DMV SEMINAR 1

Algebraic Theory of Quadratic Forms

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Notes taken by
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1. Introduction to quadratic forms and Witt rings.

We start by recalling some of the basic results in quadratic forms theory, which will motivate much of the material we would like to cover.

Throughout these lectures, a field always means a field of characteristic different from 2. Let k be a field. A (n-ary) quadratic form over k is by definition a homogeneous polynomial of degree 2

$$q(x) = q(x_1,\dots,x_n) = \sum_{i,j} a_{ij}\, x_i x_j \, , \; a_{ij} \in k.$$

The number n is often called the dimension of q, written $n = \dim q$. Since char $k \neq 2$, q corresponds uniquely to a symmetric square matrix (a_{ij}) and this matrix determines the bilinear space (V,b) by defining $b(e_i, e_j) = a_{ij}$ for a basic $\{e_1,\dots,e_n\}$ of V. One can see easily that there is a one-to-one correspondence between the isomorphic classes of quadratic forms (p is isomorphic to q, $p \cong q$, if $p(x) = q(Cx)$ for some invertible matrix C) and the isometry classes of bilinear spaces ((V,b) is isometric to (V', b') if there is a linear isomorphism $\tau : V \to V'$ such that $b'(\tau(x), \tau(y)) = b(x,y)$ for all $x,y \in V$).

In the sequel we often use the matrix notation to describe a bilinear space and a quadratic form. We also say simply form instead of quadratic form.

Since any form can be orthogonally decomposed into a nonsingular form (det $(a_{ij}) \neq 0$) and a form which is identically equal zero, we shall only consider nonsingular forms.

Remark 1.1. Any form can be diagonalized , i.e. $q \cong a_1 x_1^2 + \dots + a_n x_n^2$, and then we shall write $q \cong \langle a_1,\dots,a_n \rangle$. If all $a_i = 1$ then we further write $q \cong n \times \langle 1 \rangle$.

Now we state the following theorem of Witt, which is considered as the most central result in quadratic forms theory over fields:

Witt cancellation theorem 1.2. If $q \perp q_1 \cong q \perp q_2$ then $q_1 \cong q_2$.

The proof can be found in Lam [7] among many other sources.

Definition 1.3 Let (V,b) be a bilinear space. We say V is isotropic if there exists a nonzero vector $u \in V$ such that $b(u,u)=0$. Otherwise V is called anisotropic.

The simplest example of an isotropic space is $<1,-1>$ and the isometry class of such forms is called a hyperbolic plane and is denoted by H. An orthogonal sum $r \times H$ of r copies of H is called a hyperbolic space. The following fundamental theorem of Witt shows that an arbitrary quadratic form decomposes into an anisotropic and a hyperbolic space.

Witt decomposition theorem 1.4 Every form g has an orthogonal decomposition $g \cong rxH \perp q_o$ with q_o anisotropic. Moreover q_o (up to isometry) and r are uniquely determined by q.

Such a decomposition is called Witt decomposition. We call q_o the anisotropic part or kernel form of q and r the index of q. We also write $q_o = \ker(q)$ and $r = \mathrm{ind}(q)$.

Definition 1.5 Let q_1 and q_2 be forms. We say $q_1 \sim q_2$ (Witt equivalent) if $\ker(q_1) \cong \ker(q_2)$. Let W(k) denote the set of all equivalence classes of forms over k with respect to this equivalence relation $\sim$. Define an addition on W(k) by $[p] + [q] = [p \perp q]$ and a product $[p]\cdot[q] = [p \otimes q]$. Then these operations are well-defined (straight forward) and it is easily checked that W(k) is a commutative ring with identity given by $[<1>]$, 0 element = class of hyperbolic spaces, and additive inverse of $[<a_1,\ldots,a_n>] = [<-a_1,\ldots,-a_n>]$. W(k) is called the Witt ring of k.

We now mention several fundamental problems.

Problem 1. When is a form φ isotropic? Or equivalently, when is $\varphi \cong H \perp \Psi$ with some form Ψ?

Problem 2. When is a form φ hyperbolic? This is related to the question when two forms φ and Ψ are isomorphic, since $\varphi \perp \langle -\Psi \rangle$ is hyperbolic if and only if $\varphi \cong \Psi$.

Problem 3. How can one determine $W(k)$?

Usually, Problem 2 is much easier than Problem 1. Problem 3 is apparently very difficult in general and we may consider the following problem concerning the behaviour of the forms under field extension.

Problem 4. Let L be an extension of k. Any form φ over k may be viewed as a form over L and will then be denoted by φ_L or $\varphi \otimes L$. The map $[\varphi] \to [\varphi_L]$ is a well defined ring homomorphism $W(k) \to W(L)$. What is the kernel of $W(k) \to W(L)$?

We consider an arbitrary quadratic extension.

Proposition 1.6 Let $L = k(\sqrt{d})$. Then $\ker(W(k) \to W(L)) = \langle 1,-d \rangle W(k)$.*)

Proof. Clearly, $\langle 1,-d \rangle W(k) \subset \ker(W(k) \to W(L))$, since $\langle 1,-d \rangle \otimes L \cong H$. To prove the other direction let $\varphi = \langle b_1,\dots,b_n \rangle$ be an anisotropic form over k which becomes isotropic over L. Then $\Sigma b_i (x_i + y_i\sqrt{d})^2 = 0$, where x_i, $y_i \in k$ and not both $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n)$ are zero. Considering the rational and irrational parts,

$$\Sigma b_i x_i^2 + \Sigma b_i d y_i^2 = 0 \quad \text{and} \quad \Sigma b_i x_i y_i = 0$$

By the latter equation, x and y are orthogonal in (k^n,φ). The first equation says that $\varphi(x) = -d\,\varphi(y)$. Since φ is anisotropic, both $\varphi(x)$ and $\varphi(y)$ are not zero and hence

$$\varphi \cong \langle a,-da \rangle \perp \Psi, \text{ where } a = \varphi(y).$$

*) For convenience we write $\langle 1,-d \rangle$ instead of $[\langle 1,-d \rangle]$.

For an anisotropic form φ which becomes hyperbolic over L, applying the above argument repeatedly, we get

$$\varphi \cong \langle\alpha_1, \ldots, \alpha_n\rangle \langle 1, -d\rangle.$$

2. Generic theory of quadratic forms.

We consider a form over k, say $\varphi = \langle\alpha_1, \ldots, \alpha_n\rangle$. Then the polynomical $\varphi(x) = \varphi(x_1, \ldots, x_n)$ is usually regarded as the quadratic form φ itself. In the "generic" theory the indeterminates x_i are to be considered as elements of $k(x) = k(x_1, \ldots, x_n)$ and in this way $\varphi(x)$ is thought of as a "generic" value of φ over the rational function field $k(x)$. In this section we shall consider the behaviour of quadratic forms under transcendental extensions of the base field.

Before stating the main result, we introduce the following subset of k^* arising from a form (V,φ).

$D(\varphi) = \{\alpha \in k^* \mid \varphi(x) = \alpha$ for some $x \in V\}$ is the set of values of k represented by φ.

The main theorem of the generic theory is the following subform theorem, which characterizes the subforms of a given form. We say ψ is a subform of φ or φ represents ψ and write $\psi < \varphi$ if there exists a form ρ such that $\varphi \perp \rho$.

Theorem 2.1. (Subform theorem of Pfister). Let φ and $\psi = \langle\beta_1, \ldots, \beta_m\rangle$ be forms over k with φ anisotropic. Then the following are equivalent:

(i) ψ is isomorphic to a subform of φ.

(ii) For every field extension L of k $D(\psi_L) \subset D(\varphi_L)$.

(iii) $\psi(x) \in D(\varphi \otimes k(x))$, where $x = (x_1, \ldots, x_m)$,

$m = \dim \Psi$ (i.e. $\beta_1 x_1^2 + \ldots + \beta_m x_m^2$ is represented over the rational function field $k(x)$ by φ).

<u>Remarks 2.2.</u>

1) The equivalent conditions (i),(ii),(iii) of this theorem imply in particular that $\dim \Psi \leq \dim \varphi$.

2) It is important that φ should be anisotropic (see (ii)).

3) The criterion (ii) is nice but not practical, since it is not possible to calculate D(q) in general. The main part of the subform theorem is that the condition (ii) can be replaced by the much weaker condition (iii). For (iii), all we need is to check that the "generic" value $\Psi(x)$ is represented by φ over the single field, namely the rational function field $k(x)$.

4) The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial.

To prove the implication (iii) $\Rightarrow$ (i), we present the following important theorem due to Pfister.

<u>Theorem 2.3.</u> Let $\varphi = \langle \alpha_1, \ldots, \alpha_n \rangle$ be a form over k and let $f(x) \in k[x]$ be a nonzero polynomial. If φ represents $f(x)$ over $k(x)$ then φ represents $f(x)$ already over $k[x]$.

<u>Proof.</u> We may assume φ anisotropic, since

$$f = \left(\frac{f+1}{2}\right)^2 - \left(\frac{f-1}{2}\right)^2. \quad \text{Let } f(x) = \sum_{\tau=1}^{n} \alpha_i \frac{p_i(x)^2}{p_o(x)^2}$$

with polynomials $p_i(x)$ and $\deg(p_o) = d$ minimal. We shall show $d = 0$. Then the conclusion follows.

Suppose that $d > 0$. By Euclidean algorithm on $k[x]$,

$$p_i(x) = q_i(x)\, p_o(x) + r_i(x),$$

$i = 0, \ldots, n$, $q_o = 1$, $r_o = 1$, $\deg(r_i) < \deg(p_o)$, $1 \leq i \leq n$. Let Ψ denote the form $\langle -f(x), \alpha_1, \ldots, \alpha_n \rangle$ over $k(x)$. Then $\Psi = \langle -f \rangle \perp \varphi$ is isotropic.

Let $p = (p_o, \ldots, p_n)$ $q = (q_o, \ldots, q_n)$ and

$r = (r_o,\dots,r_n)$. Then $\Psi(p) = -fp_o^2 + \alpha_1 p_1^2 + a_n p_n^2 = 0$.
But $\Psi(q) \neq 0$. Define

$$h = (h_o,\dots,h_n) = \Psi(q)p - 2b_\Psi(p,q)q,$$

with b_Ψ the associated bilinear form to Ψ.
Then $\Psi(h) = \Psi(q)^2\,\Psi(p) - 4\Psi(q)\,b_\Psi(p,q)^2 + 4b_\Psi(p,q)^2\,\Psi(q) = 0$.

Since φ remains anisotropic over $k(x)$, $h_o \neq 0$.
From $\Psi(h) = 0$ we obtain

$$f(x) = \Sigma\,\alpha_i \frac{h_i(x)^2}{h_o(x)^2}$$

We claim deg $(h_o) <$ deg (p_o). This will contradict the minimal choice of d and hence d should be zero.

To prove the claim, we calculate h_o.
$h_o = \Psi(q)p_o - 2b_\Psi(p,q)q_o = p_o^{-1}\Psi(p_oq - q_op)$, since $\Psi(p) = 0$. Recalling the definition of Ψ we obtain

$$h_o = \frac{1}{p_o}\sum_{i=1}^{n}\alpha_i(p_oq_i - q_op_i)^2.$$

Since $q_o = 1$, finally

$$h_o = \frac{1}{p_o}\sum_{i=1}^{n}\alpha_i r_i^2$$

Thus deg $h_o \leq d - 2$. Theorem 2.3 is proven.

<u>Remark 2.4.</u> This theorem is a generalization of Cassels' theorem that if $0 \neq f(x) \in k\,[x]$ is a sum of squares in the rational function field then $f(x)$ is already a sum of squares in $k\,[x]$, cf. Cassels, Acta Arithm. 9 (1964).

<u>Corollary 2.5.</u> Let $\varphi = \langle a_1\rangle \perp \Psi$ be anisotropic over k. Then $d \in D(\Psi)$ if and only if $a_1\,x^2 + d \in D(\varphi \otimes k(x))$.

