β) be a bilinear bundle. Starting with β we define a symmetric bilinear form B on the vector bundle $U \oplus U^*$ as follows. For Z an open set in X and sections u_1,u_2 in $U(Z)$, u_1^*,u_2^* in $U^*(Z)$

$$B(u_1+u_1^*,u_2+u_2^*) := \beta(u_1,u_2) + \langle u_1 u_2^*\rangle + \langle u_2,u_1^*\rangle.$$

Thus B coincides with β on $U \times_X U$, is zero on $U^* \times_X U^*$, and is the natural pairing on $U \times_X U^*$ and $U^* \times_X U$. Let $\varphi : U \to U^*$ be the homomorphism associated with β. Then the homomorphism

$$U \oplus U^* \to (U \oplus U^*)^* = U^* \oplus U$$

associated with B is given by the matrix

$$\begin{pmatrix} \varphi & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

This homomorphism is always an isomorphism and thus B is non degenerate. We denote the space $(U \oplus U^*,B)$ by $M(U,\beta)$. A space isomorphic to some $M(U,\beta)$ will be called <u>split</u> <u>metabolic</u>. {"Metabolic spaces" will be introduced below.}

<u>Example.</u> If (U,β) is a free diagonal bilinear bundle $\langle a_1,\dots,a_r\rangle$, $a_i \in \mathcal{O}(X)$, then clearly

$$M(U,\beta) \cong \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \perp \dots \perp \begin{pmatrix} a_r & 1 \\ 1 & 0 \end{pmatrix}.$$

On any vector bundle U we can introduce the bilinear form $\beta = 0$, and obtain a space $M(U,0)$ which we denote by $H(U)$. A space isomorphic to some $H(U)$ will be called <u>hyperbolic</u>.

<u>Proposition 1.</u> For any symmetric bilinear form α on a vector bundle U the space $M(U,2\alpha)$ is isomorphic to $H(U)$. In particular all split metabolic spaces are hyperbolic if 2 is a unit in $\mathcal{O}(X)$.

This follows immediately from the following more general fact. (Put $\beta = 2\alpha$, $\gamma = -\alpha$).

<u>Lemma.</u> Let $\beta : U \times_X U \to \mathcal{O}$ be a symmetric bilinear form on the vector bundle U and let $\gamma : U \times_X U \to \mathcal{O}$ be an arbitrary (not necessarily symmetric) bilinear form on U. Let γ^t denote the bilinear form defined by

$$\gamma_Z^t(u,v) = \gamma_Z(v,u)$$

$(u,v \in U(Z)$, Z open in $X)$. Then

$$M(U,\beta) \cong M(U,\beta+\gamma+\gamma^t).$$

<u>Proof.</u> Let $\varphi : U \to U^*$ denote the homomorphism "$u \to \gamma(-,u)$" associated with γ. Then the automorphism

$$u + u^* \to u + u^* + \varphi(u)$$

of the vector bundle $U \oplus U^*$ $\{u \in U(Z),\ u^* \in U^*(Z),\ Z$ open in $X\}$ is an isometry from $M(U,\beta+\gamma+\gamma^t)$ onto $M(U,\beta)$. Indeed, denoting the bilinear form of the first space by B' and the bilinear form of the second space by B, we have for sections u,v in $U(Z)$, u^*,v^* in $U^*(Z)$:

$$B(u+u^*+\omega(u),v+v^*+\omega(v)) = \beta(u,v) + \langle u,\omega(v)\rangle + \langle v,\omega(u)\rangle +$$
$$+ \langle u,v^*\rangle + \langle v,u^*\rangle = \beta(u,v) + \gamma(u,v) + \gamma^t(u,v) + \langle u,v^*\rangle + \langle v,u^*\rangle =$$
$$= B'(u+u^*,v+v^*).$$

q.e.d.

If 2 is not a unit in $\mathcal{O}(X)$ then not every split metabolic space is hyperbolic. For example the free space $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ can not be hyperbolic, since for any global section e of a hyperbolic space $(E,B) = H(U)$ we have $B(e,e) \in 2\,\mathcal{O}(X)$. Nevertheless split metabolic spaces are "stably hyperbolic" by the following proposition.

<u>Proposition 2.</u> For any bilinear bundle (U,β)

$$M(U,\beta) \perp M(U,-\beta) \cong H(U) \perp M(U,-\beta).$$

<u>Proof.</u> We consider the space $E := M(U,\beta) \perp M(U,-\beta)$. Let U_1 and U_2 be two copies of U. We think of $M(U,\beta)$ as the bundle $U_1 \oplus U_1^*$ and of $M(U,-\beta)$ as the bundle $U_2 \oplus U_2^*$. Thus

$$E = (U_1 \oplus U_1^*) \perp (U_2 \oplus U_2^*).$$

U_1^* and the diagonal Δ of $U_1 \oplus U_2$ are both sublagrangians of E. The bilinear form B of E gives a perfect duality between Δ and U_1^*. Thus $\Delta \oplus U_1^*$ is a bilinear subspace of E and

$$\Delta \oplus U_1^* \cong H(\Delta) \cong H(U).$$

According to § 2, Prop.2,

$$E \cong H(U) \perp (\Lambda \cap U_1^*)^{\perp}.$$

One know has to compute $(\Lambda \cap U_1^*)^{\perp}$. This bilinear space turns out to be isomorphic to $M(U,-\beta)$. We leave the details as an exercise (or cf. [K,p.19 f]).

<u>Definition.</u> We call a bilinear space <u>anisotropic</u> if it has no totally isotropic subbundle different from zero. Otherwise we call the space <u>isotropic</u>.

If V is a maximal totally isotropic subbundle of a space (E,B) then clearly the space $(V^{\perp}/V,\overline{B})$ studied at the end of § 2 is anisotropic.

<u>Proposition 3.</u> If X is <u>affine</u> then every bilinear space E over X has a decomposition

$$E = E_0 \perp M$$

with E_0 anisotropic and M split metabolic.

Such a decomposition of E will be called a <u>Witt decomposition</u>. In general the isomorphism class of the anisotropic part E_0 is <u>not</u> uniquely determined by E. An example over a local ring can be extracted from [K,Satz 9.3.8]. Also the isomorphism class of the split metabolic part M is in general not uniquely determined by E. Here counter-examples are quickly obtained. For example over any scheme X

$$\langle 1\rangle \perp \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cong \langle 1\rangle \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

as is easily verified.

Proposition 3 follows immediately from the following more general proposition.

<u>Proposition 3 a.</u> Assume X is affine. Let (E,B) be a space over X and V a sublagrangian of (E,B). Let U be a subbundle of E with

$$E \cong V^{\perp} \oplus U.$$

(Such a subbundle exists, since X is affine.) Then U and V are in duality under B, hence

$$(U \oplus V, B|U \oplus V) \cong M(U, B|U).$$

Moreover the space $(U \oplus V)^{\perp}$ is isomorphic to $(V^{\perp}/V, \overline{B})$, hence by § 2, Prop.2

$$\boxed{(E,B) \cong (V^{\perp}/V, \overline{B}) \perp M(U, B|U).}$$

<u>Proof.</u> The assertion that U and V are in duality is trivial. With $G := (U \oplus V)^{\perp}$ we have

$$E = (U \oplus V) \perp G.$$

From this we deduce

$$V^{\perp} = V \perp G.$$

Thus clearly $(G, B|G)$ is isomorphic to $(V^{\perp}/V, \overline{B})$.

q.e.d.

<u>Definition.</u> A subbundle V of a space (E,B) is called a <u>Lagrangian</u> if $V^{\perp} = V$. A space which has a Lagrangian is called <u>metabolic</u>.

Clearly any split metabolic space $M(U,\beta)$ has the Lagrangian U^*, hence is metabolic.

From Proposition 3a we obtain immediately the following two statements about metabolic spaces and Lagrangians.

Corollary 1. Every metabolic space over an affine scheme is split metabolic.

Corollary 2. Let V be a sublagrangian of a space (E,B) over any scheme X. Then

i) $\operatorname{rk} E \geqslant 2 \operatorname{rk} V$.

ii) V is a Lagrangian if and only if $\operatorname{rk} E = 2 \operatorname{rk} V$.

Indeed, it suffices to check these statements i), ii) over affine open subsets of X, where they are evident by Proposition 3 a.

Already over an elliptic curve there exists infinitely many metabolic spaces which are not split metabolic, cf. [K, § 13.1].

We discuss the behaviour of metabolic spaces with respect to tensor products.

Proposition 4. If V is a Lagrangian of the space (E,B) then for any other space (E',B') the submodule $V \otimes E'$ of $E \otimes E'$ is a Lagrangian of $(E,B) \otimes (E',B')$. For a split metabolic space $M(U,\beta)$ more explicitly

$$M(U,\beta) \otimes (E',B') \cong M((U,\beta) \otimes (E',B')).$$

In particular

$$H(U) \otimes (E',B') \cong H(U \otimes E').$$

<u>Proof.</u> $V \otimes F$ injects into $E \otimes E'$ since E' is locally free. $(E/V) \otimes E'$ can be identified with $E \otimes E'/V \otimes E'$. Thus $V \otimes E'$ is a subbundle of $E \otimes E'$. Clearly $V \otimes E'$ is totally isotropic. Since

$$\operatorname{rk}(V \otimes E') = \operatorname{rk} V \cdot \operatorname{rk} E' = \frac{1}{2} \operatorname{rk}(E \otimes E')$$

$V \otimes E$ must be a Lagrangian.

We now consider the case

$$(E,B) = M(U,\beta) = (U \oplus U^*,B).$$

Then the subbundle $U^* \otimes E'$ of $E \otimes E'$ is a Lagrangian which under $B \otimes B'$ is in duality with $U \otimes E'$. Since $E \otimes E'$ is the direct sum of $U \otimes E'$ and $U^* \otimes E'$ we indeed have

$$(E,B) \otimes (E',B') \cong M(U \otimes E',\beta \otimes B').$$

As an example we consider the space

$$M(\langle 1\rangle) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let e_1,e_2 be a basis of $M(\langle 1\rangle)$ with value matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ under the bilinear form of $M(\langle 1\rangle)$. Then e_1,e_1-e_2 has the value matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus

$$M(\langle 1\rangle) \cong \langle 1,-1\rangle.$$

Multiplying this relation with an arbitrary space (E,B) we obtain by the preceding proposition 4 the following

<u>Corollary.</u> $(E,B) \perp (E,-B) \cong M(E,B)$.

This can also easily be verified in a direct way.

§ 4 The Grothendieck-Witt ring L(X).

We now often denote a space (E,B) by a single letter E for short. We look at the set Bil(X) of isomorphism classes (E) of bilinear spaces E over X. On Bil(X) we have an addition and a multiplication given by the orthogonal sum and the tensor product of spaces. In this way Bil(X) is a commutative semiring. It has the isomorphism class of the space $\langle 1\rangle = (\mathcal{O}_X, m)$ as unit element with $m: \mathcal{O}_X \times_X \mathcal{O}_X \to \mathcal{O}_X$ the multiplication on $\mathcal{O}_X$.

We now consider the Grothendieck-ring K Bil(X) of the semiring Bil(X). This is a well known construction. The elements of K Bil(X) are formal differences $[E] - [F]$ of classes $[E],[F]$ of spaces E,F with the rule that two differences $[E_1] - [F_1]$ and $[E_2] - [F_2]$ are equal if and only if

$$E_1 \perp F_2 \perp G \cong E_2 \perp F_1 \perp G$$

for some space G over X. Recall that the map $(E) \to [E]$ *) from Bil(X) to K Bil(X) is a universal map from Bil(X) to a ring, i.e. for every semiring homomorphism $\lambda: \mathrm{Bil}(X) \to A$ to a ring A there exists a unique ring homomorphism $\mu: \mathrm{K\,Bil}(X) \to A$ such that the diagram

$$\begin{tikzcd} \mathrm{Bil}(X) \arrow[r] \arrow[dr, "\lambda"'] & \mathrm{K\,Bil}(X) \arrow[d, "\mu"] \\ & A \end{tikzcd}$$

*) Of course $[E]$ is identified with $[E] - [0]$.

commutes.

Let $\mathfrak{a}$ denote the additive subgroup of K Bil(X) generated by the elements $[E] - [H(V)]$ with E metabolic and V a Lagrangian of E. If E_1 and E_2 are metabolic spaces with Lagrangians V_1 and V_2 then $E_1 \perp E_2$ is metabolic with the Lagrangian $V_1 \oplus V_2$. From this remark we obtain immediately that $\mathfrak{a}$ is the set of differences $[E] - [F]$ of metabolic spaces E and F having isomorphic Lagrangians.

$\mathfrak{a}$ is an ideal of K Bil(X). Indeed if E is metabolic with Lagrangian V and F is an arbitrary space, then according to § 3 Prop.4

$$[F]([E]-[H(V)]) = [F \otimes E] - [H(F \otimes V)],$$

and $F \otimes V$ is a Lagrangian of $F \otimes E$.

We now define the <u>Grothendieck-Witt ring</u> L(X) of the scheme X as the ring K Bil(X)/$\mathfrak{a}$. The image of a class $[E] \in$ K Bil(X) in L(X) will again be denoted by $[E]$. In L(X) we have by definition

$$[E] = [H(V)]$$

for E a metabolic space with Lagrangian V.

<u>Proposition 1.</u> If X is affine then $\mathfrak{a} = 0$, hence L(X) = K Bil(X).

<u>Proof.</u> Let (E,B) be a space with Lagrangian V. Then, as explained in § 3, $E = U \oplus V$ with some subbundle U which is dual to V with respect to B. We have

$$(E,B) \cong M(U,B|U) = M(U,\beta),$$

and, again by § 3,

$$M(U,\beta) \perp M(U,-\beta) \cong H(U) \perp M(U,-\beta).$$

Thus in K Bil(X)

$$[E,B] = [M(U,\beta)] = [H(U)] = [H(V^*)] = [H(V)].$$

This proves $\alpha = 0$.

For hyperbolic spaces we have the following general fact.

Proposition 2. Let V be a vector bundle over a scheme X and W be a subbundle of V. Then in L(X)

$$[H(V)] = [H(W)] + [H(V/W)].$$

Proof. We work in the bilinear space $H(V) = V \oplus V^*$. Let W' denote the submodule $W^{\perp} \cap V^*$ of V^*. This is the sheaf of linear forms on V which are zero on W, i.e. the kernal of the restriction homomorphism $V^* \to W^*$. On the other hand we obtain from the exact sequence

$$0 \to W \to V \to V/W \to 0$$

of vector bundles a dual exact sequence

$$0 \to (V/W)^* \to V^* \to W^* \to 0.$$

Thus we have canonical isomorphisms

$$(V/W)^* \xrightarrow{\sim} W', \quad V^*/W' \xrightarrow{\sim} W^*.$$

In particular W' is a subbundle of V^*. Clearly

$$U := W \oplus W'$$

is a totally isotropic subbundle of $H(V)$. It has the rank

$$\mathrm{rk}\ W + \mathrm{rk}\ W' = \mathrm{rk}\ W + \mathrm{rk}(V/W)^* = \mathrm{rk}\ V.$$

Thus U is a Lagrangian of $H(V)$. We have in $L(X)$

$$[H(V)] = [H(U)] = [H(W)] + [H(W')] = [H(W)] + [H((V/W)^*)] =$$

$$= H(W) + H(V/W),$$

as claimed above.

We now can prove the following weak analogue of the Witt decomposition in § 3 for an arbitrary scheme X. (The Witt decomposition in § 3 could only be done over affine X, cf. § 3, Prop. 3a.)

<u>Theorem 3.</u> Let V be a sublagrangian of a space (E,B) over X. Then we have in $L(X)$ the equation (cf. § 3 for notations)

$$[E,B] = [H(V)] + [V^{\perp}/V,\bar{B}].$$

Notice that for V a maximal sublagrangian of (E,B) the space $(V^{\perp}/V,\bar{B})$ is anisotropic.

<u>Proof.</u> We work in the space

$$(F,B') := (E,B) \perp (V^{\perp}/V,-\bar{B}).$$

Let

$$\alpha: V^{\perp} \to E \oplus V^{\perp}/V$$

denote the "diagonal injection" of $V^{\perp}$ into this space. The submodule $\alpha(V^{\perp})$ of F is clearly totally isotropic. We want to show that $\alpha(V^{\perp})$ is a Lagrangian of (F,B'). It suffices to check this over affine open subsets of X, and thus we assume now that X itself is affine. As explained in § 3 (proof of Prop. 3a) we have decompositions

$$E = (V \oplus U) \perp G, \ V^{\perp} = V \oplus G$$

with U dual to V under B. The second decomposition yields a canonical isomorphism from G onto $V^{\perp}/V$. We have

$$\alpha(V^{\perp}) = V \oplus \Delta$$

with Δ the "diagonal" of the subbundle $G \oplus V^{\perp}/V$ of F. Clearly

$$G \oplus V^{\perp}/V = \Delta \oplus V^{\perp}/V.$$

Thus $V \oplus \Delta$ is a direct summand of

$$F = (V \oplus U) \perp (\Delta \oplus V^{\perp}/V),$$

and we have verified that $\alpha(V^{\perp})$ is a subbundle of F. We have

$$\text{rk } \alpha(V^{\perp}) = \text{rk}(V^{\perp}) = \text{rk } V + \text{rk } V^{\perp}/V.$$

On the other hand

$$\text{rk } F = 2 \text{ rk } V + \text{rk } G + \text{rk } V^{\perp}/V = 2(\text{rk } V + \text{rk } V^{\perp}/V).$$

Thus $\alpha(V^{\perp})$ is indeed a Lagrangian of (F,B').

Since now X is again an arbitrary scheme. The vector bundle $\alpha(V^{\perp})$ is isomorphic to $V^{\perp}$. Thus we obtain in L(X) the relation

$$(*) \qquad [E,B] + [V^{\perp}/V,-\bar{B}] = [H(V^{\perp})].$$

By the preceding Proposition 2

$$[H(V^{\perp})] = [H(V)] + [H(V^{\perp}/V)].$$

On the other hand

$$[V^{\perp}/V,-\overline{B}] + [V^{\perp}/V,\overline{B}] = [M(V^{\perp}/V,\overline{B})] = [H(V^{\perp}/V)].$$

Thus adding $[V^{\perp}/V,\overline{B}]$ on both sides in (*) we arrive at the desired relation

$$[E,B] = [H(V)] + [V^{\perp}/V,\overline{B}].$$

§ 5 Definition of the Witt ring $W(X)$.

First we recall Grothendieck's definition of his ring $K(X)$ of vector bundles on X in a way adapted to our present study. Let $\mathrm{Vect}(X)$ denote the set of isomorphism classes (V) of vector bundles V on X. The direct sum and the tensor product of vector bundles give on $\mathrm{Vect}(X)$ an addition and a multiplication which turn $\mathrm{Vect}(X)$ into a commutative semiring. This semiring has the unit element $(\mathcal{O}_X)$.

Let $K\,\mathrm{Vect}(X)$ denote the Grothendieck ring of this semiring $\mathrm{Vect}(X)$. We denote the class of a vector bundle V in $K\,\mathrm{Vect}(X)$ by $[V]$. We introduce the additive subgroup $\mathfrak{c}$ of $K\,\mathrm{Vect}(X)$ generated by the elements

$$[V] - [W] - [V/W]$$

with V running through the vector bundles of X and W running through the subbundles of V. Clearly $\mathfrak{c}$ is an ideal of $K\,\mathrm{Vect}(X)$. The ring $K(X)$ is defined as the quotient $K\,\mathrm{Vect}(X)/\mathfrak{c}$ by this ideal. We shall denote the class of a vector bundle V in $K(X)$ again by $[V]$.

Notice that if X is affine we have $\mathfrak{c} = 0$, hence $K(X) = K\,\mathrm{Vect}(X)$.

We have a "hyperbolic map"

$$H_o : \mathrm{Vect}(X) \rightarrow \mathrm{Bil}(X)$$

sending the isomorphism class of a vector bundle V to the isomorphism class of the hyperbolic space $H(V)$. Clearly H_o is additive. Thus H_o induces an additive map

$$H_1 : K\ \mathrm{Vect}(X) \to K\ \mathrm{Bil}(X).$$

We now consider the composition

$$H_2 : K\ \mathrm{Vect}(X) \longrightarrow K\ \mathrm{Bil}(X) \twoheadrightarrow L(X)$$

of H_1 with the natural surjection from $K\ \mathrm{Bil}(X)$ to $L(X)$. According to Proposition 2 of the preceding section H_2 vanishes on c. Thus we obtain finally an additive map

$$H : K(X) \to L(X).$$

This map H sends an element $[U_1] - [U_2]$ of $K(X)$ to $[H(U_1)] - [H(U_2)]$.

The image $H\ K(X)$ of H is an ideal of $L(X)$ since for F a space and U_1, U_2 vector bundles over X

$$[F]([H(U_1)] - [H(U_2)]) = [H(F \otimes U_1)] - [H(F \otimes U_2)].$$

We define the Witt ring $W(X)$ of the scheme X as the quotient of $L(X)$ by this ideal,

$$W(X) = L(X)/H\ K(X).$$

Recalling the definition of $L(X)$ we may also write

$$W(X) = K\ \mathrm{Bil}(X)/\mathfrak{M}$$

with $\mathfrak{M}$ the ideal of $K\ \mathrm{Bil}(X)$ additively generated by the classes of all metabolic spaces on X. This ideal is the set of differences $[M_1] - [M_2]$ of metabolic spaces M_1, M_2.

We denote the class of a space (E,B) in $W(X)$ by $\{E,B\}$. We call $\{E,B\}$ the <u>Witt class</u> of the space (E,B). We often write $\{E\}$ instead of $\{E,B\}$ for short. Since

$$(E,B) \perp (E,-B) \cong M(E,B),$$

cf. § 3, we have in $W(X)$

$$-\{E,B\} = \{E,-B\}.$$

This implies that every element of $W(X)$ is the Witt class of some space, and not just the difference of two Witt classes.

<u>Definition.</u> We call two spaces E_1 and E_2 over X <u>equivalent</u> (or "Witt-equivalent"), and write $E_1 \sim E_2$, if $\{E_1\} = \{E_2\}$.

Clearly $W(X)$ is the quotient of the semiring $Bil(X)$ by this equivalence relation,

$$W(X) = Bil(X)/\sim.$$

We have the following simple description of Witt equivalence.

<u>Proposition.</u> Two spaces E_1,E_2 over X are equivalent if and only if there exist metabolic spaces M_1,M_2 over X such that

$$E_1 \perp M_1 \cong E_2 \perp M_2.$$

<u>Proof.</u> If $E_1 \perp M_1 \cong E_2 \perp M_2$ with metabolic spaces M_1,M_2 then clearly

$$\{E_1\} = \{E_1 \perp M_1\} = \{E_2 \perp M_2\} = \{E_2\}.$$

Assume now that $\{E_1\} = \{E_2\}$. Then we have in $K\,Bil(X)$ an equation

$$[E_1] - [E_2] = [N_2] - [N_1]$$

with metabolic spaces N_1 and N_2. Thus there exists a space (G,B) over X such that

$$E_1 \perp N_1 \perp G \cong E_2 \perp N_2 \perp G.$$

We add on both sides the space $(G,-B)$ and obtain

$$E_1 \perp N_1 \perp M(G,B) \cong E_2 \perp N_2 \perp M(G,B).$$

This finishes the proof of the proposition.

In the present lectures we are mainly interested in $W(X)$ and not in $L(X)$. We could have defined $W(X)$ directly by introducing on $Bil(X)$ the equivalence relation described in the proposition. But for technical reasons also $L(X)$ and the precise relationship between $L(X)$ and $W(X)$ will be needed.

§ 6 Functoriality

We now discuss the functorial behaviour of our rings $L(X)$ and $W(X)$. Let $f: Y \to X$ be a morphism of schemes. For E a vector bundle on X the inverse image $f^*(E)$ (cf. [EGA, Chap. 0, § 4]) is a vector bundle on Y. Indeed, if U is an open set of X with $E|U \cong \mathcal{O}_U^n$ then $f^{-1}(U)$ is an open set of Y with

$$f^*(E)|f^{-1}(U) \cong \mathcal{O}^n_{f^{-1}(U)}.$$

A useful description of this vector bundle is as follows. Let $U \subset X$ and $V \subset Y$ be affine open subsets of X and Y with $f(V) \subset U$. Then f yields a ring homomorphism from $\mathcal{O}_X(U)$ to $\mathcal{O}_Y(V)$, and

$$f^*(E)(V) = E(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_Y(V),$$

cf. [EGA I, § 1.6].

For any vector bundle E on X there exists a canonical isomorphism of vector bundles

$$\kappa : f^*(E^*) \xrightarrow{\sim} f^*(E)^*.$$

This isomorphism can be described as follows. Let again U and V be affine open subsets of X and Y with $f(V) \subset U$. Then

$$f^*(E^*)(V) = E^*(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_Y(V) =$$

$$= \mathrm{Hom}_{\mathcal{O}_X(U)}(E(U), \mathcal{O}_X(U)) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_Y(V).$$

On the other hand

$$f^*(E)^*(V) = \mathrm{Hom}_{\mathcal{O}_Y(V)}(f^*(E)(V), \mathcal{O}_Y(V)) =$$

$$= \mathrm{Hom}_{\mathcal{O}_Y(V)}(E(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_Y(V), \mathcal{O}_Y(V)).$$

Thus we have an obvious natural isomorphism

$$\kappa_{V,U} : f^*(E^*)(V) \xrightarrow{\sim} f^*(E)^*(V)$$

since $E(U)$ is a finitely presented $\mathcal{O}_X(U)$-module, cf. [Bb, I § 2 no 9]. If (V_1, U_1) is a second pair of affine open subsets of Y and X with $f(V_1) \subset U_1$ and $V_1 \subset V$, $U_1 \subset U$, then the diagram

$$\begin{tikzcd}
f^*(E^*)(V) \arrow[r, "\kappa_{V,U}"] \arrow[d] & f^*(E)^*(V) \arrow[d] \\
f^*(E^*)(V_1) \arrow[r, "\kappa_{V_1,U_1}"'] & f^*(E)^*(V_1)
\end{tikzcd}$$

with the restriction homomorphisms as vertical arrows commutes, as is easily checked. In particular $(V = V_1)$ the homomorphism $\kappa_{V,U}$ does not depend on U but only on V. Since now we write κ_V instead of $\kappa_{V,U}$. The family of isomorphisms

$$\kappa_V : f^*(E^*)(V) \xrightarrow{\sim} f^*(E)^*(V)$$

with V running through the affine open subsets of Y is compatible with the restriction maps by the diagram above. Thus by general sheaf theory there exists a unique isomorphism

$$\kappa : f^*(E^*) \xrightarrow{\sim} f^*(E)^*$$

of vector bundles which induces κ_V on the sections over V for every affine open subset V of Y.

If B is a symmetric bilinear form on E then we obtain a symmetric bilinear form $f^*(B)$ on $f^*(E)$ as follows. Let $\varphi: E \to E^*$ denote the homomorphism associated with B, and let ψ denote the composite map

$$f^*(E) \xrightarrow{f^*(\varphi)} f^*(E^*) \xrightarrow[\kappa]{\sim} f^*(E)^*.$$

If V and U are affine open subsets on Y and X with $f(V) \subset U$, then

$$\psi_V : f^*(E)(V) \to f^*(E)^*(V)$$

yields a bilinear form $\{e,f \in f^*(E)(V)\}$

$$\beta_V(e,f) = \langle e, \psi_V(f)\rangle$$

on $f^*(E)(V)$. It is easily verified that β_V is simply the $\mathcal{O}_Y(V)$-bilinear form on $E(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_Y(V)$ obtained from the $\mathcal{O}_X(U)$-bilinear form B_U on $E(U)$ by scalar extension. In particular β_V is symmetric. Thus $\psi = \psi^t$, and ψ yields a symmetric bilinear form β on $f^*(E)$ which yields the bilinear forms β_V above on the affine open subsets V of Y. We denote β by $f^*(B)$.

If y is a point on Y and $x = f(y)$ then the $\mathcal{O}_{Y,y}$-bilinear form $f^*(B)_y$ on

$$f^*(E)_y = E_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Y,y}$$

is clearly the bilinear form obtained from the $\mathcal{O}_{X,x}$-bilinear form B_x on E_x by scalar extension.

If B is non degenerate then φ is an isomorphism. Thus also ψ is an isomorphism, i.e. $f^*(B)$ is again non degenerate.

In this way every space $E = (E,B)$ over X yields a space

$$f^*(E) = (f^*(E), f^*(B))$$

over Y. Clearly for spaces E and F over X

$$f^*(E \perp F) \cong f^*(E) \perp f^*(F),$$

$$f^*(E \otimes F) \cong f^*(E) \otimes f^*(F).$$

Thus we have a homomorphism

$$f^*: \mathrm{Bil}(X) \to \mathrm{Bil}(Y)$$

of semirings.

If V is a subbundle of a space E then $f^*(V)$ is a subbundle of $f^*(E)$, since

$$f^*(E)/f^*(V) \cong f^*(E/V)$$

is a vector bundle. More specifically, if V is a Lagrangian of E then $f^*(V)$ is a Lagrangian of $f^*(E)$. Indeed, $f^*(V)$ is totally isotropic and

$$\mathrm{rk}\, f^*(V) = \mathrm{rk}\, V = \frac{1}{2}\,\mathrm{rk}\, E = \frac{1}{2}\,\mathrm{rk}\, f^*(E).$$

Thus our map f^* from $\mathrm{Bil}(X)$ to $\mathrm{Bil}(Y)$ induces ring homomorphisms

$$L(f): L(X) \to L(Y), \quad W(f): W(X) \to W(Y).$$

Clearly $L(f)$ and $W(f)$ both map unit elements to unit elements.

In this way $X \mapsto L(X)$ and $X \mapsto W(X)$ are turned into contravariant functors L and W from the category of schemes to the category of commutative rings. We usually simply write f^* for $L(f)$ and also for $W(f)$.

Exercise. If V is a subbundle of a space E over X and $f:Y \to X$ a morphism then in the space $f^*(E)$ the subbundles $f^*(V)^\perp$ and $f^*(V^\perp)$ coincide.

In the case of affine schemes $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(C)$ a morphism $f:Y \to X$ corresponds to a ring homomorphism $\alpha:A \to C$. Instead of $f^*:W(X) \to W(Y)$ we then most often use the notation

$$\alpha_* : W(A) \to W(C).$$

For a space (P,β) over A clearly

$$\alpha_*\{P,\beta\} = \{P \otimes_A C,\ \beta \otimes_A C\}$$

with $\beta \otimes_A C$ denoting the scalar extension of the bilinear form β on the projective A-module P to a C-bilinear form on $P \otimes_A C$.

Usually the homomorphism from $L(A)$ to $L(C)$ induced by f will also be denoted by α_*.

§ 7 The rank homomorphism.

Assume the scheme X is connected. Then every vector bundle E on X has a constant rank rk E. In particular we have for spaces an evident rank homomorphism

$$\mathrm{rk}:\mathrm{Bil}(X) \to \mathbb{N} \cup \{0\}$$

of semirings. This yields a homomorphism

$$\mathrm{rk}:K\,\mathrm{Bil}(X) \to \mathbb{Z}$$

of rings. Clearly for E a metabolic space with Lagrangian V

$$\mathrm{rk}([E]-[H(V)]) = \mathrm{rk}(E) - \mathrm{rk}\, H(V) = 0.$$

Thus we obtain a rank homomorphism, again denoted by "rk", from L(X) to $\mathbb{Z}$,

$$\mathrm{rk}:L(X) \to \mathbb{Z}.$$

Unfortunately this homomorphism does not vanish on the class $[H(V)]$ of any hyperbolic space $H(V) \neq 0$. But rk H(V) is an even number. Thus we obtain nevertheless a ring homomorphism

$$\nu:W(X) \to \mathbb{Z}/2\,\mathbb{Z}$$

defined by

$$\nu(\{E\}) = \mathrm{rk}\, E \bmod 2.$$

We usually write $\nu(E)$ instead of $\nu(\{E\})$, and we call $\nu(E)$ the <u>rank index</u> of E.

If X is not connected then we have to replace $\mathbb{Z}$ and $\mathbb{Z}/2\,\mathbb{Z}$ by the rings of locally constant functions on X with values in $\mathbb{Z}$ and $\mathbb{Z}/2\,\mathbb{Z}$ respectively.

Chapter II Local theory

§ 1 Connection with Witt's theory.

We have to study the Witt rings of the local rings $\mathfrak{o}_x$. Slightly more generally we study the Witt ring $W(A)$ of a semi-local ring A, i.e. a commutative ring A which has only finitely many maximal ideals $\mathfrak{m}_1,\ldots,\mathfrak{m}_s$. This will not cause additional difficulties, and semilocal rings turn out to be a more appropriate category for our purposes than local rings (cf. in this connection e.g. the theory of "real closures" [K_4]).

If A is not connected *) then we have an orthogonal system of primitive idempotents $e_1,\ldots,e_t$,

$$\sum_{i=1}^{t} e_i = 1, \qquad e_i e_j = e_i \delta_{ij}.$$

Introducing the connected rings $A_i := Ae_i$ we may write

$$A = \prod_{i=1}^{t} A_i.$$

Let (E,B) be a space over A. In particular then E is a projective module over A, and $E_i := e_iE$ is a projective module over A_i, hence a free module. Moreover B induces non degenerate symmetric bilinear forms

$$B_i : E_i \times E_i \to A_i$$

*) This means Spec(A) is not connected.

We write symbolically

$$(E,B) = \prod_{i=1}^{t} (E_i,B_i).$$

Conversely we can prescribe over each ring A_i a space (E_i,B_i) and fit these spaces together to a space (E,B) over A which has the (E_i,B_i) as components. It is now easily checked that we obtain in this way natural isomorphisms

$$L(A) \cong \prod_{i=1}^{t} L(A_i), \quad W(A) \cong \prod_{i=1}^{t} W(A_i).$$

All this is intuitively obvious since the $\mathrm{Spec}(A_i)$ are the finitely many connected components of Spec(A).

Justified by this observation we assume since now always without loss of generality <u>that</u> A <u>is</u> <u>connected</u>. Thus all projective modules over A are free. Let H denote the hyperbolic standard plane $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ over A. Every hyperbolic space over A is of the shape $H(A^r)$ and thus an orthogonal sum $r \times H$ of r copies of H. In particular

$$W(A) = L(A)/\mathbb{Z}[H].$$

The following proposition is now obvious.

<u>Proposition 1.</u> Two spaces E and F over A have the same image in L(A) if and only if $E \sim F$ and $\mathrm{rk}E = \mathrm{rk}F$.

Assume now 2 is a unit in A. Then all metabolic spaces are hyperbolic (I,§ 3) and hence of the form $t \times H$. Moreover the cancellation law holds true over A, i.e.

$$E \perp G \cong F \perp G \Rightarrow E \cong F$$

for any spaces E,F,G over A. A proof can be found in [R] or [K_1]. Now recalling the Witt decomposition of spaces from I, § 3 (Prop.3) we meet the following situation.

<u>Proposition 2.</u> ($2 \in A^*$) Every space E over A has a decomposition

$$E \cong E_0 \perp t \times H$$

with E_0 anisotropic. The isomorphism class of E_0 and the number $t \geq 0$ are uniquely determined by E.

We call E_0 a <u>kernel space</u> of E. Proposition 2 has the following consequence.

<u>Corollary.</u> Two spaces E and F over A are equivalent ($E \sim F$) if and only if E and F have isomorphic kernel spaces.

In particular we see that for A a field of characteristic $\neq 2$ our notion of Witt ring coincides with Witt's definition in [W].

<u>Exercise.</u> A vector x of a space E over our semilocal ring A is called <u>primitive</u> if x is not contained in $\mathfrak{m}_i E$ for any maximal ideal $\mathfrak{m}_i$ of A. Show that E is anisotropic if and only if E contains no primitive vector x with $B(x,x) = 0$. (2 not necessarily a unit of A).

§ 2 The signed determinant.

Recall that A is a connected semilocal ring. Let $Q(A)$ denote the group A^*/A^{*2} of "square classes" of A.

If $E \cong (a_{ij})$ and $E' \cong (a'_{ij})$ are isomorphic spaces over A, then

$$(a'_{ij}) = {}^tS(a_{ij})S$$

with some S in $GL(n,A)$, cf. I § . Taking determinants we obtain

$$\det(a'_{ij}) = b^2 \det(a_{ij})$$

with b a unit of A. We call the square class $\det(a_{ij})A^{*2}$ the <u>determinant</u> $\det(E)$ of the space E, and we have shown that $\det(E)$ is a well defined invariant of E.

<u>Remark.</u> In the same way we have determinants for free spaces over an arbitrary scheme X with values in $\mathcal{O}(X)^*/\mathcal{O}(X)^{*2}$. For determinants of other spaces see Chapter IV, § 3.

For two spaces E,F over A we have

$$\det(E \perp F) = \det(E)\det(F).$$

Thus our invariant yields a determinant map

$$\det\colon K\,\mathrm{Bil}(A) = L(A) \to Q(A)$$

defined by

$$\det([E]-[F]) = \det(E)\det(F).$$

Unfortunately $\det(H) = (-1)A^{*2}$. Thus det does not factor through

W(A). We resort to a trick well known from the theory over fields. Let $(\mathbb{Z}/2\mathbb{Z})\circ Q(A)$ denote the abelian group consisting of the pairs (ν,d) in $(\mathbb{Z}/2\mathbb{Z})\times Q(A)$ with "twisted" multiplication

$$(\nu_1 d_1)(\nu_2,d_2) := (\nu_1+\nu_2,(-1)^{\nu_1\nu_2}d_1d_2)).$$

We consider the map

$$z \mapsto (n \bmod 2,\ (-1)^{\frac{n(n-1)}{2}} \det z))$$

from L(A) to $(\mathbb{Z}/2\mathbb{Z})\circ Q(A)$, with n denoting the rank of z. This map is a group homomorphism and vanishes on H. Thus it induces a map

$$(\nu,d)\colon W(A) \to (\mathbb{Z}/2\mathbb{Z})\circ Q(A)\ z \to (\nu(z),d(z)).$$

We call the second component

$$d\colon W(A) \to Q(A)$$

of this map the <u>signed determinant</u>. For E a space of rank n over A we have by definition

$$d(\{E\}) = (-1)^{\frac{n(n-1)}{2}} \det(E).$$

Of course we denote this square class by d(E) for short. For spaces E_1,E_2 over A we have

$$d(E_1 \perp E_2) = (-1)^{\nu(E_1)\nu(E_2)} d(E_1)d(E_2).$$

Q(A) can be regarded as the group of isomorphy classes $\langle a\rangle$ of spaces of rank 1 over A with the tensor product as multiplication. We have a natural map $\langle a\rangle \to \{\langle a\rangle\}$ from Q(A) to W(A).

Since

$$d(\{<a>\}) = <a>$$

this map is <u>injective</u>. Henceforth we regard $Q(A)$ as a subset of $W(A)$, i.e. we identify a square class $<a>$ with the Witt class $\{<a>\}$. Now $Q(A)$ is a subgroup of the group of units $W(A)^*$ of the ring $W(A)$.

§ 3 Orthogonal bases.

We denote the radical $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_s$ of A by $\mathfrak{r}$. Every space (E,B) over A yields by reduction modulo $\mathfrak{r}$ a space $(E/\mathfrak{r}E,\overline{B})$ over $A/\mathfrak{r}$. Now according to the Chinese remainder theorem $A/\mathfrak{r}$ can be identified with the direct product $\prod_{i=1}^{s} A/\mathfrak{m}_i$. Thus

$$(*) \qquad (E/\mathfrak{r}E,\overline{B}) = \prod_{i=1}^{s} (E/\mathfrak{m}_i E,\beta_i)$$

with β_i the reduction of B modulo $\mathfrak{m}_i$.

Some well known theorems for spaces over fields can be transfered to spaces over $A/\mathfrak{r}$ according to this relation and then to spaces over A.

Lemma 1. Let (E,B) be a space over A. Every orthogonal decomposition

$$E/\mathfrak{r}E = \overline{F} \perp \overline{G}$$

of $E/\mathfrak{r}E$ into free spaces $\overline{F},\overline{G}$ over $A/\mathfrak{r}$ can be lifted to an orthogonal decomposition

$$E = F \perp G.$$

Proof. Let $\overline{y}_1,\ldots,\overline{y}_r$ be a basis of $\overline{F}$ over $A/\mathfrak{r}$. We choose pre-images $y_1,\ldots,y_r$ of the $\overline{y}_i$ in E, and we define

$$F := Ay_1 + \ldots + Ay_r.$$

The determinant of the matrix $(B(y_i,y_j))$ is a unit in A since it is modulo $\mathfrak{r}$ a unit in $A/\mathfrak{r}$. Thus $y_1,\ldots,y_r$ is a free basis of F over A, and the bilinear form $B|F$ is nondegenerate. We obtain

$$E = F \perp F^{\perp}.$$

We have $F/rF = \overline{F}$, hence

$$F^{\perp}/rF^{\perp} = (\overline{F})^{\perp} = \overline{G}.$$

Definition. A basis $x_1,\dots,x_n$ of the space (E,B) over A is an orthogonal basis if E is the orthogonal sum of the subspaces $Ax_1,\dots,Ax_n$. Notice that then all $B(x_i,x_i)$ are units.

Proposition 1. Assume the space (E,B) contains for every maximal ideal m_i a vector z_i with $B(z_i,z_i)$ not in m_i. Then (E,B) has an orthogonal basis.

Proof. It suffices to find an orthogonal basis of E/rE, since this basis can be lifted to an orthogonal basis of E according to the preceding lemma. Recalling the decomposition (*) of E/rE into spaces over the fields A/m_i above we see that if suffices to prove the proposition over fields.

Thus we assume now that A is a field. If x is a vector in E with $B(x,x) \neq 0$ then

$$E = (Ax) \perp E_1$$

with $E_1 = (Ax)^{\perp}$. Repeating this procedure we obtain a decomposition

$$E = Ax_1 \perp \dots \perp Ax_r \perp F$$

with $B(x,x) = 0$ for all x in F. If A has characteristic $\neq 2$ this implies $F = 0$. Assume now that A has characteristic 2. For any vector $x \neq 0$ of F there exists some y in F with $B(x,y) = 1$. Then

$Ax + Ay$ is the hyperbolic standard plane $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and we have

$$F = (Ax + Ay) \perp F_1$$

with some subspace F_1. Repeating the procedure we learn that $F \cong t \times H$ with some $t \geq 0$. (These spaces indeed do not represent elements $\neq 0$.) We now obtain by induction on the rank of E that E has an orthogonal basis if we verify that every space

$$G := \langle a \rangle \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with some a in A^* has an orthogonal basis. Let z,x,y be a basis corresponding to the given presentation of G. Then

$$G_1 := A(z+x) + Ay$$

is a subspace with matrix $\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$, and we have

$$G \cong \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \perp \langle b \rangle$$

with some unit b. (Comparing the determinants of both spaces we see $\langle b \rangle = \langle a \rangle$.) Moreover

$$G_1 = A(z+x) \perp (A(z+x))^{\perp} \cong \langle a \rangle \perp \langle c \rangle$$

for some unit $\langle c \rangle$. (Comparing determinants we see $\langle c \rangle = \langle -a \rangle$.) Thus

$$G \cong \langle a,c,b \rangle,$$

and G has an orthogonal basis.

<u>Corollary.</u> If 2 is a unit in A then every space over A has an orthogonal basis.

This is evident from the arguments in the proof just completed.

Proposition 2. (cf.[W, Satz 7] for fields of Char. $\neq$ 2.) Let $\mathfrak{B}$ and $\mathfrak{B}'$ be two orthogonal bases of a space E over A.

i) There exists a finite sequence

$$\mathfrak{B}_o = \mathfrak{B}, \mathfrak{B}_1, \dots, \mathfrak{B}_t = \mathfrak{B}'$$

of orthogonal bases $\mathfrak{B}_i$ of E connecting $\mathfrak{B}$ with $\mathfrak{B}'$ such that $\mathfrak{B}_i$ and $\mathfrak{B}_{i+1}$ differ at most at four places.

ii) If all residue class fields $A/\mathfrak{m}_i$ contain at least 3 elements then there exists such a sequence with $\mathfrak{B}_i$ and $\mathfrak{B}_{i+1}$ differing at most at 2 places.

The proof roughly runs as follows. We first verify the proposition over fields. Then we can connect the bases $\overline{\mathfrak{B}}$ and $\overline{\mathfrak{B}'}$ of E/rE coming from $\mathfrak{B}, \mathfrak{B}'$ by a sequence as above. We lift this sequence to a sequence of orthogonal basis of E starting with $\mathfrak{B}$ and arriving at a basis $\mathfrak{B}''$ which is congruent to $\mathfrak{B}'$ mod r. Finally we join $\mathfrak{B}'$ and $\mathfrak{B}''$ by a sequence of orthogonal basis with alterations at each step at two or less places. For the details cf. [KRW § 1]. The easier part (ii) is already contained in [K,§ 5.5].

§ 4 Generators and relations for W(A).

We denote the group of square classes $Q(A) = A^*/A^{*2}$ by G and we regard G as a subset of W(A), cf. § 2.

Proposition 1. W(A) is additively generated by G.

Proof. Let E be a space over A. The space $E \perp \langle 1\rangle$ certainly has an orthogonal basis, cf. § 3, Prop.1. Thus

$$E \perp \langle 1\rangle \cong \langle b_1\rangle \perp \ldots \perp \langle b_n\rangle$$

with some units b_i, and we have in W(A) the equation

$$\{E\} = \langle b_1\rangle + \ldots + \langle b_n\rangle + \langle -1\rangle.$$

According to this proposition we have a surjective homomorphism from the integral group ring $\mathbb{Z}[G]$ to W(A),

$$\Phi : \mathbb{Z}[G] \to W(A)$$

induced by the inclusion map from G to W(A). If we consider a square class $\langle a\rangle$ as an element of $\mathbb{Z}[G]$ we denote this square class by [a]. The homomorphism Φ maps [a] to $\langle a\rangle$. Let $\mathfrak{K}$ denote the kernel of Φ.

Theorem 2. The ideal $\mathfrak{K}$ is additively generated by the element $[1] + [-1]$ and all elements

$$z = \sum_{i=1}^{h} [a_i] - \sum_{i=1}^{h} [b_i]$$

with h = 4 and

$$\perp_{i=1}^{h} \langle a_i \rangle \cong \perp_{i=1}^{h} \langle b_i \rangle .$$

If all the finitely many residue class fields $A/\mathfrak{m}_i$ of A contain at least 3 elements, then already the elements z of this type with h = 2, together with $[1] + [-1]$, suffice to generate $\mathfrak{R}$ additively.

<u>Proof.</u> Clearly all these elements lie in $\mathfrak{R}$. Let now z be a given element of $\mathfrak{R}$. We have

$$z = [a_1] + \ldots + [a_r] - [b_1] - \ldots - [b_s]$$

with some units a_i, b_j of A. Eventually replacing z by -z we assume $r \geqslant s$. The spaces

$$E := \langle a_1, \ldots, a_r \rangle, \ F := \langle b_1, \ldots, b_s \rangle$$

are Witt-equivalent. In particular r-s is an even number 2t. According to § 1 the spaces E and $F \perp t \times \langle 1,-1 \rangle$ have the same image in $L(A) = K\,\mathrm{Bil}(A)$. Thus there exists a space G with

$$E \perp G \cong F \perp t \times \langle 1,-1 \rangle \perp G.$$

Adding for safety the space $\langle 1 \rangle$ to G we may assume that

$$G \cong \langle a_{r+1}, \ldots, a_n \rangle$$

with some further units a_i. Now introducing notations $b_i = \pm 1$ for $s < i \leqslant r$ and $b_i = a_i$ for $r < i \leqslant n$ we can write

$$E \perp G = \langle a_1, \ldots, a_n \rangle,$$

$$F \perp t \times \langle 1,-1 \rangle \perp G = \langle b_1, \ldots, b_n \rangle,$$

and we have

$$z = t([1] + [-1]) + \sum_{i=1}^{n} [a_i] - \sum_{i=1}^{n} [b_i].$$

For any orthogonal basis $\mathfrak{B} = \{x_1,\ldots,x_m\}$ of a space T over A we introduce the element

$$[\mathfrak{B}] := \sum_{i=1}^{m} [B(x_i,x_i)]$$

of $\mathbb{Z}[G]$. Using this notation we have

$$z = t([1] + [-1]) + [\mathfrak{B}] - [\mathfrak{B}']$$

for suitable orthogonal basis $\mathfrak{B}$ and $\mathfrak{B}'$ of $E \perp G$. As shown in § 3 there exists a sequence

$$\mathfrak{B}_0 = \mathfrak{B}, \mathfrak{B}_1,\ldots,\mathfrak{B}_m = \mathfrak{B}'$$

of orthogonal bases of $E \perp G$ such that subsequent bases differ at most at h places with h = 4 in general and h = 2 if A has no residue class fields containing only two elements. We have

$$[\mathfrak{B}] - [\mathfrak{B}'] = \sum_{i=0}^{m-1} ([\mathfrak{B}_i] - [\mathfrak{B}_{i+1}])$$

and every summand $[\mathfrak{B}_i] - [\mathfrak{B}_{i+1}]$ is a difference $[\mathfrak{L}] - [\mathfrak{L}']$ with $\mathfrak{L},\mathfrak{L}'$ orthogonal bases of some space of rank h. This finishes the proof of the theorem.

§ 5 The prime ideals of W(A).

We have seen that W(A) is in a natural way isomorphic to the quotient $\mathbb{Z}[Q(A)]/\mathfrak{P}$ of the group ring $\mathbb{Z}[Q(A)]$ by a more or less explicitely determined ideal $\mathfrak{P}$. Thus the prime ideals of W(A) correspond uniquely with those prime ideals of $\mathbb{Z}[Q(A)]$ which contain $\mathfrak{P}$. We now shall determine all prime ideals of $\mathbb{Z}[Q(A)]$ and then we shall look which of them contain $\mathfrak{P}$.

Let G be an arbitrary group of exponent 2, i.e. with $g^2 = 1$ for all g in G. If P is a prime ideal of $\mathbb{Z}[G]$ then clearly $g \equiv \pm 1 \bmod P$ for every g in G. Thus $\mathbb{Z}[G]/P$ is isomorphic either to $\mathbb{Z}$ or to a prime field $\mathbb{F}_p$ with p elements. Since the rings $\mathbb{Z}$ and $\mathbb{F}_p$ do not have automorphisms except the identity we obtain

<u>Lemma 1.</u> For P a prime ideal of $\mathbb{Z}[G]$ with $P \cap \mathbb{Z} = \{0\}$ there exists a unique ring homomorphism φ from $\mathbb{Z}[G]$ to $\mathbb{Z}$ which has the kernel P. Similarly for P a prime ideal of $\mathbb{Z}[G]$ with $\mathbb{Z} \cap P = p\mathbb{Z}$, p of course a prime number, there exists a unique ring homomorphism ψ from $\mathbb{Z}[G]$ to $\mathbb{F}_p$ which has kernel P.

Thus we need only to describe these homomorphisms φ and ψ. Every ring homomorphism from $\mathbb{Z}[G]$ to $\mathbb{Z}$ maps G into $\{\pm 1\}$. Thus the restriction of φ to G is a character χ on G. Conversely every character $\chi: G \to \{\pm 1\}$ extends in a unique way to a ring homomorphism $\varphi: \mathbb{Z}[G] \to \mathbb{Z}$ by the universal property of the group ring. Since now we <u>identify</u> φ and χ, i.e. we regard a character of G also as a ring homomorphism from $\mathbb{Z}[G]$ to $\mathbb{Z}$.

Assume now that p is an odd prime. Then the group $\{\pm 1\}$ embeds into $\mathbb{F}_p$ and is the subgroup of all elements of $\mathbb{F}_p^*$ of order at most 2. Thus the restriction of a ring homomorphism ψ from $\mathbb{Z}[G]$ to $\mathbb{F}_p$ yields again a character χ. This means that there exists a unique homomorphism χ from $\mathbb{Z}[G]$ to $\mathbb{Z}$ such that the diagram

$$\begin{tikzcd} \mathbb{Z}[G] \arrow[rr, "\chi"] \arrow[dr, "\psi"'] & & \mathbb{Z} \arrow[dl, "\kappa"] \\ & \mathbb{F}_p & \end{tikzcd}$$

with κ the canonical map from $\mathbb{Z}$ onto $\mathbb{F}_p$ commutes.