<u>Proof.</u> The direction "⇒" is trivial. Assume now that $a_1\,x^2 + d$ is represented by $\varphi \otimes k(x)$. According to Theorem 2.3. $a_1\,x^2 + d = a_1\,f_1(x)^2 + \dots + a_n\,f_n(x)^2$, where

$\Psi = \langle a_2,\dots,a_n\rangle$ and $f_i(x) \in k[x]$. Since φ is anisotropic, all $f_i(x)$ are linear. Write $f_1(x) = a\,x + b$. There is some $c \in k$ such that $ac + b = \pm c$. Then

$$a_1 c^2 + d = a_1 c^2 + a_2 f_2(c)^2 + \dots a_n f_n(c)^2$$

and hence $d = a_2 f_2(c)^2 + \dots + a_n f_n(c)^2 \in D(\Psi)$.

Corollary 2.6. $1 + x_1{}^2 + \dots + x_n{}^2$ can not be a sum of n squares in $k(x_1, \dots, x_n)$.

Corollary 2.7. (Pfister's Substitution Principle). Let Ψ be a form over k, and $p(x) = p(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$. Let $e = (e_1, \dots, e_n) \in k^n$ with $p(e) \neq 0$.
If $p(x) \in D(\varphi \otimes k(x))$ then $p(e) \in D(\varphi)$.

Proof. According to Pfister's theorem φ represents $p(x)$ over $k(x_1, \dots, x_{n-1})[x_n]$. Plugging in $x_n = e_n$, we see that φ represents $p(x_1, \dots, x_{n-1}, e_n)$ over $k(x_1,\dots,x_{n-1})$. The conclusion now follows by induction on $\dim \varphi = n$.

Later in §6 we shall establish a more general substitution principle.

We now enter the proof of the subform theorem. We work by induction on $\dim \varphi$. By substituting $x_1 = 1$, $x_2 = \dots = x_m = 0$, we see that $\beta_1 \in D(\varphi)$. Hence, $\varphi = \langle\beta_1\rangle \perp \varphi'$ and this remains anisotropic over $k(x') = k(x_2, \dots, x_m)$. If we write $\Psi = \langle\beta_1\rangle \perp \Psi'$ then from

$$\beta_1 x_1{}^2 + \Psi'(x) \in D(\varphi \otimes k(x)) = D((\langle\beta_1\rangle \perp \varphi') \otimes k(x')(x_1))$$

we obtain $\Psi'(x') \in D(\varphi' \otimes k(x'))$ by Corollary 2.5. By induction hypothesis, $\varphi' \cong \Psi' \perp X$ and hence $\varphi \cong \Psi \perp X$.

3. Elementary theory of Pfister forms.

An n-fold Pfister form over k means a quadratic form of the shape

$$\bigotimes_{i=1}^{n} \langle 1, a_i \rangle, \quad a_i \in k^*$$

and is denoted by $\langle\langle a_1, \ldots, a_n \rangle\rangle$. This is of dimension 2^n and is given by

$$\langle 1, a_1, \ldots, a_n, \quad a_1 a_2, \ldots, a_1 a_2 a_3, \ldots, \; a_1 \ldots a_n \rangle$$

We note the following special cases:

1. $\langle\langle 1, a_2, \ldots a_n \rangle\rangle = \langle 1,1 \rangle \langle\langle a_2, \ldots, a_n \rangle\rangle$

2. $\langle\langle -1, a_2 \ldots a_n \rangle\rangle = 2^{n-1} \times H$.

3. The n-fold Pfister form $\langle\langle 1,1,\ldots,1 \rangle\rangle$ is $2^n \times \langle 1 \rangle$.

4. The 1-fold Pfister form $\langle\langle -a \rangle\rangle$ is the norm form of the quadratic extension $k(\sqrt{a})$, $a \in k^*$, if a is not a square. Similarly the 2-fold Pfister form $\langle\langle -a, -b \rangle\rangle$ can be obtained as the norm form on the quaternion algebra$\left(\frac{a,b}{k}\right)$, and the 3-fold Pfister form $\langle\langle -a,-b,-c \rangle\rangle$ is the norm form of the Cayley-Algebra over k with the structure constants a, b, c.

Definition 3.1. For a form φ over k, we define

$$G(\varphi) = \{ \alpha \in k^* \mid \alpha\varphi \cong \varphi \}.$$

Such an element α is called a similarity factor of the form φ. Clearly $G(\varphi)$ is a group and contains k^{*2}, so we may consider $G(\varphi)$ as a subgroup of k^*/k^{*2}.

Definition 3.2. An anisotropic form φ is called multiplicative or "round" if $D(\varphi) = G(\varphi)$. An isotropic form φ is called multiplicative if φ is hyperbolic.

Remark 3.3. If $1 \in D(\varphi)$, $G(\varphi) \subset D(\varphi)$.

Examples

1) A one dimensional form $\langle a \rangle$ is multiplicative if and only if $\langle a \rangle \cong \langle 1 \rangle$.
2) Every form $\langle 1, \alpha \rangle$ is multiplicative.

Theorem 3.4. (Pfister). If φ is multiplicative then $\varphi \otimes \langle 1, \alpha \rangle$ is also multiplicative. In particular any Pfister form $\langle\langle a_1, \ldots, a_n \rangle\rangle$ is multiplicative.

Proof.(Witt) If $\varphi \sim 0$, $\varphi \otimes \langle 1, \alpha \rangle \sim 0$. This is the trivial case. Now let φ be anisotropic and $\varphi \otimes \langle 1, \alpha \rangle$ be isotropic. Then we need to show that $\varphi \otimes \langle 1, \alpha \rangle$ is hyperbolic. Since $\varphi \otimes \langle 1, \alpha \rangle = \varphi \perp \alpha \varphi$ is isotropic, $\gamma + \alpha\delta = 0$ for some γ, $\delta \in D(\varphi)$. From $G(\varphi) = D(\varphi)$, it follows

$$\varphi \perp \alpha\varphi \cong \gamma\varphi \perp \alpha\delta\varphi = \gamma\varphi \perp (-\gamma\varphi).$$

This shows that $\varphi \otimes \langle 1, \alpha \rangle$ is hyperbolic.

We now consider the case that both φ and $\varphi \otimes \langle 1, \alpha \rangle$ are anisotropic. It suffices to show $D(\varphi \otimes \langle 1, \alpha \rangle) \subset G(\varphi \otimes \langle 1, \alpha \rangle)$. Let $\delta \in D(\varphi \otimes \langle 1, \alpha \rangle)$. Then we distinguish the following 3 cases:

Case 1: $\delta \in D(\varphi)$

Case 2: $\delta \in D(\alpha\varphi)$ i.e. $\delta = \alpha\beta$, $\beta \in D(\varphi)$

Case 3: $\delta = \beta + \gamma\alpha$, β, $\gamma \in D(\varphi)$.

In the first two cases it can be shown easily that $\delta(\varphi \perp \alpha\varphi) \cong \varphi \perp \alpha\varphi$, since $G(\varphi) = D(\varphi)$. For the last case,

$$\begin{aligned} \delta(\varphi \perp \alpha\varphi) &= (\beta + \gamma\alpha)(\varphi \perp \alpha\varphi) = \beta\left(1 + \frac{\alpha\gamma}{\beta}\right)(\varphi \perp \alpha\varphi) \\ &\cong \beta\left(1 + \frac{\alpha\gamma}{\beta}\right)\left\langle 1, \frac{\alpha\gamma}{\beta}\right\rangle \varphi \\ &\overset{(*)}{\cong} \beta \left\langle 1, \frac{\alpha\gamma}{\beta} \right\rangle \varphi \end{aligned}$$

$$\delta(\varphi \perp \alpha\varphi) \cong \beta(\varphi \perp \frac{\alpha\gamma}{\beta}\varphi)$$

$$= \beta\varphi \perp \alpha\gamma\varphi$$

$$\cong \varphi \perp \alpha\varphi$$

(*) Here we use the fact that 1-fold Pfister forms are multiplicative.

Remark 3.5. As a special case of the theorem we note that any isotropic Pfister form is hyperbolic.

Corollary 3.6. (Pfister). If $\alpha, \beta \in k^*$ are both sums of 2^n squares in k then $\alpha\beta$ is also a sum of 2^n squares in k.

Proof. We apply the theorem to the n-fold Pfister form $\langle\langle 1,1,\ldots,1\rangle\rangle$.

We now give an interesting application of the above corollary.

Definition 3.7. The level $s(k)$ of a field k is the smallest natural number such that -1 is sum of $s(k)$ squares in k. If k is formally real then we put $s(k) = \infty$.

Theorem 3.8. (Pfister). The level of a field is either ∞ or power of 2.

Proof. Let $s = s(k)$ and $2^n \leq s < 2^{n+1}$. We consider the multiplicative form

$$\varphi = 2^{n+1} \times \langle 1\rangle = s \times \langle 1\rangle \perp (2^{n+1} - s) \times \langle 1\rangle.$$

Since $-1 \in D(s \times \langle 1\rangle)$, φ is isotropic and hence hyperbolic. From $2^{n+1} \times \langle 1\rangle \cong 2^n \times \langle 1\rangle \perp 2^n \times \langle 1\rangle$, we obtain $2^n \times \langle 1\rangle \sim 2^n \times \langle -1\rangle$ and then $2^n \times \langle 1\rangle \cong 2^n \times \langle -1\rangle$. In particular $-1 \in D(2^n \times \langle 1\rangle)$ i.e. -1 is a sum of 2^n squares. Thus $s = 2^n$.

We leave here aside most of the elementary theory (i.e. theory not involving transcendental field extensions) of Pfister forms and refer the reader to Lam [7] Chap.X §1 for that.

4. Generic theory of Pfister forms

As an application of the subform theorem we characterize the anisotropic Pfister forms by "multiplicative" properties.

Theorem 4.1. (Pfister). Let φ be a n-dimensional anisotropic form over K. Then the following are equivalent:

(i) φ is a Pfister form.

(ii) For all field extensions L/K φ_L is multiplicative.

(iii) For all field extensions L/K $D(\varphi_L)$ is a group.

(iv) For the vectors of indeterminates $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n)$, $\varphi(x)\varphi(y)$ is represented by φ over $k(x_1, \dots, x_n, y_1, \dots, y_n)$.

(v) $\varphi(x)$ is a similarity factor of φ over $k(x_1,\dots,x_n)$ i.e. $\varphi(x) \in G(\varphi_{k(x)})$.

Proof. (i) $\Rightarrow$ (ii): This is clear since for any field L over K the form φ_L is again Pfister.

(ii) $\Rightarrow$ (v): trivial

(v) $\Rightarrow$ (iv): Since $\varphi(x)\varphi_{K(x)} \cong \varphi_{K(x)}$ and $\varphi(y) \in D(\varphi \otimes K(y))$, $\varphi(x)\varphi(y) \in D(\varphi \otimes K(x,y)$.

(iv) $\Rightarrow$ (iii): Consider two elements $\varphi(u)$ and $\varphi(v)$ in $D(\varphi_L)$, $u, v \in L^n$. Since $\varphi(x)\varphi(y)$ is represented by φ over $L(x_1, \dots, x_n, y_1, \dots, y_n)$, the element $\varphi(u)\varphi(v)$ is represented by φ_L according to the substitution principle.

(iii) ⇒ (i): φ clearly contains <1>, the 0-fold Pfister form. Now let r be the largest integer such that φ contains an r-fold Pfister form Ψ. We shall show $2^r = n$. Then we will be finished.