Consider finally p = 2. Every homomorphism from $\mathbb{Z}[G]$ to $\mathbb{F}_2$ maps every g in G to 1. Thus there exists a unique homomorphism ψ_0 from $\mathbb{Z}[G]$ to $\mathbb{F}_2$. We obtain this homomorphism by composing any character $\chi: \mathbb{Z}[G] \to \mathbb{Z}$ with the canonical map from $\mathbb{Z}$ onto $\mathbb{F}_2$.

According to these simple observations Lemma 1 implies

Proposition 1. i) For every prime ideal P of $\mathbb{Z}[G]$ with $P \cap \mathbb{Z} = \{0\}$ there exists a unique character χ of G such that P is the kernel P_χ of the ring homomorphism χ from $\mathbb{Z}[G]$ onto $\mathbb{Z}$.
ii) For every prime ideal M of $\mathbb{Z}[G]$ with $M \cap \mathbb{Z} = p\,\mathbb{Z}$, p an odd prime, there exists a unique character χ of G such that M coincides with the set

$$M_{\chi,p} := p\,\mathbb{Z} + P_\chi$$

consisting of all z in $\mathbb{Z}[G]$ with $\chi(z) \equiv \text{mod } p$.
iii) There exists a unique prime ideal M_0 of $\mathbb{Z}[G]$ with

$M_o \cap \mathbb{Z} = 2\mathbb{Z}$. We have

$$M_o = \chi^{-1}(2\mathbb{Z}) = 2\mathbb{Z} + P_\chi$$

for every character χ of G.

Clearly the P_χ are the minimal prime ideals of $\mathbb{Z}[G]$ and the ideals $M_{\chi,p}$ and M_o are the maximal ideals of $\mathbb{Z}[G]$.

We now denote by G the group $Q(A)$ of square classes of our connected semilocal ring A and consider the prime ideals of $W(A) \cong \mathbb{Z}[G]/\mathfrak{R}$. Recall that we denoted by $I(A)$ the kernel of the rank index

$$\nu : W(A) \to \mathbb{F}_2.$$

From part (iii) of the just proved Proposition 1 we obtain

<u>Proposition 2a.</u> $I(A)$ is the unique prime ideal of $W(A)$ which contains $2\cdot 1_{W(A)}$.

<u>Definition.</u> A <u>signature</u> σ of A is a ring homomorphism from $W(A)$ to $\mathbb{Z}$.

We denote the kernel of a signature σ by P_σ and we have $W(A)/P_\sigma \cong \mathbb{Z}$. Part (i) of Proposition 1 implies

<u>Proposition 2b.</u> For every prime ideal P of $W(A)$, which does not contain $p\cdot 1_{W(A)}$ for any prime number p, there exists a unique signature σ such that $P = P_\sigma$.

To analyse the prime ideals M of $W(A)$ which contain $p\cdot 1_{W(A)}$ for p an odd prime we need the following information about the ideal $\mathfrak{R}$ which will be verified later.

<u>Lemma 2.</u> For every character χ of G either $\chi(\mathfrak{F}) = 0$ or $\chi(\mathfrak{F}) = 2^n\mathbb{Z}$ with some $n \geq 1$ (actually $n \leq 3$).

From this lemma it is clear that if $\chi(\mathfrak{F}) \subset p\,\mathbb{Z}$ for our odd prime p then $\chi(\mathfrak{F}) = 0$. Thus we obtain from part (ii) of Proposition 1 the following:

<u>Proposition 2c.</u> Let p be an odd prime. Then for every prime ideal M of W(A) with $p\cdot 1_{W(A)}$ in M there exists a unique signature σ of A such that M coincides with the set

$$M_{\sigma,p} := p\,\mathbb{Z} + P_\sigma$$

consisting of all z in W(A) with $\sigma(z) \equiv \bmod p$.

Thus the P_σ, the $M_{\sigma,p}$, and I(A) are all the prime ideals of W(A). We call the ring A <u>real</u> (or <u>formally real</u>) if A has at least one signature. Otherwise we call A <u>non real</u>. Our description of the prime ideals of W(A) implies the following

<u>Corollary.</u> Assume A is real. Then the P_σ are the minimal prime ideals of W(A). The ideals $M_{\sigma,p}$ and I(A) are the maximal ideals of W(A). Every $M_{\sigma,p}$ contains a unique minimal prime ideal, and this is P_σ. The ideal I(A) contains all minimal prime ideals.

On the other hand we have

<u>Proposition 3.</u> The following are equivalent:

a) A is non real.

b) I(A) is the unique prime ideal of W(A).

c) $2^n W(A) = 0$ for some natural number n.

Here the equivalence a) $\Leftrightarrow$ b) is evident from our analysis of the prime ideals of $W(A)$ in general. The implication c) $\Rightarrow$ a) is trivial since $W(A)$ certainly does not admit homomorphisms to $\mathbb{Z}$ if $W(A)$ entirely consists of torsion elements. It remains to prove b) $\Rightarrow$ c). By assumption b) and elementary commutative algebra $I(A)$ is the nilradical of $W(A)$. In particular $2\cdot 1_{W(A)}$ is nilpotent, hence $2^n \cdot 1_{W(A)} = 0$ for some n. This implies $2^n W(A) = 0$.

We still have to prove Lemma 2 about the sets $\chi(\mathfrak{P})$. By the preceding § 4 the ideal $\mathfrak{P}$ is additively generated by $[1] + [-1]$ and by the elements

$$z = \sum_{i=1}^{4} [a_i] - \sum_{i=1}^{4} [b_i]$$

with

$$\perp_{i=1}^{4} \langle a_i \rangle \cong \perp_{i=1}^{4} \langle b_i \rangle .$$

On $[1] + [-1]$ every χ has value 0 or 2. We claim that χ has on a fixed element z as above a value 0, $\pm$ 4, or $\pm$ 8. Then the Lemma will be evident.

Let s denote the number of square classes $[a_i]$ with $\chi([a_i]) = -1$ and t the number of square classes $[b_i]$ with $\chi([b_i]) = -1$. Then

$$\chi(z) = (-s) + (4-s) + t - (4-t) = 2(t-s).$$

Now observe that the spaces $\langle a_1, a_2, a_3, a_4\rangle$ and $\langle b_1, b_2, b_3, b_4\rangle$ have the same determinant, since they are isomorphic.

$$\prod_{i=1}^{4} [a_i] = \prod_{i=1}^{4} [b_i].$$

Applying χ we obtain $(-1)^s = (-1)^t$, hence $t-s$ is even. This implies

$$\chi(z) \equiv 0 \bmod 4.$$

On the other hand clearly $|\chi(z)| \leq 8$. Thus indeed $\chi(z) = 0$ or ± 4 or ± 8.

§ 6 Nilpotent and torsion elements.

If A is non real then we know already that all elements of W(A) are torsion and in fact killed by a fixed power of 2. Moreover I(A) is the set of all nilpotent elements.

Since now we assume that A is real. Since the P_σ are precisely all niminal prime ideals of W(A) we have

Proposition 1. An element z of W(A) is nilpotent if and only if $\sigma(z) = 0$ for every signature σ of A.

We now look at the torsion elements of W(A).

Proposition 2. An element z of W(A) is a torsion element if and only if z is nilpotent.

Proof. Assume $nz = 0$ for some $n \geqslant 1$. Then certainly $\sigma(z) = 0$ for all signatures σ of A, hence z is nilpotent.

Assume now that z is nilpotent. There exists a finite subgroup H of Q(A) such that z is contained in the subring R of W(A) generated by H. Now by Maschke's theorem the group ring $\mathbb{Q}[H]$ over the field $\mathbb{Q}$ of rational numbers is semisimple. The tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} R$ is a homomorphic image of $\mathbb{Q}[H]$ and thus is again semisimple. The image $1 \otimes z$ of z in $\mathbb{Q} \otimes R$ is nilpotent and therefore must be zero. This implies $nz = 0$ for some natural number n.

Proposition 3. For every torsion element z of W(A) there exists a 2-power 2^r with $2^r z = 0$.

<u>Proof.</u> Again z lies in the subring R of W(A) generated by some finite subgroup H of Q(A). Let p be any odd prime. The ring R/pR is a homomorphic image of the group ring $\mathbb{F}_p[H]$. By Maschke's theorem R/pR is semisimple. By Proposition 2 the element z is nilpotent. Thus the image of z in R/pR must be zero. This means that there exists an element y of R with z = py. Clearly y is again a torsion element. Thus we have shown that the finite abelian group R_t consisting of the torsion elements in R is divisible by every odd prime p. Therefore R_t must be 2-primary, and $2^r z = 0$ for some r.

§ 7 A closer look at signatures.

Let σ be a signature of A. Then σ yields a homomorphism

$$A^* \twoheadrightarrow Q(A) \xrightarrow{\sigma | Q(A)} \{\pm 1\}$$

the first arrow denoting the canonical map from A^* to A^*/A^{*2}. Our signature σ is completely determined by this homomorphism from A^* to $\{\pm 1\}$, since W(A) is generated by Q(A). We identify henceforth a signature σ and the corresponding map from A^* to $\{\pm 1\}$, i.e. for any a in A^* we simply write $\sigma(a)$ instead of $\sigma(\{<a>\})$.

Proposition 1. If a map σ from A^* to $\{\pm 1\}$ is a signature then the following three properties hold true:

(i) $\sigma(ab) = \sigma(a)\sigma(b)$ for a,b in A^*;

(ii) $\sigma(-1) = -1$;

(iii) If $a_1, \dots, a_r$ are units with $\sigma(a_1) = \sigma(a_2) = \dots = \sigma(a_r) = +1$, then for any unit

$$b = \lambda_1^2 a_1 + \dots + \lambda_r^2 a_r$$

with some λ_i in A again $\sigma(b) = +1$.

Here the properties (i) and (ii) are evident. To prove property (iii) we consider the bilinear space $<a_1, \dots, a_r>$. This space contains a vector x with value $B(x,x) = b$. Thus

$$<a_1, \dots, a_n> \cong <b> \perp G$$

with G the orthogonal complement of Ax in the whole space. The space $G \perp \langle 1\rangle$ certainly has an orthogonal bases, cf. Prop. 1 of § 3. Thus

$$\langle 1,a_1,\ldots,a_r\rangle \cong \langle b,b_1,\ldots,b_r\rangle$$

with some units b_i. Computing the values of σ on the classes of these two spaces we obtain

$$r+1 = \sigma(b) + \sigma(b_1) + \ldots + \sigma(b_r).$$

Since all summands on the right hand side are ± 1 they actually must be + 1. In particular $\sigma(b) = +1$.

Proposition 2. Assume A has no residue class fields A/m_i with only two elements. Let σ be a map from A^* to $\{\pm 1\}$ fulfilling the conditions (i) and (ii) in the preceding Proposition 1 and the following condition weaker than (iii):

(iii)' If a is a unit of A with $\sigma(a) = +1$ then for any unit $b = \lambda^2 + \mu^2 a$ with λ,μ in A again $\sigma(b) = +1$. Then σ is a signature of A.

Proof. By property (i) we have $\sigma(a^2) = +1$ for every a in A^*. Thus σ yields a character on $G := Q(A)$ which extends in a unique way to a homomorphism

$$\sigma : \mathbb{Z}[G] \to \mathbb{Z}.$$

We have to show that σ vanishes on the ideal $\mathfrak{P}$ described in § 4. Clearly σ vanishes on $1 + [-1]$ since $\sigma(-1) = -1$. It remains to show $\sigma(z) = 0$ for an element

$$z = [a_1] + [a_2] - [b_1] - [b_2]$$

with

$$\langle a_1,a_2\rangle \cong \langle b_1,b_2\rangle.$$

Introducing the units $a := a_1^{-1}a_2$, $c := a_1^{-1}b_1$, $c' := a_1^{-1}b_2$, we write

$$z = [a_1]([1]+[a]-[c]-[c']).$$

We have

$$\langle 1,a\rangle \cong \langle c,c'\rangle.$$

Thus

$$c = \lambda^2 + a\mu^2$$

with some λ,μ in A. Moreover comparing determinants we see that $[c'] = [a][c]$, hence

$$z = [a_1]([1]+[a])([1]-[c]).$$

If $\sigma(a) = -1$ then clearly $\sigma(z) = 0$. If $\sigma(a) = +1$ then again $\sigma(z) = 0$ by condition (iii)'. Thus indeed $\sigma(^\sigma) = 0$.

q.e.d.

We now study signatures in the special case that A is a <u>field.</u> Let σ be a signature of A. Let P denote the set of all a in A* with $\sigma(a) = +1$, and let -P denote the set (-1)P of negatives of these elements. Since $\sigma(-1) = -1$ the set -P is just the set of all b in A* with $\sigma(b) = -1$, and we have

$$P \cap (-P) = \emptyset,\ P \cup (-P) = A^*.$$

Moreover exploiting the properties (i) and (iii) of signatures (cf.Prop.1) we see that P is closed under addition and multiplication. Thus P is the set of positive elements of a total ordering α of A compatible with addition and multiplication. Henceforth we refer to these orderings simply as "the orderings" of the field A.

Let now an ordering α of A be given and define a map

$$\sigma_\alpha : A^* \to \{\pm 1\}$$

as follows. For a in A^* the value $\sigma(a)$ is +1 if and only if a is positive with respect to α. Observe that A has certainly more than 2 elements, since no field of positive characteristic can be ordered. We can apply Proposition 2 and we check immediately that σ_α is a signature of A. Thus we have arrived at the following important theorem, found independently by Harrison [H] and Leicht-Lorenz [LL]:

<u>Theorem 3.</u> The orderings α of a field A correspond uniquely with the signatures σ of A by the relation ($a \in A^*$)

$$a > 0 \text{ with respect to } \alpha \Leftrightarrow \sigma(a) = +1.$$

Notice that for a bilinear space E over A the value $\sigma_\alpha(E) = \sigma_\alpha(\{E\})$ of the signature σ_α corresponding to an ordering α is just the classical "Sylvester-signature" of E with respect to α. Indeed, choosing a diagonalization

$$E \cong \langle a_1, \ldots, a_n \rangle$$

we have

$$\sigma_\alpha(E) = \sigma_\alpha(a_1) + \ldots + \sigma_\alpha(a_n).$$

Coming to the end of this chapter on local theory we prove the following theorem [KRW_1, § 4].

<u>Theorem 4.</u> Let A be a connected semilocal ring in which 2 is a unit. Let $a_1,\dots,a_r$ be units of A. Then for any unit b of A the following statements are equivalent:

a) For every signature σ of A with $\sigma(a_1) = \dots = \sigma(a_r) = +1$ also $\sigma(b) = +1$.

b) The unit b can be expressed in the form

$$b = \sum_{0 \leq i_k \leq 1} d_{i_1,\dots,i_r}\, a_1^{i_1} \dots a_r^{i_r}$$

with coefficients $d_{i_1,\dots,i_r}$ which are sums so squares of elements in A.

Here the implication b) ⇒ a) is evident from Proposition 1. To prove the implication a) ⇒ b) we consider the "Pfister spaces"

$$F := \langle 1,a_1\rangle \otimes \dots \otimes \langle 1,a_r\rangle$$

and

$$E := \langle 1,-b\rangle \otimes F$$

Our assumption a) implies $\sigma(E) = 0$ for all signatures σ of A. Thus the class of E in W(A) is nilpotent, hence torsion by § 6, and there exists some natural number m - actually a 2-power - such that $m \times E \sim 0$.[1] From this we obtain

$$m \times F \sim m \times \langle b\rangle \otimes F.$$

[1] $m \times E$ denotes the orthogonal sum of m copies of the space E.

Since the spaces on both sides have the same rank, and since cancellation holds true if 2 is a unit, we deduce that

$$m \times F \cong m \times \langle b \rangle \otimes F,$$

cf. § 1. Since F represents the element 1, i.e. $B(x,x) = 1$ for some x in F, the space $m \times F$ represents the element b. This gives the desired expression for b with sums of m squares as coefficients.

In the special case $r = 1$, $a_1 = 1$, our theorem says that the units of A which have value +1 under all signatures are precisely the units which are sums of squares. This is a well known theorem of Artin in the field case [A, Satz 1]. Our more general theorem has been observed over fields by Pfister [P, Satz 21].

<u>Proposition 5.</u> Let A be an arbitrary semilocal ring. Then A is non real if and only if -1 is a sum of squares.

Indeed, if -1 is a sum of squares then A has no signatures as is already clear from the property (iii) of signatures in Proposition 1. Assume now that A is non real and in addition that 2 is a unit. Then taking $r = 1$, $a_1 = 1$, $b = -1$ in Theorem 4 we see that -1 is a sum of squares. If 2 is not a unit a proof that -1 is a sum of squares is contained in [K_2, § 1] by a method involving quadratic forms.

Guide to the literature: Local theory.

To § 1: If 2 is not a unit in the semilocal ring A then the cancellation theorem stated in § 1 remains true for quadratic spaces, i.e. for free modules of finite rank equipped with a non degenerate quadratic form [K_1]. For bilinear spaces only a restricted - but nevertheless useful - cancellation theorem holds true, cf. [K,§ 6]. A surprisingly general theorem about extensions of isometries in quadratic modules over local rings, containing the first mentioned theorem (over local rings) as a very special case, has been proved by M.Kneser, cf. Nachr.Akad. Wiss. Göttingen Math. Phys. Kl II 1972, 195-203.

To § 3 - § 6: The material has been taken from the paper [KRW]. There a much more thorough study of W(A) and related rings has been made than in our lecture *), and also Witt rings of hermitian forms are included. For a study of the reduced Witt ring W(A)/Nil W(A) see § 4 of the paper [KRW_1].

To § 7: For further details and for "signatures of semilocal rings with involutions" cf. [KRW_1] and [K_2]. In Chapter V, § 1 of these lectures we shall add an important theorem to our local theory of signatures.

There exists an extension theory for signatures to certain types of overrings of a semilocal ring. For this cf. § 3, § 4,

*) cf. Appendix 3

and § 8 of $[K_3]$ as well as a recent paper "Signatures on Frobenius extensions" by myself (to appear in the book "Number theory and Algebra" in honour of H.B.Mann, A.E.Ross, and O.Taussky-Todd, to be published by Academic Press).

If you want to take a glance at other techniques, more working with forms themselves, see the paper "Annullatoren von Pfisterformen über semilokalen Ringen" by R.Baeza and myself (Math.Z. 140, 1974), and some of the literature cited there.

Roughly all those statements about bilinear spaces over fields of characteristic $\neq 2$, which in their proof do not involve transcendental field extensions ("function field methods", cf. e.g. Arason-Pfister, Invent.math. 12, 1971, or Lam's Kingston lectures), can be transfered in some way to bilinear spaces over semilocal rings with 2 a unit. If 2 is not a unit, bilinear spaces are much less understood. The trouble is that then a bilinear space usually has few automorphisms. For example an anisotropic bilinear space over a field of characteristic 2 has no automorphisms at all except the identity. (Exercise!) It sometimes seems to be useful to work both with bilinear and quadratic spaces. {A bilinear space can be multiplied with a quadratic space yielding a quadratic space.} A thorough exposition of the "elementary theory" of quadratic spaces over semilocal rings has recently been given by R.Baeza in his Habilitationsschrift "Quadratische Formen über semilokalen Ringen" (Fachbereich Mathematik der Universität, 66 Saarbrücken, West Germany, to be published). Supplements to this treatise can be extracted from two papers of K.Mandelberg, "A note on quadratic forms over arbitrary semi-local rings" (Canad.J.Math. 27, 1975), and "On the classification of quadratic forms over semi-local rings" (to appear).

Of course this guide to the literature is incomplete. In particular I only mentioned papers with a strong focus on the general theory.

Chapter III. The Prime Ideal Theorem.

§ 1 Divisorial schemes.

Since now our scheme X is always tacitly assumed to be <u>separated</u>. Then the intersection of any two affine open subsets of X is again affine.

Let $\mathcal{L}$ be a line bundle on X, i.e. a vector bundle of rank one, and let $f \in \mathcal{L}(X)$ be a global section of $\mathcal{L}$. For any point x of X the "value" f(x) of f at x is defined as the image of f in the <u>fibre</u> $\mathcal{L}(x) = \mathcal{L}_x/m_x\mathcal{L}_x$ of $\mathcal{L}$ at x. We denote by X_f the set of all x in X with $f(x) \neq 0$. These open sets X_f are important to us, since the following extension theorem for sections of quasicoherent sheaves - and in particular for sections of vector bundles - holds true.

<u>Extension theorem</u> [EGA I § 9.3].
Assume X is quasicompact, i.e. X can be covered by finitely many affine open subsets. Let $\mathcal{F}$ be a quasicoherent sheaf on X.
i) For every section u of $\mathcal{F}$ over X_f there exists some $n > 0$ such that the section $f^{\otimes n} \otimes u$ of $\mathcal{L}^{\otimes n} \otimes_{\mathcal{O}} \mathcal{F}$ over X_f can be extended to a section of $\mathcal{L}^{\otimes n} \otimes \mathcal{F}$ over X.
ii) If v is a section of F over X with $v|X_f = 0$ then there exists some $n > 0$ such that the section $f^{\otimes n} \otimes v$ over $\mathcal{L}^{\otimes n} \otimes \mathcal{F}$ vanishes.

<u>Definition.</u> A <u>special open set</u> of X is an <u>affine</u> open set Z for which there exists a global section f of some line bundle $\mathcal{L}$ on X with $Z = X_f$. The scheme X is called <u>divisorial</u> if X is

(separated) quasicompact, and the special open sets are a <u>basis</u> of X.

The notion "divisorial scheme" has been introduced and studied by M.Borelli in two papers [Bo],[Bo_1]. Divisorial schemes seem to be a natural category for K-theoretic enterprises, since the extension theorem cited above can there widely be applied.

To prove that a scheme X is divisorial it suffices to show that X can be covered by finitely many special open subsets. Even a still weaker condition suffices, cf. [Bo, Th.3.3]. In particular for any commutative ring A the projective n-space P_A^n, i.e. the homogeneous spectrum of the polynomial ring $A[T_o,\ldots,T_n]$ with its standard grading, is divisorial. Here already the sets X_f with f running through the global sections of the powers $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$ $(n \geq 1)$ of the canonical line bundle $\mathcal{O}(1)$ are a basis of X.

Clearly every locally closed subscheme of a divisorial scheme is again divisorial. In particular every quasiprojective scheme, i.e. locally closed subscheme of a projective space P_A^n, is divisorial. More generally every factorial and in particular every regular, quasicompact scheme is divisorial, cf. [Bo, Th. 4.1], hence also every locally closed subscheme of such a scheme. It is also easily verified that a fibre product of two divisorial schemes over an arbitrary scheme is divisorial. Thus the category of divisorial schemes is indeed very extensive.

§ 2 Consequences of the Prime Ideal Theorem.

The central theorem of the present chapter is the following beautiful

Prime Ideal Theorem. Let X be a divisorial scheme and let P be a prime ideal of $W(X)$. Then there exists some closed point x of X such that P is the inverse image of a prime ideal Q of $W(\mathcal{O}_x)$ under the homomorphism from $W(X)$ to $W(\mathcal{O}_x)$ induced by the inclusion morphism from $\mathrm{Spec}(\mathcal{O}_x)$ to X.

Notice that this homomorphism $W(X) \to W(\mathcal{O}_x)$ just maps a class $\{E\}$ to the class $\{E_x\}$ of the stalk of E at x.

The Prime Ideal Theorem has been proved for X affine by A.Dress [D]. In our proof for divisorial schemes we shall follow the ideas in [D] closely. The proof will be given in the later sections of this chapter. We now point out some consequences of the Prime Ideal Theorem. We always assume tacitly that X is divisorial.

Corollary. (Weak local global principle.)
Let z be an element of $W(X)$ with image zero in $W(\mathcal{O}_x)$ for every closed point x of X. Then $z^n = 0$ for some $n \geqslant 1$.

Indeed, let P be a prime ideal of $W(X)$. Then by the Prime Ideal Theorem there exists a closed point x of X and a prime ideal Q of $W(\mathcal{O}_x)$ lying over P. Since z has image zero in $W(\mathcal{O}_x)$ this element lies in P. Thus z lies in all prime ideals of $W(X)$ and must be nilpotent.

<u>Remark.</u> The analogous statement for the ring $K(X)$ is well known. We have $K(\mathcal{O}_x) = \mathbb{Z}$ for every x in X, and the natural map from $K(X)$ to $K(\mathcal{O}_x)$ is the local rank at x. An element z if $K(X)$ of rank zero everywhere is known to be nilpotent. Indeed, a stronger statement holds true if X is quasiprojective and has finite dimension d [Ma, Theorem 9.1]: The $(d+1)$-th power of the ideal

$$\tilde{K}(X) = \ker(K(X) \to \prod_x K(\mathcal{O}_x))$$

with x running through the closed points of X is zero. To obtain a similar result for $W(X)$ still some "semilocal difficulties" have to be surmounted, cf. § 8.

From our study of the prime ideals of the Witt rings $W(\mathcal{O}_x)$ in Chapter II we obtain some insight into the spectrum of $W(X)$ by use of the Prime Ideal Theorem.

<u>Definition.</u> A <u>signature</u> σ of X is a ring homomorphism from $W(X)$ to $\mathbb{Z}$.

The kernel of σ is a prime ideal P_σ with $W(X)/P_\sigma = \mathbb{Z}$, and clearly in this way the signatures correspond uniquely with the prime ideals P of $W(X)$ such that $W(X)/P$ is isomorphic to $\mathbb{Z}$, since the ring $\mathbb{Z}$ has no automorphisms except the identity.

As in the local theory we call X <u>real</u> (or formally real) if X has signatures, and we call X <u>non real</u> if X has no signatures.

<u>Theorem 2.</u> Assume X is real and connected.

a) The kernel $I(X)$ of the rank homomorphism from $W(X)$ to $\mathbb{Z}/2\mathbb{Z}$ is the unique prime ideal P of $W(X)$ with $2\cdot 1_{W(X)} \in P$.

b) The prime ideals P_σ with σ a signature are precisely all prime ideals P of $W(X)$ with $p\cdot 1_{W(X)} \notin P$ for all prime numbers p. They correspond uniquely with the signatures of X.

c) Let P be a prime ideal of $W(X)$ with $p\cdot 1_{W(X)} \in P$ for some odd prime number p. Then there exists a signature σ of X such that P coincides with the maximal ideal

$$M_{\sigma,p} := \{z \in W(X) \mid \sigma(z) \in p\,\mathbb{Z}\} = p\,\mathbb{Z} + P_\sigma .$$

Notice that a) and b) are precisely the same statements as obtained in the local theory in II, § 5 Prop.2 part a) and b), but that part c) of the present theorem is weaker then the corresponding part c) of that proposition.