Assume $n > 2^r$ and write $\varphi = \Psi \perp \rho$ with $\rho \neq 0$. Let us fix a value $\alpha \in D(\rho)$ and consider the form $\Psi \perp \alpha\Psi$. We claim that $\Psi \perp \alpha\Psi$ is a subform of φ. To prove the claim, consider

$$(*) \quad \Psi(Z) + \alpha\Psi(T) = \Psi(T)\,[\Psi(Z)\,\Psi(T)^{-1} + \alpha],$$

where $Z = (Z_1, \ldots, Z_{2^r})$ and $T = (T_1, \ldots, T_{2^r})$ are 2 independent sequences of 2^r indeterminates. Since Ψ is a Pfister form, $\Psi(Z)\,\Psi(T)^{-1} \in D(\Psi_{K(Z,T)})$ and it follows that the expression in the brackets belongs to $D(\varphi_{K(Z,T)})$. Since $D(\varphi_{K(Z,T)})$ is a group by hypothesis, both sides of (*) are represented by φ over $K(Z,T)$. By the subform theorem $\Psi \perp \alpha\Psi$ is a subform of φ. But $\Psi \perp \alpha\Psi = \langle\langle\alpha\rangle\rangle\,\Psi$ is an (r+1)-fold Pfister form, and this contradicts the choice of r.

5. Fields with prescribed level.

We have seen that $D(m \times \langle 1\rangle)$ is a group if $m = 2^n$. Here we shall give a more refined result.

Lemma 5.1. Let $m = 2^n$ and let $a_1, \ldots, a_m$ be elements of K. Put $a_1^2 + \ldots + a_m^2 = a$. Then there exists a $m \times m$-matrix A with coefficients in K and first row $(a_1, \ldots, a_m)$ such that $AA^t = A^tA = aI_m$, I_m the $m \times m$-identity matrix.

Proof. We proceed by induction on n. The case $n = 0$ is trivial. Assume $n > 0$ and write $a = b+c$ with

$$b = a_1^2 + \dots + a_{2^{n-1}}^2 , \quad c = a_{2^{n-1}+1}^2 + \dots + a_{2^n}^2.$$

By induction hypothesis there exists $2^{n-1} \times 2^{n-1}$ matrices B and C over k with first rows $(a_1, \dots, a_{2^{n-1}})$, $(a_{2^{n-1}+1}, \dots, a_{2^n})$ such that

$$B \cdot B^t = B^t \cdot B = b\, I_{2^{n-1}}, \quad C \cdot C^t = C^t C = c\, I_{2^{n-1}}.$$

If $b \neq 0$ the matrix

$$A = \begin{pmatrix} B & , & C \\ -b^{-1} B^t \cdot C^t \cdot B & , & B^t \end{pmatrix}$$

solves our problem, as can be checked by easy computation. If $b = 0$, but $c \neq 0$ then the matrix

$$A = \begin{pmatrix} B & C \\ C^t & -a^{-1} C^t B^t C \end{pmatrix}$$

has the desired porperties. Finally if $b = c = 0$ then the matrix

$$A = \begin{pmatrix} B & C \\ B & C \end{pmatrix}$$

has the desired properties.

<u>Lemma 5.2.</u> Let $m = 2^n$ and $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m$ be elements of K. Then there exist elements $\gamma_2, \dots, \gamma_m$ in K such that

$$(\alpha_1^2 + \dots + \alpha_m^2)(\beta_1^2 + \dots + \beta_m^2 =$$

$$= (\alpha_1 \beta_1 + \dots + \alpha_m \beta_m)^2 + \gamma_2^2 + \dots + \gamma_m^2$$

Proof. Let $\alpha := \alpha_1^2 + \ldots + \alpha_m^2$, $\beta := \beta_1^2 + \ldots + \beta_m^2$. By the above lemma there exist m x m-matrices A and B over k with $AA^t = A^tA = \alpha I_m$ and $BB^t = B^tB = \beta I_m$ and with first rows $(\alpha_1, \ldots, \alpha_m)$ and $(\beta_1, \ldots, \beta_m)$ respectively.
Let $(\gamma_1, \ldots, \gamma_m)$ be the first row of the matrix $C = AB^t$. Then $CC^t = \alpha\beta I_m$ and hence $\alpha\beta = \gamma_1^2 + \ldots + \gamma_m^2$ with $\gamma_1 = \alpha_1\beta_1 + \ldots + \alpha_m\beta_m$.

Lemma 5.3. Let K be formally real and $L = K(\sqrt{-d})$ be a quadratic extension. If the level of L is $s(<\infty)$ then d is a sum of 2s-1 squares in K.

Proof. Since $s(L) = s$, there exists a representation

$$-1 = \sum_{i=1}^{s} (\alpha_i + \beta_i \sqrt{-d})^2 , \quad \alpha_1\ \beta_i \in K$$

This yields $-1 = \sum_{i=1}^{s} \alpha_i^2 - d \sum_{i=1}^{s} \beta_i^2$ and $\sum_{i=1}^{s} \alpha_i\beta_i = 0$.

Hence $d(\sum_{i=1}^{s} \beta_i^2)^2 = (\sum_{i=1}^{s} \alpha_i^2)(\sum_{i=1}^{s} \beta_i^2) + (\sum_{i=1}^{s} \beta_i^2)$.

By Lemma 5.2 $(\Sigma\, \alpha_i^2)(\Sigma\, \beta_i^2)$ is a sum of s-1 squares and consequently d is a sum of 2s-1 squares.

Now we present the theorem, due to Pfister, which answers the title of this section.

Theorem 5.4. Let K be a formally real field and let $d \in K$ be a sum of n squares but not sum of n-1 squares.
Let $2^k \leq n < 2^{k+1}$ and $L = K(\sqrt{-d})$. Then the level of L is 2^k.

Proof. Let $\alpha = \sqrt{-d}$ then $-\alpha^2 = d = \alpha_1^2 + \ldots + \alpha_n^2$, $\alpha_i \in K$. Hence $s(L) \leq n$ which implies $s(L) \leq 2^k$. To prove $s(L) = 2^k$ assume that $s(L) \leq 2^{k-1}$. Then by Lemma 5.3 d is a sum of 2^k-1 squares in K, which is a contradiction.

<u>Example 5.5.</u> Let K be formally real and let $(X_1, \ldots, X_n)$ be a sequence of indeterminates over K.
Let $2^k \leqslant n < 2^{k+1}$. Then the field

$$K(X_1, \ldots, X_n)(\sqrt{-X_1^2 - \ldots - X_n^2}) \text{ has level } 2^k.$$

Indeed, we know from the subform theorem, that $X_1^2 + \ldots + X_n^2$ is not a sum of less than n squares in $K(X_1, \ldots, X_n)$, cf. § 2.

6. <u>Specialization of quadratic forms.</u>

We first recall some definitions from valuation theory. A subring R of a field K is called a <u>valuation ring of</u> K if for all $x \in K^*$ $x \in R$ or $x^{-1} \in R$. Let Γ be a totally ordered abelian group. Then a <u>valuation on</u> K is a mapping $v: K \to \Gamma \cup \{\infty\}$ such that $v(x+y) \geqslant \min\{v(x), v(y)\}$ $v(xy) = v(x)v(y)$, $v(1) = 0$ and $v(0) = \infty$.

If v is a valuation on K then $R_v = \{x \in K \mid v(x) \geqslant 0\}$ is a valuation ring of K with maximal ideal $\mathfrak{m}_v = \{x \in K \mid v(x) > 0\}$.

Conversely let R be a valuation ring of K. We define a relation < on K by $\alpha < \beta$ if $\beta\alpha^{-1} \in R$.
Let $R^* = \{\alpha \in K^* \mid 1 < \alpha < 1\}$. Then < gives a total ordering on $\Gamma = K^*/R^*$ and the canonical projection $v: K^* \cup 0 \to \Gamma \cup \infty$ is a valuation on K. Hence "valuation ring of K" and "valuation on K" are equivalent notions.

Now we come to the third equivalent notion. Let L be a field and $L^\infty = L \cup \{\infty\}$. The laws of composition of L extend to L^∞ by setting $a + \infty = \infty$ for $a \in L$ and $\infty \cdot a = a \cdot \infty = \infty$ for $a \in L^\infty$, $a \neq 0$. The compositions $\infty + \infty$ and $\infty \cdot 0$ are not defined.

Let K and L be fields. A <u>place</u> $\lambda : K \to L^\infty$ is a homomorphism from K to L^∞, i.e. $\lambda(x+y) = \lambda(x) + \lambda(y)$ and $\lambda(xy) = \lambda(x)\lambda(y)$ whenever the right hand sides are defined. Any homomorphism $\varphi : K \to L$ from K to another field L - automatically injective - is called a <u>trivial place</u>. Let $R_\lambda = \{x \in K \mid \lambda(x) \neq \infty\}$. Then R_λ is a valuation ring of K with maximal ideal $m_\lambda = \{x \in K \mid \lambda(x) = 0\}$. The residue class field $k_\lambda = R_\lambda/m_\lambda$ can be considered as a subfield of L. We have $\lambda(K) = k_\lambda^\infty$ if λ is nontrivial.

Conversely let R be a valuation ring in K with residue class field k. Then $\lambda : K \to k^\infty$, with $\lambda|R$ = canonical projection and $\lambda(x) = \infty$ for all $x \notin R$, is a place. Hence we have three essentially equivalent notions, namely valuations, valuation rings and places.

We recall some standard notions and well known facts about symmetric bilinear forms over a local ring R with 2 a unit in R.

A <u>bilinear form</u> $\varphi = (E,\varphi)$ over R is a finitely generated free R-module E together with a symmetric bilinear form $\varphi : E \times E \to R$. We often denote $\varphi = (E,\varphi)$ by the symmetric matrix (a_{ij}) which is uniquely determined by φ for a fixed basis of E over R. We say $\varphi = (a_{ij})$ is a <u>nonsingular</u> bilinear form, or <u>bilinear space</u>, if det (a_{ij}) is a unit in R. For any ring homomorphism $\lambda : R \to S$ we denote by $\lambda_*(\varphi)$ the bilinear form over S obtained from φ by base extension with λ.

<u>Theorem 6.1.</u> (Cancellation theorem) If φ, ψ, χ are nonsingular bilinear forms over R with $\varphi \perp \chi \cong \psi \perp \chi$ then $\varphi \cong \psi$.

A proof can be found in Knebusch [6] or Roy [10]. From now on let R be a valuation ring of a field K with maximal ideal m and 2 a unit in R.

<u>Lemma 6.2.</u> Let $\varphi = (M,\varphi)$ be a bilinear space over R such that $\varphi_K = (M \otimes_R K, \varphi_K)$ becomes isotropic.
Then $\varphi \cong \langle 1,-1\rangle \perp \psi$ with some bilinear space ψ over R.

Proof. By the hypothesis we may choose some $x \in M$ such that $x \neq 0$ and $\varphi(x,x) = 0$. We regard M as an R-submodule of the K-vector space $M \otimes_R K = KM$. Choosing a basis of M we see easily that $Kx \cap M = Rax$ with some $a \in K^*$. Replacing x by ax we may assume that $Kx \cap M = Rx$. Then x is part of a basis of M. In particular there exists an R-linear form $M \to R$ which maps x to 1. Since φ is nonsingular we have a vector y in M with $\varphi(x,y) = 1$. Replacing y by $y - \frac{1}{2}\varphi(y,y)x$ we may assume in addition that $\varphi(y,y) = 0$. Since the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has unit determinant, the module $U := Rx + Ry$ is free with basis x,y, and M is the orthogonal sum $U \perp U^{\perp}$ of U and the module $U^{\perp} := \{z \in M \mid \varphi(z,U) = 0\}$. The bilinear space $(U, \varphi|U)$ has the orthogonal basis $u := x + \frac{1}{2}y$, $v := x - \frac{1}{2}y$ with $\varphi(u,u) = 1$, $\varphi(v,v) = -1$.

Corollary 6.3.

i) If φ is a nonsingular bilinear form over R such that φ_K is hyperbolic then $\varphi \cong m \times \langle 1,-1 \rangle$ with some $m \in \mathbb{N}$ (over R).

ii) If φ and ψ are nonsingular bilinear forms over R such that $\varphi_K \cong \psi_K$, then $\varphi \cong \psi$.

Proof. The first statement follows by repeated application of Lemma 6.2. Let now φ and ψ be nonsingular forms over R with $\varphi_K \cong \psi_K$. Then $\varphi \perp (-\psi)$ becomes hyperbolic over K, hence $\varphi \perp (-\psi) \cong m \times \langle 1,-1 \rangle$, with m the rank of the bilinear module ψ. We also have $\psi \perp (-\psi) \cong m \times \langle 1,-1 \rangle$. From the cancellation theorem we obtain $\varphi \cong \psi$.

It is convenient to know also the following fact.

Proposition 6.4. Every bilinear space (M, φ) over R has an orthogonal basis, i.e. $(M,\varphi) \cong \langle a_1, \ldots, a_n \rangle$ with some units a_i of R.

Proof. By induction on the rank n of M. The case $n = 1$ is trivial. Assume $n > 1$. It suffices to find a vector x in M

such that $\varphi(x,x) = a$, a unit. Then the restriction of φ to Rx will be nonsingular, hence

$$M = Rx \perp (Rx)^{\perp} \cong \langle a \rangle \perp (Rx)^{\perp},$$

and we can apply the induction hypothesis to $(Rx)^{\perp}$. Now start with any vector u of M which is part of a basis of M. If $\varphi(u,u) \in R^*$ we are through. Otherwise $\varphi(u,u) \in \mathfrak{m}$. Since φ is nonsingular there exists some vector v in M with $\varphi(u,v) = 1$. If $\varphi(v,v) \in R^*$ we are through again. If $\varphi(v,v) \in \mathfrak{m}$ then $x := u+v$ is a vector with $\varphi(x,x) \in R^*$, since 2 is a unit.

<u>Definition 6.5.</u> Let $\lambda : K \to U \infty$ be a given place with valuation ring R. Let φ be a bilinear space over K. We say φ has <u>good reduction with respect to</u> λ if $\varphi \cong (a_{ij})$ with $\lambda(a_{ij}) \neq \infty$ and $\det(\lambda(a_{ij})) \neq 0$. This means that there exists a bilinear space Ψ over R such that $\Psi_K \cong \varphi$. Let $\lambda_*(\varphi)$ denote the bilinear form $(\lambda(a_{ij}))$ over L. Because of the above Corollary 6.3. $\lambda_*(\varphi)$ is uniquely determined by φ up to isometry. We call $\lambda_*(\varphi)$ the <u>reduction</u> or <u>specialization</u> of φ with respect to λ. Note also that if φ has good reduction then the matrix (a_{ij}) above can be chosen as a diagonal matrix according to Proposition 6.4.

<u>Lemma 6.6.</u> Let φ, Ψ and ρ be forms over K with $\varphi \cong \Psi \perp \rho$. If φ and Ψ have good reduction with respect to a given place $\lambda: K \to L \cup \infty$, then ρ has good reduction with respect to λ and $\lambda_*(\varphi) \cong \lambda_*(\Psi) \perp \lambda_*(\rho)$.

<u>Proof.</u> By hypothesis $\varphi \perp (-\Psi)$ has good reduction and

$$\varphi \perp (-\Psi) \cong (\Psi \perp (-\Psi)) \perp \rho \cong n \times \langle 1,-1 \rangle \perp \rho, \quad n = \dim \Psi.$$

By Lemma 6.2. there exists a space ρ_o over the valuation ring R_λ associated to λ such that $\varphi \perp (-\Psi) \cong \rho_o \otimes K \perp n \times \langle 1,-1 \rangle$. Hence $\rho_o \otimes K \cong \rho$, i.e. ρ

has good reduction and

$$\lambda_*(\varphi) \cong \lambda_*(\Psi) \perp \lambda_*(\rho) .$$

Corollary 6.7. (Substitution Principle)
Let $\varphi = (f_{ij}(X))_{1\leq i,\ j\leq n}$ and $\Psi = (g_{kl}(X))_{1\leq k,\ l\leq m}$ be symmetric matrices of polynomials in $K[X] = K[X_1, \ldots, X_n]$, Let $x = (x_1, \ldots, x_r)$ be an r-tuple with coordinates x_i in a field extension L of K. If the forms $\varphi_x := (f_{ij}(x))$ and $\Psi_x := (g_{kl}(x))$ over L are not singular and if Ψ is a subform of φ over K(X), then Ψ_x is a subform of φ_x over L.

Proof. Since Ψ is a subform of φ over L(X), we may assume K = L. There exists(*) a place (in fact many places)

$$\lambda : K(X) \to K \cup \infty \quad \text{with} \quad \lambda(X_i) = x_i,\ 1 \leq i \leq n$$

over K. Then φ and Ψ have good reduction with respect to λ. From hypothesis, $\varphi \cong \Psi \perp \rho$ over k(X). By Lemma 6.5. ρ has good reduction with respect to λ and $\lambda_*(\varphi) \cong \lambda_*(\Psi) \perp \lambda_*(\rho)$ i.e. $\varphi_x \cong \Psi_x \perp \rho_x$.

We finally state a theorem, which we need in the next section in proving the norm theorem 7.2. For the proof we refer to theorem 3.1. in Knebusch [4]. Actually the part of the norm theorem for which this theorem is needed, is not essential for the later sections.

Theorem 6.8. Let $\lambda : K \to L \cup \infty$ be a place. Then there exists a well-defined additive map

$$\lambda_* : W(K) \to W(L)$$

(*) For the convenience of the reader we have included a proof of this standard fact of valuation theory in §9 (Lemma 9.3.).

such that for each $a \in K^*$

$$\lambda_*(\langle a\rangle) = \begin{cases} \langle \lambda(a)\rangle & \text{if } \lambda(a) \neq 0, \neq \infty \\ 0 & \text{if } \lambda(ac^2) = 0 \text{ or } \infty \text{ for all } c \in K^*. \end{cases}$$

Remark 6.9. If φ is a form over K with good reduction with respect to λ then clearly $\lambda_*[\varphi]$ is the class $[\lambda_*(\varphi)]$ of the specialization $\lambda_*(\varphi)$ defined above.

7. A norm theorem.

We fix some further notations. From now on k always denotes the ground field. For any form $\varphi = \langle a_1,\ldots,a_n\rangle$ over k the (signed) determinant of φ is defined as the square class of

$(-1)^{\frac{n(n-1)}{2}} a_1 \ldots a_n$ and denoted by $d(\varphi)$. Notice that $d(\varphi)$ depends only on the Witt class $[\varphi]$. We often regard $d(\varphi)$ as a one dimensional quadratic form. Let $X_1, \ldots, X_n$ be indeterminates and $X = (X_1, \ldots, X_n)$. Let φ be a quadratic form over k of dimension $n(\geq 2)$ which is not isomorphic to H. Then $\varphi(X) \in k[X]$ is irreducible in $k[X]$ and we may regard the function field $k(\varphi)$ of the affine quadratic $\varphi(X) = 0$, i.e. the quotient field of $k[X]/\varphi(X)$. Let x_i denote the image of X_i in $k(\varphi)$. Then we have $k(\varphi) = k(x_1, \ldots, x_n)$. The function field $k(\varphi)_o$ of the projective quadratic $\varphi(X) = 0$ is

$$k(\varphi)_o = k\left(\frac{x_2}{x_1} \ldots\ldots \frac{x_n}{x_1}\right).$$

In particular $k(\varphi)_0 \subset k(\varphi)$ and $k(\varphi) = k(\varphi)_0\ (x_1)$ with x_1 transcendental over $k(\varphi)_0$.

Remark 7.1. Let φ be isotropic but not the hyperbolic plane H. Then $k(\varphi)_0$ is a purely transcendental extension of k.

Proof. By a linear coordinate change, we may assume

$$\varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \psi .$$

Thus $\varphi(X_1, \ldots, X_n) = X_1X_2 + \psi(X_3, \ldots, X_n)$.

Since $x_2\, x_1^{-1} = -\psi(x_3x_1^{-1}, \ldots, x_nx_1^{-1})$ we have $k(\varphi)_0 = k(x_3x_1{-1}, \ldots, x_nx_1{-1})$, which is a purely transcendental extension of k.

For $\varphi \cong H$ we write conventionally $k(\varphi)_0 = k$ and $k(\varphi) = k(x)$.

Theorem 7.2. (Norm theorem) Let φ and ψ be forms over k with $\dim \psi = n \geq 2$. We further assume that ψ represents 1. Then $\varphi \otimes k(\psi) \sim 0$ if and only if $\psi(X) \in G(\varphi \otimes k(X))$, where $X = (X_1, \ldots, X_n)$.

Proof. First assume $\psi(X) \in G(\varphi \otimes k(X))$ i.e. $\psi(X)(\varphi \otimes k(X)) \cong \varphi \otimes k(X)$. We consider the canonical place

$$\lambda : k(X) \to k(\psi) \cup \infty$$

over k associated to the prime polynomial $\psi(X)$. (Remember that $k[X]$ is a unique factorization domain)

Let $\lambda_* : W(k(X)) \to W(k(\psi))$ be the additive map induced by λ as described at the end of §6. Let $\varphi = \langle a_1, \ldots, a_m\rangle$, $a_i \in k^*$. Then $\lambda_*(\varphi \otimes k(X)) = \varphi \otimes k(\psi) = \langle a_1, \ldots, a_m\rangle$. By hypothesis $\langle a_1\, \psi(X), \ldots, a_m\, \psi(X)\rangle \cong \langle a_1, \ldots, a_m\rangle$ over $k(X)$ and we learn from theorem 6.8. that $\lambda_*\, [\varphi \otimes k(X)] = 0$ i.e.

$\varphi \otimes k(\Psi) \sim 0$.

To prove the converse we may assume $\Psi = \langle 1\rangle \perp \Psi'$. Then $\Psi(X) = X_1^2 + \Psi'(X_2, \ldots, X_n)$ and $k(\Psi) = k(X')[\sqrt{-\Psi'(X')}]$, $X' = (X_2, \ldots, X_n)$. Since $\varphi \otimes k(\Psi) \sim 0$, $\varphi \otimes k(X') \cong \cong \langle 1, \Psi'(x')\rangle \otimes \gamma$, for some form γ over $k(x')$ by Proposition 1.6. Then $\varphi \otimes k(X) \cong (\langle 1, \Psi'(X')\rangle \otimes \gamma) \otimes k(X)$ and hence $G(\langle 1, \Psi'(X')\rangle \otimes k(X)) \subset G(\varphi \otimes k(X))$. Now $\Psi(X) = X_1^2 + \Psi'(X') \in D(\langle 1, \Psi'(X')\rangle \otimes k(X)) = G(\langle 1, \Psi'(X)\rangle \otimes k(X))$, and the conclusion follows.

We have the following interesting corollaries:

<u>Corollary 7.3.</u> Let φ be anisotropic over k with $1 \in D(\varphi)$. Then φ is a Pfister form if and only if $\varphi \otimes k(\varphi) \sim 0$.

<u>Proof.</u> "$\Rightarrow$": This is clear since $\varphi \otimes k(\varphi)$ is isotropic and isotropic Pfister forms are hyperbolic. "$\Leftarrow$": According to the Norm Theorem $\varphi(X) \in G(\varphi \otimes k(X))$. Hence φ is a Pfister form by Theorem 4.1.

<u>Corollary 7.4.</u> Let φ and Ψ be forms over k such that φ is anisotropic, $\dim \Psi \geq 1$, $a \in D(\varphi)$ and $b \in D(\Psi)$. If $\varphi \otimes k(\Psi) \sim 0$ then $ab\Psi < \varphi$. In particular $\dim \Psi \leq \dim \varphi$.