<u>Proof.</u> a) Let P be a prime ideal containing $2\cdot 1_{W(X)}$. Let x be a point of X and Q a prime ideal of $W(\mathcal{O}_x)$ lying over P. Then Q contains $2\cdot 1_{W(\mathcal{O}_x)}$. According to II, § 5 we have $Q = I(\mathcal{O}_x)$, and P is the kernel of the map

$$W(X) \to W(\mathcal{O}_x) \xrightarrow{\nu_x} \mathbb{Z}/2\,\mathbb{Z},$$

with ν_x the rank function (I,§ 7) on $W(\mathcal{O}_x)$. This map is the rank function on X since X is connected, hence $P = I(X)$.

b) Let now P be a prime ideal of $W(X)$ not containing $p\cdot 1_{W(X)}$ for any prime number p. Then $\mathbb{Z}$ embeds into the ring $W(X)/P$. Let again Q be a prime ideal of $W(\mathcal{O}_x)$ for some point x of X lying over P.

Then $\mathbb{Z}$ also embeds into $W(\mathcal{O}_x)/Q$. According to Chapter II, § 5 there exists a unique signature τ of $\mathcal{O}_x$ with kernel Q. Thus P is the kernel of the signature

$$\sigma : W(X) \to W(\mathcal{O}_x) \xrightarrow{\tau} \mathbb{Z}.$$

As already mentioned above different signatures of X have different kernels.

c) The proof runs in the same way as the proof for part b).

Our proof of part b) applied to a prime ideal P_σ also shows the following

<u>Corollary 1.</u> Let σ be a signature of X. Then there exists a closed point x of X and a signature τ of $\mathcal{O}_x$ such that σ coincides with the composite map

$$W(X) \to W(\mathcal{O}_x) \xrightarrow{\tau} \mathbb{Z}.$$

According to Theorem 2 the minimal prime ideals of W(X) are precisely the kernels P_σ of the signatures σ of X. We thus obtain

<u>Corollary 2.</u> Assume X is connected and real. An element z of W(X) is nilpotent if and only if $\sigma(z) = 0$ for every signature σ of X. In particular all torsion elements of W(X) are nilpotent.

This corollary is considerably weaker then the results obtained in the local theory in II § 6. In fact there exist affine schemes X such that W(X) contains nilpotent elements which are not torsion. Also elements of odd torsion can occur in W(X).

Let for example A be the ring of continuous $\mathbb{R}$-valued functions on a compact Hausdorff space S. Then W(A) is known to be isomorphic to the Grothendieck ring KO(S) of topological real vector bundles on S, cf. [MH, p.106]. KO(S) contains already for S the 4-sphere nilpotent elements which are not torsion, cf. [Hu, Chap.15, § 12.3]. There also exist compact spaces S such that KO(S) contains p-torsion elements for an arbitrary prescribed odd prime number p, e.g. suitable lens spaces [Km].

Passing from A to an appropriate finitely generated subring of A we see that there exist noetherian affine schemes with Witt rings containing non torsion nilpotent elements or non zero p-torsion elements for p an arbitrary odd prime number.

On the other hand, if X is non real we meet precisely the same situation as in the local theory.

Theorem 3. Let X be connected. The following statements are equivalent:

a) X is non real

b) I(X) is the only prime ideal of W(X).

c) $2^n W(X)$ for some $n \geqslant 1$.

The proof of the implications b) ⇒ c) and c) ⇒ a) runs as in the local theory, cf. II § 5. We are left to show a) ⇒ b). Let P be a prime ideal of W(X), and let Q be a prime ideal of $W(\mathcal{O}_x)$ lying over P for some point x of X. The ring $\mathcal{O}_x$ cannot have any signatures since they would yield signatures of X. Thus according to the local theory $Q = I(\mathcal{O}_x)$. Since X is connected this implies $P = I(X)$.

In the case of affine schemes we have the following criterion for non reality.

<u>Proposition 4.</u> A commutative ring A is non real if and only if -1 is a sum of squares in A.

<u>Proof.</u> If -1 is a sum of squares in A then -1 is also a sum of squares in every local ring $\mathcal{O}_x$ of $X = \mathrm{Spec}(A)$. Thus all these local rings do not have signatures, cf. II § 7 Prop. 5. This implies that A has no signatures by Corollary 1 above.

Assume now that A is non real. For every maximal ideal $\mathfrak{m}$ of A the local ring $A_{\mathfrak{m}}$ is again non real, hence -1 is a sum of squares in $A_{\mathfrak{m}}$ by Chapter II, § 7. This yields an equation

$$-f^2 = g_1^2 + \dots + g_n^2$$

in A with elements $g_1, \dots, g_n$ of A and f in $A \setminus \mathfrak{m}$, the number n depending on $\mathfrak{m}$. We choose for every maximal ideal $\mathfrak{m}$ such an equation. The ideal generated by the elements f with $\mathfrak{m}$ running through all maximal ideals must be the whole of A. Thus we have in A an equation

$$1 = h_1 f_1 + \dots + h_r f_r$$

together with equations

$$-f_i^2 = \sum_{j=1}^{n(i)} g_{ij}^2 .$$

Multiplying the presentation of $-f_i^2$ as sum of squares with h_i^2 we assume in addition $h_1 = \dots = h_r = 1$. We have the identity

$$2^{r-1}(f_1^2 + \dots + f_r^2) = 1 + \sum_{\varepsilon}{}'(f_1 + \varepsilon_2 f_2 + \dots + \varepsilon_r f_r)^2$$

with $\varepsilon = (\varepsilon_2,\dots,\varepsilon_r)$ running through all $(r-1)$-tuples with entries $\varepsilon_i = \pm 1$ which are different from $(1,\dots,1)$. Thus

$$-1 = 2^{r-1}(-f_1^2 - \dots - f_r^2) + \sum_{\varepsilon}{}'(f_1+\varepsilon_2 f_2 + \dots + \varepsilon_r f_r)^2.$$

Inserting the expressions for the elements $-f_i^2$ above we obtain a presentation of -1 as sum of squares in A.

§ 3 Bilinear complexes

Our proof of the Prime Ideal Theorem will depend on some machinery involving "bilinear complexes".

Definition. A bilinear complex *) over a scheme X is a triple (E,∂,B) consisting of a graded vector bundle

$$E = \bigoplus_{i\in\mathbb{Z}} E_i,$$

with only finitely many components $E_i \neq 0$ of course, a non degenerate symmetric bilinear form

$$B: E \times_X E \to \mathcal{O},$$

and an endomorphism

$$\partial: E \to E$$

of the vector bundle E with $\partial(E_i) \subset E_{i+1}$ for all i, such that the following three properties hold true:

a) $\partial\circ\partial = 0$;

b) ∂ is selfadjoint with respect to B, i.e. $B(\partial u,v) = B(u,\partial v)$ for sections u,v over an arbitrary open subset Z of X;

c) $B(E_i \times_X E_j) = 0$ for $i + j \neq 0$, i.e. B is homogenous of degree zero if we regard $\mathcal{O}$ as a graded vector bundle concentrated at the zero term.

By condition c) the bilinear form B induces a duality between E_i and E_{-i}, and we have the following orthogonal de-

*) We say "bilinear complex" instead of "symmetric bilinear complex" for short.

composition of the bilinear space (E,B):

$$E = E_0 \perp \coprod_{i=1}^{\infty} (E_i \oplus E_{-i}).$$

Every summand $E_i \oplus E_{-i}$, $i \geq 1$, is isomorphic to the hyperbolic space $H(E_i)$.

As usual we denote the restriction $E_i \to E_{i+1}$ of ∂ by ∂_i. There is another way to look at bilinear complexes. Let (E^*, ∂^*) denote the <u>dual complex</u> of the complex (E,∂), defined as follows: $(E^*)_i$ is the dual E^*_{-i} of the vector bundle E_{-i} and ∂^*_i is the transpose $(\partial_{-i-1})^t$ of $\partial_{-i-1} : E_{-i-1} \to E_{-i}$. The non degenerate symmetric bilinear form B corresponds to an isomorphism φ from E to the dual bundle E^* with $\varphi = \varphi^t$. Condition (c) means that φ is homogeneous of degree zero, and condition (b) means $\partial^* \circ \varphi = \varphi \circ \partial$. Thus (b) and (c) together just mean that φ is a morphism in the category of complexes of vector bundles.

Let us look at this isomorphism $\varphi : E \to E^*$ of complexes more closely. Assume $E_i = 0$ for $i > n \geq 0$, hence also for $i < -n$. We have a commutative diagram (A) with $\varphi_{-i} = (\varphi_i)^t$.

$$\begin{tikzcd}
0 \arrow[r] & E_{-n} \arrow[r, "\partial_{-n}"] \arrow[d, "\varphi_{-n}"] & E_{-n+1} \arrow[r] \arrow[d, "\varphi_{-n+1}"] & \cdots \arrow[r, "\partial_{-1}"] & E_0 \arrow[r, "\partial_0"] \arrow[d, "\varphi_0"] & E_1 \arrow[r, "\partial_1"] \arrow[d, "\varphi_1"] & \cdots \arrow[r, "\partial_{n-1}"] & E_n \arrow[r] \arrow[d, "\varphi_n"] & 0 \\
0 \arrow[r] & E^*_n \arrow[r, "\partial^t_{n-1}"] & E^*_{n-1} \arrow[r] & \cdots \arrow[r, "\partial^t_0"] & E^*_0 \arrow[r, "\partial^t_{-1}"] & E^*_{-1} \arrow[r, "\partial^t_{-2}"] & \cdots \arrow[r, "\partial^t_{-n}"] & E^*_{-n} \arrow[r] & 0
\end{tikzcd} \qquad (A)$$

Now observe that this diagram (A) is essentially determined by the maps ∂_i with $i < 0$ and the selfadjoint isomorphism $\varphi_0 : E_0 \xrightarrow{\sim} E_0^*$. Indeed, starting from these maps we define a new bilinear complex (E', ∂', B') in the following way:

$$\text{(B1)} \qquad E_i' := \begin{cases} E_i & i \leq 0 \\ (E_{-i})^* & i > 0 \ ; \end{cases}$$

$$\text{(B2)} \qquad (\partial_i' : E_i' \to E_{i+1}') := \begin{cases} \partial_i & i < 0 \\ (\partial_{-1})^t \circ \varphi_0 & i = 0 \\ (\partial_{-i-1})^t & i > 0 \ ; \end{cases}$$

$$\text{(B3)} \qquad (B' | E_i' \times_X E_j') := \begin{cases} 0 & i+j \neq 0 \\ \text{natural pairing} & i+j = 0,\ i \neq 0 \\ B | E_0 \times_X E_0 & i = j = 0. \end{cases}$$

To check that we have obtained an honest bilinear complex notice that the isomorphism $\varphi' : E' \to E'^*$ associated with B' is homogeneous of degree zero with components

$$\text{(B4)} \qquad \varphi_i' = \begin{cases} \mathrm{id} & i \neq 0 \\ \varphi_0 & i = 0. \end{cases}$$

From (B2) and (B4) we obtain immediately that $\varphi' \circ \partial' = \partial'^* \circ \varphi'$ and $\partial_{i+1}' \circ \partial_i' = 0$ for $i \neq -1$. The equation $\partial_0' \circ \partial_{-1}' = 0$ follows from the commutativity of the central square in the diagram (A).

We now have an isomorphism

$$\alpha : (E, \partial, B) \xrightarrow{\sim} (E', \partial', B')$$

between bilinear complexes in the obvious sense with components $\alpha_i : E_i \to E_i'$ defined by

$$\alpha_i := \begin{cases} \mathrm{id} & i \leq 0 \\ \varphi_i & i > 0, \end{cases}$$

as is easily verified.

<u>Proposition.</u> i) The restriction of a bilinear complex (E,∂,B) to the sequence

$$(C) \qquad 0 \to E_{-n} \xrightarrow{\partial_{-n}} E_{-n+1} \to \dots \to E_{-1} \xrightarrow{\partial_{-1}} E_o$$

gives a bijection from the set of isomorphism classes of bilinear complexes to the set of isomorphism classes of sequences (C) with vector bundles E_{-n}, $E_{-n+1},\dots,E_{-1}$ and a bilinear space E_o fulfilling the following two conditions:

(D1) $\partial_{i+1}\circ\partial_i = 0$ for $-n \leq i \leq -2$, if $n \geq 2$, i.e. (C) is a "half-complex";

(D2) $\partial_{-1}(E_{-1})$ is a totally isotropic $\mathcal{O}$-submodule of E_o.

ii) The complex (E,∂) is exact if and only if $\partial_{-1}(E_{-1})$ is a lagrangian of E_o and (C) is exact.

Thus <u>exact</u> bilinear complexes correspond uniquely with metabolic spaces equipped with a finite resolution of some lagrangian by vector bundles.

To prove part (i) of this proposition all that remains to be done is the following. Start with a sequence (C) having the property (D1). Let $\varphi_o : E_o \xrightarrow{\sim} E_o^*$ be the isomorphism associated with the bilinear form B_o pregiven on E_o. Define a triple (E',∂',B') by (B1)-(B3) and show that (E',∂',B') is a bilinear complex if and only if (D2) is fulfilled.

The map $\varphi':E' \to E'^*$ associated with the symmetric bilinear form B' is given by (B4). The equation

$$\varphi'\circ\partial' = \partial'^*\circ\varphi'$$

follows from (B2) and (B4). The equation

$$\partial'_{i+1}\circ\partial'_i = 0$$

follows for $i \neq -1$ from (D1) and (B2). Thus we only have to analyze the meaning of

$$\partial'_0\circ\partial'_{-1} = 0.$$

Let V denote the image $\partial_{-1}(E_{-1})$ of ∂'_{-1} and W denote the kernel of

$$\partial'_0 = (\partial_{-1})^t\circ\varphi_0:E_0 \to E^*_{-1}.$$

These are quasicoherent $\mathcal{O}$-submodules of E_0. An element u in some stalk E_{0x} lies in W_x if and only if for every v in E_{-1x}

$$\langle v,\partial^t_{-1}\circ\varphi_0(u)\rangle = B_0(\partial_{-1}v,u) = 0.$$

Thus $W = V^\perp$, and we see that $\partial'_0\circ\partial'_{-1} = 0$ means $V \subset V^\perp$. Moreover the complex (E',∂') is exact at E'_0 if and only if $V = V^\perp$.

Assume now that the half complex (C) is exact. Then we have an exact sequence

$$0 \to E_{-n} \to E_{-n+1} \to \dots \to E_{-1} \to V \to 0$$

of vector bundles *). The dual sequence

*) Also V is now a vector bundle!

$$0 \to V^* \to E^*_{-1} \to E^*_{-2} \to \dots \to E^*_{-n} \to 0$$

is again exact. Thus (E',∂') is exact at all places E'_i with $i \neq 0,1$. If V is a lagrangian of E_o then in addition (E',∂') is exact at E'_o. Moreover then φ_o induces an isomorphism from E_o/V onto V^*, and we see that (E',∂') is also exact at E'_1.

Assume finally that (E',∂') is exact. Then the cokernel E_o/V of ∂'_{-1} is mapped isomorphically onto the kernel V^* of ∂'_1. Thus V is a subbundle of E_o. As we have seen above $V^\perp = V$. Thus V is a lagrangian of E_o and the proof of our proposition is complete.

We call a bilinear complex (E,∂,B) an <u>n-complex</u>, if $E_i = 0$ for $i > n$.

<u>Example.</u> The exact bilinear 1-complexes are up to isomorphism of the form

$$0 \to V \xrightarrow{\partial} E_o \xrightarrow{\partial^t \circ \varphi_o} V^* \to 0$$

where E_o is a bilinear metabolic space with associated homomorphism $\varphi_o : E_o \to E_o^*$, and ∂ is the inclusion map of some lagrangian V of E_o into E_o. The bilinear form on $E_{-1} \oplus E_1 = V \oplus V^*$ is of course the natural hyperbolic form.

For bilinear complexes (E',∂',B') and (E'',∂'',B'') over X there exists an <u>orthogonal sum</u>

$$(E',\partial',B') \perp (E'',\partial'',B'') := (E' \oplus E'', \partial' \oplus \partial'', B' \perp B'').$$

We also have a <u>tensor product</u>

$$(E',\partial',B') \otimes (E'',\partial'',B'') = (E,\partial,B)$$

defined as follows: E is the tensor product $E' \otimes E''$ of the vector bundles E' and E'' with the grading

$$E_k := \bigoplus_{i+j=k} E'_i \otimes E''_j;$$

$$B := B' \otimes B'';$$

$$\partial := \partial' \otimes \mathrm{id} + \alpha' \otimes \partial''$$

with α' denoting the involution of E' associated with the grading, i.e.

$$\alpha'(u) = (-1)^i u$$

for u a section of E'_i over some open set. (E,∂) is the usual tensor product of the complexes (E',∂') and (E'',∂''). Denoting by $\varphi':E' \to E'^*$, $\varphi'':E'' \to E''^*$,

$$\varphi:E' \otimes E'' \to (E' \otimes E'')^* = E'^* \otimes E''^*$$

the homomorphisms associated with the bilinear forms B',B'' and B, we have $\varphi = \varphi' \otimes \varphi''$, and the equation $\partial\circ\varphi = \varphi\circ\partial$ is easily verified.

§ 4 Euler characteristics.

Let (E,∂,B) be a bilinear complex over X, denoted in the sequel by E for short. We define the Euler characteristic $\chi(E)$ of E as the following element of the Witt-Grothendieck ring L(X):

$$\chi(E) := [E_o,B_o] + \sum_{i=1}^{\infty} (-1)^i \, [H(E_i)].$$

Here of course B_o is the restriction B/E_o. Let E^+ denote the direct sum of the vector bundles E_i with i even and E^- denote the direct sum of the E_i with i odd. The bilinear space (E,B) has the orthogonal decomposition

$$(E,B) = (E^+,B^+) \perp (E^-,B^-)$$

with $B^{\pm}$ the restriction of B to $E^{\pm}$. A second description of the Euler characteristic is given by

$$\chi(E) = [E^+,B^+] - [E^-,B^-].$$

For two bilinear complexes E and F we have

$$\chi(E \perp F) = \chi(E) + \chi(F),$$

$$\chi(E \otimes F) = \chi(E)\ \chi(F).$$

The first equation is obvious. The second equation can be proved as follows. The bilinear spaces $(E \otimes F)^{\pm}$ have decompositions

$$(E \otimes F)^+ = E^+ \otimes F^+ \perp E^- \otimes F^-,$$

$$(E \otimes F)^- = E^+ \otimes F^- \perp E^- \otimes F^+.$$

Thus

$$\chi(E \otimes F) = [E^+ \otimes F^+] + [E^- \otimes F^-] - [E^+ \otimes F^-] - [E^- \otimes F^+] =$$
$$= ([E^+] - [E^-])\ ([F^+] - [F^-]).$$

Theorem. If the bilinear complex E is exact then $\chi(E) = 0$.

Proof. This holds true for 1-complexes by the very definition of $L(X)$. Let now E be an exact n-complex with $n \geq 2$, and let V denote the image of ∂_{-1}. Then V is a Lagrangian of E_o. Thus in L(X)

(1) $$[E_o] = [H(V)].$$

We have an exact sequence

$$0 \to E_{-n} \xrightarrow{\partial_{-n}} E_{-n+1} \longrightarrow \dots \xrightarrow{\partial_{-2}} E_{-1} \to V \to 0.$$

This sequence yields short exact sequences

$$0 \to Z(E_i) \to E_i \to Z(E_{i+1}) \to 0$$

$(-n+1 \leqslant i \leqslant -1)$ with $Z(E_i)$ the kernel of ∂_i and

$$Z(E_{-n+1}) \cong E_{-n},\ Z(E_o) = V.$$

We see successively that all $Z(E_i)$ are vector bundles (an argument already used in § 3), and we have in K(X) for $-n+1 \leqslant i \leqslant -1$

$$[E_i] = [Z(E_i)] + [Z(E_{i+1})].$$

Taking the alternating sum of these $[E_i]$ we obtain in K(X)

$$\sum_{i=1}^{n} (-1)^{i-1} [E_{-i}] = [V].$$

Applying the hyperbolic map

$$H:K(X) \to L(X)$$

we obtain in $L(X)$ the equation

$$(2) \qquad [H(V)] = \sum_{i=1}^{n} (-1)^{i-1} [H(E_{-i})]$$

$$= \sum_{i=1}^{n} (-1)^{i-1} [H(E_i)].$$

From (1) and (2) we deduce $\chi(E) = 0$.

§ 5 Proof of the prime ideal theorem, part I.

If ξ is an element of $L(X)$ and Z is an open subset of X let $\xi | Z$ denote the image of ξ under the "restriction map" from $L(X)$ to $L(Z)$ induced by the inclusion morphism $Z \to X$. A similar notation will be used for the restriction of elements of $W(X)$ to open subsets of X. The following lemma will be very helpful to us.

Fundamental Lemma. Let X be quasicompact and Z be a special open subset (cf. § 1) of X. Let ξ be an element of $L(X)$ with $\xi | Z = 0$. Then there exists a bilinear complex E over X such that $\chi(E) = \xi$ and the restriction $E | Z$ of E to Z is exact.

Notice that conversely for every bilinear complex E over X which is exact over Z we have

$$\chi(E) | Z = \chi(E | Z) = 0$$

according to § 4.

This fundamental lemma will be proved in the next section. We now take its truth for granted and deduce from it the Prime Ideal Theorem stated in § 2. We first recall a well known lemma about complexes of vector bundles.

Lemma 1. Let E' and E'' be finite complexes of vector bundles over an arbitrary scheme X. Assume E' is exact. Then also $E' \otimes E''$ is exact.

Proof. It suffices to check the exactness of $E' \otimes E''$ on affine open subsets. Thus we assume from the beginning that X itself is affine. We write the complex E' as an exact sequence

$$\ldots \to 0 \to 0 \to E'_r \xrightarrow{\partial'_r} E'_{r+1} \xrightarrow{\partial'_{r+1}} \ldots \quad \to E'_s \to 0 \to 0 \to \ldots \quad .$$

Let Z'_i denote the kernel of ∂'_i. This is a subbundle of E'_i (cf. proof of the theorem in § 4). Since X is affine there exist decomposition

$$E'_i = Z'_i \oplus F'_i$$

for $r \leq i \leq s$, and we see that E' is the direct sum of "elementary complexes"

$$\ldots \to 0 \to 0 \to F'_i \xrightarrow{\sim} Z'_{i+1} \to 0 \to 0 \to \ldots \quad .$$

Thus there exists an endomorphism $D':E' \to E'$ of the vector bundle E', homogeneous of degree -1, such that

$$\partial'D' + D'\partial' = id_{E'} \ .$$

Let ∂ denote the boundary map of $E' \otimes E''$, i.e.

$$\partial = \partial' \otimes id_{E''} + \alpha' \otimes \partial''$$

with α' the involution of E' corresponding to the grading. Then

$$(D' \otimes id_{E''})\partial + \partial(D' \otimes id_{E''}) = id_{E' \otimes E''}$$

since $\partial'\alpha' + \alpha'\partial' = 0$. Thus $E' \otimes E''$ is exact.

<u>Theorem 1.</u> Assume the scheme X is covered by special open subsets $Z_1, \ldots, Z_n$. Let $\xi_1, \ldots, \xi_n$ be elements of L(X) with $\xi_i | Z_i = 0$ for $1 \leq i \leq n$. Then

$$\xi_1 \cdot \ldots \cdot \xi_n = 0.$$

<u>Proof.</u> According to the fundamental lemma there exist bilinear complexes $E_1,\dots,E_n$ over X such that E_i has the Euler characteristic ξ_i and is exact over Z_i. We have

$$\chi(E_1 \otimes \dots \otimes E_n) = \xi_1 \dots \xi_n.$$

By the preceding lemma $E_1 \otimes \dots \otimes E_n$ is exact on each open subset Z_i, since E_i is exact on Z_i. Thus $E_1 \otimes \dots \otimes E_n$ is exact on the whole of X, and $\xi_1 \dots \xi_n = 0$.

We now can prove the prime ideal theorem for L(X) instead of W(X).

<u>Theorem 2.</u> Let X be a divisorial scheme and P be a prime ideal of L(X). Then there exists a closed point x of X and a prime ideal Q of $L(\mathfrak{O}_x)$ such that P is the preimage of Q under the restriction map from L(X) to $L(\mathfrak{O}_x)$.

<u>Proof.</u> i) It suffices to show that there exists a closed point x of X such that P contains the kernel A of the restriction map r from L(X) to $L(\mathfrak{O}_x)$. Indeed, then we have a natural factorization

$$\begin{array}{ccc} L(X) & \xrightarrow{\quad r \quad} & L(\mathfrak{O}_x) \\ & {\scriptstyle\alpha}\searrow \quad \nearrow{\scriptstyle\beta} & \\ & L(X)/A & \end{array}$$

and a prime ideal $\bar{P}$ of L(X)/A with $\alpha^{-1}(\bar{P}) = P$. The ring $L(\mathfrak{O}_x)$ is generated by the classes of the spaces of rank one over $\mathfrak{O}_x$, cf. II § 4. For every such class g we have $g^2 - 1 = 0$. Thus $L(\mathfrak{O}_x)$ is integral over $\mathbb{Z}$. A fortiori β is an integral ring homomorphism.

By a well known theorem of Cohen-Seidenberg there exists a prime ideal Q of $L(\mathfrak{o}_x)$ with $\beta^{-1}(Q) = \overline{P}$, hence $r^{-1}(Q) = P$.

ii) Suppose P does <u>not</u> contain the kernel of the restriction map from $L(X)$ to $L(\mathfrak{o}_x)$ for any closed point x of X. We choose for every closed point x of X an element $\xi(x)$ of $L(X)$ which has image zero in $L(\mathfrak{o}_x)$ but does not lie in P.