<u>Proof.</u> Since $b \in D(\Psi)$, $1 \in D(b\Psi)$ and then $b\Psi(X) \in G(\varphi \otimes k(X))$ by the Norm Theorem. Since $a \in D(\varphi) \subset D(\varphi \otimes k(X))$, $ab\Psi(X) \in D(\varphi \otimes k(X))$. Since φ is anisotropic, we obtain $ab\Psi < \varphi$ using the subform theorem.

<u>Corollary 7.5.</u> Let τ be a Pfister form and let φ be anisotropic over k. Then $\varphi \otimes k(\tau) \sim 0$ if and only if $\varphi \cong \tau \otimes \eta$ for some η. In particular $\mathrm{Ker}(W(k) \to W(k(\tau))) = \tau W(k)$.

<u>Proof.</u> "$\Leftarrow$": Since τ is a Pfister form, this direction is clear.
"$\Rightarrow$": We prove inductively on $\dim \varphi$. If $\dim \varphi = 0$ then there is nothing to prove.

We choose $a_1 \in D(\varphi)$. Then $\varphi \cong a_1\tau \perp \varphi_1$ by Corollary 7.4. Since $\varphi_1 \otimes k(\tau) \sim 0$, by induction hypothesis $\varphi_1 \cong \tau \otimes \eta_1$. Hence the conclusion $\varphi \cong \tau \otimes (\langle a_1\rangle \perp \eta_1)$ follows.

8. The generic splitting problem

Let φ be a form over k. In this section we are interested in the following question. What are the indices $\mathrm{ind}(\varphi_L)$ and kernel forms $\ker(\varphi_L)$ if L runs through all extensions of k in some universal domain?

Let K and L be field extensions of k. Let Ψ be the kernel of $\varphi \otimes K$ and η be the kernel form of $\varphi \otimes L$. Let $\lambda: K \to L \cup \infty$ be any place over k (i.e. $\lambda(x) = x$ for all x in k). Then $\varphi \otimes K$ has good reduction with respect to every k-place λ from K to L and $\lambda_*(\varphi \otimes K) = \varphi \otimes L$. Let $r = \mathrm{ind}(\varphi \otimes K)$. Since $r \times H$ has good reduction with respect to λ, Ψ has also good reduction and $\varphi \otimes L \cong$ $\cong r \times H \perp \lambda_*(\Psi)$ by Lemma 6.6. In particular $\mathrm{ind}(\varphi \otimes L) \geq$ $\geq r$ and $\eta \sim \lambda_*(\Psi)$.

Definition 8.1. We call two field extensions K and L equivalent over k if there exist places

$$\lambda: K \to L \cup \infty \quad \text{and} \quad \mu: L \to K \cup \infty$$

over k. We then write $K \sim L$ or more precisely $K \underset{k}{\sim} L$.

From the discussion above the following is clear:

Theorem 8.2. Let $K \underset{k}{\sim} L$ and φ be a form over k.

Then

(i) $\mathrm{ind}(\varphi \otimes K) = \mathrm{ind}(\varphi \otimes L)$

(ii) $\ker(\varphi \otimes K) = \Psi$ has good reduction with respect to λ and $\lambda_*(\Psi) = \ker(\varphi \otimes L)$ for every k-place $\lambda : K \to L \cup \infty$. Thus over equivalent fields φ has the same splitting behavior.

We may ask:

Problem. Let φ be a form over k. For any given integer $r \geq 1$ we look for a field extension K over k with the following properties:

(i) $\mathrm{ind}(\varphi \otimes K) \geq r$

(ii) For any field L over k with $\mathrm{ind}(\varphi \otimes L) \geq r$ there exists a k-place $\lambda : K \to L \cup \infty$.

Such a field K is called a partial generic splitting field of φ, more precisely a generic field for splitting off r hyperbolic planes. In particular if $r = 1$, K is called a generic zero field of φ, and if $r = \frac{\dim \varphi}{2}$, a (total) generic splitting field of φ.

We say that φ splits if $\dim(\ker(\varphi)) \leq 1$, i.e.

$$\varphi \sim \begin{cases} 0 & \text{if } \dim \varphi \text{ even} \\ d(\varphi) & \text{if } \dim \varphi \text{ odd.} \end{cases}$$

9. Generic zero fields

We want to show that $k(\varphi)$ is a generic zero field of φ. If φ is isotropic, $k(\varphi)$ is a purely transcendental extension of k and hence $k(\varphi) \sim k$. Therefore in this case $k(\varphi)$ certainly is a generic zero field. To deal with the anisotropic case we need three lemmas:

Lemma 9.1. Let K be a quadratic field extension of E, say $K = E(\sqrt{a}) = E(\alpha)$, $\alpha^2 = a$. Let $\mu: E \to L \cup \infty$ be a given place with $\mu(a) = \beta^2$, $\beta \in L$, $\beta \neq 0$. Then there exists a unique extension $\lambda: K \to L \cup \infty$ of μ with $\lambda(\alpha) = \beta$.

Proof. (Here we give an elementary proof without referring to general principles of valuation theory.) By the general extension theorem of places there exists a place

$$\lambda: K \to \bar{L} \cup \infty, \quad \bar{L} = \text{algebraic closure of } L,$$

which extends μ. Since $\lambda(\alpha)^2 = \beta^2$, we have $\lambda(\alpha) = \pm \beta$. Eventually composing λ with the involution of K/E, we obtain an extension λ of μ with $\lambda(\alpha) = \beta$ and values in $\bar{L}$. Now let $\lambda: K \to \bar{L} \cup \infty$ be any extension of μ with $\lambda(\alpha) = \beta$. Given $x, y \in E$, we want to show that the value $\lambda(x + y\alpha)$ lies in $L \cup \infty$ and is uniquely determined by μ.

Let $v: E \twoheadrightarrow \Gamma$ be the associated valuation to μ. Then we have the following 3 cases:

1. $v(y) \geq v(x)$, $x \neq 0$, $y \neq 0$: Now

$$\lambda(x+y\alpha) = \lambda(x(1 + \tfrac{y}{x}\alpha))$$

$$\lambda(1 + \tfrac{y}{x}\alpha) = 1 + \mu(\tfrac{y}{x})\beta.$$

a) If $1 + \mu(\frac{y}{x})\beta \neq 0$, $\lambda(x+y\alpha) = \mu(x)(1 + \mu(\frac{y}{x})\beta) \in L \cup \infty$.

b) If $1 + \mu(\frac{y}{x})\beta = 0$, $2 = 1 - \mu(\frac{y}{x})\beta = \lambda(1 - \frac{y}{x}\alpha)$

$$2\,\lambda(x+y\alpha) = \lambda(x(1 - \frac{y^2}{x^2}a)) = \mu(x - \frac{y^2}{x}a) \in L \cup \infty$$

2. $v(y) < v(x)$, $x \neq 0$, $y \neq 0$:

$$\lambda(x+y\alpha) = \lambda(y(\frac{x}{y} + \alpha)) = \mu(y)\beta \in L \cup \infty, \text{ since } \mu(\frac{x}{y}) = 0.$$

3. $x = 0$ or $y = 0$: trivial.

Lemma 9.2. Let $\gamma: K \to L \cup \infty$ be a place. Let $X_1, \ldots, X_r$ be indeterminates over K and $U_1, \ldots, U_r$ be indeterminates over L. Then there exists a unique extension

$$\mu: K(X_1, \ldots, X_r) \to L(U_1, \ldots, U_r) \cup \infty$$

of γ with $\mu(X_i) = U_i$, $1 \leq i \leq r$.

Proof. Without loss of generality we may assume $r = 1$ and write $X_1 = X$, $U_1 = U$. Let $\mathfrak{o}$ be the valuation ring associated to γ with the maximal ideal $\mathfrak{m}$. Define

$$\mathfrak{O} = \{\frac{f(X)}{g(X)} \mid f(X) \in \mathfrak{o}[X],\ g(X) \in \mathfrak{o}[X] \setminus \mathfrak{m}[X]\}$$

Then $\mathfrak{O}$ is a valuation ring of $K(X)$ with the maximal ideal

$$\mathfrak{M} = \{\frac{f(X)}{g(X)} \mid f(X) \in \mathfrak{m}[X],\ g(X) \in \mathfrak{o}[x] \setminus \mathfrak{m}[X]\}$$

The ring homomorphism $\mathfrak{o} \to L$ obtained by restriction of γ has a unique extension

$$\alpha: \mathfrak{o}[X] \longrightarrow L[U] \text{ with } \alpha(X) = U.$$

If $g(X) \in \mathfrak{o}[X] \setminus \mathfrak{m}[X]$, then $\alpha(g(x)) \neq 0$. Hence α has a well defined extension

$$\mu: \mathfrak{O} \longrightarrow L(U),\quad \mu(\mathfrak{M}) = 0.$$

This is the place we looked for.

Lemma 9.3. Let $y_1, \ldots, y_r$ be the given elements in K and let $X_1, \ldots, X_r$ be indeterminates. Then there exists a K-place $\lambda: K(X_1, \ldots, X_r) \to K \cup \infty$ such that $\lambda(X_i) = y_i$, $1 \leq i \leq r$.

Proof. For any transcendental field extension E(U)/E with one indeterminate U and any $a \in E^*$ there exists a unique place $\gamma: E(U) \to E \cup \infty$ over E with $\gamma(U) = a$, namely the canonical place corresponding to the prime polynomial U-a of E[U]. Thus for every $i \in \{1, \ldots, r\}$ we have a unique place $\lambda_i: K(X_1, \ldots, X_i) \to K(X_1, \ldots, X_{i-1}) \cup \infty$ over $K(X_1, \ldots, X_{i-1})$ which maps X_i to y_i. (Read K for $K(X_1 \ldots X_{i-1})$ if $i = 1$). Composing all the places λ_i we obtain a K-place $\lambda: K(X_1, \ldots, X_r) \to K \cup \infty$ with $\lambda(X_i) = y_i$.

Theorem 9.4. Let φ be a form over k with dim ≥ 2. Further assume that φ has good reduction with respect to a given place $\gamma: k \to L \cup \infty$. Then $\gamma_*(\varphi)$ is isotropic if and only if γ can be extended to a place $\lambda: k(\varphi) \to L \cup \infty$.

Remark 9.5. If γ is a trivial place, i.e. γ is injective, then $\gamma_*(\varphi) = \varphi \otimes L$. Thus Theorem 9.4. states in particular that $k(\varphi)$ is a generic zero field of φ.

Proof. Without loss of generality we may assume $\varphi \not\cong H$.

$\Leftarrow$: We have a decomposition $\varphi \otimes k(\varphi) \cong H \perp \chi$.
Then $\gamma_*(\varphi) = \lambda_*(\varphi \otimes k(\varphi)) = H \perp \lambda_*(\chi)$ i.e. $\gamma_*(\varphi)$ is isotropic.

$\Rightarrow$: Let $\varphi = \langle a_1, \ldots, a_n\rangle$, $a_i \in k^*$.
Since φ has good reduction with respect to γ, we have after linear change of coordinates $\varphi = \langle a_1, \ldots, a_n\rangle$, $\gamma(a_i) = b_i \neq 0, \infty$, cf Proposition 6.4. Then $\gamma_*(\varphi) = \langle b_1, \ldots, b_n\rangle$.
Since γ_* is isotropic, there exist $y_1, \ldots, y_n$ such that not all of them are zero with $b_1y_1^2 + \ldots + b_ny_n^2 = 0$.
Let $y_n \neq 0$. We have $k(\varphi) = k(x_1, \ldots, x_n)$ with the

relation $a_1x_1{}^2 + \dots + a_n x_n{}^2 = 0$. Thus $k(\varphi) = E(x_n)$, where $E = k(x_y, \dots, x_{n-1})$ and $x_1, \dots, x_{n-1}$ are algebraically independent over k.