We verify that there exists an open neighbourhood $Z(x)$ of x in X such that already the restriction $\xi(x)|Z(x)$ is zero. Indeed, $\xi(x) = [E_1] - [E_2]$ with bilinear spaces E_1, E_2 over X. There exists a free space $G = (a_{ij})$ over $\mathfrak{o}_x$ such that the spaces $E_{1x} \perp G$ and $E_{2x} \perp G$ over $\mathfrak{o}_x$ are isomorphic. We have an affine open neighbourhood $U(x)$ of x such that

$$a_{ij} = (b_{ij})_x$$

with some $b_{ij} \in \mathfrak{o}(U(x))$, $b_{ij} = b_{ji}$, and such that the determinant of the symmetric matrix (b_{ij}) is a unit in $\mathfrak{o}(U(x))$. Introducing the space

$$G := (b_{ij})$$

over $U(x)$ we have some isomorphism of spaces over $\mathfrak{o}_x$

$$\lambda : (E_1 \perp G)_x \xrightarrow{\sim} (E_2 \perp G)_x .$$

This isomorphism λ can be extended to an isomorphism

$$\mu : (E_1 \perp G)|Z(x) \xrightarrow{\sim} (E_2 \perp G)|Z(x)$$

over some open neighbourhood $Z(x)$ of x in $U(x)$. Clearly $\xi(x)|Z(x) = 0$.

Since X is divisorial we may assume that all $Z(x)$ are special open sets. The complement of the union of all $Z(x)$ in X is a closed subset Y. But every non empty closed subset of X contains closed points (since every non zero commutative ring has maximal ideals). Thus $Y = \emptyset$, i.e. X is covered by the $Z(x)$. Since X is quasicompact we have

$$X = Z(x_1) \cup \ldots \cup Z(x_n)$$

for suitable closed points $x_1, \ldots, x_n$. Now the preceding Theorem 1 yields

$$\xi(x_1) \ldots\ldots \xi(x_n) = 0.$$

This is the desired contradiction since none of the $\xi(x_i)$ lies in P.

We finally deduce the prime ideal theorem for $W(X)$ from the now proved prime ideal theorem for $L(X)$. Let P be a prime ideal of $W(X)$. Let $\widetilde{P}$ denote the preimage of P in $L(X)$. By our Theorem 2 there exists a closed point x of X and a prime ideal $\widetilde{Q}$ of $L(\mathcal{O}_x)$ lying over $\widetilde{P}$. Now $\widetilde{P}$ contains the class of the free space $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ over X. Thus $\widetilde{Q}$ contains the class of the space $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ over $\mathcal{O}_x$. This implies that $\widetilde{Q}$ is the inverse image of a prime ideal Q of $W(\mathcal{O}_x)$, since

$$W(\mathcal{O}_x) = L(\mathcal{O}_x) / \mathbf{Z}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right],$$

cf. II, § 1. The natural commutative diagram

$$\begin{array}{ccc} L(X) & \longrightarrow & L(\mathcal{O}_x) \\ \downarrow & & \downarrow \\ W(X) & \longrightarrow & W(\mathcal{O}_x) \end{array}$$

shows that Q lies over the original prime ideal P of W(X).

§ 6 Proof of the fundamental lemma.

Assume Z is a special open set X_f with f a global section of a line bundle $\mathcal{L}$ on a quasicompact scheme X. Assume further ξ is an element of $L(X)$ with $\xi|Z = 0$. We have to find a bilinear complex E on X which is exact on Z and has the Euler characteristik ξ.

We shall use the following standard notation. For an integral number n we denote by $\mathcal{L}^n$ the tensor product $\mathcal{L}^{\otimes n}$ of n copies of $\mathcal{L}$ over $\mathcal{O}$ if $n > 0$, the trivial bundle $\mathcal{O}$ if $n = 0$, and the dual bundle of $\mathcal{L}^{\otimes(-n)}$ if $n < 0$. For sections n in $\mathcal{L}^r(W)$ and v in $\mathcal{L}^s(W)$ over some open set W of X we have an obvious product uv lying in $\mathcal{L}^{r+s}(W)$.

Step 1. Our first goal is to write ξ as a difference $[E_1] - [E_2]$ with E_2 metabolic and $E_1|Z \cong E_2|Z$. We start with any presentation

$$\xi = [F_1] - [F_2],$$

with bilinear spaces F_1, F_2 over X. Replacing F_1 and F_2 by $F_1 \perp (-F_2)$ and $F_1 \perp (-F_2)$ *) we assume that F_2 is split metabolic. We then have $[F_2] = [H(U_o)]$ with some vector bundle U_o over X. Since $\xi|Z = 0$ and Z is affine we have

$$(1) \qquad F_1|Z \perp G' \cong H(U_o)|Z \perp G'$$

*) If F_2 has the bilinear form B_2 then $(-F_2)$ means the vector bundle F_2 equipped with the bilinear form $-B_2$.

for some space G' over Z. The main problem now is to replace G' by the restriction $G|Z$ of some space G over X. We have

$$G' \perp (-G') \cong M(G').$$

We choose a vector bundle U'' on Z such that the vector bundle $G' \oplus U''$ is free, which is possible since Z is affine. Then

$$(2) \qquad M(G') \perp H(U'') \cong M(U',\beta')$$

for some <u>free</u> bilinear bundle (U',β') over Z. We assume U' is the direct sum $n \times \mathcal{O}_Z$ of n copies of $\mathcal{O}_Z$ choosing some fixed trivialization of U'. Let (a'_{ij}) denote the symmetric matrix of β' with respect to the standard basis of $n \times \mathcal{O}_Z$ with coefficients a'_{ij} in $\mathcal{O}(Z)$. By the extension theorem stated in § 1 there exists a natural number m and sections a_{ij} in $\mathcal{L}^{2m}(X)$, $a_{ij} = a_{ji}$, such that

$$a_{ij}|Z = f^{2m}\, a'_{ij}.$$

Here we simply wrote $f^{2m} a'_{ij}$ instead of $(f|Z)^{2m} a'_{ij}$. We now establish a symmetric bilinear form β on the vector bundle

$$U := n \times \mathcal{L}^{-m}$$

over X. Let

$$u = (u_1,\ldots,u_n), \ v = (v_1,\ldots,v_n)$$

be sections in $U(W)$ for some open set W in X. $\{u_i, v_j \in \mathcal{L}^{-m}(W).\}$ Put

$$\beta(u,v) := \sum_{i,j} (a_{ij}|W)u_i v_j.$$

We then have an isomorphism of spaces

$$(3) \qquad \psi : M(U,\beta)\big|Z \xrightarrow{\sim} M(U',\beta')$$

defined as follows. $M(U,\beta)\big|Z$ has the underlying vector bundle

$$(U \oplus U^*)\big|Z = (\mathcal{L}^{-m}\big|Z) \otimes U' \oplus (\mathcal{L}^{m}\big|Z) \otimes U'^*,$$

and $M(U',\beta')$ has the underlying vector bundle $U' \oplus U'^*$. Let u and u^* be sections of U and U^* respectively over some open set $W \subset Z$. Put

$$\psi(u) := f^m u, \quad \psi(u^*) = f^{-m}u^*.$$

It is then easily checked that ψ is an isomorphism from $M(U,\beta)\big|Z$ onto $M(U',\beta')$. For example for sections u,v in $U(W)$, $W \subset Z$, the compatibility with the bilinear forms is verified as follows. Let $u_1,\dots,u_n$ and $v_1,\dots,v_n$ denote the coordinates of u and v in $\mathcal{L}^{-m}(W)$.

$$\beta'(f^m u, f^m v) = \sum_{i,j} a'_{ij} \, f^{2m} u_i v_j$$

$$= \sum_{ij} a_{ij} u_i v_j = \beta(u,v),$$

since W is contained in Z, and thus

$$a_{ij}\big|W = f^{2m} a'_{ij}\big|W.$$

Introducing the spaces

$$E_1 := F_1 \perp M(U,\beta); \; E_2 := H(U_o) \perp M(U,\beta)$$

we have

$$\xi = [E_1] - [E_2].$$

Moreover by (1), (2), and (3)

$$E_1|Z \cong E_2|Z,$$

and finally clearly E_2 has the Lagrangian

$$V := U_0^* \oplus U^*.$$

In particular

$$\xi = [E_1] - [H(V)].$$

Step 2. We now want to construct the bilinear complex with the properties stated in the fundamental lemma. Actually this complex will consist of only three terms. Choosing an isomorphism from $E_2|Z$ onto $E_1|Z$ and restricting this to $V|Z$ we obtain an injection

$$0 \to V|Z \xrightarrow{\alpha'} E_1|Z$$

whose image is a Lagrangian of E_1. Let B_1 denote the bilinear form on E_1 and $\varphi: E_1 \to E_1^*$ the associated linear map. By our considerations in § 3 we know that

$$0 \to V|Z \xrightarrow{\alpha'} E_1|Z \xrightarrow{(\alpha')^t \circ (\varphi|Z)} V^*|Z \to 0$$

is an exact complex. We further know from § 3 how to establish a bilinear form B' on this complex such that we obtain a bilinear complex: regard $E_1|Z$ as the zero term of our complex, hence $V|Z$ as the (-1) -term and $V^*|Z$ as the (+1) -term. We then define B' as the orthogonal sum of the form $B_1|Z$ on $E_1|Z$ and the hyperbolic standard form on $(V \oplus V^*)|Z$.

Starting from this bilinear complex we want to establish a similar bilinear complex over X. We regard α' as a global section of the vector bundle $\mathrm{Hom}(V,E_1)$ over Z. By the extension theorem, part (i), stated in § 1, there exists a natural number r and a global section α of $L^r \otimes \mathrm{Hom}(V,E_1)$ such that

$$\alpha|Z = (f|Z)^r \otimes \alpha'.$$

Now we have an obvious isomorphism between $L^r \otimes \mathrm{Hom}(V,E_1)$ and $\mathrm{Hom}(\mathcal{L}^{-r} \otimes V,E_1)$, and we regard α via this isomorphism as a global section of the second bundle, i.e. as a linear map from $\mathcal{L}^{-r} \otimes V$ to E_1. The restriction $\alpha|Z$ is then determined by the following commutative triangle

$$\begin{tikzcd}
(\mathcal{L}^{-r} \otimes V)|Z \arrow[rr, "\alpha|Z"] \arrow[dr, "\cdot f^r"'] & & E_1|Z \\
& V|Z \arrow[ur, "\alpha'"'] &
\end{tikzcd}$$

We briefly write

$$\alpha|Z = \alpha'\cdot f^r$$

to describe this factorization of $\alpha|Z$. We want to compute the restriction of the transposed map

$$\alpha^t : E_1^* \to \mathcal{L}^r \otimes V^*$$

to Z. Notice for this that our map from $(\mathcal{L}^{-r} \otimes V)|Z$ to $V|Z$ in the triangle above is the tensor product of the map

$$\mathcal{L}^{-r}|Z \xrightarrow{\cdot f^r} \mathcal{O}|Z$$

and the identity on $V|Z$. The transpose of the first map is the map

$$\mathcal{O}|Z \xrightarrow{f^r} \mathcal{L}^r|Z$$

and the transpose of the identity on $V|Z$ is the identity on $V^*|Z$. Thus we have the commutative triangle

$$\begin{tikzcd}
E_1^*|Z \arrow[rr, "\alpha^t|Z"] \arrow[dr, "(\alpha')^t"'] & & (\mathcal{L}^r \otimes V^*)|Z \\
& V^*|Z \arrow[ur, "\cdot f^r"'] &
\end{tikzcd}$$

and we obtain

$$(\alpha^t \circ \varpi \circ \alpha)|Z = f^r \cdot (\alpha')^t \circ (\varpi|Z) \circ \alpha' \cdot f^r = 0$$

Again by the extension theorem, this time part (ii), there exists some $s > 0$ such that the global section

$$f^{2s} \otimes (\alpha^t \circ \varpi \circ \alpha)$$

of

$$\mathcal{L}^{2s} \otimes \mathrm{Hom}(\mathcal{L}^{-r} \otimes V,\ \mathcal{L}^r \otimes V^*)$$

vanishes. This vector bundle is canonically isomorphic to

$$\mathrm{Hom}(\mathcal{L}^{-h} \otimes V,\ \mathcal{L}^h \otimes V^*)$$

with $h := r+s$. The global section of this second vector bundle corresponding to the global section above is

$$f^s \cdot \alpha^t \circ \varpi \circ \alpha \cdot f^s,$$

which hence also vanishes. Now we introduce the linear map

$$\beta : \mathcal{L}^{-h} \otimes V \xrightarrow{\cdot f^s} \mathcal{L}^{-r} \otimes V \xrightarrow{\alpha} E_1 .$$

As above we see that the transpose of β is

$$\beta^t : E_1^* \xrightarrow{\alpha^t} \mathcal{L}^r \otimes V^* \xrightarrow{\cdot f^s} \mathcal{L}^h \otimes V^* .$$

Thus we have

$$\beta^t \circ \varphi \circ \beta = f^s \cdot \alpha^t \circ \varphi \circ \alpha \cdot f^s = 0,$$

and we can write down the complex

$$T : \qquad 0 \to \mathcal{L}^{-h} \otimes V \xrightarrow{\beta} E_1 \xrightarrow{\beta^t \circ \varphi} \mathcal{L}^h \otimes V^* \to 0 .$$

This complex is exact on Z. Indeed, on Z multiplication with a power of the global section f of $\mathcal{L}$ is an isomorphism. Thus we easily establish an isomorphism between $T|Z$ and the complex over Z' introduced above. We turn T into a bilinear complex - with zero term E_1 - by choosing as bilinear form the orthogonal sum of the pregiven bilinear form on E_1 and the hyperbolic standard form on the direct sum of the other two terms. Then

$$\chi(T) = [E_1] - [H(\mathcal{L}^h \otimes V^*)],$$

which is not quite what we want. Let

$$\varphi_0 : (\mathcal{L}^h \otimes V^*) \oplus (\mathcal{L}^{-h} \otimes V) \to (\mathcal{L}^{-h} \otimes V) \oplus (\mathcal{L}^h \otimes V^*)$$

denote the linear map associated with the hyperbolic stadard form on this direct sum, i.e.

$$\varphi_0 = \begin{pmatrix} 0 & id \\ id & 0 \end{pmatrix} .$$

Let further γ denote the map

$$V^* \xrightarrow{\cdot f^h} \mathcal{L}^h \otimes V^* \hookrightarrow H(\mathcal{L}^h \otimes V^*)$$

with the second arrow denoting the inclusion map. Then γ^t is the map

$$H(\mathcal{L}^h \otimes V^*) \twoheadrightarrow \mathcal{L}^h \otimes V \xrightarrow{\cdot f^h} V$$

with the first arrow denoting the canonical projection. Thus we see that

$$T_o : 0 \to V^* \xrightarrow{\gamma} H(\mathcal{L}^h \otimes V^*) \xrightarrow{\gamma^t \circ \varphi_o} V \to 0$$

is a complex which is exact on Z. We turn T_o into a bilinear complex in the same way as we did twice above. Then

$$\chi(T_o) = [H(\mathcal{L}^h \otimes V^*)] - [H(V)].$$

The bilinear complex $T \perp T_o$ is exact on Z and has the Euler characteristic ξ.

* § 7 An example : Projective spaces.

A major step in our proof of the Prime Ideal Theorem has been Theorem 1 in § 5. Starting from this theorem the proof of the Prime Ideal Theorem was easy. But Theorem 1 gives in many cases a more precise information about Witt rings than the Prime Ideal Theorem does. This will be illustrated in the present and the next section.

We shall make use of the following important theorem of Karoubi without giving the proof here.

Theorem 1. [Kr, II p.139]
Let A be a commutative ring in which 2 is a unit, and let $A[t_1,\dots,t_n]$ denote the polynomial ring in $n \geqslant 1$ variables t_i over A. The natural map from W(A) to $W(A[t_1,\dots,t_n])$ is bijective.

Let now X be the projective n-space P^n_A over some commutative ring A, i.e. the homogenous spectrum of the polynomial ring $A[T_o,\dots,T_n]$ in n+1 variables T_i with its standard grading. X is covered by the affine open sets

$$Z_i \cong \operatorname{Spec} A\left[\frac{T_o}{T_i},\dots,\frac{T_n}{T_i}\right]$$

consisting of the homogeneous prime ideals of $A[T_o,\dots,T_n]$ which do not contain T_i. Clearly Z_i is the special open set X_{T_i} coming from the global section T_i of the canonical line bundle $\mathcal{O}(1)$ on X.

We denote the structure morphism from X to Spec A by f. We further introduce the ideal $\mathfrak{N}$ of W(X) consisting of all ξ in W(X) with $\xi | Z_i = 0$ for $0 \leq i \leq n$.

Starting from Karoubi's theorem we prove

Theorem 2. i) The map f^* from $W(A)$ to $W(X)$ is injective. If 2 is a unit in A

$$W(X) = f^*W(A) \oplus \mathfrak{m}.$$

ii) If 2 is a unit in A and A is regular then $\mathfrak{m}^{n+1} = 0$.

Proof. i) The intersection

$$W := Z \cap Z_1 \cap \ldots \cap Z_n$$

can be identified in an obvious way with the spectrum of the ring $A[T_o,T_o^{-1},\ldots,T_n,T_n^{-1}]_o$ consisting of the elements of degree zero in the localization $A[T_o,T_o^{-1},\ldots,T_nT_n^{-1}]$ of $A[T_o,\ldots,T_n]$. We introduce the closed immersion

$$\sigma:\mathrm{Spec}(A) \to W$$

corresponding to the ring epimorphism from $A[T_o,T_o^{-1},\ldots,T_n,T_n^{-1}]_o$ to A over the base ring A which maps all $T_iT_j^{-1}$ to 1. Let s denote the composition of σ with the inclusion morphism from W to X and s_i the composition of σ with the inclusion morphism from W to Z_i, $0 \leq i \leq n$. Then s is a section of the structure morphism $f:X \to \mathrm{Spec}\ A$ and s_i is a section of the restriction $f_i:Z_i \to \mathrm{Spec}\ A$ of f, i.e. we have

$$f\circ s = \mathrm{id}, \quad f_i\circ s_i = \mathrm{id} \quad (0 \leq i \leq n).$$

From the first equation we obtain that f^* is injective and

$$W(X) = f^*W(A) \oplus \mathrm{Ker}(s^*).$$

Since for ξ in $W(X)$

$$s^*(\xi) = s_i^*(\xi | Z_i)$$

clearly $\mathfrak{N}$ is contained in $\mathrm{Ker}(s^*)$. But by Karoubi's theorem cited above the maps $f_i^* : W(A) \to W(Z_i)$ are bijective provided 2 is a unit in A. Thus also the maps s_i^* are bijective, and we obtain that $\mathfrak{N}$ coincides with the kernel of s^*. Actually the kernel of <u>every</u> restriction map $W(X) \to W(Z_i)$ coincides with the kernel of s^*.

ii) Assume now that A is regular. We choose n+1 elements $\eta_0, \ldots, \eta_n$ in $\mathfrak{N}$ and we want to show that their product is zero. Let E_i be a space over X representing η_i. Since $\eta_i | Z_i = 0$ we have an isomorphism

$$(*) \qquad (E_i | Z_i) \perp H(U_i') \cong H(U_i'')$$

with some vector bundles U_i', U_i'' over Z_i. Since A is regular the natural map from $K(A)$ to $K(Z_i)$ is bijective according to a theorem of Grothendieck, cf.[BHS]. Thus there exist a natural number m and vector bundles V_i, W_i over $\mathrm{Spec}(A)$ such that

$$U_i' \perp m \times \mathcal{O}_{Z_i} \cong f_i^*(V_i) = f^*(V_i) | Z_i,$$

$$U_i'' \perp m \times \mathcal{O}_{Z_i} \cong f_i^*(W_i) = f^*(W_i) | Z_i.$$

Adding in (*) at both sides m copies of the free space $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ over Z_i we obtain

$$(E_i \perp f^*(H(V_i)) | Z_i \cong f^*H(W_i) | Z_i.$$

Introducing the elements

$$\xi_i := [E_i] + f^*[H(V_i)] - f^*[H(W_i)]$$

of L(X) we have $\xi_i | Z_i = 0$ for $i \leq i \leq n$, hence by § 5 Theorem 1

$$\xi_o \cdot\cdots\cdot \xi_n = 0.$$

Taking images in W(X) we obtain

$$\eta_o \cdot\cdots\cdot \eta_n = 0.$$

<u>Remark.</u> If A is not regular but still 2 is a unit in A we have the weaker statement

$$2^{n+1}\,\mathfrak{N}^{n+1} = 0$$

according to the following general lemma.

<u>Lemma.</u> Let X be a scheme which is covered by special open sets $Z_1,\dots,Z_n$. Let $\eta_1,\dots,\eta_n$ be elements of W(X) with $\eta_i | Z_i = 0$. Then

$$2^n\eta_1 \cdot\cdots\cdot\eta_n = 0.$$

Let E_i be a space over X representing η_i. We have isomorphisms

$$E_i | Z \perp M(U_i', \beta_i') \cong M(U_i, \beta_i)$$

with some bilinear bundles (U_i', β_i') and (U_i, β_i) over Z_i. Adding a suitable space $H(U_i'')$ over Z_i on both sides we may assume that U_i' is free. We now find by the method displayed in § 6 (step I of the proof) bilinear bundles (V_i, γ_i) over X with

$$M(V_i, \gamma_i) | Z_i \cong M(U_i', \beta_i').$$

Replacing E_i by $E_i \perp M(V_i, \gamma_i)$ we may assume

$$E_i | Z_i \cong M(U_i, \beta_i).$$

The right hand side is isomorphic to $M(U_i, -\beta_i)$, cf. I § 3 Prop. 1. The map

$$\mathrm{id} \oplus (-\mathrm{id}) : U_i \oplus U_i^* \xrightarrow{\sim} U_i \oplus U_i^*$$

is an isomorphism from $M(U_i, -\beta_i)$ to $- M(U_i, \beta_i)$. Thus

$$E_i | Z_i \cong (-E_i) | Z_i,$$

and we obtain

$$(2 \times E_i) | Z_i \cong M(E_i) | Z_i.$$

According to § 5 Theorem 1 we have in $L(X)$ the relation

$$\prod_{i=1}^{n} (2[E_i] - [H(E_i)]) = 0.$$

Taking images in $W(X)$ we obtain, as wanted,

$$2^n \eta_1 \,\ldots\ldots\, \eta_n = 0.$$

It is tempting to conjecture in view of Karoubi's theorem that actually $\mathfrak{N} = 0$, even if A is not regular. If A is a field I showed many years ago for $n = 1$ that indeed $\mathfrak{N} = 0$ [K, Th. 13.2.2]. There the characteristic of the field was allowed to be 2. Thus it is not clear whether the assumption that 2 is a unit in A is necessary.

* § 8 A semilocal-global principle.

If X is a noetherian scheme of finite dimension d then, as mentioned in § 2, the kernel $\tilde{K}(X)$ of the natural map

$$K(X) \to \prod_{x} K(\mathcal{O}_x)$$

with x running through all closed points of X is nilpotent with

$$\tilde{K}(X)^{d+1} = 0.$$

It would be interesting to have a similar result about the kernel $\tilde{W}(X)$ of the analogous map

$$W(X) \to \prod_{x} W(\mathcal{O}_x).$$

We prove in this section such a statement with the $\mathcal{O}_x$ replaced by certain semilocal rings $\mathcal{O}_S$. We first give a description of these semilocal rings.

Let S be a finite non empty set consisting of closed points of some scheme X. We assume that S has an affine open neighbourhood. We define $\mathcal{O}_S$ as the inductive limit of the rings $\mathcal{O}(U)$ with U running through all open neighbourhoods of S,

$$\mathcal{O}_S := \varinjlim_{U \supset S} \mathcal{O}(U).$$

For every point s in S the natural map

$$\mathcal{O}(U) \to \mathcal{O}_s/\mathfrak{r}_s = k(s)$$

yields a ring homomorphism from $\mathcal{O}_S$ to k(s). This homomorphism is surjective since for U an affine neighbourhood of S the map from $\mathcal{O}(U)$ to $k(s)$ is surjective. We denote the kernel of this epimorphism from $\mathcal{O}_S$ to $k(s)$ by $\mathfrak{p}_s$. It is a maximal ideal of $\mathcal{O}_S$. We claim that the $\mathfrak{p}_s$ are already all maximal ideals of $\mathcal{O}_S$ and therefore $\mathcal{O}_S$ is semilocal.

To prove this it suffices to show that an element φ of $\mathcal{O}_S$ which does not lie in any of the ideals $\mathfrak{p}_s$ is a unit of $\mathcal{O}_S$. Let $f \in \mathcal{O}(U)$ be a representative of φ. Then the open set $V := U_f$ contains S. The function $f|V$ is a unit of $\mathcal{O}(V)$. Thus φ is indeed a unit of $\mathcal{O}_S$.

We assume since now that X is <u>quasiprojective</u>. Then <u>every</u> finite set S of closed points of X has an affine open neighbourhood. More precisely we meet the following situation. X is embedded as an open subset into a scheme Y which has a line bundle $\mathcal{L}$ such that no point of Y is a common zero of all global sections of $\mathcal{L}$ and such that the natural morphism from Y to the homogeneous spectrum of the graded ring

$$R = \bigoplus_{i=P}^{\infty} R_i, \qquad R_i := \mathcal{L}^{\otimes i}(Y),$$

is an isomorphism. Let S be a finite non empty set of closed points in X, and let T denote the set of all homogeneous elements f of R of positive degree with $f(s) \neq 0$ for every s in S, i.e. S contained in Y_f. The open sets Y_f are known to be affine. Thus they are special open subsets of Y.

<u>Lemma 1.</u> The special open sets Y_f with f running through T are a fundamental system of neighbourhoods of S. We have a natural

isomorphism

$$(T^{-1}R)_o \xrightarrow{\sim} \mathfrak{o}_S$$

from the ring of elements of degree zero in $T^{-1}R$ to $\mathfrak{o}_S$.

<u>Proof.</u> The last assertion is evident if the first one is proved, since for f in T

$$\mathfrak{o}(Y_f) \cong R[f^{-1}]_o$$

according to the extension theorem in III § 1. Let S consist of the points $x_1,\ldots,x_n$, and let U be a pregiven open neighbourhood of S in Y. There exist elements $h_1,\ldots,h_n$ of T such that

$$x_i \in Y_{h_i} \subset U \qquad (1 \leq i \leq n).$$

Raising the h_i to suitable powers we assume that all h_i have the same degree d. There exist homogeneous elements $g_1,\ldots,g_n$ of R such that

$$g_i(x_i) \neq 0, \quad g_j(x_i) = 0 \quad \text{for } j \neq i.$$

Again we may assume that the g_i all have the same degree e. Consider the element

$$f = g_1h_1 + \ldots + g_nh_n$$

of degree d+e. We have $Y_f \subset U$ since for any point x of Y with $f(x) \neq 0$ at least one of the sections h_i does not vanish at x. Moreover $f(x_i) \neq 0$ for $1 \leq i \leq n$. Our lemma is proved.