Let $U_1, \dots, U_{n-1}$ be indeterminates. By Lemma 9.2. there exists an extension

$$\tilde{\gamma} : E \to L(U_1, \dots, U_{n-1}) \cup \infty$$

of γ with $\tilde{\gamma}(x_i) = U_i$, $1 \leq i \leq n-1$. By Lemma 9.3. there exists an L-place

$$L(U_1, \dots, U_{n-1}) \cup \infty \to L \cup \infty, \; U_i \to y_i, \; 1 \leq i \leq n-1.$$

By composing the two places we get a place μ: $E \to L \cup \infty$ with $\mu(x_i) = y_i$ $1 \leq i \leq n-1$, hence

$$\mu(x_n{}^2) = - b_n^{-1} (b_1y_1{}^2 + \dots + b_{n-1} y_{n-1}^2) = y_n{}^2.$$

Then μ can be extended to a place

$$\lambda : k(\varphi) \to L \cup \infty \quad \text{with } \lambda(x_n) = y_n$$

by Lemma 9.1.

At first glance Theorem 9.4. might appear as some "general abstract nonsense". To illustrate that this is actually not true we mention some results of A. Heuser, obtained in his thesis (Univ. Regensburg 1976). For any form φ of higher degree over k we also have the notion of a generic zero field. Now let $\mathfrak{D}$ be a central division algebra over k of dimension d^2. Let $\varphi(T_1, \dots, T_{d^2})$ be the reduced norm of $\mathfrak{D}$ over k with respect to a basis. This is a form of degree d.

(i) If d is not a prime power then φ has no generic zero field.

(ii) $d = p^\alpha$ is a prime power with $\alpha > 1$ then φ has a generic zero field but $k(\varphi)$ is not a generic zero field

(iii) If $d = p$ then $k(\varphi)$ is a generic zero field.

We conclude this section with some remarks on generic zero fields.

(i) $k(\varphi) \underset{k}{\sim} k(\varphi)_0$. Therefore the theorem holds for $k(\varphi)_0$ instead of $k(\varphi)$. Note that the transcendental degree $tr(k(\varphi)_0/k) = n - 2$ if $\dim \varphi = n$.

(ii) φ is isotropic if and only if $k(\varphi)$ is purely transcendental extension over k.

Proof.

$\Rightarrow$: We have seen this before.

$\Leftarrow$: Since $k(\varphi) \sim k$ and $\varphi \otimes k(\varphi)$ is isotropic, φ is isotropic over k.

(iii) Let φ be an anisotropic form of dimension n. The transcendental degree of a generic zero field of φ can be less than $n - 2$. For example let φ be an anisotropic n-fold Pfister form and let $\varphi \cong \Psi \perp \eta$ with $\dim \Psi \geq 2^{n-1} + 1$. Then for any L over k the following are equivalent:

(a) $\varphi \otimes L$ is isotropic

(b) $\varphi \otimes L$ is hyperbolic

(c) $\Psi \otimes L \sim (-\eta) \otimes L$.

Since $\dim \eta < \dim \Psi$, in this case $\Psi \otimes L$ is isotropic. Thus φ and Ψ become isotropic over the same fields and in particular $k(\Psi)_0$ is a generic zero field of φ. If $\dim \Psi = 2^{n-1} + 1$, $tr(k(\Psi)_0/k) = 2^{n-1} - 1$.

10. Generic splitting towers

Let φ be a form over k. We construct a tower of fields

$$k = K_0 \subset K_1 \subset \dots \subset K_h$$

in the following way. Decompose φ into a hyperbolic and a kernel form

$$\varphi \cong i_0 \times H \perp \varphi_0 .$$

If φ_0 splits (i.e. dim $\varphi_0 \leq 1$), we stop with $K_0 = k$. Otherwise we choose a generic zero field K_1 of φ_0 and decompose

$$\varphi_0 \otimes K_1 \cong i_1 \times H \perp \varphi_1$$

with φ_1 anisotropic. If dim $\varphi_1 \leq 1$ we stop. Otherwise we choose a generic zero field K_2 of φ_1 and decompose

$$\varphi_1 \otimes K_2 \cong i_2 \times H \perp \varphi_2$$

with φ_2 anisotropic and so on. Thus we obtain a tower

$$k = K_0 \subset K_1 \subset \dots \subset K_h ,$$

a system of anisotropic forms φ_r over K_r, and a system of indices i_r such that $\varphi \cong i_0 \times H \perp \varphi_0$, $\varphi_{r-1} \otimes K_r \cong$ $\cong i_r \times H \perp \varphi_r$ $(1 \leq r \leq h)$ and dim $\varphi_h \leq 1$. The tower constructed in this way is called a <u>generic splitting tower</u> of φ.

<u>Theorem 10.1.</u> Let $(K_r : 0 \leq r \leq h)$ be a generic splitting tower of φ with indices i_r and kernel forms φ_r. Let $\gamma: k \to L \cup \infty$ be a place and let $\lambda: K_m \to L \cup \infty$ be an extension of γ for some $m \in \{0, \dots, h\}$, which can not be further extended to K_{m+1} in the case $m < h$. If φ has good reduction with respect to γ, then φ_m has good reduction with respect to λ, $\ker(\gamma_*(\varphi)) = \lambda_*(\varphi_m)$ and $\mathrm{ind}(\gamma_*(\varphi)) =$

$= i_o + \dots + i_m$.

<u>Proof.</u> Since φ has good reduction with respect to γ, $\varphi \otimes K_m$ has good reduction with respect to λ and $\lambda_*(\varphi \otimes K_M) = \gamma_*(\varphi)$.
Thus φ_m has good reduction with respect to λ and $\gamma_*(\varphi) \cong (i_o + \dots + i_m) \times H \perp \lambda_*(\varphi_m)$. If $\lambda_*(\varphi_m)$ were isotropic, λ could be extended to K_{m+1} by Theorem 9.4. Hence $\lambda_*(\varphi_m)$ is anisotropic.

In particular we obtain for a field extension L/k, i.e. a trivial place γ: $k \to L$, the following:

There is a unique $m \in [0, h]$(*) with $\text{ind}(\varphi \otimes L) = i_o + \dots\dots + i_m$ and $\ker(\varphi \otimes L) = \varphi_m$, namely m is the maximal number such that there exists a place λ: $K_m \to L \cup \infty$ over k.

1) Thus the possible indices of the form $\varphi \otimes L$ for varying L are precisely

$$i_o, \; i_o + i_1, \; \dots, \; i_o + \dots + i_h \,.$$

In particular h and all i_r are uniquely determined by φ $(0 \leq r \leq h)$. We call h the <u>height</u> $h(\varphi)$ of φ and i_r the <u>r-th index</u> $i_r(\varphi)$ <u>of</u> φ.

2) If $(K'_r \;:\; 0 \leq r \leq h)$ is another generic splitting field field of φ, then $K_m \sim K'_m$ over k. For any place λ: $K_m \to K'_m \cup \infty$ over k the form φ_m has good reduction with respect to λ and $\lambda_*(\varphi_m) = \varphi'_m$, where $\varphi'_m = \ker(\varphi \otimes K'_m)$. In this sense φ_m does not depend on the choice of a generic splitting tower of φ and we call φ_m the <u>m-th kernel form</u> of φ. We note that

$$\varphi_h = \begin{cases} 0 & \text{if} \quad \dim \varphi = \text{even} \\ d(\varphi) & \text{if} \quad \dim \varphi = \text{odd.} \end{cases}$$

(*) We here denote by $[0, h]$ the set $\{0, 1, \dots, h\}$.

3) For any s with $0 \leq s \leq [\frac{\dim \varphi}{2}]$ there exists a minimal r with $0 \leq r \leq h$ and
$i_o + \ldots + i_r \geq s$. The field K_r is a generic field for splitting off s hyperbolic planes from φ.

Theorem 10.1. contains the following information about the behaviour of these invariants under specialization:

<u>Corollary 10.2.</u> Let φ be a form over k with good reduction with respect to a given place $\lambda: k \to k' \cup \infty$. Let
$(K_r: 0 \leq r \leq h)$ and $(K'_s: 0 \leq s \leq h')$ be generic splitting towers of φ and $\varphi' = \lambda_*(\varphi)$ respectively. For any
$s \in [0, h']$ let $r(s)$ denote the maximal number r in $[0, h]$ such that λ can be extended to a place from K_r to K'_s.
Then $h' \leq h$ and $0 \leq r(0) < \ldots. < r(h') = h$.
Moreover $i_o(\varphi') = i_o(\varphi) + \ldots + i_{r(0)}(\varphi)$ and
$i_s(\varphi') = i_{r(s-1)+1}(\varphi) + \ldots + i_{r(s)}(\varphi)$, $1 \leq s \leq h'$.
For any place $\mu: K_{r(s)} \to K'_s$ extended from λ,
$\mu_*(\ker(\varphi \otimes K_{r(s)}) \cong \ker(\varphi' \otimes K'_s)$.

<u>Proof.</u> Apply the above theorem to the places
$k \xrightarrow{\lambda} k' \cup \infty \hookrightarrow K'_s \cup \infty$.

<u>Excercise.</u> Determine the height and indices of a "generic form": Let $U_1, \ldots, U_n$ be indeterminates over a field k.
Consider the form $\varphi = \langle U_1, \ldots, U_n \rangle$ over $K = k(U_1, \ldots, U_n)$.
Then show that $h(\varphi) = [\frac{n}{2}]$, $i_o(\varphi) = 0$, $i_1(\varphi) = \ldots = i_h(\varphi) =$
$= 1$.

<u>Solution.</u> Clearly φ is anisotropic. We want to construct a tower

$$L_o = K \subset L_1 \subset \ldots \subset L_m, \quad m = h(\varphi)$$

with $\operatorname{ind}(\varphi \otimes L_r) = r$ for $1 \leq r \leq m$. Then we are done.
Consider the field

$$k_r := k(U_n, \sqrt{-U_n U_{n-1}}, \ldots, U_{n-2r+2}, \sqrt{-U_{n-2r+2} U_{n-2r+1}})$$

in the algebraic closure $\bar{K}$. Choose L_r as the purely transcendental extension $k_r(U_1, \ldots, U_{n-2r})$ of k_r.
Then $\varphi \otimes L_r \sim \langle U_1, \ldots, U_{n-2r} \rangle$ and $\mathrm{ind}(\varphi \otimes L_r) = r$ as required.

11. The leading form.

In this section we determine all the forms of height 1. First we recall that any Pfister form τ has a decomposition $\langle 1 \rangle \perp \tau'$. We call τ' the pure part of τ.

Theorem 11.1. Let φ be anisotropic.
(a) If $\dim \varphi$ is even then $h(\varphi) = 1$ if and only if $\varphi \cong a\tau$ for some Pfister form τ of dimension at least 2 and some a in k^*.
(b) If $\dim \varphi$ is odd then $h(\varphi) = 1$ if and only if $\varphi \cong a\tau'$ with τ' the pure part of a Pfister form τ of dimension at least 4 and some a in k^*.

Proof.
(a) If φ represents a then $1 \in D(a\varphi)$. Let $\tau = a\varphi$.
If $h(\varphi) = 1$ then $\tau \otimes k(\tau) \sim 0$. By Corollary 7.3 of the norm theorem τ is a Pfister form. The converse is clear.
(b) Assume $\varphi \cong a\tau'$, τ' the pure part of a Pfister form τ, $\dim \tau \geq 4$. Then $h(\varphi) = 1$, since $k(\tau') \sim k(\tau)$ (cf §10) and $\tau' \otimes k(\tau) \sim \langle -1 \rangle \otimes k(\tau)$.

To prove the converse we may assume $d(\varphi) = 1$, replacing φ by $d(\varphi)\varphi$, if necessary. Consider the form $\tau := \langle 1 \rangle \perp \langle -\varphi \rangle$.