We now study the kernels $\approx\!\!L(X)$ and $\approx\!\!W(X)$ of the natural maps

$$L(X) \to \prod_S L(\mathfrak{o}_S), \quad W(X) \to \prod_S W(\mathfrak{o}_S),$$

with S running through all finite non empty sets of closed points in X.

Let M(X) denote the set of all closed points in X equipped with its topology as a subspace of the topological space X. We assume that X is quasiprojective and M(X) is a noetherian space, i.e. every descending chain of closed subsets of M(X) has finite length. Then every closed subset of M(X) has a unique decomposition into irreducible components. We further assume that M(X) has finite dimension d. This means that for chains of irreducible closed subsets of M(X) the supremum of the lengths is the finite number d.

Theorem. ([D]) for X affine).

$$\approx\!\!\!\!L(X)^{d+1} = 0 \quad \text{and} \quad \approx\!\!\!\!W(X)^{d+1} = 0.$$

We first verify the claim about $\approx\!\!\!\!L(X)$. Let $\xi_0,\dots,\xi_d$ be given elements of $\approx\!\!\!\!L(X)$. We choose a finite set S_0 consisting of one point on each irreducible component of M(X). Since ξ_0 has image zero in $L(\mathcal{O}_{S_0})$ there exists an open neighbourhood U_0 of S_0 in X with $\xi_0|U_0 = 0$, cf. step (ii) in the proof of § 5 Theorem 2. By the preceding Lemma 1 we may assume that U_0 is a special open set. Let X_1 denote the complement of U_0 in X and let $M(X_1)$ denote the set of closed points in X_1. Then $M(X_1)$ is a closed subset of M(X) which has dimension at most d-1 or is empty. If $M(X_1)$ is not empty we choose a finite set S_1 consisting of one point on each irreducible component of $M(X_1)$. Then we find a special open neighbourhood U_1 of S_1 in X with $\xi_1|U_1 = 0$, and we define X_2 as the complement of $U_o \cup U_1$ in X. Repeating this procedure we obtain a descending chain of closed sets

$$X = X_o \supset X_1 \supset \dots \supset X_r \supset X_{r+1}$$

with $r \leq d$, $M(X_{r+1}) = \emptyset$, and special open sets $U_o, U_1, \dots, U_r$ such that

$$X \smallsetminus (U_o \cup \dots \cup U_i) = X_{i+1}$$

and $\xi_i | U_i = 0$ for $0 \leq i \leq r$. Since any non empty closed subset of X contains closed points the set X_{r+1} must be empty. Thus $U_o, \dots, U_r$ cover the whole of X. By our chief tool § 5 Theorem 1 the product $\xi_o \cdot \dots \cdot \xi_r$ is zero, hence

$$\xi_o \cdot \dots \cdot \xi_d = 0.$$

The claim about $\approx{W}(X)$ now follows from

<u>Lemma 2.</u> The natural map from $\overset{\approx}{L}(X)$ to $\overset{\approx}{W}(X)$ is surjective.

<u>Proof.</u> Let E be a space over X whose Witt class $\{E\}$ lies in $\overset{\approx}{W}(X)$. Then the rank of E is a locally constant function on X taking even values. Thus $rkE = 2m$ with m a locally constant function taking values in non negative integers. The element

$$[E] - m\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

of L(X) is a lifting of $\{E\}$ and lies in $\overset{\approx}{L}(X)$ according to II § 1 Prop.1.

This completes the proof of our theorem. To obtain a similar result for the kernels $\tilde{L}(X)$ and $\tilde{W}(X)$ of the natural maps

$$L(X) \to \prod_x L(\mathcal{O}_x), \quad W(X) \to \Pi_x W(\mathcal{O}_x)$$

with x running through all closed points of X it would be sufficient to prove for A a semilocal ring with the maximal ideals $m_1,\ldots,m_n$ a statement

$$[\mathrm{Ker}(L(A) \to \prod_{i=1}^{n} L(A_{m_i}))]^a = 0$$

with some universal constant a. For the functor K the analogous kernel clearly is zero and no problem arises.

Chapter IV Spaces of rank one

§ 1 The group of square classes Q(X).

Let X be an arbitrary scheme.

Definition. A square class of X is the isomorphism class of a bilinear space $(\mathcal{L},B)$ of rank one over X.

Let Q(X) denote the set of square classes of X. Obviously Q(X) is a commutative semigroup under tensor multiplication with unit element $\langle 1\rangle = (\mathcal{O},m)$, m denoting the multiplication $\mathcal{O} \times \mathcal{O} \to \mathcal{O}$ on $\mathcal{O}$. {We often use the same notation for a space and its isomorphism class for short.} Actually Q(X) is even a group of exponent 2. Indeed, let $(\mathcal{L},B)$ be a bilinear space of rank one. Then B induces a linear map

$$\psi:\mathcal{L} \otimes \mathcal{L} \to \mathcal{O},$$

and ψ is an isomorphism from the space $(\mathcal{L} \otimes \mathcal{L}, B \otimes B)$ to the space $(\mathcal{O},m)$. This can be easily verified on the stalks using the fact that every stalk $\mathcal{L}_x$ is a free $\mathcal{O}_x$-module of rank one.

The isomorphism classes $\langle a\rangle$ of free spaces of rank one, a in $\mathcal{O}(X)^*$, clearly form a subgroup of Q(X) which can be identified with $\mathcal{O}(X)^*/\mathcal{O}(X)^{*2}$. Denoting as usual by Pic(X) the group of isomorphism classes of line bundles on X, we have a "forget"-homomorphism

$$v:Q(X) \to_2 \operatorname{Pic}(X)$$

into the subgroup of elements of order ≤ 2 of Pic(X), which maps the isomorphism class of a space $(\mathcal{L},B)$ to the isomorphism class of $\mathcal{L}$.

<u>Proposition.</u> The sequence

$$1 \to \mathcal{O}(X)^*/\mathcal{O}(X)^{*2} \to Q(X) \xrightarrow{v} {}_2\operatorname{Pic}(X) \to 1$$

is exact.

This statement is already evident up to the surjectivity of v. Let $\mathcal{L}$ be a line bundle on X such that $\mathcal{L} \otimes \mathcal{L}$ is isomorphic to $\mathcal{O}$. We choose an isomorphism

$$\alpha:\mathcal{L} \otimes \mathcal{L} \to \mathcal{O}.$$

α corresponds by the definition of the tensor product to a bilinear form

$$B:\mathcal{L} \times_X \mathcal{L} \to \mathcal{O}.$$

B is automatically symmetric since $\mathcal{L}$ is locally free of rank one. Moreover it is immediately checked on the stalks that B is non degenerate. This proves the surjectivity of v.

§ 2 Construction of spaces of rank one.

We now assume that X is irreducible and reduced. Let ξ be the generic point of X. Then

$$\mathfrak{O}_\xi = \varinjlim_{Z} \mathfrak{O}(Z)$$

with Z running through all non empty open subsets of X, since all these sets contain ξ. Moreover for $Z' \subset Z$ the restriction map from $\mathfrak{O}(Z)$ to $\mathfrak{O}(Z')$ is injective. Thus all $\mathfrak{O}(Z)$ inject into $\mathfrak{O}_\xi$ and we simply regard $\mathfrak{O}_\xi$ as the union of the $\mathfrak{O}(Z)$, replacing every $\mathfrak{O}(Z)$ by its image in $\mathfrak{O}_\xi$. Clearly $\mathfrak{O}_\xi$ is a field F, the "function field" of X, cf. [EGA,I § 7]. We call the elements f of F the <u>rational functions</u> on X, and we say that a function f <u>is defined</u> on Z if $f \in \mathfrak{O}(Z)$.

Assume now we are given an element f in F^*, a covering $X = \bigcup_\alpha Z_\alpha$ of X by open sets Z_α with α running through an arbitrary index set, and we are given for every Z_α an equation

$$f = \varepsilon_\alpha g_\alpha^2$$

in F with ε_α in $\mathfrak{O}(Z_\alpha)^*$ and g_α in F. Then we define on each Z_α a bilinear space $(\mathfrak{L}_\alpha, B^\alpha)$ as follows. For an open set $W \subset Z_\alpha$ we put

$$\mathfrak{L}_\alpha(W) := g_\alpha^{-1}\,\mathfrak{O}(W)$$

which is an $\mathfrak{O}(W)$-submodule of F. If $W' \subset W$ then $\mathfrak{L}_\alpha(W) \subset \mathfrak{L}_\alpha(W')$, and we define the restriction map from $\mathfrak{L}_\alpha(W)$ to $\mathfrak{L}_\alpha(W')$ as the inclusion map. We thus obtain on $\mathfrak{O}_{Z_\alpha}$ -module $\mathfrak{L}_\alpha$. Clearly $\mathfrak{L}_\alpha$ is a free $\mathcal{O}_{Z_\alpha}$ -module of rank one with basis element $g_\alpha^{-1} \in \mathcal{L}^\alpha(Z_\alpha)$. We define the bilinear form B^α on $\mathcal{L}_\alpha$ as follows: If u,v are sections in $\mathcal{L}_\alpha(W)$, $W \subset Z_\alpha$, then

$$B^\alpha(u,v) := fuv = \varepsilon_\alpha(g_\alpha u)(g_\alpha v),$$

i.e. B^α is the unique symmetric bilinear form on $\mathcal{L}_\alpha$ with

$$B^\alpha(g_\alpha^{-1}, g_\alpha^{-1}) = \varepsilon_\alpha.$$

Since ε_α is a unit B^α is non degenerate.

If we base our construction on another equation

$$f = \varepsilon'_\alpha \, g'^2_\alpha$$

with the same f and of course again ε'_α a unit in $\mathcal{O}(Z_\alpha)$ then we obtain precisely the same sheaf $\mathcal{L}_\alpha$ and the same bilinear form B^α. In particular, if $Z_\alpha \cap Z_\beta$ is not empty then

$$(\mathcal{L}_\alpha, B_\alpha)|Z_\alpha \cap Z_\beta = (\mathcal{L}_\beta, B_\beta)|Z_\alpha \cap Z_\beta$$

since both spaces over $Z_\alpha \cap Z_\beta$ come from equations

$$f = \varepsilon_\alpha g_\alpha^2, \quad f = \varepsilon_\beta g_\beta^2$$

with $\varepsilon_\alpha, \varepsilon_\beta$ units of $\mathcal{O}(Z_\alpha \cap Z_\beta)$ and the same f. Thus all $(\mathcal{L}_\alpha, B_\alpha)$ fit together and yield a bilinear space $(\mathcal{L}, B)$ on X. We have

$$(\mathcal{L}_\xi, B_\xi) \cong \langle f \rangle.$$

We denote this bilinear space $(\mathcal{L}, B)$ by $\mathcal{L}(f)$. From our construction of $\mathcal{L}(f)$ it is clear that $\mathcal{L}(f)$ does not depend on the chosen covering $\{Z_\alpha\}$ and the equations $f = \varepsilon_\alpha g_\alpha^2$. We only have to be shure that for f at least one such covering and set of equations exists. This happens if and only if for every closed point x of X there exists an equation $f = \varepsilon_x g_x^2$ with g_x in F^* and ε_x in $\mathcal{O}_x^*$.

<u>Proposition.</u> Let $f \in F^*$. Then we have an open covering $\{Z_\alpha\}$ of X and equations $f = \varepsilon_\alpha g_\alpha^2$ in F^* with every ε_α a unit of $\mathcal{O}(Z_\alpha)$ if and only if $\langle f\rangle$ lies in the image of the natural map from $Q(X)$ to $Q(F)$. In this case $\mathcal{L}(f)$ is up to isomorphism the unique bilinear space $\mathcal{L}$ of rank one over X with $\mathcal{L}_\xi \cong \langle f\rangle$. Thus the natural map from $Q(X)$ to $Q(F)$ is injective.

<u>Proof.</u> It suffices to show that if $(\mathcal{L},B)$ is a bilinear space of rank one over X and $(\mathcal{L},B)_\xi \cong \langle f\rangle$ then f admits a covering $\{Z_\alpha\}$ of X with equations as above, and $(\mathcal{L},B)$ is isomorphic to $\mathcal{L}(f)$.

This is easy. For every open Z in X the restriction map from $\mathcal{L}(Z)$ to $\mathcal{L}_\xi$ is injective and we regard $\mathcal{L}_\xi$ as the union of all $\mathcal{L}(Z)$ as we did above for $\mathcal{O}$. Let (F,B_o) be the free space $\langle f\rangle$ over F, i.e. the vector space F over F, equipped with the bilinear form $B_o(u,v) = fuv$. We choose an isomorphism of spaces

$$\lambda:(\mathcal{L}_\xi,B_\xi) \xrightarrow{\sim} (F,B_o).$$

Let $\{Z_\alpha\}$ be an open covering of X such that $\mathcal{L}|Z_\alpha$ is free with basis element $s_\alpha \in \mathcal{L}(Z_\alpha)$ for every α. We introduce the elements

$$g_\alpha := \lambda(s_\alpha)^{-1}$$

of F^*, and the units

$$\varepsilon_\alpha := B(s_\alpha, s_\alpha).$$

We then have

$$\varepsilon_\alpha = B_o(\lambda(s_\alpha), \lambda(s_\alpha)) = f g_\alpha^{-2} ,$$

hence

$$f = \varepsilon_\alpha g_\alpha^2.$$

Thus we can construct the space $\mathcal{L}(f)$. Clearly λ maps the subset $\mathcal{L}(Z_\alpha)$ of $\mathcal{L}_\xi$ bijectively onto the subset $\mathcal{L}(f)(Z_\alpha) = g_\alpha^{-1}\,\mathcal{O}(Z_\alpha)$ of F and is isometric with respect to the bilinear forms. Thus λ induces an isomorphism from $(\mathcal{L}, B)$ onto $\mathcal{L}(f)$.

§ 3 Determinants

Spaces of rank one come up in a natural way as the highest non vanishing exterior powers of arbitrary bilinear spaces. Thus let us first define exterior powers. For every vector bundle E over X and $r > 0$ we have a vector bundle $\overset{r}{\Lambda}E$ which for Z an <u>affine</u> open set has the section module

$$(\overset{r}{\Lambda}E)(Z) := \overset{r}{\Lambda}E(Z)$$

where the right hand side means the r-th exterior power of the projective $\mathcal{O}(Z)$-module $E(Z)$. For any open set Z and sections $u_1,\ldots,u_r$ in $E(Z)$ we then have a well defined product $u_1 \wedge \ldots \wedge u_r$ in $(\overset{r}{\Lambda}E)(Z)$. There exists a natural isomorphism

$$\kappa : \overset{r}{\Lambda}E^* \xrightarrow{\sim} (\overset{r}{\Lambda}E)^* ,$$

characterized by the following property. If $u_1,\ldots,u_r$ are sections in $E(Z)$ and $u_1^*,\ldots,u_r^*$ sections in $E^*(Z)$ for some open set Z then

$$\langle u_1 \wedge\ldots\wedge u_r, \kappa(u_1^* \wedge\ldots\wedge u_r^*)\rangle = \det(\langle u_i, u_j^*\rangle)_{1 \leq i,\, j \leq r}$$

We identify $\overset{r}{\Lambda}E^*$ and $(\overset{r}{\Lambda}E)^*$ by this canonical isomorphism κ.

Let now B be a symmetric bilinear form on E and $\varphi: E \to E^*$ the associated linear map. Then φ induces a linear map

$$\overset{r}{\Lambda}(\varphi): \overset{r}{\Lambda}E \to \overset{r}{\Lambda}E^*$$

which is again selfadjoint. Let $\overset{r}{\Lambda}B$ denote the symmetric bilinear form on $\overset{r}{\Lambda}E$ associated with $\overset{r}{\Lambda}\varphi$. If B is non degenerate then $\overset{r}{\Lambda}\varphi$ is an isomorphism, hence $\overset{r}{\Lambda}B$ is non degenerate. We call $(\overset{r}{\Lambda}E,\overset{r}{\Lambda}B)$ the r-th exterior power of the bilinear bundle (E,B). For sections $u_1,\dots,u_r$, $v_1,\dots,v_r$ of E over some open set Z we clearly have the formula

$$(\overset{r}{\Lambda}B)(u_1\wedge\dots\wedge u_r,\ v_1\wedge\dots\wedge v_r) = \det(B(u_i,v_j)).$$

The exterior power $(\overset{o}{\Lambda}E,\overset{o}{\Lambda}B)$ is for all spaces (E,B) defined as the space $\langle 1\rangle = (\mathfrak{O},m)$.

One easily verifies for two bilinear modules E and F over X (cf.[K,§ 2]).

<u>Proposition 1.</u> The natural map

$$\coprod_{i=0}^{r} (\overset{i}{\Lambda}E)\otimes(\overset{r-i}{\Lambda}F)\to\overset{r}{\Lambda}(E\perp F)$$

which maps a section $(u_1\wedge\dots\wedge u_i)\otimes(v_1\wedge\dots\wedge v_{r-i})$ onto the section $u_1\wedge\dots\wedge u_i\wedge v_1\wedge\dots\wedge v_{r-i}$ is an isomorphism of bilinear bundles.

Now we define the <u>determinant</u> det E of a bilinear space E of rank n as the isomorphism class of the exterior power $\overset{n}{\Lambda}E$. This makes sense also if X is not connected, since then X is a disjoint union of open subschemes with n a constant on each of these subschemes. $\overset{n}{\Lambda}E$ is a space of rank one, hence det(E) is a square class, and we have obtained a function

$$\text{"det"} : \mathrm{Bil}(X)\to Q(X).$$

As a special case of the proposition above we have the formula

$$\det(E \perp F) = \det(E)\det(F).$$

Thus we obtain from this determinant function a group homomorphism

$$\det' : K\,\mathrm{Bil}(X) \to Q(X).$$

We want to show that this homomorphism vanishes on the element $[E] - [H(V)]$ for E a metabolic space and V a Lagrangian of E. Then we know that det' yields a group homomorphism

$$\det : L(X) \to Q(X).$$

<u>Proposition 2.</u> If E is a metabolic space of rank $2m$ then $\det(E) = \langle(-1)^m\rangle$.

From this proposition it will be clear that the homomorphisms det' vanishes on the elements $[E] - [H(V)]$ above.

<u>Proof.</u> Let V be a Lagrangian of E. We have a natural isomorphism of vector bundles

$$\alpha : \overset{m}{\Lambda} V \otimes \overset{m}{\Lambda}(E/V) \xrightarrow{\sim} \overset{2m}{\Lambda} E$$

which for sections $v_1,\dots,v_m$ of $V(Z)$ and images $\bar{u}_1,\dots,\bar{u}_m$ of sections $u_1,\dots,u_m \in E(Z)$ in $(E/V)(Z)$ maps $v_1 \wedge\dots\wedge v_m \otimes \bar{u}_1 \wedge\dots\wedge \bar{u}_m$ to $v_1 \wedge\dots\wedge v_m \wedge u_1 \wedge\dots\wedge u_m$. Notice that this map is well defined, since the section $v_1 \wedge\dots\wedge v_m \wedge u_1 \wedge\dots\wedge u_m$ does not depend on the choice of preimages u_i of $\bar{u}_i$. {If Z is not affine then not every section of

(E/V)(Z) has necessarily a preimage in E(Z), but this does not bother us, since the affine open sets are enough to define a homomorphism of vector bundles.} Now B gives a perfect duality between the vector bundles V and E/V since $V = V^{\perp}$. Thus we have via B an isomorphism

$$\beta : (\overset{m}{\Lambda} V)^* \xrightarrow{\sim} \overset{m}{\Lambda}(E/V).$$

For the line bundle $\mathcal{L} := \overset{m}{\Lambda} V$ we obtain an isomorphism

$$\alpha \circ (1 \otimes \beta) : \mathcal{L} \otimes \mathcal{L}^* \xrightarrow{\sim} \overset{m}{\Lambda} E.$$

On the other hand we have the usual canonical isomorphism

$$\gamma : \mathcal{L} \otimes \mathcal{L}^* \to \mathcal{O}$$

which for sections $e \in \mathcal{L}(Z)$, $e^* \in \mathcal{L}^*(Z)$ maps $e \otimes e^*$ onto $\langle e, e^* \rangle$. I now claim that

$$\chi := \alpha \circ (1 \otimes \beta) \circ \gamma^{-1}$$

is an isomorphism of the space $\langle (-1)^m \rangle$ onto the space $\overset{m}{\Lambda} E$. It suffices to check this locally. Let x be a point of X. We choose a decomposition

$$E_x = V_x \oplus U_x$$

and bases $v_1, \ldots, v_m$ of V_x, $u_1, \ldots, u_m$ of U_x such that $B(v_i, u_j) = \delta_{ij}$. Let $v_1^*, \ldots, v_m^*$ denote the basis of V^* which is dual to $v_1, \ldots, v_m$. Then $\gamma(v_1 \wedge \ldots \wedge_m \otimes v_1^* \wedge \ldots \wedge v_m^*) = 1$. Furthermore

$$\beta(v_1^* \wedge \ldots \wedge v_m^*) = \bar{u}_1 \wedge \ldots \wedge \bar{u}_m,$$

hence

$$\alpha\circ(1\otimes\beta)(v_1\wedge\ldots\wedge v_m\otimes v_1^*\wedge\ldots\wedge v_m^*) = v_1\wedge\ldots\wedge v_m\wedge u_1\wedge\ldots\wedge u_m.$$

Let us denote this element by e. That χ is isometric on the stalks over x means that

$$(\overset{2m}{\Lambda} B)(e,e) = (-1)^m.$$

This is indeed true:

$$(\overset{2m}{\Lambda} B)(e,e) = \det\begin{pmatrix} 0 & I_m \\ I_m & * \end{pmatrix}.$$

We thus have established a determinant homomorphism

$$\det\colon\ L(X) \to Q(X),$$

and we have

$$\det[H(V)] = \langle(-1)^m\rangle$$

for V a vector bundle of rank m. By th usual trick (cf.II,§ 2) we obtain a well defined <u>signed</u> <u>determinant</u>

$$d\colon W(X) \to Q(X)$$

defining

$$d(E) := \langle(-1)^{\frac{n(n-1)}{2}}\rangle \det E$$

for E a bilinear space of rank n. For spaces E and F over X we have

$$d(E\perp F) = d(E)d(F)$$

if at least one of the spaces has even rank. But we have

$$d(E\perp F) = \langle -1\rangle\, d(E)d(F)$$

if both spaces have odd rank.

§ 4 The units of W(X).

If $\mathcal{L}$ is a bilinear space of rank one then $d(\mathcal{L}) = \mathcal{L}$. Thus the natural map $\mathcal{L} \to \{\mathcal{L}\}$ from Q(X) to W(X) is injective. Since now we regard Q(X) as a subset of W(X). Then Q(X) is a subgroup of the group $W(X)^*$ of units of W(X).

Theorem 1. For every element z of $W(X)^*$ we have

$$z = d(z)(1+u)$$

with u a nilpotent element. Thus $W(X)^*$ is generated by the subgroups Q(X) and 1 + Nil W(X).

Proof. Define u by the equation

$$d(z)\cdot z = 1 + u.$$

We have to show $\sigma(u) = 0$ for every signature σ on X. This means, we have to show

(*) $$\sigma(z) = \sigma(d(z))$$

Now there exists a closed point x on X and a factorisation of σ through a signature τ of $\mathcal{O}_x$. Thus it suffices to verify the equation (*) for z a unit of W(A), A a local ring, and σ a signature of A. Write

$$z = \langle a_1\rangle + \ldots + \langle a_n\rangle$$

with units a_i of A. Since $\sigma(z)$ is a unit of $\mathbb{Z}$ we have $\sigma(z) = \varepsilon$ with $\varepsilon = +1$ or -1. In particular n is an odd number 2k+1. Let s denote the number of a_i with $\sigma(a_i) = -1$. Then we have

$$\sigma(d(z)) = (-1)^{\frac{n(n-1)}{2}} \cdot (-1)^s = (-1)^{k+s}.$$

On the other hand

$$\varepsilon = \sigma(z) = n-2s = 1+2(k-s).$$

Now observe that

$$\varepsilon = (-1)^{\frac{\varepsilon-1}{2}} .$$

This yields

$$\sigma(z) = (-1)^{k-s} = \sigma(d(z)).$$

<u>Definition.</u> A square class $\mathcal{L}$ is <u>totally positive</u> if $\sigma(\mathcal{L}) = +1$ for every signature σ of X.

We denote the group of all totally positive square classes of X by $Q^+(X)$.

<u>Proposition 2.</u>

$$Q^+(X) = Q(X) \cap (1 + \text{Nil } W(X)).$$

Indeed, a square class $\mathcal{L}$ is positive definite if and only if all signatures vanish on $\mathcal{L}-1$, which means that $\mathcal{L}-1$ is nilpotent.

Here is an application of Theorem 1.

<u>Proposition 3.</u> For every nilpotent element u of W(X) the signed determinant d(u) is totally positive.

<u>Proof.</u> We apply the theorem to the unit 1+u. Since u has even rank we have d(1+u) = d(u). Thus

$$1 + u = d(u) + v$$

with v again nilpotent. From this equation we obtain $\sigma(d(u)) = 1$ for every signature σ.

Chapter V Algebraic schemes over $\mathbb{R}$.

§ 1 Factorization of local signatures.

Let A be a connected semilocal ring and $\sigma: W(A) \to \mathbb{Z}$ a signature of A. For $\mathfrak{p}$ a prime ideal of A we denote by $A(\mathfrak{p})$ the quotient field of $A/\mathfrak{p}$. We want to find a factorization

$$(*) \qquad \begin{tikzcd} W(A) \arrow[rr, "\sigma"] \arrow[dr] & & \mathbb{Z} \arrow[dl, "\tau"] \\ & W(A(\mathfrak{p})) & \end{tikzcd}$$

with a suitable prime ideal $\mathfrak{p}$ of A and some signature τ of the field $A(\mathfrak{p})$. Here of course $W(A) \to W(A(\mathfrak{p}))$ is the homomorphism induced by the natural map from A to $A(\mathfrak{p})$.