We claim τ is anisotropic. To prove the claim, assume that τ is isotropic. Then $\varphi \cong \langle 1 \rangle \perp \eta$ for some η with dim $\eta \geq 2$. Since $h(\varphi) = 1$ and $d(\varphi) = 1$, $\varphi \otimes k(\varphi) \sim \langle 1\rangle$. Hence $\eta \otimes k(\varphi) \sim 0$. Let $a \in D(\eta)$. Then $a\varphi < \eta$ by Corollary 7.4. But this is impossible since $\dim \eta < \dim \varphi$.

Now to prove τ is a Pfister form, it is sufficient to show that $\tau \otimes k(\tau) \sim 0$. Assume that $\varphi \otimes k(\tau)$ is anisotropic. Then $h(\varphi \otimes k(\tau)) = 1$, since $h(\varphi \otimes k(\tau)) \leq h(\varphi) = 1$. Then $\langle 1\rangle \perp (-\varphi \otimes k(\tau)) = \tau \otimes k(\tau)$ is anisotropic by applying the previous argument to $\varphi \otimes k(\tau)$ instead of φ. But this is impossible and thus $\varphi \otimes k(\tau)$ is isotropic. By hypothesis $\varphi \otimes k(\tau) \sim \langle 1\rangle$ and $\tau \otimes k(\tau) \sim 0$ follows.

<u>Definition 11.2.</u> Let φ be a form over k which does not split. Let $(K_i \mid 0 \leq i \leq h)$ be a generic splitting tower of φ with $\varphi_i = \ker(\varphi \otimes K_i)$. Then $h(\varphi_{h-1}) = 1$. By the above theorem $\varphi_{h-1} \cong a\tau$ or $\varphi_{h-1} \cong a\tau'$, where τ is a Pfister form and τ' is the pure part of a Pfister form τ over K_{h-1}. Of course τ is uniquely determined by φ over K_{h-1}. We call K_{h-1} a <u>leading field</u> of φ and τ the <u>leading form</u> of φ over K_{h-1}.

Let φ be a form over k of dimension n. Let $c(\varphi)$ denote the Clifford invariant of φ, which is defined as follows: If n is even, $c(\varphi)$ is the class $[C(\varphi)]$ of the Clifford algebra $C(\varphi)$ in the Brauer group $Br(k)$. If n is odd $c(\varphi)$ is the class $[C^+(\varphi)]$ of the subalgebra of elements of even degree in $C(\varphi)$. For a more detailed discussion of $c(\varphi)$ see Lam [7] Chap. V §3.

The leading form has the following connection with the determinant $d(\varphi)$ and the Clifford invariant $c(\varphi)$.

<u>Theorem 11.3.</u> Let φ be a form over k which does not split. Let τ be the leading form of φ over a leading field K_{h-1} of φ.

(1) If dim φ is even and $d(\varphi) \neq 1$, then $\tau \cong \langle 1, -d(\varphi)\rangle$ over K_{h-1}.

(2) If dim φ is even, $d(\varphi) = 1$ and $c(\varphi) \neq 1$, then $\dim \tau = 4$.

(3) If dim φ is even, $d(\varphi) = c(\varphi) = 1$, then dim $\tau \geq 8$.
(4) If dim φ is odd and $c(\varphi) \neq 1$, then dim $\tau = 4$.
(5) If dim φ is odd and $c(\varphi) = 1$, then dim $\tau \geq 8$.

Proof. We prove (1),(2) and (3). The rest can be proved analogously.

(1) Let $r \in [o, h]$ be maximal such that $d(\varphi) \otimes K_r \neq \langle 1 \rangle$. Then $r \leq h-1$. If dim $\varphi_r > 2$, K_r is algebraically closed in $K_r(\varphi_r)$. Hence $r = h-1$ and dim $\varphi_{h-1} = 2$.
Let $d(\varphi) = \langle d \rangle$. Since φ_{h-1} has again discriminant $\langle d \rangle$, clearly $\tau = \langle 1,-d \rangle$.

(2) Since $d(\varphi_{h-1}) = d(\tau) = 1$, dim $\tau \geq 4$. Let $r \in [0, h]$ be maximal such that $[\mathfrak{D}] = c(\varphi) \otimes K_r \neq 1$, where $\mathfrak{D}$ is a central division algebra over K_r. $\mathfrak{D}$ splits by $K_r(\varphi_r)$. But $K_r(\varphi_r)$ has the form $E(\sqrt{g})$, where E is a purely transcendental extension of K_r and $g \in E$. The division algebra $\mathfrak{D} \otimes E$ splits by $E(\sqrt{g})$. Thus $\mathfrak{D} \otimes E$ has dimension 4, and $\mathfrak{D}$ has dimension 4 over K_r. Let σ be the norm form of $\mathfrak{D}$. Then σ is an anisotropic Pfister form and $\sigma \otimes K_r(\varphi_r) \sim 0$. By Corollary 7.4. $a\varphi_r < \sigma$ for some $a \in D(\varphi_r)$. In particular dim $\varphi_r \leq 4$. Since dim $\varphi_{h-1} \geq 4$, $r = h-1$, dim $\varphi_{h-1} = 4$ and $\sigma = \tau$.

(3) Let $d(\varphi) = c(\varphi) = 1$ and let $\varphi_{h-1} \cong a\tau$. If dim $\tau = 2$, $d(\varphi_{h-1}) = d(\tau) \neq 1$. This contradicts the assumption $d(\varphi) = 1$.

If dim $\tau = 4$, $c(\varphi_{h-1}) = [Q]$ with Q the quaternion algebra over K_{h-1} corresponding to τ. Since τ is anisotropic the algebra Q has no zero-divisors, hence $c(\varphi_{h-1}) \neq 1$. This contradicts the assumption that $c(\varphi) = 1$. Thus dim $\tau \geq 8$.

12. The degree of a quadratic form.

Let φ be a form over k which is not hyperbolic. Let L run through all field extensions of k in a universal domain such that $\varphi \otimes L$ is not hyperbolic. Let τ be the leading form of φ. Then

$$\min\{\dim(\ker(\varphi \otimes L)) \mid \varphi \otimes L \not\sim 0\} = \begin{cases} 1 & \text{if } \dim \varphi \text{ is odd} \\ \dim \tau & \text{if } \dim \varphi \text{ is even.} \end{cases}$$

This is an immediate consequence of our theory in §10 and § 11. Hence the minimum of the dimensions of the kernel forms of these $\varphi \otimes L$ is a 2-power 2^d ($d = 0$ if and only if $\dim \varphi$ is odd). We call d the <u>degree</u> $\deg(\varphi)$ of φ. If $\varphi \sim 0$ we put $\deg(\varphi) = \infty$. If $\dim \varphi$ is even then $\deg(\varphi)$ is the number of binary factors in the leading form τ of φ.

Since the degree of a form depends only on the Witt class, we have a well-defined map

$$\deg : W(k) \longrightarrow \mathbb{N} \cup \infty.$$

For each $n \geq 0$ let $J_n(k)$ denote the set of all $[\varphi]$ in $W(k)$ with $\deg(\varphi) \geq n$. Clearly $J_1(k) = \{[\varphi] \in W(k) \mid \dim \varphi \text{ is even}\}$. Using the end of §11 we also see that

$$J_2(k) = \{[\varphi] \in J_1(k) \mid d(\varphi) = 1\}$$

$$J_3(k) = \{[\varphi] \in J_2(k) \mid c(\varphi) = 1\}$$

We want to prove that $J_n(k)$ is an ideal of $W(k)$ for every $n \geq 0$. For this we need the following lemma.

<u>Lemma 12.1.</u> Let τ be an anisotropic n-fold Pfister form over k with $n \geq 1$. Let ψ be a form over k with $\deg(\psi) \geq n + 1$. Let a be an element in k^*. Then

$$\deg(a\tau \perp \psi) = n.$$

Proof. If $\Psi \sim 0$, there is nothing to prove. Assume now that Ψ is not hyperbolic and let $(L_i \mid 0 \leq i \leq l)$ be a generic splitting tower of Ψ. Then we claim that $\tau \otimes L_l$ is still anisotropic. To prove the claim, assume that $\tau \otimes L_l \sim 0$. Let $r \in [0, l-1]$ be maximal with $\tau \otimes L_r \not\sim 0$ and hence $a\Psi_r < \tau \otimes L_r$ by Corollary 7.4. But this contradicts the assumption that $\deg(\Psi) > n$. Thus $\tau \otimes L_l$ is indeed anisotropic.

Now let $\varphi := a\tau \perp \Psi$. Then $\varphi \otimes L_l \sim a\tau \otimes L_l$, since $\Psi \otimes L_l \sim 0$. Hence $\deg(\varphi) = m \leq n$.

Assume $m < n$ and let $(K_j \mid 0 \leq j \leq h)$ be a generic splitting tower of φ. Let ρ be the leading form of φ, hence $\ker(\varphi \otimes K_{h-1}) = b\rho$ with $\deg(\rho) = m$, $b \in K^*_{h-1}$. Then

$$\Psi \otimes K_{h-1} \sim b\rho \perp (-a)(\tau \otimes K_{h-1}) .$$

But the right hand side has dimension $2^m + 2^n < 2^{n+1}$. Thus $\Psi \otimes K_{h-1} \sim 0$ and we obtain $b\rho \sim a(\tau \otimes K_{h-1})$.

Since $\dim \rho < \dim \tau$, we have $\tau \otimes K_{h-1} \sim 0$ and hence also $\rho \sim 0$, which is a contradiction. This proves that $\deg(\varphi) = n$.

Theorem 12.2. $J_n(k)$ is an ideal of $W(k)$ for every $n \geq 0$.

Proof. We shall first prove that all $J_n(k)$ are closed under addition, i.e. $\deg(\varphi_1 \perp \varphi_2) \geq \min(\deg(\varphi_1), \deg(\varphi_2))$ for any φ_1 and φ_2 over k. We exclude the following trivial cases:

φ_1 or φ_2 has odd dimension,
$\varphi_1 \sim 0$ or $\varphi_2 \sim 0$,
$\varphi_1 \perp \varphi_2 \sim 0$.

Let $n \geq 1$ be the degree of $\varphi_1 \perp \varphi_2$. We choose a field extension L of k such that $\ker(\varphi_1 \otimes L \perp \varphi_2 \otimes L) = a\rho$ with ρ a Pfister form of degree n and $a \in L^*$.

Then

$$\deg(\varphi_i \otimes L) \geq \deg(\varphi_i) \quad (i = 1,2)$$

by definition of the degree function. Thus it suffices to prove the assertion for the forms $\tilde{\varphi}_i = \varphi_i \otimes \tilde{L}$ instead of φ_i. If $\deg(\tilde{\varphi}_2) > n$ then we obtain $\deg \tilde{\varphi}_1 = n$ from the fact that $\tilde{\varphi}_1 \sim a\rho \perp (-\tilde{\varphi}_2)$ using Lemma 12.1. Thus in any case $\min(\deg(\tilde{\varphi}_1), \deg(\tilde{\varphi}_2)) \leq n$.

Since $\deg(a\varphi) = \deg(\varphi)$, $\langle a\rangle J_n(k) = J_n(k)$. Now $W(k)$ is is additively generated by 1-dimensional forms $\langle a\rangle$. Thus $J_n(k)$ is stable under the multiplication by elements of $W(k)$. This shows that $J_n(k)$ is an ideal in $W(k)$.

As an immediate consequence of the theorem we have the following corollary:

<u>Corollary 12.3.</u> $\deg(\varphi_1 \perp \varphi_2) = \min(\deg(\varphi_1), \deg(\varphi_2))$ if $\deg \varphi_1 \neq \deg \varphi_2$.

Let $I(k)$ be the fundamental ideal of $W(k)$, consisting of the Witt classes $[\varphi]$ over k with dim φ even. $I(k)$ is additively generated by 1-fold Pfister forms. Thus the n-th power $I^n(k)$ of $I(k)$ is additively generated by n-fold Pfister forms and hence the theorem yields another corollary:

<u>Corollary 12.4.</u> $I^n(k) \subset J_n(k)$ for all n.

In [2] Arason and Pfister obtained the following:

<u>Hauptsatz.</u> If φ is an anisotropic form $\neq 0$ over k with $[\varphi] \in I^n(k)$ then dim $\varphi \geq 2^n$.

Notice that our Corollary 12.4. is just another way to state this Hauptsatz. It is clear from the definition of the ideals $J_n(k)$ that $\bigcap_{n=1}^{\infty} J_n(k) = 0$. Thus also $\bigcap_{n=1}^{\infty} I^n(k) = 0$, a highly nontrivial fact, since $W(k)$ is usually not noetherian.

<u>Question.</u> Is $I^n(k) = J_n(k)$ for every n?

It is evident that this holds true for $n \leq 1$, and it is not difficult to prove $I^n(k) = J_n(k)$ for $n = 2$ (Pfister [9]). Until now no counter example has been found, but nevertheless there is not much evidence for an affirmative answer in general. We mention some of the known results.

(1) If $\varphi \in J_3$ with $\dim \varphi \leq 12$ then $\varphi \in I^3$ (Pfister [9]).

(2) Let F be a function field over a real closed field R.
If $tr(F/R) \leq 3$, $J_n(F) = I^n(F)$ for all n.
If $tr(F/R) = d \geq 4$, $J_n(F) = I^n(F)$ for all $n \geq d + 1$

(3) $J_n(k) = I^n(k)$ for all n if k is a global or local field in the sense of algebraic number theory.

(4) If $J_3(k) = I^3(k)$ then $J_3(k(X_1, \ldots, X_n)) = I^3(k(X_1, \ldots, X_n))$ for any set $X_1, \ldots, X_n$ of indeterminates.

The statements (2), (3), (4) are proved in Arason - Knebusch [1], a paper which moreover contains a detailed study of the behaviour of the degree function, nevertheless leaving many questions open.

Theorem 12.5. For arbitrary $m > 0$ and $n > 0$, and for any field k,

$$I^m(k)\, J_n(k) \subset J_{m+n}(k).$$

Proof. It suffices to prove this for $m = 1$. For a form φ over k which is not hyperbolic, we have to show that

$$(*) \quad \deg(\langle 1,-a\rangle \otimes \varphi) > \deg \varphi, \qquad a \in K^*.$$

If φ has odd dimension then this is trivial. Now we assume that $\dim \varphi$ is even and we proceed by induction on $h(\varphi)$. If $h(\varphi) = 1$ the assertion (*) is clear. Assume $h(\varphi) > 1$. Without loss of generality further assume that $\alpha := \langle 1,-a\rangle \otimes \varphi$ does not split. Let F be a leading field of α and ρ be the leading form of α over F. Since $\deg(\varphi \otimes F) \geq \deg(\varphi)$, it suffices to prove that $\deg(\rho) > \deg(\varphi \otimes F)$. Since $h(\varphi \otimes F) \leq h(\varphi)$, we may replace k by F and φ by the form $\varphi \otimes F$. Thus we may assume from the beginning that $\langle 1,-a\rangle \otimes \varphi \sim \rho$ with an anisotropic Pfister form ρ.

Assume first that $\rho \otimes k(\varphi)$ is anisotropic. Since $(\langle 1,-a\rangle \otimes \varphi) \otimes k(\varphi) \sim \rho \otimes k(\varphi)$ and $h(\varphi \otimes k(\varphi)) \leq h(\varphi) - 1$, by induction hypothesis $\deg(\rho \otimes k(\varphi)) > \deg(\varphi \otimes k(\varphi))$. But $\deg(\rho) = \deg(\rho \otimes k(\varphi))$ and $\deg(\varphi) = \deg(\varphi \otimes k(\varphi))$.

Now assume that $\rho \otimes k(\varphi)$ splits. Then $\rho \cong b\varphi \perp \eta$ over k by Corollary 7.4. It follows then $2^{\deg(\varphi)} < \dim \varphi \leq 2^{\deg(\rho)}$ and hence $\deg \varphi < \deg \rho$.

One may ask: Is $J_m(k)\, J_n(k) \subset J_{m+n}(k)$?

<u>Corollary 12.6.</u> Let φ be an odd dimensional form and ψ be an arbitrary form over k. Then $\deg(\varphi \otimes \psi) = \deg(\psi)$.

<u>Proof.</u> Since $\varphi \sim \langle 1\rangle \perp (\varphi \perp \langle -1\rangle)$, we have

$$\varphi \otimes \psi \sim \psi \perp (\varphi \perp \langle -1\rangle) \otimes \psi .$$

By the theorem $\deg((\varphi \perp \langle -1\rangle) \otimes \psi) > \deg \psi$ and the conclusion follows from Corollary 12.3.

It can be shown by elementary methods, i.e. methods not involving transcendental field extensions over k, that any form φ of odd dimension over k yields a non zero divisor $[\varphi]$ of W(k), cf. Pfister [9] or Lam [7]. Corollary 12.6. is a deepening of this fact.

13. Subforms of Pfister forms.

Definition 13.1. A form φ over k is called a Pfister neighbour if there exists a Pfister form ρ such that $\varphi < a\rho$ with $\dim \varphi > \frac{1}{2} \dim \rho$ for some $a \in k^*$. Let φ be a Pfister neighbour, $\varphi \perp \eta \cong a\rho$ as above. Then $k(\varphi) \sim k(\rho)$ over k as has been observed at the end of § 9. We claim that ρ is uniquely determined by φ. Let σ be another Pfister form with $\varphi < b\sigma$ and $\dim \varphi > \frac{1}{2} \dim \sigma$. Then $\sigma \otimes k(\rho) \sim 0$ and $\rho \otimes k(\sigma) \sim 0$ i.e. $\rho < \sigma$ and $\sigma < \rho$. (Notice that ρ and σ both represent 1.) Hence $\sigma \cong \rho$. If b is any element of $D(\varphi)$ then $a = bc$ with $c \in D(\rho)$. Thus $b\rho \cong a\rho$. We see that the form $a\rho$ and hence also η is uniquely determined by φ. η is called the complementary form of φ and φ is called a neighbour of ρ.

Without proof we quote the following rather astonishing theorem due to Robert Fitzgerald [3].

Theorem 13.2. Let φ and η be anisotropic over k with $\varphi \not\cong -\eta$ and $\dim \eta < 2 \dim \varphi$. We further assume $\varphi \otimes k(\varphi) \sim (-\eta) \otimes k(\varphi)$. Then $\varphi \perp \eta \cong a\sigma$ with σ an anisotropic Pfister form and $a \in k^*$.

Remark 13.3. Actually Fitzgerald has an even stronger theorem, replacing the assumption $\dim \eta < 2 \dim \varphi$ by $\dim \eta < \dim \varphi + 2^{\deg q}$ with $q := \varphi \perp \eta$. That this theorem is indeed stronger than Theorem 13.2. can be seen as follows: We have $q \otimes k(\varphi) \sim 0$. Let F be a leading field of q with the leading form τ. Then $q \otimes F \sim a\tau$ with $a \in F^*$. The field $F(\varphi \otimes F)$ is a composite $k(\varphi) \cdot F$ and thus contains $k(\varphi)$. Therefore

$$0 \sim q \otimes F(\varphi \otimes F) \sim a\tau \otimes F(\varphi \otimes F).$$

This implies $b(\varphi \otimes F) < \tau$ for some $b \in F^*$, and we see that $\dim \varphi \leq 2^{\deg q}$.

Corollary 13.4. Let q be an anisotropic form with $1 \in D(q)$.

Let φ be a form over k with dim $\varphi > \frac{1}{3}$ dim q and assume that $\varphi(t) \in G(q \otimes k(t))$, with a set of n = dim φ indeterminates $t = (t_1, \ldots, t_n)$. Then q is a Pfister form.

<u>Proof.</u> By the norm theorem $q \otimes k(\varphi) \sim 0$. Choose $a \in D(\varphi)$. Then $q \cong a\varphi \perp a\eta$ by Corollary 7.4. Let

$$\rho: = aq \cong \varphi \perp \eta .$$

Then $\varphi \nleq -\eta$, since q is anisotropic. Since $\varphi \otimes k(\varphi) \sim -\eta \otimes k(\varphi)$, ρ is a Pfister form by the theorem.
Since $1 \in D(q) = a\, D(\rho)$ it follows that $\rho \cong q$.

<u>Example.</u> The restriction dim $\varphi > \frac{1}{3}$ dim q is the best possible.

Consider $q: = \langle 1, a, b\rangle \otimes \tau$ with an anisotropic Pfister form τ. Further assume that q is anisotropic. Then dim $\tau = \frac{1}{3}$ dim q and $\tau(t) \in G(q \otimes k(t))$. But q is not a Pfister form.

<u>Definition 13.4.</u> An anisotropic form φ over k is called <u>excellent</u> if for every field extension L/k there exists a form ψ over k such that $\psi \otimes L \cong \ker(\varphi \otimes L)$.

<u>Theorem 13.5.</u> The following are equivalent:

(i) φ is excellent

(ii) There exists a sequence of anisotropic forms $\varphi = \eta_0, \eta_1, \ldots, \eta_t$ $(t \geq 0)$ over k such that dim $\eta_t \leq 1$ and $\eta_i (0 \leq i < t)$ is a Pfister neighbour with complementary form η_{i+1}.

Moreover then the following holds true:

(a) $h(\varphi) = t$

(b) If $\eta_i \perp \eta_{i+1} \cong a_i \sigma_i$, with σ_i a Pfister form then $(K_i : 0 \leq i \leq t)$, with $K_0 = k$, $K_i = k(\sigma_0, \ldots, \sigma_{i-1})$ $(1 \leq i \leq t)$, is a generic splitting tower of φ and $\ker(\varphi \otimes K_i) \cong (-1)^i\, \eta_i \otimes K_i$.

<u>Proof.</u> First we show that (a) and (b) hold assuming (ii). We proceed by induction on t. If $t \leq 1$ the assertion is trivial. Assume $t > 1$.

Since $k(\sigma_0) \underset{k}{\sim} k(\varphi)$, $k(\sigma_0)$ is a generic zero field of φ. If $\eta_1 \otimes k(\varphi)$ were isotropic, $\varphi \otimes k(\varphi) \sim (-\eta_1) \otimes k(\varphi) \sim \eta_2 \otimes k(\varphi)$. We would obtain from Theorem 13.2. that φ is a Pfister neighbour and both η_1 and η_2 are complementary forms of φ. This is impossible, since the complementary form is uniquely determined by φ.

The sequence

$$\eta_1 \otimes K_1,\ \eta_2 \otimes K_2,\ \dots\ \eta_t \otimes K_1 \quad \text{with } K_1 = k(\sigma_0)$$

is as in (ii) and has length t-1. Applying the induction hypothesis to $\eta_1 \otimes K_1$ we obtain $h(\varphi) = t$, a generic splitting tower $(K_i : 0 \leq i \leq t)$ of φ with $K_i = k(\sigma_0, \dots, \sigma_{i-1})$ and $\ker(\varphi \otimes K_i) \cong (-1)^i \eta_i \otimes K_i$. It is now clear that φ is excellent.

The proof of the implication (i) $\Rightarrow$ (ii) is not hard with Fitzgerald's theorem 13.2. Actually much weaker results(essentially also necessary to prove Fitzgerald's theorem) suffice to establish this implication, cf. Knebusch [5], part II §7.

One of the difficulties of the algebraic theory of quadratic forms over fields is that the dimensions of the "building stones" of this theory, the Pfister forms, are 2-powers and that the difference between consecutive 2-powers grows exponentially. Excellent forms exist in any dimension if k is formally real.
(Example: $\varphi = n \times \langle 1 \rangle$, $n \in \mathbb{N}$. Exercise: Prove that these φ are indeed excellent.) They are closely related to Pfister forms and have a well understood splitting behaviour.*)

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*) The remarks on excellent forms concluding this lecture had been omitted in the oral version in September 1979 for lack of time.

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