There is an obvious condition to fulfill for any candidate $\mathfrak{p}$. Let $Q(\sigma)$ denote the set of all sums

$$s = \lambda_1^2 a_1 + \ldots + \lambda_r^2 a_r$$

of arbitrary length r with units a_i of A and elements λ_i of A such that

$$\sigma(a_1) = \ldots = \sigma(a_r) = 1$$

and

$$\lambda_1 A + \ldots + \lambda_r A = A.$$

$Q(\sigma)$ is a multiplicative subset of A. Suppose we have a factorization (*) as above. Then I claim that $\mathfrak{p}$ does not meet

$Q(\sigma)$. Indeed, assume an element s as above lies in $\mathfrak{p}$. Then denoting the images of elements of A in $A(\mathfrak{p})$ by bars we obtain in $A(\mathfrak{p})$ the equation

$$(**) \qquad \overline{\lambda}_1^2\overline{a}_1 + \dots + \overline{\lambda}_r^2\overline{a}_r = 0$$

and not all $\overline{\lambda}_i$ are zero since the λ_i generate A. The square class $<\overline{a}_i>$ of $A(\mathfrak{p})$ is the image of the square class $<a_i>$ of under the map from $W(A)$ to $W(A(\mathfrak{p}))$. Thus

$$\tau(\overline{a}_i) = \sigma(\overline{a}_i) = 1.$$

This means that all $\overline{a}_i$ are positive under the ordering of $A(\mathfrak{p})$ corresponding to τ, and the equation (**) is a contradiction.

Thus it is natural to look for the maximal ideals $\mathfrak{p}$ of A which do not meet the multiplicative set $Q(\sigma)$, since such ideals are automatically prime. Let $-Q(\sigma)$ denote the set of all elements $-s$ with s in $Q(\sigma)$.

<u>Lemma 1.</u> The sets $Q(\sigma)$ and $-Q(\sigma)$ are disjoint.

<u>Proof.</u> Suppose we have a relation

$$\lambda_1^2a_1 + \dots + \lambda_r^2a_r = - \mu_1^2b_1 \mu \dots \mu_s^2b_s$$

with units a_i, b_j such that

$$\sigma(a_1) = \dots = \sigma(a_r) = \sigma(b_1) = \dots = \sigma(b_s) = 1$$

and

$$\lambda_1A + \dots + \lambda_rA = 0.$$

(We actually do not need that the μ_i generate A too.) Then the space $<a_1,\ldots,a_r,\ b_1,\ldots,b_s>$ over A has the <u>primitive</u> isotropic vector $(\lambda_1,\ldots,\lambda_r,\ \mu_1,\ldots,\mu_s)$. Thus (cf. II § 1)

$$<a_1,\ldots,a_r,\ b_1,\ldots,b_s> \cong M \perp G$$

with M some metabolic plane $\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$ and some space G. Adding for safety $<1>$ we may write

$$<1,a_1,\ldots,a_r,\ b_1,\ldots,b_s> \cong M \perp <c_1,\ldots,c_{r+s-1}>$$

with some units c_i, cf. II § 3. Applying σ we obtain

$$r+s+1 = \sigma(c_1) + \ldots + \sigma(c_{r+s-1}).$$

But such an equation is impossible since the right hand side has less than r+s+1 summands $\pm$ 1.

Now a little miracle comes up with the following lemma.

<u>Lemma 2.</u> The complement of the set $Q(\sigma) \cup (-Q(\sigma))$ in A is a prime ideal $\mathfrak{p}$.

The proof of this lemma can be found in [K_3, I Appendix B]. We call this prime ideal $\mathfrak{p}$ the <u>associated prime ideal of the signature</u> σ.

The ring $A/\mathfrak{p}$ is the disjoint union of the images $\bar{Q}$ and $-\bar{Q}$ of $Q(\sigma)$ and $-Q(\sigma)$ and the set $\{0\}$. The set $\bar{Q}$ is closed under addition and multiplication. Thus there exists a unique total ordering on $A/\mathfrak{p}$, compatible with addition and multiplication, such that $\bar{Q}$ is the set of positive elements of this ordering. This ordering has a unique extension to an ordering of the quotient field $A(\mathfrak{p})$. The positive elements of $A(\mathfrak{p})$ are the fractions $\bar{a}\bar{b}^{-1}$ with a and b in $Q(\sigma)$.

Let τ denote the signature of $A(\mathfrak{p})$ corresponding to this ordering. Then σ is indeed the composite

$$W(A) \to W(A(\mathfrak{p})) \xrightarrow{\tau} \mathbb{Z}$$

as is easily checked on the square classes in $W(A)$. It is also clear that τ is the unique signature on $A(\mathfrak{p})$ with this property. Indeed, for any such signature on $A(\mathfrak{p})$ all elements of $\overline{Q}$ must be positive under the corresponding ordering.

Thus we have obtained the following factorization theorem, first observed by Kanzaki and Kitamura [KK] (in the case A local and 2 invertible).

<u>Factorization-theorem.</u> Given a signature σ on A there exists a unique signature τ on the residue class field $A(\mathfrak{p})$ of the associated prime ideal $\mathfrak{p}$ of σ such that the diagram

$$\begin{tikzcd} W(A) \arrow[r, "\sigma"] \arrow[dr] & \mathbb{Z} \\ & W(A(\mathfrak{p})) \arrow[u, "\tau"'] \end{tikzcd}$$

commutes. $\mathfrak{p}$ contains every other prime ideal of A for which such a factorization is possible.

Using Corollary 1 in III § 2 we deduce from this factorization theorem immediately a less precise factorization theorem over schemes.

Corollary. Let X be a divisorial scheme and σ a signature on X. Then there exists a point y on X and a signature τ on the field $k(y) = \mathcal{O}_y/\mathfrak{m}_y$ such that the diagram

$$\begin{tikzcd} W(X) \arrow[rr, "\sigma"] \arrow[dr] & & \mathbb{Z} \\ & W(k(y)) \arrow[ur, "\tau"'] & \end{tikzcd}$$

commutes.

Here of course $W(X) \to W(k(y))$ is the functorial homomorphism induced by the inclusion morphism from Spec $k(y)$ to X. This homomorphism maps the Witt class of a space E over X to the Witt class of the fibre

$$E(y) = E_y/\mathfrak{m}_y E_y.$$

§ 2 Signatures and real points.

Assume now that X is an algebraic scheme [EGA, I § 6.4] over the field $\mathbb{R}$ of real numbers. Then for any open set Z of X the ring $\mathcal{O}(Z)$ is an $\mathbb{R}$-algebra and for open sets $Z' \subset Z$ the restriction map from $\mathcal{O}(Z)$ to $\mathcal{O}(Z')$ is an $\mathbb{R}$-algebra homomorphism. Moreover X is covered by finitely many affine open subsets $Z_1, \ldots, Z_N$ such that every $\mathbb{R}$-algebra $\mathcal{O}(Z_i)$ is finitely generated.

It may be helpful to look again at Example 2 in Chapter I, § 1. The schemes considered there are precisely the reduced affine and projective algebraic schemes over $\mathbb{R}$.

For x a closed point of X the residue class field $k(x) = \mathcal{O}_x/m_x$ is an algebraic extension of $\mathbb{R}$, hence we have either $k(x) = \mathbb{R}$ or $k(x)$ is isomorphic to the $\mathbb{R}$-algebra $\mathbb{C}$. The closed points x with $k(x) = \mathbb{R}$ are called the <u>real points</u> of X and the other closed points are called the <u>complex points.</u>

We denote the set of real points of X by γ. Let Z be an open subset of X and f an element of $\mathcal{O}(Z)$. Then f yields an $\mathbb{R}$-valued function

$$f: Z \cap \gamma \to \mathbb{R}$$

defined by

$$f(x) := \text{image of } f \text{ in } k(x).$$

We equip the set γ with the coarsest topology such that $\gamma \cap Z$ is open for every open subset Z of X and f is continuous for every f in $\mathcal{O}(Z)$ with respect to **the usual** topology on $\mathbb{R}$. For a discussion of this topology the reader may consult [S, p.309 ff].

An important theorem of Whitney (cf.[M, Appendix A]) states that the topological space γ has only <u>finitely many connected components</u>. We denote these components by $\gamma_1,\dots,\gamma_r$ and put $r = 0$ if γ is empty.

The Witt ring $W(\mathbb{R})$ is isomorphic to $\mathbb{Z}$. Thus every real point x of X yields a signature

$$\tau_x : W(X) \to W(k(x)) \xrightarrow{\sim} \mathbb{Z}.$$

Here the first arrow is the natural homomorphism induced by the inclusion morphism from $\mathrm{Spec}(k(x))$ to X. The second arrow is the unique isomorphism from $W(k(x))$ to $\mathbb{Z}$. A more explicit description of τ_x is as follows: Let E be a bilinear space over X. We choose some diagonalization of the stalk E_x, which is a space over $\mathcal{O}_x$,

$$E \cong \langle f_1,\dots,f_n\rangle$$

with units f_i of $\mathcal{O}_x$. The value of τ_x at the Witt class $\{E\}$ is

$$\tau_x(E) = \sum_{i=1}^{n} \operatorname{sign} f_i(x)$$

with $\operatorname{sign} f_i(x)$ denoting the sign ± 1 of the real number $f_i(x) \neq 0$.

<u>Lemma.</u> For any bilinear space E over X the $\mathbb{Z}$-valued function $x \mapsto \tau_x(E)$ on γ is locally constant.

<u>Proof.</u> Let E be a fixed bilinear space over X and x be a fixed point on γ. There exists an open neighbourhood Z of x in X and units $f_1,\dots,f_n$ in $\mathcal{O}(Z)$ such that

$$E|Z \cong \langle f_1,\dots,f_n\rangle.$$

The set W consisting of all points y in $Z \cap \gamma$ with sign $f_i(y) =$ $=$ sign $f_i(x)$ for $1 \leq i \leq n$ is an open neighbourhood of x in γ on which our function $y \mapsto \tau_y(E)$ is constant.

As a consequence of this lemma we have $\tau_y = \tau_x$ for any points x and y on the same component γ_i. Thus we define signatures $\tau_1,\dots,\tau_r$ of X by $\tau_i = \tau_x$ with x arbitrary in γ_i. Our main result now is the following

<u>Theorem 1.</u> Let X be a divisorial algebraic scheme over Γ. Then $\tau_1,\dots,\tau_r$ are the only signatures of X. In particular X is non real if and only if γ is empty.

<u>Remark.</u> I do not know whether always $\tau_i \neq \tau_j$ for $i \neq j$.

For the proof of this theorem it suffices to show:
(*) Let σ be a given signature of X and let $\xi_1,\dots,\xi_m$ be finitely many elements of W(X). Then there exists some index k, $1 \leq k \leq r$, such that

$$\sigma(\xi_i) = \tau_k(\xi_i)$$

simultaneously for $1 \leq i \leq m$.

Indeed, suppose σ is different from all the signatures τ_i. Then choose for every i, $1 \leq i \leq r$, an element ξ_i with $\sigma(\xi_i) \neq \tau_i(\xi_i)$. For this set $\xi_1,\dots,\xi_r$ the assertion (*) would be wrong.

We now start the proof of (*). As explained in § 1 there exists a factorization

$$\begin{tikzcd} W(X) \arrow[rr, "\sigma"] \arrow[dr] & & \mathbb{Z} \\ & W(F) \arrow[ur, "\tilde{\sigma}"'] & \end{tikzcd}$$

with F the residue class field k(y) at some point y of X and $W(X) \to W(F)$ the natural homomorphism, mapping a Witt class $\xi = \{E\}$ to the class $\{E(y)\}$ of the fibre E(y) at y. We denote this fibre class here by $\xi|F$.

Let Y be the closure $\overline{\{y\}}$ of y in X. We equip Y with the unique structure sheaf $\mathcal{O}_Y$ such that Y becomes a reduced closed subscheme of X. This subscheme is irreducible with generic point y and has the function field F.

We now choose a non empty affine open subset Z of Y which does not contain any singular points. This is possible since the singular set of Y is known to be closed [EGA,IV 6.12.6]. The $\mathbb{R}$-algebra $A := \mathcal{O}_Y(Z)$ is finitely generated [EGA,I § 6.3] and has no zero divisors. F is the quotient field of A. We choose diagonalizations

$$\xi_i|F \cong \langle f_{i1},\ldots,f_{in_i}\rangle$$

over F with functions f_{ij} in $A\setminus\{0\}$. We introduce on F the ordering corresponding to the signature σ above. By a well known specialization theorem of Artin and Lang there exists a place $\lambda: F \to \mathbb{R} \cup \infty$ over $\mathbb{R}$ which does not map any member of a fixed set of generators of A to ∞ and moreover maps every f_{ij} to an element in $\mathbb{R}\setminus\{0\}$ of the same sign as f_{ij} has with respect to our ordering on F, cf. [L, Theorem 8] and for a proof using quadratic form techniques [K_5].

Let α denote the restriction of λ to A. This is a homomorphism from A to $\mathbb{R}$ over $\mathbb{P}$. The kernel of α is a maximal ideal of A which corresponds to a point x in $Z \cap \gamma$. We have $\alpha(f) = f(x)$ for every f in A. By our choice of α we obtain for every f_{ij}

$$\tilde{\sigma}(f_{ij}) = \operatorname{sign} f_{ij}(x),$$

hence

$$\sigma(\xi_i) = \tilde{\sigma}(\xi_i|F) = \sum_{j=1}^{n_i} \operatorname{sign} f_{ij}(x).$$

Our point x lies on a component γ_k. We want to show that $\tau_k(\xi_i)$ coincides with $\sigma(\xi_i)$ for $1 \leq i \leq m$. We have a natural factorization of τ_k as follows:

$$\tau_k : W(X) \to W(\mathcal{O}_{Y,x}) \to W(k(x)) \xrightarrow{\sim} \mathbb{Z}$$

Let ρ denote the composite of the second and the third arrow. ρ is a signature of $\mathcal{O}_{Y,x}$. Now observe that all f_{ij} lie in $\mathcal{O}_{Y,x}$ and that the space $\langle f_{ij},\dots,f_{in_i}\rangle$ over $\mathcal{O}_{Y,x}$ has the same image in W(F) as the element $\xi_i|\mathcal{O}_{Y,x}$, i.e. the natural image of ξ_i in $W(\mathcal{O}_{Y,x})$. This implies in $W(\mathcal{O}_{Y,x})$ an equation

$$(**) \qquad \xi_i|\mathcal{O}_{Y,x} = \{\langle f_{i1},\dots,f_{in_i}\rangle\} + \eta$$

with η nilpotent, according to the following theorem due to

Craven-Rosenberg-Ware [CRW].

<u>Theorem 2.</u> Let A be a regular local ring and F the quotient field of A. Every element in the kernel of the natural map from W(A) to W(F) is nilpotent.

This theorem will be proved in the next section. Applying ρ to the equation (**) we obtain

$$\tau_k(\xi_i) = \sum_{j=1}^{n_i} \rho(f_{ij}) = \sum_{j=1}^{n_i} \operatorname{sign} f_{ij}(x).$$

Thus indeed $\tau_k(\xi_i) = \sigma(\xi_i)$ for $1 \leq i \leq m$.

§ 3 Proof of the theorem of Craven-Rosenberg-Ware.

This is Theorem 2 at the end of the preceding section. To prove it we need the following tool.

Theorem $[K_6, \S\ 3]$.
Given a place $\lambda: F \to k \cup \infty$ from a field F to a field k there exists a unique additive map λ_* from the Witt ring $W(F)$ to $W(k)$ such that $\lambda_*(\langle a\rangle) = \langle\lambda(a)\rangle$ for any element a of F with $\lambda(a) \neq 0,\infty$, and $\lambda_*(\langle a\rangle) = 0$ if $\lambda(ac^2) = 0$ or ∞ for every c in F.

In other terms, if a square class $\langle a\rangle = aF^{*2}$ of F contains a unit b of the valuation ring of λ then the element $\langle a\rangle$ of $W(A)$ is mapped to $\langle\lambda(b)\rangle$. Otherwise $\langle a\rangle$ is mapped to zero.

For the proof of this theorem we recall from II § 4 that there exists a natural epimorphism

$$\Phi : \mathbb{Z}[G] \twoheadrightarrow W(F)$$

with G the group of square classes of F, and that we determined in II § 4 Theorem 2 a set of generators of the kernel $\mathfrak{R}$ of Φ as an additive group. *) Now define an additive map

$$\Lambda : \mathbb{Z}[G] \to W(k)$$

by prescribing the image $\Lambda(g)$ of a square class $g = aF^{*2}$ as follows: If g contains an element b with $\lambda(b) \neq 0,\infty$ put

*) Our theorem is trivial for F a finite field. Thus the generators of length 4 suffice.

$\Lambda(g) = \langle\lambda(b)\rangle$. Otherwiese put $\Lambda(g) = 0$. It is only an exercise to verify that Λ vanishes on the generators of $\mathcal{F}$. For the details cf. [K_6,p.289 f]. This gives the proof of the theorem.

The map λ_* is usually not a ring homomorphism but it is multiplicative on certain subrings of $W(F)$.

Lemma 1. Let A be a semilocal subring of F on which λ does not take the value ∞ (i.e. A is contained in the valuation ring of λ). Let $\mu : A \to k$ be the ring homomorphism obtained from λ by restriction to A. The triangle

$$\begin{tikzcd} W(A) \arrow[rr, "\kappa"] \arrow[dr, "\mu_*"'] & & W(F) \arrow[dl, "\lambda_*"] \\ & W(k) & \end{tikzcd}$$

with κ the natural map from $W(A)$ to $W(F)$ commutes.

Proof. It suffices to check the commutativity on the square classes $\langle a\rangle$ of $A (a \in A^*)$, since they generate $W(A)$. Since a is a unit of A we have $\lambda(a) = \mu(a) \neq 0$ hence indeed

$$(\lambda_* \circ \kappa)(\langle a\rangle) = \lambda_*(\langle a\rangle_F) = \langle\lambda(a)\rangle = \mu_*(\langle a\rangle).$$

Remark. Lemma 1 remains true for arbitrary subrings A of F but we do not need this.

The possibility to use these maps λ_* for a proof of the theorem of Craven-Rosenberg-Ware now comes from the following fact:

<u>Lemma 2.</u> Let A be a regular local ring with quotient field F and residue class field $A/\mathfrak{m} = k$. There exists a place $\lambda : F \to k \cup \infty$ which extends the natural epimorphism from A to k.

The proof is easy, cf. [K_6,p.285].

After these preliminaries our proof of the theorem of Craven-Rosenberg-Ware can be done. Let A be a regular local ring with quotient field F, and let ξ be an element of $W(A)$ with image $\xi|F = 0$ in $W(F)$. We have to show that ξ is nilpotent. This is clear if A is non real, since ξ has even rank. We assume since now that A is real.

Suppose ξ is not nilpotent. Then there exists a signature σ on A with $\sigma(\xi) \neq 0$, cf. II § 6. According to our factorization theorem in § 1 there exists a prime ideal $\mathfrak{p}$ of A and a signature τ of the residue class field $\overline{B} := A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ of the localization $B := A_{\mathfrak{p}}$ such that the triangle

$$\begin{tikzcd} W(A) \arrow[rr] \arrow[dr, "\sigma"'] & & W(\overline{B}) \arrow[dl, "\tau"] \\ & \mathbb{Z} & \end{tikzcd}$$

with canonical map from $W(A)$ to $W(\overline{B})$ commutes. Thus we have $\tau(\xi|\overline{B}) \neq 0$ for the image $\xi|\overline{B}$ of ξ in $W(\overline{B})$.

On the other hand, since B is again regular, there exists by Lemma 2 a place $\lambda : F \to \overline{B} \cup \infty$ which extends the natural map $B \to \overline{B}$. Applying Lemma 1 to the subring B of F and the element

$\xi | B$ we obtain.

$$\xi | \overline{B} = \lambda_*(\xi | F) = 0.$$

This contradicts the fact that $\tau(\xi | \overline{B}) \neq 0$, and the theorem of Craven-Rosenberg-Ware is proved.

Remark. Using the weak local global principle in III § 2 it is now clear that for any regular commutative ring A with quotient field F the kernel of the natural map from W(A) to W(F) contains only nilpotent elements [CRW].

It is an open question whether for A a regular local ring the natural map from W(A) to W(F) is actually injective. We shall see in the next section that this is true if dim A = 1, i.e. A is a discrete valuation ring, and more generally if A is any valuation ring.

§ 4 Curves over $\mathbb{R}$.

We first consider a regular connected curve X over an arbitrary ground field k. This means X is an algebraic irreducible scheme over k of dimension 1 all whose points have regular local rings. Thus the local ring $\mathcal{O}_\xi$ at the generic point ξ of X is a field F, the function field of X, and the local rings $\mathcal{O}_x$ at the points $x \neq \xi$, i.e. the closed points x of X, are discrete valuation rings. For Z an open set we regard as usual (cf. IV,§ 2) $\mathcal{O}(Z)$ as a subring of F. For Z affine $\mathcal{O}(Z)$ is a Dedekind domain.

We also regard the local rings $\mathcal{O}_x$ at the closed points x of X as subrings of F. For f an element in F^* we can speak at every point $x \neq \xi$ of the <u>order</u> $\operatorname{ord}_x f$ of f, which means the value of f under the normed valuation $\operatorname{ord}_x : F^* \to \mathbb{Z}$ associated with $\mathcal{O}_x$.

Life is easy over curves since we have the following

<u>Theorem 1.</u> Let E be a space over X with E_ξ metabolic. Then E itself is metabolic.

<u>Proof.</u> For every open set Z we regard the $\mathcal{O}(Z)$mmodule E(Z) as a subset of the vectorspace E_ξ over F. This is possible since X is irreducible. Let W be a Lagrangian of E_ξ. We then obtain an $\mathcal{O}$-submodule V of E by defining

$$V(Z) := W \cap E(Z)$$

for every open set Z. Now fix some affine open set Z_o. The $\mathcal{O}(Z_o)$-

module $E(Z_o)/V(Z_o)$ is finitely generated and torsion free. Since $\mathfrak{O}(Z_o)$ is Dedekind this implies that $E(Z_o)/V(Z_o)$ is projective. Thus we have a splitting

$$E(Z_o) = V(Z_o) \oplus P$$

with P some $\mathfrak{O}(Z_o)$-submoldule of $E(Z_o)$. Define on Z_o an $\mathfrak{O}_{Z_o}$-submodule U of $E|Z_o$ by

$$U(Z) := E(Z) \cap P$$

for Z open in Z_o. Then $E|Z_o$ is the direct sum of $V|Z_o$ and U. Since Z_o has been an arbitrary affine open subset of X we now know that V is a subbundle of E. The bilinear form of E is totally isotropic on V, since all sections of V are contained in W. Moreover $V_\xi = W$, hence

$$2 \text{ rank } V = 2 \text{ rank } W = \text{rk } E.$$

Thus V is a Lagrangian subbundle of E.

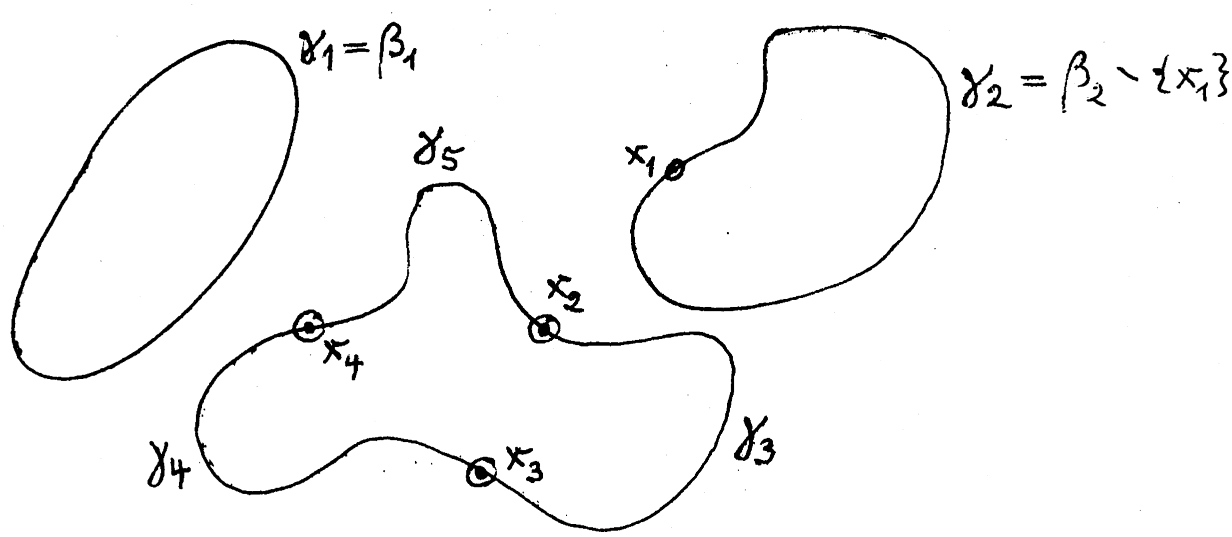
Corollary. The natural map from W(X) to W(F) is injective.

Proof. Let E be a space over X such that $E_\xi \sim 0$. Then E_ξ is metabolic. This is well known if the field F has characteristic $\neq 2$ (cf.II,§ 1), but is also true if char(F) = 2, cf.[MH,p.58] or [K,§ 8.2]. Thus E is metabolic, and a fortiori $E \sim 0$.

It should be mentioned that in our proof we never used the assumption that a ground field k is present. We only needed that X is irreducible and that for affine open subsets Z of X the rings $\mathfrak{O}(Z)$ are Dedekind domains. More generally it suffices to know that the $\mathfrak{O}(Z)$ are Prüfer domains, since the Prüfer domains are precisely the rings without zero divisors over which every finitely generated torsion free module is projective, cf.[CE, p.133]. In particular we have in the affine case the following result.

<u>Remark.</u> For every Prüfer domain A - in particular for a valuation ring - the natural map from W(A) to the Witt ring W(F) of the quotient field F of A is injective.

Assume now that our base field is $\mathbb{R}$, and let $\gamma_1,\dots,\gamma_r$ denote the components of the set γ of real points of X. (We allow γ to be empty. Then $r = 0$.) X may be regarded as an open set of a <u>complete</u> regular connected curve Y, as is well known. This allows us to visualize the γ_i. Indeed the set β of real points of Y is a compact C^∞-manifold of dimension 1, cf. [S,p.89], hence a disjoint union of "circles" in the C^∞-sense. We obtain X from Y by omitting finitely many points, hence also γ from β by omitting finitely many points. Here is a picture where β consists of three components β_1,β_2,β_3, and γ consists of five components.



We shall use Witt's classification of bilinear forms over the function field F in his fundamental paper [W]. For every real point x of X we denote by λ_x the canonical place associated with the valuation ring $\mathcal{O}_x$,

$$\lambda_x : F \to (\mathcal{O}_x/m_x) \cup \infty = \mathbb{R} \cup \infty.$$

As explained in the preceding § 3, λ_x yields an additive map

$$(\lambda_x)_* : W(F) \to W(\mathbb{R}) \xrightarrow{\sim} \mathbb{Z}.$$

<u>Theorem 2.</u> [W, Satz 23].
Let E_1 and E_2 be bilinear spaces over F with same determinant, same rank index, and $(\lambda_x)_*(E_1) = (\lambda_x)_*(E_2)$ for almost all x in γ. Then $E_1 \sim E_2$.

{Actually Witt proves a much stronger theorem [W,Satz 22] dealing with isotropy instead of equivalence.} Now let $\tau_1,\ldots,\tau_r$ denote the signatures of X corresponding to $\gamma_1,\ldots,\gamma_r$. For E a space over X and x a point of γ_i we have

$$(*) \qquad \tau_i(E) = (\lambda_x)_*(E_\xi).$$

Indeed, choose a diagonalisation

$$E_x \cong \langle f_1,\ldots,f_n\rangle$$

over $\mathcal{O}_x$, all f_i in $\mathcal{O}_x^*$. This is possible since 2 is a unit in $\mathcal{O}_x$. Then by definition of τ_i and $(\lambda_x)_*$ both sides of (*) coincide with

$$\sum_{i=1}^{n} \operatorname{sign} f_i(x)$$

Since W(X) injects into W(F) we obtain from Witt's classification theorem

<u>Theorem 2':</u> Let z_1 and z_2 be elements of $W(X)$ with $d(z_1) = d(z_2)$, $\nu(z_1) = \nu(z_2)$, and $\tau_i(z_1) = \tau_i(z_2)$ for $1 \leq i \leq r$. Then $z_1 = z_2$.

Assume now f is an element of F^* which has even order everywhere on X. Then except zeros and poles f has on each γ_i a constant sign $\varepsilon_i = \pm 1$, since at a point of sign change f would have odd order. Witt proved the following

<u>Theorem 3.</u> ([W_1], cf. also [G])
Given on each γ_i a sign $\varepsilon_i = \pm 1$ there exists a function f which has even order everywhere on X and sign ε_i on γ_i for $1 \leq i \leq r$.

Actually Witt only shows the existence of such functions which have even order everywhere on γ. But it is easy to change his functions by multiplication with a suitable sum of two squares into functions which in addition have even order everywhere on X, cf.[K_4,I Prop.2.4].

We choose for every i, $1 \leq i \leq r$ a function $f_i \neq 0$ which has even order everywhere on X, sign -1 on γ_i and sign $+1$ on all other γ_j. Let $\mathcal{L}_i$ denote the square class $\mathcal{L}(f_i)$ constructed in IV, § 2. We clearly have

$$\tau_j(\mathcal{L}_i) = \delta_{ij}, \qquad 1 \leq i,\ j \leq r.$$

We now can give a very explicit description of $W(X)$. For convenience we describe the ideal $I(X)$ of spaces of even rank, which means nearly the same, since $W(X) = \mathbb{Z}\langle 1\rangle + I(X)$. Let $I_t(X)$

denote the torsion part of $I(X)$.

Theorem 4.

i) $I(X) = I_t(X) \oplus \bigoplus_{i=1}^{r} \mathbb{Z}(1-\mathcal{L}_i)$.

ii) $I_t(X)^2 = 2I_t(X) = 0$,

$I_t(X)(1-\mathcal{L}_i) = 0$ for $1 \leq i \leq r$,

$(1-\mathcal{L}_i)(1-\mathcal{L}_j) = 2\delta_{ij}(1-\mathcal{L}_i)$ for $1 \leq i,\ j \leq r$.

iii) The map $\mathcal{L} \to 1-\mathcal{L}$ from the group $Q^+(X)$ of totally positive square classes (cf.IV,§ 4) to $I(X)$ is an additive isomorphism from $Q^+(X)$ onto $I_t(X)$.

Proof. Let z be an element of $I(X)$. Then all signatures $\tau_i(z)$ are even numbers $2n_i$. Write

$$(*) \qquad z = \sum_{i=1}^{r} n_i(1-\mathcal{L}_i) + u$$

with some other element u. Then $\tau_i(u) = 0$ for $1 \leq i \leq r$, hence u is nilpotent. Since $I(X)$ injects into $I(F)$ and the nilpotent elements of $I(F)$ are torsion (II,§ 6), we see that u lies in $I_t(X)$. Moreover (*) is clearly the unique possible equation with last term u in $I_t(X)$. This proves assertion (i). The second assertion is now verified checking every equation by the basic invariants $d, \nu, \tau_1, \ldots, \tau_r$. If $\mathcal{L}$ is a totally positive square class, then $1-\mathcal{L}$ is nilpotent (a general fact, cf.IV,§ 4), and thus by the same argument as above $1-\mathcal{L}$ lies in $I_t(X)$ (actually again a general fact, cf. Appendix 3). If $\mathcal{L}'$ is a second totally positive square class then by use of the basic invariants we check that

$$(1-\mathcal{L}) + (1-\mathcal{L}') = 1 - \mathcal{L}\mathcal{L}'.$$

Thus our map from $Q^+(X)$ to $I_t(X)$ is indeed additive. It remains to be shown that this map is onto. Let u be an element of $I_t(X)$. As has been proved in IV,§ 4 the square class d(u) is totally positive. The elements u and 1-d(u) have the same basic invariants and thus are equal. This finishes the proof.

Starting from the exact sequence for Q(X) in IV,§ 1 it is easy to show that the group Q(X) is finite, cf. [K_4,I § 2]. Clearly $Q^+(X)$ has index 2^r in Q(X). One would like to know the precise order of $Q^+(X)$. The answer is known in the case that X is complete, cf.[Al,Th.5.9] and [K_4,II Th.10.12]:

Theorem 5. If γ is not empty then $Q^+(X)$ has order 2^g with g the genus of X. If γ is empty then $Q^+(X)$ has the order 2^{g+1}, provided $\mathbb{R}$ is the precise field of constants of X, i.e. F does not contain $\sqrt{-1}$.

Our computation of W(X) can be generalized to the case that the base field k is real closed, cf. § 10 of [K_4,II].

Exercise. Assume the base field k is an algebraic closed field of arbitrary characteristic. Show that $I(X)^2 = 0$ and that the map $\mathcal{L} \to 1 - \mathcal{L}$ gives an isomorphism of abelian groups from Q(X) onto I(X). Show that W(X) consists of 2^{2g+1} elements.

Hint. Use the fact that an element of W(F) is determined by rank index and determinant, since every space of rank 3 over F is isotropic.

* Appendix 1. Level and height of a non real commutative ring.

The level $s(A)$ of a commutative ring A is defined as the least natural number s such that -1 can be written as a sum of s squares in A. If -1 is not a sum of squares in A we put $s(A) = \infty$. The height $h(A)$ of A is defined as the least natural number h with $h \cdot W(A) = 0$, i.e. with $h \times \langle 1 \rangle \sim 0$ over A. If $W(A)$ is not a torsion group we put $h(A) = \infty$. We have seen in III § 2 that

$$(*) \qquad s(A) < \infty \Leftrightarrow h(A) < \infty.$$

We called the rings A with $h(A) < \infty$ non real. We have also seen in III § 8 that the height of a non real ring is always a 2-power. In this appendix we prove in the case that 2 is a unit in A inequalities involving $s(A)$ and $h(A)$ which improve the purely qualitative result (*).

We make use of three theorems not proved in these lectures but well known. We assume that the space $\max(A)$ of maximal ideals of A is noetherian and has finite dimension d. All occuring modules are assumed to be finitely generated.

Theorem 1. (J.P.Serre, cf. [B,Prop.10.1])
If U is a projective A-module of rank $\geqslant d+1$ then U has a decomposition $U \cong A \oplus U'$.

Theorem 2. (A.Roy [R])
Assume E_1, E_2, E are bilinear spaces over A with

$$E_1 \perp E \cong E_2 \perp E.$$

Assume further that E_1 has a hyperbolic subspace $H(U)$ with U projective of rank $\geqslant d+1$, and that 2 is a unit in A. Then $E_1 \cong E_2$.

<u>Theorem 3.</u> (Baeza [Ba,p.125]) *)
If $d = 0$, i.e. A is semilocal, and 2 is a unit in A then $h(A) = 2s(A)$.

<u>Remark.</u> If A is semilocal but 2 is not a unit in A the following is known: Let $h(A) = 2^t < \infty$. Then $s(A)$ is one of the four numbers 2^{t-1}, $2^{t-1}-1$, 2^t, 2^t-1, cf. [K_2,§ 3] and [Ba_1].

Assume now $d > 0$ and let δ denote the natural number with

$$2^{\delta-1} \leqslant d < 2^{\delta}.$$

Our first statement is the following

<u>Theorem 4.</u> If 2 is a unit in A then

$$s(A) \leqslant \mathrm{Max}(h(A), 2^{\delta+1}).$$

<u>Proof.</u> Put

$$\mathrm{Max}(h(A), 2^{\delta+1}) = 2^n.$$

Then $2^n \times \langle 1 \rangle \sim 0$ over A and $2d < 2^n$. We have in $L(A)$

*) The proof of this theorem in [Ba] can be greatly simplified by use of § 2 of the paper "Annullatoren von Pfisterformen", Math. Z. 140 (1974) by Baeza and myself.

$$[2^n \times \langle 1\rangle] = [H(U)]$$

with some projective A-module U of rank $2^{n-1} > d$. Using Theorem 2 we obtain

$$2^n \times \langle 1\rangle \cong H(U).$$

According to Theorem 1 the space H(U) has a subspace isomorphic to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong \langle 1,-1\rangle.$$

Thus -1 is a sum of 2^n squares in A, q.e.d.

We now give a bound from above for h(A) in terms of s(A). Slightly more generally we consider a quasiprojective scheme X such that the ring $A := \mathcal{O}(X)$ of global functions on X has finite level. Then of course all local rings $\mathcal{O}_x$ have finite levels and thus X is non real. Let h(X) denote the height of X, i.e. the smallest natural number h with $h \cdot W(X) = 0$. We know from III, § 2 that h(X) is a 2-power. We assume that the space M(X) of closed points of X is noetherian and of finite dimension d.

<u>Theorem 5.</u> If 2 is a unit in $A := \mathcal{O}(X)$ then

$$h(X) \leqslant (2s(A))^{d+1}.$$

<u>Proof.</u> We use the contents of III,§ 8. Let

$$2^{r-1} \leqslant s(A) < 2^r.$$

For every finite set S of closed points in X the level of the semilocal ring $\mathfrak{o}_S$ is bounded from above by $s(A)$, hence by 2^{r-1} according to Theorem 3. By the same theorem

$$2^r \times \langle 1 \rangle \sim 0 \qquad \text{over } \mathfrak{o}_S$$

for every set S as above. According to III,§ 8 this implies

$$2^{r(d+1)} \times \langle 1 \rangle \sim 0 \quad \text{over } X,$$

hence

$$h(A) \leq 2^{r(d+1)} \leq (2s(A))^{d+1}.$$

A similar result can be proved for 2 not a unit by use of the remark above on the levels of arbitrary semilocal rings.

* Appendix 2. The prime ideals of L(X).

Let X be a connected divisorial scheme. We proved in Chapter III the prime ideal theorem not only for W(X) but also for L(X) (cf. III § 5, Th.2). In fact we proved it for L(X) first. Thus it is possible to determine the prime ideals of L(X) in the same way as we did for W(X) in III § 2. Notice that for L(X) we are always in the "real case", since the rank homomorphism

$$\mathrm{rk} : L(X) \to \mathbb{Z}$$

always exists. We denote the kernel of the rank homomorphism by $L(X)^{\circ}$. We further denote for a signature σ of X the induced homomorphism from L(X) to $\mathbb{Z}$ by $\tilde{\sigma}$,

$$\tilde{\sigma}: L(X) \to W(X) \xrightarrow{\sigma} \mathbb{Z}$$

Theorem.

i) The homomorphisms $\tilde{\sigma}$ and the rank homomorphism are precisely all homomorphisms from the ring L(X) to $\mathbb{Z}$.

ii) For every prime ideal P of L(X) with $P \cap \mathbb{Z} = \{0\}$ there exists a unique homomorphism φ from L(X) to $\mathbb{Z}$ such that P coincides with the kernel P_{φ} of φ.

iii) For every prime ideal M of L(X) with $M \cap \mathbb{Z} = p\,\mathbb{Z}$, p an odd prime number, there exists a homomorphism φ from L(X) to $\mathbb{Z}$ such that M coincides with

$$M_{\varphi,p} := \varphi^{-1}(p\,\mathbb{Z}) = P_{\varphi} + p\,\mathbb{Z}.$$

iv) The prime ideal

$$M_o := L(X)^o + 2\mathbb{Z},$$

consisting of all elements of even rank, is the unique prime ideal M of $L(X)$ with $M \cap \mathbb{Z} = 2\mathbb{Z}$.

<u>Proof.</u> Let P be a prime ideal of $L(X)$. By the prime ideal theorem there exists a closed point x of X such that $L(\mathfrak{o}_x)$ has a prime ideal Q lying over P.

Assume first P does not contain the space $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_X$ over X. Then Q does not contain the space $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\mathfrak{o}_x}$ over $\mathfrak{o}_x$. For every space E over $\mathfrak{o}_x$ we have

$$E \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\mathfrak{o}_x} \cong H(E) \cong (\mathrm{rk} E) \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\mathfrak{o}_x}.$$

Thus

$$\xi \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = (\mathrm{rk}\,\xi)\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

for every ξ in $L(\mathfrak{o}_x)$. If $Q \cap \mathbb{Z} = \{0\}$ then we learn from this equation that ξ lies in Q if and only if $\mathrm{rk}\,\xi = 0$. If $Q \cap \mathbb{Z} = p\,\mathbb{Z}$, p an arbitrary prime number, then we learn that ξ lies in Q if and only if $\mathrm{rk}\,\xi$ is divisible by p. Thus P is either the ideal P_φ or the ideal $\varphi^{-1}(p\,\mathbb{Z})$ with φ the rank homomorphism.

Assume now that P contains the space $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_X$. Then Q contains the space $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\mathfrak{o}_x}$, and according to Chapter II, § 1 the ideal Q is the inverse image of a prime ideal $\overline{Q}$ of $W(\mathfrak{o}_x)$.

Let $\overline{P}$ denote the preimage of $\overline{Q}$ in W(X). Then P is the preimage of the prime ideal $\overline{P}$ of W(X) in L(X).

Now our theorem immediately follows from the description of the prime ideals of W(X) in § 2 of Chapter III.

<u>Corollary.</u> An element ξ of L(X) is nilpotent if and only if $\mathrm{rk}\,\xi = 0$ and $\sigma(\xi) = 0$ for every signature σ of X.

<u>Example.</u> Let E be a metabolic space over X of rank 2m. Then the element

$$[E] - m\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

of L(X) is nilpotent.

* Appendix 3. Abstract Witt rings.

Let R be a commutative ring with $R \neq 0$. Assume there is given a ring epimorphism

$$\Phi : \mathbb{Z}[G] \longrightarrow\!\!\!\!\!\rightarrow R$$

with G a group of exponent 2. Let $\mathfrak{K}$ denote the kernel of Φ.

Definition. R is called an abstract Witt ring with respect to Φ if for every character $\chi : \mathbb{Z}[G] \to \mathbb{Z}$ (cf. II § 5) either $\chi(\mathfrak{K}) = 0$ or $\chi(\mathfrak{K})$ is a power $2^{n(\chi)}\mathbb{Z}$ with $n(\chi) \geq 0$.

If R is an abstract Witt ring with respect to Φ then actually all occuring exponents $n(\chi)$ are ≥ 1. Indeed, if $\chi_1(\mathfrak{K})$ is not contained in $2\mathbb{Z}$ for some character χ_1 then $\chi(\mathfrak{K}) = \mathbb{Z}$ for every character χ, since $\chi(z) \equiv \chi_1(z) \bmod 2$ for all z in $\mathbb{Z}[G]$. According to the prime ideal theory of $\mathbb{Z}[G]$ (cf. II § 5 Prop.1) this implies $\mathfrak{K} = \mathbb{Z}[G]$ which is impossible.

Notice that our definition of abstract Witt rings is an axiomatization of Lemma 2 in II § 5. In this section we developed the prime ideal for the Witt ring W(A) of a semilocal ring A using only the presentation of W(A) as a homomorphic image of the group ring $\mathbb{Z}[Q(A)]$ in II § 4 and this lemma. Thus the prime ideal theory for W(A) in II § 5 can be transfered to the present more axiomatic setting, and we obtain the following theorem.

Theorem 1. Assume R is an abstract Witt ring with respect to Φ. Let $\overline{M}_o$ denote the image $\Phi(M_o)$ of the unique prime ideal M_o of

$\mathbb{Z}[G]$ with $\mathbb{Z}[G]/M_o \cong \mathbb{Z}/2\mathbb{Z}$ (cf. II § 5 Prop.1).

a) $\overline{M}_o$ is the unique prime ideal of R which contains $2\cdot 1_R$.

b) For any prime ideal P of R which does not contain $p\cdot 1_R$ for any prime number p there exists a unique homomorphism $\sigma: R \to \mathbb{Z}$ such that P coincides with the kernel P_σ of σ.

c) Let p be an odd prime. Then for any prime ideal M of R containing $p\cdot 1_R$ there exists a unique homomorphism $\sigma: R \to \mathbb{Z}$ such that M coincides with the ideal

$$M_{\sigma,p} := p\,\mathbb{Z} + P_\sigma$$

consisting of all z in R with $\sigma(z) \equiv 0 \mod p$.

We also have an obvious analogue of Proposition 3 in II § 5 ("non real" abstract Witt rings).

Moreover the results about torsion and nilpotent elements in II § 6 can be transfered to abstract Witt rings, since they are entirely based on the prime ideal theory in II § 5. A more thorough investigation, done in [KRW], yields the following <u>characterizations</u> of abstract Witt rings.

<u>Theorem 2.</u> [KRW , § 3]
The following are equivalent.

i) R is an abstract Witt ring with respect to Φ.
ii) $\mathfrak{R} \subset M_o$ and all zero divisors of R lie in $\overline{M}_o$.
iii) All torsion elements of R are 2-primary.

iv) All torsion elements of R are nilpotent.

v) If P is a prime ideal of R containing $p\cdot 1_R$ for some odd prime number p then P is <u>not</u> a minimal prime ideal.

In connection with condition (iv) we mention that a priori all nilpotent elements of R are torsion elements as a consequence of Maschke's theorem, cf. the proof of Prop.2 in II § 6.

Notice that the condition (iii), (iv) and (v) in Theorem 2 do not depend on the given epimorphism Φ. Thus we learn that if $\Phi':\mathbb{Z}[G'] \twoheadrightarrow R$ is a second epimorphism with G' again of exponent 2 then R is an abstract Witt ring with respect to Φ if and only if R is an abstract Witt ring with respect to Φ'. Consequently we call since now a commutative ring R an <u>abstract Witt ring</u> if there exists an epimorphism $\Phi:\mathbb{Z}[G] \twoheadrightarrow R$ with G of exponent 2 and if the equivalent conditions (i) - (v) in Theorem 2 are fulfilled. The existence of an epimorphism Φ means that R is generated by the set of all x in R with $x^2 = 1$.

The theory of abstract Witt rings is a comfortable language to have a unified description of various phenomena in rings which are relatives of the classical Witt rings. For some examples see [KRW ,§ 1]; W.Scharlau "Quadratische Formen und Galois-Cohomologie", Invent.Math. 4 (1967); A.A.Belskii, "Cohomological Witt rings", Izv.Akad.Nauk SSSR Ser.Math.32 (1968), 1147-1161 = Math.USSR Izv. 2 (1968), 1101 - 1115. I mention here still another example relevant to our global theory.

For X an arbitrary scheme we denote by $W'(X)$ the subring of $W(X)$ generated by the set $Q(X)$ of square classes, which we consider as in IV § 4 a subset of $W(X)$. Recall that $Q(X)$ is a group of exponent 2.

Theorem 3. If X is divisorial then $W'(X)$ is an abstract Witt ring.

Proof. We verify condition (v) in Theorem 2. Suppose P is a minimal prime ideal of $W'(X)$ containing $p\cdot 1_{W(X)}$ for some odd prime number p. By general commutative algebra the overring $W(X)$ of $W'(X)$ contains a minimal prime ideal Q with

$$Q \cap W'(X) = P.$$

But we have seen in III § 2 that no minimal prime ideal of $W(X)$ contains $p\cdot 1_{W(X)}$. Thus an ideal P as above cannot exist.

Here is an application of our theory.

Corollary. Let Ω be a totally positive square class of X (cf. IV § 4). Then we have in $W(X)$ an equation

$$2^n(1-\Omega) = 0$$

with some $n \geqslant 0$.

Indeed, $1-\Omega$ is a nilpotent element of the abstract Witt ring $W'(X)$ and hence a 2-torsion element.

Guide to the literature: Global theory.

We leave aside the classical arithmetic theory (bilinear modules over rings of algebraic numbers and over rings of algebraic functions in one variable over finite fields). We also do not deal with the many papers written in recent years about quadratic and hermitian forms over integral group rings of finite groups, since we regard this area a new province of the arithmetic theory. We then can safely say that the global theory of bilinear spaces as well as of quadratic spaces (not studied in our lectures) still is in the state of infancy with an adequately meager literature.

There exists a cancellation theorem for quadratic spaces over commutative rings with a noetherian maximal spectrum of finite dimension (and also certain rings with involution), due to A.Bak, which is an analogue of Serre's well known cancellation theorem for projective modules, and which generalizes the theorem of A.Roy used in Appendix 1. For this cancellation theorem see Chapter IV of the article "Unitary algebraic K-theory" by H.Bass in volume III of the proceedings of the Battelle conference on algebraic K-theory 1972 (Springer Lecture Notes vol. 343). Chapter III of the same article contains a unitary Mayer-Vietoris sequence and other useful exact sequences.

M.Karoubi has developed in [Kr] two localization sequences for Grothendieck-Witt and Witt groups of quadratic spaces over commutative rings (more generally rings with involution). According to this theory for example the Witt ring $W(A[x_1,x_1^{-1},\ldots,x_n,x_n^{-1}])$ of the Laurent ring in n variables x_i over a regular commutative ring A with 2 invertible is a free $W(A)$-module with the square classes $\langle x_1\rangle,\ldots,\langle x_n\rangle$ as basis, cf. [Kr II], Th.3.11. More recently W.Pardon has written a useful paper on localization sequences: "The exact sequence of a localization for Witt groups" (to appear, preprint Columbia Univ.).

In the papers cited up to now beside the Grothendieck-Witt groups $L_o(A)$ and the Witt groups $W_o(A)$ of quadratic spaces over A also groups $L_1(A)$ and $W_1(A)$ play an important role (in the article of Bass also certain L_2 -groups, there called KU_2). M.Karoubi, C.T.C.Wall, A.S.Miščenko, A.Ranicki, and others have developed over rings with involution theories of groups $L_i(A)$, $W_i(A)$ for quadratic spaces (Karoubi, Wall, Ranicki) and bilinear spaces (Miščenko, Ranicki) with arbitrary positive and in some papers negative indices i. The articles of Karoubi, Wall, and Ranicki in volume III of the Battelle conference on algebraic K-theory, cited above, and a recent paper of Ranicki "The algebraic theory of surgery" (to appear, preprint Univ. Cambridge) give a good impression of the state of art in this area. Since few explicit computations for commutative rings of dimension $\geqslant 2$ have been done on the basis of any of these theories it is not yet clear how useful they are to obtain information about our Witt rings $W(A)$, even if 2 is invertible. But the existence of reasonable exact sequences connecting our rings $W(A)$ and $L(A)$ with these higher Witt groups and L-groups is encouraging.

Concerning the theory of bilinear spaces on a complete algebraic scheme X over a field k of characteristic $\neq 2$ there exists a very readable paper "Quadratische Formen in additiven Kategorien" by H.G.Quebbemann, R.Scharlau, W.Scharlau, and M.Schulte (to appear, preprint Univ. Münster).*) From this axiomatic paper it follows that over these schemes X the Krull-Schmidt theorem holds true for bilinear spaces and hence also the cancellation law. Moreover a rather explicit description of the bilinear spaces over X can be extracted from the paper. In the case that k is algebraically closed this description boils down to an old theorem of Grothendieck stating that any two non degenerate symmetric bilinear forms on a given vector bundle over X are isomorphic, cf. A.Grothendieck, "Sur la classification des fibres holomorphes sur la sphère de Riemann" p.130 (Amer. J. Math. 79, 1957).

For X a smooth connected <u>curve</u> over a field k with function field F there exists an exact sequence

$$0 \to W(X) \to W(F) \xrightarrow{\partial} \bigoplus_x W(k(x))$$

with x running through all closed points of X, cf. [K,§ 13] and for X affine [MH,Chap.IV § 3]. This sequence demonstrates the importance of our global Witt rings W(X) for the theory of W(F) if F is a function field in one variable. For x complete the cokernel of ∂ has been explicitly computed if k is algebraically closed (easy), if k is real closed [K_4,II § 11], and if

*) W.Scharlau gave a talk on this material at the conference.

k is finite (an exercise using the arguments in the paper "Quadratische Formen und quadratische Reziprozitätsgesetze über algebraischen Zahlkörpern" by W.Scharlau and myself, Math.Z. 121, 1971). For X affine and characteristic $\neq 2$ Karoubi has given a general description of the cokernel of ∂, cf. Theorem 2.8 in [Kr II]. A general "sum formula" about the image of ∂ for X complete can be found in the paper "Ein Residuensatz für symmetrische Bilinearformen" by W.D.Geyer, G.Harder, W.Scharlau and myself (Invent.math.11, 1970).

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